İSTANBUL TECHNICAL UNIVERSITY ★ INSTITUTE OF SCIENCE AND TECHNOLOGY

TOWARDS THE CLASSIFICATION OF SCALAR INTEGRABLE EVOLUTION EQUATIONS IN (1+1)-DIMENSIONS

Ph.D. Thesis by
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JUNE 2008

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Date of submission : 19 February 2008

Date of defence examination: 18 June 2008

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JUNE 2008

İSTANBUL TEKNİK ÜNİVERSİTESİ ★ FEN BİLİMLERİ ENSTİTÜSÜ

SINIFLANDIRMA YOLUNDA (1+1)-BOYUTTA İNTEGRE EDİLEBİLİR SKALER EVRİM DENKLEMLERİ

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Tezin Enstitüye Verildiği Tarih : 19 Şubat 2008
Tezin Savunulduğu Tarih : 18 Haziran 2008

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HAZİRAN 2008

ACKNOWLEDGEMENTS

I would like to express my heartfelt thanks to my supervisor Prof. Ayşe H. Bilge, for her continuous support, encouragement and assistance in every stage of the thesis.

I would also like to express my heartfelt thanks to Professor Avadis Hacınlıyan, for his help and support in programming languages.

I owe my deepest gratitude to my husband Jeki for his caring throughout my life and for encouraging me in everything I decided to do.

I also wish to give my very special thanks to my daughter Leyla, my son Vedat and my daughter in law Suzi for living with the thesis as well as me and giving me love and support all through the years.

June 2008 Eti Mizrahi

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LIST OF THE SYMBOLS

u	: Dependent variable.
u_k	: kth Derivative of u w.r.t x
$K^{(i)}$	: Constant labeled by i.
s	: Scaling weight.
$wt(u)$	: Weight of u.
$F[u]$	: Differential function.
F_*	: Frechet derivative of F.
R	: Recursion operator.
D	: Total derivative with respect to x.
D_t	: Total derivative with respect to t.
D^{-1}	: Inverse of D.
σ	: Symmetry of a differential equation.
ρ	: Conserved density of a differential equation.
G	: Group.
$\mathcal{K}$	: Ring.
$\mathcal{S}$	: Commutative ring.
$\mathcal{C}$	: Field.
V	: Vector space.
W	: Associative algebra.
M	: Graded, filtered algebra.
M^i	: Subspace of the graded algebra.
$g(M^i)$	: Grade of the subspace M^i.
m_i	: Monomial.
$d(m_i)$	: Degree of the monomial m_i.
$l(m_i)$	: Level of the monomial m_i.
M_d	: Free module generated by monomials.
M_d^l	: Submodules of free module M_d generated. by monomials of level l .
$\overline{M_d^l}$	: Quotient submodules.

TOWARDS THE CLASSIFICATION OF SCALAR INTEGRABLE EVOLUTION EQUATIONS IN (1+1)-DIMENSIONS

SUMMARY

In the literature, integrable equations are meant to be non-linear equations which are solvable by a transformation to a linear equation or by an inverse spectral transformation [22]. The difficulty in constructing an inverse spectral transformation had motivated the search for other methods which would identify the equations expected to be solvable by an inverse spectral transformation. These methods which consist of finding a property shared by all known integrable equations are called "integrability tests". The existence of an infinite number of conserved quantities, infinite number of symmetries, soliton solutions, Hamiltonian and bi-Hamiltonian structures, Lax pairs, Painleve property, are well known integrability tests.

"The classification problem" is defined as the classification of families of integrable differential equations. Recently Wang and Sanders used the existence of infinitely many symmetries to solve this problem for polynomial scale invariant, scalar equations, by proving that scale invariant scalar integrable evolution equations of order greater than seven are symmetries of third and fifth order equations [3].

The first result towards a classification for arbitrary m'th order evolution equations is obtained in [1] where it is shown that scalar evolution equations $u_t = F[u]$, of order $m = 2k + 1$ with $k \geq 3$, admitting a nontrivial conserved density $\rho = Pu_n^2 + Qu_n + R$ of order $n = m + 1$, are quasi-linear. This result indicates that essentially non-linear classes of integrable equations arising at the third order are absent for equations of order larger than 7 and one may hope to give a complete classification in the non-polynomial case. This is the motivation of the present work where the problem of classification of scalar integrable evolution equations in (1+1) (1 spatial and 1 temporal)-dimensions is further analyzed.

In this thesis, we use the existence of a formal symmetry introduced by Mikhailov et al. as the integrability test [2]. We introduce a graded algebra structure "the level grading" on the derivatives of differential polynomials. Our main result is the proof that arbitrary (non-polynomial) scalar, integrable evolution equations of order m, are polynomial in top three derivatives, namely u_{m-i}, $i = 0, 1, 2$. In the proof of this result, explicit computations are needed at lower orders and computations for equations of order 7 and 9 are given as examples.

In our computations we used three conserved densities, $\rho^{(1)}, \rho^{(2)}, \rho^{(3)}$ obtained in [1]. Computations for the general case and for the lower orders showed that it is impossible to obtain polynomiality in u_{m-3} by using only these three conserved

densities. Thus the investigation of polynomials beyond u_{m-3} is postponed to future work.

The first section is devoted to a general introduction with a literature review, on conservation laws, symmetries, integrability and classification, beginning from the discovery of the soliton.

Notation used in this study, basic definitions and preliminary notions about integrability tests, symmetries and recursion operators with examples on KdV equations are given in the second section.

Main results are gathered in sections three and four. In section three, we give basic definitions and properties of graded and filtered algebras, and define the "level grading" while in section four we present the polynomiality results.

In section three, a graded algebra structure on the polynomials in the derivatives u_{k+i} over the ring of functions depending on $x, t, u, \dots, u_k$ is introduced. This grading, called "level grading", is motivated by the fact that derivatives of a function depending on $x, t, u, \dots, u_k$ are polynomial in higher order derivatives and have a natural scaling by the order of differentiation above the "base level k". The crucial point is that, equations relevant for obtaining polynomiality results involve only the term with top scaling weight with respect to level grading. This enables to consider top level term only, disregarding the lower ones, and to reduce symbolic computations to a feasible range.

Polynomiality results on the classification of scalar integrable evolution equations of order m are given in section four. In our computations we proved that arbitrary scalar integrable evolution equations of order $m \geq 7$ are polynomial in the derivatives u_{m-i} for $i = 0, 1, 2$.

Section five is devoted to explicit computations for the classification of 7th and 9th order evolution equations. The purpose of this section is to give an information about explicit computations and compare with the solutions for general m. In particular, at order 7, it is shown that no further information is obtained by the use of all conserved densities $\rho^{(i)}$, $i = -1, 1, 2, 3$.

The discussion of the results and directions for future research are given in section six.

Appendices A,B,C,D, give respectively the submodules and quotient submodules with their generating monomials, used in the classification of 7th and 9th order evolution equations. One can derive easily the monomials for evolution equations of order higher than nine using these lists.

SINIFLANDIRMA YOLUNDA (1+1)-BOYUTTA İNTEGRE EDİLEBİLİR SKALER EVRİM DENKLEMLERİ

ÖZET

Literatürde, "integre edilebilen denklemler", lineer denklemlere dönüştürülebilen ya da ters spektral dönüşüm ile çözülebilen denklemler olarak tanımlanır [22]. Ters spektral dönüşümlerin inşaasının çok zor olması, ters spektral dönüşüm ile çözülmeye aday denklemleri belirleyebilecek yöntemlerin geliştirilmesine yolaçmıştır. Bilinen tüm integre edilebilen denklemlerin ortak bir özelliğini bulmaya dayalı bu yöntemlere "integrabilite testleri" adı verilir. Sonsuz sayıda korunan nicelikler, sonsuz sayıda simetriler, soliton çözümleri, Hamiltonyen ve bi-Hamiltonyen yapı, Lax çiftleri veya Painlevé özelliğinin varlığı, yaygın kullanılan integrabilite testleri olarak bilinir.

"Sınıflandırma problemi", integre edilebilir diferansiyel denklem ailelerinin sınıflandırılması olarak bilinir. Yakın geçmişte Wang ve Sanders, sonsuz sayıda simetrilerin varlığını kullanarak, ölçek bağımsız skaler integre edilebilir, 7 inci mertebeden büyük, evrim denklemlerinin, 3 üncü ve 5 inci mertebeden denklemlerin simetrileri olduğunu göstererek sınıflandırma problemini, polinom ölçek bağımsız skaler denklemler için çözmüşlerdir. [3].

Keyfi m inci mertebeden evrim denklemlerinin sınıflandırılması hakkında, ilk sonuç, [1]'de elde edilmişir. Bu sonuç, $n = m + 1$ mertebeden, trivial olmayan korunan yoğunluk (conserved density) olarak $\rho = Pu_n^2 + Qu_n + R$ yu kabul eden, $m = 2k + 1$, ve $k \geq 3$ mertebeden, $u_t = F[u]$ evrim denklemlerinin kuazilineer olmasıdır. Elde edilen sonuca göre özellikle 3 üncü mertebede ortaya çıkan, lineer olmayan, integre edilebilir evrim denklemlerinin sınıfları, 7 den büyük mertebelerde gözükmez. Bu nedenle polinom olmayan durumlar için bir sınıflandırma yapılabileceği düşünülebilir. Bu düşünceden yola çıkarak, bu çalışmada, (1+1) boyutta (1 uzaysal 1 zamansal) integre edilebilir evrim denklemlerinin sınıflandırılması problemi ele alınmıştır.

Bu tezde, integrabilite testi olarak, Mikhailov ve diğerleri tarafından ortaya konan, biçimsel simetrilerinin varlığı kabul edilmiştir [2]. Ayrıca "Level grading" adı verilen, diferansiyel polinomların türevleri üzerine bir kademeli cebir (graded algebra) yapısı tanımlanmıştır. Bu çalışmanın esas sonucu, keyfi polinom olmayan skaler integre edilebilir m inci mertebeden evrim denklemlerinin u_{m-i}, $i = 0, 1, 2$ olmak üzere, en üst üç mertebeden türeve göre polinom olduğunun ispatıdır. Bu sonucun ispatı, düşük mertebelerde açık hesaplamaların yapılmasını gerektirdiğinden, 7 inci ve 9 uncu mertebeden keyfi skaler evrim denklemleri, örnek olarak, açık şekilde hesaplanmıştır.

Bu çalışmada, [1]'de hesaplanan ve biçimsel simetrinin varlığının bir sonucu olan, üç korunan yoğunluk, $\rho^{(1)}, \rho^{(2)}, \rho^{(3)}$ (conserved densities) kullanılmıştır.

Genel m ve $m \leq 19$ için yapılan hesaplamalarda, sadece bu üç korunan yoğunluk kullanılarak, daha alt mertebeler (örneğin u_{m-3}) için polinomluğun elde edilmesinin imkansız olduğu görülmüştür. Böylece problem ile ilgili bundan başka yapılacak olan tartışmalar ileriki çalışmalara ertelenmiştir.

Birinci bölümde genel bir giriş ile birlikte, soliton dalgaların keşfi ile başlayan ve günümüze dek gelen, korunum yasaları, simetriler, integre edilebilirlik ve sınıflandırma ile ilgili literatür özeti verilmiştir.

İkinci bölüm, çalışmada kullanılan notasyon, temel tanımlar, integre edilebilirlik testleri, simetriler ve rekürsyon operatörleri hakkında ön bilgilere ve KdV denklemleri ile ilgili örneklere ayrılmıştır.

Esas sonuçlar üçüncü ve dördüncü bölümde toplanmıştır. Üçüncü bölümde, kademeli (graded) ve filtrelenmiş (filtered) cebir ile ilgili temel tanımlar ve özellikler ile birlikte "level grading" in tanımı yapılmıştır. Dördüncü bölümde ise polinom sonuçlar verilmiştir.

Katsayıları, $x, t, u, \ldots, u_k$ ya bağlı fonksiyonlar halkası üzerinde, türevleri u_{k+i} olan polinomlara kademeli cebir (graded algebra) yapısı oturtulmuştur. "Level grading" olarak adlandıracağımız bu yapının oluşma nedeni; $x, t, u, \ldots, u_k$ ya bağlı fonksiyonların türevlerinin yüksek mertebe türevlerde polinom olması ve türevlenme sırasına göre baz seviye k üzerinde doğal bir ölçeklemeye sahip olmasıdır. Bu yapının oluşmasındaki can alıcı nokta, polinom sonuçlar içeren denklemlerin "level grading"'e göre, sadece en yüksek mertebeden ölçekleme ağırlığına sahip terimleri içermesidir. Bu durum, düşük seviyedeki terimleri gözardı ederek ve sadece yüksek seviyedeki terimleri dikkate alarak sembolik hesaplamaların yapılabilir bir seviyeye indirgenmesini sağlamıştır.

Skaler, integre edilebilir m inci mertebeden evrim denklemlerinin sınıflandırılması üzerine polinom sonuçlar dördüncü bölümde verilmiştir. Hesaplamalarda, keyfi skaler integre edilebilir $m \geq 7$ mertebe evrim denklemlerinin u_{m-i}, $i = 0, 1, 2$ türevlerine göre polinom olduğu ispatlanmıştır.

Beşinci bölüm ise 7 inci ve 9 uncu mertebeden evrim denklemlerinin sınıflandırılması için yapılan açık hesaplamalara ayrılmıştır. Bu bölümün amacı, açık hesaplamaların sonuçlarını genel m için elde edilen sonuçlarla karşılaştırmak olmuştur. Özel olarak 7 inci mertebede, bilinen tüm korunan yoğunlukların, $\rho^{(i)}, i = -1, 1, 2, 3$, kullanılmasının elde edilen sonucu değiştirmediği gösterilmiştir.

Altıncı bölümde sonuçlar üzerine tartışmalar ve ileriki araştırmalar için yönlendirmeler verilmiştir.

A,B,C ve D, eklerinde sırasıyla, 7 inci ve 9uncu mertebeden evrim denklemlerinin hesaplamalarında kullanılan alt modül ve kalan alt modülleri üreten monomların listeleri verilmiştir. Bu listelerin yardımı ile dokuzuncu mertebeden büyük evrim denklemleri için monomialların kolaylıkla türetilebildiği görülmüştür.

1 INTRODUCTION

1.1 Introduction

In the literature, integrable equations refer to non-linear equations for which explicit solutions can be obtained by means of a transformation to linear equations or by the inverse scattering method. Burgers' and Korteweg de Vries (KdV) equations are respectively prototypes for these two cases. For example the Cole-Hopf transformation $v_x = uv$, which is local, linearizes the Burgers equation $u_t - 2uu_x - u_{xx} = 0$, to the heat equation $v_t = v_{xx}$, while the KdV equation requires an inverse spectral transformation [29], [28].

Investigations of the solutions of nonlinear partial differential equations motivated the discovery of mathematical "soliton" known as solitary wave which asymptotically preserves its shape and velocity upon nonlinear interaction with other solitary waves [6]. In 1834 J. Scott Russel was the first who observes, riding on horse back beside a narrow canal, the formation of a solitary wave. In 1895 Korteweg and de Vries derived the equation for water waves in shallow channels which bears their name and which confirmed the existence of solitary waves. The discovery of additional properties of solitons began with the appearance of computers followed by the numerical calculations carried out on the Maniac I computer by Fermi, Pasta and Ulam in 1955. They took a chain of harmonic oscillators coupled with a quadratic nonlinearity and investigated how the energy in one mode would spread to the rest. They found the system cycled periodically, implying it was much more integrable than they had thought. The continuum limit of their model was the KdV equation [30].

The exact solutions of the KdV equation were the solitary wave and cnoidal wave solutions. While the exact solution of the KdV equation $u_t + 6uu_x + u_{xxx} = 0$ subject to the initial condition $u(x, 0) = f(x)$ where $f(x)$ decays sufficiently rapidly as $|x| \to \infty$, was developed by Gardner, Green, Kruakal and Miura in 1967. The basic idea for this solution is to relate the KdV equation to the time-independent Schrödinger scattering problem [10]. Gardner, Miura and Kruskal found out that the eigenvalues of the Schrödinger operator are integrals of the Korteweg-de Vries equation. This discovery were succeeded by Laxs' principle which associates nonlinear equations of evolutions with linear operators so that the eigenvalues of the linear operator are integrals of the nonlinear equation [24].

The interest to the integrability problem increased by the discovery of soliton behavior of the KdV equation and of the inverse spectral transformation for its analytical solution. In 1965 Zabusky and Kruskal, in numerical studies, re-derived the KdV equation and they found the remarkable property of solitary waves [21]. They give the conjecture that the double wave solutions of the KdV equation with large $|t|$ behave as the superposition of two solitary waves travelling at different speeds [24]. These numerical results lead them to search some analytic explanation. They found out that this behavior can be explained by the existence of many conservation laws. Therefore the search for the conservation laws for the KdV equation started. A conservation law has the following form where U is the conserved density and F is the conserved flux.

$$D_t U + D_x F = 0.$$

First Zabusky and Kruskal found conserved densities of order 2 and 3, after Miura found a conserved density of order 8. Finally it is proved that there exist an infinite number of conservation laws and conserved densities at each order [7].

Methods for selecting equations that are considered to be "integrable" among a general class are called "integrability tests". Integrability tests use the fact that integrable equations have a number of remarkable properties such as the existence of an infinite number of conserved quantities, infinite number of symmetries, soliton solutions, Hamiltonian and bi-Hamiltonian structure, conserved covariant (co-symmetries), Lax pairs, Painlevé property, conservation laws etc. Usually the requirement of sharing a certain property with known integrable equations leads to the selection of a finite number of equations from a general class and the selected equations are expected to be integrable. This method leads to a "classification" and the criterion used in is called an "integrability test".

The Korteweg-deVries (KdV) equation

$$u_t = u_3 + u u_1 \tag{1.1}$$

is the prototype of integrable evolution equations. There are a number of other equations related to it involving first order derivatives. These are called the "modified KdV" or "potential KdV" equations and they are also integrable. Miura found that the Modified Korteweg de Vries equation

$$v_t = v_3 + v^2 v_1 \tag{1.2}$$

turn to KdV equation under the transformation

$$u = v^2 + \sqrt{-6} v_1.$$

Therefore he proved that if $v(x,t)$ is a solution of (1.2), $u(x,t)$ is a solution of (1.1).

Miura transformations map symmetries to symmetries hence those equations that are related to a known integrable equation by Miura transformations are also considered integrable and belonging to the same class.

Sophus Lie was the first who studied the symmetry groups of differential equations. A symmetry group of a system can be defined as the geometric transformations of its dependent and independent variables which leave the system invariant. Geometric transformations on the space of independent and dependent variables of the system are called geometric symmetries. In 1918

Emmy Noether proved the one-to-one correspondence between one-parameter symmetry groups and conservation laws for the Euler Lagrange equations [12]. This result can not explain the existence of infinitely many conserved densities for the KdV equation which possessed only a four-parameter symmetry group. This observation leads to reinterpret higher order analogs of the KdV equation as "higher order symmetries". Then the search began for the hidden symmetries called "generalized symmetries", which are groups whose infinitesimal generators depend not only on the dependent and independent variables of the system but also the derivatives of the dependent variables [5].

In the classical theory, the "symmetry of a differential equation" is defined in terms of the invariance groups of the differential equation. This definition is essentially equivalent to defining symmetries as solutions of the linearized equation. That is if σ is a symmetry of the evolution equation $u_t = F(x, t, u, u_x, \ldots, u_{x\ldots x})$, then

$$\sigma_t = F_* \sigma \tag{1.3}$$

where F_* is the Frechet derivative of F. A function $f(x, t, u, u_x, u_t, \ldots)$ is called symmetry of the partial differential equation $H(x, t, u, u_x, u_t, u_{xx}, u_{xt}, u_{tt}, \ldots) = 0$, if it satisfies the following "linearization":

$$\begin{aligned} &\left(\frac{\partial H}{\partial u} + \frac{\partial H}{\partial u_x} \frac{\partial}{\partial x} + \frac{\partial H}{\partial u_t} \frac{\partial}{\partial t} + \frac{\partial H}{\partial u_{xx}} \left(\frac{\partial}{\partial x} \right)^2 \right. \\ &\left. + \frac{\partial H}{\partial u_{xt}} \frac{\partial}{\partial t} \frac{\partial}{\partial x} + \frac{\partial H}{\partial u_{tt}} \left(\frac{\partial}{\partial t} \right)^2 + \ldots \right) (f) = 0 \end{aligned} \tag{1.4}$$

For a nonlinear evolution equation $u_t = F(x, t, u, u_x, \ldots, u_{x\ldots x})$, symmetries of the form $\sigma = \sigma(x, t, u, u_x, u_t)$, linear in u_x, u_t are called "classical symmetries", while symmetries depending on higher order derivatives of the dependent variable $\mathbf{u}$ with respect to x are called "generalized symmetries". For example one of the simplest general symmetries of the KdV equation $u_t = u_{xxx} + 6uu_x$ is: $f = u_{xxxxx} + 10uu_{xxx} + 20u_x u_{xx} + 30u^2 u_x$ [2].
The existence of infinitely many generalized symmetries is tied to the existence of a recursion operator which maps symmetries to symmetries [9].

The recursion operator is in general an integro-differential operator say R such that $R\sigma$ is a symmetry whenever σ is a symmetry, i.e.,

$$u_t = F\left[u\right]. \tag{1.5}$$

$(R\sigma)_t = F_*(R\sigma)$. It follows that for any symmetry σ,

$$\left(R_t + [R, F_*]\right) \sigma = 0. \tag{1.6}$$

where $\mathbf{F}_*$, is the Frechet derivative of $\mathbf{F}$. Given a recursion operator R, one may expand the integral terms in R in an infinite formal series in terms of the inverse powers of the operator $\mathbf{D=d/dx}$. A "formal recursion operator" is defined as a formal series in inverse powers of D satisfying the operator equation (1.6). A truncation of the formal series satisfying equation (1.6) for a given evolution equation or $\mathbf{F}_*$ is defined to be a formal symmetry in [2].

The solvability of the coefficients of R in the class of local functions requires that certain quantities denoted as $\rho^{(i)}$ be conserved densities. The existence of one higher symmetry permits to construct not one but many conservation laws. These conservation laws contain a lot of knowledge about the equation under consideration.

A function $\rho = \rho(x, t, u, u_1, u_2, \ldots, u_n)$ is called a density of a conservation law of $u_t = F(x, t, u, u_x, \ldots)$, if there exists a local function σ such that $\frac{d}{dt}(\rho) = D(\sigma)$. If $\rho = D(h)$ for any h and $\sigma = h_t$ then σ is a trivial conserved density. These conserved density conditions give over-determined systems of partial differential equations for F and lead to a classification.

Integrability tests based on the existence of a formal symmetry has lead to many strong results on the classification of 3th and 5th order equations [14] [15]. Among polynomial equations, at the third order the KdV class is unique, while at the fifth order there are in addition the Sawada-Kotera and Kaup equations [2].

The classification problem for scalar integrable equations is solved in the work of Wang and Sanders [3], for the polynomial scale invariant case. Their method is based on the search of higher symmetries and uses number theoretical techniques. They proved that if λ homogeneous (with respect to the scaling $xu_x + \lambda u$, with $\lambda > 0$) equations of the form $u_t = u_m + f(u, \ldots, u_{m-1})$ have one generalized symmetry, they have infinitely many and these can be found using recursion operators or master symmetries [3]. They proved also that if an equation has a generalized symmetry, it is enough to be able to solve the symmetry equation up till quadratic terms to find other symmetries [3]. They showed also that if the order of the symmetry is > 7, there exists a nontrivial symmetry of order ≤ 7[3]. Their main result is that scale invariant, scalar integrable evolution equations of order greater than seven are symmetries of third and fifth order equations [3] and similar results are obtained in the case where negative powers are involved [4]. The problem of classification of arbitrary evolution equations is thus reduced to proving that such equations have desired polynomiality and scaling properties. In a recent work, the non-polynomial case is studied in [1] and it is shown that the existence of a conserved density of order $m + 1$ leads to quasi-linearity. This is a first step in proving polynomiality, and our motivation is to prove step by step further polynomiality results and give a complete classification in the non-polynomial case.

We shall summarize recent works done on the similar field. In 1993 Roberto Camassa and Darryl Holm derive a new completely integrable dispersive shallow water equation

$$u_t + 2\kappa u_x - u_{xxt} + 3uu_x = 2u_x u_{xx} + u_{xxx}$$

where u is the fluid velocity in the x direction and κ is a constant related to the critical shallow water wave speed. The equation is obtained by using an asymptotic expansion directly in the Hamiltonian for Euler's equation. This equation is bi-Hamiltonian it can be expressed in Hamiltonian form in two different ways. The ration of its two Hamiltonian operators is a recursion operator that produces an infinite sequence of conservation laws [31]. In 2001 Artur Sergyeyev extended the recursion operators with nonlocal terms of special form for evolution systems in (1+1)-dimensions, to well-defined operators on the space of nonlocal symmetries and showed that these extended recursion operators leave this space invariant [33]. Another study of Sergyeyev published in 2002 is about conditionally integrable evolution systems. They describe all (1+1)-dimensional evolution systems that admit a generalized (Lie-B$\ddot{a}$cklund) vector field satisfying certain non-degeneracy assumptions, as a generalized conditional symmetry [34]. Two of several studies about systems of evolution equations are produced by Wolf. With Sokolov they extend the simplest version of the symmetry approach to the classification of integrable evolution equations for (1+1)-dimensional nonlinear PDEs., to the case of vector evolution equations. They considered systems of evolution equations with one or two vector unknowns and systems with one vector and one scalar unknown. They gave the list of all equations having the simplest higher symmetry for these classes [32]. Tsuchida and Wolf performed a classification of integrable systems mixed scalar and vector evolution equations with respect to higher symmetries. They consider polynomial systems that are homogeneous under suitable weighting of variables. They gave the complete lists of second order systems with a third order or fourth order symmetry and third order systems with a fifth order symmetry using the KdV, the Burgers, the Ibrahimov-Shabat and two unfamiliar weightings [35]. Partial differential equations of second order (in time) that possess a hierarchy of infinitely many higher symmetries are studied in [36]. The classification of homogeneous integrable evolution equations of fourth and sixth order (in the space derivative) equations has been done applying the perturbative symmetry approach in symbolic representation. Three new tenth order integrable equations has been found. The integrability condition has been proved providing the corresponding bi-Hamiltonian structures and recursion operators [36]. Recently number theory results on factorization of polynomials has been used to classify symmetries of integrable equations [37].

In the present work, the classification of quasi-linear evolution equations of order $m \geq 7$, using the existence of a "formal symmetry" as an integrability test proposed in [2], is studied.

The presentation is organized as follows: Notation used in this study, basic definitions and preliminary notions about integrability tests, symmetries and recursion operators with examples on KdV equations are given in Section 2. A new structure called "level grading" based on the structure of graded and filtered algebra accompanied by related definitions, properties and examples are introduced in Section 3. Polynomiality results in top three derivatives on classification of scalar integrable evolution equations of order m are given in Section 4. Section 5 is devoted to the classification of 7th and 9th order evolution equations. Two different methods are given for the computations of evolution equation of order 7. Discussions and conclusions are given in Section 6. The submodules and quotient submodules with their generating monomials, used in the classification of 7th and 9th order evolution equations are respectively given in Appendices A,B,C and D.

2 PRELIMINARIES

The purpose of this section is to introduce notations used in this study. We also give a brief knowledge about integrability tests, symmetries and recursion operators which constitute the fundamental part of this study.

2.1 Notation and Basic Definitions

In this study we work with scalar evolution equations in one space dimension where the independent space and time variables are respectively $\mathbf{x}$ and $\mathbf{t}$ while the dependent variable is $\mathbf{u} = \mathbf{u}(\mathbf{x}, \mathbf{t})$.

Definition 2.1.1. A differential function $\mathbf{F}[\mathbf{u}]$ is a smooth function of x,t, u and of the derivatives of $\mathbf{u}$ with respect to $\mathbf{x}$, up to an arbitrary but finite order.

Evolution equations are of the form

$$\frac{\partial}{\partial t}u(x,t) = F[u] \tag{2.1.1}$$

where $\mathbf{F}[\mathbf{u}]$ is a differential function.

We simplify the notation for the derivatives of differential polynomials as follows:

The partial derivative of $\mathbf{u}$ with respect to $\mathbf{t}$ is denoted by $\mathbf{u_t} = \frac{\partial \mathbf{u}}{\partial \mathbf{t}}$. The partial derivatives with respect to $\mathbf{x}$ are denoted by $\mathbf{u_i} = \frac{\partial^{\mathbf{i}} \mathbf{u}}{\partial \mathbf{x}^{\mathbf{i}}}$.
This agreement emphasizes that these quantities are considered as independent variables.
If $F[u] = F(x, t, u, u_1, ..., u_m)$ is a differential function, the total derivative with respect to $\mathbf{t}$ denoted by D_t is

$$D_t(F) = \sum_{i=0}^{m} \frac{\partial F}{\partial u_i}\frac{\partial u_i}{\partial t} + \frac{\partial F}{\partial t}, \tag{2.1.2}$$

and the total derivative with respect to $\mathbf{x}$ denoted by $\mathbf{D}$ is

$$DF = \sum_{i=0}^{m} \frac{\partial F}{\partial u_i}u_{i+1} + \frac{\partial F}{\partial x} \tag{2.1.3}$$

We agree on the convention that the operator inverse of D is D^{-1} defined as $D^{-1}\varphi = \int \varphi$.

Definition 2.1.2. The differential polynomial $F[u]$ is said to have fixed scaling weight s if it transforms as $F[u] \to \lambda^s F[u]$ under the scaling $(x, u) \to (\lambda^{-1}x, \lambda^d u)$, where d is called the weight of u and denote it as $\mathbf{wt(u)}$.

Definition 2.1.3. A differential polynomial is called KdV-like if it is a sum of polynomials with odd scaling weight s, with $\mathbf{wt}(\mathbf{u}) = \mathbf{2}$.

Definition 2.1.4. . A Laurent series in D is a formal series

$$L = \sum_{i=1}^{n} L_i D^i + \sum_{i=1}^{\infty} L_{-i} D^{-i} \tag{2.1.4}$$

and it is called a pseudo-differential operator. The order of the operator is the highest index n with $L_n \neq 0$. The operators given in closed form that involve integral operations will be called integro-differential operators.
In order to define the products of pseudo-differential operators, we need to define the operator

$$\begin{aligned} D^{-1}\varphi &= \varphi D^{-1} - D\varphi D^{-2} + D^2\varphi D^{-3} + \ldots \\ &= \sum_{i=0}^{m} (-1)^i \left(D^i\varphi\right) D^{-i-1} + (-1)^{m+1} D^{-1} \left[\left(D^{m+1}\varphi\right) D^{-m-1}\right] \end{aligned} \tag{2.1.5}$$

this formula is just the expression of integration by parts, for example

$$\begin{aligned} D^{-1}\left(\varphi D^k \psi\right) &= \int \varphi D^k \psi = \varphi D^{k-1}\psi - \int D\varphi D^{k-1}\psi \\ &= \varphi D^{k-1}\psi - D\varphi D^{k-2}\psi + \int D^2\varphi D^{k-2}\psi \end{aligned} \tag{2.1.6}$$

The action of D^{-k} is computed by repeated applications of (2.1.6), up to any desired order.

2.2 Integrability Tests

In this part we shall briefly discuss integrability tests. First of all one needs a definition of "integrability" for nonlinear partial differential evolution equations, in order to understand the integrability of known equations, to test the integrability of new equations and to obtain new integrable equations. This subject has been discussed in several papers gathered on the book "What is integrability?"[23]. "There is no precise definition besides the two notions of "C-integrability" and "S-integrability" " as stated by Calogero in [22]. The first one corresponds to the possibility of linearization via an appropriate "change of variables", while the second denotes solvability via the "Spectral transform technique" or the "inverse Scattering method". The transformation of a nonlinear equation via an invertible change of coordinates into a linear equation can also be defined as "C-integrability" [38]. Indirect methods are used to identify the equations expected to be solvable by an inverse spectral transformation and they are commonly called "integrability tests".

A well known example for equations solvable by an inverse spectral transformation is the Korteweg de Vries equation. Integrability tests are inspired by the remarkable properties of the KdV equation such as an infinite number of conserved quantities, infinite number of symmetries, soliton solutions, Hamiltonian and bi-Hamiltonian structure, Lax pairs and Painlevé property.

The existence of an infinite set of conservation laws for the KdV equation suggests that certain nonlinear partial differential evolution equations might have similar properties. The existence of the infinite set of conservation laws motivated the search for a simple way of generating the conserved quantities[11]. This search led to the Miura-Gardner one parameter family of Backlaund transformations between the solutions of KdV and of the modified KdV equations. Therefore the inverse scattering transform for directly linearizing the equation was developed. In 1968 Lax put the inverse scattering method for solving the KdV equation into a more general form and found the Lax pair [24].

The Backlaund transformation was an important key to check a given equation for symmetries. The nonexistence of an infinite number of conservation laws does not obstruct integrability. There exist integrable equations which have a finite number of conservation laws, which are not Hamiltonian and which are dissipative. An appropriate example is the Burgers equation which has only one conservation law, although an infinite number of symmetries. It can be integrated using the one parameter family of Backlaund transformation between solutions of the heat and Burgers equation.

