ISTANBUL TECHNICAL UNIVERSITY ★ GRADUATE SCHOOL OF SCIENCE ENGINEERING AND TECHNOLOGY

SPECTRAL PROPERTIES OF ASYMPTOTICALLY COMPACT OPERATOR SEQUENCES

M.Sc. THESIS

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Department of Mathematics

Mathematics Engineering Programme

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Thesis Co-advisor: Assis. Prof. Dr. R. Tunç MISIRLIOĞLU

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ASİMPTOTİK KOMPAKT OPERATÖR DİZİLERİNİN SPEKTRAL ÖZELLİKLERİ

YÜKSEK LİSANS TEZİ

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To my parents

PREFACE

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SPECTRAL PROPERTIES OF ASYMPTOTICALLY COMPACT OPERATOR SEQUENCES

SUMMARY

A set of linear operators on a normed linear space is collectively compact if the union of the images of the unit ball has compact closure. A necessary and sufficient condition for a set to be collectively compact established by J.A. Higgins shows that, in a sense, collectively compact sets can be characterized in terms of diagonal sequences. We reprove it in a different and a shorter way without using the diagonalization. A sequence $\{T_n\}$ of bounded linear operators on a Banach space is asymptotically compact if for any bounded sequence $\{x_n\}$ in X, every subsequence of $\{T_n x_n\}$ has a convergent subsequence. Every collectively compact sequence is asymptotically compact but the converse is not true. In this work, we mainlystudy on spectral properties of asymptotically compact operator sequences. The results are closely parallel to the more completely studied case with $\{T_n\}$ collectively compact. We prove that for bounded linear operators T and T_n, n=1,2,..., such that $T_n \to T$ strongly, $\{T_n - T\}$ asymptotically compact, if f(T) is defined by the operational calculus, then $f(T_n)$ is eventually defined, $f(T_n) \to f(T)$ strongly, and$\{f(T_n) - f(T)\}$ is asymptotically compact. Also, we generalize the upper semi-continuity of the spectrum function for asymptotically compact operator sequences under the union of disjoint open sets.

ASİMPTOTİK KOMPAKT OPERATÖR DİZİLERİNİN SPEKTRAL ÖZELLİKLERİ

ÖZET

Bu tez çalışmasında asimptotik kompakt operatör dizilerinin spektral özellikleri araştırılmıştır. Bu tez çalışmasının temel amaçlarından biri, asimptotik kompakt operatör dizileri için spektrum fonksiyonunun, iki ayrık açık kümenin birleşimi altında üst yarı sürekli olduğuna genelleştirilmesidir. Ayrıca tez çalışmasının son bölümünde, sınırlı lineer T ve T_n, n=1,2,…, operatörleri için $T_n \to T$ (noktasal), $\{T_n\text{-}T\}$ asimptotik kompakt**,** f(T) fonksiyonu tanımlı iken $f(T_n)$ fonksiyonu tanımlanmış, $f(T_n) \to f(T)$ noktasal yakınsaklığı gösterilmiş ve $\{f(T_n) - f(T)\}$ nin asimptotik kompakt olduğu ispatlanmıştır. Tez çalışmasının son bölümünde yer alan bu bulgular için gerekli olan tanım ve teoremler ilk bölümlerde verilmiştir.

Tezimiz dört bölümden oluşmaktadır ve ilk bölümde tezimizin temelini oluşturan lineer sınırlı operatörlerin birlikte kompakt kümelerin gelişimi hakkında kısaca bilgi verilmiştir.

Tezimizin ikinci bölümünde, lineer sınırlı operatörlerin birlikte kompakt kümeleri hakkında temel bilgi ve teoremler verilmiştir. Bu bölümün ilk kısmında birlikte kompakt kümelerin tanımı ve temel özellikleri verilmiştir. Normlu lineer uzaylar üzerinde tanımlı lineer operatörlerin kümesine, birim yuvarın bu kümeye ait operatörler altındaki görüntülerinin birleşiminin kapanışı kompakt ise birlikte kompakt küme adı ile tanımlandığı söylenmiştir. Normlu lineer uzaylar üzerinde tanımlı kolektifli kompakt küme kavramı P.M. Anselone ve Moore tarafından f ve g fonksiyonları $0 \leq x \leq 1$ için sürekli, k fonksiyonu $0 \leq x,y \leq 1$ için sürekli iken

$$f(x) - \int_0^1 k(x,y)f(y)dy=g(x)$$

denkleminin yaklaşık çözümleriyle bağlantı kurarak, normlu uzay üzerinde tanımlı birlikte kompakt küme kavramı verilmiştir. C[0,1]üzerinde

$$Kf(x) = \int_0^1 \mathbf{k(x,y)f(y)dy}$$

ile tanımlı K operatörü, C[0,1] üzerinde $\{K_n\}$ operatörlerinin tanımlanmasıyla yukarıda verilen integral denkleminin yaklaşık çözümü $(I - K_n)f_n=g$ formundaki eşitliklerin çözümüyle sağlanabilir.

Çalışmalarının kaynağını bu tür problemlerin oluşturduğu Anselone ve Moore, bir Banach uzayı üzerinde $K_n \rightarrow K$ (noktasal) ve $\{K_n\}$birlikte kompakt olan operatörler için

$$(I - K)x = y \text{ ve } (I - K_n)x_n = y \text{ , } n=1,2,...$$

ile lineer operatör denklemlerini çalışmışlardır. Birlikte kompakt olma durumu, sürekli bir çekirdeğe sahip olan integral denklemlerinin nümerik integrasyon yaklaşımlarıyla verilmiştir. Birim yuvardan alınan her $\{x_n\}$ dizisi için, $\{K_n x_n\}$ d-kompakt, yani $\{K_n x_n\}$ in her alt dizisinin yakınsak bir alt dizisi varsa,$\{K_n\}$dizisinin asimptotik kompakt olarak adlandırıldığı söylenmiştir. Bu durumda zayıf tekil çekirdekler asimptotik kompakt operatör yaklaşımları sonucunu doğurmuştur.

Birlikte kompakt bir kümenin asimptotik kompakt olduğu fakat tersinin genel olarak doğru olmadığı bir örnekle belirtilmiştir. Buna rağmen, asimptotik kompakt dizideki herbir operatörün kompakt olması bu dizinin birlikte kompakt olmasını gerektirir. Birlikte kompakt olma durumundaki yapılan çalışmaların birçoğu asimptotik kompakt olma durumuna genişletilmiştir.

İkinci bölümde, öncelikle Banach uzay özelliklerinin verilmesiyle birlikte kompakt kümelerin bazı genel özellikleri üzerinde çalışılmıştır. J.A . Higgins' in bir kümenin birlikte kompakt olması için vermiş olduğu gerekli ve yeterli koşul gösterilmiştir. Bu koşul, bir anlamda birlikte kompakt dizilerin köşegen diziler şeklinde karakterize edilebileceğini göstermektedir. Bu sonuç, bizim tarafımızdan köşegenleştirmeyi kullanmadan çok daha kısa ve farklı bir yolla tekrar ispat edilmiştir. Asimptotik ve birlikte kompakt operatör dizilerinin spektral analizi üzerinde çalıştığımız için, bu bölümün son kısmında spektrum, spektrumun sürekliliği ve fonksiyonel hesaplamaların temel kavramları verilmiştir.

Üçüncü bölümde, birlikte kompakt olan $\{T_n\}$ dizisi için, T_n operatörünün T operatörüne noktasal yakınsaması durumunda, T_n operatörünün T operatörüne birlikte kompakt olarak yakınsadığı söylenmiştir. Bu durumda, birlikte kompakt T_n operatörünün yakınsadığı T operatörü de kompakt olmaktadır. Ayrıca bu bölümde, birlikte kompakt operatör dizilerinin spektral özellikleri detaylı olarak araştırılmıştır. Sınırlı lineer operatörler T ve T_n, n=1,2,..., için, $T_n \to T$ (noktasal) ve $\{T_n\text{-}T\}$birlikte kompakt iken, $\| T_n - T \| \to 0$ olduğu durumda var olan spektrumun üst yarı sürekliliği teorisi oldukca benzerlik göstermektedir. Aslında T_n nin spektrumunun, T nin spektrumunun herhangi bir komşuluğunda olduğu gösterilmiştir. Üstelik, f(T) işlevsel hesaplamalar kullanılarak tanımlanırsa $f(T_n)$ nin de tanımlanabileceği, $f(T_n) \to f(T)$(noktasal) ve $\{f(T_n) - f(T)\}$ nin birlikte kompakt olduğu gösterilmiştir.

Tezimizin son bölümü olan dördüncü bölümde, asimptotik kompakt operatör dizilerinin temel tanım ve özelliklerine detaylı olarak yer verilmiştir. Ayrıca bu bölümde, asimptotik kompakt dizilerin spektral analizi üzerinde çalışılmış ve yeni sonuçlar elde edilmiştir. P.M. Anselone ve M.L. Treuden'in makalesi öncelikli olarak temel alınmasına rağmen, bu bölümde sonuçlara birkaçı daha eklenmiştir. Bu bölümün ilk üç kısmında, asimptotik kompakt operatör dizileri ve onların bazı özellikleri, kompakt olmama ölçüsü ve yoğunlaştırıcı operatörlerin yardımıyla çalışılmıştır. Ek olarak bu bölümde, asimptotik kompakt operatör yakınsaklığı kavarmı verilmiş ve bu yakınsaklığın son kısımda gösterilmesi için gerekli olan temel teorem ve özellikler verilmiştir. Asimptotik kompakt olan $\{T_n\}$ dizisi için, T_n operatörünün T operatörüne noktasal yakınsaması durumunda, T_n operatörünün T operatörüne asimptotik kompakt olarak yakınsadığı söylenmiştir. Bu durumda, asimptotik kompakt T_n operatörünün yakınsadığı T operatörü de kompakt olmaktadır. Bu bölümde, T ve T_n nin, $n \to \infty$ iken spektrumları karşılaştırılmıştır. Bu bölümdeki sonuçların, birliktekompakt kümeler için varolan durumlarla oldukça benzerlik gösterdiği belirtilmiştir. Sınırlı lineer operatörler T ve T_n, n=1,2,..., için, $T_n \to T$ noktasal ve $\{T_n\text{-}T\}$ asimptotik kompakt iken ve, f(T) işlevsel hesaplamalar kullanılarak tanımlanıyorsa, $f(T_n)$ nin de tanımlanabileceği, $f(T_n) \to f(T)$ noktasal ve $\{f(T_n) - f(T)\}$ nin asimptotik kompakt olduğu gösterilmiştir. Bu bulgulara ek olarak asimptotik kompakt operatör dizileri için spektrum fonksiyonunun, iki ayrık açık kümenin birleşimi altında üst yarı sürekli olduğuna genelleştirilmiştir.

1. INTRODUCTION

A set of linear operators on a normed linear space is called collectively compact if the union of the images of the unit ball has compact closure. The concept of collectively compact sets on normed linear spaces was introduced by Anselone and Moore in [5] in connection with approximate solutions of the equation

$$f(x) - \int_0^1 k(x,y)f(y)dy = g(x),$$

where f and g are continuous for $0 \leq x \leq 1$ and k is continuous for $0 \leq x, y \leq 1$. If K is the operator on $C[0,1]$ defined by

$$Kf(x) = \int_0^1 k(x,y)f(y)dy,$$

then by defining operators $\{K_n\}$ on $C[0,1]$ in terms of a quadrature formula the problem of obtaining approximate solution to the integral equation above can be transformed into working with equations of the form $(I - K_n)f_n = g$. So motivated by such problems, Anselone and Moore deeply studied linear operator equations

$$(I - K)x = y \quad \text{and} \quad (I - K_n)x_n = y, \; n = 1, 2, \ldots$$

in a Banach space X, where $K_n \to K$ pointwise and $\{K_n\}$ collectively compact. The collectively compact case is typified by numerical integration approximations of integral equations with continuous kernels. If $\{K_n\}$ is asymptotically compact, i.e., for any sequence $\{x_n\}$ in the closed unit ball, $\{K_n x_n\}$ is d-compact, that is, every subsequence of $\{K_n x_n\}$ has a convergent subsequence, then weakly singular kernels lead to asymptotically compact operator approximations. We note that each collectively compact sequence is asymptotically compact but the converse is not true in general; however, if each operator is compact in any asymptotically compact sequence, then this sequence is collectively compact. We extend much of the theory for the collectively compact case to the asymptotically compact case.

In Chapter II, first by giving a Banach space fundamentals, some general properties of collectively compacts are studied. J.A. Higgins established an important necessary and sufficient condition for a set to be collectively compact in [12]. This condition shows that, in a sense, collectively compact sets can be characterized in terms of diagonal sequences. We reprove it in a different and a very shorter way without using the diagonalization. Since we study on the spectral analysis of collectively and asymptotically compact operator sequences, we also give some basic concepts of spectrum, the continuity of the spectrum, and functional calculus in the last part of this chapter.

Spectral properties of collectively compact operator sequences are investigated in Chapter III. For bounded linear operators T and T_n, $n = 1, 2, \ldots$, such that $T_n \to T$ strongly and $\{T_n - T\}$ is collectively compact, the theory of the upper semi-continuity of the spectrum somewhat resembles that for $\| T_n - T \| \to 0$. In fact, the spectrum of T_n is eventually contained in any neighborhood of the spectrum of T. Moreover, if $f(T)$ is defined by the operational calculus, then $f(T_n)$ is eventually defined, $f(T_n) \to f(T)$ strongly, and $\{f(T_n) - f(T)\}$ is collectively compact.

The last chapter, Chapter IV, concerns with the spectral analysis of asymptotically compact sequences. Although it is primarily based on the paper of P.M. Anselone and M.L. Treuden in [7], this chapter provides a few further results. In the first three sections of this chapter, asymptotically compact operator sequences and their elementary properties in general Banach spaces are studied with the aid of measures of non-compactness and condensing operators. By writing $T_n \stackrel{ac}{\to} T$ whenever $\{T_n\}$ is asymptotically compact and $T_n \to T$ strongly-in this case T is compact-, we compare the spectra of T and T_n as $n \to \infty$ in the last section. The results closely parallel the more completely studied case with $\{T_n\}$ collectively compact. Precisely, we prove that for bounded linear operators T and T_n, $n = 1, 2, \ldots$, such that $T_n \to T$ strongly, $\{T_n - T\}$ is asymptotically compact, if $f(T)$ is defined by the operational calculus, then $f(T_n)$ is eventually defined, $f(T_n) \to f(T)$ strongly, and $\{f(T_n) - f(T)\}$ is asymptotically compact. We also generalize the upper semi-continuity of spectrum for asymptotically compact convergence sequence of operators under the union of disjoint open sets.

2. BASIC DEFINITIONS AND CONCEPTS

In this section, we will introduce basic definitions and concepts associated with the collectively compact sets of linear operators.

Now let X be an arbitrary real or complex Banach space, $U_X = \{x \in X, \| x \| \leq 1\}$ the closed unit ball, $B(X)$ the space of linear and bounded operators $T : X \to X$ with the usual operator norm

$$\| T \| = \sup_{x \in X} \| Tx \| . \qquad \textbf{(2.1)}$$

In particular, I denotes the identity operator on X. Convergence in norm of an operator sequence will be expressed by $\| T_n - T \| \to 0$ as $n \to \infty$ through the positive integers. Pointwise convergence (or equivalently strong convergence) will be denoted simply by $T_n \to T$.

Since in this chapter we are heavily concerned with compactness properties, we will give some basic concepts about compactness of sets. Let us begin with the definition of a compact set.

Definition 2.1 A set $B \subset X$ is a compact set if for any open cover of B has a finite subcover.

Definition 2.2 A set $B \subset X$ is sequentially compact if each sequence in B has a convergent subsequence with the limit in X.

Definition 2.3 A set $B \subset X$ is totally bounded if $\forall \epsilon > 0$ $\exists$ a finite set (an ϵ- net) $\{x_1, \ldots, x_k\}$ such that for any $x \in B$, $\min\{\| x - x_i \| : i = 1, \ldots, k\} < \epsilon$.

We shall give the relations between these three concepts defined above. First, in a complete metric space, sequentially compactness and totally boundedness are equivalent. Secondly, compactness is the same as either one or the other concepts if the set B is also closed. And finally, a set B is totally bounded (or sequentially compact) if and only if the closure of B is compact (called precompact). Therefore, in a Banach space X the following three concepts are equivalent for a subset $B \subset X$:

- B is precompact
- B is sequentially compact
- B is totally bounded.

Totally boundedness will be used in the proofs of theorems much more than either of the equivalent concepts.

2.1 Banach Space Fundamentals

The standard theory of approximate solutions is based on the operator norm convergence. Before developing it, we will give three elementary propositions without their proofs.

Proposition 2.1 [3] *Let* $A \in B(X)$ *and* $\| A \| < 1$. *Then there exists* $(I - A)^{-1} \in B(X)$,

$$(I - A)^{-1} = \sum_{m=o}^{\infty} A^m \qquad \textbf{(2.2)}$$

where the (Neumann) series converges in operator norm, and

$$\| (I - A)^{-1} \| \leq \frac{1}{1 - \| A \|}. \qquad \textbf{(2.3)}$$

Proposition 2.2 [3] *Let* $T, B \in B(X)$. *If* $BT = I - A$, $\| A \| < 1$, *then* T^{-1} *exists (as an operator defined on* TX*), and* T^{-1} *is bounded*

$$T^{-1} = (I - A)^{-1} B, \qquad (T^{-1} - B) = (I - A)^{-1} AB, \qquad \textbf{(2.4)}$$

$$\| T^{-1} \| \leq \frac{\| B \|}{1 - \| A \|}, \qquad \| T^{-1} - B \| \leq \frac{\| A \| \| B \|}{1 - \| A \|} \qquad \textbf{(2.5)}$$

Thus, if T has a sufficiently good approximate left inverse B, then T^{-1} exists and is near B. By a similar argument, if $TB = I - L$ with $\| L \| < 1$, then $TX = X$.

Proposition 2.3 [3] *Let* $S, \ T \ \in B(X)$ *. Assume that there exists* $S^{-1} \in B(X)$ *and*

$$\Delta = \| S^{-1} \| \ \| S - T \| < 1. \tag{2.6}$$

Then there exists $T^{-1} \in B(X)$ *and*

$$\| T^{-1} \| \leq \frac{\| S^{-1} \|}{1 - \Delta}, \qquad \| T^{-1} - S^{-1} \| \leq \frac{\Delta \| S^{-1} \|}{1 - \Delta}. \tag{2.7}$$

We will develop in this section the properties of pointwise convergence and the relations between norm convergence. For this aim let us assume $T_n \to T$, as $n \to \infty$. Then the sequence $\{T_n\}$ is bounded, therefore $\| T \| \leq \lim_{n\to\infty} \| T_n \|$. Indeed, since $\lim_{n\to\infty} \| T_n x \| = \| Tx \| < \infty$, we have $\sup_n \| T_n x \| < \infty$. So by the principle of uniform boundedness, which states that for $x \in X$, if $\sup_n \| T_n x \| < \infty$, then $\| T_n \| < L$ for some $L > 0$, that is $\{\| T_n \|\}$ is bounded. Hence,

$$\| Tx \| = \lim_{n\to\infty} \| T_n x \| \leq \lim_{n\to\infty} \| T_n \| \| x \| \tag{2.8}$$

and therefore we have $\| T \| \leq \lim_{n\to\infty} \| T_n \|$.

Therefore, if $T_n \to T$, then the convergence is uniform on each compact set (or each totally bounded set) in X. If B is a totally bounded set then there is a finite ϵ - net $\{x_1, x_2 \ldots, x_k\}$ such that for any $x \in B$, $\min\{\| x - x_i \| : i = 1, \ldots, k\} < \epsilon$. Since $T_n \to T$, it follows that there is an integer $N > 0$ such that $\| T_n x_i \| < \epsilon$ for $i = 1, \ldots, k$ and $n > N$. Using the triangle inequality we have;

$$\| T_n x \| \leq \| T_n x_i \| + \| T_n(x - x_i) \| . \tag{2.9}$$

Here, we have pointwise convergence of uniformly bounded operators. Recall that a linear operator T on X is bounded if and only if it is continuous. Analogously, a set of linear operators is bounded (uniformly) if and only if it is equicontinuous. Thus a bounded sequence $\{T_n\}$ is equicontinuous. So pointwise convergence of an equicontinuous sequence of functions from one metric space to an other is uniform on any totally bounded set.

The following theorem is very important to distinguish between pointwise and norm convergence of operators.

Theorem 2.1 [3] *Let* $T,\ T_n \in B(X)$, $n = 1, 2, \ldots$ *Then* $\| T_n - T \| \to 0$ *if and only if* $T_n \to T$ *and* $\{T_n\}$ *is precompact, with respect to the norm topology on* $B(X)$.

2.2 Collectively Compact Sets of Linear Operators

In this section, we shall concern with collectively compact sets of linear operators where the idea of it is originated and with some properties of this sets.

A linear operator $T : X \to Y$ is called compact if T maps bounded sets in X into relatively compact sets. In general, it is enough to check that T maps the unit ball U_X of X into a relatively compact set; thus, T is compact if $T(U_X)$ has compact closure in Y. It is clear that a linear map $T : X \to Y$ is compact if and only if for every bounded sequence $\{x_n\} \in X$, the sequence $\{Tx_n\}$ in Y has a convergent subsequence in Y. Clearly, any compact operator is bounded.

An integral operator with a continuous kernel $k(x, y)$ on $[0, 1] \times [0, 1]$ defined by

$$Kx(s) = \int_0^1 k(s, t)x(t)dt \qquad \textbf{(2.10)}$$

is the easiest non-trivial example of a compact operator on $C[0, 1]$, the space of real or complex continuous functions $x(t)$, $0 \leq t \leq 1$, with the uniform norm $\| x \| = \max \mid x(t) \mid$. It is clear that K is linear and bounded. By Arzela - Ascoli theorem, which says that a totally bounded set in $C[0, 1]$ is a bounded equicontinuous set of functions, we only need to check that T maps bounded sets into (bounded) equicontinuous sets. If $\| x \| \leq 1$, then

$$\mid Kx(r) - Kx(q) \mid \leq \int_0^1 \mid k(r, t) - k(q, t) \mid \mid x(t) \mid dt \leq \sup \mid k(r, t) - k(q, t) \mid . \qquad \textbf{(2.11)}$$

Now, since $k(s, t)$ is uniformly continuous on $[0, 1] \times [0, 1]$, it follows that the family of functions $TU_{C[0,1]}$is uniformly continuous, that is, T maps the unit ball into an equicontinuous set.

It is known that if $\{T_n\}$ is a sequence of compact operators in $B(X, Y)$, $T \in B(X, Y)$, and $\| T_n - T \| \to 0$, then T is compact. If we replace the norm convergence with the strong convergence, then T need not to be compact. To show this, we give the following example.

Example 2.1 Let H be the infinite dimensional Hilbert space, $\{u_k\}_{k=1}^{\infty}$ be an orthonormal basis in H and P_n be the orthogonal projections on the subspaces spanned by $\{u_k\}_{k=1}^{n}$. Then for any $x \in H$, $\| P_n x - x \| \to 0$ which means that the strong limit of P_n is the identity operator I which is not compact. Here all P_n's are compact since the ranges of them are finite- dimensional.

Let $T \in B(X)$. The practical solution of an equation $Tx = y$ often involves operator approximations $T_n \in B(X)$, $n = 1, 2, \ldots$, such that the equations $T_n x_n = y$ can be solved by some means. P.M. Anselone and R.H. Moore in [5], by introducing the concept of collectively compact sets of linear operators on normed linear spaces in connection with the approximate solution of a Fredholm integral equation of the second kind

$$\lambda x(s) - \int_0^1 k(s,t)x(t)dt = y(s), \qquad 0 \le s \le 1, \tag{2.12}$$

where x, y and k are continuous real or complex - valued functions for $0 \le s, t \le 1$ and $\lambda \neq 0$, dealt with such an approximation.

Numerical integration with a general quadrature formula yields equations

$$\lambda x_n(s) - \sum_{k=1}^{n} w_{nk} k(s, t_{nk}) x(t_{nk}) = y(s), \qquad 0 \le s \le 1, \qquad n = 1, 2 \ldots. \tag{2.13}$$

for approximate solutions of the integral equation above.

The method used here for their numerical solutions was to replace the integral operator with an approximating numerical integration operator. More explicitly, if we define the integral operator K on $C[0, 1]$ by

$$Kx(s) = \int_0^1 k(s,t)x(t)dt, \tag{2.14}$$

we then have $(\lambda I - K) = y$. By defining operators K_n on $C[0, 1]$ with respect to a quadrature formula

$$K_n x(s) = \sum_{k=1}^{n} w_{nk} k(s, t_{nk}) x(t_{nk}), \quad \textbf{(2.15)}$$

the problem of obtaining approximate solutions to the integral equation was transformed into working with equations of the form $(\lambda I - K_n)x_n = y$.

Define linear bounded functionals φ and φ_n on $C[0, 1]$ by

$$\varphi x = \int_0^1 x(t)dt, \qquad \varphi_n x = \sum_{k=1}^{n} w_{nk} x(t_{nk}). \quad \textbf{(2.16)}$$

and their bounds are

$$\| \varphi \| = 1, \qquad \| \varphi_n \| = \sum_{k=1}^{n} | w_{nk} | . \quad \textbf{(2.17)}$$

Suppose that $\varphi_n \to \varphi$, i.e., $\varphi_n x \to \varphi x$ for each $x \in C[0, 1]$ as $n \to \infty$. Banach-Steinhaus theorem, the sequence $\{\varphi_n\}$ is bounded. Moreover, the convergence is pointwise so it is uniform on totally bounded sets.

By the Arzela - Ascoli theorem we know that a set of operators is uniformly bounded if and only if it is an equicontinuous set of functions. Contrary to pointwise convergence $\| \varphi_n - \varphi \| \nrightarrow 0$.

With this, it is obtained that $K_n \to K$ but $\| K_n \to K \| \nrightarrow 0$ [1, Proposition 2.1].

Let us define $k_s(t) = k(s, t)$. Then we get the equations for K and K_n above

$$Kx(s) = \varphi(k_s x) \qquad and \qquad K_n x(s) = \varphi_n(k_s x). \quad \textbf{(2.18)}$$

Since we know $k(s, t)$ is a continuous function on the unit square, the set of functions $\{k_s : 0 \leq s \leq 1\}$ is equicontinuous so $k_s x$ are. It is known that pointwise convergence is uniform on such a set therefore we get uniform convergence for each $x \in C[0, 1]$. Since $\{K_n x : \quad x \in U_X \quad \text{and} \quad n = 1, 2, \ldots, \}$ are bounded and equicontinuous, they also obtained that K is compact and $\{K_n\}$ is collectively compact (i.e., the set $\{K_n x : \| x \| \leq 1 \quad for \quad x \in X, \ n = 1, 2, \ldots, \}$ has compact closure in X) [1, Proposition 2.2]. Now let us give the formal definition of collectively compact sets of operators.

Definition 2.4 A family $M \subset B(X)$ is called *collectively compact* if the set $M(U_X) = \{Tx : T \in M, x \in U_X\}$ has compact closure in X.

Here, we want to give some basic properties of collectively compact sets.

Proposition 2.4 *Every operator in a collectively compact set is compact.*

Proof. Let $M \subset B(X)$ be a collectively compact set. Then $\overline{M(U_X)}$ is compact. Take an arbitrary operator $T \in M$. We will show $\overline{T(U_X)}$ is compact to prove the theorem. We claim that $T(U_X) \subset \overline{M(U_X)}$. Indeed, if $y \in T(U_X)$, then $y = Tx$, $\forall x \in (U_X)$. Also, $T \in M \Rightarrow \forall x \in U_X, Tx \in M(U_X) \Rightarrow y \in M(U_X) \Rightarrow y \in \overline{M(U_X)}$. Thus, $T(U_X) \subset \overline{M(U_X)}$, and so $\overline{T(U_X)} \subset \overline{M(U_X)}$ (compact). Since every closed subset of compact set is compact, it follows that $T(U_X)$ is compact, and so T is compact. □

Proposition 2.5 *Any finite set of compact operators is collectively compact.*

Proof. Let $M := \{T_1, \ldots, T_n\}$ be a finite set of compact operators. Then,

$$\overline{M(U_X)} = \overline{\{T_k\}_{k=1}^{n}(U_X)} = \overline{\{T_k x : x \in U_X, k = 1, \ldots, n\}} =$$
$$= \overline{\bigcup_{k=1}^{n} T_k(U_X)} = \underbrace{\overline{T_1(U_x)}}_{\text{compact}} \cup \ldots \cup \underbrace{\overline{T_n(U_x)}}_{\text{compact}}$$

is compact. □

Proposition 2.6 *Every subset of a collectively compact set is collectively compact.*

Proof. Let $M \subset B(X)$ be a collectively compact set and let $N \subset M$ be a subset.
$\underbrace{\overline{N(U_X)}}_{\text{closed}} \subset \overline{M(U_X)}$ (compact) $\Rightarrow \overline{N(U_X)}$ is compact $\Rightarrow N$ is compact. □

We will need some properties of collectively compact sets in the subsequent analysis. Any scalar multiple of a collectively compact set is collectively compact. Any finite union or finite sum of collectively compact sets is collectively compact. Further properties are given in the following proposition, which is given in [3, Proposition 4.2].

Proposition 2.7 *Let M be a collectively compact set in $B(X)$. Then each of the following sets is collectively compact:*

(a) $\Lambda M = \{\lambda T : \lambda \in \Lambda,\ T \in M\}$ *for any bounded scalar set* Λ,

(b) $M\mathcal{N} = \{TN : T \in M,\ N \in \mathcal{N}\}$ *for any bounded set* $\mathcal{N} \subset B(X)$,

(c) $\mathcal{N}M = \{NT : N \in \mathcal{N},\ T \in M\}$ *for any relatively compact set* $\mathcal{N} \subset B(X)$,

(d) *The strong closure* $\overline{M}^{\,s}$ *and the norm closure* $\overline{M}$ *of* M,

(e) *The convex hull of* M,

(f) $(\left\{\sum_{j=1}^{J} \lambda_j K_j : K_j \in M, \sum_{j=1}^{J} \mid \lambda_j \mid \leq b\right\}$ *for any* $b < \infty$ *and any* $J \leq \infty$,

(g) $\left\{\int_\Gamma K(\lambda)d\lambda : K(\lambda) \in M, \ell(\Gamma) \leq b\right\}$ *for any* $b < \infty$, *where* Γ *is an interval or rectifiable arc of length* $\ell(\Gamma)$ *and the integrals are strong or norm norm limits of the usual approximating sums*

$$\sum_{j=1}^{J} K(\lambda'_j)(\lambda_j - \lambda_{j-1}).$$

Proposition 2.8 [3] *Let* $T, T_n \in B(X),\ n = 1, 2, \ldots,$. *If* $T_n \to T$ *and* $\{T_n\}$ *is collectively compact, then* T *is compact.*

Proof. Let $T_n \to T$ $n = 1, 2, \ldots$ and $\{T_n\}$ be collectively compact.

$$\overline{\{T_n\}(U_X)} = \{T_n\}(U_X) \cup T(U_X),\ \forall n \in \mathbb{N}$$

$$\Rightarrow \underbrace{\overline{T(U_X)}}_{\text{closed}} \subset \overline{\{T_n(U_X)\}},\ \forall n \in \mathbb{N}$$

$$\Rightarrow \overline{T(U_X)}\,\text{compact} \Rightarrow T\ \text{is compact.}$$

Moreover, we know every operator in a collectively compact set is compact therefore each operator T_n is compact. □

Now, we will need the following three theorems.