In 1971 Hirota developed a direct method, known as Bilinear Representation, for finding N-soliton solutions of nonlinear evolution equations [6]. It is shown that KdV-like equations with non-zero $3rd$ order part, viewed as perturbations of the KdV equations can be transformed to the KdV equations up to a certain order provided that the coefficients satisfy certain conditions. These conditions are obtained by requiring that the conserved densities of the KdV equations be extended to higher orders [26], [27].

Kruskal and Zabusky after the re-derivation of the KdV equation discovered the interaction properties of the soliton and they explain the existence of infinitely many conservation laws by suggesting the existence of hidden symmetries in this equation. In 1987, Fokas proposed the existence of one generalized symmetry as an integrability test [25]. MSS develop a new symmetry, called "formal symmetry", using as a base, the locality concept of the Sophus Lie classical theory of contact transformations and the inverse scattering transform. The existence of formal symmetries of sufficiently high order is proposed as an integrability test [2]. A formal symmetry is a pseudo-differential operator which agrees up to a certain order with some fractional power of a recursion operator expanded in inverse powers of D, which is the total derivative with respect to x. The existence of a formal symmetry gives certain conserved density conditions which in turn lead to a classification. The Painlevé method, which can be applied to systems of ordinary and partial differential equations alike, is one of the methods used to identify integrable systems. The basic idea is to expand each dependent variable in the system of equations as a Laurent series about a pole manifold [8].

2.3 Symmetries and Recursion Operators

In this section we shall give the definitions and interrelations between symmetries, recursion operators conserved densities and discuss the formal symmetry method.

Definition 2.3.1. A local one parameter group of transformations acting on the space of variables (x, t, u) is called a symmetry group of the equation $u_t = F[u]$, if it transforms all solutions to solutions.

Definition 2.3.2. The Frechet derivative denoted by F_* is a linearized operator associated with the differential function $F[u]$ and is defined as

$$F_* = \sum_{i=0}^{m} \left(\frac{\partial F}{\partial u_i} \right) D^i \tag{2.3.1}$$

where $u_i = \left(\frac{\partial^i u}{\partial^i x} \right)$

Definition 2.3.3. A differential function σ is called a symmetry of the equation $u_t = F[u]$ if it satisfies the linearized equation, $\sigma_t = F_* \sigma$.

Definition 2.3.4. The symmetries depending linearly on the first derivatives of the unknown function are called classical symmetries or Lie-point symmetries. All other symmetries are called non Lie-point or generalized symmetries.

Definition 2.3.5. A differential function ρ is called a conserved density, if there exists a differential polynomial φ such that $\rho_t = D\varphi$.

In the solution of the KdV equation via the inverse spectral transformation, it appears that not only the KdV equation, but the sequence of odd order equations called the KdV hierarchy are all solvable by the same method. These equations are symmetries of the KdV equation and they can be defined recursively.

Definition 2.3.6. A recursion operator is a linear operator R such that $R\sigma$ is a symmetry whenever σ is a symmetry. It can also be defined as a solution of the operator equation $R_t + [R, F_*] = 0$.

Assuming that σ is a symmetry and using the equation above we have

$$\begin{aligned} (R\sigma)_t &= R_t \sigma + RF_* \sigma \\ &= (-RF_* \sigma + F_* R\sigma) + RF_* \sigma \\ &= F_* R\sigma \end{aligned} \tag{2.3.2}$$

hence $R\sigma$ is a symmetry. We can conclude that R sends symmetries to symmetries.

Example 2.3.7: The Recursion operator for the KdV equation of the form:

$$u_t = u_3 + uu_1 \tag{2.3.3}$$

is

$$\Re_{KdV} = D_x^2 + \frac{2}{3}u + \frac{1}{3}u_1 D_x^{-1} \tag{2.3.4}$$

where D^{-1} is the left inverse of D_x [13].
Applying (2.3.3) to (2.3.4) we get:

$$(D_x^2+\frac{2}{3}u+\frac{1}{3}u_1 D_x^{-1})(u_3+uu_1) = D_x^2(u_3+uu_1)+\frac{2}{3}u(u_3+uu_1)+\frac{1}{3}u_1 D_x^{-1}(u_3+uu_1).$$

Computations give fifth order KdV equation:

$$u_t = u_5 + \frac{10}{3}u_1u_2 + \frac{5}{3}uu_3 + \frac{5}{6}u^2u_1$$

Since recursion operators send symmetries to symmetries, it can be said that fifth order KdV equation is the symmetry of the third order equation.

Definition 2.3.8. A formal recursion operator for the evolution equation $u_t = F[u]$, is as pseudo-differential operator R satisfying the equation

$$R_t + [R, F_*] = 0 \tag{2.3.5}$$

If R is not purely differential operator, it is difficult to determine its integro-differential part, but the last one can be expand in inverse powers of D and obtain a pseudo-differential operator. Now the equation (2.3.5) is an infinite series in inverse powers of D, and only a finite number of these equations can be solved. If a finite number of terms in the equation (2.3.5) hold, then the equation will hold identically.

Definition 2.3.9. A formal symmetry is a pseudo-differential operator which satisfies the operator equation $R_t + [R, F_*] = 0$ up to a certain order.

If $Ord(R) = n$, $Ord(F_*) = m$ and in the symmetry equation the coefficients of D^{n+m-1} up to $D^{n+m-1-N}$ are zero, the highest N terms in the symmetry equation are satisfied, we say that the formal symmetry has order N.

3 BASIC ALGEBRAIC STRUCTURES

In this section we give the algebraic structures that will be used in our problem. In the first part we give basic algebraic definitions with some examples, in the second and third part we give respectively the structure of the graded algebra and the "level grading" structure that we introduce in this study.

3.1 Basic Definitions

In this part we give fundamental definitions concerning the graded and filtered algebra.

Definition 3.1.1 A **ring** $\langle \mathcal{K}, +, . \rangle$ is a set $\mathcal{K}$ together with two binary operations $(+, .)$, which we call addition and multiplication, defined on $\mathcal{K}$ such that the following axioms are satisfied:
1) $\langle \mathcal{K}, + \rangle$ is an abelian group.
2) Multiplication is associative.
3) For all $a, b, c, \in \mathcal{K}$, the left distributive law, $a\,(b + c) = (ab) + (ac)$ and
4)The right distributive law, $(a + b)\,c = (ac) + (bc)$, hold [17].

Definition 3.1.2. Let $\mathcal{K}$ be a ring. A (left) $\mathcal{K}$**-module** consists of an abelian group G together with an operation of external multiplication of each element of G by each element of $\mathcal{K}$ on the left such that for all $\alpha, \beta, \in G$ and $r, s, \in \mathcal{K}$, the following conditions are satisfied:
1) $(r\alpha) \in G$.
2) $r(\alpha + \beta) = r\alpha + r\beta$.
3) $(r + s)\alpha = r\alpha + s\alpha$.
4) $(rs)\alpha = r(s\alpha)$.

A $\mathcal{K}$-module is very much like a vector space except that the scalars need only form a ring [17]. In any left $\mathcal{K}$-module, a family of elements $x_1, x_2, ..., x_n$ is called linearly independent if for any $\alpha_i \in \mathcal{K}$ the relation $\sum \alpha_i x_i = 0$ holds only when $\alpha_1 = ... = \alpha_n = 0$. A linearly independent generating set is called a basis.

Definition 3.1.3. A module is said to be **free** if it has a basis.
It is clear that any basis of a free module is a minimal generating set, i.e. a generating set such that no proper subset generates the whole module [19].

Definition 3.1.4. An **algebra** consists of a vector space V over a field $\mathcal{C}$, together with a binary operation of multiplication on the set V of vectors, such that for all $a \in \mathcal{C}$ and $\alpha, \beta, \gamma \in V$, the following conditions are satisfied:
1)$(a\alpha)\beta = a(\alpha\beta) = \alpha(a\beta)$.
2)$(\alpha + \beta)\gamma = \alpha\gamma + \beta\gamma$.
3)$\alpha(\beta + \gamma) = \alpha\beta + \alpha\gamma$.
W is an **associative algebra** over $\mathcal{C}$ if, in addition to the preceding three conditions:
4)$(\alpha\beta)\gamma = \alpha(\beta\gamma)$ for all $\alpha, \beta, \gamma \in W$ [17].

Definition 3.1.5. If for a field $\mathcal{C}$ a positive integer n exists such that $n.a = 0$ for all $a \in \mathcal{C}$, then the least such positive integer is the **characteristic of the field** $\mathcal{C}$. If no such positive integer exists, then $\mathcal{C}$ is of characteristic 0 [17].

Definition 3.1.6. Let $\mathcal{C}$ be a field of characteristic 0. A vector space V over $\mathcal{C}$ is called a **Lie Algebra** over $\mathcal{C}$ if there is a map
$(X, Y) \mapsto [X, Y],\ (X, Y, [X, Y] \in V)$
of $V \times V$ into V with the following properties:
$(i)(X, Y) \mapsto [X, Y]$ is bilinear
$(ii)[X, Y] + [Y, X] = 0,\ (X, Y \in V)$
$(iii)[X, [Y, Z]] + [Y, [Z, X]] + [Z, [Y, X]] = 0,\ (X, Y, Z \in V)$ [20].

The following definitions of the graded and filtered algebra are based on the previous preliminary definitions.

Definition 3.1.7. Let M be an associative algebra over a field $\mathcal{C}$ of characteristic 0. M is said to be graded if for each integer $n \geq 0$ there is a subspace M_n of M such that
$(i)1 \in M_{(0)}$ and M is the direct sum of the M_n,
$(ii)M_{(n_i)}M_{(n_j)} \subseteq M_{(n_i+n_j)}$
for all $n_i, n_j \geq 0$
In this case the elements of $\bigcup_{n_i=0}^{\infty} M_{n_i}$ are called homogeneous, and those of M_n are called homogeneous of degree n; if $v = \sum_{n\geq 0} v_n(v_n \in M_n, v \in M)$, then v_n is called the homogeneous component of v of degree n [20].

Definition 3.1.8. M is said to be filtered if for each integer $n \geq 0$ there is a subspace $M^{(n)}$ of M such that:
$(i)1 \in M^{(0)},\ M^0 \subseteq M^1 \subseteq ...,\ \bigcup_{n=0}^{\infty} M^{(n)} = M$ and
$(ii)M^{(n_i)}M^{(n_j)} \subset M^{(n_i+n_j)}$ for all $n_i, n_j \geq 0$.
It is convenient to use the convention that $M^{(-1)} = \emptyset$. For $v \in M$, the integer $s \geq 0$ such that $M \in M^{(s)}$ but $\notin M^{(s-1)}$ is called the degree of v, and written $deg(v)$. For $n \geq 0$, $M^{(n)}$ is then the set of all $v \in M$ with $deg(v) \leq n$ [20].
We give now certain examples of gradings on polynomial rings. The polynomials in a single variable x over a field $\mathcal{C}$ is a standard example of graded and filtered algebra.

Example 3.1.1. Let M be the algebra of polynomials in one indeterminate x over a field of coefficients $\mathcal{C}$. The degree of the indeterminate x gives a grading where each submodule $M^n = span(x^n)$.

Table 3.1.1:Submodules and their graded elements.

M^1	M^2	M^3	M^4	M^5	M^6
x	x^2	x^3	x^4	x^5	x^6

Example 3.1.2. Let M be the algebra of polynomials in two independent variables x, y over a field of coefficients $\mathcal{C}$. The total degree gives a grading on this algebra. Alternatively, we can induce a grading by choosing a weight, $wt(x)$, $wt(y)$, for each of the variables and then use total degree. For example if the weights of x and y are respectively 1 and 2, then the grade of $x^\alpha y^\beta$ will be $\alpha + 2\beta$. The generators of the graded modules are given in the table below.

Table 3.1.2: Generators of graded modules

M^1	M^2	M^3	M^4	M^5	M^6
x	x^2	x^3	x^4	x^5	x^6
	y	xy	x^2y	x^3y	x^4y
			y^2	xy^2	x^2y^2
					y^3

Definition 3.1.9. Let $\mathcal{S} = \mathcal{K}[x_1, ..., x_n]$ be a commutative ring of polynomials where x_k's are indeterminates and

$$f = \Sigma a_{i_1 \dots i_n} x_1^{i_1}, ..., x_n^{i_n} \tag{3.1.1}$$

where $a_{i_1 \dots i_n}$ are real numbers. Each product $m_{(i)} = x_1^{i_1}, ..., x_n^{i_n}$ in f is called a **monomial** and the corresponding term $a_{i_1 \dots i_n}$ is called a monomial term. The total degree of the monomial m_i is

$$d(m_i) = \sum i_k \tag{3.1.2}$$

where i_k is the degree of x_k [18].

Example 3.1.3. $\mathcal{S} = \mathcal{K}[x_1, ..., x_n]$ is a commutative ring of polynomials in k indeterminates. We use a grading by the total degree, then the grades of the following monomials $x_1^3 x_2^5$, $x_1^5 x_2^4 x_3$ and $x_2^7 x_3^5$ are respectively 8, 10 and 12.

If we work with polynomial evolution equations, we can work with a polynomial algebra where the indeterminates are the derivatives u_i's and the coefficient ring is $\mathcal{K} = C^\infty(x, t)$. In the case of non-polynomial evolution equations, the expressions of the time derivatives of the conserved densities are polynomial in higher order derivatives. For example for quasilinear equations, the indeterminates are $\{u_m, u_{m+1}, \dots\}$ and the coefficient ring is the ring of C^∞ functions of x,t, u, and the derivatives of u up to order $m - 1$.

On this polynomial algebra, the order of the derivative gives a natural grading. In dealing with this grading, as shown in [1], in order to obtain top two terms of a time derivative, it is necessary to take top four terms of all expressions, and this necessitates complicated computations.

The key feature of our work is the use of a different type of grading, that we shall call the "level" of a monomial. The advantage of using a grading by "levels" is that the computation of the top terms in time derivatives necessitates the knowledge of the dependence on a single highest term.

3.2 The Structure of The Graded Algebra

Let $\mathcal{K}^{(k)}$ be the ring of C^∞ functions of $x, t, u, \ldots, u_k$ and $M^{(k)}$ be the polynomial algebra over $\mathcal{K}^{(k)}$ generated by $S^{(k)} = \{u_{k+1}, u_{k+2}, \ldots\}$. A monomial m_i in $M^{(k)}$ is a product of a finite number of elements of $S^{(k)}$. Monomials are of the form

$$m_i = \prod_{j=1}^{n} u_{m-1+k_j} \tag{3.2.1}$$

We shall call $m-1$ as the "base level" and the "level of a derivative term u_n will be the number of derivatives above the base level, i.e., $n-(m-1)$.

Let's now fix a base level d and denote the submodules generated by elements of level j above this base level by M_d^j. Then the polynomial algebra M_d over $\mathcal{K}$ will be a direct sum of these modules given as

$$M_d = \bigoplus_{l \geq 0} M_d^l \tag{3.2.2}$$

or explicitly as

$$M_d = M_d^0 \bigoplus M_d^1 \bigoplus M_d^2 \bigoplus M_d^3 \bigoplus M_d^4 \ldots$$

The first few submodules can be expressed in terms of their generators as follows.

$$M_d = \mathcal{K} \bigoplus \langle u_m \rangle \bigoplus \langle u_{m+1}, u_m^2 \rangle \bigoplus \langle u_{m+2}, u_{m+1}u_m, u_m^3 \rangle \bigoplus \ldots.$$

Full sets of generators are given in the Appendices.

The differentiation with respect to x induces a map on these modules compatible with the grading as follows. For example, a general term in M_d^1 is of the form ϕu_{d+1} where ϕ is a function of x, t, u and the u_i's for $i \leq d$. Then

$$\begin{aligned} D(\phi u_{d+1}) &= \phi u_{d+2} + D\phi u_{d+1} \\ &= \phi u_{d+2} + u_{d+1}[\phi_d u_{d+1} + \phi_{d-1} u_d + \ldots] \end{aligned} \tag{3.2.3}$$

Thus it has parts in M_d^1 and inM_d^2. Similarly it can be seen that

$$\pi : \quad M_d^i \rightarrow M_d^i \bigoplus M_d^{i+1}$$

and

$$\pi^j : \quad M_d^i \rightarrow M_d^i \bigoplus M_d^{i+1} \bigoplus \ldots \bigoplus M_d^{i+j}$$

Note that not all monomials appear in a total derivative, i.e, in each submodule m_d^i there are monomials that are not in the image of π. These will be the "non-integrable" terms that we shall be searching for. It can be seen that a monomial is non-integrable if and only if it is nonlinear in the highest derivative. We describe the structure as follows. Let

$$\mathcal{R} = Im(\pi^j) \bigcap M_d^{i+j}$$

and define the quotient submodule $\overline{M}_d^{i+j}$ by

$$\overline{M}_d^{i+j} = M_d^{i+j} / \mathcal{R}.$$

It can be seen that the quotient module is generated by the the non-integrable monomials. These are listed in the Appendices.

The most important feature of this grading is that in the intersection of the image of π^j with the top module only the dependencies on u_d appear. That is practically if we work on the top module, we may assume that our functions depend on u_d only. This allows a considerable reduction in the computational requirements.

3.3 The Ring of Polynomials and "Level-Grading"

In preliminary analytic computations as well as in explicit symbolic computations, we have noticed that, although we used dependencies in top four derivatives, polynomiality results involved only the dependencies on the top derivatives. This remark led to the definition of an unusual grading on the monomials in the derivatives of u, called the "level grading". More precisely, if the unknown functions depend on the derivatives of u up to order k, the expressions of $F[u]$ and the conserved densities $\rho^{(i)}$ are polynomial in u_{k+j}, for $j \geq 1$, hence they look like polynomials over functions depending on the derivatives u_j with $j \leq k$. By defining the "level of u_{k+j} above k" to be j, we can give a graded algebra structure described below. In this setup, the crucial point is that the level above k is preserved under differentiations and integrations by parts hence we can work with the top level part of the expressions. We define the "level" of a monomial as follows:

Definition 3.3.1: Let $m_i = u_{k+j_1}^{a_1} u_{k+j_2}^{a_2} \dots u_{k+j_n}^{a_n}$ be a monomial in $M^{(k)}$ and u_k be the base term of the monomial. The level above k of m_i is defined by :

$$lev_k(m_i) = a_1 j_1 + a_2 j_2 + \dots + a_n j_n \tag{3.3.1}$$

The "level above k" gives a graded algebra structure to $M^{(k)}$. Monomials of a fixed level p form a free module over $\mathcal{K}^{(k)}$ that we denote by $M_p^{(k)}$. By definition $M_0^{(k)} = \mathcal{K}^{(k)}$ and $M^{(k)}$ is the direct sum of these modules, i.e.

$$S_p^{(k)} = \{\mu \in S^{(k)} | lev_k(\mu) = p\}$$

$$M^{(k)} = M_0^{(k)} \oplus M_1^{(k)} \oplus \dots \oplus M_{k-1}^{(k)} \oplus \dots.$$

Starting from this graded algebra structure and defining the modules

$$\tilde{M}_p^k = \bigoplus_{i=0}^{p} M_i^{(k)}$$

and $\tilde{M}^{(k)} = \tilde{M}_0^{(k)} + ... + \tilde{M}_p^{(k)}$ we obtain a corresponding filtered algebra. We illustrate these structures by an example.

Example 3.3.1. Let $k = 5$. Then $M_0^{(5)} = \mathcal{K}^{(5)}$ is the ring of functions depending on x, t, $u, \dots, u_5$ and $M_i^{(5)}$, $i = 1, 2, 3, 4$ are spanned by the monomials

$$\begin{aligned}
M_1^{(5)} &= \langle u_6 \rangle, \\
M_2^{(5)} &= \langle u_7, u_6^2 \rangle, \\
M_3^{(5)} &= \langle u_8, u_7 u_6, u_6^3 \rangle, \\
M_4^{(5)} &= \langle u_9, u_8 u_6, u_7^2, u_7 u_6^2, u_6^4 \rangle.
\end{aligned}$$

The modules $\tilde{M}_i^{(5)}$ are obtained as direct sums of the modules above and they are spanned by

$$\begin{aligned}
\tilde{M}_1^{(5)} &= \langle u_6 \rangle, \\
\tilde{M}_2^{(5)} &= \langle u_7, u_6^2, u_6 \rangle, \\
\tilde{M}_3^{(5)} &= \langle u_8, u_7 u_6, u_6^3, u_7, u_6^2, u_6 \rangle, \\
\tilde{M}_4^{(5)} &= \langle u_9, u_8 u_6, u_7^2, u_7 u_6^2, u_6^4, u_8, u_7 u_6, u_6^3, u_7, u_6^2, u_6 \rangle.
\end{aligned}$$

Now we take a specific monomial and illustrate the effects of differentiation and integration by parts.

Example 3.3.2. Let $k = 5$ and $M^{(5)}$ be as in example 3.3.1. We take the following polynomial of level 3 above 5 in $M_3^{(5)}$

$$v = \varphi u_8 + \psi u_7 u_6 + \eta u_6^3 \qquad \varphi, \psi, \eta \in \mathcal{K}^{(5)}.$$

Then, the total derivative Dv given by

$$\begin{aligned} Dv &= \varphi u_9 + \psi u_8 u_6 + \psi u_7^2 + 3\eta u_6^2 u_7 + (\varphi_5 u_6 + \varphi_4 u_5 + \ldots + \varphi_x) u_8 \\ &+ (\psi_5 u_6 + \psi_4 u_5 + \ldots + \psi_x) u_7 u_6 + (\eta_5 u_6 + \eta_4 u_5 + \ldots + \eta_x) u_6^3 \end{aligned}$$

belongs to $\tilde{M}_4^{(5)}$ as seen below.

$$\begin{aligned} Dv = & \underbrace{\varphi u_9 + (\varphi_5 + \psi) u_8 u_6 + \psi u_7^2 + (\psi_5 + 3\eta) u_7 u_6^2 + \eta_5 u_6^4}_{M_4^{(5)}} \\ + & \underbrace{(\varphi_4 u_5 + \ldots + \varphi_x) u_8 + (\psi_4 u_5 + \ldots + \psi_x) u_7 u_6 + (\eta_4 u_5 + \ldots + \eta_x) u_6^3}_{M_3^{(5)}} \end{aligned}$$

and the projection to $M_4^{(5)}$ depends only on the derivatives with respect to u_5.

Integrations by parts are treated using (2.1.6) in the preliminaries as below

$$\int v dx = \varphi u_7 + \frac{1}{2}\psi u_6^2 + \int \left[-D(\varphi) u_7 - \frac{1}{2} D(\psi) u_6^2 + \eta u_6^3 \right]$$

$$\begin{aligned} \int v dx = & \underbrace{\varphi u_7 + \left[\frac{1}{2}\psi - \frac{1}{2}\varphi_5 \right] u_6^2}_{M_3^{(5)}} \\ + & \underbrace{\int \left[\frac{1}{2}\varphi_{5,5} - \frac{1}{2}\psi_5 + \eta \right] u_6^3}_{M_2^{(5)}} \end{aligned}$$

and we see that the projection on $M_3^{(5)}$ depends on the u_5 dependencies only as before.

To see the behavior of the level grading under time derivatives, we let $u_t = F[u]$ where F is of order m and level $i = m - k$ above k. Let ρ be an arbitrary differential function of order n and level $j = n - k$ above k. Then since

$$D_t \rho = \sum_{h=0}^{n} \frac{\partial \rho}{\partial u_h} D^h(F) + \rho_t$$

and $\frac{\partial \rho}{\partial u_n}$ and $D^n F$ have orders n and $n + m$ respectively,

$$|D_t \rho| = n + m.$$

Similarly since the level of the partial derivatives of ρ are at most j, and the level of $D^n F$ is $i + n$, it follows that

$$lev_k(D_t \rho) \leq j + i + n$$

These results allow us to facilitate the computations using top dependencies only.

4 CLASSIFICATION OF m'th ORDER EVOLUTION EQUATIONS

4.1 Notation and Terminology, Conserved Densities

Let $u = u(x,t)$. A function φ of x, t, u and the derivatives of u up to a fixed but finite order will be called a "differential function" [9] and denoted by $\varphi[u]$. We shall assume that φ has partial derivatives of all orders. We shall denote indices by subscripts or superscripts in parenthesis such as in $\alpha_{(i)}$ or $\rho^{(i)}$ and reserve subscripts without parentheses for partial derivatives, i.e., for $u = u(x,t)$,

$$u_t = \frac{\partial u}{\partial t}, \quad u_k = \frac{\partial^k u}{\partial x^k}$$

and for $\varphi = \varphi(x,t,u,u_1,\ldots,u_n)$,

$$\varphi_t = \frac{\partial \varphi}{\partial t}, \quad \varphi_x = \frac{\partial \varphi}{\partial x}, \quad \varphi_k = \frac{\partial \varphi}{\partial u_k}.$$

If φ is a differential function, the total derivative with respect to x is denoted by $D\varphi$ and it is given by

$$D\varphi = \sum_{i=0}^{n} \varphi_i u_{i+1} + \varphi_x. \tag{4.1.1}$$

Higher order derivatives can be computed by applying the binomial formula as

$$D^k\varphi = \sum_{i=0}^{n} \left[\sum_{j=0}^{k-1} \binom{k-1}{j} \left(D^j \varphi_i\right) u_{i+k-j} \right] + D^{k-1}\varphi_x. \tag{4.1.2}$$

In the computation of $\int D_t\rho$, we shall use only top two order nonlinear terms, which come from top 4 derivatives. For this purpose, we need the expression of (4.1.2) only up to top 4 derivatives which are given in $(4.1.3 - 4.1.6)$. The general expression for $D^k\varphi$ given by (4.1.6) is valid for $k \geq 7$. It follows that in the present thesis general formulas are valid for equations of order $m \geq 19$, and we have done explicit computations for equations of lower orders.

We shall denote generic functions φ that depend on at most u_n by $O(u_n)$ or by $|\varphi| = n$. That is

$$\varphi = O(u_n) \quad \text{or} \quad |\varphi| = n \quad \text{if and only if} \quad \frac{\partial \varphi}{\partial u_{n+k}} = 0 \quad \text{for} \quad k \geq 1.$$

If $\varphi = O(u_n)$, then $D\varphi$ is linear in u_{n+1} and $D^k\varphi$ is polynomial in u_{n+i} for $i \geq 1$. In order to distinguish polynomial functions we use the notation $\varphi = P(u_n)$, i.e.,

$$\varphi = P(u_n) \quad \text{if and only if} \quad \varphi = O(u_n) \quad \text{and} \quad \frac{\partial^k \varphi}{\partial u_n^k} = 0 \quad \text{for some } k.$$

This distinction is used in the expression of derivatives given in $(4.1.3 - 4.1.6)$.

Note that even if $\varphi = O(u_n)$, and φ has an arbitrary functional form, $D^k\varphi$ is polynomial in u_{n+i} for $i \geq 1$. The total derivative with respect to x increases the order by one, thus if $|\varphi| = n$ then $|D^k\varphi| = n + k$.

When $u_t = F$, and $|\varphi| = n$, then the total derivative with respect to t is given by

$$D_t\varphi = \sum_{i=0}^{n} \varphi_i D^i F \ + \ \varphi_t,$$

thus if $|F| = m$, D_t increases the order by m.

Equalities up to total derivatives with respect to x will be denoted by $\cong$, i.e.,

$$\varphi \cong \psi \quad \text{if and only if} \quad \varphi = \psi + D\eta$$

Integration by parts of monomials is defined as follows. Let $p_1 < p_2 < \ldots < p_l < s - 1$. Then

$$\begin{aligned} \varphi u_{p_1}^{a_1} \ldots u_{p_l}^{a_l} u_s &\cong -D\left(\varphi u_{p_1}^{a_1} \ldots u_{p_l}^{a_l}\right) u_{s-1}, \\ \varphi u_{p_1}^{a_1} \ldots u_{p_l}^{a_l} u_{s-1}^{p} u_s &\cong -\tfrac{1}{p+1} D\left(\varphi u_{p_1}^{a_1} \ldots u_{p_l}^{a_l}\right) u_{s-1}^{p+1}. \end{aligned}$$

The integration by parts is repeated until one encounter a monomial which is nonlinear in the highest derivative,

$$u_{p_1}^{a_1} \ldots u_{p_l}^{a_l} u_s^p, \quad p > 1.$$

The order of a differential monomial is not invariant under integration by parts, but it is possible to compute when the integration by parts will give a non-integrable term.

Higher order derivatives are computed in [1] as follows:

$$D^k\varphi = \varphi_n u_{n+k} + P(u_{n+k-1}), \quad k \geq 1 \tag{4.1.3}$$

$$\begin{aligned} D^k\varphi &= \varphi_n u_{n+k} + \left[\varphi_{n-1} + kD\varphi_n\right] u_{n+k-1} \\ &+ P(u_{n+k-2}), \quad k \geq 3 \end{aligned} \tag{4.1.4}$$

$$\begin{aligned} D^k\varphi &= \varphi_n u_{n+k} + \left[\varphi_{n-1} + kD\varphi_n\right] u_{n+k-1} \\ &+ \left[\varphi_{n-2} + kD\varphi_{n-1} + \binom{k}{2} D^2\varphi_n\right] u_{n+k-2} \\ &+ P(u_{n+k-3}), \quad k \geq 5 \end{aligned} \tag{4.1.5}$$

$$\begin{aligned} D^k\varphi &= \varphi_n u_{n+k} + \left[\varphi_{n-1} + kD\varphi_n\right] u_{n+k-1} \\ &+ \left[\varphi_{n-2} + kD\varphi_{n-1} + \binom{k}{2} D^2\varphi_n\right] u_{n+k-2} \\ &+ \left[\varphi_{n-3} + kD\varphi_{n-2} + \binom{k}{2} D^2\varphi_{n-1} + \binom{k}{3} D^3\varphi_n\right] u_{n+k-3} \\ &+ P(u_{n+k-4}), \quad k \geq 7 \end{aligned} \tag{4.1.6}$$

If the evolution equation $u_t = F[u]$ is integrable, it is known that the quantities

$$\rho^{(-1)} = F_m^{-1/m}, \quad \rho^{(0)} = F_{m-1}/F_m, \tag{4.1.7}$$

where

$$F_m = \frac{\partial F}{\partial u_m}, \quad F_{m-1} = \frac{\partial F}{\partial u_{m-1}} \tag{4.1.8}$$

are conserved densities for equations of any order [2].

Higher order conserved densities are computed in [1] as below, with the following notation

$$a = F_m^{1/m}, \quad \alpha_{(i)} = \frac{F_{m-i}}{F_m}, \quad i = 1, 2, 3, 4 \tag{4.1.9}$$

$$\begin{aligned} \rho^{(1)} &= a^{-1}(Da)^2 - \frac{12}{m(m+1)} Da\alpha_{(1)} \\ &+ a\left[\frac{12}{m^2(m+1)}\alpha_{(1)}^2\right. \\ &- \left.\frac{24}{m(m^2-1)}\alpha_{(2)}\right], \end{aligned} \tag{4.1.10}$$

$$\begin{aligned} \rho^{(2)} &= a(Da)\left[D\alpha_{(1)} + \frac{3}{m}\alpha_{(1)}^2 - \frac{6}{(m-1)}\alpha_{(2)}\right] \\ &+ 2a^2\left[-\frac{1}{m^2}\alpha_{(1)}^3 + \frac{3}{m(m-1)}\alpha_{(1)}\alpha_{(2)}\right. \\ &- \left.\frac{3}{(m-1)(m-2)}\alpha_{(3)}\right], \end{aligned} \tag{4.1.11}$$

$$\begin{aligned} \rho^{(3)} &= a(D^2a)^2 - \frac{60}{m(m+1)(m+3)} a^2 D^2 a D\alpha_{(1)} \\ &+ \frac{1}{4}a^{-1}(Da)^4 + 30a(Da)^2\left[\frac{(m-1)}{m(m+1)(m+3)}D\alpha_{(1)}\right. \\ &+ \left.\frac{1}{m^2(m+1)}\alpha_{(1)}^2 - \frac{2}{m(m^2-1)}\alpha_{(2)}\right] \\ &+ \frac{120}{m(m^2-1)(m+3)}a^2 Da\left[-\frac{(m-1)(m-3)}{m}\alpha_{(1)}D\alpha_{(1)}\right. \\ &+ (m-3)D\alpha_{(2)} - \frac{(m-1)(2m-3)}{m^2}\alpha_{(1)}^3 \\ &+ \left.\frac{6(m-2)}{m}\alpha_{(1)}\alpha_{(2)} - 6\alpha_{(3)}\right] \\ &+ \frac{60}{m(m^2-1)(m+3)}a^3\left[\frac{(m-1)}{m}(D\alpha_{(1)})^2\right. \\ &- \frac{4}{m}D\alpha_{(1)}\alpha_{(2)} + \frac{(m-1)(2m-3)}{m^3}\alpha_{(1)}^4 \\ &- 4\frac{(2m-3)}{m^2}\alpha_{(1)}^2\alpha_{(2)} + \frac{8}{m}\alpha_{(1)}\alpha_{(3)} \\ &+ \left.\frac{4}{m}\alpha_{(2)}^2 - \frac{8}{(m-3)}\alpha_{(4)}\right]. \end{aligned} \tag{4.1.12}$$

4.2 General Results on Classification

In [1], the criterion for integrability is the existence of a formal symmetry in the sense of [2]. The existence of a formal symmetry requires the existence of certain conserved densities $\rho^{(i)}$, $i = -1, 0, 1, \ldots$. It is well known that for any m, the first two conserved densities are

$$\rho^{(-1)} = F_m^{-1/m} \quad \text{and} \quad \rho^{(0)} = F_{m-1}/F_m.$$

The explicit expressions of $\rho^{(1)}$ and $\rho^{(2)}$ for $m \geq 5$ and of $\rho^{(3)}$ for $m \geq 7$ obtained in [1] are given in Appendix A. In our computations we shall use only conserved densities which look like $\rho^{(1)}$, $\rho^{(2)}$ and $\rho^{(3)}$, but all conserved densities have been used in computer algebra computations at lower orders for cross checking purposes.

The coefficients of top two nonlinear terms in $D_t\rho^{(1)}$ give a linear homogeneous system of equations for $\frac{\partial^2 F}{\partial u_m^2}$ and $\frac{\partial^2 \rho^{(1)}}{\partial u_n^2}$, with coefficients depending on m. The coefficient matrix is nonsingular for $m \neq 5$, hence it follows that for $m \geq 7$, an evolution equation of order m admitting a nontrivial conserved density of order $m+1$ has to be quasi-linear. In [1], it is shown that u_{3k+l+1}^2 is the top nonlinear term in $\int D_t\rho$, for $\rho = \rho(x, t, u, \ldots, u_n)$ and $u_t = F(x, t, u, \ldots, u_m)$ where $m = 2k+1$, $n \geq m$ and $n = 2k + l + 1$.

In this section first we shall show that the contribution to the top two nonlinear terms, come from the top 4 derivatives in the expansion of $\int D_t\rho$. Then we shall give the expression of the coefficients of top two nonlinearities u_{3k+l+1}^2 and u_{3k+l}^2 in the expansion of $\int D_t\rho$.

This result is based on the expression of the derivatives as in [1].

Proposition 4.2.1. *Let $\rho = \rho(x, t, u, \ldots, u_n)$ and $u_t = F(x, t, u, \ldots, u_m)$ where $m = 2k+1$, $n = 2k+l+1$ and $k+l-1 \geq 0$. Then*

$$\begin{aligned}(-1)^{k+1} D_t\rho &\cong [D^{k+1}\rho_n - D^k\rho_{n-1}]D^{k+l}F - [D^k\rho_{n-2} - D^{k-1}\rho_{n-3}]D^{k+l-1}F \\ &+ O(u_{3k+l-1}).\end{aligned} \tag{4.2.1}$$

Proof:

$$D_t\rho = \sum_{i=0}^{n} \rho_i D^i F + \rho_t \tag{4.2.2}$$

In $D_t\rho$, the highest order derivative comes from $\rho_n D^n F$, where ρ_n and $D^n F$ are of orders $2k+1+l$ and $4k+2+l$ respectively. If we integrate by parts $k+1$ times we obtain

$$\rho_n D^n F \cong (-1)^{k+1} D^{k+1}\rho_n \; D^{k+l}F$$

where $D^{k+1}\rho$ and $D^{k+l}F$ are now respectively of orders $3k+2+l$ and $3k+1+l$. One more integration by parts gives a term nonlinear in u_{3k+1+l}. Similarly one can see that in $\rho_{n-1}D^{n-1}F$, ρ_{n-1} and $D^{n-1}F$ are of orders $2k+1+l$ and $4k+1+l$. This time, integrating by parts k times, we have

$$\rho_{n-1}D^{n-1}F \cong (-1)^k D^k\rho_n \; D^{k+l}F,$$

where $D^k\rho_{n-1}$ and $D^{k+l}F$ are both of orders $3k+1+l$. Thus the highest order nonlinear term, in u_{3k+l+1}, comes from top two derivatives in $\rho_n D^n$ and $\rho_{n-1}D^{n-1}F$.

By similar counting arguments, one can easily see that top two nonlinear terms are obtained from top four derivatives and the remaining terms are of order $3k+l-1$.

$$\begin{aligned}\rho_{n-2p}D^{n-2p}F+\rho_{n-2p-1}D^{n-2p-1}F &= \rho_{2k+l+1-2p}D^{2k+l+1-2p}F \\ &+ \rho_{2k+l-2p}D^{2k+l-2p}F \end{aligned} \tag{4.2.3}$$

with $2k+l-2p>0$. We integrate by parts until the order of the product differ by one and we get:

$$(-1)^{k+1-p}[D^{k+1-p}\rho_{n-2p}-D^{k-p}\rho_{n-2p-1}]D^{k+l-p}F \tag{4.2.4}$$

Here, the first and the second term in the brackets are of orders $3k+l+2-p$ and $3k+l+1-p$ respectively, and $D^{k+l-p}F$ has order $3k+l+1-p$. Thus after integration by parts again we get the term $u^2_{3k+l+1-p}$. It follows that the top two nonlinear terms for $p=0$, and $p=1$ come from ρ_n, ρ_{n-1}, ρ_{n-2} and ρ_{n-3}. □

Remark 4.2.2 As the general expressions for the derivatives given in (4.1.3 – 4.1.6) are valid for large k, there are restrictions on the validity of the formula (4.2.1). Since the top four terms of $D^{k+l}F$ and $D^{k+1}\rho_n$ are needed in (4.2.1), from (4.1.6) it follows that $k+1$ and $k+l$ should be both larger than or equal to 7. On the other hand, at most top two terms of the expressions in the second bracket in (4.2.1) contribute to the top nonlinearities and it turns out that the restrictions coming from (4.1.3, 4.1.4) are always satisfied and the crucial restriction is $k+1\geq 7$ and $k+l\geq 7$. Thus for $l=1,\ 0,\ -1$ and -2, we need respectively $k\geq 6,\ 7,\ 8$ and 9 hence $m\geq 13,\ 15,\ 17$ and 19.

We shall now give the explicit expressions of the coefficients of top two nonlinear terms for $m\geq 19$ and $k+l\geq 7$.

Proposition 4.2.3 *Let $u_t=F(x,t,u,\dots,u_m)$, $m=2k+1$, be an evolution equation and $\rho=\rho(x,t,u,\dots,u_n)$ with $n=m+l$, $-2\leq l\leq 2$, be a conserved density for $u_t=F$. For $m\geq 13$ and $k+l\geq 7$, the coefficients of the top two nonlinear terms u^2_{3k+l+1} for $k+l>0$ and u^2_{3k+l} for $k+l-1>0$ are respectively as follows*

$$(k+\frac{1}{2})F_mD\rho_{n,n}-(k+l+\frac{1}{2})DF_m\rho_{n,n}-F_{m-1}\rho_{n,n}=0, \tag{4.2.5}$$

$$\begin{aligned}
&\rho_{n,n}\ D^3F_m\ \left[\frac{1}{12}\ (2k^3+6k^2l+6kl^2+2l^3+3k^2+3l^2+6kl+k+l)\right]\\
+&\ \rho_{n,n}\ D^2F_{m-1}\ \left[\frac{1}{2}\ (k^2+2kl+2k+2l+l^2+1)\right]\\
+&\ \rho_{n,n}\ DF_{m-2}\ \left[\frac{1}{2}\ (3+2k+2l)\right]\\
+&\ \rho_{n,n}\ F_{m-3}\\
+&\ D\rho_{n,n}\ D^2F_m\ \left[\frac{1}{4}\ (-2k^3-4k^2l-2kl^2+k^2+l^2+2kl+k+l)\right]\\
+&\ D\rho_{n,n}\ DF_{m-1}\ \left[\frac{1}{2}\ (1+l-2k^2-2kl)\right]\\
+&\ D\rho_{n,n}\ F_{m-2}\ \left[\frac{1}{2}\ (1-2k)\right]\\
+&\ D^2\rho_{n,n}\ DF_m\ \left[\frac{1}{4}(2k^3+2k^2l-k^2-k)\right]
\end{aligned}$$

$$
\begin{aligned}
&+ \quad D^2\rho_{n,n}\ F_{m-1}\ \left[\frac{1}{2}\ k^2\right] \\
&+ \quad D^3\rho_{n,n}\ F_m\ \left[\frac{1}{12}\ (-2k^3 - 3k^2 - k)\right] \\
&+ \quad D\rho_{n,n-1}\ DF_m\ \left[\frac{1}{2}\ (-1 + 2k + 2l)\right] \\
&+ \quad D\rho_{n,n-1}\ F_{m-1} \\
&+ \quad D^2\rho_{n,n-1}\ F_m\ \left[\frac{1}{2}\ (-1 - 2k)\right] \\
&+ \quad \rho_{n,n-2}\ DF_m\ [2k + 2l - 1] \\
&+ \quad \rho_{n,n-2}\ F_{m-1}\ [2] \\
&+ \quad D\rho_{n,n-2}\ F_m\ [-2k - 1] \\
&+ \quad \rho_{n-1,n-1}\ DF_m\ \left[\frac{1}{2}\ (1 - 2k - 2l)\right] \\
&+ \quad \rho_{n-1,n-1}\ F_{m-1}\ [-1] \\
&+ \quad D\rho_{n-1,n-1}\ F_m\ \left[\frac{1}{2}\ (1 + 2k)\right] = 0.
\end{aligned}
\tag{4.2.6}
$$

Proof. The proof is a straightforward computation of the integrations indicated in Proposition 4.2.1. Writing the first four terms in $D_t\rho$ and keeping only the terms which contribute to the nonlinearities u^2_{3k+l+1}, u^2_{3k+l}, we get

$$
\begin{aligned}
(-1)^{k+1}D_t\rho \quad \cong \quad & \rho_{n,n}\ F_m u_{3k+l+1}\ u_{3k+l+2} \\
+ \quad & \rho_{n,n}\ [F_{m-1} + (k+l)DF_m]\ u_{3k+l}\ u_{3k+l+2} \\
+ \quad & \rho_{n,n}\ \left[F_{m-2} + (k+l)DF_{m-1} + \binom{k+l}{2}D^2F_m\right]\ u_{3k+l-1}\ u_{3k+l+2} \\
+ \quad & \rho_{n,n}\ \left[F_{m-3} + (k+l)DF_{m-2} + \binom{k+l}{2}D^2F_{m-1}\right. \\
+ \quad & \left.\binom{k+l}{3}D^3F_m\right]\ u_{3k+l-2}\ u_{3k+l+2} \\
+ \quad & (k+1)\ D\rho_{n,n}\ F_m\ u_{3k+l+1}\ u_{3k+l+1} \\
+ \quad & (k+1)\ D\rho_{n,n}\ [F_{m-1} + (k+l)DF_m]\ u_{3k+l}\ u_{3k+l+1} \\
+ \quad & (k+1)\ D\rho_{n,n}\ \left[F_{m-2} + (k+l)DF_{m-1}\right. \\
+ \quad & \left.\binom{k+l}{2}D^2F_m\right]\ u_{3k+l-1}\ u_{3k+l+1} \\
+ \quad & \left[\rho_{n,n-2} + D\rho_{n,n-1} + \binom{k+1}{2}D^2\rho_{n,n}\right. \\
- \quad & \left.\rho_{n-1,n-1}\right]\ F_m u_{3k+l+1}\ u_{3k+l} \\
+ \quad & \left[\rho_{n,n-2} + D\rho_{n,n-1} + \binom{k+1}{2}D^2\rho_{n,n} - \rho_{n-1,n-1}\right] \\
& \qquad \times [F_{m-1} + (k+l)DF_m]\ u_{3k+l}\ u_{3k+l} \\
+ \quad & \left[\rho_{n,n-3} + (k+1)D\rho_{n,n-2} + kD^2\rho_{n,n-1} + \binom{k+1}{3}D^3\rho_{n,n}\right. \\
- \quad & \left.\rho_{n-1,n-2} - kD\rho_{n-1,n-1}\right] F_m\ u_{3k+l+1}\ u_{3k+l-1} \\
- \quad & \rho_{n-2,n}\ F_m\ u_{3k+l}\ u_{3k+l+1} \\
- \quad & \rho_{n-2,n}\ [F_{m-1} + (k+l-1)DF_m]\ u_{3k+l-1}\ u_{3k+l+1}
\end{aligned}
$$

$$- \quad [\rho_{n-2,n-1} + kD\rho_{n-2,n} - \rho_{n-3,n}] \; F_m \; u_{3k+l} \; u_{3k+l}. \tag{4.2.7}$$

After integrations by parts we get (4.2.5) as the coefficient of the first nonlinear term u^2_{3k+l+1} and (4.2.6) as the coefficient of the second nonlinear term u^2_{3k+l}. □

From equation (4.2.5) we can easily get a number of results pertaining the form of the conserved densities. In particular we can see that higher order conserved densities should be quadratic in the highest derivative and top coefficients of the conserved densities at every order are proportional to each other [1].

Corollary 4.2.4 *Let $\rho = \rho(x, t, u, \ldots, u_n)$ and $u_t = F(x, t, u, \ldots, u_m)$, $m \geq 7$ and $n > m$. Then*

$$\rho_{n,n,n} = 0 \tag{4.2.8}$$

Proof. It can be seen that (4.2.5) uses only top two terms and for $l > 0$ it is valid for $k + 1 \geq 3$. Writing it in the form

$$(k + \frac{1}{2})\frac{D\rho_{n,n}}{\rho_{n,n}} - (k + l + \frac{1}{2})\frac{DF_m}{F_m} = \frac{F_{m-1}}{F_m}, \tag{4.2.9}$$

we can see that for $n > m$ the highest order term is $D\rho_{n,n}$ and it follows that $\rho_{n,n,n} = 0$. □

Remark 4.2.5 From (4.2.9) one can easily see that if ρ and η are both conserved densities of order n, with $\rho_{n,n} = P$ and $\eta_{n,n} = Q$, then $\frac{DP}{P} = \frac{DQ}{Q}$, hence the ratio of the top coefficients is independent of x. If ρ and η are conserved densities of consecutive orders say, $|\rho| = n$ and $|\eta| = n + 1$ with $\rho_{n,n} = P$ and $\eta_{n+1,n+1} = Q$, then

$$(k + \frac{1}{2})\left(\frac{DQ}{Q} - \frac{DP}{P}\right) = \frac{DF_m}{F_m},$$

hence $Q = F_m^{2/m} P$.

Remark 4.2.6 If the partial derivatives of F and ρ in (4.2.5) and (4.2.6) depend at most on u_j, then these equations are polynomial in u_{j+i}, $i > 0$. In all the subsequent computations we have used only the coefficient of the top order derivatives.

4.3 Polynomiality Results in Top Three Derivatives

In this section we give our final results which are the polynomiality in top three derivatives. We apply Proposition 4.2.3 either directly to a canonical density $\rho^{(i)}$, $i = 1, 2, 3$(step 3 and 6), or to generic conserved densities ρ, ν. We also give Tables showing that at least one of the canonical densities is of the generic form (Tables (4.3.1) – (4.3.5))

We used generic conserved densities in steps one, two and four. The first step is to obtain the quasilinearity result for $m > 5$, which follows from the fact that the coefficient matrix of a homogeneous system is non-singular for $m > 5$. At the second and fourth steps we have a similar structure; we show that the coefficient of u_m is independent of u_{m-1} and u_{m-2} respectively, by obtaining nonsingular homogeneous systems of linear equations.

The third and sixth steps are based on relatively straightforward computations using the canonical densities. At the third step we complete polynomiality in u_{m-1} while at the fifth and sixth steps we complete polynomiality in u_{m-2}, by using the explicit form of the canonical densities $\rho^{(1)}$ and $\rho^{(3)}$.

Step 0:Quasilinearity $F_{m,m} = 0$

First we obtain again the quasilinearity result given in [1], for $m > 5$. Thanks to the general expressions in Proposition 4.2.3, applying (4.2.5) to $\rho^{(1)}$ the proof given here is very neat.

Proposition 4.3.1 Let $u_t = F(x, t, u, \ldots, u_m)$, $m = 2k + 1 > 13$, $|F| = m$, be an arbitrary evolution equation and $\rho^{(1)} = P^{(0)}u_{m+1}^2 + Q^{(0)}u_{m+1} + R^{(0)}$, $l = 1$, $|P^{(0)}| = |Q^{(0)}| = |R^{(0)}| = m$, $P^{(0)} \neq 0$ its conserved density, then $F_{m,m} = 0$

Proof: The coefficient of u_{m+1} in (4.2.5) and the coefficient of u_{m+3} in (4.2.6) are respectively as follows:

$$(2k+1)\,\frac{P_m^{(0)}}{P^{(0)}} - (2k+3)\,\frac{F_{m,m}}{F_m} = 0 \tag{4.3.1}$$

$$(2k+1)(k^2+k+6)\frac{P_m^{(0)}}{P^{(0)}} - (2k+3)(k+1)(k+2)\frac{F_{m,m}}{F_m} = 0 \tag{4.3.2}$$

From (4.3.1) and (4.3.2) we get:

$$\begin{bmatrix} 1 & -1 \\ (k^2+k+6) & -(k^2+3k+2) \end{bmatrix} \begin{bmatrix} \frac{P_m^{(0)}}{P^{(0)}} \\ \frac{F_{m,m}}{F_m} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{4.3.3}$$

Since $k \neq 2$ from (4.3.3) we conclude that

$$F_{m,m} = P_m^{(0)} = 0 \tag{4.3.4}$$

□

The following steps are valid for $m \geq 19$

The aim of the first and second steps is to investigate the dependency on u_{m-1} of the coefficients A and B in the quasi-linear integrable evolution equation $u_t = Au_m + B$. For this purpose we use the coefficients of the top two nonlinearities in $\int D_t\rho$ which are respectively equations 4.2.5 and 4.2.6. The results with their proof are given below.

Step 1:Polynomiality in u_{m-1} 1st Result $A_{m-1} = P_{m-1} = 0$

Proposition 4.3.2 Let

$$u_t = Au_m + B,$$

with $A = a^m$ and $|A| = |B| = m - 1$. Then the canonical density $\rho^{(1)}$ reduces to

$$\rho^{(1)} = P^{(1)}u_m^2 + Q^{(1)}u_m + R^{(1)},$$

with $|P^{(1)}| = |Q^{(1)}| = |R^{(1)}| = m - 1$, And if $\rho^{(1)}$ is conserved density then

$$A_{m-1} = P_{m-1}^{(1)} = 0 \tag{4.3.5}$$

Proof: Substituting $u_t = Au_m + B$ in (4.1.10) and integrating by parts if necessary we get $\rho^{(1)}$ with

$$P^{(1)} = \frac{a_{m-1}^2}{a} \tag{4.3.6}$$

The coefficient of u_m in (4.2.5) is:

$$(2k+1)\,\frac{P^{(1)}_{m-1}}{P^{(1)}} - (2k+3)\,\frac{A_{m-1}}{A} = 0 \tag{4.3.7}$$

While the coefficient of u_{m+2} in (4.2.6) is:

$$\begin{aligned} 2P^{(1)}A_{m-1}\frac{1}{12}\left[2k^3+9k^2+13k+6\right] &- 2P^{(1)}_{m-1}A\frac{1}{12}\left[2k^3+3k^2+13k+6\right] = 0 \\ \left[2k^3+9k^2+13k+6\right]\frac{A_{m-1}}{A} &- \left[2k^3+3k^2+13k+6\right]\frac{P^{(1)}_{m-1}}{P^{(1)}} = 0 \end{aligned} \tag{4.3.8}$$

From (4.3.7) and (4.3.8) we get:

$$\begin{bmatrix} 2k+1 & -(2k+3) \\ 2k^3+3k^2+13k+6 & -(2k^3+9k^2+13k+6) \end{bmatrix} \begin{bmatrix} \frac{P^{(1)}_{m-1}}{P^{(1)}} \\ \frac{A_{m-1}}{A} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{4.3.9}$$

Since $k \neq 2$, $A_{m-1} = P^{(1)}_{m-1} = 0$. □

Now we shall see that the existence of a conserved density determines the form of B.

Step 3:Polinomiality in u_{m-1} 2nd Result $B_{m-1,m-1,m-1} = 0$

Proposition 4.3.3 Let

$$u_t = Au_m + B,$$

with $|A| = m-2$, $|B| = m-1$ and the canonical densities

$$\rho^{(1)} = P^{(1)}u_{m-1}^2 + Q^{(1)}u_{m-1} + R^{(1)}$$

$$\rho^{(3)} = P^{(3)}u_m^2 + Q^{(3)}u_m + R^{(3)}$$

with $|P^{(1)}| = m-2$ and $|P^{(3)}| = m-1$ And if $\rho^{(1)}$ and $\rho^{(3)}$ are conserved densities,then

$$B_{m-1,m-1,m-1} = 0 \tag{4.3.10}$$

Proof: We substitute $u_t = Au_m + B$ and $A_{m-1} = 0$ in (4.1.10), (4.1.12) and integrate by parts then we get the coefficients $P^{(1)}$ and $P^{(3)}$ respectively:

$$P^{(1)} = \frac{24}{m^2-1}a_{m-2,m-2} + a_{m-2}^2 a^{-1}(m^2-1) \tag{4.3.11}$$

$$\begin{aligned} P^{(3)} &= \frac{a}{m^3+3m^2-m-3}\left[a_{m-2}^2\left(m^3+3m^2-121m+597\right)\right. \\ &+ a^{-m+1}a_{m-2}B_{m-1,m-1}\left(\frac{180}{m}-60\right) \\ &+ \left. a^{-2m+2}B_{m-1,m-1}^2\left(\frac{60}{m}-\frac{60}{m^2}\right)\right] \end{aligned} \tag{4.3.12}$$

We compute (4.2.5) using $\rho^{(1)}$ where $l=-1$ and the top dependency of $P^{(1)}$ is u_{m-2} and we get:

$$(k+\frac{1}{2})2AP^{(1)}_{m-2}u_{m-1} - (k-\frac{1}{2})2P^{(1)}A_{m-2}u_{m-1} - 2P^{(1)}B_{m-1} = 0 \tag{4.3.13}$$

We compute also (4.2.5) using $\rho^{(3)}$ where $l=0$ and the top dependency of $P^{(3)}$ is u_{m-1} and we get:

$$(k+\frac{1}{2})2AP^{(3)}_{m-1}u_m-(k+\frac{1}{2})2P^{(3)}A_{m-2}u_{m-1}-2P^{(3)}B_{m-1}=0 \tag{4.3.14}$$

Differentiating twice (4.3.13) with respect to u_{m-1} we obtain:

$$2P^{(1)}B_{m-1,m-1,m-1}=0 \tag{4.3.15}$$

On the other hand the coefficient of u_m in (4.3.14) gives:

$$(k+\frac{1}{2})2AP^{(3)}_{m-1}=0 \tag{4.3.16}$$

and $P^{(3)}_{m-1}$ can be written as

$$B_{m-1,m-1,m-1}(p^1A_{m-2}+p^2B_{m-1,m-1}) \tag{4.3.17}$$

where p^1 and p^2 are independent from u_{m-1}. Then from (4.3.15) and (4.3.17) we conclude that:

$$B_{m-1,m-1,m-1}=0. \tag{4.3.18}$$

□

Here is the form of the integrable evolution equation that we obtain which is polynomial in u_{m-1}. $u_t=Au_m+Cu^2_{m-1}+Du_{m-1}+E$. The following three steps investigate the polynomiality in u_{m-2}.

Step 4:Polinomiality in u_{m-2} 1st Result $A_{m-2}=P_{m-2}=C=0$

Proposition 4.3.4 *Let* $u_t=Au_m+Cu^2_{m-1}+Du_{m-1}+E$, *where* $a=A^{\frac{1}{m}}$ *and the canonical densities*

$$\rho_{(1)}=P^{(1)}u^2_{m-1}+Q^{(1)}u_{m-1}+R^{(1)} \tag{4.3.19}$$

and

$$\rho_{(2)}=P^{(2)}u^2_m+Q^{(2)}u_m+R^{(2)} \tag{4.3.20}$$

where $A,C,D,E,P^{(1)},P^{(2)},a$ *depend on* $x,t,u,\ldots,u_{m-2},$. If $\rho_{(1)}$ and $\rho_{(2)}$ are conserved densities, then

$$A_{m-2}=P^{(1)}_{m-2}=C=0 \tag{4.3.21}$$

Proof: We substitute $u_t=Au_m+Cu^2_{m-1}+Du_{m-1}+E$, in (4.1.10) and (4.1.12q) and notice that $\rho^{(1)}$ and $\rho^{(3)}$ have the same form as $\rho_{(1)}$ and $\rho_{(2)}$. We recomputed (4.2.5), and (4.2.6) for $l=-1,0$ and we get respectively:
The coefficient of u_{m-1} in (4.2.5) for $\rho^{(1)}$, $l=-1$

$$4C-(2k+1)\,a^m\frac{P^{(1)}_{m-2}}{P^{(1)}}+\left(4k^2-1\right)a^{m-1}a_{m-2}=0 \tag{4.3.22}$$

The coefficient of u_{m+1} in (4.2.6) for $\rho^{(1)}$, $l=-1$

$$\begin{aligned}&12k^2C-\left(2k^3+3k^2+13k+6\right)a^m\frac{P^{(1)}_{m-2}}{P^{(1)}}\\ +\quad&\left(4k^4-4k^3+23k^2+25k+6\right)a^{m-1}a_{m-2}=0\end{aligned} \tag{4.3.23}$$

Thus (4.3.22) and (4.3.23) give:

$$(8k^2 - 4k - 24)C + (-8k^3 + 26k + 12)a^{m-1}a_{m-2} = 0 \tag{4.3.24}$$

The coefficient of u_{m-1} in (4.2.5) for $\rho^{(2)}, \;\; l = 0$

$$4C - (2k+1)\, a^m \frac{P^{(2)}_{m-2}}{P^{(2)}} + \left(4k^2 + 4k + 1\right) a^{m-1} a_{m-2} = 0 \tag{4.3.25}$$

The coefficient of u_{m+1} in (4.2.6) for $\rho^{(2)}, \;\; l = 0$

$$\begin{aligned} 12 \quad & (k^2 + 2k + 1)C - (2k^3 + 3k^2 + 25k + 12)a^m \frac{P^{(2)}_{m-2}}{P^{(2)}} \\ + \quad & (4k^4 + 9k^3 + 28k^2 + 49k + 18)a^{m-1}a_{m-2} = 0 \end{aligned} \tag{4.3.26}$$

Thus (4.3.25) and (4.3.26) give:

$$(8k^2 + 20k - 36)C + (k^3 - 25k^2 + 6)a^{m-1}a_{m-2} = 0 \tag{4.3.27}$$

Finally equations (4.3.24) and (4.3.27) give:

$$\begin{bmatrix} (8k^2 - 4k - 24) & (-8k^3 + 26k + 12) \\ (8k^2 + 20k - 36) & (k^3 - 25k^2 + 6) \end{bmatrix} \begin{bmatrix} C \\ a^{m-1}a_{m-2} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{4.3.28}$$

Since the determinant of the matrix (4.3.28) is different than zero for $k \neq 2$ and $a \neq 0$ we have:

$$a_{m-2} = A_{m-2} = C = 0 \tag{4.3.29}$$

Thus equations (4.3.22) and (4.3.25) give:

$$P^{(1)}_{m-2} = P^{(2)}_{m-2} = 0 \tag{4.3.30}$$

Step 5:Polinomiality in u_{m-2} 2nd Result $D_{m-2,m-2} = 0$

Proposition 4.3.5 *Let $u_t = Au_m + Du_{m-1} + E$, where $A_{m-2} = 0$, and assume that there exist a conserved density $\rho = Pu^2_{m-1} + Qu_{m-1} + R$, where $P_{m-2} = 0$.* Then

$$D_{m-2,m-2} = 0 \tag{4.3.31}$$

Proof: We recomputed (4.2.5) under these conditions and we get:

$$\left[k + \frac{1}{2}\right] 2P_{m-3}(A)u_{m-2} - \left[k - \frac{1}{2}\right] 2P(A_{m-3})u_{m-2} = 2P(D) \tag{4.3.32}$$

We differentiate (4.3.32) twice with respect to u_{m-2} and we obtain:

$$D_{m-2,m-2} = 0 \tag{4.3.33}$$

□

Step 6:Polinomiality in u_{m-2} 3rd Result $E_{m-2,m-2,m-2,m-2} = 0$.

Proposition 4.3.6 *Let* $u_t = Au_m + Gu_{m-1}u_{m-2} + Hu_{m-1} + E$, $A = a^m$, $|A| = |G| = |H| = m-3$, $|E| = m-2$ *be the evolution equation and* $\rho^{(1)}$ *its conserved density.* Then

$$E_{m-2,m-2,m-2,m-2} = 0. \tag{4.3.34}$$

Proof: In this case $\rho^{(1)}$ should be quadratic in u_{m-2}.

$$\begin{aligned}
\rho^{(1)} &= \frac{u_{m-2}^2}{m^2-1}\left[\frac{a_{m-3}^2}{a}(m^2-1) + 36\frac{a_{m-3}G}{a^m}(\frac{1}{m}-1)\right.\\
&+ \left. 24\frac{aG_{m-3}}{ma^m} + 12\frac{aG^2}{ma^{2m}}(1-\frac{1}{m})\right]\\
&+ \frac{2u_{m-2}u_{m-3}}{m^2-1}\left[\frac{a_{m-3}a_{m-4}}{a}(m^2-1)\right.\\
&+ \left. 18\frac{a_{m-4}G}{a^m}(\frac{1}{m}-1) + 12\frac{G_{m-4}a}{ma^m}\right]\\
&+ 12u_{m-2}\frac{H}{a^{2m}m^2(m+1)}\left[2aG - ma^m a_{m-3}\right]\\
&+ u_{m-3}^2\frac{a_{m-4}^2}{a} - 12u_{m-3}\frac{a_{m-4}H}{m(m+1)a^m}\\
&+ 12\frac{a}{m^2(m^2-1)a^{2m}}\left[-2ma^m E_{m-2} + H^2(m-1)\right]
\end{aligned} \tag{4.3.35}$$

Thus the third derivative of $\rho^{(1)}$ with respect to u_{m-2} is:

$$\frac{\partial^3\rho^{(1)}}{\partial {u_{m-2}}^3} = -\frac{24a^{-m+1}}{m(m^2-1)}E_{m-2,m-2,m-2,m-2} = 0 \tag{4.3.36}$$

Finally the integrable evolution equation of order m where the coefficients depend on $x, t, u, , \ldots, u_{m-3}$ has the following form:

$$F = Au_m + Gu_{m-2}u_{m-1} + Hu_{m-1} + Ju_{m-2}^3 + Lu_{m-2}^2 + Nu_{m-2} + S \tag{4.3.37}$$

where A, G, H, J, L, N, S depend on $x, t, u, \ldots, u_{m-3}$

The following Tables show the correspondence between the generic and canonical densities for each Step.

Table 4.3.1: **Step 1:** $u_t = Au_m + B$, Polynomiality in u_{m-1} First Result

$m = 2k+1,\ k = k$	$\rho(u_m) = Pu_m^2$	$Eq(7), Eq(8)$	$u_{3k+1}^2 u_m,\ u_{3k}^2 u_{m+2}$	$A_{m-1} = 0$
$m = 7, k = 3$	$\rho^{(1)} = Pu_7^2$	$Eq(7), Eq(8)$	$u_{10}^2 u_7,\ u_9^3$	$A_6 = 0$
$m = 9, k = 4$	$\rho^{(1)} = Pu_9^2$	$Eq(7), Eq(8)$	$u_{13}^2 u_9,\ u_{12}^2 u_{11}$	$A_8 = 0$
$m = 11, k = 5$	$\rho^{(1)} = Pu_{11}^2$	$Eq(7), Eq(8)$	$u_{16}^2 u_{11},\ u_{15}^2 u_{13}$	$A_{10} = 0$
$m = 13, k = 6$	$\rho^{(1)} = Pu_{13}^2$	$Eq(7), Eq(8)$	$u_{19}^2 u_{13},\ u_{18}^2 u_{15}$	$A_{12} = 0$
$m = 15, k = 7$	$\rho^{(1)} = Pu_{15}^2$	$Eq(7), Eq(8)$	$u_{22}^2 u_{15},\ u_{21}^2 u_{17}$	$A_{14} = 0$
$m = 17, k = 8$	$\rho^{(1)} = Pu_{17}^2$	$Eq(7), Eq(8)$	$u_{25}^2 u_{17},\ u_{24}^2 u_{19}$	$A_{16} = 0$
$m = 19, k = 9$	$\rho^{(1)} = Pu_{19}^2$	$Eq(7), Eq(8)$	$u_{28}^2 u_{19},\ u_{27}^2 u_{21}$	$A_{18} = 0$

Table 4.3.2: **Step 2:** $u_t = Au_m + B, \quad A_{m-1} = 0$ Polynomiality in u_{m-1} Second Result

$m = 2k+1,\ k = k$	$\rho(u_m) = Pu_m^2$	$Eq(7)$	$u_{3k+1}^2 u_m$	$B_{m-1,m-1,m-1} = 0$
$m = 7, k = 3$	$\rho^{(3)} = Pu_7^2$	$Eq(7)$	$u_{10}^2 u_7$	$B_{6,6,6} = 0$
$m = 9, k = 4$	$\rho^{(3)} = Pu_9^2$	$Eq(7)$	$u_{13}^2 u_9$	$B_{8,8,8} = 0$
$m = 11, k = 5$	$\rho^{(3)} = Pu_{11}^2$	$Eq(7)$	$u_{16}^2 u_{11}$	$B_{10,10,10} = 0$
$m = 13, k = 6$	$\rho^{(3)} = Pu_{13}^2$	$Eq(7)$	$u_{19}^2 u_{13}$	$B_{12,12,12} = 0$
$m = 15, k = 7$	$\rho^{(3)} = Pu_{15}^2$	$Eq(7)$	$u_{22}^2 u_{15}$	$B_{14,14,14} = 0$
$m = 17, k = 8$	$\rho^{(3)} = Pu_{17}^2$	$Eq(7)$	$u_{25}^2 u_{17}$	$B_{16,16,16} = 0$
$m = 19, k = 9$	$\rho^{(3)} = Pu_{19}^2$	$Eq(7)$	$u_{28}^2 u_{19}$	$B_{18,18,18} = 0$

Table 4.3.3: **Step 3:** $u_t = Au_m + Cu_{m-1}^2 + Du_{m-1} + E,\ A_{m-1} = 0$ Polynomiality in u_{m-2} First Result.