Theorem 2.2 [3] *Let* $T, T_n \in B(X)$, *X is Banach space,* $n = 1, 2, \ldots,$ *and* $T_n \to T$. *Then* $\| (T_n - T)K \| \to 0$ *for each compact operator* $K \in B(X)$. *The convergence is uniform, with respect to the* K, *for* K *in any collectively compact set set* $M \subset B(X)$.

Theorem 2.3 [3] *Let* $T, T_n \in B(X)$, *X is Banach space,* $n = 1, 2, \ldots,$. *Assume that* $T_n \to T$, $\{T_n\}$ *is collectively compact. Then*

$$\| (T_n - T)T \| \to 0 \quad \textit{and} \quad \| (T_n - T)T_n \| \to 0. \qquad \textbf{(2.19)}$$

Theorem 2.4 [3] *Let $T, T_n \in B(X)$, X is Banach space, $n = 1, 2, \ldots,$. Assume that $T_n \to T$, $\{T_n\}$ is collectively compact and T is compact. Then $(\lambda I - T)^{-1}$ exits if and only if for some N all $n \geq N$ the operators $(\lambda I - T_n)^{-1}$ exists and bounded uniformly, in which case, $(\lambda I - T_n)^{-1} \to (\lambda I - T)^{-1}$.*

The idea behind the proof of the last theorem above is the Fredholm alternative which states tat for any compact operator $T \in B(x)$ $\lambda \neq 0$, $(\lambda I - T)^{-1}$ exists if and only if $(\lambda I - T)X = X$ exists and by the Banach‘s isomorphism theorem it is bounded.

We will compare collectively compact sets of linear operators in $B(X)$ with bounded sets and also with totally bounded sets of compact operators in $B(X)$. A set $M \subset B(X)$ is bounded if and only if $M(U_X)$ is bounded, whereas M is collectively compact if and only if $M(U_X)$ is precompact. Therefore, every collectively compact set $M \subset B(X)$ is a bounded set of compact operators. The converse fails in general.

Example 2.2 [3] Let X be an infinite dimensional Hilbert space (e.g. ℓ_2) with orthonormal basis $\{\psi_\alpha : \alpha \in A\}$. Let P_α be the orthogonal projection onto the one-dimensional subspace spanned by ψ_α. Then

$$P_\alpha x = < x, \psi_\alpha > \psi_\alpha, \qquad \| P_\alpha \| = 1, \tag{2.20}$$

and P_α is compact since $\dim P_\alpha(X) < \infty$. Thus $\mathcal{P} = \{P_\alpha : \alpha \in A\}$ is a bounded set of compact operators in $B(X)$. However, $\mathcal{P}$ is not collectively compact because

$$\psi_\alpha = P_\alpha \psi_\alpha \in \mathcal{P}(U_X) \quad \text{for} \quad \alpha \in A, \tag{2.21}$$

$$\| \psi_\alpha - \psi_\beta \| = \sqrt{2} \quad \text{for} \quad \alpha, \beta \in A \text{ and } \alpha \neq \beta \tag{2.22}$$

which implies that $\mathcal{P}(U_X)$ is not precompact.

Proposition 2.9 [3] *Every precompact set of compact operators is collectively compact in $B(X)$.*

The converse of the theorem above is not true in general.

Example 2.3 [3] Let M be the set of operators $\{T_n\}$ on ℓ_p $(1 \leq p \leq \infty)$ defined by $T_n x = x_n e_1$ where $x = (x_1, \ldots, x_n, \ldots)$ and $e_1 = (1, 0, 0, \ldots)$. Since $M(U_X)$ is bounded and is contained in a one-dimensional space, M is collectively compact. But M is not precompact, for $\| T_m - T_n \| = 2^{\frac{1}{p}}$ if $m \neq n$.

The following example shows that a bounded family of compact operators, compact in the topology of pointwise convergence, may also fail to be a collectively compact set even if the pointwise limit of them is compact.

Example 2.4 [12] Define $\{T_n\}$ on ℓ_1 by $T_n x = x_n e_n$ where $x = (x_1, \ldots, x_n, \ldots)$ and $e_n = (0, \ldots, 0, 1, 0, \ldots)$ in which 1 is in the n^{th} coordinate. Each T_n is compact since $T_n(U_X)$ is a bounded subset of a one-dimensional subspace of l_1. Furthermore, each sequence in l_1 converges to *zero*, so $\{T_n\}$ converges pointwise to the operator 0. However, $M = \{T_n\} \cup \{0\}$ is not collectively compact, for $\{e_n\} \subset M(U_X)$, and $\{e_n\}$ fails to have a limit point since $\| e_n - e_m \| = 2$ for $n \neq m$.

J.A. Higgins established the following two important necessary and sufficient conditions for a set to be collectively compact in [12]. We omit the proof of the first theorem. In the second one, he showed that, in some sense, collectively compact sets can be characterized in terms of diagonal sequences. We shall prove it in a different and a very shorter way without using the diagonalization.

Theorem 2.5 *A set $M \subset B(X)$ is collectively compact if and only if each sequence $\{T_n\} \subset M$ is collectively compact.*

Theorem 2.6 *Let $\{T_n\} \subset B(X)$. Then*

$$\{T_n\} \text{ collectively compact} \Leftrightarrow \begin{cases} \forall n, T_n \text{ compact} \\ \text{and} \\ \forall (x_n) \subset U_X, \{T_n x_n\} \quad \text{is relatively compact.} \end{cases}$$

Proof. Let $\{T_n\}$ be collectively compact. Then for each n, T_n is compact.
Since $\forall n,\ T_n(U_X) \subset \bigcup T_n(U_X) \Rightarrow \overline{T_n(U_X)} \subset \overline{\bigcup T_n(U_X)}$ (compact), it follows that $T_n(U_X)$ is relatively compact.
An arbitrary sequence in $\bigcup T_n(U_X)$ has the form $y_n = T_{\sigma(n)} x_n$, $\forall\, x_n \in U_X$, where σ is a mapping from $\mathbb{N}$ into itself. So, $\bigcup T_n(U_X)$ is relatively compact if and only if any sequence $\{y_n\}$ in $\bigcup T_n(U_X)$ has a convergent subsequence in X.
Conversely, let $\forall (x_n) \subset U_X, \{T_n x_n\}$ be relatively compact and let $\sigma : \mathbb{N} \to \mathbb{N}$ be an arbitrary mapping.
If $\sigma(\mathbb{N})$ is not bounded, then $\exists N_1$(infinite)$\subset \mathbb{N}$ such that $\sigma_1 := \sigma|_{N_1}$ is injective. σ_1 is bijective on $\sigma(N_1)$.

We define $x'_n = \begin{cases} 0, & \text{if } n \in \mathbb{N}\backslash\sigma(N_1) \\ x_{\sigma_1^{-1}(n)}, & \text{if } n \in \sigma(N_1). \end{cases}$ and $y'_n := T_n x'_n$.

$n \in N_1 \Rightarrow y_n = T_{\sigma(n)}x_n = T_{\sigma(n)}x_{\sigma_1^{-1}(\sigma(n))} = T_{\sigma(n)}x'_{\sigma(n)} = y'_{\sigma)(n)}$. Hence,

$$\bigcup_{n\in N_1} y_n = \bigcup_{\sigma(n)\in\sigma(N_1)} y'_{\sigma(n)} \subset \bigcup_{n\in\mathbb{N}} y'_n. \qquad \textbf{(2.23)}$$

Since $\{T_n x_n\}$ is relatively compact, it follows that $\bigcup_{n\in\mathbb{N}} y'_n$ is relatively compact and so, (y_n) has a convergent subsequence. So, $\{T_n\}$ is collectively compact.

If $\sigma(\mathbb{N})$ is bounded, then $\exists N_1$ (infinite) $\subset \mathbb{N}$, $\exists p \in \mathbb{N}$ such that $\sigma(N_1) = \{p\}$. For $n \in N_1,\ y_n = T_p x_n$, where T_p is compact. □

In view of the Theorem 2.6, $\{T_n - T\}$ is collectively compact if and only if $\{(T_n - T)x_n\}$ is relatively compact for each sequence $\{x_n\} \subset U_X$. The following proposition indicates that the present hypotheses are implied by those assumed above.

Proposition 2.10 [3] *Let* $T, T_n \in B(X)$, $n = 1, 2, \ldots$. *Then the following are equivalent.*

(a) $\{T_n\}$ *collectively compact,* T *compact;*

(b) $T_n - T$ *collectively compact,* T *compact;*

(c) $T_n - T$ *collectively compact, some* T_n *compact;*

(d) $T_n - T$ *collectively compact, every* T_n *compact.*

2.3 Spectrum and its Continuity

Since we shall investigate the upper semicontinuity of the spectrum for a certain family of operators, we will introduce basic definitions and some properties of the spectrum of an operator $T \in B(X)$. For $T \in B(X)$ let $\mathcal{N}(T) = \{x \in X : Tx = 0\}$ be the *null space* of T. Real or complex scalars will be denoted by λ and μ. A scalar λ is an *eigenvalue* of T if and only if $\mathcal{N}(\lambda - T) \neq \emptyset$, in which case $\mathcal{N}(\lambda - T)$ is the corresponding *eigenmanifold*. The *resolvent set* for T is

$$\rho(T) = \{\lambda : \exists(\lambda - T)^{-1} \in B(X)\}. \qquad \textbf{(2.24)}$$

The *spectrum* of T is the complement of $\rho(T)$ which is denoted by

$$\sigma(T) = \{\lambda : (\lambda - T) \text{ is not invertible}\}. \qquad \textbf{(2.25)}$$

The set of all eigenvalues of T is known as the *point spectrum* of T and is denoted by

$$\sigma_p(T) = \{\lambda \in \sigma(T) : \ \lambda - T \ \text{ is not one-to-one}\}. \qquad \textbf{(2.26)}$$

By Proposition 2.3, if $\lambda \in \rho(T)$ and $|\mu - \lambda| < \| (\lambda - T)^{-1} \|^{-1}$, then $\mu \in \rho(T)$. Therefore $\rho(T)$ is open, $\sigma(T)$ is closed and

$$\| (\lambda - T)^{-1} \| \ \text{ dist } (\lambda, \ \sigma(T)) \geq 1, \quad \text{ for } \lambda \in \rho(T). \qquad \textbf{(2.27)}$$

By Proposition 2.1, if $|\lambda| > \| T \|$, then $\lambda \in \rho(T)$ and

$$(\lambda - T)^{-1} = \sum_{k=0}^{\infty} \frac{T^k}{\lambda^{k+1}} \qquad \textbf{(2.28)}$$

$$\| (\lambda - T)^{-1}) \| \leq \frac{1}{|\lambda| - \| T \|}, \qquad \textbf{(2.29)}$$

$$\| (\lambda - T)^{-1}) \| \to 0 \text{ as } \ |\lambda| \to \infty. \qquad \textbf{(2.30)}$$

Hence $|\lambda| \leq \| T \|$ for $\lambda \in \sigma(T)$, $\sigma(T)$ is bounded, and $\sigma(T)$ is compact.

The *spectral radius* of T will be denoted by $r(T)$ and for each $T \in B(X)$, there exists

$$r(T) = \lim_{k \to \infty} \| T^k \|^{1/k}, \ r(T) \ \leq \| T \|, \ |\lambda| \leq \ r(T) \text{ for } \lambda \in \sigma(T), \qquad \textbf{(2.31)}$$

and $r(T)$ is the radius of convergence of the power series for $(\lambda - T)^{-1}$ displayed above, i.e., the series converges for $|\lambda| > r(T)$ and diverges for $|\lambda| < r(T)$. If X is complex and $X \neq \emptyset$, then $\sigma(T)$ is nonvoid and

$$r(T) = \max\{|\lambda| : \lambda \in \sigma(T)\}. \qquad \textbf{(2.32)}$$

It will be convenient to compactify the scalar field with the point $\lambda = \infty$ when X is complex and with $\lambda = \pm\infty$ when X is real. Define $(\lambda - T)^{-1} = 0$ for such λ and any $T \in B(X)$. The *extended* resolvent set $\widetilde{\rho}(T)$ consists of $\rho(T)$ and the compactification point(s). In the extended scalar field a set is compact if and only if it is closed.
The *resolvent* for $T \in B(X)$ is the function $\lambda \mapsto (\lambda - T)^{-1}$ from $\widetilde{\rho}(T)$ to $B(X)$. By Proposition 2.4, it is a continuous function (in fact, analytic). So, $\{(\lambda - T)^{-1} : \lambda \in \Lambda\}$ is compact for each closed set $\Lambda \subset \widetilde{\rho}(T)$. The *resolvent equation*,

$$(\lambda - T)^{-1} - (\mu - T)^{-1} = (\mu - \lambda)(\lambda - T)^{-1}(\mu - T)^{-1} \quad \lambda, \mu \in \rho(T), \qquad \textbf{(2.33)}$$

is frequently useful.

One of the main questions in spectral theory is when the spectrum function $T \to \sigma(T)$ for $T \in B(X)$ is continuous. In order to measure the continuity of the spectrum we introduce a distance on the set of compact subsets of $\mathbb{C}$, called the Hausdorff distance and defined by

$$\Delta(K_1, K_2) = \max\{\sup_{\lambda\in K_2} d(\lambda, K_1)\ ,\ \sup_{\lambda\in K_1} d(\lambda, K_2)\}, \qquad \textbf{(2.34)}$$

where $d(\lambda, K_1) = \inf_{\mu\in K_1} \mid \lambda - \mu \mid$ for compact subsets K_1, K_2 of $\mathbb{C}$.

The function $T \to \sigma(T)$ is called continuous at an element S of $B(X)$ if for every $\epsilon > 0$ there exists a $\delta > 0$ such that $\| T - S \| < \delta$ implies $\Delta(\sigma(T), \sigma(S)) < \epsilon$. As usual , we say that $T \to \sigma(T)$ is continuous on $B(X)$ if it is continuous at every point of $B(X)$.

Let $f : X \to [-\infty, \infty]$ be a function on a topological space X. The function f is called *lower semi-continuous* (or *upper semi-continuous*) if

$$\liminf_{n\to\infty} f(x_n) \geq f(x) \qquad (or \limsup_{n\to\infty} f(x_n) \leq f(x)) \qquad \textbf{(2.35)}$$

whenever $x_n \to x$.

We know that $\liminf_{n\to\infty} f(x_n) \leq \limsup_{n\to\infty} f(x_n)$. So, if f is both upper and lower semi-continuous, then lim inf and lim sup will be equal, which means that f is continuous.

The following theorem due to Newburgh [16] is referred to as upper semi-continuity of the spectrum.

Theorem 2.7 *Let $T, S \in B(X)$ and U be an open subset of $\mathbb{C}$.*

If $\sigma(T) \subseteq U$, then there exists a $\delta > 0$ such that $\| S - T \| < \delta$ implies $\sigma(S) \subseteq U$.

Proof. Assume that there exists sequences $\{S_n\} \subset B(X)$ and $\{\lambda_n\} \subset \mathbb{C}$ such that $T = \lim_{n\to\infty} S_n$ and $\lambda_n \in \sigma(S_n) \cap (\mathbb{C}\backslash U)$, $\forall n$. Then $\mid \lambda_n \mid \leq \| S_n \|$, so $\{\lambda_n\}$ is a bounded sequence.

By the Bolzano - Weierstrass Theorem we may suppose without loss of generality that λ_n converges to λ.

Since $\mathbb{C}\backslash U$ is closed, $\lambda \notin U$, so $\lambda \notin \sigma(T)$ and hence $\lambda I - T$ is invertible and hence $\lambda_n I - S_n$ is invertible for n large, that is $\lambda \notin \sigma(S_n)$ which is a contradiction. □

As an application of the upper semi-continuity of the spectrum, we have the following interesting result.