$m = 2k+1,$ $k = k$	$\rho(u_{m-1}) = Pu_{m-1}^2$ $\rho(u_m) = Pu_m^2$	$Eq(7), Eq(8)$ $Eq(7), Eq(8)$	$u_{3k}^2 u_{m-1},\ u_{3k-1}^2 u_{m+1}$ $u_{3k+1}^2 u_{m-1},\ u_{3k}^2 u_{m+1}$	$A_{m-2} = 0$ $C = 0$
$m = 7,\ k = 3$ $k = 3$	$\rho^{(1)} = Pu_6^2$ $\rho^{(3)} = Pu_7^2$	$Eq(7), Eq(8)$ $Eq(7), Eq(8)$	$u_9^2 u_6,\ u_8^3$ $u_{10}^2 u_6,\ u_9^2 u_8$	$A_5 = 0$ $C = 0$
$m = 9,\ k = 4$ $k = 4$	$\rho^{(1)} = Pu_8^2$ $\rho^{(3)} = Pu_9^2$	$Eq(7), Eq(8)$ $Eq(7), Eq(8)$	$u_{12}^2 u_8,\ u_{11}^2 u_{10}$ $u_{13}^2 u_8,\ u_{12}^2 u_{10}$	$A_7 = 0$ $C = 0$
$m = 11,\ k = 5$ $k = 5$	$\rho^{(1)} = Pu_{10}^2$ $\rho^{(3)} = Pu_{11}^2$	$Eq(7), Eq(8)$ $Eq(7), Eq(8)$	$u_{15}^2 u_{10},\ u_{14}^2 u_{12}$ $u_{16}^2 u_{10},\ u_{15}^2 u_{12}$	$A_9 = 0$ $C = 0$
$m = 13,\ k = 6$ $k = 6$	$\rho^{(1)} = Pu_{12}^2$ $\rho^{(3)} = Pu_{13}^2$	$Eq(7), Eq(8)$ $Eq(7), Eq(8)$	$u_{18}^2 u_{12},\ u_{17}^2 u_{14}$ $u_{19}^2 u_{12},\ u_{18}^2 u_{14}$	$A_{11} = 0$ $C = 0$
$m = 15,\ k = 7$ $k = 7$	$\rho^{(1)} = Pu_{14}^2$ $\rho^{(3)} = Pu_{15}^2$	$Eq(7), Eq(8)$ $Eq(7), Eq(8)$	$u_{21}^2 u_{14},\ u_{20}^2 u_{16}$ $u_{22}^2 u_{14},\ u_{21}^2 u_{16}$	$A_{13} = 0$ $C = 0$
$m = 17,\ k = 8$ $k = 8$	$\rho^{(1)} = Pu_{16}^2$ $\rho^{(3)} = Pu_{17}^2$	$Eq(7), Eq(8)$ $Eq(7), Eq(8)$	$u_{24}^2 u_{16},\ u_{23}^2 u_{18}$ $u_{25}^2 u_{16},\ u_{24}^2 u_{18}$	$A_{15} = 0$ $C = 0$
$m = 19,\ k = 9$ $k = 9$	$\rho^{(1)} = Pu_{18}^2$ $\rho^{(3)} = Pu_{19}^2$	$Eq(7), Eq(8)$ $Eq(7), Eq(8)$	$u_{27}^2 u_{18},\ u_{26}^2 u_{20}$ $u_{28}^2 u_{18},\ u_{27}^2 u_{20}$	$A_{17} = 0$ $C = 0$

Table 4.3.4: **Step 4:** $u_t = Au_m + Du_{m-1} + E,\ A_{m-1} = A_{m-2} = 0$
Polynomiality in u_{m-2} Second Result

$m = 2k+1,\ k = k$	$\rho(u_{m-1}) = Pu_{m-1}^2$	$Eq(7)$	$u_3k^2u_{m-1}$	$D_{m-2,m-2} = 0$
$m = 7, k = 3$	$\rho^{(3)} = Pu_6^2$	$Eq(7)$	$u_9^2u_6$	$D_{5,5} = 0$
$m = 9, k = 4$	$\rho^{(3)} = Pu_8^2$	$Eq(7)$	$u_{12}^2u_8$	$D_{7,7} = 0$
$m = 11, k = 5$	$\rho^{(3)} = Pu_{10}^2$	$Eq(7)$	$u_{15}^2u_{10}$	$D_{9,9} = 0$
$m = 13, k = 6$	$\rho^{(3)} = Pu_{12}^2$	$Eq(7)$	$u_{18}^2u_{12}$	$D_{11,11} = 0$
$m = 15, k = 7$	$\rho^{(3)} = Pu_{14}^2$	$Eq(7)$	$u_{21}^2u_{14}$	$D_{13,13} = 0$
$m = 17, k = 8$	$\rho^{(3)} = Pu_{16}^2$	$Eq(7)$	$u_{24}^2u_{16}$	$D_{15,15} = 0$
$m = 19, k = 9$	$\rho^{(3)} = Pu_{18}^2$	$Eq(7)$	$u_{27}^2u_{18}$	$D_{17,17} = 0$

Table 4.3.5:**Step 5:** $u_t = Au_m + Gu_{m-2}u_{m-1} + Hu_{m-1} + E, \quad A_{m-1} = A_{m-2} = 0$
Polynomiality in u_{m-2} Third Result

$m = 2k+1,$ $k = k$	$\rho(u_{m-2}) = Pu_{m-2}^2$	$Eq(8)$	$u_{3k-1}^2u_{m-1}$	$E_{m-2,m-2,m-2,m-2} = 0$
$m = 7, k = 3$	$\rho^{(1)} = Pu_5^2$	$Eq(8)$	$u_8^2u_6$	$E_{5,5,5,5} = 0$
$m = 9, k = 4$	$\rho^{(1)} = Pu_7^2$	$Eq(8)$	$u_{11}^2u_8$	$E_{7,7,7,7} = 0$
$m = 11, k = 5$	$\rho^{(1)} = Pu_9^2$	$Eq(8)$	$u_{14}^2u_{10}$	$E_{9,9,9,9} = 0$
$m = 13, k = 6$	$\rho^{(1)} = Pu_{11}^2$	$Eq(8)$	$u_{17}^2u_{12}$	$E_{11,11,11,11} = 0$
$m = 15, k = 7$	$\rho^{(1)} = Pu_{13}^2$	$Eq(8)$	$u_{20}^2u_{14}$	$E_{13,13,13,13} = 0$
$m = 17, k = 8$	$\rho^{(1)} = Pu_{15}^2$	$Eq(8)$	$u_{23}^2u_{16}$	$E_{15,15,15,15} = 0$
$m = 19, k = 9$	$\rho^{(1)} = Pu_{17}^2$	$Eq(8)$	$u_{26}^2u_{18}$	$E_{17,17,17,17} = 0$

5 SPECIAL CASES

In the application of the integrations by parts to $D_t\rho$ we use the expression of the kth derivatives given by (4.1.6). This expression, which involves top 4 derivatives is valid for $k \geq 7$. In our computations $D^{k+l-1}F$ for $l = -2, -1, 0, 1, 2$ appears in (4.2.1). Although at this place only top 2 derivatives are used, we assume that the general results obtained in the previous section are valid for $k \geq 9$, i.e, for $m \geq 19$, and treat $m = 7, 9, 11, 13, 15, 17, 19$ as special cases and conserved density conditions are computed directly, *without using* 4.2.5 and 4.2.6. The derivations for $m = 7$ and $m = 9$ are given in some detail below. At all other orders, computations are carried out explicitly. One can compute it using the tables below (Tables 5.1.1−5.1.4) where it has been shown the steps and quadratic terms to be considered on each computation.

5.1 Classification of 7th order evolution equation

5.1.1 First Method

In this section we give the explicit computations for $m = 7$ using all canonical densities $\rho^{(i)}$, $i = -1, 1, 2, 3$ computed in [1].

Step 1. We start with the quasilinear case where

$$u_t = A(x, t, u, \ldots, u_6)u_7 + B(x, t, u, \ldots, u_6).$$

Recalling that in the grading by levels we need only the dependency on u_6 we write

$$u_t \sim A(u_6)u_7 + B(u_6).$$

In the following $\sim$ will denote equality using the dependency on the top base level only.

The conserved densities are

$$\rho^{(-1)} = a^{-1} = A^{-1/7} \tag{5.1.1.1}$$

$$\rho^{(1)} = u_7^2 a_6^2 a^{-1} + \frac{3}{14} u_7 a_6 B_6 a^{-7} + \frac{3}{98} B_6^2 a^{-13}, \tag{5.1.1.2}$$

$$\begin{aligned}\rho^{(2)} &= u_7^2 a_6 a^{-7}\left[-7a_6 B_6 + B_6 a\right] - \frac{3}{7} u_7 a_6 B_6^2 a^{-13} \\ &- \frac{2}{49} B_6^3 a^{-19},\end{aligned} \tag{5.1.1.3}$$

$$\begin{aligned}\rho^{(3)} &= u_8^2 a_6^2 a + u_7^4\left[-\frac{2}{3}a_{666}a_6 a + \frac{1}{3}a_{66}^2 a - \frac{2}{3}a_{66}a_6^2 + \frac{1}{4}a_6^4 a^{-1}\right] \\ &+ u_7^3 a^{-6}\left[\frac{3}{56}a_{66}B_{66}a + \frac{9}{28}a_6^2 B_{66} - \frac{3}{56}a_6 B_{666} a\right] \\ &+ u_7^2 a^{-13}\left[\frac{165}{98}a_6^2 B_6^2 - \frac{33}{98}a_6 B_{66} B_6 a + \frac{3}{196}B_{66}^2 a^2\right] \\ &+ \frac{33}{686}u_7 a_6 B_6^3 a^{-19} + \frac{33}{9604}B_6^4 a^{-25}.\end{aligned} \tag{5.1.1.4}$$

The structure of non-integrable terms is given by the following table. The coefficients of the nonintegrable terms shown with bold face are the ones that lead to $a_6 = 0$.

Table 5.1.1: Structure of non-integrable terms in Step 1. for order $m = 7$.

	$\rho^{(-1)}$	$\rho^{(1)}$	$\rho^{(2)}$	$\rho^{(3)}$
Top terms in $\rho^{(i)}$	u_6	u_7^2	u_7^2	u_8^2
Order of $\rho^{(i)}$	6	7	7	8
Level of $\rho^{(i)}$	0	2	2	4
Top term in $\int \rho_t^{(i)}$	u_{13}	$u_{14}u_7$	$u_{14}u_7$	$u_{15}u_8$
Level of top term in $\int \rho_t^{(i)}$	7	9	9	11
Non-integrable terms in $\int \rho_t^{(i)}$	$u_9^2u_7$, $u_8^3u_7$, $u_8^2u_7^3$, u_7^7,	$\mathbf{u_{10}^2u_7}$, $\mathbf{u_9^3}$, $u_9^2u_8u_7$, $u_9^2u_8u_7$, $u_8^4u_7$, $u_8^3u_7^3$, $u_8^2u_7^5$, u_7^9,	u_{11}^2, $u_9^3u_7$, $u_9^2u_8^2$, $u_9^2u_8u_7$, $u_9^2u_7^4$, u_8^5, $u_8^4u_7^2$, $u_8^3u_7^4$, $u_8^2u_7^6$, u_7^{10},	$u_{11}^2u_7$, $u_{10}^2u_9$, $u_{10}^2u_8u_7$, $u_{10}^2u_7^3$, $u_9^3u_8$, $u_9^3u_7^2$, $u_9^2u_8^2u_7$, $u_9^2u_8u_7^3$, $u_9^2u_7^5$, $u_8^5u_7$, $u_8^4u_7^3$, $u_8^3u_7^5$, $u_8^2u_7^7$, u_7^{11}

The conserved density $\rho^{(1)}$ is of order 7, level 2, $\rho^{(1)} = \rho^{(1)}(..., u_7, u_7^2)$,

$$\rho_t^{(1)} = \rho_7^{(1)}(u_7)_t + ... \tag{5.1.1.5}$$

where $\rho_t^{(1)}$ is of order $(1 + 7 + 7) = 15$ and level 9. The integrability conditions for $D^{-1}\rho_t^{(1)}$ will be provided by equating to zero the coefficients of the non-integrable terms in $\overline{M_6^9}$.

The coefficients $u_{10}^2u_7$ and u_9^3 in $\rho_t^{(1)}$ gave respectively the following conditions: for $a \neq 0$

$$\mathbf{u_{10}^2u_7} \quad : \quad \mathbf{14a_6a^5(a_{66}a - 5a_6^2) = 0} \tag{5.1.1.6}$$
$$\mathbf{u_9^3} \quad : \quad \mathbf{7a_6a^5(-2a_{66}a + 11a_6^2) = 0} \tag{5.1.1.7}$$

The compatibility of (5.1.1.6), (5.1.1.7) gives:

$$\mathbf{a_6 = 0} \tag{5.1.1.8}$$

Since the coefficient a has to be independent of u_6 and $A = a^7$, the dependency of A is restricted to u_5.

Step 2. In our second step the evolution equation is:

$$u_t \sim A(u_5)\, u_7 + B(u_5, u_6) \tag{5.1.1.9}$$

and the conserved densities are:

$$\rho^{(-1)} = a^{-1} = A^{-1/7} \tag{5.1.1.10}$$

$$\begin{aligned}\rho^{(1)} &= -\frac{1}{2}u_7 a_5 + u_6^2 a_5^2 a^{-1} - \frac{3}{14}u_6 a_5 B_6 a^{-7} \\ &+ a^{-13}\left[-\frac{1}{14}B_5 a^7 + \frac{3}{98}B_6^2\right],\end{aligned} \tag{5.1.1.11}$$

$$\begin{aligned}\rho^{(2)} &= u_7 a_5 B_6 a^{-6} + u_7 u_6 a_5\left[-7a_5 + B_{66}a^{-6}\right] \\ &+ u_6^2 a_5 a^{-7}\left[-7a_5 B_6 + B_{56}a\right] \\ &+ u_6 a_5 a^{-13}\left[-B_5 a^7 + \frac{3}{7}B_6^2\right] \\ &+ B_6 a^{-19}\left[\frac{1}{7}B_5 a^7 - \frac{2}{49}B_6^2\right],\end{aligned} \tag{5.1.1.12}$$

$$\begin{aligned}\rho^{(3)} &= u_7^2 a^{-11}\left[\frac{1}{2}a_5^2 a^{12} - \frac{5}{28}a_5 B_{66}a^6 + \frac{3}{196}B_{66}^2\right] \\ &+ u_7 u_6^2\left[a_{55}a_5 a - \frac{3}{28}a_{55}B_{66}a^{-5} - \frac{13}{4}a_5^3\right. \\ &+ \left.\frac{9}{28}a_5^2 B_{66}a^{-6}\right] + u_7 u_6 a^{-12}\left[\frac{65}{28}a_5^2 B_6 a^6\right. \\ &- \left.\frac{1}{28}a_5 B_{56}a^7 - \frac{33}{98}a_5 B_{66}B_6 + \frac{3}{98}B_{56}B_{66}a\right] \\ &+ u_7 a^{-12}\left[\frac{1}{7}a_5 B_5 a^7 - \frac{11}{98}a_5 B_6^2\right. \\ &- \left.\frac{1}{98}B_5 B_{66}a\right] + u_6^4 a^{-1}\left[a_{55}^2 a^2 + \frac{1}{4}a_5^4\right] \\ &+ u_6^3 a^{-7}\left[\frac{3}{4}a_{55}a_5 B_6 a - \frac{3}{28}a_{55}B_{56}a - \frac{9}{4}a_5^3 B_6\right. \\ &+ \left.\frac{9}{28}a_5^2 B_{56}a\right] + u_6^2 a^{-13}\left[-\frac{33}{28}a_5^2 B_5 a^7 + \frac{165}{98}a_5^2 B_6^2\right. \\ &- \left.\frac{33}{98}a_5 B_{56}B_6 a + \frac{1}{7}a_5 B_{55}a^8 + \frac{3}{196}B_{56}^2 a^2\right] \\ &+ u_6 a^{-19}\left[\frac{11}{49}a_5 B_5 B_6 a^7 - \frac{33}{686}a_5 B_6^3 - \frac{1}{98}B_{56}B_5 a^8\right] \\ &+ a^{-25}\left[\frac{1}{98}B_5^2 a^{14} - \frac{11}{686}B_5 B_6^2 a^7 + \frac{33}{9604}B_6^4\right]\end{aligned} \tag{5.1.1.13}$$

Then we repeated the same computations for (5.1.1.9) with $\rho^{(-1)}, \rho^{(1)}, \rho^{(2)}, \rho^{(3)}$. The conserved density $\rho^{(3)}$ is of order 7 and level 2, $\rho^{(3)} = \rho^{(3)}(..., u_7^2)$.

$$\rho_t^{(3)} = \rho_7^{(3)}(u_7)_t + ... \tag{5.1.1.14}$$

$\rho_t^{(3)}$ is of order $(7+7+1) = 15$ and level 9. The following conditions came up from the coefficients of the non-integrable term $u_{10}^2 u_7$ in $\overline{M_6^9}$:

$$\mathbf{u_{10}^2 u_7}: \quad \mathbf{B_{666}a^{-4}\left(-\frac{5}{4}a_5 a^6 + \frac{3}{14}B_{66}\right) = 0} \tag{5.1.1.15}$$

Therefore

$$\mathbf{B_{666} = 0 \ \Rightarrow\ B = Bu_6^2 + Cu_6 + D} \tag{5.1.1.16}$$

Table 5.1.2: Structure of non-integrable terms in Step 2. for order $m = 7$.

	$\rho^{(-1)}$	$\rho^{(1)}$	$\rho^{(2)}$	$\rho^{(3)}$
Top terms in $\rho^{(i)}$	u_5	u_7	u_7	u_7^2
Order of $\rho^{(i)}$	5	7	7	7
Level of $\rho^{(i)}$	0	1	1	2
Top term in $\int \rho_t^{(i)}$	u_{12}	u_{14}	u_{14}	$u_{14}u_7$
Level of top term in $\int \rho_t^{(i)}$	6	8	8	9
Non-integrable terms in $\int \rho_t^{(i)}$	u_9^2, u_8^3, $u_8^2u_7^2$, u_7^6	u_{10}^2, $u_9^2u_8$, $u_9^2u_7^2$, u_8^4 $u_8^3u_7^2$, $u_8^2u_7^4$, u_7^8	u_{10}^2, $u_9^2u_8$, $u_9^2u_7^2$, u_8^4, $u_8^3u_7^2$, $u_8^2u_7^4$, u_7^8	$\mathbf{u_{10}^2u_7}$, u_9^3, $u_9^2u_8u_7$, $u_9^2u_7^3$, $u_8^4u_7$, $u_8^3u_7^3$, $u_8^2u_7^5$, u_7^9

Step 3. The evolution equation is:

$$u_t \sim Au_7 + Bu_6^2 + Cu_6 + D \tag{5.1.1.17}$$

where $A = A(u_5)$, $B = B(u_5)$, $C = C(u_5)$, $D = D(u_5)$. Here we noticed that the coefficients A, B, C, D, in u_t are of order 5, then we choose u_5 as our basic term with level zero. The conserved densities are:

$$\rho^{(-1)} = a^{-1} = A^{-1/7} \tag{5.1.1.18}$$

$$\begin{aligned} \rho^{(1)} &= u_6^2 \left[\frac{1}{2}a_{55} + a_5^2a^{-1} - \frac{3}{7}a_5a^{-7}B \right. \\ &- \left. \frac{1}{14}B_5a^{-6} + \frac{6}{49}a^{-13}B^2\right] + u_6a^{-13}\left[-\frac{3}{14}a_5a^6C \right. \\ &- \left. \frac{1}{14}C_5a^7 + \frac{6}{49}BC\right] + a^{-13}\left[-\frac{1}{14}D_5a^7 + \frac{3}{98}C^2\right] \end{aligned} \tag{5.1.1.19}$$

$$\begin{aligned} \rho^{(2)} &= u_6^3\left[7a_{55}a_5 - 2a_{55}a^{-6}B - 2a_5^2a^{-7}B - a_5B_5a^{-6}\right. \\ &+ \left. \frac{12}{7}a_5a^{-13}B^2 + \frac{2}{7}B_5a^{-12}B - \frac{16}{49}a^{-19}B^3\right] \\ &+ u_6^2a^{-19}\left[-a_{55}a^{13}C - a_5^2a^{12}C - a_5C_5a^{13}\right. \\ &+ \left. \frac{12}{7}a_5a^6BC + \frac{2}{7}C_5a^7B + \frac{1}{7}B_5a^7C - \frac{24}{49}CB^2\right] \\ &+ u_6a^{-19}\left[-a_5D_5a^{13} + \frac{3}{7}a_5a^6C^2 + \frac{2}{7}D_5a^7B\right. \\ &+ \left. \frac{1}{7}C_5a^7C - \frac{12}{49}C^2B\right] + a^{-19}C\left[\frac{1}{7}D_5a^7 - \frac{2}{49}C^2\right] \end{aligned} \tag{5.1.1.20}$$

$$\begin{aligned} \rho^{(3)} &= u_7^2a^{11}\left[\frac{1}{2}a_5^2a^{12} - \frac{5}{14}a_5a^6B + \frac{3}{49}B^2\right] \\ &+ u_6^4\left[-\frac{1}{3}a_{555}a_5a + \frac{1}{14}a_{555}a^{-5}B + \frac{2}{3}a_{55}^2a + \frac{35}{12}a_{55}a_5^2\right. \end{aligned}$$

$$
\begin{aligned}
- &\ \frac{50}{21}a_{55}a_5a^{-6}B - \frac{1}{6}a_{55}B_5a^{-5} + \frac{88}{147}a_{55}a^{-12}B^2 \\
+ &\ \frac{1}{4}a_5^4a^{-1} + \frac{85}{14}a_5^3a^{-7}B - \frac{61}{28}a_5^2B_5a^{-6} \\
- &\ \frac{22}{49}a_5^2a^{-13}B^2 + \frac{5}{42}a_5B_{55}a^{-5} + \frac{33}{49}a_5B_5a^{-12}B \\
- &\ \frac{132}{343}a_5a^{-19}B^3 - \frac{5}{147}B_{55}a^{-11}B + \frac{5}{294}B_5^2a^{-11} \\
- &\ \frac{22}{343}B_5a^{-18}B^2 + \frac{132}{2401}a^{-25}B^4\Big] + u_6^3a^{-25}\Big[-\frac{11}{7}a_{55}a_5a^{19}C \\
- &\ \frac{9}{56}a_{55}C_5a^{20} + \frac{55}{98}a_{55}a^{13}BC + \frac{33}{7}a_5^3a^{18}C - \frac{7}{4}a_5^2C_5a^{19} \\
+ &\ \frac{5}{56}a_5C_{55}a^{20} + \frac{55}{98}a_5C_5a^{13}B + \frac{11}{98}a_5B_5a^{13}C - \frac{198}{343}a_5a^6CB^2 \\
- &\ \frac{1}{49}C_{55}a^{14}B + \frac{3}{98}C_5B_5a^{14} - \frac{22}{343}C_5a^7B^2 - \frac{22}{343}B_5a^7BC \\
+ &\ \frac{264}{2401}CB^3\Big] + u_6^2a^{-25}\Big[-\frac{1}{7}a_{55}D_5a^{20} + \frac{11}{98}a_{55}a^{13}C^2 \\
- &\ \frac{13}{28}a_5^2D_5a^{19} + \frac{33}{98}a_5^2a^{12}C^2 + \frac{11}{49}a_5D_5a^{13}B + \frac{11}{98}a_5C_5a^{13}C \\
- &\ \frac{99}{343}a_5a^6C^2B + \frac{1}{49}D_{55}a^{14}B + \frac{1}{49}D_5B_5a^{14} - \frac{22}{343}D_5a^7B^2 \\
+ &\ \frac{3}{196}C_5^2a^{14} - \frac{22}{343}C_5a^7CB - \frac{11}{686}B_5a^7C^2 + \frac{198}{2401}C^2B^2\Big] \\
+ &\ u_6a^{-25}\Big[\frac{11}{49}a_5D_5a^{13}C - \frac{33}{686}a_5a^6C^3 + \frac{1}{98}D_5C_5a^{14} \\
- &\ \frac{22}{343}D_5a^7CB - \frac{11}{686}C_5a^7C^2 + \frac{66}{2401}C^3B\Big] \\
+ &\ a^{-25}\Big[\frac{1}{98}D_5^2a^{14} - \frac{11}{686}D_5a^7C^2 + \frac{33}{9604}C^4\Big] \qquad (5.1.1.21)
\end{aligned}
$$

In this step the computations has been done for (5.1.1.17) with $\rho^{(-1)},\ \rho^{(1)},\ \rho^{(2)},\ \rho^{(3)}$. These conserved densities are respectively of order 5, 6, 7 and level 0, 2, 3, and 4. The orders of $\rho_t^{(-1)},\ \rho_t^{(1)},\ \rho_t^{(2)},\ \rho^{(3)}$ are respectively 12, 13, 14 and 14 and their levels are 7, 9, 10, and 11. The integrability conditions came up from the coefficients of the non-integrable terms: $u_8^2u_6$, $u_7^3u_6$ in $\rho^{(-1)}$, $u_9^2u_6$, u_8^3 in $\rho^{(1)}$, $u_9^2u_7$ in $\rho^{(2)}$ and $u_{10}^2u_6$, $u_9^2u_8$, $u_9^2u_7u_6$, $u_9^2u_6^3$ in $\rho^{(3)}$.

$$
\begin{aligned}
u_8^2u_6 \ &: \ \frac{1}{2a^3}\Big(-7a_{555}a^8 + 63a_{55}a_5a^7 + 4a_{55}aB \\
&- \ 84a_5^3a^6 - 8a_5^2B\Big) \qquad (5.1.1.22) \\
u_9^2u_6 \ &: \ \frac{1}{98a^{13}}\Big(343a_{555}a^{20} - 343a_{55}a_5a^{19} - 490a_{55}a^{13}B \\
&- \ 4116a_5^3a^{18} + 3136a_5^2a^{12}B + 245a_5B_5a^{13} \\
&- \ 1344a_5a^6B^2 - 49B_{55}a^{14} + 196B_5a^7B - 48B^3\Big) \qquad (5.1.1.23) \\
u_8^3 \ &: \ \frac{1}{98a^{13}}\Big(-343a_{555}a^{20} + 588a_{55}a^{13}B + 3430a_5^3a^{18} \\
&- \ 2646a_5^2a^{12}B - 196a_5B_5a^{13} + 1176a_5a^6B^2 \\
&+ \ 49B_{55}a^{14} - 210B_5a^7B + 72B^3\Big) \qquad (5.1.1.24) \\
u_9^2u_7 \ &: \ \frac{3}{7a^{12}}\Big(343a_{55}a_5a^{19} - 98a_{55}a^{13}B - 98a_5^2a^{12}B
\end{aligned}
$$

$$
\begin{aligned}
& - \quad 49a_5B_5a^{13} + 84a_5a^6B^2 + 14B_5a^7B - 16B^3\Big) && (5.1.1.25)\\
u_{10}^2u_6 \quad & : \quad \frac{1}{98a^{11}}\Big(686a_{55}a_5a^{19} - 245a_{55}a^{13}B - 2058a_5^3a^{18}\\
& + \quad 2744a_5^2a^{12}B - 245a_5B_5a^{13} - 616a_5a^6B^2\\
& + \quad 84B_5a^7B - 24B^3\Big) && (5.1.1.26)\\
u_9^2u_8 \quad & : \quad \frac{1}{98a^{11}}\Big(-2744a_{55}a_5a^{19} + 980a_{55}a^{13}B + 6517a_5^3a^{18}\\
& - \quad 8965a_5^2a^{12}B + 980a_5B_5a^{13} + 1694a_5a^6B^2\\
& - \quad 336B_5a^7B + 192B^3\Big) && (5.1.1.27)
\end{aligned}
$$

We solve (5.1.1.22) for a_{555} and replace in (5.1.1.23) - (5.1.1.24). Then we get:

$$
\begin{aligned}
u_9^2u_6 \quad & : \quad \frac{1}{98a^{13}}\Big(2744a_{55}a_5a^{19} - 294a_{55}a^{13}B - 8232a_5^3a^{18}\\
& + \quad 2744a_5^2a^{12}B + 245a_5B_5a^{13} - 1344a_5a^6B^2\\
& - \quad 49B_{55}a^{14} + 196B_5a^7B - 48B^3\Big) && (5.1.1.28)\\
u_8^3 \quad & : \quad \frac{1}{98a^{13}}\Big(-3087a_{55}a_5a^{19} + 392a_{55}a^{13}B + 7546a_5^3a^{18}\\
& - \quad 2254a_5^2a^{12}B - 196a_5B_5a^{13} + 1176a_5a^6B^2\\
& + \quad 49B_{55}a^{14} - 210B_5a^7B + 72B^3\Big) && (5.1.1.29)
\end{aligned}
$$

We solve (5.1.1.25) for a_{55} and replace in (5.1.1.22)- (5.1.1.26) - (5.1.1.27) - (5.1.1.28) - (5.1.1.29) and we get:

$$
\begin{aligned}
u_8^2u_6 \quad & : \quad \frac{1}{98a^{15}}\Big(-4116a_5^3a^{18} + 1176a_5^2a^{12}B + 637a_5B_5a^{13}\\
& - \quad 1204a_5a^6B^2 - 49B_{55}a^{14} + 126B_5a37B - 16B^3\Big) && (5.1.1.30)\\
u_9^2u_6 \quad & : \quad \frac{1}{14a^7}\Big(-1176a_5^3a^{12} + 504a_5^2a^6B + 91a_5B_5a^7\\
& - \quad 268a_5B^2 - 7B_{55}a^8 + 22B_5aB\Big) && (5.1.1.31)\\
u_8^3 \quad & : \quad \frac{1}{98a^{13}}\Big(7546a_5^3a^{18} - 3136a_5^2a^{12}B - 637a_5B_5a^{13}\\
& + \quad 1792a_5a^6B^2 + 49B_{55}a^{14} - 154B_5a^7B + 8B^3\Big) && (5.1.1.32)\\
u_{10}^2u_6 \quad & : \quad \frac{1}{98a^{11}}\Big(-2058a_5^3a^{18} + 2940a_5^2a^{12}B - 147a_5B_5a^{13}\\
& - \quad 798a_5a^6B^2 + 49B_5a^7B + 16B^3\Big) && (5.1.1.33)\\
u_9^2u_8 \quad & : \quad \frac{1}{98a^{11}}\Big(-2744a_{55}a_5a^{19} + 980a_{55}a^{13}B + 6517a_5^3a^{18}\\
& - \quad 8965a_5^2a^{12}B + 980a_5B_5a^{13} + 1694a_5a^6B^2\\
& - \quad 336B_5a^7B + 192B^3\Big) && (5.1.1.34)
\end{aligned}
$$

We solve (5.1.1.31) for B_{55} and replace in (5.1.1.30) - (5.1.1.32) and we get:

$$
\begin{aligned}
u_8^2u_6 \quad & : \quad \frac{2}{49a^{15}}\Big(1029a_5^3a^{18} - 588a_5^2a^{12}B + 168a_5a^6B^2\\
& - \quad 7B_5a^7B - 4B^3\Big) && (5.1.1.35)
\end{aligned}
$$

$$u_8^3 \quad : \quad \frac{1}{49a^{13}}\Big(-343a_5^3a^{18} + 196a_5^2a^{12}B$$
$$- \quad 42a_5a^6B^2 + 4B^3\Big) \tag{5.1.1.36}$$

We solve (5.1.1.35) for B_5, we replace in (5.1.1.31) - (5.1.1.33)−(5.1.1.34) and we get:

$$u_9^2u_6 \quad : \quad \frac{3}{14a^{13}B^3}\Big(50421a_5^6a^{36} - 50421a_5^5a^{30}B$$
$$+ \quad 21266a_5^4a^{24}B^2 - 5684a_5^3a^{18}B^3 + 1148a_5^2a^{12}B^4$$
$$- \quad 148a_5a^6B^5 + 8B^6\Big) \tag{5.1.1.37}$$
$$u_{10}^2u_6 \quad : \quad \frac{3}{98a^{11}B}\Big(-7203a_5^4a^{24} + 5831a_5^3a^{18}B$$
$$- \quad 1568a_5^2a^{12}B^2 + 154a_5a^6B^3 - 4B^4\Big) \tag{5.1.1.38}$$
$$u_9^2u_8 \quad : \quad \frac{1}{98a^{11}B}\Big(86436a_5^4a^{24} - 71687a_5^3a^{18}B$$
$$+ \quad 20825a_5^2a^{12}B^2 - 2618a_5a^6B^3 + 144B^4\Big) \tag{5.1.1.39}$$

Computations between (5.1.1.36), (5.1.1.38) and (5.1.1.39) end up with

$$\frac{1}{49a^{11}}B\Big(1029a_5^2a^{12} - 560a_5a^6B^+76B^2\Big) = 0. \tag{5.1.1.40}$$

We solve (5.1.1.40) and find three roots:

$$B = 0,\ B = \frac{147a_5a^6}{38},\ B = \frac{7a_5a^6}{2} \tag{5.1.1.41}$$

1)For $B = 0$ and $B = \frac{147a_5a^6}{38}$ the coefficients of $u_{10}^2u_6$ and $u_9^2u_8$ are respectively

$$u_{10}^2u_6 \quad : \quad 7a_5a^7\Big(a_{55}a - 3a_5^2\Big) \tag{5.1.1.42}$$
$$u_9^2u_8 \quad : \quad \frac{7a_5a^7}{2}\Big(-8a_{55}a + 19a_5^2\Big) \tag{5.1.1.43}$$

Computations between (5.1.1.42) and (5.1.1.43) end up with the condition

$$a_5 = 0 \tag{5.1.1.44}$$

2)For $B = \frac{7a_5a^6}{2}$, The coefficients of $u_8^2u_6$, $u_7^3u_6$, $u_9^2u_7u_6$, and $u_9^2u_6^3$ are as follows:

$$u_8^2u_6 \quad : \quad \frac{7a^3}{2}\Big(-a_{555}a^2 + 11a_{55}a_5a - 16a_5^3\Big) \tag{5.1.1.45}$$
$$u_7^3u_6 \quad : \quad \frac{a^2}{2}\Big(18a_{5555}a^3 + 178a_{555}a_5a^2 + 123a_{55}^2a^2$$
$$+ \quad 5688a_{55}a_5^2a + 4776a_5^4\Big) \tag{5.1.1.46}$$
$$u_9^2u_7u_6 \quad : \quad \frac{7a^6}{2}\Big(-2a_{555}a_5a^2 + 7a_{55}^2a^2 - 29a_{55}a_5^2a + 60a_5^4\Big) \tag{5.1.1.47}$$
$$u_9^2u_6^3 \quad : \quad \frac{7a^5}{4}\Big(-2a_{5555}a_5a^3 + 12a_{555}a_{55}a^3 - 17a_{555}a_5^2a^2$$
$$- \quad 100a_{55}^2a_5a^2 + 443a_{55}a_5^3a - 480a_5^5\Big) \tag{5.1.1.48}$$

We solve (5.1.1.45) for a_{555}, replace in (5.1.1.46), (5.1.1.47), (5.1.1.48) and we get:

$$u_7^3 u_6 \quad : \quad \frac{a^2}{2}\left(321a_{55}^2 a^2 + 8762 a_{55} a_5^2 a - 664 a_5^4\right) \tag{5.1.1.49}$$

$$u_9^2 u_7 u_6 \quad : \quad \frac{7a^6}{2}\left(7a_{55}^2 a^2 - 51 a_{55} a_5^2 a + 92 a_5^4\right) \tag{5.1.1.50}$$

$$u_9^2 u_6^3 \quad : \quad \frac{35 a_5 a^5}{2}\left(a_{55}^2 a^2 - 6 a_{55} a_5^2 a + 8 a_5^4\right) \tag{5.1.1.51}$$

After computations between (5.1.1.49), (5.1.1.50) and (5.1.1.49), (5.1.1.51) we get

$$\frac{713583}{5190694} a_5^5 = 0 \quad \Rightarrow a_5 = 0 \tag{5.1.1.52}$$

Thus for all values of B obtained in (5.1.1.41) we get $a_5 = 0$. Then we conclude that our third and fourth conditions are: $\mathbf{a_5 = 0}$ **and** $\mathbf{B = 0}$

Table 5.1.3: Structure of non-integrable terms in Step 3. for order $m = 7$.

	$\rho^{(-1)}$	$\rho^{(1)}$	$\rho^{(2)}$	$\rho^{(3)}$
Top terms in $\rho^{(i)}$	u_5	u_6^2	u_6^3	u_7^2
Order of $\rho^{(i)}$	5	6	6	7
Level of $\rho^{(i)}$	0	2	3	4
Top term in $\int \rho_t^{(i)}$	u_{12}	u_{14}	$u_{14}u_6$	$u_{15}u_6$
Level of top term in $\int \rho_t^{(i)}$	7	9	10	11
Non-integrable terms in $\int \rho_t^{(i)}$	$\mathbf{u_8^2 u_6}$, $\mathbf{u_7^3 u_6}$, $u_7^2 u_6^3$, u_6^7	$\mathbf{u_9^2 u_6}$, $\mathbf{u_8^3}$, $u_8^2 u_7 u_6$, $u_8^2 u_6^3$, $u_7^4 u_6$, $u_7^3 u_6^3$, $u_7^2 u_6^5$, u_6^9	u_{10}^2, $\mathbf{u_9^2 u_7}$, $u_9^2 u_6^2$, $u_8^3 u_6$, $u_8^2 u_7^2$, $u_8^2 u_7 u_6^2$, $u_8^2 u_6^4$, u_7^5, $u_7^4 u_6^2$, $u_7^3 u_6^4$, $u_7^2 u_6^6$, u_6^{10}	$\mathbf{u_{10}^2 u_6}$, $\mathbf{u_9^2 u_8}$, $\mathbf{u_9^2 u_7 u_6}$, $\mathbf{u_9^2 u_6^3}$, $u_8^3 u_7$, $u_8^3 u_6^2$, $u_8^2 u_7^2 u_6$, $u_8^2 u_7 u_6^3$, $u_8^2 u_6^5$, $u_7^5 u_6$, $u_7^4 u_6^3$, $u_7^3 u_6^5$, $u_7^2 u_6^7$, u_6^{11}

Step 4. The evolution equation is:

$$u_t \sim A u_7 + C u_6 + D \tag{5.1.1.53}$$

where $A = A(u_4),\ C = C(u_4, u_5),\ D = D(u_4, u_5)$.
In this step the basic term with level zero is u_5 and the conserved densities are as follows:

$$\rho^{(-1)} = a^{-1} = A^{-1/7} \tag{5.1.1.54}$$

$$\rho^{(1)} = -\frac{1}{14} u_6 C_5 a^{-6} + a^{-13}\left[a_4^2 a^{12} u_5^2 - \frac{3}{14} a_4 a^6 C u_5 - \frac{1}{14} D_5 a^7 + \frac{3}{98} C^2\right] \tag{5.1.1.55}$$

$$\begin{aligned}\rho^{(2)} &= u_6\left[\frac{7}{5}a_{44}au_5+\frac{7}{5}a_4^2u_5-\frac{1}{5}C_4a^{-5}+\frac{1}{7}C_5a^{-12}C\right]\\&+ a^{-19}\Big[a_4C_4a^{13}u_5^2-7a_4^2a^{12}Cu_5^2-a_4D_5a^{13}u_5\\&+ \frac{3}{7}a_4a^6C^2u_5-\frac{1}{5}D_4a^{14}+\frac{1}{7}D_5a^7C-\frac{2}{49}C^3\Big]\end{aligned}\tag{5.1.1.56}$$

$$\begin{aligned}\rho^{(3)} &= u_6^2a^{-11}\left[\frac{5}{2}a_4^2a^{12}-\frac{11}{28}a_4C_5a^6+\frac{3}{196}C_5^2\right]\\&+ u_6\Big[5a_{44}a_4au_5^2-\frac{1}{4}a_{44}C_5a^{-5}u_5^2-\frac{1}{7}a_{44}a^{-5}Cu_5\\&+ \frac{3}{2}a_4^3u_5^2-\frac{1}{7}a_4^2C_5a^{-6}u_5^2+\frac{41}{28}a_4^2a^{-6}Cu_5\\&+ \frac{1}{7}a_4D_{55}a^{-5}u_5-\frac{13}{28}a_4C_4a^{-5}u_5-\frac{11}{98}a_4C_5a^{-12}Cu_5\\&+ \frac{1}{98}D_5C_5a^{-11}+\frac{1}{49}C_4C_5a^{-11}u_5+\frac{1}{49}C_4a^{-11}C\\&- \frac{11}{686}C_5C^2a^{-18}\Big]+a^{-25}\Big[a_{44}^2a^{26}u_5^4+\frac{3}{4}a_{44}a_4a^{19}Cu_5^3\\&- \frac{3}{28}a_{44}C_4a^{20}u_5^3+\frac{1}{4}a_4^4a^{24}u_5^4\\&- \frac{9}{4}a_4^3a^{18}Cu_5^3-\frac{33}{28}a_4^2D_5a^{19}u_5^2+\frac{9}{28}a_4^2C_4a^{19}u_5^3\\&+ \frac{165}{98}a_4^2a^{12}C^2u_5^2+\frac{1}{7}a_4D_{45}a^{20}u_5^2-\frac{3}{14}a_4D_4a^{20}u_5\\&+ \frac{11}{49}a_4D_5a^{13}Cu_5-\frac{33}{98}a_4C_4a^{13}Cu_5^2-\frac{33}{686}a_4a^6C^3u_5\\&+ \frac{1}{49}D_4a^{14}C+\frac{1}{98}D_5^2a^{14}-\frac{1}{98}D_5C_4a^{14}u_5\\&- \frac{11}{686}D_5a^7C^2+\frac{3}{196}C_4^2a^{14}u_5^2+\frac{33}{9604}C^4\Big]\end{aligned}\tag{5.1.1.57}$$

Computations has been done for (5.1.1.53 – 5.1.1.57). We noticed that the conserved densities $\rho^{(-1)}$, $\rho^{(1)}$, $\rho^{(2)}$ and their total derivatives with respect to t $\rho_t^{(-1)}$, $\rho_t^{(1)}$, $\rho_t^{(2)}$ don't agree with our basic term and level admission; except the conserved density $\rho^{(3)}$ which is of order 7 and level 2 and its total derivative with respect to t ($\rho_t^{(3)}$) which is of order 14 and level 9. The integrability condition came from the non-integrable term $u_9^2u_6$ of $\rho_t^{(3)}$ which is:

$$C_{55}=0 \;\Rightarrow\; C=Su_5+T \tag{5.1.1.58}$$

Step 5. The evolution equation is:

$$u_t \sim Au_7+Su_5u_6+Tu_6+D \tag{5.1.1.59}$$

where $A=A(u_4),\ S=S(u_4),\ T=T(u_4),\ D=D(u_4,u_5)$. In this step the basic term with level zero is u_4 and the conserved densities are as follows:

$$\rho^{(-1)}=a^{-1}=A^{-1/7} \tag{5.1.1.60}$$

$$\begin{aligned}\rho^{(1)} &= -\frac{1}{14}u_6a^{-6}S+a^{-13}\Big[a_4^2a^{12}u_5^2-\frac{3}{14}a_4a^6Su_5^2\\&- \frac{3}{14}a_4a^6Tu_5-\frac{1}{14}D_5a^7+\frac{3}{98}S^2u_5^2\\&+ \frac{3}{49}STu_5+\frac{3}{98}T^2\Big]\end{aligned}\tag{5.1.1.61}$$

$$\begin{aligned}
\rho^{(2)} &= u_6\left[\frac{7}{5}a_{44}au_5+\frac{7}{5}a_4^2u_5-\frac{1}{5}S_4a^{-5}u_5-\frac{1}{5}T_4a^{-5}\right. \\
&+ \left.\frac{1}{7}a^{-12}S^2u_5+\frac{1}{7}a^{-12}ST\right]+a^{-19}\left[-7a_4^2a^{12}Su_5^3\right. \\
&- 7a_4^2a^{12}Tu_5^2-a_4D_5a^{13}u_5+a_4S_4a^{13}u_5^3 \\
&+ a_4T^4a^{13}u_5^2+\frac{3}{7}a_4a^6S^2u_5^3+\frac{6}{7}a_4a^6STu_5^2 \\
&+ \frac{3}{7}a_4a^6T^2u_5-\frac{1}{5}D_4a^{14}+\frac{1}{7}D_5a^7Su_5+\frac{1}{7}D_5a^7T \\
&- \left.\frac{2}{49}S^3u_5^3-\frac{6}{49}S^2Tu_5^2-\frac{6}{49}ST^2u_5-\frac{2}{49}T^3\right] \qquad (5.1.1.62)
\end{aligned}$$

$$\begin{aligned}
\rho^{(3)} &= u_6^2a^{-11}\left[\frac{5}{2}a_4^2a^{12}-\frac{11}{28}a_4a^6S+\frac{3}{196}S^2\right]+u_6\left[5a_{44}a_4au_5^2\right. \\
&- \frac{11}{28}a_{44}a^{-5}Su_5^2-\frac{1}{7}a_{44}a^{-5}Tu_5+\frac{3}{2}a_4^3u_5^2+\frac{37}{28}a_4^2a^{-6}Su_5^2 \\
&+ \frac{41}{28}a_4^2a^{-6}Tu_5+\frac{1}{7}a_4D_{55}a^{-5}u_5-\frac{13}{28}a_4S_4a^{-5}u_5^2 \\
&- \frac{13}{28}a_4T_4a^{-5}u_5-\frac{11}{98}a_4a^{-12}S^2u_5^2-\frac{11}{98}a_4a^{-12}STu_5 \\
&+ \frac{1}{98}D_5a^{-11}S+\frac{2}{49}S_4a^{-11}Su_5^2+\frac{1}{49}S_4a^{-11}Tu_5 \\
&+ \frac{2}{49}T_4a^{-11}Su_5+\frac{1}{49}T_4a^{-11}T-\frac{11}{686}a^{-18}S^3u_5^2 \\
&- \left.\frac{11}{343}a^{-18}S^2Tu_5-\frac{11}{686}a^{-18}ST^2\right]+a^{-25}\left[a_{44}^2a^{26}u_5^4\right. \\
&+ \frac{3}{4}a_{44}a_4a^{19}Su_5^4+\frac{3}{4}a_{44}a_4a^{19}Tu_5^3-\frac{3}{28}a_{44}S_4a^{20}u_5^4 \\
&- \frac{3}{28}a_{44}T_4a^{20}u_5^3+\frac{1}{4}a_4^4a^{24}u_5^4-\frac{9}{4}a_4^3a^{18}Su_5^4 \\
&- \frac{9}{4}a_4^3a^{18}Tu_5^3-\frac{33}{28}a_4^2D_5a^{19}u_5^2+\frac{9}{28}a_4^2S_4a^{19}u_5^4 \\
&+ \frac{9}{28}a_4^2T_4a^{19}u_5^3+\frac{165}{98}a_4^2a^{12}S^2u_5^4+\frac{165}{49}a_4^2a^{12}STu_5^3 \\
&+ \frac{165}{98}a_4^2a^{12}T^2u_5^2+\frac{1}{7}a_4D_{45}a^{20}u_5^2-\frac{3}{14}a_4D_4a^{20}u_5 \\
&+ \frac{11}{49}a_4D_5a^{13}Su_5^2+\frac{11}{49}a_4D_5a^{13}Tu_5-\frac{33}{98}a_4T_4a^{13}Su_5^3 \\
&- \frac{33}{98}a_4T_4a^{13}Tu_5^2-\frac{33}{686}a_4a^6S^3u_5^4-\frac{99}{686}a_4a^6S^2Tu_5^3 \\
&- \frac{99}{686}a_4a^6ST^2u_5^2-\frac{33}{686}a_4a^6T^3u_5+\frac{1}{49}D_4a^{14}Su_5 \\
&+ \frac{1}{49}D_4a^{14}T+\frac{1}{98}D_5^2a^{14}-\frac{1}{98}D_5S_4a^{14}u_5^2 \\
&- \frac{1}{98}D_5T_4a^{14}u_5-\frac{11}{686}D_5a^7T^2+\frac{3}{196}S_4^2a^{14}u_5^4 \\
&+ \frac{3}{98}S_4T_4a^{14}u_5^3+\frac{3}{196}T_4^2a^{14}u_5^2+\frac{33}{9604}S^4u_5^4 \\
&+ \frac{33}{2401}S^3Tu_5^3+\frac{99}{4802}S^2C^2u_5^2+\frac{33}{2401}ST^3u_5 \\
&+ \left.\frac{33}{9604}T^4\right] \qquad (5.1.1.63)
\end{aligned}$$

Computations showed that the integrability condition came from the coefficient of $u_8^2u_6$ in $\rho_t^{(1)}$. Here we noticed that the level of $u_8^2u_6$ is 10 while $\rho_t^{(1)}$ is of order 13 and level 9. The condition is:

$$D_{5555} = 0 \;\Rightarrow\; D = Eu_5^3 + Fu_5^2 + Gu_5 + H \tag{5.1.1.64}$$

where $E = E(u_4),\; F = F(u_4),\; G = G(u_4),\; H = H(u_4)$
Step 6. The evolution equation is:

$$u_t \sim Au_7 + Su_5u_6 + Tu_6 + Eu_5^3 + Fu_5^2 + Gu_5 + H \tag{5.1.1.65}$$

where all the coefficients depend on u_4. The basic term with level zero is still u_4 and the conserved densities are as follows:

$$\rho^{(-1)} = a^{-1} = A^{-1/7} \tag{5.1.1.66}$$

$$\begin{aligned}\rho^{(1)} &= a^{-13}\Big[a_4^2a^{12}u_5^2 - \frac{9}{14}a_4a^6Su_5^2 - \frac{3}{14}a_4a^6Tu_5\\ &+ \frac{1}{14}S_4a^7u_5^2 - \frac{3}{14}a^7Eu_5^2 - \frac{1}{7}a^7Fu_5 - \frac{1}{14}a^7G\\ &+ \frac{3}{98}S^2u_5^2 + \frac{3}{49}STu_5 + \frac{3}{98}T^2\Big]\end{aligned} \tag{5.1.1.67}$$

$$\begin{aligned}\rho^{(2)} &= u_5^3\Big[-\frac{7}{10}a_{444}a - \frac{21}{10}a_{44}a_4 - 7a_4^2a^{-7}S\\ &+ \frac{1}{2}a_4S_4a^{-6} - 3a_4a^{-6}E + \frac{9}{7}a_4a^{-13}S^2 + \frac{1}{10}S_{44}a^{-5}\\ &- \frac{1}{7}S_4a^{-12}S - \frac{1}{5}E_4a^{-5} + \frac{3}{7}a^{-12}SE - \frac{2}{49}a^{-19}S^3\Big]\\ &+ u_5^2a^{-19}\Big[-7a_4^2a^{12}T - 2a_4a^{13}F + \frac{18}{7}a_4a^6ST\\ &- \frac{1}{7}S_4a^7T + \frac{1}{5}T_{44}a^{14} - \frac{1}{7}T_4a^7S\\ &- \frac{1}{5}F_4a^{14} + \frac{2}{7}a^7SF + \frac{3}{7}a^7TE - \frac{6}{49}S^2T\Big]\\ &+ u_5a^{-19}\Big[-a_4a^{13}G + \frac{3}{7}a_4a^6T^2 - \frac{1}{5}G_4a^{14} + \frac{1}{7}a^7SG\\ &+ \frac{2}{7}a^7TF - \frac{6}{49}ST^2\Big] + a^{-19}\Big[-\frac{1}{5}H_4a^{14}\\ &+ \frac{1}{7}a^7TG - \frac{2}{49}T^3\Big]\end{aligned} \tag{5.1.1.68}$$

$$\begin{aligned}\rho^{(3)} &= u_6^2a^{-11}\Big[\frac{5}{2}a_4^2a^{12} - \frac{11}{28}a_4a^6S + \frac{3}{196}S^2\Big]\\ &+ u_5^4\Big[-\frac{5}{3}a_{444}a_4a + \frac{11}{84}a_{444}a^{-5}S - \frac{2}{3}a_{44}^2a\\ &- \frac{19}{6}a_{44}a_4^2 - \frac{11}{14}a_{44}a_4a^{-6}S + \frac{5}{28}a_{44}S_4a^{-5}\\ &- \frac{2}{7}a_{44}a^{-5}E + \frac{11}{294}a_{44}a^{-12}S^2 + \frac{1}{4}a_4^4a^{-1}\\ &+ \frac{11}{28}a_4^3a^{-7}S - \frac{25}{28}a_4^2S_4a^{-6} - \frac{59}{28}a_4^2a^{-6}E\end{aligned}$$

$$
\begin{aligned}
&+ \frac{121}{98}a_4^2a^{-13}S^2 + \frac{13}{84}a_4S_{44}a^{-5} - \frac{11}{98}a_4S_4a^{-12}S \\
&- \frac{1}{14}a_4E_4a^{-5} + \frac{11}{14}a_4a^{-12}SE - \frac{99}{686}a_4a^{-19}S^3 \\
&- \frac{2}{147}S_{44}a^{-11}S + \frac{1}{588}S_4^2a^{-11} - \frac{2}{49}S_4a^{-11}E \\
&+ \frac{11}{686}S_4a^{-18}S^2 + \frac{1}{98}E_4a^{-11}S + \frac{9}{98}a^{-11}E^2 \\
&- \left.\frac{33}{686}a^{-18}S^2E + \frac{33}{9604}a^{-25}S^4\right] \\
&+ u_5^3a^{-25}\left[\frac{1}{4}a_{444}a^{20}T - \frac{15}{14}a_{44}a_4a^{19}T\right. \\
&+ \frac{11}{56}a_{44}T_4a^{20} - \frac{1}{7}a_{44}a^{20}F + \frac{11}{196}a_{44}a^{13}ST \\
&+ \frac{15}{7}a_4^3a^{18}T - \frac{11}{7}a_4^2T_4a^{19} - \frac{23}{14}a_4^2a^{19}F \\
&+ \frac{132}{49}a_4^2a^{12}ST - \frac{33}{196}a_4S_4a^{13}T + \frac{13}{56}a_4T_{44}a^{20} \\
&- \frac{11}{196}a_4T_4a^{13}S - \frac{1}{14}a_4F_4a^{20} + \frac{55}{98}a_4a^{13}SF \\
&+ \frac{33}{49}a_4a^{13}TE - \frac{297}{686}a_4a^6S^2T - \frac{1}{98}S_{44}a^{14}T \\
&- \frac{3}{98}S_4a^{14}F + \frac{11}{343}S_4a^7ST - \frac{1}{49}T_{44}a^{14}S \\
&- \frac{3}{98}T_4a^{14}E + \frac{11}{686}T_4a^7S^2 + \frac{1}{49}E_4a^{14}T \\
&+ \frac{1}{98}F_4a^{14}S + \frac{6}{49}a^{14}EF - \frac{11}{343}a^7S^2F \\
&- \left.\frac{33}{343}a^7STE + \frac{33}{2401}S^3T\right] \\
&+ u_5^2a^{-25}\left[-\frac{33}{28}a_4^2a^{19}G + \frac{165}{98}a_4^2a^{12}T^2\right. \\
&- \frac{11}{98}a_4T_4a^{13}T - \frac{1}{14}a_4G_4a^{20} + \frac{33}{98}a_4a^{13}SG \\
&+ \frac{22}{49}a_4a^{13}TF - \frac{297}{686}a_4a^6ST^2 - \frac{1}{49}S_4a^{14}G \\
&+ \frac{11}{686}S_4a^7T^2 - \frac{1}{49}T_{44}a^{14}T - \frac{1}{196}T_4^2a^{14} \\
&- \frac{1}{49}T_4a^{14}F + \frac{11}{343}T_4a^7ST + \frac{1}{49}F_4a^{14}T \\
&+ \frac{1}{98}G_4a^{14}S + \frac{3}{49}a^{14}EG + \frac{2}{49}a^{14}F^2 \\
&- \frac{11}{686}a^7S^2G - \frac{22}{343}a^7STF - \frac{33}{686}a^7T^2E \\
&+ \left.\frac{99}{4802}S^2T^2\right] + u_5a^{-25}\left[-\frac{3}{14}a_4H_4a^{20}\right. \\
&+ \frac{11}{49}a_4a^{13}TG - \frac{33}{686}a_4a^6T^3 - \frac{1}{98}T_4a^{14}G \\
&+ \frac{1}{49}G_4a^{14}T + \frac{1}{49}H_4a^{14}S + \frac{2}{49}a^{14}FG \\
&- \left.\frac{11}{343}a^7STG - \frac{11}{343}a^7T^2F + \frac{33}{2401}ST^3\right]
\end{aligned}
$$

$$
\begin{aligned}
&+ \quad a^{-25}\left[\frac{1}{49}H_4a^{14}T+\frac{1}{98}a^{14}G^2-\frac{11}{686}a^7T^2G\right.\\
&+ \quad \left.\frac{33}{9604}T^4\right]
\end{aligned}
\tag{5.1.1.69}
$$

The integrability conditions came from the coefficients of $u_7^2u_5$ in $\rho_t^{(-1)}$ with order 11 and level 7, the coefficients of u_7^3 in $\rho_t^{(1)}$ with order 12 and level 9, the coefficient of $u_8^2u_6$ in $\rho_t^{(2)}$ with order 12 and level 10 and the coefficients of $u_9^2u_5$, $u_8^2u_7$, $u_8^2u_6u_5$, $u_8^2u_5^3$ in $\rho_t^{(3)}$ with order 12 and level 11.
First we solve a_{444} from the coefficient of $u_7^2u_5$

$$
\begin{aligned}
u_7^2u_5 \quad &: \quad a^{-3}\left(-\frac{7}{2}a_{444}a^8+\frac{49}{2}a_{44}a_4a^7+a_{44}aS\right.\\
&- \quad \left.28a_4^3a^6-2a_4^2S\right)
\end{aligned}
\tag{5.1.1.70}
$$

$$
a_{444}=a^{-8}\left(7a_{44}a_4a^7+\frac{2}{7}a_{44}aS-8a_4^3a^6-\frac{4}{7}a_4^2S\right)
\tag{5.1.1.71}
$$

Then we get the coefficients of $u_9^2u_5$ and $u_8^2u_7$.

$$
\begin{aligned}
u_9^2u_5 \quad &: \quad a^{-11}\left(35a_{44}a_4a^{19}-\frac{11}{4}a_{44}a^{13}S-70a_4^3a^{18}\right.\\
&+ \quad \frac{45}{2}a_4^2a^{12}S-\frac{11}{4}a_4S_4a^{13}-\frac{13}{14}a_4a^6S^2\\
&+ \quad \left.\frac{3}{14}S_4a^7S-\frac{3}{98}S^3\right)
\end{aligned}
\tag{5.1.1.72}
$$

$$
\begin{aligned}
u_8^2u_7 \quad &: \quad a^{-11}\left(-140a_{44}a_4a^{19}+11a_{44}a^{13}S+\frac{105}{2}a_4^3a^{18}\right.\\
&- \quad \frac{137}{4}a_4^2a^{12}S+11a_4S_4a^{13}-\frac{23}{28}a_4a^6S^2\\
&- \quad \left.\frac{6}{7}S_4a^7S+\frac{12}{49}S^3\right)
\end{aligned}
\tag{5.1.1.73}
$$

We compute (5.1.1.72) and (5.1.1.73) and end up with:

$$
a^{-11}\left(-\frac{455}{2}a_4^3a^{18}+\frac{223}{4}a_4^2a^{12}S-\frac{127}{28}a_4a^6S^2+\frac{6}{49}S^3\right)
\tag{5.1.1.74}
$$

We solve S from (5.1.1.74) and we get three roots:

$$
S = \frac{91}{8}a_4a^6 \tag{5.1.1.75}
$$

$$
S = \frac{35}{3}a_4a^6 \tag{5.1.1.76}
$$

$$
S = 14a_4a^6 \tag{5.1.1.77}
$$

We choose $S=14a_4a^6$ and we solve E from the coefficient of $u_8^2u_6$.

$$
u_8^2u_6 \quad : \quad a\left(\frac{1029}{5}a_{44}a_4a^6-\frac{5586}{5}a_4^3a^5+63a_4E-\frac{21}{5}E_4a\right)
\tag{5.1.1.78}
$$

$$
E_4=a_4a^{-1}\left(49a_{44}a^6-266a_4^2a^5+15E\right)
\tag{5.1.1.79}
$$

Then we solve E from the coefficient of u_7^3.

$$u_7^3 : a_4\left(-\frac{221}{2}a_{44}a^6 - 63a_4^2a^5 + 3E\right) \tag{5.1.1.80}$$

$$E = a^5\left(\frac{7}{2}a_{44}a + 21a_4^2\right) \tag{5.1.1.81}$$

Computations between the coefficients of $u_8^2u_6u_5$ and $u_8^2u_5^3$ end up with:

$$a_4^2a^6\left(-\frac{9}{7}a_{44}a + \frac{36}{7}a_4^2\right) = 0 \tag{5.1.1.82}$$

If $df(a, u_4) = 0$ then $E = 0$.

If $df(a, u_4) \neq 0$ we get the following conditions:

$$a_{44} = \frac{4a_4^2}{a} \tag{5.1.1.83}$$

$$E = 35a_4^2a^5 \tag{5.1.1.84}$$

We did the same computations for $S = \frac{35}{3}a_4a^6$ and $T = \frac{91}{8}a_4a^6$ respectively and we get:

$$a_{44} = \frac{7438a_4^2}{2603a} \tag{5.1.1.85}$$

$$E = \frac{1559299a_4^2a^5}{70281} \tag{5.1.1.86}$$

$$a_{44} = \frac{240953683457a_4^2}{64582469384a} \tag{5.1.1.87}$$

$$E = \frac{829201536681a_4^2a^5}{133332887424} \tag{5.1.1.88}$$

We choose $S = 14a_4a^6$ for computational simplicity. Then the integrable evolution equation of $7th$ order has the following form:

$$u_t = a^7u_7 + 14a_4a^6u_5u_6 + Tu_6 + 35a_4^2a^5u_5^3 + Fu_5^2 + Gu_5 + H \tag{5.1.1.89}$$

where a, T, F, G, H depend on u_4.

Table 5.1.4: Structure of non-integrable terms in Step 6. for order $m = 7$.

	$\rho^{(-1)}$	$\rho^{(1)}$	$\rho^{(2)}$	$\rho^{(3)}$
Top terms in $\rho^{(i)}$	u_4	u_5^2	u_5^3	u_6^2
Order of $\rho^{(i)}$	4	5	5	5
Level of $\rho^{(i)}$	0	2	3	4
Top term in $\int \rho_t^{(i)}$	u_{11}	$u_{12}u_5$	$u_{12}u_5^2$	$u_{13}u_6$
Level of top term in $\int \rho_t^{(i)}$	7	9	10	11
Non-integrable terms in $\int \rho_t^{(i)}$	$\mathbf{u_7^2u_5}$, $u_6^3u_5$, $u_6^2u_5^3$, u_5^7	$u_8^2u_5$, $\mathbf{u_7^3}$ $u_7^2u_6u_5$, $u_7^2u_5^3$, $u_6^4u_5$, $u_6^3u_5^3$, $u_6^2u_5^5$, u_5^9	u_9^2, $\mathbf{u_8^2u_6}$, $u_8^2u_5^2$, $u_7^3u_5$, $u_7^2u_6^2$, $u_7^2u_6u_5^2$, $u_7^2u_5^4$, u_6^5, $u_6^4u_5^2$, $u_6^3u_5^4$, $u_6^2u_5^6$, u_5^{10}	$\mathbf{u_9^2u_5}$, $\mathbf{u_8^2u_7}$, $\mathbf{u_8^2u_6u_5}$, $\mathbf{u_8^2u_5^3}$, $u_7^3u_6$, $u_7^3u_5^2$, $u_7^2u_6^2u_5$, $u_7^2u_6u_5^3$, $u_7^2u_5^5$, $u_6^5u_5$, $u_6^4u_5^3$, $u_6^3u_5^5$, $u_6^2u_5^7$, u_5^{11}

5.1.2 Second Method

We observed that the use of the canonical densities $\rho^{(1)}$ and $\rho^{(3)}$ is sufficient to get the same results. In this section we give the explicit computations for $m = 7$ using only the canonical densities $\rho^{(1)}$ and $\rho^{(3)}$.