Corollary 2.1 *Let $T_n, T \in B(X)$ and $\| T_n - T \| \to 0$. If $\{\lambda_n\} \subset \sigma(T_n)$ for all n and $\lambda_n \to \lambda$, then $\lambda \in \sigma(T)$.*

Proof. Assume that λ is not in $\sigma(T)$. Then there exists a $\delta > 0$ such that $\sigma(T) \subset U$ with $U := B(\lambda, \delta)$. Since $\| T_n - T \| \to 0$, it follows from the upper semi-continuity of the spectrum that there exists an $N \in \mathbb{N}$ such that $\sigma(T_n) \subset U$ for all $n \geq N$. Therefore, $\lambda_n \notin U$ for all $n \geq N$. This contradicts with $\lambda_n \to \lambda$. □

Since the spectrum is upper semi-continuous so is the spectral radius. That is, to each operator S and to each $\epsilon > 0$ there exists a $\delta > 0$ such that if $\| S - T \| \leq \delta$ then $r(T) < r(S) + \epsilon$.

Recall that a connected space is a topological space which cannot be represented as the union of two or more disjoint nonempty open subsets. A subset of a topological space X is a connected set if it is a connected space when viewed as a subspace of X. The space resulting from the deletion of an infinite line from the plane is an example to disconnected spaces (that is, spaces which are not connected). A totally disconnected space is a topological space which is maximally disconnected, in the sense that it has no non-trivial connected subsets. In every topological space the empty set and the one-point sets are the only connected subsets. An important example of a totally disconnected space is Cantor set.

Newburgh in [16] also proved that if the spectrum of S is totally disconnected, then $T \mapsto \sigma(T)$ is continuous at S. This implies that the spectral function is continuous at all elements having finite or countable spectrum. We gather from this point that the continuity of the spectral radius holds for compact operators since their spectrum are at most countable. However, the spectrum and the spectral radius are not continuous, in general. See, for instance, in [11, Problem 87].

2.4 Functions of Operators

We begin this section with well-known properties of the spectrum of an operator and we shall be concerned with functions of operators defined in terms of analytic functions of a complex variable. In this section we refer to in [1, Section 6.4] for details. Let again X be a Banach space and for each $T \in B(X)$, let $\mathcal{F}(T)$ denote the family of all complex functions f which are analytic on open, not necessarily connected, domains $\mathcal{D}(f) \supset \sigma(T)$. It is well-known that if T is an operator on a complex Banach space and $\lambda \in \sigma(T)$, then $\lambda^n \in \sigma(T^n)$ for each n. More generally, if $p(z) = a_n z^n + \cdots + a_1 z + a_0$ is a polynomial and $p(T)$ denotes the operator obtained by the formal substitution of an operator $T \in B(X)$ for z in $p(z)$, then it can be shown that $\sigma(p(T)) = p(\sigma(T))$. This result is a very special case of a powerful theory that allows one to associate with each operator $T \in B(X)$ a broad class $\mathcal{F}(T)$ of analytic functions (that includes all polynomials) such that for each $f \in \mathcal{F}(T)$ an operator $f(T)$ can be defined whose spectrum $\sigma(f(T))$ coincides with $f(\sigma(T))$.

Recall that a *neighborhood* of a subset A of $\mathbb{C}$ is any open set G such that $A \subseteq G$. We shall denote by $\mathcal{F}(T)$ the collection of all complex-valued functions that are analytic on neighborhoods of the spectrum $\sigma(T)$ of T. That is, $f \in \mathcal{F}(T)$ if and only if there exists a neighborhood G (depending on f) of $\sigma(T)$ such that $f : G \to \mathbb{C}$ is analytic- the neighborhood G can always be assumed to be bounded.

More precisely, $\mathcal{F}(T)$ consists of pairs (f, V), where V is an open bounded set including $\sigma(T)$ and $f : V \to \mathbb{C}$ is an analytic function. If we define the relation $\sim$ on $\mathcal{F}(T)$ by letting $(f, V) \sim (g, W)$ whenever $f = g$ on some neighborhood U of $\sigma(T)$ satisfying $U \subseteq V \cap W$, then $\sim$ is an equivalence relation on $\mathcal{F}(T)$. With this equivalence relation in mind, we shall interpret the elements in $\mathcal{F}(T)$ as equivalence classes rather than as individual functions.

Lemma 2.1 [1] *For every operator $T \in B(X)$ the collection $\mathcal{F}(T)$ of all equivalence classes of functions equipped with the pointwise operations of addition and multiplication is a unital algebra.*

It is very important fact that to each function $f \in \mathcal{F}(T)$ we can associate an operator $f(T) \in B(X)$. We will describe the definition of the operator $f(T)$ next. But first, we say that a subset C of $\mathbb{C}$ is *Jordan contour* if there exists a finite number of pairwise disjoint (closed simple) rectifiable Jordan curves $C_1, C_2, \ldots, C_m$ in $\mathbb{C}$ such that $C = \cup_{n=1}^{m} C_m$.

Definition 2.5 Let V be a neighborhood of a subset A of $\mathbb{C}$. A Jordan contour C in V *surrounds* A if there exists an open subset W of V such that $A \subseteq W \subseteq \overline{W} \subseteq V$ and $C = \partial W$.

The Jordan contour C, viewed as the boundary of W, is always assumed to be positively oriented.

The following is a very useful result.

Lemma 2.2 [1] *Every neighborhood V of a compact set A admits a Jordan contour in V surrounding A.*

Now let $T \in B(X)$ and $f \in \mathcal{F}(T)$. Pick any neighborhood V of $\sigma(T)$ on which f is analytic, and let C be a Jordan contour in V surrounding $\sigma(T)$. Then the operator-valued function $\lambda \mapsto f(\lambda)R(\lambda, T)$ is continuous on C, and hence it is Riemann integrable over C. Moreover, Cauchy's Integral Theorem shows that if C_1 is another Jordan contour surrounding $\sigma(T)$ in the domain of analyticity of f, then

$$\int_C f(\lambda)R(\lambda, T)d\lambda = \int_{C_1} f(\lambda)R(\lambda, T)d\lambda. \tag{2.36}$$

Thus, the integral $\int_C f(\lambda)R(\lambda, T)d\lambda$ is well defined, i.e., it is independent of the Jordan contour C surrounding $\sigma(T)$ in the domain of analyticity of f. This observation gives to the operator $f(T)$ in the following definition.

Definition 2.6 Let $T \in B(X)$ and fix $f \in \mathcal{F}(T)$. If C is any Jordan contour surrounding $\sigma(T)$ in the domain of analyticity of f, then the operator $f(T)$ in $B(X)$ is defined by

$$f(T) = \frac{1}{2\pi i}\int_C f(\lambda)R(\lambda, T)d\lambda. \tag{2.37}$$

The algebraic properties of the operators $f(T)$ are included in the next result. These properties are referred to collectively as *functional calculus*.

Theorem 2.8 [1] *The mapping $f \mapsto f(T)$, from $\mathcal{F}(T)$ to $B(X)$, is an algebraic homomorphism. That is, for each pair f, g in $\mathcal{F}(T)$ and all scalars α and β in $\mathbb{C}$ we have*

$$(\alpha f + \beta g)(T) = \alpha f(T) + \beta g(T) \qquad \text{and} \qquad (fg)(T) = f(T)g(T).$$

Moreover;

1. *If $S \in B(X)$ commutes with T, then S commutes with $f(T)$. In particular, $Tf(T) = f(T)T$ for each $f \in \mathcal{F}(T)$.*

2. *If a function f satisfies $f(\lambda) = \sum_{n=0}^{\infty} a_n\lambda^n$ for all λ in a neighborhood of $\sigma(T)$, then $f \in \mathcal{F}(T)$ and $f(T) = \sum_{n=0}^{\infty} a_nT^n$.*

Corollary 2.2 [1] If $T \in B(X)$ an V is a neighborhood of $\sigma(T)$, then for any Jordan contour C in V surrounding $\sigma(T)$ we have

$$\frac{1}{2\pi i}\int_C R(\lambda, T)d\lambda = I. \tag{2.38}$$

Theorem 2.9 [1] (The Spectral Mapping Theorem). If $T \in B(X)$, then for every function $f \in \mathcal{F}(T)$ we have $\sigma(f(T)) = f(\sigma(T))$. That is,

$$\sigma(f(T)) = \{f(\lambda) : \lambda \in \sigma(T)\}.$$

We have the following for a Jordan contour Γ. Given $T$and f, there is an open set Ω such that

$$\sigma(T) \subset \Omega \subset \overline{\Omega} \subset \mathcal{D}(f) \tag{2.39}$$

and the boundary Γ of Ω consists of a finite number of rectifiable Jordan curves. The operator $f(T) \in B(X)$ is defined by

$$f(T) = \frac{1}{2\pi i}\int_\Gamma f(\lambda)(\lambda - T)^{-1}d\lambda, \tag{2.40}$$

where Γ has positive orientation and the integral is the limit in operator norm of the usual approximating sums. Then

$$f(T)x = \frac{1}{2\pi i}\int_\Gamma f(\lambda)(\lambda - T)^{-1}xd\lambda, \qquad x \in X. \tag{2.41}$$

The operator $f(T)$ is independent of the particular choice of the contour Γ. Hence $f, g \in \mathcal{F}(T)$ and $f(\lambda) = g(\lambda)$ on a neighborhood of $\sigma(T)$, then $f(T) = g(T)$.

Important examples are

$$f(T) = 0 \qquad \text{when} \qquad f(\lambda) \equiv 0,$$

$$f(T) = I \qquad \text{when} \qquad f(\lambda) \equiv 1,$$

$$f(T) = T \qquad \text{when} \qquad f(\lambda) \equiv \lambda.$$

Hence if $f(\lambda)$ is any polynomial in λ then $f(T)$ is the corresponding polynomial in T.

3. SPECTRAL ANALYSIS OF COLLECTIVELY COMPACT OPERATOR SEQUENCES

3.1 Collectively Compact Convergence

In this chapter we define various types of convergence of a sequence of bounded operators that cover most practical methods and we shall use them in the later sections of this chapter. Let X be an arbitrary real or complex Banach space.

Definition 3.1 Given $\{x_n\}_{\mathbb{N}}$, a sequence of finite dimensional subspaces of X such that there exists a sequence of projections π_n on X_n, which satisfy $\pi_n x \to x, \ \forall x \in X$. $(\pi_n : X_n \to X_n, \ \pi_n(x) = x, \ \forall x \in X_n)$. Given $D \subseteq X$, set $D_n := \pi_n D$ and let $\mathcal{T}_n : X_n \to X_n$ be a linear operator with domain D_n. The operator $T_n := \mathcal{T}_n \pi_n$ is said to be of *class* $\mathfrak{D}$.

In particular, if $D = X$ and $D_n = X_n$,, and $\mathcal{T}_n$ is a linear operator on X_n, then $T_n = \mathcal{T}_n \pi_n$ is bounded with domain X; i.e., $T_n \in B(X)$.

Definition 3.2 $T_n \overset{cc}{\to} T$ is said to be *collectively compact convergence* if $T_n \to T$ and the set $\bigcup_{n=1}^{\infty}(T - T_n)(U_X)$ is relatively compact in X.

Definition 3.3 $T_n \overset{c}{\to} T$ is said to be *compact convergence* if $T_n \to T$ and for any sequence $\{x_n\}_{\mathbb{N}} \in U_X$, the sequence $\{(T - T_n)x_n\}_{\mathbb{N}}$ is relatively compact in X.

Example 3.1 [8] Let $X = \ell_2, \ \{e_i\}_1^{\infty}$ be the canonical basis, and $X_n = \{e_1, e_2, \ldots, e_n\}$, $n = 1, 2, \ldots$. If we take $Tx = 0, T_n x = (1/n)x$ then $\| T_n - T \| = 1/n \to 0$, and $T_n \overset{c}{\to} T$, but $T_n \overset{cc}{\nrightarrow} T$.

Definition 3.4 $T_n \overset{d-c}{\to} T$ is said to be *discrete - compact convergence* if for $T_n = \mathcal{T}_n \pi_n$ *of class* $\mathfrak{D}$; that is, $T_n \to T$ and for any sequence $\{x_n\}_{\mathbb{N}}$ such that $x_n \in X_n$, $\| x_n \| \leq c$, the sequence $\{(T - T_n)x_n\}_{\mathbb{N}} = \{(T - \mathcal{T}_n)x_n\}_{\mathbb{N}}$ is relatively compact in X.

Example 3.2 [8] Let $X = \ell_2$, $\{e_i\}_1^\infty$ be the canonical basis, and $X_n = \{e_1, e_2, \ldots, e_n\}$, $n = 1, 2, \ldots$. Assume $x = \sum_{i=1}^{\infty} x_i e_i$, with $x_i = (x, e_i)$. $x \mapsto Tx = \sum_{i=1}^{\infty} x_{i+1} e_i$. For $x \in X_n$, let $x \mapsto \mathcal{T}_n x = \sum_{i=1}^{n-1} x_{i+1} e_i$. Let π_n be orthogonal projection on X_n, $T_n = \mathcal{T}_n \pi_n$ with $x \mapsto T_n x = \sum_{i=1}^{n-1} x_{i+1} e_i = T\pi_n$. For $x_n \in X_n$, $(T - T_n)x_n = 0 : T_n \overset{d-c}{\to} T$. Note that T is not compact therefore $T_n \overset{cc}{\nrightarrow} T$.

We note that $T_n \overset{cc}{\to} T$ implies $T_n \overset{c}{\to} T$, but the converse is not true in general, as shown by Example 3.1. We first note that $T_n \overset{cc}{\to} T$ implies that each $T - T_n$ is compact, since for $n \in \mathbb{N}$, $(T - T_n)U_X \subset \bigcup_{n=1}^{\infty}(T - T_n)U_X$, which is relatively compact. An arbitrary sequence in $K := \bigcup_{n=1}^{\infty}(T - T_n)U_X$ has the form $y_n = (T - T_{\sigma(n)})\, x_n \in U_X$, where σ is a mapping from $\mathbb{N}$ into itself. K is relatively compact if and only if any sequence $\{y_n\}_{\mathbb{N}}$ in K has a convergent subsequence in X. We will show under which conditions $T_n \overset{c}{\to} T$ would imply that $T_n \overset{cc}{\to} T$.

Proposition 3.1 [8] *The following are equivalent:*

(a) $T - T_n$ *is compact for any integer n and* $T_n \overset{c}{\to} T$ *; and*

(b) $T_n \overset{cc}{\to} T$.

Proposition 3.2 [8] *Suppose that* $V_n \overset{cc}{\to} V$, *V compact and for* $L_n \in B(X)$, $L_n \to L$, *then* $V_n L_n \overset{cc}{\to} VL$.

Proposition 3.3 [8] *Let* T_n *be compact,* $T_n \overset{cc}{\to} T$ *implies T compact.*

Proof. $T_n \overset{cc}{\to} T$ implies $T - T_n$ compact for $n \in \mathbb{N}$, which in turn implies that $T = T_n - (T_n - T)$ is compact. □

In general, $\| T_n - T \| \to 0$ does not imply $T_n \overset{cc}{\to} T$, so we want to give following proposition to distinguish between uniform convergence and collectively compact convergence.

Proposition 3.4 [8] *If* $T_n - T$ *is compact,* $n \in \mathbb{N}$ *then* $\| T_n - T \| \to 0$ *implies* $T_n \overset{cc}{\to} T$.

Proof. $\| T_n - T \| \to 0$ implies $T_n \to T$ and $(T - T_n)x_n$ is relatively compact. With applying Proposition 3.1 we have the desired result. □

It is clear to see that $T_n \overset{cc}{\to} T$ implies $T_n \overset{dc}{\to} T$, but converse is not true in general as shown in Example 3.2 so we give the following proposition in order to have under which condition $T_n \overset{dc}{\to} T$ implies $T_n \overset{cc}{\to} T$.

Proposition 3.5 [8] *If T is compact, T_n of class $\mathfrak{D}$, then $T_n \stackrel{dc}{\to} T$ implies $T_n \stackrel{cc}{\to} T$.*

Proof. $T - T_n$ is compact then for any sequence $\{x_n\}_{\mathbb{N}} \in U_X$, the sequence $\{(T - T_n)x_n\}_{\mathbb{N}}$ is relatively compact, which says that the set $K := \bigcup_{n=1}^{\infty}(T - T_n)(U_x)$ is relatively compact in X. We then show for any sequence $\{x_n\}_{\mathbb{N}}$ such that $x_n \in X_n$, $\| x_n \| \leq c$, the sequence $\{(T - T_n)x_n\}_{\mathbb{N}} = \{(T - \mathcal{T}_n)x_n\}_{\mathbb{N}}$ is relatively compact in X implies compact convergence . Recall that $\mathcal{T}_n\pi_n = T_n$. Let $x_n \in U_X$,

$$(T - T_n)x_n = (T - T_n)\pi_n x_n + (T - T_n)(1 - \pi_n)x_n.$$

Since $\{\pi_n x_n\}_{\mathbb{N}}$ is a bounded sequence in X_n, $\{(T - T_n)\pi_n x_n\}$ is relatively compact by the definition of discrete-compact convergence and $\bigcup_{n=1}^{\infty}(T - T_n)\pi_n(U_x)$ is relatively compact by Proposition 3.1. Since T is compact, and $\{(1-\pi_n)x_n\}$ is a bounded sequence in X, $\bigcup_{n=1}^{\infty} T(1 - \pi_n)U_X$ is relatively compact. With $T_n(1 - \pi_n)x_n = 0$, this establishes that $\bigcup_{n=1}^{\infty}(T - T_n)(U_x)$ is relatively compact. □

3.2 Spectral Approximation Theorems

We shall give the results in this section related to the spectra and the resolvents of operators $T, T_n \in B(X)$, $n = 1, 2, \ldots,$, which satisfy Proposition 2.10.

Theorem 3.1 [3] *Let $T, T_n \in B(X)$, $n = 1, 2, \ldots,$. Assume $T_n \to T$ and $\{T_n - T\}$ collectively compact (i.e., $T_n \stackrel{cc}{\to} T$). Then*

(a) $\lambda \in \rho(T)$

if and only if

(b) $\exists N \in \mathbb{N}$ *such that $\lambda \in \rho(T_n)$ for $n \geq N$ and the set $\{(\lambda - T_n)^{-1} : n \geq N\}$ is bounded, in which case;*

(c) $(\lambda - T_n)^{-1} \to (\lambda - T)^{-1}$,

(d) $\{(\lambda - T_n)^{-1} - (\lambda - T)^{-1} : n \geq N\}$ *is collectively compact.*

Proof. First assume $\lambda \in \rho(T)$. Then

$$\lambda - T_n = (I - K_n)(\lambda - T), \qquad K_n = (T_n - T)(\lambda - T)^{-1},$$

$$K_n \to 0, \qquad \{K_n\} \text{ is collectively compact}$$

By Theorem 2.4 with $K = 0$, there exists N such that $(I - K_n)^{-1} \in B(X)$ for $n \geq N$, $\{(I - K_n)^{-1} : n \geq N\}$ is bounded, and $(I - K_n)^{-1} \to I$. Consequently, $\lambda \in \rho(T_n)$ for $n \geq N$ and

$$(\lambda - T_n)^{-1} = (\lambda - T)^{-1}(I - K_n)^{-1} \to (\lambda - T)^{-1},$$

$$(\lambda - T_n)^{-1} - (\lambda - T)^{-1} = (\lambda - T)^{-1}(T_n - T)(\lambda - T_n)^{-1}.$$

Thus by Proposition 2.7 $(a) \Rightarrow (b), (c), (d)$.

Now assume (b). $(\lambda - T)^{-1}$ exists as an operator defined on $(\lambda - T)X$. For $n \geq N$,

$$(\lambda - T) = (I - L_n)(\lambda - T_n), \qquad L_n = (T - T_n)(\lambda - T_n)^{-1},$$

$$(I - L_n)^{-1} \text{ exists}, \qquad L_n \text{ is compact.}$$

By the Fredholm alternative, $(I - L_n)^{-1} \in B(X)$ and

$$(\lambda - T)^{-1} = (\lambda - T_n)^{-1}(I - L_n)^{-1} \in B(X), \qquad n \geq N.$$

Thus , $(b) \Rightarrow (a)$. □

Proposition 3.6 [3] *Let $A \in B(X)$ and $\| A^2 \| < 1$. Then there exists $(I - A)^{-1} \in B(X)$ and*

$$\| (I - A)^{-1} \| \leq \frac{\| I + A \|}{1 - \| A^2 \|}.$$

The next theorem gives us another proof to show $(a) \Rightarrow (b), (c)$ in Theorem 3.1.

Proposition 3.7 [3] *Let $T, T_n \in B(X)$, $n = 1, 2, \ldots$. Assume $T_n \to T$ and $\{T_n - T\}$ is collectively compact. Let $\lambda \in \rho(T)$ and*

$$K_n = (T_n - T)(\lambda - T)^{-1}.$$

Then $K_n \to 0$ and $\| K_n^2 \| \to 0$. Whenever $\| K_n^2 \| < 1$ then $\lambda \in \rho(T_n)$,

$$\| (\lambda - T_n)^{-1} \| \leq \frac{\| (\lambda - T)^{-1} \| \| I + K_n \|}{1 - \| K_n^2 \|}$$

and

$$\| (\lambda - T_n)^{-1}x - (\lambda - T)^{-1}x \| \leq \| (\lambda - T_n)^{-1} \| \| K_n x \| \to 0, \qquad x \in X.$$

Theorem 3.2 [6] *Let $T, T_n \in B(X)$, $n = 1, 2, \ldots$. Assume $T_n \to T$ and $\{T_n - T\}$ is collectively compact. Then for each closed set $\Lambda \subset \widetilde{\rho}(T)$ there exists N such that :*

(a) $\Lambda \subset \widetilde{\rho}(T_n)$ *for* $n \geq N$;

(b) $\{(\lambda - T_n)^{-1} : \lambda \in \Lambda,\ n \geq N\}$ *is bounded*;

(c) *for each* $x \in X$, $(\lambda - T_n)^{-1}x \to (\lambda - T)^{-1}x$ *uniformly for* $\lambda \in \Lambda$;

(d) $\{(\lambda - T_n)^{-1} - (\lambda - T)^{-1} :\ \lambda \in \Lambda,\ n \geq N\}$ *is collectively compact*;

(e) *the resolvents* $\lambda \mapsto (\lambda - T_n)^{-1}$, $n \geq N$, *are equicontinuous on* Λ.

We now give an important theorem proved by P.M. Anselone which states that the spectrum function is upper semi-continuous when we have collectively compact convergence of operators instead of norm convergence.

Theorem 3.3 [3] *Let $T, T_n \in B(X)$, $n = 1, 2, \ldots$. Assume $T_n \to T$ and $\{T_n - T\}$ is collectively compact (i.e.,$T_n \stackrel{cc}{\to} T$). Then for each open set $\Omega \supset \sigma(T)$ there exists N such that $\Omega \supset \sigma(T_n)$ for $n \geq N$.*

In [3], P.M. Anselone also shows that Corollary 2.1 is valid if we replace norm convergence by collectively compact convergence.

Corollary 3.1 [3] *Let $T, T_n \in B(X)$, $n = 1, 2, \ldots$. Assume $T_n \to T$ and $\{T_n - T\}$ is collectively compact. If $\{\lambda_n\} \subset \sigma(T_n)$, $\forall n$ and $\lambda_n \to \lambda$, then $\lambda \in \sigma(T)$.*

Proof. Without loss of generality, $\lambda_n \neq 0$, for all n. Let $K = T/\lambda$ and $K_n = T_n/\lambda_n$ in Theorem 2.4 to obtain the assertions involving $x_{n_i} \to x \in \mathcal{N}(\lambda - T)$. Hence, $\lambda \in \sigma(T)$.□

Theorem 3.4 [6] *Let $T, T_n \in B(X)$, $n = 1, 2, \ldots$. Assume $T_n \to T$ and $\{T_n - T\}$ is collectively compact and, $f \in \mathcal{F}(T)$.Then there exists $N \in \mathbb{N}$ such that for all $n \geq N$,*

(a) $f \in \mathcal{F}(T_n)$ *for* $n \geq N$,

(b) $f(T_n)x \to f(T)x$ *for each* $x \in X$,

(c) $\{f(T_n) - f(T) : n \geq N\}$ *is collectively compact.*

Proof. Choose $\Gamma \subset \mathfrak{D}(f) \cap \rho(T)$ with $\sigma(T)$ inside Γ. By Theorem 3.3, there exists an N such that $\forall n \geq N$, $\sigma(T_n)$ inside Γ. Thus (a) follows.
Now if $n \geq N$ and $x \in X$, then

$$f(T_n)x - f(T)x = \frac{1}{2\pi i} \int_\Gamma f(\lambda)[(\lambda I - T_n)^{-1}x - (\lambda I - T)^{-1}x]d\lambda \qquad (\mathbf{3.1})$$

By Theorem 3.2 (c), it follows that $f(\lambda)[(\lambda I - T_n)^{-1}x - (\lambda I - T)^{-1}x]) \to 0$ uniformly for $\lambda \in \Gamma$ as $n \to \infty$. This implies (b).
We have for $n \geq N$,

$$f(T_n) - f(T) = \frac{1}{2\pi i} \int_\Gamma f(\lambda)[(\lambda I - T)^{-1}(T_n - T)(\lambda I - T_n)^{-1}]d\lambda.$$

Indeed,

$$(\lambda I - T_n)^{-1} - (\lambda I - T)^{-1} = (\lambda I - T)^{-1}(T_n - T)(\lambda I - T_n)^{-1}.$$

In fact, since $T_n - T = T_n - T$, it follows that $(\lambda I - T) - (\lambda I - T_n) = T_n - T$. From this, we have $I - (\lambda I - T)^{-1}(\lambda I - T_n) = (\lambda I - T)^{-1}(T_n - T)$, and so

$$(\lambda I - T_n)^{-1} - (\lambda I - T)^{-1} = (\lambda I - T)^{-1}(T_n - T)(\lambda I - T_n)^{-1}.$$

Moreover, the set $\{f(\lambda)(\lambda I - T)^{-1}(T_n - T)(\lambda I - T_n)^{-1} : \lambda \in \Gamma, n \geq N\}$ is collectively compact. Indeed, since we know that $\{(\lambda - T)^{-1} : \lambda \in \Lambda\}$ is compact for each closed set $\Lambda \subset \widetilde{\rho}(T)$, then $\{(\lambda I - T)^{-1}(T_n - T)(\lambda I - T_n)^{-1} : \lambda \in \Gamma, n \geq 1\}$ is collectively compact which follows from Proposition 2.4 (b) and (c). Now, consider integrals of operators valued functions. Let Γ be a finite interval if X is real and rectifiable arc if X is complex. Suppose $T_\alpha(\lambda) \subset B(X, Y)$ for $\lambda \in \Gamma$ and α in an index set A. For each $\alpha \in A$, assume that $\int_\Gamma T_\alpha d\lambda$ is the strong or norm limit of the usual approximating sums $\sum_{j=1}^m T_\alpha(\lambda'_j)(\lambda_j - \lambda_{j-1})$. If $\{T_\alpha \lambda : \alpha \in A, \lambda \in \Gamma\}$ is collectively compact, then $\{\int_\Gamma T_\alpha d\lambda : \alpha \in A\}$ is collectively compact. It follows from Proposition 2.7 (d), (f) and (g). This implies (c). □

The next theorem yields that the spectrum function is upper semi-continuous under the union of disjoint open sets when replacing norm convergence to collectively compact convergence.

Theorem 3.5 [2] *Let $T, T_n \in B(X)$, $T_n \to T$, and $\{T_n\}$ be collectively compact. If U and V are disjoint open sets with $\sigma(T) \subset U \cup V$ and $\sigma(T) \cap U \neq \emptyset$, then there exists $N \in \mathbb{N}$ such that $\sigma(T_n) \cap U \neq \emptyset$ for all $n \geq N$.*

Proof. Let us observe that the family $\{T_n - T\}$ is collectively compact. Let f be the analytic function defined on $U \cup V$ as $f(\lambda) \equiv 1$ on U and $f(\lambda) \equiv 0$ on V. By Theorem 3.3, $\sigma(T_n) \subset U \cup V$ for all $n \geq N$ for some $N \in \mathbb{N}$. Assume that the claim is not true. Then for each N, there exists $n \geq N$ such that $\sigma(T_n) \subset V$. In this way, we construct a subsequence of operators $\{T_{n_k}\}$ with the property that $\sigma(T_{n_k}) \subset V$ for each k.

On the other hand, as $f(T_n)x \to f(T)x$ from Theorem 3.4, we have $f(T_{n_k})x \to f(T)x$. As $f(T_{n_k})x = 0$ for each k, $f(T)x = 0$ for each $x \in X$. By the spectral mapping theorem, $\sigma(f(T)) = f(\sigma(T))$ and 1 must be in $\sigma(f(T))$, $f(T)$ can not be the zero operator. This contradiction yields $\sigma(T_n) \cap U \neq \emptyset$ for all $n \geq N$ for some N. □

4. SPECTRAL ANALYSIS OF ASYMPTOTICALLY COMPACT OPERATOR SEQUENCES

In this section we will extend some of the collectively compact approximation theory to the asymptotically compact case. Asymptotically compact operator sequences arise from the approximate solution of various integral equations. Let us begin with some concepts and results for the set theory needed in later sections.

Let (X, ρ) be a metric space, x_n, $x \in X$ and S_n, $S \subset X$ where $n \in \mathbb{N} = \{1, 2, \ldots\}$. Assume that N' and N'' denote infinite subsets of $\mathbb{N}$.
We introduce ϵ- neighborhoods of sets,

$$\Omega_\epsilon(S) = \bigcup_{x \in S} \{y \in X : \| x - y \| < \epsilon\}, \qquad \epsilon > 0. \tag{4.1}$$

By analogy with point convergence, $x_n \to x$ as $n \to \infty$, set convergence is defined by

$$S_n \to S \text{ if } S_n \subset \Omega_\epsilon(S), \ \forall n \text{ large } \forall \epsilon > 0, \tag{4.2}$$

i.e., for $n \geq n_\epsilon$ with some n_ϵ. Such set limits are not unique:

$$S_n \to S \subset S' \Rightarrow S_n \to S' \tag{4.3}$$

The void set $\emptyset$ plays a special role: $\Omega_\epsilon(\emptyset) = \emptyset$ and

$$S_n \to \emptyset \Rightarrow S_n = \emptyset, \ \forall n \text{ large}, \tag{4.4}$$

$$S_n = \emptyset \ \forall n \text{ large} \Rightarrow S_n \to S, \ \forall S \subset X. \tag{4.5}$$

For S compact,

$$S_n \to S \ \Leftrightarrow \ \forall \text{ open } \Omega \supset S, \ \Omega \supset S_n, \ \forall n \text{ large}, \tag{4.6}$$

i.e., for $n \geq n_\Omega$ with some n_Ω.