Step 1:

$$u_t = Au_7 + B, \quad A = a^7 \tag{5.1.2.1}$$

We compute $\int D_t(\rho^{(1)})$ for (5.1.2.1).
The coefficients of the top two nonlinear terms, $u_{10}^2u_7$ and u_9^3 give respectively:

$$14a_6a^5(a_{66}a - 5a_6^2) = 0 \tag{5.1.2.2}$$

$$7a_6a^5(-2a_{66}a + 11a_6^2) = 0 \tag{5.1.2.3}$$

$$\begin{bmatrix} 1 & -5 \\ -2 & 11 \end{bmatrix} \begin{bmatrix} a_{66}a \\ a_6^2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{5.1.2.4}$$

Since the determinant is different than zero

$$a_6 = 0.$$

Step 2: We compute $\int D_t(\rho^{(3)})$ for (5.1.2.1), and consider $a_6 = 0$. The coefficient of the top nonlinear term, $u_{10}^2u_7$ gives:

$$B_{666}a^{-4}(-\frac{5}{4}a_5a^6 + \frac{3}{14}B_{66}) = 0 \tag{5.1.2.5}$$

Then

$$B_{666} = 0 \tag{5.1.2.6}$$

or

$$\frac{5}{4}a_5 a^6 - \frac{3}{14}B_{66} = 0 \tag{5.1.2.7}$$

Since $a_6 = 0$, the derivative of (5.1.2.7) with respect to u_6 gives:

$$B_{666} = 0 \tag{5.1.2.8}$$

Step 3:

$$u_t = Au_7 + Cu_6^2 + Du_6 + E \tag{5.1.2.9}$$

We compute $\int D_t(\rho^{(1)})$ and $\int D_t(\rho^{(3)})$ for (5.1.2.9). The coefficients of $u_9^2 u_6,\ \ u_8^3$ in $\int D_t(\rho^{(1)})$, and the coefficient of $u_{10}^2 u_6,\ \ u_9^2 u_8$ in $\int D_t(\rho^{(3)})$ are respectively:

$$\begin{aligned}\frac{7}{2}a_{555}a^7 - \frac{7}{2}a_{55}a_5 a^6 - 5a_{55}C - 42a_5^3 a^5 \\ +32a_5^2 a^{-1}C + \frac{5}{2}a_5 C_5 - \frac{96}{7}a_5 a^{-7}C^2 \\ -\frac{1}{2}C_{55}a + 2C_5 a^{-6}C - \frac{24}{49}a^{-13}C^3 = 0\end{aligned} \tag{5.1.2.10}$$

$$\begin{aligned}-\frac{7}{2}a_{555}a^7 + 6a_{55}C + 35a_5^3 a^5 - 27a_5^2 a^{-1}C - 2a_5 C_5 \\ +12a_5 a^{-7}C^2 + \frac{1}{2}C_{55}a - \frac{15}{7}C_5 a^{-6}C + \frac{36}{49}a^{-13}C^3 = 0\end{aligned} \tag{5.1.2.11}$$

$$\begin{aligned}a^{-11}(7a_{55}a_5 a^{19} - \frac{5}{2}a_{55}a^{13}C - 21a_5^3 a^{18} + 28a_5^2 a^{12}C \\ -\frac{5}{2}a_5 C_5 a^{13} - \frac{44}{7}a_5 a^6 C^2 + \frac{6}{7}C_5 a^7 C - \frac{12}{49}C^3) = 0\end{aligned} \tag{5.1.2.12}$$

$$\begin{aligned}a^{-11}(-28a_{55}a_5 a^{19} + 10a_{55}a^{13}C + \frac{133}{2}a_5^3 a^{18} - \frac{183}{2}a_5^2 a^{12}C \\ +10a_5 C_5 a^{13} + \frac{121}{7}a_5 a^6 C^2 - \frac{24}{7}C_5 a^7 C + \frac{96}{49}C^3) = 0\end{aligned} \tag{5.1.2.13}$$

We compute (5.1.2.12) and (5.1.2.13) and we solve C:

$$\begin{aligned}C &= \frac{35}{16}a_5 a^6 \\ C &= \frac{7}{3}a_5 a^6 \\ C &= \frac{7}{2}a_5 A^6\end{aligned} \tag{5.1.2.14}$$

The coefficient of u_7^2 in $\rho^{(3)}$ vanishes for $C = \frac{7}{3}a_5 a^6$ and $\frac{7}{2}a_5 a^6$. This contradicts our assumption thus we use only $C = \frac{35}{16}a_5 a^6$. We substitute $C = \frac{35}{16}a_5 a^6$ in (5.1.2.10), (5.1.2.11) and (5.1.2.13) and we get respectively:

$$a^5(\frac{77}{32}a_{555}a^2 - \frac{2443}{128}a_{55}a_5 a + \frac{7511}{512}a_5^3) = 0 \tag{5.1.2.15}$$

$$a^5(-\frac{77}{32}a_{555}a^2 + \frac{4655}{256}a_{55}a_5 a - \frac{14245}{1024}a_5^3) = 0 \tag{5.1.2.16}$$

$$a_5 a^7(-\frac{21}{32}a_{55}a + \frac{609}{256}a_5^2) = 0 \tag{5.1.2.17}$$

Then (5.1.2.15) and (5.1.2.16) give:

$$a_5 a^5(-\frac{231}{256}a_{55}a + \frac{777}{1024}a_5^2) = 0 \tag{5.1.2.18}$$

From (5.1.2.17) and (5.1.2.18) we get:

$$\begin{bmatrix} -8 & 29 \\ -44 & 37 \end{bmatrix} \begin{bmatrix} a_{55}a \\ a_5^2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{5.1.2.19}$$

Since the determinant is different than zero $a_5 = 0$. Thus $C = 0$.
Step 4:

$$u_t = Au_7 + Du_6 + E, a_5 = 0 \tag{5.1.2.20}$$

We compute $\int D_t(\rho^{(1)})$ for (5.1.2.20).

The coefficient of $u_9^2 u_6$ gives:

$$D_{55} = 0 \tag{5.1.2.21}$$

Step 5:

$$u_t = Au_7 + Fu_6u_5 + Gu_6 + E \tag{5.1.2.22}$$

We compute $\int D_t(\rho^{(3)})$ for (5.1.2.22). The coefficient of $u_8^2 u_6$ gives:

$$-\frac{1}{4}E_{5555}a = 0 \tag{5.1.2.23}$$

Hence the integrable evolution equation of order 7 has to be as follows:

$$u_t = Au_7 + Fu_6u_5 + Gu_6 + Hu_5^3 + Ju_5^2 + Lu_5 + M \tag{5.1.2.24}$$

5.2 Classification of 9th order evolution equation

We aim to obtain a general formula for an evolution equation of order m. Then we did the same computations and the same assumptions for an evolution equation of order 9 using the same conserved densities in (**??**) for $m = 9$. The general form of the quasi-linear evolution equation of order 9 is:

$$u_t = A(x, t, u, ..., u_8)u_9 + B(x, t, u, ..., u_8) \tag{5.2.1}$$

In this section we used "level grading". The corresponding submodules and quotient submodules are given in **Appendix C and Appendix D**.

Step 1. As we used the same grading by levels we observed that we need only the dependency on u_8 which is our basic term with level zero. The evolution equation and the conserved densities used in the first step are:

$$u_t \sim A(u_8)u_9 + B(u_8). \tag{5.2.2}$$

$$\rho^{(-1)} = a^{-1} = A^{-1/7} \tag{5.2.3}$$

$$\rho^{(1)} = u_9^2 a_8^2 a^{-1} + \frac{2}{19} u_9 a_8 B_8 a^{-9} + \frac{2}{135} B_8^2 a^{-17} \tag{5.2.4}$$

$$\rho^{(2)} = u_9^2 a_8 a^{-9} \left[-9 a_8 B_8 + B_{88} a\right] - \frac{1}{3} u_9 a_8 B_8^2 a^{-17} - \frac{2}{81} B_8^3 a^{-25} \tag{5.2.5}$$

$$\begin{aligned} \rho^{(3)} &= u_{10}^2 a_8^2 a + u_9^4 \left[-\frac{2}{3} a_{888} a_8 a + \frac{1}{3} a_{88}^2 a - \frac{2}{3} a_{88} a_8^2 + \frac{1}{4} a_8^4 a^{-1}\right] \\ &+ u_9^3 a^{-8} \left[\frac{1}{36} a_{88} B_{88} a + \frac{2}{9} a_8^2 B_{88} - \frac{1}{36} a_8 B_{888} a\right] \\ &+ u_9^2 a^{-17} \left[\frac{65}{54} a_8^2 B_8^2 - \frac{5}{27} a_8 B_{88} B_8 a + \frac{1}{162} B_{88}^2 a^2\right] \\ &+ \frac{5}{243} u_9 a_8 B_8^3 a^{-25} + \frac{5}{4374} B_8^4 a^{-33} \end{aligned} \tag{5.2.6}$$

The integrability condition came from the coefficients of non-integrable terms, $u_{13}^2 u_9$ and $u_{12}^2 u_{11}$ in $\rho_t^{(1)}$.

$$u_{13}^2 u_9 \;:\; 18 a_8 a^7 \left(-a_{88} a + 6 a_8^2\right) \tag{5.2.7}$$

$$u_{12}^2 u_{11} \;:\; 6 a_8 a^7 \left(13 a_{88} a - 89 a_8^2\right) \tag{5.2.8}$$

The compatibility of (5.2.7), (5.2.8) gives:

$$-\frac{24}{13} a_8^3 a^7 = 0 \tag{5.2.9}$$

Since $a \neq 0$ we get:

$$a_8 = 0 \tag{5.2.10}$$

Table 5.2.1: Structure of non-integrable terms in Step 1. for order $m = 9$.

	$\rho^{(-1)}$	$\rho^{(1)}$	$\rho^{(2)}$	$\rho^{(3)}$
Top terms in $\rho^{(i)}$	u_8	u_9^2	u_9^2	u_{10}^2
Order of $\rho^{(i)}$	8	9	9	10
Level of $\rho^{(i)}$	0	2	2	4
Top term in $\int \rho_t^{(i)}$	u_{17}	$u_{18}u_9$	$u_{18}u_9$	$u_{19}u_{10}$
Level of top term in $\int \rho_t^{(i)}$	9	11	11	13
Non-integrable terms in $\int \rho_t^{(i)}$	$u_{12}^2u_9$, u_{11}^3, $u_{11}^2u_{10}u_9$, $u_{11}^2u_9^3$, $u_{10}^4u_9$, $u_{10}^3u_9^3$, $u_{10}^2u_9^5$, u_9^9	$\mathbf{u_{13}^2u_9}$, $\mathbf{u_{12}^2u_{11}}$, $u_{12}^2u_{10}u_9$, $u_{12}^2u_9^3$, $u_{11}^3u_{10}$, $u_{11}^3u_9^2$, $u_{11}^2u_{10}^2u_9$, $u_{11}^2u_{10}u_9^3$, $u_{11}^2u_9^5$, $u_{10}^5u_9$, $u_{10}^4u_9^3$, $u_{10}^3u_9^5$, $u_{10}^2u_9^7$, u_9^{11}	$u_{13}^2u_9$, $u_{12}^2u_{11}$, $u_{12}^2u_{10}u_9$, $u_{12}^2u_9^3$, $u_{11}^3u_{10}$, $u_{11}^3u_9^2$, $u_{11}^2u_{10}^2u_9$, $u_{11}^2u_{10}u_9^3$, $u_{11}^2u_9^5$, $u_{10}^5u_9$, $u_{10}^4u_9^3$, $u_{10}^3u_9^5$, $u_{10}^2u_9^7$, u_9^{11}	$u_{14}^2u_9$, $u_{13}^2u_{11}$, $u_{13}^2u_{10}u_9$, $u_{13}^2u_9^3$, $u_{12}^3u_9$, $u_{12}^2u_{11}u_{10}$, $u_{12}^2u_{11}u_9^2$, $u_{12}^2u_{10}^2u_9$, $u_{12}^2u_{10}u_9^3$, $u_{12}^2u_9^5$, $u_{11}^4u_9$, $u_{11}^3u_{10}^2$, $u_{11}^3u_9^4$, $u_{11}^3u_9^4$, $u_{11}^2u_{10}^3u_9$, $u_{11}^2u_{10}^2u_9^3$, $u_{11}^2u_{10}u_9^5$, $u_{11}^2u_9^7$, $u_{10}^6u_9$, $u_{10}^5u_9^3$, $u_{10}^4u_9^5$, $u_{10}^3u_9^7$, $u_{10}^2u_9^9$, u_9^{13}

Step 2. In this step the evolution equation is:

$$u_t \sim Au_9 + B \tag{5.2.11}$$

where $A = A(u_7)$ and $B = B(u_7, u_8)$. Here the basic term with level zero is still u_8. The conserved densities under this condition are as follows:

$$\rho^{(-1)} = a^{-1} = A^{-1/7} \tag{5.2.12}$$

$$\begin{aligned} \rho^{(1)} &= u_8^2\left[\frac{3}{10}a_{77} + a_7^2a^{-1}\right] - \frac{2}{15}u_8a_7B_8a^{-9} \\ &+ a^{-17}\left[-\frac{1}{30}B_7a^9 + \frac{2}{135}B_8^2\right] \end{aligned} \tag{5.2.13}$$

$$\begin{aligned} \rho^{(2)} &= u_9u_8a_7\left[-\frac{27}{4}a_7 + B_{88}a^{-8}\right] + \frac{3}{4}u_9a_7B_8a^{-8} \\ &+ u_8^2a_7a^{-9}\left[-9a_7B_8 + B_{78}a\right] + u_8a_7a^{-17}\left[-\frac{3}{4}B_7a^9\right. \\ &+ \left.\frac{1}{3}B_8^2\right] + B_8a^{-25}\left[\frac{1}{12}B_7a^9 - \frac{2}{81}B_8^2\right] \end{aligned} \tag{5.2.14}$$

$$
\begin{aligned}
\rho^{(3)} &= u_9^2 a^{-15}\left[\frac{1}{2}a_7^2 a^{16} - \frac{1}{12}a_7 B_{88} a^8 + \frac{1}{162}B_{88}^2\right] \\
&+ u_9 u_8^2\left[\frac{5}{4}a_{77}a_7 a - \frac{1}{18}a_{77}B_{88}a^{-7} - \frac{9}{4}a_7^3\right. \\
&+ \left.\frac{2}{9}a_7^2 B_{88} a^{-8}\right] + u_9 u_8 a^{-16}\left[\frac{4}{3}a_7^2 B_8 a^8\right. \\
&- \left.\frac{5}{27}a_7 B_{88} B_8 + \frac{1}{81}B_{78}B_{88}a\right] + u_9 a^{-16}\left[\frac{1}{18}a_7 B_7 a^9\right. \\
&- \left.\frac{5}{108}a_7 B_8^2 - \frac{1}{324}B_7 B_{88} a\right] + u_8^4 a^{-1}\left[a_{77}^2 a^2\right. \\
&+ \left.\frac{1}{4}a_7^4\right] + u_8^3 a^{-9}\left[\frac{1}{2}a_{77}a_7 B_8 a - \frac{1}{18}a_{77}B_{78}a^2\right. \\
&- \left.2a_7^3 B_8 + \frac{2}{9}a_7^2 B_{78} a\right] + u_8^2 a^{-17}\left[-\frac{5}{6}a_7^2 B_7 a^9\right. \\
&+ \frac{65}{54}a_7^2 B_8^2 - \frac{5}{27}a_7 B_{78} B_8 a + \frac{1}{2}a_7 B_{77} a^{10} \\
&+ \left.\frac{1}{162}B_{78}^2 a^2\right] + u_8 a^{-25}\left[\frac{5}{54}a_7 B_7 B_8 a^9 - \frac{5}{243}a_7 B_8^3\right. \\
&- \left.\frac{1}{324}B_{78}B_7 a^{10}\right] + a^{-33}\left[\frac{1}{324}B_7^2 a^{18}\right. \\
&- \left.\frac{5}{972}B_7 B_8^2 a^9 + \frac{5}{4374}B_8^4\right] \qquad (5.2.15)
\end{aligned}
$$

In this step the integrability condition came from the coefficient of the non-integrable term $u_{13}^2 u_9$ in $\rho_t^{(3)}$:

$$
u_{13}^2 u_9 : B_{888} a^{-6}\left(\frac{3}{7}a_7 a^8 - \frac{1}{9}B_{88}\right) = 0 \qquad (5.2.16)
$$

Then we get:

$$
B_{888} = 0 \qquad (5.2.17)
$$

Step 3. Here our evolution equation is:

$$
u_t \sim A u_9 + B u_8^2 + C u_8 + D \qquad (5.2.18)
$$

where A,B,C,D depend only on u_7. Since all the coefficients depend only on u_7, the basic term with level zero is u_7. And the conserved densities are as follows:

$$
\rho^{(-1)} = a^{-1} = A^{-1/7} \qquad (5.2.19)
$$

$$
\begin{aligned}
\rho^{(1)} &= u_8^2\left[\frac{3}{10}a_{77} + a_7^2 a^{-1} - \frac{4}{15}a_7 a^{-9} B - \frac{1}{30}B_7 a^{-8}\right. \\
&+ \left.\frac{8}{135}a^{-17}B^2\right] + u_8 a^{-17}\left[-\frac{2}{15}a_7 a^8 C - \frac{1}{30}C_7 a^9\right. \\
&+ \left.\frac{8}{135}BC\right] + a^{-17}\left[-\frac{1}{30}D_7 a^9 + \frac{2}{135}C^2\right] \qquad (5.2.20)
\end{aligned}
$$

$$
\begin{aligned}
\rho^{(2)} &= u_8^3\left[\frac{27}{4}a_{77}a_7 - \frac{7}{4}a_{77}a^{-8}B - 4a_7^2 a^{-9} B\right. \\
&- \frac{1}{2}a_7 B_7 a^{-8} + \frac{4}{3}a_7 a^{-17} B^2 + \frac{1}{6}B_7 a^{-6} B \\
&- \left.\frac{16}{81}a^{-25}B^3\right] + u_8^2 a^{-25}\left[-\frac{3}{4}a_{77}a^{17}C - 3a_7^2 a^{16} C\right.
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{1}{2}a_7C_7a^{17} + \frac{4}{3}a_7a^8BC + \frac{1}{6}C_7a^9B + \frac{1}{12}B_7a^9C \\
&- \frac{8}{27}B^2C\bigg] + u_8a^{-25}\left[-\frac{3}{4}a_7D_7a^{17} + \frac{1}{3}a_7a^8C^2\right. \\
&+ \left.\frac{1}{6}D_7a^9B + \frac{1}{12}C_7a^9C - \frac{4}{27}BC^2\right] \\
&+ a^{-25}C\left[\frac{1}{12}D_7a^9 - \frac{2}{81}C^2\right]
\end{aligned}
\tag{5.2.21}
$$

$$
\begin{aligned}
\rho^{(3)} &= u_9^2a^{-15}\left[\frac{1}{2}a_7^2a^{16} - \frac{1}{6}a_7a^8B + \frac{2}{81}B^2\right] \\
&+ u_8^4\left[-\frac{5}{12}a_{777}a_7a + \frac{1}{27}a_{777}a^{-7}B + \frac{7}{12}a_{77}^2a\right. \\
&+ \frac{11}{6}a_{77}a_7^2 - \frac{4}{3}a_{77}a_7a^{-8}B - \frac{5}{54}a_{77}B_7a^{-7} \\
&+ \frac{25}{81}a_{77}a^{-16}B^2 + \frac{1}{4}a_7^4a^{-1} + \frac{116}{27}a_7^3a^{-9}B \\
&- \frac{35}{27}a_7^2B_7a^{-8} - \frac{10}{81}a_7^2a^{-17}B^2 + \frac{7}{108}a_7B_{77}a^{-7} \\
&+ \frac{5}{18}a_7B_7a^{-16}B - \frac{40}{243}a_7a^{-25}B^3 - \frac{7}{486}B_{77}a^{-5}B \\
&+ \left.\frac{7}{972}B_7^2a^{-15} - \frac{5}{243}B_7a^{-24}B^2 + \frac{40}{2187}a^{-33}B^4\right] \\
&+ u_8^3a^{-33}\left[-\frac{5}{6}a_{77}a_7a^{25}C - \frac{1}{12}a_{77}C_7a^{26} + \frac{5}{18}a_{77}a^{17}BC\right. \\
&+ \frac{10}{3}a_7^3a^{24}C - \frac{13}{12}a_7^2C_7a^{25} + \frac{10}{27}a_7^2a^{16}BC \\
&+ \frac{1}{18}a_7C_{77}a^{26} + \frac{25}{108}a_7C_7a^{17}B - \frac{20}{81}a_7a^8CB^2 \\
&- \frac{1}{108}C_{77}a^{18}B + \frac{1}{81}C_7B_7a^{18} - \frac{5}{243}C_7a^9B^2 \\
&- \left.\frac{5}{243}B_7a^9BC + \frac{80}{2187}B^3C\right] + u_8^2a^{-33}\left[-\frac{1}{18}a_{77}D_7a^{26}\right. \\
&+ \frac{5}{108}a_{77}a^{17}C^2 - \frac{4}{9}a_7^2D_7a^{25} + \frac{25}{54}a_7^2a^{16}C^2 \\
&+ \frac{1}{36}a_7D_{77}a^{26} + \frac{5}{54}a_7D_7a^{17}B - \frac{10}{81}a_7a^8C^2B \\
&+ \frac{1}{162}D_{77}a^{18}B + \frac{1}{162}D_7B_7a^{18} - \frac{5}{243}D_7a^9B^2 \\
&+ \left.\frac{1}{162}C_7^2a^{18} - \frac{5}{243}C_7a^9BC - \frac{5}{972}B_7a^9C^2 + \frac{20}{729}C^2B^2\right] \\
&+ u_8a^{-33}\left[\frac{5}{54}a_7D_7a^{17}C - \frac{5}{243}a_7a^8C^3 + \frac{1}{324}C_7D_7a^{18}\right. \\
&- \left.\frac{5}{243}D_7a^9BC - \frac{5}{972}C_7a^9C^2 + \frac{20}{2187}BC^3\right] \\
&+ a^{-33}\left[\frac{1}{324}D_7^2a^{18} - \frac{5}{972}D_7a^9C^2 + \frac{5}{4374}C^4\right]
\end{aligned}
\tag{5.2.22}
$$

Table 5.2.2: Structure of non-integrable terms in Step 3. for order $m = 9$.

	$\rho^{(-1)}$	$\rho^{(1)}$	$\rho^{(2)}$	$\rho^{(3)}$
Top terms in $\rho^{(i)}$	u_7	u_8^2	u_8^3	u_9^2
Order of $\rho^{(i)}$	7	8	8	9
Level of $\rho^{(i)}$	0	2	3	4
Top term in $\int \rho_t^{(i)}$	u_{16}	$u_{17}u_8$	$u_{17}u_8^2$	$u_{18}u_9$
Level of top term in $\int \rho_t^{(i)}$	9	11	12	13
Non-integrable terms in $\int \rho_t^{(i)}$	$\mathbf{u_{11}^2u_8}$, $\mathbf{u_{10}^3}$, $u_{10}^2u_9u_8$, $u_{10}^2u_8^3$ $u_9^4u_8$, $u_9^3u_8^3$, $u_9^2u_8^5$, u_8^9	$u_{12}^2u_8$, $u_{11}^2u_{10}$, $u_{11}^2u_9u_8$, $u_{11}^2u_8^3$, $u_{10}^3u_9$, $u_{10}^3u_8^2$, $u_{10}^2u_9^2u_8$, $u_{10}^2u_9u_8^3$, $u_{10}^2u_8^5$ $u_9^5u_8$, $u_9^4u_8^3$, $u_9^3u_8^5$, $u_9^2u_8^7$, u_8^{11}	u_{13}^2, $u_{12}^2u_9$, $u_{12}^2u_8^2$, u_{11}^3, $u_{11}^2u_{10}u_8$, $u_{11}^2u_9^2$, $u_{11}^2u_9u_8^2$, $u_{11}^2u_8^4, u_{10}^4$, $u_{10}^3u_9u_8$, $u_{10}^3u_8^3$, $u_{10}^2u_9^3$, $u_{10}^2u_9^2u_8^2$, $u_{10}^2u_9u_8^4$, $u_{10}^2u_8^6$, $u_9^6, u_9^5u_8^2$, $u_9^4u_8^4$, $u_9^3u_8^6$, $u_9^2u_8^8$, u_8^{12}	$\mathbf{u_{13}^2u_8}, \mathbf{u_{12}^2u_{10}}$, $u_{12}^2u_9u_8$, $u_{12}^2u_8^3, u_{11}^3u_8$, $u_{11}^2u_{10}u_9$, $u_{11}^2u_{10}u_8^2$, $u_{11}^2u_9^2u_8$, $u_{11}^2u_9u_8^3$, $u_{11}^2u_8^5$, $u_{10}^4u_8$, $u_{10}^3u_9^2$, $u_{10}^3u_9u_8^2$, $u_{10}^3u_8^4$, $u_{10}^2u_9^3u_8$, $u_{10}^2u_9^2u_8^3$, $u_{10}^2u_9u_8^5$, $u_{10}^2u_8^7$, $u_9^6u_8, u_9^5u_8^3$, $u_9^4u_8^5, u_9^3u_8^7$, $u_9^2u_8^9, u_8^{13}$

The integrability conditions came from the coefficients of $u_{11}^2u_8$, u_{10}^3 in $\rho_t^{(-1)}$ and $u_{13}^2u_8$, $u_{12}^2u_{10}$ in $\rho_t^{(3)}$.

$$\begin{aligned} u_{11}^2u_8 \quad &: \quad a^{-3}\left(\frac{9}{2}a_{777}a^{10} - \frac{99}{2}a_{77}a_7a^9 - 2a_{777}aB\right. \\ &+ \quad \left. 72a_7^3a^8 + 4a_7^2B\right) \end{aligned} \tag{5.2.23}$$

$$\begin{aligned} u_{10}^3 \quad &: \quad a^{-3}\left(-5a_{777}a^{10} + 48a_{77}a_7a^9 + 3a_{77}aB\right. \\ &- \quad \left. 66a_7^3a^8 - 6a_7^2B\right) \end{aligned} \tag{5.2.24}$$

Computations of (5.2.23), (5.2.24) give:

$$a^{-3}\left(-\frac{7}{5}a_{77}a_7a^9 + \frac{7}{45}a_{77}aB + \frac{14}{5}a_7^3a^8 - \frac{14}{45}a_7^2B\right) \tag{5.2.25}$$

We solve B from (5.2.25):

$$B = 9a_7a^8 \tag{5.2.26}$$

The coefficients of $u_{13}^2u_8$ and $u_{12}^2u_{10}$ have the following form after equating the value of B:

$$u_{13}^2u_8 \quad : \quad 18a_7a^9\left(-a_{77}a + 6a_7^2\right) \tag{5.2.27}$$

$$u_{12}^2u_{10} \quad : \quad 3a_7a^9\left(32a_{77}a - 257a_7^2\right) \tag{5.2.28}$$

Compatibility of (5.2.27), (5.2.28) gives:

$$-\frac{65}{32}a_7^3a^9 = 0 \tag{5.2.29}$$

Since we accept that $a \neq 0$, we conclude that

$$a_7 = 0 \; and \; B = 0 \tag{5.2.30}$$

Step 4. The evolution equation and the conserved densities under the preceding conditions are as follows:

$$u_t \sim Au_9 + Cu_8 + D \tag{5.2.31}$$

where $A = A(u_6)$, $C = C(u_6, u_7)$, $D = D(u_6, u_7)$ and the basic term with level zero is u_7.

$$\rho^{(-1)} = a^{-1} = A^{-1/7} \tag{5.2.32}$$

$$\begin{aligned}\rho^{(1)} &= \frac{1}{30}u_8u_7C_{77}a^{-8} + u_7^2a^{-9}\left[a_6^2a^8 - \frac{4}{15}a_6C_7\right.\\ &+ \left.\frac{1}{30}C_{67}a\right] - \frac{2}{15}u_7a_6a^{-9}C\\ &+ a^{-17}\left[-\frac{1}{30}D_7a^9 + \frac{2}{135}C^2\right]\end{aligned} \tag{5.2.33}$$

$$\begin{aligned}\rho^{(2)} &= u_8u_7\left[\frac{27}{28}a_{66}a + \frac{27}{28}a_6^2 + \frac{1}{4}a_6C_7a^{-8}\right]\\ &+ u_8a^{-16}\left[-\frac{3}{28}C_6a^9 + \frac{1}{12}C_7C\right]\\ &+ u_7^2a_6a^{-9}\left[-9a_6C + C_6a\right]\\ &+ u_7a_6a^{-17}\left[-\frac{3}{4}D_7a^9 + \frac{1}{3}C^2\right]\\ &+ a^{-25}\left[-\frac{3}{28}D_6a^{18} + \frac{1}{12}D_7a^9C - \frac{2}{81}C^3\right]\end{aligned} \tag{5.2.34}$$

$$\begin{aligned}\rho^{(3)} &= u_8^2a^{-15}\left[\frac{7}{4}a_6^2a^{16} - \frac{7}{36}a_6C_7a^8 + \frac{1}{162}C_7^2\right]\\ &+ u_8u_7^2\left[\frac{7}{2}a_{66}a_6a - \frac{5}{36}a_{66}C_7a^{-7} + \frac{3}{4}a_6^3\right.\\ &- \left.\frac{1}{36}a_6^2C_7a^{-8}\right] + u_8u_7a^{-16}\left[-\frac{1}{18}a_{66}a^9C\right.\\ &+ \frac{8}{9}a_6^2a^8C + \frac{1}{12}a_6D_{77}a^9 - \frac{7}{36}a_6C_6a^9 - \frac{5}{54}a_6C_7C\\ &+ \left.\frac{1}{108}C_6C_7a\right] + u_8a^{-24}\left[\frac{1}{324}D_7C_7a^9 + \frac{1}{162}C_6a^9C\right.\\ &- \left.\frac{5}{972}C_7C^2\right] + u_7^4a^{-1}\left[a_{66}^2a^2 + \frac{1}{4}a_6^4\right]\\ &+ u_7^3a^{-9}\left[\frac{1}{2}a_{66}a_6aC - \frac{1}{18}a_{66}C_6a^2 - 2a_6^3C + \frac{2}{9}a_6^2C_6a\right]\\ &+ u_7^2a^{-17}\left[-\frac{5}{6}a_6^2D_7a^9 + \frac{65}{54}a_6^2C^2 + \frac{1}{12}a_6D_{67}a^{10}\right.\\ &- \left.\frac{5}{27}a_6C_6aC + \frac{1}{162}C_6^2a^2\right] + u_7a^{-25}\left[-\frac{1}{12}a_6D_6a^{18}\right.\end{aligned}$$

$$
\begin{aligned}
&+ \quad \frac{5}{54}a_6 D_7 a^9 C - \frac{5}{243}a_6 C^3 - \frac{1}{324}D_7 C_6 a^{10}\Big] \\
&+ \quad a^{-33}\left[\frac{1}{162}D_6 a^{18} C + \frac{1}{324}D_7^2 a^{18} - \frac{5}{972}D_7 a^9 C^2\right. \\
&+ \quad \left.\frac{5}{4374}C^4\right]
\end{aligned}
\tag{5.2.35}
$$

The integrability condition is obtained from the coefficient of $u_{12}^2 u_8$ in $\rho_t^{(3)}$:

$$
u_{12}^2 u_8 : C_{77} a^{-6}\left(\frac{7}{4}a_6 a^8 - \frac{1}{9}C_7\right) \tag{5.2.36}
$$

$$
C_{77} = 0 \quad \Rightarrow C = S u_7 + T \tag{5.2.37}
$$

Step 5. In this step the evolution equation and the conserved densities are as follows:

$$
u_t \sim A u_9 + S u_7 u_8 + T u_8 + D \tag{5.2.38}
$$

where $A = A(u_6)$, $S = S(u_6)$, $T = T(u_6)$ and $D = D(u_6, u_7)$.