Definition 4.1 Define the cluster point sets:

$$\{x_n\}^* = \{x \in X : \ x_n \to x, \ n \in N'\}$$

$$\{S_n\}^* = \{x \in X : \ x_n \to x, \ x_n \in S_n, \ n \in N'\}.$$

The sequence $\{x_n\} = \{x_n : n \in \mathbb{N}\}$ is *d-compact (discretely compact)* if each subsequence has a convergent subsequence, i.e., $\{x_n : \ n \in N'\}^* \neq \emptyset, \ \forall N' \subset \mathbb{N}$. Similarly, $\{S_n\}$ is *d-compact* if $\{S_n : \ n \in N'\}^* \neq \emptyset \ \forall N' \subset \mathbb{N}$ such that $S_n \neq \emptyset, \ n \in N'$ All cluster point sets are closed. Elementary properties include,

$$\{S_n\} \text{ d-compact}, \{S_n\}^* = \emptyset \Rightarrow S_n = \emptyset, \ \forall n \text{ large}, \quad \textbf{(4.7)}$$

$$\{\overline{S_n}\}^* = \{S_n\}^*, \quad \textbf{(4.8)}$$

$$\{\overline{S_n}\} \text{ d-compact} \Leftrightarrow \{S_n\} \text{ d-compact}, \quad \textbf{(4.9)}$$

$$\overline{S_n} \to S \Leftrightarrow S_n \to S, \quad \textbf{(4.10)}$$

$$\overline{\cup S_n} = (\cup \overline{S}_n) \cup \{S_n\}^*, \quad \textbf{(4.11)}$$

$$\overline{\cup S_n} \text{ compact} \Leftrightarrow \overline{S}_n \text{ compact } \forall n, \ \{S_n\} \text{ d-compact}. \quad \textbf{(4.12)}$$

We give the following result to note that the notions of d-compactness and the aforementioned set convergence are related.

Theorem 4.1 [18] *$\{S_n\}$ d-compact $\Leftrightarrow S_n \to \{S_n\}^*$ and $\{S_n\}^*$ compact.*

Definition 4.2 $\{S_n\}$ is *asymptotically totally bounded* if $\forall \epsilon > 0$ $n_\epsilon \in \mathbb{N}$ *such that* $\bigcup_{n \geq n_\epsilon} S_n$ has a finite ϵ-net (in X).

Theorem 4.2 [18] *$\{S_n\}$ d-compact $\Rightarrow \{S_n\}$ asymptotically totally bounded.*

The converse of the Theorem 4.2 holds if X is complete.

4.1 Measure of Non-compactness

Let (X, ρ) be a complete metric space. In this part we will mention about the Kuratowski measure of non-compactness and the Hausdorff measure of non-compactness of a bounded set $S \subset X$.

It is well known that the *diameter* of a subset of a metric space is the least upper bound of the distances between pairs of points in the subset. So, if S is the subset, the diameter is

$$\sup\{\, d(x,y) |\, x,\ y \in S\,\}.$$

Definition 4.3 For $S \subset X$ bounded, define

$$\alpha(S) = \inf\{\epsilon > 0 : \ S \subset \bigcup_{finite} U_k \ \text{with} \ \ \text{diam} \ \ U_k < \epsilon\}.$$

α is called the *Kuratowski measure of non-compactness*.

We give the following lemma in [13] to show the basic properties of the Kuratowski measure of non-compactness.

Lemma 4.1 For bounded subsets U, S of X, we have

1. $0 \leq \alpha(S) = \alpha(\overline{S})$,
2. $\alpha(S) = 0 \Leftrightarrow \overline{S}\ compact$,
3. $S \subset U \Rightarrow \alpha(S) \leq \alpha(U)$
4. $\alpha(S \cup U) = \max\{\alpha(S), \alpha(U)\}$,
5. $\alpha(S \cap U) \leq \min\{\alpha(S), \alpha(U)\}$.

Kuratowski uses α to show that if $\{S_n\}$ is a decreasing sequence of nonempty, closed and bounded sets with $\alpha(S_n) \to 0$, then $\emptyset \neq \bigcap_{n=1}^{\infty} S_n$ is compact.
Now assume $(X, \|\ .\ \|)$ is a Banach space. We then have

1. $\alpha(S + U) \leq \alpha(S) + \alpha(U)$ for any $S,\ U \subset X$,
2. $\alpha(\lambda S) =| \lambda | \alpha(S)$, λ complex.

Definition 4.4 Let S be a bounded subset of X. The*Hausdorff measure of non-compactness* $\mathcal{X}(S)$ of S is the infimum of all $r > 0$ such that S can be covered by a finite number of open balls of radius r. That is,

$$\mathcal{X}(S) = \| S \|_{\mathcal{X}} = \inf\{r : \exists\ x_1, x_2, \ldots, x_n \text{ such} \quad \text{that } S \subseteq \bigcup_{i=1}^{n} B(x_i, r)\}. \qquad \textbf{(4.13)}$$

The above definition was given by Goldenstein, Gohberg, and Markus in [9]. The terminology is motivated by the following observations. For $x \in X$, $r > 0$, set $B(x, r) = \{x' \in X : \| x - x' \| < r\}$. Define $d(S, U) = \inf\{r > 0 : S \subset U + B(0, r)\}$, and set $D(S, U) = \max\{d(S, U), d(U, S)\}$, the Hausdorff metric.
If $F = \{U \subset X : U \neq \emptyset, \overline{U} \text{ compact}\}$, then for a bounded set $S \subset X$

$$\mathcal{X}(S) = \inf_{U \in F} D(S, U).$$

Note that for an unbounded subset S of a metric space, we let $\mathcal{X}(S) = \infty$. The following lemma says that $\mathcal{X}$ enjoys properties in Lemma 4.1 along with α.
Lemma 4.2 We have the following properties for bounded subsets U, S of X.

1. $0 \leq \mathcal{X}(S) = \mathcal{X}(\overline{S})$,
2. $\mathcal{X}(S) = 0 \Leftrightarrow \overline{S} compact$,
3. $S \subset U \Rightarrow \mathcal{X}(S) \leq \mathcal{X}(U)$,
4. $\mathcal{X}(S \cup U) = \max\{\mathcal{X}(S), \mathcal{X}(U)\}$,
5. $\mathcal{X}(S \cap U) \leq \min\{\mathcal{X}(S), \mathcal{X}(U)\}$.

If X is a Banach space and $T \in B(X)$, then the *Hausdorff measure of non-compactness* $\mathcal{X}(T)$ of T is defined by

$$\mathcal{X}(T) = \mathcal{X}(T(U_X)).$$

The next result which is [1, Lemma 7.56] contains several elementary properties of the Hausdorff measure of non-compactness of operators.
Lemma 4.3 Let $T \in B(X, Y)$ and $K \in B(Y, Z)$. Then we have followings;

1. $T \in B(X)$ compact $\Leftrightarrow \mathcal{X}(T) = 0$.
2. $T \in B(X, Y),\ K \in B(Y, Z) \Rightarrow \mathcal{X}(KT) \leq \mathcal{X}(K)\mathcal{X}(T)$.

Now we return to our subject and will give the relations between the Hausdorff measure of non-compactness and asymptotically totally boundedness.The next result is a direct application of the definitions.

Theorem 4.3 [18] $\{S_n\}$ *asymptotically totally bounded* $\Leftrightarrow \mathcal{X}(\bigcup_{j\geq n} S_j) \to 0$ *as* $n \to \infty$.
Corollary 4.1 [18] $\{S_n\}$ *d-compact* $\Rightarrow \mathcal{X}(\bigcup_{j\geq n} S_j) \to 0$.
The converse of Corollary 4.1 holds if X is complete.

It immediately follows that $\mathcal{X}(S_n) \to 0$ when $\{S_n\}$ is d-compact. But $\mathcal{X}(S_n) \to 0$ is not enough to guarantee $\{S_n\}$ is d-compact.
Example 4.1 [18] Let $S_n = [2n,\ 2n+1] \subset \mathbb{R},\ n = 1, 2, \ldots$. Then $\mathcal{X}(S_n) = 0$ but S_n is not d-compact.

Lemma 4.4 [18] *Let X be a Banach space and $T \in B(X)$. Then*

$$\| T \|_{\mathcal{X}} = \mathcal{X}(T(U_X)).$$

Proof. Since $\mathcal{X}(U_X) \leq 1$, we have

$$\mathcal{X}(T(U_X)) \leq \| T \|_{\mathcal{X}} \mathcal{X}(U_X) \leq \| T \|_{\mathcal{X}} .$$

To show the reverse inequality, let $S \subset X$ be bounded and let $\epsilon > \mathcal{X}(S)$. Then there exists a finite ϵ -net $\{x_1, x_2, \ldots, x_n\} \subset X$ of S, i.e., $TS \subset \bigcup_{k=1}^{n} TB(x_k, \epsilon)$.
For any $1 \leq k \leq n$, $TB(x_k, \epsilon) = \epsilon\, TB(\frac{x_k}{\epsilon}, 1) = \epsilon\, TU_X + \frac{x_k}{\epsilon}$ gives
$\mathcal{X}(TS) \leq \max_{1\leq k\leq n}\{\mathcal{X}[TB(x_k, \epsilon)]\} = \epsilon\ \mathcal{X}(TU_X)$. Since $\epsilon > \mathcal{X}(S)$ was arbitrary, the proof is completed. □

Definition 4.5 Let X, Y be the normed linear spaces and let $A \in B(X, Y)$. The operator $A' : Y' \to X'$ defined by $A'f : X \to \mathbb{F}$, where $\mathbb{F}$ is a scalar field, $(A'f)(x) = f(Ax)$ for each $x \in X$ and $f \in Y'$, is called a *dual operator* (or *conjugate operator*) of A.

It is shown that the relation between the Kuratowski measure and the Hausdorff measure of non-compactness in [18]. We give it briefly as a property.

$$\| T' \|_{\alpha} \leq \| T \|_{\mathcal{X}} \quad \text{and} \quad \| T \|_{\alpha} \leq \| T' \|_{\mathcal{X}} . \tag{4.14}$$

We now define $\alpha(T) = \dim \mathcal{N}(T)$ and $\beta(T) = \dim(X \backslash \mathcal{R}(T))$ =codim$\mathcal{R}(T)$. We remark that when $\mathcal{R}(T)$ is closed and $\beta(T) < \infty$, it can be shown that $\beta(T) = \dim \mathcal{N}(T')$. Then,

$$\alpha(T) = 0 \Leftrightarrow \exists T^{-1}, \tag{4.15}$$

$$\beta(T) = 0 \Leftrightarrow \mathcal{R}(T) = X. \tag{4.16}$$

We note that T is bijective if and only if $\alpha(T) = 0$ and $\beta(T) = 0$. The next definition gives a measure of how far a given operator is from being bijective.

Definition 4. 6 The *index* of T is defined by

$$ind(T) = \alpha(T) - \beta(T),$$

when the right side is well-defined.

We give some information about Fredholm operator which will be used in the sequel.

Definition 4.7 An operator T is a *Fredholm operator* if $\mathcal{N}(T)$, the null space of T, has finite dimension and $\mathcal{R}(T)$, the range of T, has finite codimension. If T has closed range and either $\alpha(T)$ or $\beta(T)$ is finite, then T is a *semi-Fredholm operator*. In this case, the index of T is defined by

$$ind(T) = dim\mathcal{N}(T) - codim\mathcal{R}(T).$$

In this section we also introduce regular operators since we characterize semi-Fredholm operators A with $\mathcal{R}(A)$ closed and dim $\mathcal{N}(T) < \infty$.

Definition 4.8 $A \in B(X)$ *is regular* if $\overline{S}$ is compact whenever $S \subset X$ is bounded and $\overline{AS}$ is compact.

$$\exists A^{-1} \text{ bounded } \Rightarrow A \text{ regular.} \quad \textbf{(4.17)}$$

It follows from the definition that restrictions and products of regular operators are regular. Equivalent definitions are the following.

$$A \text{ regular } \Leftrightarrow \{x_n\} \text{ bounded, } \{Ax_n\} \text{ d-compact} \Rightarrow \{x_n\} \text{ d-compact,} \quad \textbf{(4.18)}$$

$$A \text{ regular } \Leftrightarrow \{x_n\} \text{ bounded, } Ax_n \to y \Rightarrow \{x_n\}^* \neq \emptyset. \quad \textbf{(4.19)}$$

If A is regular and $Ax_n \to y$, where $\{x_n\}$ is bounded, then (4.16) gives $Ax = y,\ x \in \{x_n\}^*$. Also,

$$A \text{ regular, S closed and bounded} \Rightarrow AS \text{ closed .} \quad \textbf{(4.20)}$$

Consequently, A regular $\Rightarrow$ AU closed, where $U = \{x \in X : \| x \| = 1\}$ is the unit sphere. We know that from operator fundamentals if AU closed, $\exists T^{-1}$ bounded $\Leftrightarrow T^{-1}$ bounded. Moreover, let $\exists T^{-1}$. Then T^{-1} bounded $\Leftrightarrow \mathcal{R}(T)$ closed. Hence, we have

$$\text{Whenever } \exists A^{-1},\ A \text{ regular} \Leftrightarrow A^{-1} \text{ bounded} \Leftrightarrow \mathcal{R}(A) \text{ closed.} \quad \textbf{(4.21)}$$

We also have the following for regular operators in terms of the Hausdorff measure of non-compactness,

$$A \text{ regular} \Leftrightarrow S \text{ bounded}, \ \mathcal{X}(AS) = 0 \text{ implies } \mathcal{X}(S) = 0. \tag{4.22}$$

Theorem 4.4 [18] *A regular* $\Leftrightarrow$ $\mathcal{R}(A)$ *closed and* $\dim \mathcal{N}(A) < \infty$.

Definition 4.9 Let $A_n, A \in B(X)$. $\{A_n\}$ is *asymptotically regular* if $\{S_n\}$ uniformly bounded, $\{A_n S_n\}$ d-compact $\Rightarrow$ $\{S_n\}$ d-compact. Equivalent definitions are:

$$A_n \text{ asymptotically regular} \Leftrightarrow \{x_n\} \text{ bounded}, \ \{A_n x_n\} \text{ d-compact} \Rightarrow \{x_n\} \text{ d-compact}, \tag{4.23}$$

$$A_n \text{ asymtotically regular} \Leftrightarrow \{x_n\} \text{ bounded}, \ A_n x_n \to y \Rightarrow \{x_n\}^* \neq \emptyset. \tag{4.24}$$

Definition 4.10 $A_n \xrightarrow{r} A$ is said to be regular convergence if $A_n \to A$ and $\{A_n\}$ asymptotically regular.

In view of (4.16) we obtain

$$A_n \to A, \{x_n\} \text{ bounded}, A_n x_n \to y \Rightarrow y \in \mathcal{R}(A), \ Ax = y, \ \forall x \in \{x_n\}^*. \tag{4.25}$$

We also note that any subsequence of an asymptotically regular sequence is asymptotically regular.

4.2 Condensing operators

It is well known that when X is complete, the vector space $\mathcal{K}$ of all compact operators on X is a closed (two-sided) ideal in $B(X)$. The quotient vector space $B(X)/\mathcal{K}$ is a unital algebra under the algebraic operations :

- $[S] + [T] = [S + T]$,
- $\lambda[T] = [\lambda T]$,
- $[ST] = [S][T]$

with the element [I] as its unit. Moreover $B(X)/\mathcal{K}$ is a Banach space under the quotient norm

$$\| [T] \| = \inf_{S\in[T]} \| S \| = \inf_{S-T\in\mathcal{K}} \| S \| .$$

Since the quotient norm also satisfies the properties $\| [S][T] \| \leq \| [S] \| \, \| [T] \|$ and $\| [I] \| = 1$, it follows that $B(X)/\mathcal{K}$ is a unital algebra. This Banach algebra is called the *Calkin algebra* of X and is denoted $\mathfrak{C}(X)$.

In this section we will extend previous results for compact operators to the more general case when the operators are condensing with respect to some measure of non-compactness.

Definition 4.11 An operator $T \in B(X)$ is called *condensing* if $\mathcal{X}(T(U_X)) < 1$.

Now again let $\mathcal{K}$ be the set of all compact operators in $B(X)$.

Definition 4.12 For $T \in B(X)$, define $\| T \|_{\mathcal{K}} = \inf\{\| T + K \| : \ K \in \mathcal{K}\}$.

Then $\| T \|_{\mathcal{K}} = 0 \Leftrightarrow T$ compact. Therefore, we can think of $\| T \|_{\mathcal{K}}$ as measuring the non-compactness of T.