$$
\rho^{(-1)} = a^{-1} = A^{-1/7} \tag{5.2.39}
$$

$$
\begin{aligned}
\rho^{(1)} \quad &= \quad u_7^2 a^{-17}\left[a_6^2 a^{16} - \frac{2}{5}a_6 a^8 S + \frac{1}{30}S_6 a^9\right. \\
&+ \quad \left.\frac{2}{135}S^2\right] + u_7 a^{-17} T\left[-\frac{2}{5}a_6 a^8\right. \\
&+ \quad \left.\frac{4}{135}S\right] + a^{-17}\left[-\frac{1}{30}D_7 a^9 + \frac{2}{135}T^2\right]
\end{aligned}
\tag{5.2.40}
$$

$$
\begin{aligned}
\rho^{(2)} \quad &= \quad u_7^3\left[-\frac{27}{56}a_{666} a - \frac{81}{56}a_{66} a_6 - \frac{1}{8}a_{66} a^{-8} S\right. \\
&- \quad 8 a_6^2 a^{-9} S + \frac{1}{2}a_6 S_6 a^{-8} + a_6 a^{-17} S^2 + \frac{3}{56}S_{66} a^{-7} \\
&- \quad \left.\frac{1}{12}S_6 a^{-16} S - \frac{2}{81}a^{-25} S^3\right] \\
&+ \quad u_7^2 a^{-25}\left[-9 a_6^2 a^{16} T + \frac{1}{4}a_6 T_6 a^{17}\right. \\
&+ \quad 2 a_6 a^8 S T + \frac{3}{28}T_{66} a^{18} - \frac{1}{12}T_6 a^9 S - \frac{1}{12}S_6 a^9 T \\
&- \quad \left.\frac{2}{27}S^2 T\right] + u_7 a^{-25}\left[-\frac{3}{4}a_6 D_7 a^{17} + \frac{1}{3}a_6 a^8 T^2\right. \\
&+ \quad \left.\frac{1}{12}D_7 a^9 S - \frac{2}{27}S T^2\right] + a^{-25}\left[-\frac{3}{28}D_6 a^{18}\right. \\
&+ \quad \left.\frac{1}{12}D_7 a^9 T - \frac{2}{81}T^3\right]
\end{aligned}
\tag{5.2.41}
$$

$$
\begin{aligned}
\rho^{(3)} \quad &= \quad u_8^2 a^{-15}\left[\frac{7}{4}a_6^2 a^{16} - \frac{7}{36}a_6 a^8 S + \frac{1}{162}S^2\right] \\
&+ \quad u_8 u_7^2\left[\frac{7}{2}a_{66} a_6 a - \frac{7}{36}a_{66} a^{-7} S + \frac{3}{4}a_6^3\right. \\
&+ \quad \frac{31}{36}a_6^2 a^{-8} S - \frac{7}{36}a_6 S_6 a^{-7} - \frac{5}{54}a_6 a^{-16} S^2
\end{aligned}
$$

$$
\begin{aligned}
&+ \quad \frac{5}{324}S_6a^{-15}S - \frac{5}{972}a^{-24}S^3\Big] \\
&+ \quad u_8u_7a^{-24}\Big[-\frac{1}{18}a_{66}a^{17}T + \frac{8}{9}a_6^2a^{16}T + \frac{1}{12}a_6D_{77}a^{17} \\
&- \quad \frac{7}{36}a_6T_6a^{17} - \frac{5}{54}a_6a^8ST + \frac{5}{324}T_6a^9S \\
&+ \quad \frac{1}{162}S_6a^9T - \frac{5}{486}TS^2\Big] + u_8a^{-24}\Big[\frac{1}{324}D_7a^9S \\
&+ \quad \frac{1}{162}T_6a^9T - \frac{5}{972}ST^2\Big] + u_7^4a^{-33}\Big[a_{66}^2a^{34} \\
&+ \quad \frac{1}{2}a_{66}a_6a^{25}S - \frac{1}{18}a_{66}S_6a^{26} + \frac{1}{4}a_6^4a^{32} \\
&- \quad 2a_6^3a^{24}S + \frac{2}{9}a_6^2S_6a^{25} + \frac{64}{54}a_6^2a^{16}S^2 - \frac{5}{27}a_6S_6a^{17}S \\
&- \quad \frac{5}{243}a_6a^8S^3 + \frac{1}{162}S_6^2a^{18} + \frac{5}{4374}S^4\Big] \\
&+ \quad u_7^3a^{-33}\Big[\frac{1}{2}a_{66}a_6a^{25}T - \frac{1}{18}a_{66}T_6a^{26} - 2a_6^3a^{24}T \\
&+ \quad \frac{2}{9}a_6^2T_6a^{25} + \frac{65}{27}a_6^2a^{16}ST - \frac{5}{27}a_6T_6a^{17}S \\
&- \quad \frac{5}{27}a_6S_6a^{17}T - \frac{5}{81}a_6a^8TS^2 + \frac{1}{81}T_6S_6a^{18} \\
&+ \quad \frac{10}{2187}TS^3\Big] + u_7^2a^{-33}\Big[-\frac{5}{6}a_6^2D_7a^{25} \\
&+ \quad \frac{65}{54}a_6^2a^{16}T^2 + \frac{1}{12}a_6D_{67}a^{26} + \frac{5}{54}a_6D_7a^{17}S \\
&- \quad \frac{5}{27}a_6C_6'a^{17}T - \frac{5}{81}a_6a^8ST^2 - \frac{1}{324}D_7S_6a^{18} \\
&- \quad \frac{5}{972}D_7a^9S^2 + \frac{1}{162}T_6^2a^{18} + \frac{5}{729}S^2T^2\Big] \\
&+ \quad u_7a^{-33}\Big[-\frac{1}{2}a_6D_6a^{26} + \frac{5}{54}a_6D_7a^{17}T - \frac{5}{243}a_6a^8T^3 \\
&+ \quad \frac{1}{162}D_6a^{18}S - \frac{1}{324}D_7T_6a^{18} - \frac{5}{486}D_7a^9ST \\
&+ \quad \frac{10}{2187}ST^3\Big] + a^{-33}\Big[\frac{1}{162}D_6a^{18}T + \frac{1}{324}D_7^2a^{18} \\
&- \quad \frac{5}{972}D_7a^9T^2 + \frac{5}{4374}T^4\Big]
\end{aligned} \tag{5.2.42}
$$

The integrability condition came from the coefficient of $u_{11}^2u_8$ in $\rho_t^{(1)}$.

$$u_{11}^2u_8 : \frac{3}{20}D_{7777}a \tag{5.2.43}$$

Here the basic term with level zero is u_6. We noticed that the level of $u_{11}^2u_8$ is 12 while the level of $\rho_t^{(1)}$ is 11. The top term of $\rho_t^{(1)}$ is u_{17} which has a level of 11. The condition is:

$$D_{7777} = 0 \quad \Rightarrow D = Eu_7^3 + Fu_7^2 + Gu_7H \tag{5.2.44}$$

Step 6. In this step the evolution equation and the conserved densities are as follows:

$$u_t \sim Au_9 + Su_7u_8 + Tu_8 + Eu_7^3 + Fu_7^2 + Gu_7 + H \tag{5.2.45}$$

where A, S, T, E, F, G, H depend only on u_6.

$$\rho(-1) = a^{-1} = A^{-1/7} \tag{5.2.46}$$

$$\begin{aligned}
\rho^{(1)} &= u_7^2 a^{-17}\left[a_6^2 a^{16} - \frac{2}{5}a_6 a^8 S + \frac{1}{30}S_6 a^9\right. \\
&- \left.\frac{1}{10}a^9 E + \frac{2}{135}S^2\right] + u_7 A^{-17}\left[-\frac{2}{15}a_6 a^8 T\right. \\
&- \left.\frac{1}{15}a^9 F + \frac{4}{135}ST\right] + a^{-17}\left[-\frac{1}{30}a^9 G\right. \\
&+ \left.\frac{2}{135}T^2\right] \qquad (5.2.47)
\end{aligned}$$

$$\begin{aligned}
\rho^{(2)} &= u_7^3\left[-\frac{27}{56}a_{666}a - \frac{81}{56}a_{66}a_6 - \frac{1}{8}a_{66}a^{-8}S\right. \\
&- 8a_6^2 a^{-9}S + \frac{1}{2}a_6 S_6 a^{-8} - \frac{9}{4}a_6 a^{-8}E \\
&+ a_6 a^{-17}S^2 + \frac{3}{56}S_{66}a^{-7} - \frac{1}{12}S_6 a^{-16}S \\
&- \left.\frac{3}{28}E_6 a^{-7} + \frac{1}{4}a^{-16}SE - \frac{2}{81}a^{-25}S^3\right] \\
&+ u_7^2 a^{-25}\left[-9a_6^2 a^{16}T + \frac{1}{4}a_6 T_6 a^{17} - \frac{3}{2}a_6 a^{17}F\right. \\
&+ 2a_6 a^8 ST - \frac{1}{12}S_6 a^9 T + \frac{3}{28}T_{66}a^{18} \\
&- \frac{1}{12}T_6 a^9 S - \frac{3}{28}F_6 a^{18} + \frac{1}{6}a^9 SF + \frac{1}{4}a^9 TE \\
&- \left.\frac{2}{27}S^2 T\right] + u_7 a^{-25}\left[-\frac{3}{4}a_6 a^{17}G\right. \\
&+ \frac{1}{3}a_6 a^8 T^2 - \frac{3}{28}G_6 a^{18} + \frac{1}{12}a^9 SG \\
&+ \left.\frac{1}{6}a^9 TF - \frac{2}{27}ST^2\right] + a^{-25}\left[-\frac{3}{28}H_6 a^{18}\right. \\
&+ \left.\frac{1}{12}a^9 TG - \frac{2}{81}T^3\right] \qquad (5.2.48)
\end{aligned}$$

$$\begin{aligned}
\rho^{(3)} &= u_8^2 a^{-15}\left[\frac{7}{4}a_6^2 a^{16} - \frac{7}{36}a_6 a^8 S + \frac{1}{162}S^2\right] \\
&+ u_7^4\left[-\frac{7}{6}a_{666}a_6 a + \frac{7}{108}a_{666}a^{-7}S - \frac{1}{6}a_{66}^2 a\right. \\
&- \frac{23}{12}a_{66}a_6^2 - \frac{19}{36}a_{66}a_6 a^{-8}S + \frac{2}{27}a_{66}S_6 a^{-7} \\
&- \frac{1}{6}a_{66}a^{-7}E + \frac{5}{162}a_{66}a^{-16}S^2 + \frac{1}{4}a_6^4 a^{-1} \\
&+ \frac{8}{27}a_6^3 a^{-9}S - \frac{14}{27}a_6^2 S_6 a^{-8} - \frac{4}{3}a_6^2 a^{-8}E \\
&+ \frac{115}{162}a_6^2 a^{-17}S^2 + \frac{7}{108}a_6 S_{66}a^{-7} - \frac{5}{108}a_6 S_6 a^{-16}S \\
&+ \frac{35}{108}a_6 a^{-16}SE - \frac{5}{81}a_6 a^{-25}S^3 - \frac{5}{972}S_{66}a^{-15}S \\
&+ \frac{1}{972}S_6^2 a^{-15} - \frac{1}{81}S_6 a^{-15}E + \frac{5}{972}S_6 a^{-24}S^2 \\
&+ \frac{1}{324}E_6 a^{-15}S + \frac{1}{36}a^{-15}E^2 - \frac{5}{324}a^{-24}S^2 E \\
&+ \left.\frac{5}{4374}a^{-33}S^4\right] + u_7^3 a^{-33}\left[\frac{1}{36}a_{666}a^{26}T\right.
\end{aligned}$$

$$
\begin{aligned}
&- \frac{7}{12}a_{66}a_6a^{25}T + \frac{5}{72}a_{66}T_6a^{26} - \frac{1}{12}a_{66}a^{26}F \\
&+ \frac{5}{108}a_{66}a^{17}ST + \frac{14}{9}a_6^3a^{24}T - \frac{65}{72}a_6^2T_6a^{25} \\
&- \frac{13}{12}a_6^2a^{25}F + \frac{5}{3}a_6^2a^{16}ST - \frac{5}{54}a_6S_6a^{17}T \\
&+ \frac{7}{72}a_6T_{66}a^{26} - \frac{5}{216}a_6T_6a^{17}S + \frac{25}{108}a_6a^{17}SF \\
&+ \frac{5}{18}a_6a^{17}TE - \frac{5}{27}a_6a^8S^2T - \frac{1}{324}S_{66}a^{18}T \\
&+ \frac{1}{648}S_6T_6a^{18} - \frac{1}{108}S_6a^{18}F + \frac{5}{486}S_6a^9ST \\
&- \frac{5}{648}T_{66}a^{18}S - \frac{1}{108}T_6a^{18}E + \frac{5}{972}T_6a^9S^2 \\
&+ \frac{1}{162}E_6a^{18}T + \frac{1}{324}F_6a^{18}S + \frac{1}{27}a^{18}EF \\
&- \left.\frac{5}{486}a^9S^2F - \frac{5}{162}a^9STE + \frac{10}{2187}S^3T\right] \\
&+ u_7^2a^{-33}\left[-\frac{5}{6}a_6^2a^{25}G + \frac{65}{54}a_6^2a^{16}T^2\right. \\
&- \frac{5}{54}a_6T_6a^{17}T + \frac{5}{36}a_6a^{17}SG + \frac{5}{27}a_6a^{17}TF \\
&- \frac{5}{27}a_6a^8ST^2 - \frac{1}{162}S_6a^{18}G + \frac{5}{972}S_6a^9T^2 \\
&- \frac{1}{162}T_{66}a^{18}T - \frac{1}{162}T_6a^{18}F + \frac{5}{486}T_6a^9ST \\
&+ \frac{1}{162}F_6a^{18}T + \frac{1}{324}G_6a^{18}S + \frac{1}{54}a^{18}EG \\
&+ \frac{1}{81}a^{18}F^2 - \frac{5}{972}a^9S^2G - \frac{5}{243}a^9STF \\
&- \left.\frac{5}{324}a^9T^2E + \frac{5}{729}S^2T^2\right] + u_7a^{-33}\left[-\frac{1}{12}a_6H_6a^{26}\right. \\
&+ \frac{5}{54}a_6a^{17}TG - \frac{5}{243}a_6a^8T^3 - \frac{1}{324}T_6a^{18}G \\
&+ \frac{1}{162}G_6a^{18}T + \frac{1}{162}H_6a^{18}S + \frac{1}{81}a^{18}FG \\
&- \left.\frac{5}{486}a^9SC'G - \frac{5}{486}a^9T^2F + \frac{10}{2187}ST^3\right] \\
&+ a^{-33}\left[\frac{1}{162}H_6a^{18}T + \frac{1}{324}a^{18}G^2\right. \\
&- \left.\frac{5}{972}a^9T^2G + \frac{5}{4374}T^4\right] \qquad (5.2.49)
\end{aligned}
$$

Table 5.2.3: Structure of non-integrable terms in Step 6. for order $m = 9$.

	$\rho^{(-1)}$	$\rho^{(1)}$	$\rho^{(2)}$	$\rho^{(3)}$
Top terms in $\rho^{(i)}$	u_6	u_7^2	u_7^3	u_8^2
Order of $\rho^{(i)}$	6	7	7	8
Level of $\rho^{(i)}$	0	2	3	4
Top term in $\int \rho_t^{(i)}$	u_{15}	$u_{16}u_7$	$u_{16}u_7^2$	$u_{17}u_8$
Level of top term in $\int \rho_t^{(i)}$	9	11	12	13
Non-integrable terms in $\int \rho_t^{(i)}$	$\mathbf{u_{10}^2 u_7}$, u_9^3, $u_9^2u_8u_7$, $u_9^2u_7^3$, $u_8^4u_7$, $u_8^3u_7^3$, $u_8^2u_7^5$, u_7^9	$u_{11}^2u_7$, $\mathbf{u_{10}^2 u_9}$ $u_{10}^2u_8u_7$, $u_{10}^2u_7^3$, $u_9^3u_8$, $u_9^3u_7^2$, $u_9^2u_8^2u_7$, $u_9^2u_8u_7^3$, $u_9^2u_7^5$, $u_8^5u_7$, $u_8^4u_7^3$, $u_8^3u_7^5$, $u_8^2u_7^7$, u_7^{11}	u_{12}^2, $\mathbf{u_{11}^2 u_8}$, $u_{11}^2u_7^2$, u_{10}^3, $u_{10}^2u_9u_7$, $u_{10}^2u_8^2$, $u_{10}^2u_8u_7^2$, $u_{10}^2u_7^4$, u_9^4, $u_9^3u_8u_7$, $u_9^3u_7^3$, $u_9^2u_8^3$, $u_9^2u_8^2u_7^2$ $u_9^2u_8u_7^4$, $u_9^2u_7^6$, u_8^6, $u_8^5u_7^2$, $u_8^4u_7^4$, $u_8^3u_7^6$, $u_8^2u_7^8$, u_7^{12}	$\mathbf{u_{12}^2 u_7}$, $\mathbf{u_{11}^2 u_9}$, $\mathbf{u_{11}^2 u_8 u_7}$, $\mathbf{u_{11}^2 u_7^3}$, $u_{10}^3u_7$, $u_{10}^2u_9u_8$, $u_{10}^2u_9u_7^2$, $u_{10}^2u_8^2u_7$, $u_{10}^2u_8u_7^3$, $u_{10}^2u_7^5$, $u_9^4u_7$, $u_9^3u_8^2$, $u_9^3u_8u_7^2$, $u_9^3u_7^4$, $u_9^2u_8^3u_7$, $u_9^2u_8^2u_7^3$, $u_9^2u_8u_7^5$, $u_9^2u_7^7$, $u_8^6u_7$, $u_8^5u_7^3$, $u_8^4u_7^5$, $u_8^3u_7^7$, $u_8^2u_7^9$, u_7^{13}

The integrability conditions came from the coefficients of $u_{10}^2 u_7$ in $\rho_t^{(-1)}$, $u_{10}^2 u_9$ in $\rho_t^{(1)}$, $u_{11}^2 u_8$ in $\rho_t^{(2)}$, $u_{12}^2 u_8$, $u_{11}^2 u_9$, $u_{11}^2 u_8 u_7$, $u_{11}^2 u_7^3$ in $\rho_t^{(3)}$. First we solve a_{666} from the coefficient of $u_{10}^2 u_7$:

$$\begin{aligned} u_{10}^2 u_7 \quad &: \quad a^{-3}\left(\frac{9}{2}a_{666}a^{10} - \frac{81}{2}a_{66}a_6a^9 - a_{66}aS\right. \\ &+ \quad \left. 54a_6^3a^8 + 2a_6^2S\right) \end{aligned} \tag{5.2.50}$$

$$a_{666} = a^{-10}\left(9a_{66}a_6a^9 + \frac{2}{9}a_{66}aS - 12a_6^3a^8 - \frac{4}{9}a_6^2S\right) \tag{5.2.51}$$

Then we get the coefficients of $u_{12}^2 u_7$ and $u_{11}^2 u_9$:

$$\begin{aligned} u_{12}^2 u_7 \quad &: \quad a^{-15}\left(-\frac{63}{2}a_{66}a_6a^{25} + \frac{7}{4}a_{66}a^{17}S\right. \\ &+ \quad \frac{189}{2}a_6^3a^{24} - 21a_6^2a^{16}S + \frac{7}{4}a_6S_6a^{17} \\ &+ \quad \left.\frac{5}{6}a^8S^2 - \frac{1}{9}S_6a^9S + \frac{1}{81}S^3\right) \end{aligned} \tag{5.2.52}$$

$$\begin{aligned} u_{11}^2 u_9 \quad &: \quad a^{-15}\left(168a_{66}a_6a^{25} - \frac{28}{3}a_{66}a^{17}S - 168a_6^3a^{24}\right. \\ &+ \quad \frac{595}{12}a_6^2a^{16}S - \frac{28}{3}a_6S_6a^{17} \\ &- \quad \left.\frac{17}{36}a_6a^8S^2 + \frac{16}{27}S_6a^9S - \frac{25}{162}S^3\right) \end{aligned} \tag{5.2.53}$$

We compute (5.2.52), (5.2.53) and we get:

$$a^{-15}\left(2a_6^3a^{24}-\frac{107}{288}a_6^2a^{16}S+\frac{143}{6048}a_6a^8S^2-\frac{43}{81648}S^3\right) \tag{5.2.54}$$

We solve S from (5.2.54) and choose one of the three roots:

$$S=\frac{576}{43}a_6a^8 \tag{5.2.55}$$

We solve E_6 form the coefficient of $u_{11}^2u_8$ and E from the coefficient of $u_{10}^2u_9$:

$$\begin{aligned} u_{11}^2u_8 \quad &: \quad a\left(-\frac{865323}{3698}a_{66}a_6a^8+\frac{73670553}{159014}a_6^3a^7\right. \\ &- \left.\frac{5103}{172}a_6E+\frac{81}{28}E_6a\right) \end{aligned} \tag{5.2.56}$$

$$E_6 = a_6a^{-1}\left(\frac{149562}{1849}a_{66}a^8-\frac{12733182}{79507}a_6^2a^7+\frac{441}{43}E\right) \tag{5.2.57}$$

$$\begin{aligned} u_{10}^2u_9 \quad &: \quad a_6\left(-\frac{782997}{9245}a_{66}a^8-\frac{111425793}{397535}a_6^2a^7\right. \\ &+ \left.\frac{4071}{215}E\right) \end{aligned} \tag{5.2.58}$$

$$E = a^7\left(\frac{260999}{58351}a_{66}a+\frac{37141931}{2509093}a_6^2\right) \tag{5.2.59}$$

Finally we get from the computations between the coefficients of $u_{11}^2u_8u_7$ and $u_{11}^2u_7^3$:

$$\begin{aligned} u_{11}^2u_8u_7 \quad &: \quad a^{14}\left(\frac{156358224795}{3404839201}a_{66}^2a^2\right. \\ &+ \frac{91566976338255}{146408085643}a_{66}a_6^2a \\ &- \left.\frac{235137525820554155}{6295547682649}a_6^4\right) \end{aligned} \tag{5.2.60}$$

$$\begin{aligned} u_{11}^2u_7^3 \quad &: \quad a^{14}\left(\frac{305455811181333}{292816171286}a_{66}^2a^2\right. \\ &- \frac{5549671127072697}{12591095365298}a_{66}a_6^2a \\ &- \left.\frac{335197654360042203}{2707078550353907}a_6^4\right) \end{aligned} \tag{5.2.61}$$

$$a_6^2a^{-2}\left(K^{(1)}a_{66}a-K^{(2)}a_6^2\right)=0 \tag{5.2.62}$$

where $K^{(1)}$ and $K^{(2)}$ are constants. Assuming that $a_6\neq 0$

$$a_{66}=K^{(3)}a_6^2a^{-1} \tag{5.2.63}$$

where $K^{(3)}=K^{(2)}/K^{(1)}$, and

$$E=K^{(4)}a_6^2a^7 \tag{5.2.64}$$

where $K^{(4)}$ is a constant. In conclusion, integrable evolution equations of order 9 have the following form:

$$u_t=a^9u_9+\frac{576}{43}a_6a^8u_7u_8+Tu_8+K^{(4)}a_6^2a^7u_7^3+Fu_7^2+Gu_7+H \tag{5.2.65}$$

where a,T,F,G,H depend on u_6.

6 DISCUSSIONS AND CONCLUSIONS

The aim of this thesis is to classify scalar integrable evolution equations. The constraints of the integrable equations, to be classified, are to be scalar and evolutionary. Non-evolutionary equations i.e. $u_{tt} = F[u]$, and integrable evolution systems or vector evolution equations are excluded. The problem is based on an arbitrary evolution equation $u_t = F[u]$ without any functional and scaling assumption.

The integrability test that we used is Mikhailov-Shabat-Sokolov's "formal symmetry method". This integrability test is based on the existence of a truncated expansion of a formal pseudo-differential operator R, satisfying the operator equation

$$R_t + [R, F_*] = 0,$$

where F_* is the Frechet derivative of F. The solvability of the coefficients in R requires the existence of canonical conserved densities, denoted as $\rho^{(i)},\ \ i = -1, 0, 1, 2, 3, \ldots$

We worked with mth order equations. The order $m = 2k + 1$ is odd because the conserved densities of even order equations are trivial [1].

It is known that there are essentially nonlinear integrable $3rd$ order equations [2] which need special techniques [16]. There exist also essentially nonlinear fifth order equations which are not quasi-linear. Since $3rd$ and $5th$ order equations remain still as open problems, our results are valid for $m \geq 7$.

The integrability of homogeneous and non-polynomial scalar evolution equations has been discussed in [3] and [4]. In the former one it has been shown that scale invariant polynomial evolution equations of order greater than seven are symmetries of third and fifth order equations, while in the latter similar results are obtained using negative powers. Thus the problem of classification of arbitrary evolution equations is reduced to prove the polynomiality and scaling properties of such equations.

We proved that arbitrary (non-polynomial) scalar evolution equations of order $m \geq 7$, that are integrable in the sense of admitting the canonical conserved densities $\rho^{(1)}$, $\rho^{(2)}$ and $\rho^{(3)}$, introduced in [2], and computed in [1] are polynomial in the derivatives u_{m-i} for $i = 0, 1, 2$. By the use of these conserved densities we did actually some more computations and we obtained that the coefficient of u_m is independent of u_{m-3} and the coefficient of $u_{m-2}u_{m-1}$ is zero. But as it was not possible to obtain polynomiality in u_{m-3}, we didn't present these in the thesis. The similarity of the results obtained in explicit computations for $m = 7$, $m = 9$ and symbolic computations for general m assure the consistence of our results. In computations for arbitrary m, we haven't used $\rho^{(-1)}$ and $\rho^{(0)}$, while for $m = 7$ we did the computations in two methods where respectively we used $\rho^{(-1)}$, $\rho^{(0)}$, $\rho^{(1)}$, $\rho^{(2)}$, $\rho^{(3)}$ and only the conserved densities, $\rho^{(1)}$ and $\rho^{(3)}$. These two methods showed that $\rho^{(-1)}$ and $\rho^{(0)}$ didn't give any further information. We did not use $\rho^{(2)}$ in the computations of $7th$ order whereas we needed it for the computations of the general m.

For future work the computation of few more conserved densities is necessary to settle the classification problem at low orders. But is there any end for the computations of the conserved densities? The explicit computations of the conserved densities for $7th$ order evolution equations will give an idea.

The use of the graded algebra structure named as "level grading" that we introduced in this thesis and the scale invariance property seems to be promising for an explicit formula for general m. The use of "level grading" will be a tool for the assumption of a fixed level and the form of the conserved densities $\rho^{(k)}$, without explicit computations. These assumptions will give the proof of the independence of A from u_k, ($\frac{\partial A}{\partial u_k} = 0$) which will be sufficient for the polynomiality case.

Appendix A

The submodules M_d^l and their generating monomials with level $l(m_i)$, where $d = 6,5,4,3$ and $l = 1,2,3,...,11$ used in classification of $7th$ order evolution equations:

Submodules with base level 6

$$\begin{aligned}
M_6^1 &= \langle u_7\rangle\\
M_6^2 &= \langle u_8, u_7^2\rangle\\
M_6^3 &= \langle u_9, u_8u_7, u_7^3\rangle\\
M_6^4 &= \langle u_{10}, u_9u_7, u_8^2, u_8u_7^2, u_7^4\rangle\\
M_6^5 &= \langle u_{11}, u_{10}u_7, u_9u_8, u_9u_7^2, u_8^2u_7, u_8u_7^3, u_7^5\rangle\\
M_6^6 &= \langle u_{12}, u_{11}u_7, u_{10}u_8, u_{10}u_7^2, u_9^2, u_9u_8u_7, u_9u_7^3,\\
&\quad u_8^3, u_8^2u_7^2, u_8u_7^4, u_7^6\rangle\\
M_6^7 &= \langle u_{13}, u_{12}u_7, u_{11}u_8, u_{11}u_7^2, u_{10}u_9, u_{10}u_8u_7, u_{10}u_7^3,\\
&\quad u_9^2u_7, u_9u_8^2, u_9u_8u_7^2, u_9u_7^4,\\
&\quad u_8^3u_7, u_8^2u_7^3, u_8u_7^5, u_7^7\rangle\\
M_6^8 &= \langle u_{14}, u_{13}u_7, u_{12}u_8, u_{12}u_7^2, u_{11}u_9, u_{11}u_8u_7, u_{11}u_7^3,\\
&\quad u_{10}^2, u_{10}u_9u_7, u_{10}u_8^2, u_{10}u_8u_7^2, u_{10}u_7^4,\\
&\quad u_9^2u_8, u_9^2u_7^2, u_9u_8^2u_7, u_9u_8u_7^3, u_9u_7^5,\\
&\quad u_8^4, u_8^3u_7^2, u_8^2u_7^4, u_8u_7^6, u_7^8\rangle\\
M_6^9 &= \langle u_{15}, u_{14}u_7, u_{13}u_8, u_{13}u_7^2,\\
&\quad u_{12}u_9, u_{12}u_8u_7, u_{12}u_7^3,\\
&\quad u_{11}u_{10}, u_{11}u_9u_7, u_{11}u_8^2, u_{11}u_8u_7^2, u_{11}u_7^4,\\
&\quad u_{10}^2u_7, u_{10}u_9u_8, u_{10}u_9u_7^2, u_{10}u_8^2u_7, u_{10}u_8u_7^3, u_{10}u_7^5,\\
&\quad u_9^3, u_9^2u_8u_7, u_9^2u_7^3, u_9u_8^3, u_9u_8^2u_7^2, u_9u_8u_7^4, u_9u_7^6,\\
&\quad u_8^4u_7, u_8^3u_7^3, u_8^2u_7^5, u_8u_7^7, u_7^9\rangle\\
M_6^{10} &= \langle u_{16}, u_{15}u_7, u_{14}u_8, u_{14}u_7^2,\\
&\quad u_{13}u_9, u_{13}u_8u_7, u_{13}u_7^3,\\
&\quad u_{12}u_{10}, u_{12}u_9u_7, u_{12}u_8^2, u_{12}u_8u_7^2, u_{12}u_7^4,\\
&\quad u_{11}^2, u_{11}u_{10}u_7, u_{11}u_9u_8, u_{11}u_9u_7^2, u_{11}u_8^2u_7, u_{11}u_8u_7^3, u_{11}u_7^5,\\
&\quad u_{10}^2u_8, u_{10}^2u_7^2, u_{10}u_9^2, u_{10}u_9u_8u_7, u_{10}u_9u_7^3,\\
&\quad u_{10}u_8^3, u_{10}u_8^2u_7^2, u_{10}u_8u_7^4, u_{10}u_7^6,\\
&\quad u_9^3u_7, u_9^2u_8^2, u_9^2u_8u_7^2, u_9^2u_7^4, u_9u_8^3u_7, u_9u_8^2u_7^3, u_9u_8u_7^5, u_9u_7^7,\\
&\quad u_8^5, u_8^4u_7^2, u_8^3u_7^4, u_8^2u_7^6, u_8u_7^8, u_7^{10}\rangle\\
M_6^{11} &= \langle u_{17}, u_{16}u_7, u_{15}u_8, u_{15}u_7^2,\\
&\quad u_{14}u_9, u_{14}u_8u_7, u_{14}u_7^3,\\
&\quad u_{13}u_{10}, u_{13}u_9u_7, u_{13}u_8^2, u_{13}u_8u_7^2, u_{13}u_7^4,\\
&\quad u_{12}u_{11}, u_{12}u_{10}u_7, u_{12}u_9u_8, u_{12}u_9u_7^2, u_{12}u_8^2u_7, u_{12}u_8u_7^3, u_{12}u_7^5\\
&\quad u_{11}^2u_7, u_{11}u_{10}u_8, u_{11}u_{10}u_7^2, u_{11}u_9^2, u_{11}u_9u_8u_7, u_{11}u_9u_7^3,\\
&\quad u_{11}u_8^3, u_{11}u_8^2u_7^2, u_{11}u_8u_7^4, u_{11}u_7^6,
\end{aligned}$$

$$
\begin{aligned}
& u_{10}^2u_9, u_{10}^2u_8u_7, u_{10}^2u_7^3,\\
& u_{10}u_9^2u_7, u_{10}u_9u_8^2, u_{10}u_9u_8u_7^2, u_{10}u_9u_7^4,\\
& u_{10}u_8^3u_7, u_{10}u_8^2u_7^3, u_{10}u_8u_7^5, u_{10}u_7^7,\\
& u_9^3u_8, u_9^3u_7^2, u_9^2u_8^2u_7, u_9^2u_8u_7^3, u_9^2u_7^5,\\
& u_9u_8^4, u_9u_8^3u_7^2, u_9u_8^2u_7^4, u_9u_8u_7^6, u_9u_7^8,\\
& u_8^5u_7, u_8^4u_7^3, u_8^3u_7^5, u_8^2u_7^7, u_8u_7^9, u_7^{11}\rangle\\
M_6^{12} \quad = \quad & \langle u_{18}, u_{17}u_7, u_{16}u_8, u_{16}u_7^2, u_{15}u_9, u_{15}u_8u_7, u_{15}u_7^3,\\
& u_{14}u_{10}, u_{14}u_9u_7, u_{14}u_8^2, u_{14}u_8u_7^2, u_{14}u_7^4,\\
& u_{13}u_{11}, u_{13}u_{10}u_7, u_{13}u_9u_8, u_{13}u_9u_7^2, u_{13}u_8^2u_7, u_{13}u_8u_7^3, u_{13}u_7^5,\\
& u_{12}^2, u_{12}u_{11}u_7, u_{12}u_{10}u_8, u_{12}u_{10}u_7^2, u_{12}u_9^2, u_{12}u_9u_8u_7, u_{12}u_9u_7^3,\\
& u_{12}u_8^3, u_{12}u_8^2u_7^2, u_{12}u_8u_7^4, u_{12}u_7^6,\\
& u_{11}^2u_8, u_{11}^2u_7^2, u_{11}u_{10}u_9, u_{11}u_{10}u_8u_7, u_{11}u_{10}u_7^3, u_{11}u_9^2u_7,\\
& u_{11}u_9u_8^2, u_{11}u_9u_8u_7^2, u_{11}u_9u_7^4, u_{11}u_8^3u_7, u_{11}u_8^2u_7^3, u_{11}u_8u_7^5, u_{11}u_7^7,\\
& u_{10}^3, u_{10}^2u_9u_7, u_{10}^2u_8^2, u_{10}^2u_8u_7^2, u_{10}^2u_7^4,\\
& u_{10}u_9^2u_8, u_{10}u_9^2u_7^2, u_{10}u_9u_8^2u_7, u_{10}u_9u_8u_7^3, u_{10}u_9u_7^5,\\
& u_{10}u_8^4, u_{10}u_8^3u_7^2, u_{10}u_8^2u_7^4, u_{10}u_8u_7^6, u_{10}u_7^8,\\
& u_9^4, u_9^3u_8u_7, u_9^3u_7^3, u_9^2u_8^3, u_9^2u_8^2u_7^2, u_9^2u_8u_7^4, u_9^2u_7^6,\\
& u_9u_8^4u_7, u_9u_8^3u_7^3, u_9u_8^2u_7^5, u_9u_8u_7^7, u_9u_7^9,\\
& u_8^6, u_8^5u_7^2, u_8^4u_7^4, u_8^3u_7^6, u_8^2u_7^8, u_8u_7^{10}, u_7^{12}\rangle\\
M_6^{13} \quad = \quad & \langle u_{19}, u_{18}u_7, u_{17}u_8, u_{17}u_7^2, u_{16}u_9, u_{16}u_8u_7, u_{16}u_7^3,\\
& u_{15}u_{10}, u_{15}u_9u_7, u_{15}u_8^2, u_{15}u_8u_7^2, u_{15}u_7^4,\\
& u_{14}u_{11}, u_{14}u_{10}u_7, u_{14}u_9u_8, u_{14}u_9u_7^2, u_{14}u_8^2u_7, u_{14}u_8u_7^3, u_{14}u_7^5,\\
& u_{13}u_{12}, u_{13}u_{11}u_7, u_{13}u_{10}u_8, u_{13}u_{10}u_7^2, u_{13}u_9^2, u_{13}u_9u_8u_7, u_{13}u_9u_7^3,\\
& u_{13}u_8^3, u_{13}u_8^2u_7^2, u_{13}u_8u_7^4, u_{13}u_7^6, u_{12}^2u_7, u_{12}u_{11}u_8, u_{12}u_{11}u_7^2,\\
& u_{12}u_{10}u_9, u_{12}u_{10}u_8u_7, u_{12}u_{10}u_7^3,\\
& u_{12}u_9^2u_7, u_{12}u_9u_8^2, u_{12}u_9u_8u_7^2, u_{12}u_9u_7^4,\\
& u_{12}u_8^3u_7, u_{12}u_8^2u_7^3, u_{12}u_8u_7^5, u_{12}u_7^7,\\
& u_{11}^2u_9, u_{11}^2u_8u_7, u_{11}^2u_7^3, u_{11}u_{10}^2,\\
& u_{11}u_{10}u_9u_7, u_{11}u_{10}u_8^2, u_{11}u_{10}u_8u_7^2, u_{11}u_{10}u_7^4,\\
& u_{11}u_9^2u_8, u_{11}u_9^2u_7^2, u_{11}u_9u_8^2u_7, u_{11}u_9u_8u_7^3, u_{11}u_9u_7^5,\\
& u_{11}u_8^4, u_{11}u_8^3u_7^2, u_{11}u_8^2u_7^4, u_{11}u_8u_7^6, u_{11}u_7^8,\\
& u_{10}^3u_7, u_{10}^2u_9u_8, u_{10}^2u_9u_7^2, u_{10}^2u_8^2u_7, u_{10}^2u_8u_7^3, u_{10}^2u_7^5,\\
& u_{10}u_9^3, u_{10}u_9^2u_8u_7, u_{10}u_9^2u_7^3,\\
& u_{10}u_9u_8^3, u_{10}u_9u_8^2u_7^2, u_{10}u_9u_8u_7^4, u_{10}u_9u_7^6,\\
& u_{10}u_8^4u_7, u_{10}u_8^3u_7^3, u_{10}u_8^2u_7^5, u_{10}u_8u_7^7, u_{10}u_7^9,\\
& u_9^4u_7, u_9^3u_8^2, u_9^3u_8u_7^2, u_9^3u_7^4, u_9^2u_8^3u_7, u_9^2u_8^2u_7^3, u_9^2u_8u_7^5, u_9^2u_7^7,\\
& u_9u_8^5, u_9u_8^4u_7^2, u_9u_8^3u_7^4, u_9u_8^2u_7^6, u_9u_8u_7^8, u_9u_7^{10},\\
& u_8^6u_7, u_8^5u_7^3, u_8^4u_7^5, u_8^3u_7^7, u_8^2u_7^9, u_8u_7^{11}, u_7^{13}\rangle
\end{aligned}
$$