Let $K \in \mathcal{K}$ and $T \in B(X)$. From the definitions above the following observations are straightforward.

$$\| . \|_{\mathcal{K}}, \ \| . \|_{\mathcal{X}} \ \text{are seminorms on} \ B(X), \tag{4.26}$$

$$\| T + K \|_{\mathcal{K}} = \| T \|_{\mathcal{K}}, \tag{4.27}$$

$$\| TL \|_{\mathcal{K}} = \| T \|_{\mathcal{K}} \| L \|_{\mathcal{K}}, \tag{4.28}$$

$$\| T \|_{\mathcal{X}} \leq \| T \|_{\mathcal{K}} \leq \| T \| . \tag{4.29}$$

Definition 4.13 Let $T \in B(X)$. T is called $\mathcal{K}$-condensing if $\| T \|_{\mathcal{K}} < 1$.

Definition 4.14 Let $T \in B(X)$. T is called $\mathcal{X}$-condensing if $\| T \|_{\mathcal{X}} < 1$.

By condensing , we will mean $\mathcal{X}$-condensing.

Theorem 4.5 [17] *If $T \in B(X)$ and T^n is condensing for some $n \geq 1$ then I-T is Fredholm of index zero.*

Proof. Let $S \subset X$ be bounded, $\overline{(I-T)S}$ compact. The identity $I = T^n + \sum_{k=0}^{n-1} T^k(I-T)$ gives $S \subset T^n S + \sum_{k=0}^{n-1} T^k(I-T)S$. Consequently,

$$\mathcal{X}(S) \leq \| T^n \|_{\mathcal{X}} \mathcal{X}(S) + \sum_{k=0}^{n-1} \| T^k \|_{\mathcal{X}} \mathcal{X}[(I-T)S],$$

$$= \| T^n \|_{\mathcal{X}} \mathcal{X}(S).$$

Since $\| T^n \|_{\mathcal{X}} < 1$, it follows that $\mathcal{X}(S) = 0$. Hence, I-T is a regular operator, i.e., $\dim \mathcal{N}(I-T) < \infty$ and $\mathcal{R}(I-T)$ is closed by Theorem 4.4. By (4.11),

$$\| (T^n)' \|_{\alpha} = \| (T')^n \|_{\alpha} < 1.$$

The previous arguments with $\mathcal{X}$ replaced by α and S replaced by $S' \subset X'$ show that $(I-T)'$ is also regular.

Therefore, since R(I-T) is closed, $\dim N[(I-T)'] = codimR(I-T) < \infty$. Consequently, $I-T$ is Fredholm. The same arguments show that $I-tT$ is Fredholm $\forall t \in [0,1]$. By the stability of the index, $ind(I-T) = ind(I) = 0$. □

In view of (4.23) we have the following corollary.

Corollary 4.2 If $T \in B(X)$ is a $\mathcal{K}$-condensing operator then $ind(I-T) = 0$.

4.3 Asymptotically Compact Sequences

Let X be a Banach space, U_X be the closed unit ball in X and B(X) is the space of bounded linear operators on X.

Definition 4.15 A sequence of operators $\{T_n\} = \{T_n : n = 1, 2, \ldots\}$ in $B(X)$ is *asymptotically compact* if for any sequence $\{x_n\} = \{x_n : n = 1, 2 \ldots\}$ in U_X, $\{T_n x_n\}$ is d-compact, i.e., every subsequence of $\{T_n x_n\}$ has a convergent subsequence.

Definition 4.16 $T_n \stackrel{ac}{\rightarrow} T$ is said to be *asymptotically compact convergence* if $T_n \rightarrow T$ and $\{T_n\}$ is asymptotically compact.

Recall that $\{T_n\}$ collectively compact implies that each T_n is compact. This need not to be the case when asymptotically compact sequences are considered.

Example 4.2 Let $T_n = \frac{1}{n}I$. Since $\| T_n \| \to 0$, $\{T_n\}$ asymptotically compact but $\frac{1}{n}I$ is never compact when $\dim X = \infty$.

In the following theorem, Anselone and Ansorge [4] have shown that the lack of compactness of the individual operators is the only difference between collectively and asymptotically compact sequences. We will give the detailed proof of it.

Theorem 4.6 *Let $T, T_n \in B(X)$. Then*

(a) $\{T_n\}$ *collectively compact* $\Leftrightarrow$ $\{T_n\}$ *asymptotically compact and* $\{T_n\}$ *compact* $\forall n$,

(b) $T_n \overset{cc}{\to} T \Rightarrow T_n \overset{ac}{\to} T \Rightarrow T$ *compact.*

Proof. (a)$\Rightarrow$: Let $\{T_n\}$ be collectively compact. Then $\bigcup_{n=1}^{\infty} T_n(U_X)$ is relatively compact, which means that $\overline{\bigcup T_n(U_X)}$ is compact. From (4.9) we know that $(\bigcup \overline{T_n(U_X)}) \cup \{T_n(U_X)\}^* \subset \overline{\bigcup T_n(U_X)}$ (compact) and $(\bigcup \overline{T_n(U_X)}) \cup \{T_n(U_X)\}^*$ is closed. Then, we have $(\bigcup \overline{T_n(U_X)}$ is closed. Therefore $\forall n$, $\overline{T_n(U_X)}$ is compact.From this it follows that $\forall n$, $T_n(U_X)$ is relatively compact. Hence $\forall n$, T_n compact.

Now we will show T_n is asymptotically compact.We have, $\overline{\bigcup T_n(U_X)}$ is compact then, $\{T_n(U_X)\}$ is d-compact.

Indeed, each subsequence $\{x_n \in T_n(U_X) \ : \ n \in N'\} \subset \overline{\bigcup T_n(U_X)}$. We also know that $\{T_n(U_X)\}^* = \{x \in X \ : \ x_n \to x, \ x_n \in T_n(U_X), \ n \in N'\} \neq \emptyset$. So, let $\{x_m\} \not\subset \bigcup T_n(U_X)$ and $x_m \to x$. Then $\{x_m\} \subset \bigcup_{k=1}^{N} \overline{T_k(U_X)} \Rightarrow x \in \bigcup \overline{T_n(U_X)}$ or $\{x_m\} \not\subset \bigcup_{k=1}^{N} \overline{T_k(U_X)}$ then $x \in \{T_n(U_X)\}^*$. So $\{T_n(U_X)\}$ is d-compact. Therefore $\{T_n\}$ is asymptotically compact.

$\Leftarrow$: Let $\{T_n(U_X)\}$ is d-compact and for all n, T_n is compact, i.e., $T_n(U_X)$is relatively compact. Suppose first $\{x_m\} \subset \bigcup_{k=1}^{N} \overline{T_k(U_X)}$ then we get $\overline{\bigcup T_n(U_X)}$ is compact.Now assume $\{x_m\} \not\subset \bigcup_{k=1}^{N} T_k(U_X)$then there exists an N' such that $x_m \in T_m(U_X)$, $m \in N'$. Since $T_n(U_X)$ is d-compact then for $x_m \to x$ we have $x \in \{T_n(U_X)\}^*$which means that $x \in \overline{\bigcup T_n(U_X)}$.

(b) It is known that $T_n \overset{cc}{\to} T$ means $T_n \to T$ and $\{T_n\}$ collectively compact. Therefore for each n, T_n is compact. Since $\{T_n\}$ is collectively compact then we know that $\overline{\bigcup T_n(U_X)}$ is compact $\Leftrightarrow$ $\{T_n(U_X)\}$ d-compact and for each n, T_n is compact. Hence $\{T_n\}$ is asymptotically compact and since $T_n \to T$ then we have $T_n \overset{ac}{\to} T$.

To show that $T_n \overset{ac}{\to} T \Rightarrow T$ is compact, we will use Theorem 4.1, which states $\{T_n(U_X)\}^*$ is compact. Since we know $T_n \to T \Rightarrow T(U_X) \subset \{T_n(U_X)\}^* \Rightarrow \overline{T(U_X)} \subset \{T_n(U_X)\}^*$.

$\overline{T(U_X)}$ is compact. Then T is compact. □

Theorem 4.7 [18] *Let $K, L, K_n, L_n \in B(X)$ with $K_n \overset{ac}{\to} K$, $L_n \overset{ac}{\to} L$. Suppose we have the scalar convergence $\alpha_n \to \alpha$, $\beta_n \to \beta$. Then we have following convergence; $\alpha_n K_n \overset{ac}{\to} \alpha K$, $\beta_n L_n \overset{ac}{\to} \beta L$, $\alpha_n K_n + \beta_n L_n \overset{ac}{\to} \alpha K + \beta L$, and $K_n L_n \overset{ac}{\to} KL$. Moreover, $L_n K \overset{ac}{\to} LK$ and $KL_n \overset{ac}{\to} KL$.*

As a special case of the previous theorem we note that

$$K_n \overset{cc}{\to} K \quad \text{and} \quad \| L_n \| \to 0 \quad \text{then} \quad K_n + L_n \overset{ac}{\to} K, \tag{4.30}$$

but $L_n \overset{cc}{\to} 0$ and $K_n + L_n \overset{cc}{\to} K$ only when every L_n is compact. They are not compact if $L_n = c_n I$, with $c_n \downarrow 0$ and $dim X = \infty$.

Theorem 4.8 [18] *Let $T_n \in B(X)$.*

$$\{T_n\} \textit{ asymptotically} \quad \textit{compact} \Leftrightarrow \lim_{k\to\infty} \mathcal{X}(\bigcup_{n\geq k} T_n(U_X)) = 0$$

The proof of this theorem, immediately follows from Corollary 4.1.

Corollary 4.3 [7] I*f $\{T_n\}$ is asymptotically compact then T_n is $\mathcal{X}$-condensing, $\forall n$ large. Thus, $\{T_n\}$ asymptotically compact implies that T_n is condensing and $I - T_n$ is Fredholm with index zero for all n large enough.*

We now study again the linear equations $(I-T)x = y$ and $(I-T_n)x_n = y$ where $T_n \overset{ac}{\to} T$. The organization for our study is essentially given for the collectively compact theory by Anselone [3] . If $(I - T)$ is invertible, then the equation $(I - T)x = y$ has a unique solution for each y and the solution depends continuously on y. We are investigating what can be said about the approximate solutions of the equations $(I - T_n)x_n = y$. The following lemma is useful to answer this question.

Lemma 4.5 [18] *Let$T, T_n \in B(X)$ and $T_n \overset{ac}{\to} T$. Then $(I - T_n) \overset{r}{\to} (I - T)$.*

Proof. The pointwise convergence is clear from the definitions. To show $\{I - T_n\}$ asymptotically regular it suffices to show that $\{x_n\}^* \neq \emptyset$ when $\{x_n\}$ is bounded and

$(I - T_n)x_n \to y$. Since $\{T_n\}$ is asymptotically compact $\exists N' \subset \mathbb{N},\ z \in X$ such that $T_n x_n \to z$ for $n \in N'$. Hence $x_n = T_n x_n + (I - T_n)x_n \to z + y$ for $n \in N'$, which is the desired result. □

Theorem 4.9 [18] *Let* $T, T_n \in B(X)$ *and* $T_n \overset{ac}{\to} T$. *Then there exists* $(I - T)^{-1} \Leftrightarrow$ *there exist* $(I - T_n)^{-1}$ *uniformly bounded for all n large, in which case* $(I - T_n)^{-1} \to (I - T)^{-1}$ *on X.*

Lemma 4.6 [18] Let $T_n, T \in B(X)$ with $T_n \to T$ and suppose $\{S_n\}$ is asymptotically totally bounded. Then $\| T_n x_n - T x_n \| \to 0$ uniformly in the choice of sequences $\{x_n : x_n \in S_n\}$.

Corollary 4.4 [18] Let $T,\ T_n \in B(X)$ and $T_n \overset{ac}{\to} T$. Then

$$\| (T_n - T)T \| \to 0, \qquad \| (T_n - T)T_n \| \to 0 \text{ as } n \to \infty.$$

It is shown that the results for the equation $(I - T)x = y$ when T is compact extend to the case when T has a sufficiently small measure of non-compactness (e.g.,$\| T \|_{\mathcal{X}} < | \lambda |$). We will give the difference between the operators with a small measure of non-compactness and compact operators. Hence we will consider the measure $\| \, . \, \|_{\mathcal{K}}$. For if $\epsilon > 0$ and $\| T \|_{\mathcal{K}} < \epsilon$, then there exist a compact operator L and a bounded operator K such that $T = K + L$ with $\| T - L \| = \| K \| < \epsilon$. We note that it is enough only to consider $\epsilon = 1$. We may not replace $\| \, . \, \|_{\mathcal{K}}$ with $\| \, . \, \|_{\mathcal{X}}$. In fact, if the operators are defined on a Hilbert space H, then it follows from the fact that $\| T \|_{\mathcal{K}} = \| T \|_{\mathcal{X}}$ for all $T \in B(H)$, see [18, Theorem 3.29]. On the other hand, Goldenstein and Markus [10] give an example of an operator defined on a product of sequence spaces which is $\mathcal{X}$-condensing but not $\mathcal{K}$-condensing.

Recall that

$$\{T_n\} \text{ is aymptotically compact } \Rightarrow \| T_n \| \to 0 \text{ as } n \to \infty. \tag{4.31}$$

and

$$\{T_n\} \text{ is collectively compact } \Leftrightarrow \{T_n\} \text{ is asymptotically compact, and } \forall\, n,\ T_n \text{ compact.} \tag{4.32}$$

Hence, we consider an analogous question relating asymptotically and collectively compact sequences, i.e., it is true that

$$\{T_n\} \text{ aymptotically compact } \Leftrightarrow \exists\{L_n\} \text{ collectively compact s. t. } \| T_n - L_n \| \to 0 \quad \mathbf{(4.33)}$$

Sufficiency follows from Theorem 4.7 since $\{L_n\}$, $\{T_n - L_n\}$ asymptotically compact, we have $\{T_n - L_n + L_n\}$ asymptotically compact.
To prove necessity, since we know from (4.14) $\| \,.\, \|_{\mathcal{X}} \leq \| \,.\, \|_{\mathcal{K}}$ so it is enough to find a constant $C > 0$ s.t. $\| \,.\, \|_{\mathcal{K}} \leq C \| \,.\, \|_{\mathcal{X}}$. If such a constant exists,

$$\{T_n\} \text{ ac } \Rightarrow \| T_n \|_{\mathcal{X}} \to 0 \Rightarrow \| T_n \|_{\mathcal{K}} \to 0 \Rightarrow \exists\{L_n\} \subset \mathcal{K} \text{ s.t. } \| T_n - L_n \| \to 0\ .$$

Since $\{L_n = T_n + (L_n - T_n)\}$ is asymptotically compact and each $\{L_n\}$ is compact then we have $\{L_n\}$ collectively compact by Theorem 4.6 (a). Therefore such a constant exists of the semi-norms $\| \,.\, \|_{\mathcal{X}}$ and $\| \,.\, \|_{\mathcal{K}}$ are equivalent on $B(X)/\mathcal{K}$. The equivalence of these semi-norms will be shown for a large class of spaces by the next definition.

Definition 4.17 [15] A Banach space X is said to have the *compact approximation property* (abbreviated C.A.P.) if for each $\epsilon > 0$ and finite set of points $x_1, \ldots, x_n \in X$ there exists $K \in \mathcal{K}$ s.t. $\| x_k - Kx_k \| \leq \epsilon$, $1 \leq k \leq n$. If $1 \leq \lambda \leq \infty$, then X has the *$\lambda$-compact approximation property* (abbr. λ-C.A.P.) if X has the C.A.P. with $\| K \| \leq | \lambda |$.
Lemma 4.7 [14] *Let X have the λ-C.A.P. Then*

$$\| T \|_{\mathcal{K}} \leq (\lambda + 1)\ \| T \|_{\mathcal{X}},\ \forall T \in B(X).$$

Theorem 4.10 [18] *Let X have the λ-C.A.P. and $\{T_n\}$, $\{L_n\} \subset B(X)$ Then*

$$\{T_n\} \textit{ asymptotically compact } \Leftrightarrow T_n = L_n + K_n,\ \{L_n\} \textit{ collectively compact },\ \| K_n \| \to 0.$$

Proof. First assume $T_n \subset B(X)$ is asymptotically compact. Then by Theorem 4.8 we know that $\lim_{k\to\infty} \mathcal{X}(\bigcup_{n\geq k} T_n(U_X)) = 0$ and since, $\mathcal{X}(T_n(U_X)) = \| T_n \|_{\mathcal{X}}$ we have $\| T_n \|_{\mathcal{X}} \to 0$. By Lemma 4.7 $\| T_n \|_{\mathcal{K}} \to 0$. It follows that $\exists\{L_n\} \subset \mathcal{K}$ such that $\| T_n - L_n \| \to 0$. Set $K_n = T_n - L_n$. Then $\| K_n \| \to 0$ and $\{K_n\}$ is asymptotically compact. $\{L_n = T_n - K_n\}$ is asymptotically compact. But since each L_n is compact we have $\{L_n\}$ is collectively compact.

Now suppose $T_n = L_n + K_n$, $\{L_n\}$ collectively compact, $\| K_n \| \to 0$. Then $\{L_n\}$ is asymptotically compact. So is $T_n = L_n + K_n$. $\square$

4.4 Spectral Comparisons

Let again X be a Banach space, U_X be the closed unit ball in X and $B(X)$ is the space of bounded linear operators on X. We want to recall that a sequence $\{T_n\}$ of bounded linear operators on a Banach space X is asymptotically compact if, for any bounded sequence $\{x_n\}$ in X, every subsequence of $\{T_n x_n\}$ has a convergent subsequence.

In this section we will compare the spectra of T and T_n as $n \to \infty$, particularly for X a complex Banach space. The results closely parallel the more completely studied case with $\{T_n\}$ collectively compact.

We will give the following theorem that plays an essential role in our study of the spectrum of T_n as $n \to \infty$.

Theorem 4.11 [7] *Assume $\{T_n\}$ is asymptotically compact and $\delta > 0$. Then there exists $n(\delta)$ such that for $n \geq n(\delta)$ and $|\lambda| > \delta$, $\lambda - T_n$ is a Fredholm operator with $ind(\lambda - T_n) = 0$ and $\lambda \in \rho(T_n)$ or $\lambda \in \sigma_p(T_n)$.*

Proof. Let $\{T_n\}$ be asymptotically compact. Then by Corollary 4.3 $\| T_n \|_X \to 0$. So,

$$\exists n(\delta) \ni \ \forall n \geq n(\delta),\ \| T_n \|_X < \delta.$$

Now let $S_n := \frac{1}{\lambda} T_n$. Then $\| S_n \|_X = \| \frac{1}{\lambda} T_n \|_X = \frac{1}{\lambda} \| T_n \|_X < \frac{\delta}{|\lambda|} < 1$.

By Corollary 4.3, $I - \frac{1}{\lambda} T_n$ is Fredholm with $ind(I - \frac{1}{\lambda} T_n) = 0$. Hence, $\lambda I - T_n$ is Fredholm with $ind(\lambda I - T_n) = 0$, and so, $\lambda \in \rho(T_n)$ or $\lambda \in \sigma_p(T_n)$. $\square$

The following two results are consequences of $T_n \xrightarrow{ac} T$.

Lemma 4.8 [7] *Let $T_n \xrightarrow{ac} T$, $(\lambda_n - T_n)x_n \to 0$, and $\lambda_n \to \lambda \neq 0$ with $\| x_n \| = 1$ and $n \in N'$. Then $\lambda \in \sigma(T)$, and there exists N'' and x such that $x_n \to x$ with $n \in N''$, $\| x \| = 1$, and $Tx = \lambda x$.*

Proof. We know that by Lemma 4.5 when $T_n \xrightarrow{ac} T \Rightarrow T_n \xrightarrow{r} T$. It follows that $T_n \to T$ and $\{T_n\}$ asymptotically regular. From the definition of asymptotically regular sequence we know that for $T_n \to T$, $\forall \{x_n\}$ bounded $T_n x_n \to y$, $n \in N' \subset \mathbb{N}$, then $x_n \to x$, $n \in N'' \subset N'$ and $Tx = y$. And also we are given $\lambda_n x_n - T_n x_n \to 0$ which means that $\lambda_n x_n = T_n x_n$.Therefore, $\lambda x = Tx = y$. With this we have $\lambda \in \sigma_p(T)$, so $\lambda \in \sigma(T)$. □

Theorem 4.12 [7] *Let $T_n \overset{ac}{\to} T$. Then $\lambda \in \rho(T) \Leftrightarrow \lambda \in \rho(T_n)$ and $(\lambda - T_n)^{-1}$ is bounded uniformly for all n large, in which case $(\lambda - T_n)^{-1} \to (\lambda - T)^{-1}$.*

We know that for $T, T_n \in B(X)$ when $\| T_n - T \| \to 0$, $\lambda_n \in \sigma(T_n)$ and $\lambda_n \to \lambda \neq 0$, we have $\lambda \in \sigma(T)$. In the previous chapter it is noted that we have the same result if there is collectively compact convergence instead of norm convergence by P.M. Anselone. Now we give that result for asymptotically compact convergence sequence of operators. Moreover it compares the dimensions of eigenmanifolds of T and T_n.

Theorem 4.13 [7] *Let $T, T_n \in B(X)$, $\lambda_n \in \sigma(T_n)$, and $\lambda_n \to \lambda \neq 0$. Then $\lambda \in \sigma(T)$ and $dim\mathcal{N}(\lambda_n - T_n) \leq dim\mathcal{N}(\lambda - T)$ for all large n.*

Proof. Assume $dim\mathcal{N}(\lambda_n - T_n) \geq m$ for all n in some N'. By the Riesz lemma, there exist $x_{nk} \in \mathcal{N}(\lambda_n - T_n)$ such that

$$\| x_{nk} \| = 1 \quad \text{and} \quad \| x_{nk} - \sum_{j=1}^{k-1} c_j x_j \| \geq 1$$

for $n \in N', k = 1, 2, \ldots, m$, and any c_j. By Lemma 4.8 , there exist N'' and x_k, for $k = 1, \ldots, m$, such that $x_{nk} \to x_k$ with $n \in N''$, $x_k \in \mathcal{N}(\lambda - T)$, and

$$\| x_k \| = 1, \quad \| x_k - \sum_{j=1}^{k-1} c_j x_j \| \geq 1$$

for $k = 1, 2, \ldots, m$ and any c_j. So $\{x_k\}$ is linearly independent and $dim\mathcal{N}(\lambda - T) \geq m$. Contrapositively, if $dim\mathcal{N}(\lambda - T) < m$ then $dim\mathcal{N}(\lambda_n - T_n) < m$ for all n large. Hence, $dim\mathcal{N}(\lambda_n - T_n) \leq dim\mathcal{N}(\lambda - T)$ for all n large. It follows that $\lambda \in \sigma(T)$. □

From now on, let X be a complex Banach space. Let $\mathbb{C}$ be the complex plane and $\widetilde{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, the one-point compactification. Then closed sets are compact in $\mathbb{C}$. For any $T \in B(X)$, we have $\infty \in \rho(T)$ and $(\lambda - T)^{-1} = 0$ at $\lambda = \infty$. First, we will compare resolvent sets $\rho(T)$ and $\rho(T_n)$. As in the previous chapter, we will define the below equation

$$L_n(\lambda) = (T_n - T)(\lambda - T)^{-1} \text{ for } \lambda \in \rho(T),\ n \in \mathbb{N} \qquad \textbf{(4.34)}$$

By simple calculation;

$$(\lambda - T_n) = [I - L_n(\lambda)](\lambda - T) \text{ if } \lambda \in \rho(T), \qquad \textbf{(4.35)}$$

$$(\lambda - T_n)^{-1} = (\lambda - T)^{-1}[I - L_n(\lambda)]^{-1}, \qquad \textbf{(4.36)}$$

whenever the inverses exists.

Lemma 4.9 [7] *Let* $T_n \overset{ac}{\to} T$ *and* $\lambda \in \rho(T)$. *Then* $L_n(\lambda) \overset{ac}{\to} 0$ *and* $\| L_n(\lambda)^2 \| \to 0$. *If* $\| L_n(\lambda)^2 \| < 1$, *then* $[I - L_n(\lambda)]^{-1} \in B(X)$, *and*

$$[I - L_n(\lambda)]^{-1} \leq \frac{\| I + L_n(\lambda) \|}{1 - \| L_n(\lambda)^2 \|}.$$

Theorem 4.14 [7] *Let* $T_n \overset{ac}{\to} T$ *and* $\lambda \in \rho(T)$. *Then for all n large:*

(a)

$$\lambda \in \rho(T_n),\ \| (\lambda - T_n)^{-1} \| \leq \frac{\| (\lambda - T) \| \| I + L_n(\lambda) \|}{1 - \| L_n(\lambda)^2 \|}.$$

(b)

$$(\lambda - T_n)^{-1} - (\lambda - T)^{-1} \overset{ac}{\to} 0.$$

Proof. Equation (4.21) and Lemma 4.9 yields (a).

By Theorem 4.12 $(\lambda - T_n)^{-1} - (\lambda - T)^{-1} \to 0$. It follows from the identity

$$(\lambda - T_n)^{-1} - (\lambda - T)^{-1} = (\lambda - T)^{-1}(T_n - T)(\lambda - T_n)^{-1}$$

and $T_n \overset{ac}{\to} T$ that $\{(\lambda - T_n)^{-1} - (\lambda - T)^{-1}\}$ is asymptotically compact.

Now we can allow λ to vary in a closed subset of $\sigma(T)$. □

Theorem 4.15 [7] *Let* $T_n \overset{ac}{\to} T$, $\Gamma \subset \rho(T)$, *closed (compact in* $\mathbb{C}$ *). Then,*

(a) $\{L_n(\lambda) : \lambda \in \Gamma,\ n \in N\}$ *is bounded and equicontinuous in* λ,

(b) $L_n(\lambda)x \to 0$ *as* $n \to \infty$ *for all* $x \in X$, *uniformly for* $\lambda \in \Gamma$,

(c) $\| L_n(\lambda)^2 \| \to 0$ *as* $n \to \infty$, *uniformly for* $\lambda \in \Gamma$.

Theorem 4.16 [7] *Let* $T_n \overset{ac}{\to} T$, $\Gamma \subset \rho(T)$, Γ *closed. Then there exists* n_0 *such that for* $n \geq n_0$,

(a) $\Gamma \subset \rho(T_n), \{(\lambda - T_n)^{-1} : \lambda \in \Gamma,\ n \geq n_0\}$ *is bounded,*

(b) $(\lambda - T_n)^{-1}x \to (\lambda - T)^{-1}x$ *as* $n \to \infty$ *for all* $x \in X$, *uniformly for* $\lambda \in \Gamma$.

The following theorem says that spectrum is upper semi-continuous for asymptotically compact convergence sequence of operators which is a generalization of Theorem 3.3.

Theorem 4.17 [7] *Let* $T_n \xrightarrow{ac} T$, $\Omega \supset \sigma(T)$, Ω *open. Then* $\Omega \supset \sigma(T_n)$ *for all n large.*

Proof. By taking complements in Theorem 4.16, we obtain the desired result. □

The next result involves integrals of operator valued functions. Let Γ be a finite interval if X is real and a rectifiable arc if X is complex. Suppose $T_\alpha(\lambda) \subset B(X)$ for $\lambda \in \Gamma$ and α in an index set A. For each $\alpha \in A$, assume that $\int_\Gamma T_\alpha(\lambda)d\lambda$ is the strong or norm limit of the usual approximating sums $\sum_{j=1}^{m} T_\alpha(\lambda'_j)(\lambda_j - \lambda_{j-1})$.

Lemma 4.10 *With the foregoing notation, if* $\{T_\alpha\lambda : \alpha \in A, \lambda \in \Gamma\}$ *is asymptotically compact, then* $\{\int_\Gamma T_\alpha d\lambda : \alpha \in A\}$ *is asymptotically compact.*

Proof. This follows from Theorem 4.7 and $\sum_{j=1}^{m} |\lambda_j - \lambda_{j-1}| \leq \text{length}(\Gamma)$. □

Theorem 4.18 *Let* $T, T_n \in B(X)$, $n = 1, 2, \ldots$. *Assume* $T_n \to T$ *and* $\{T_n - T\}$ *is asymptotically compact and,* $f \in \mathcal{F}(T)$. *Then there exists* $N \in \mathbb{N}$ *such that for all* $n \geq N$,

(a) $f \in \mathcal{F}(T_n)$,

(b) $f(T_n)x \to f(T)x$ for each $x \in X$,

(c) $\{f(T_n) - f(T)\}$ is asymptotically compact for $n \geq N$.

Proof. (a) Choose $\Gamma \subset \mathfrak{D}(f) \cap \rho(T)$ with $\sigma(T)$ inside Γ. By Theorem 4.17, there exists an $N \in \mathbb{N}$ such that $\forall n \geq N$, $\sigma(T_n)$ inside Γ.
(b) For $n \geq N$ and $x \in X$ we have

$$f(T_n)x - f(T)x = \frac{1}{2\pi i}\int_\Gamma f(\lambda)[(\lambda I - T_n)^{-1}x - (\lambda I - T)^{-1}x]d\lambda.$$

By Theorem 4.16(b), $f(\lambda)[(\lambda I - T_n)^{-1}x - (\lambda I - T)^{-1}x] \to 0$ uniformly for $\lambda \in \Gamma$ as $n \to \infty$.
(c) By Theorem 4.14(b), $\{(\lambda I - T_n)^{-1} - (\lambda I - T)^{-1}\}$ is asymptotically compact. Moreover, the set $\{f(\lambda)[(\lambda I - T_n)^{-1} - (\lambda I - T)^{-1}]\}$ is also asymptotically compact. So, by the argument used in the proof of Theorem 3.4(c), for $n \geq N$, we have

$$f(T_n) - f(T) = \frac{1}{2\pi i}\int_\Gamma f(\lambda)(\lambda I - T_n)^{-1}(T_n - T)(\lambda I - T)^{-1}d\lambda,$$

where $\{f(\lambda)(\lambda I - T_n)^{-1}(T_n - T)(\lambda I - T)^{-1} : \lambda \in \Gamma, n \geq N\}$ is asymptotically compact. Hence, by Lemma 4.10, $\{f(T_n) - f(T)\}$ is asymptotically compact. □

Now we can generalize Theorem 4.17 as follows.

Theorem 4.19 *Assume $T, T_n \in B(X)$ and $T_n \stackrel{ac}{\to} T$. If U and V are disjoint open sets with $\sigma(T) \subset U \cup V$ and $\sigma(T) \cap U \neq \emptyset$, then there exists $N \in \mathbb{N}$ such that $\sigma(T_n) \cap U \neq \emptyset$ for all $n \geq N$.*

Proof. Let us observe that the family $\{T_n - T\}$ is asymptotically compact. Let f be the analytic function defined on $U \cup V$ as $f(\lambda) \equiv 1$ on U and $f(\lambda) \equiv 0$ on V. By Theorem 4.17, $\sigma(T_n) \subset U \cup V$ for all $n \geq N$ for some $N \in \mathbb{N}$.

Assume that the claim is not true. Then for each N, $\exists n \geq N$ such that $\sigma(T_n) \subset V$. In this way we construct a subsequence of operators (T_{n_k}) with the property that $\sigma(T_{n_k}) \subset V$ for each k.

On the other hand, from Theorem 4.18(b) as $f(T_n)x \to f(T)x$ then, it follows that $f(T_{n_k})x \to f(T)x$.

As $f(T_{n_k}) = 0$ for each k, $f(T)x = 0$ for each x. By the spectral mapping theorem, $\sigma(f(T)) = f(\sigma(T))$ and 1 must be in $\sigma(f(T))$, $f(T)$ can not be the zero operator. This is a contradiction. This yields $\sigma(T_n) \cap U \neq \emptyset$ for all $n \geq N$ for some N. □

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