Submodules with base level 5

$$
\begin{aligned}
M_5^1 &= \langle u_6\rangle \\
M_5^2 &= \langle u_7, u_6^2\rangle \\
M_5^3 &= \langle u_8, u_7u_6, u_6^3\rangle \\
M_5^4 &= \langle u_9, u_8u_6, u_7^2, u_6^2u_7, u_6^4\rangle \\
M_5^5 &= \langle u_{10}, u_9u_6, u_8u_7, u_8u_6^2, u_7^2u_6, u_7u_6^3, u_6^5\rangle \\
M_5^6 &= \langle u_{11}, u_{10}u_6, u_9u_7, u_9u_6^2, u_8^2, u_8u_7u_6, u_8u_6^3, \\
&\quad u_7^3, u_7^2u_6^2, u_7u_6^4, u_6^6\rangle \\
M_5^7 &= \langle u_{12}, u_{11}u_6, u_{10}u_7, u_{10}u_6^2, u_9u_8, u_9u_7u_6, u_9u_6^3, u_8^2u_6, u_8u_7^2, \\
&\quad u_8u_7u_6^2, u_8u_6^4, u_7^3u_6, u_7^2u_6^3, u_7u_6^5, u_6^7\rangle \\
M_5^8 &= \langle u_{13}, u_{12}u_6, u_{11}u_7, u_{11}u_6^2, u_{10}u_8, u_{10}u_7u_6, u_{10}u_6^3, \\
&\quad u_9^2, u_9u_8u_6, u_9u_7^2, u_9u_7u_6^2, u_9u_6^4, \\
&\quad u_8^2u_7, u_8^2u_6^2, u_8u_7^2u_6, u_8u_7u_6^3, u_8u_6^5, u_7^4, u_7^3u_6^2, u_7^2u_6^4, u_7u_6^6, u_6^8\rangle \\
M_5^9 &= \langle u_{14}, u_{13}u_6, u_{12}u_7, u_{12}u_6^2, u_{11}u_8, u_{11}u_7u_6, u_{11}u_6^3, \\
&\quad u_{10}u_9, u_{10}u_8u_6, u_{10}u_7^2, u_{10}u_7u_6^2, u_{10}u_6^4, \\
&\quad u_9^2u_6, u_9u_8u_7, u_9u_8u_6^2, u_9u_7^2u_6, u_9u_7u_6^3, u_9u_6^5, \\
&\quad u_8^3, u_8^2u_7u_6, u_8^2u_6^3, u_8u_7^3, u_8u_7^2u_6^2, u_8u_7u_6^4, u_8u_6^6, \\
&\quad u_7^4u_6, u_7^3u_6^3, u_7^2u_6^5, u_7u_6^7, u_6^9\rangle \\
M_5^{10} &= \langle u_{15}, u_{14}u_6, u_{13}u_7, u_{13}u_6^2, u_{12}u_8, u_{12}u_7u_6, u_{12}u_6^3, \\
&\quad u_{11}u_9, u_{11}u_8u_6, u_{11}u_7^2, u_{11}u_7u_6^2, u_{11}u_6^4, \\
&\quad u_{10}^2, u_{10}u_9u_6, u_{10}u_8u_7, u_{10}u_8u_6^2, u_{10}u_7^2u_6, u_{10}u_7u_6^3, u_{10}u_6^5, \\
&\quad u_9^2u_7, u_9^2u_6^2, u_9u_8^2, u_9u_8u_7u_6, u_9u_8u_6^3, u_9u_7^2u_6^2, u_9u_7u_6^4, u_9u_7^3, u_9u_6^6, \\
&\quad u_8^3u_6, u_8^2u_7^2, u_8^2u_7u_6^2, u_8^2u_6^4, u_8u_7^3u_6, u_8u_7^2u_6^3, u_8u_7u_6^5, u_8u_6^7, \\
&\quad u_7^5, u_7^4u_6^2, u_7^3u_6^4, u_7^2u_6^6, u_7u_6^8, u_6^{10}\rangle \\
M_5^{11} &= \langle u_{16}, u_{15}u_6, u_{14}u_7, u_{14}u_6^2, u_{13}u_8, u_{13}u_7u_6, u_{13}u_6^3, \\
&\quad u_{12}u_9, u_{12}u_8u_6, u_{12}u_7^2, u_{12}u_7u_6^2, u_{12}u_6^4, \\
&\quad u_{11}u_{10}, u_{11}u_9u_6, u_{11}u_8u_7, u_{11}u_8u_6^2, u_{11}u_7^2u_6, u_{11}u_7u_6^3, u_{11}u_6^5, \\
&\quad u_{10}^2u_6, u_{10}u_9u_7, u_{10}u_9u_6^2, u_{10}u_8^2, u_{10}u_8u_7u_6, u_{10}u_8u_6^3, \\
&\quad u_{10}u_7^3, u_{10}u_7^2u_6^2, u_{10}u_7u_6^4, u_{10}u_6^6, \\
&\quad u_9^2u_8, u_9^2u_7u_6, u_9^2u_6^3, u_9u_8^2u_6, u_9u_8u_7^2, u_9u_8u_7u_6^2, u_9u_8u_6^4, \\
&\quad u_9u_7^3u_6, u_9u_7^2u_6^3, u_9u_7u_6^5, u_9u_6^7, \\
&\quad u_8^3u_7, u_8^3u_6^2, u_8^2u_7^2u_6, u_8^2u_7u_6^3, u_8^2u_6^5, \\
&\quad u_8u_7^4, u_8u_7^3u_6^2, u_8u_7^2u_6^4, u_8u_7u_6^6, u_8u_6^8, \\
&\quad u_7^5u_6, u_7^4u_6^3, u_7^3u_6^5, u_7^2u_6^7, u_7u_6^9, u_6^{11}\rangle \\
M_5^{12} &= \langle u_{17}, u_{16}u_6, u_{15}u_7, u_{15}u_6^2, u_{14}u_8, u_{14}u_7u_6, u_{14}u_6^3 \\
&\quad u_{13}u_9, u_{13}u_8u_6, u_{13}u_7^2, u_{13}u_7u_6^2, u_{13}u_6^4, \\
&\quad u_{12}u_{10}, u_{12}u_9u_6, u_{12}u_8u_7, u_{12}u_8u_6^2, u_{12}u_7^2u_6, u_{12}u_7u_6^3, u_{12}u_6^5 \\
&\quad u_{11}^2, u_{11}u_{10}u_6, u_{11}u_9u_7, u_{11}u_9u_6^2, u_{11}u_8u_7u_6, u_{11}u_8u_6^3, u_{11}u_7^3 \\
&\quad u_{11}u_7^2u_6^2, u_{11}u_7u_6^4, u_{11}u_6^6, u_{10}^2u_7, u_{10}^2u_6^2, \\
&\quad u_{10}u_9u_8, u_{10}u_9u_7u_6, u_{10}u_9u_6^3, u_{10}u_8^2u_6, u_{10}u_8u_7^2, u_{10}u_8u_7u_6^2, u_{10}u_8u_6^4,
\end{aligned}
$$

$$
\begin{aligned}
& u_{10}u_7^3u_6, u_{10}u_7^2u_6^3, u_{10}u_7u_6^5, u_{10}u_6^7, u_9^3, \\
& u_9^2u_8u_6, u_9^2u_7^2, u_9^2u_7u_6^2, u_9^2u_6^4, u_9u_8^2u_7, u_9u_8^2u_6^2, \\
& u_9u_8u_7^2u_6, u_9u_8u_7u_6^3, u_9u_8u_6^5, u_9u_7^4, u_9u_7^3u_6^2, u_9u_7^2u_6^4, \\
& u_9u_7u_6^6, u_9u_6^8, u_8^4, u_8^3u_7u_6, u_8^3u_6^3, u_8^2u_7^3, u_8^2u_7^2u_6^2, u_8^2u_7u_6^4, u_8^2u_6^6, \\
& u_8u_7^4u_6, u_8u_7^3u_6^3, u_8u_7^2u_6^5, u_8u_7u_6^7, u_8u_6^9, u_7^6, u_7^5u_6^2, u_7^4, u_6^4, \\
& u_7^3u_6^6, u_7^2u_6^8, u_7u_6^{10}, u_6^{12}\rangle \\
M_5^{13} \quad = \quad & \langle u_{18}, u_{17}u_6, u_{16}u_7, u_{16}u_6^2, u_{15}u_8, u_{15}u_7u_6, u_{15}u_6^3, \\
& u_{14}u_9, u_{14}u_8u_6, u_{14}u_7^2, u_{14}u_7u_6^2, u_{14}u_6^4, \\
& u_{13}u_{10}, u_{13}u_9u_6, u_{13}u_8u_7, u_{13}u_8u_6^2, u_{13}u_7^2u_6, u_{13}u_7u_6^3, u_{13}u_6^5, \\
& u_{12}u_{11}, u_{12}u_{10}u_6, u_{12}u_9u_7, u_{12}u_9u_6^2, u_{12}u_8^2, \\
& u_{12}u_8u_7u_6, u_{12}u_8u_6^3, u_{12}u_7^3, u_{12}u_7^2u_6^2, u_{12}u_7u_6^4, u_{12}u_6^6, \\
& u_{11}u_{10}u_7, u_{11}u_{10}u_6^2, u_{11}u_9u_8, u_{11}u_9u_7u_6, u_{11}u_9u_6^3, u_{11}u_8^2u_6, \\
& u_{11}u_8u_7^2, u_{11}u_8u_7u_6^2, u_{11}u_8u_6^4, u_{11}u_7^3u_6, u_{11}u_7^2u_6^3, u_{11}u_7u_6^5, u_{11}u_6^7, \\
& u_{10}^2u_8, u_{10}^2u_7u_6, u_{10}^2u_6^3, u_{10}u_9^2, \\
& u_{10}u_9u_8u_6, u_{10}u_9u_7u_6^2, u_{10}u_9u_6^4, u_{10}u_8^2u_7, u_{10}u_8^2u_6^2, \\
& u_{10}u_8u_7^2u_6, u_{10}u_8u_7u_6^3, u_{10}u_8u_6^5, u_{10}u_7^4, u_{10}u_7^3u_6^2, \\
& u_{10}u_7^2u_6^4, u_{10}u_7u_6^6, u_{10}u_6^8, u_9^3u_6, \\
& u_9^2u_8u_7, u_9^2u_8u_6^2, u_9^2u_7^2u_6, u_9^2u_7u_6^3, u_9^2u_6^5, \\
& u_9u_8^3, u_9u_8^2u_7u_6, u_9u_8^2u_6^3, u_9u_8u_7^3, u_9u_8u_7^2u_6^2, u_9u_8u_7u_6^4, \\
& u_9u_8u_6^6, u_9u_7^4u_6, u_9u_7^3u_6^3, u_9u_7^2u_6^5, u_9u_7u_6^7, u_9u_6^9, \\
& u_8^4u_6, u_8^3u_7^2, u_8^3u_7u_6^2, u_8^3u_6^4, u_8^2u_7^3u_6, u_8^2u_7^2u_6^3, u_8^2u_7u_6^5, \\
& u_8^2u_6^7, u_8u_7^5, u_8u_7^4u_6^2, u_8u_7^3u_6^4, u_8u_7^2u_6^6, u_8u_7u_6^8, u_8u_6^{10}, \\
& u_7^6u_6, u_7^5u_6^3, u_7^4u_6^5, u_7^3u_6^7, u_7^2u_6^9, u_7u_6^{11}u_6^{13}, \rangle
\end{aligned}
$$

Submodules with base level 4

$$
\begin{aligned}
M_4^1 \quad = \quad & \langle u_5\rangle \\
M_4^2 \quad = \quad & \langle u_6, u_5^2\rangle \\
M_4^3 \quad = \quad & \langle u_7, u_6u_5, u_5^3\rangle \\
M_4^4 \quad = \quad & \langle u_8, u_7u_5, u_6^2, u_6u_5^2, u_5^4\rangle \\
M_4^5 \quad = \quad & \langle u_9, u_8u_5, u_7u_6, u_7u_5^2, u_6^2u_5, u_6u_5^3, u_5^5\rangle \\
M_4^6 \quad = \quad & \langle u_{10}, u_9u_5, u_8u_5^2, u_8u_6, u_7^2, u_7u_6u_5, u_7u_5^3, \\
& u_6^3, u_6^2u_5^2, u_6u_5^4, u_5^6\rangle \\
M_4^7 \quad = \quad & \langle u_{11}, u_{10}u_5, u_9u_6, u_9u_5^2, u_8u_7, u_8u_6u_5, u_8u_5^3, \\
& u_7^2u_5, u_7u_6^2, u_7u_6u_5^2, u_7u_5^4, u_6^3u_5, u_6^2u_5^3, u_6u_5^5, u_5^7\rangle \\
M_4^8 \quad = \quad & \langle u_{12}, u_{11}u_5, u_{10}u_6, u_{10}u_5^2, u_9u_7, u_8^2, u_9u_6u_5, u_9u_5^3, \\
& u_8u_7u_5, u_8u_6^2, u_8u_6u_5^2, u_8u_5^4, u_7^2u_6, u_7^2u_5^2, u_7u_6^2u_5, u_7u_6u_5^3, u_7u_5^5, \\
& u_6^4, u_6^3u_5^2, u_6^2u_5^4, u_6u_5^6, u_5^8\rangle \\
M_4^9 \quad = \quad & \langle u_{13}, u_{12}u_5, u_{11}u_6, u_{11}u_5^2, u_{10}u_7, u_{10}u_6u_5, u_{10}u_5^3, \\
& u_9u_8, u_9u_7u_5, u_9u_6^2, u_9u_6u_5^2, u_9u_5^4, \\
& u_8^2u_5, u_8u_7u_6, u_8u_7u_5^2, u_8u_6^2u_5, u_8u_6u_5^3, u_8u_5^5, \\
& u_7^3, u_7^2u_6u_5, u_7^2u_5^3, u_7u_6^3, u_7u_6^2u_5^2, u_7u_6u_5^4, u_7u_5^6, \\
& u_6^4u_5, u_6^3u_5^3, u_6^2u_5^5, u_6u_5^7, u_5^9\rangle
\end{aligned}
$$

$$
\begin{aligned}
M_4^{10} \quad = \quad & \langle u_{14}, u_{13}u_5, u_{12}u_6, u_{12}u_5^2, u_{11}u_7, u_{11}u_6u_5, u_{11}u_5^3, u_{10}u_8, u_9^2, \\
& u_{10}u_7u_5, u_{10}u_6^2, u_{10}u_6u_5^2, u_{10}u_5^4, \\
& u_9u_8u_5, u_9u_7u_6, u_9u_7u_5^2, u_9u_6^2u_5, u_9u_6u_5^3, u_9u_5^5, \\
& u_8^2u_6, u_8^2u_5^2, u_8u_7^2, u_8u_7u_6u_5, u_8u_7u_5^3, u_8u_6^3, u_8u_6^2u_5^2, u_8u_6u_5^4, u_8u_5^6, \\
& u_7^3u_5, u_7^2u_6^2, u_7^2u_6u_5^2, u_7^2u_5^4, u_7u_6u_5^5, u_7u_6^3u_5, u_7u_6^2u_5^3, u_7u_5^7, \\
& u_6^5, u_6^4u_5^2, u_6^3u_5^4, u_6^2u_5^6, u_6u_5^8, u_5^{10}\rangle \\
M_4^{11} \quad = \quad & \langle u_{15}, u_{14}u_5, u_{13}u_6, u_{13}u_5^2, u_{12}u_7, u_{12}u_6u_5, u_{12}u_5^3, \\
& u_{11}u_8, u_{11}u_7u_5, u_{11}u_6^2, u_{11}u_6u_5^2, u_{11}u_5^4, \\
& u_{10}u_9, u_{10}u_7u_6, u_{10}u_8u_5, u_{10}u_7u_5^2, u_{10}u_6^2u_5, u_{10}u_6u_5^3, u_{10}u_5^5, \\
& u_9^2u_5, u_9u_8u_6, u_9u_8u_5^2, u_9u_7^2, u_9u_7u_6u_5, u_9u_7u_5^3, \\
& u_9u_6^2u_5^2, u_9u_6u_5^4, u_9u_6^3, u_9u_5^6, \\
& u_8^2u_7, u_8^2u_6u_5, u_8^2u_5^3, u_8u_7^2u_5, u_8u_7u_6^2, u_8u_7u_6u_5^2, u_8u_7u_5^4, \\
& u_8u_6^3u_5, u_8u_6^2u_5^3, u_8u_6u_5^5, u_8u_5^7, u_7^3u_6, u_7^3u_5^2, \\
& u_7^2u_6^2u_5, u_7^2u_6u_5^3, u_7^2u_5^5, u_7u_6^4, u_7u_6^3u_5^2, u_7u_6^2u_5^4, u_7u_6u_5^6, u_7u_5^8, \\
& u_6^5u_5, u_6^4u_5^3, u_6^3u_5^5, u_6^2u_5^7, u_6u_5^9, u_5^{11}\rangle \\
M_4^{12} \quad = \quad & \langle u_{16}, u_{15}u_5, u_{14}u_6, u_{14}u_5^2, u_{13}u_7, u_{13}u_6u_5, u_{13}u_5^3, \\
& u_{12}u_8, u_{12}u_7u_5, u_{12}u_6^2, u_{11}u_9, u_{11}u_8u_5, u_{11}u_7u_6, u_{11}u_7u_5^2, \\
& u_{11}u_6^2u_5, u_{11}u_6u_5^3, u_{11}u_5^5, u_{10}^2, u_{10}u_9u_5, \\
& u_{10}u_8u_6, u_{10}u_8u_5^2, u_{10}u_7^2, u_{10}u_7u_6u_5, u_{10}u_7u_5^3, \\
& u_{10}u_6^3, u_{10}u_6^2u_5^2, u_{10}u_6u_5^4, u_{10}u_5^6, \\
& u_9^2u_6, u_9^2u_5^2, u_9u_8u_7, u_9u_8u_6u_5, u_9u_8u_5^3, u_9u_7^2u_5, \\
& u_9u_7u_6^2, u_9u_7u_6u_5^2, u_9u_7u_5^4, u_9u_6^3u_5, u_9u_6^2u_5^3, u_9u_6u_5^5, u_9u_5^7, \\
& u_8^3, u_8^2u_7u_5, u_8^2u_6^2, u_8^2u_6u_5^2, u_8^2u_5^4, u_8u_7^2u_6, u_8u_7^2u_5^2, \\
& u_8u_7u_6^2u_5, u_8u_7u_6u_5^3, u_8u_7u_5^5, u_8u_6^4, u_8u_6^3u_5^2, u_8u_6^2u_5^4, u_8u_6u_5^6, u_8u_5^8, \\
& u_7^4, u_7^3u_6u_5, u_7^3u_5^3, u_7^2u_6^3, u_7^2u_6^2u_5^2, u_7^2u_6u_5^4, u_7^2u_5^6, \\
& u_7u_6^4u_5, u_7u_6^3u_5^3, u_7u_6^2u_5^5, u_7u_6u_5^7, u_7u_5^9, u_6^6, u_6^5u_5^2, \\
& u_6^4u_5^4, u_6^3u_5^6, u_6^2u_5^8, u_6u_5^{10}, u_5^{12}\rangle \\
M_4^{13} \quad = \quad & \langle u_{17}, u_{16}u_5, u_{15}u_5^2, u_{14}u_7, u_{14}u_6u_5, u_{14}u_5^3, \\
& u_{13}u_8, u_{13}u_7u_5, u_{13}u_6^2, u_{13}u_6u_5^2, u_{13}u_5^4, \\
& u_{12}u_9, u_{12}u_8u_5, u_{12}u_7u_6, u_{12}u_7u_5^2, u_{12}u_6^2u_5, u_{12}u_6u_5^3, u_{12}u_5^5, \\
& u_{11}u_{10}, u_{11}u_9u_5, u_{11}u_8u_6, u_{11}u_8u_5^2, u_{11}u_7u_6u_5, u_{11}u_7u_5^3, \\
& u_{11}u_6^3, u_{11}u_6^2u_5^2, u_{11}u_6u_5^4, u_{11}u_5^6, u10^2u_5, \\
& u_{10}u_9u_6, u_{10}u_9u_5^2, u_{10}u_8u_7, u_{10}u_8u_6u_5, u_{10}u_8u_5^3, \\
& u_{10}u_7u_6^2, u_{10}u_7u_6u_5^2, u_{10}u_7u_5^4, u_{10}u_6^3u_5, u_{10}u_6^2u_5^3, \\
& u_{10}u_6u_5^5, u_{10}u_7^5, u_9^2u_7, u_9^2u_6u_5, u_9^2u_5^3, u_9u_8^2, \\
& u_9u_8u_7u_5, u_9u_8u_6^2, u_9u_8u_6u_5^2, u_9u_8u_5^4, u_9u_7^2u_6, u_9u_7^2u_5^2, \\
& u_9u_7u_6^2u_5, u_9u_7u_6u_5^3, u_9u_7u_5^5, u_9u_6^4, u_9u_6^3u_5^2, u_9u_6^2u_5^4, \\
& u_9u_6u_5^6, u_9u_5^8, u_8^3u_5, u_8^2u_7u_6, u_8^2u_7u_5^2, u_8^2u_6^2u_5, u_8 32u_6u_5^3, u_8^2u_5^5 \\
& u_8u_7^3, u_8u_7^2u_6u_5, u_8u_7^2u_5^3, u_8u_7u_6^3, u_8u_7u_6^2u_5^2, u_8u_7u_6u_5^4, u_8u_7u_5^6, \\
& u_8u_6^4u_5, u_8u_6^3u_5^3, u_8u_6^2u_5^2, u_8u_6u_5^7, u_8u_5^9, u_7^4u_5, \\
& u_7^3u_6^2, u_7^3u_6u_5^2, u_7^3u_5^4, u_7^2u_6^3u_5, u_7^2u_6^2u_5^3, u_7^2u_6u_5^5, u_7^2u_5^7, \\
& u_7u_6^5, u_7u_6^4, u_5^2, u_7u_6^3u_5^4, u_7u_6^2u_5^6, u_7u_6u_5^8, u_7u_5^{10},
\end{aligned}
$$

$u_6^6 u_5, u_6^5 u_5^3, u_6^4 u_5^5, u_6^3 u_5^7, u_6^2 u_5^9, u_6 u_5^{11}, u_5^{13}\rangle$

Submodules with base level 3

$$
\begin{aligned}
M_3^1 &= \langle u_4\rangle\\
M_3^2 &= \langle u_5, u_4^2\rangle\\
M_3^3 &= \langle u_6, u_5u_4, u_4^3\rangle\\
M_3^4 &= \langle u_7, u_6u_4, u_5^2, u_5u_4^2, u_4^4\rangle\\
M_3^5 &= \langle u_8, u_7u_4, u_6u_5, u_6u_4^2, u_5^2u_4, u_5u_4^3, u_4^5\rangle\\
M_3^6 &= \langle u_9, u_8u_4, u_7u_4^2, u_7u_5, u_6^2,\\
&\quad u_6u_5u_4, u_6u_4^3, u_5^3, u_5^2u_4^2, u_5u_4^4, u_4^6\rangle\\
M_3^7 &= \langle u_{10}, u_9u_4, u_8u_5, u_8u_4^2, u_7u_6, u_7u_5u_4, u_7u_4^3,\\
&\quad u_6^2u_4, u_6u_5^2, u_6u_5u_4^2, u_6u_4^4, u_5^3u_4, u_5^2u_4^3, u_5u_4^5, u_4^7\rangle\\
M_3^8 &= \langle u_{11}, u_{10}u_4, u_9u_5, u_9u_4^2, u_8u_6, u_7^2, u_8u_5u_4, u_8u_4^3,\\
&\quad u_7u_6u_4, u_7u_5^2, u_7u_5u_4^2, u_7u_4^4, u_6^2u_5, u_6^2u_4^2, u_6u_5^2u_4, u_6u_5u_4^3, u_6u_4^5,\\
&\quad u_5^4, u_5^3u_4^2, u_5^2u_4^4, u_5u_4^6, u_4^8\rangle\\
M_3^9 &= \langle u_{12}, u_{11}u_4, u_{10}u_5, u_{10}u_4^2, u_9u_6, u_9u_5u_4, u_9u_4^3,\\
&\quad u_8u_7, u_8u_6u_4, u_8u_5^2, u_8u_5u_4^2, u_8u_4^4,\\
&\quad u_7^2u_4, u_7u_6u_5, u_7u_6u_4^2, u_7u_5^2u_4, u_7u_5u_4^3, u_7u_4^5,\\
&\quad u_6^3, u_6^2u_5u_4, u_6^2u_4^3, u_6u_5^2u_4^2, u_6u_5u_4^4, u_6u_5^3, u_6u_4^6,\\
&\quad u_5^4u_4, u_5^3u_4^3, u_5^2u_4^5, u_5u_4^7, u_4^9\rangle\\
M_3^{10} &= \langle u_{13}, u_{12}u_4, u_{11}u_5, u_{11}u_4^2, u_{10}u_6, u_{10}u_5u_4, u_{10}u_4^3,\\
&\quad u_9u_7, u_9u_6u_4, u_9u_5^2, u_9u_5u_4^2, u_9u_4^4,\\
&\quad u_8^2, u_8u_7u_4, u_8u_6u_5, u_8u_6u_4^2, u_8u_5^2u_4, u_8u_5u_4^3, u_8u_4^5,\\
&\quad u_7^2u_5, u_7^2u_4^2, u_7u_6^2, u_7u_6u_5u_4, u_7u_6u_4^3, u_7u_5^3, u_7u_5^2u_4^2, u_7u_5u_4^4, u_7u_4^6,\\
&\quad u_6^3u_4, u_6^2u_5^2, u_6^2u_5u_4^2, u_6^2u_4^4, u_6u_5u_4^5, u_6u_5^3u_4, u_6u_5^2u_4^3, u_6u_4^7,\\
&\quad u_5^5, u_5^4u_4^2, u_5^3u_4^4, u_5^2u_4^6, u_5u_4^8, u_4^{10}\rangle\\
M_3^{11} &= \langle u_{14}, u_{13}u_4, u_{12}u_5, u_{12}u_4^2, u_{11}u_6, u_{11}u_5u_4, u_{11}u_4^3,\\
&\quad u_{10}u_7, u_{10}u_5^2, u_{10}u_6u_4, u_{10}u_5u_4^2, u_{10}u_4^4,\\
&\quad u_9u_8, u_9u_7u_4, u_9u_6u_5, u_9u_6u_4^2, u_9u_5^2u_4, u_9u_5u_4^3, u_9u_4^5,\\
&\quad u_8^2u_4, u_8u_7u_5, u_8u_6^2, u_8u_7u_4^2, u_8u_6u_5u_4, u_8u_6u_4^3,\\
&\quad u_8u_5^2u_4^2, u_8u_5^3, u_8u_5u_4^4, u_8u_4^6,\\
&\quad u_7^2u_6, u_7^2u_5u_4, u_7u_6^2u_4, u_7u_6u_5^2, u_7^2u_4^3, u_7u_6u_5u_4^2, u_7u_5^3u_4, u_7u_6u_4^4,\\
&\quad u_7u_5^2u_4^3, u_7u_5u_4^5, u_7u_4^7, u_6^3u_5, u_6^3u_4^2, u_6^2u_5^2u_4, u_6^2u_5u_4^3, u_6^2u_4^5,\\
&\quad u_6u_5^2u_4^4, u_6u_5u_4^6, u_6u_5^4, u_6u_5^3u_4^2, u_6u_4^8,\\
&\quad u_5^5u_4, u_5^4u_4^3, u_5^3u_4^5, u_5^2u_4^7, u_5u_4^9, u_4^{11}\rangle\\
M_3^{12} &= \langle u_{15}, u_{14}u_4, u_{13}u_5, u_{13}u_4^2, u_{12}u_6, u_{12}u_5u_4, u_{12}u_4^3,\\
&\quad u_{11}u_7, u_{11}u_6u_4, u_{11}u_5^2, u_{11}u_5u_4^2, u_{11}u_4^4,\\
&\quad u_{10}u_8, u_{10}u_7u_4, u_{10}u_6u_5, u_{10}u_6u_4^2, u_{10}u_5^2u_4, u_{10}u_5u_4^3, u_{10}u_4^5,\\
&\quad u_9^2, u_9u_8u_4, u_9u_7u_5, u_9u_7u_4^2, u_9u_6^2, u_9u_6u_5u_4, u_9u_6u_4^3,\\
&\quad u_9u_5^3, u_9u_5^2u_4^2, u_9u_5u_4^4, u_9u_4^6,\\
&\quad u_8^2u_5, u_8^2u_4^2, u_8u_7u_6, u_8u_7u_5u_4, u_8u_7u_4^3,\\
&\quad u_8u_6^2u_4, u_8u_6u_5u_4^2, u_8u_6u_5^2, u_8u_6u_4^4,
\end{aligned}
$$

$$
\begin{aligned}
& u_8u_5^3u_4, u_8u_5^2u_4^3, u_8u_5u_4^5, u_8u_4^7, \\
& u_7^3, u_7^2u_6u_4, u_7^2u_5^2, u_7^2u_5u_4^2, u_7^2u_4^4, \\
& u_7u_6^2u_5, u_7u_6^2u_4^2, u_7u_6u_5^2u_4, u_7u_6u_5u_4^3, u_7u_6u_4^5, u_7u_5^4, u_7u_5^3u_4^2, u_7u_5^2u_4^4, \\
& u_7u_5u_4^6, u_7u_4^8, u_6^4, u_6^3u_5u_4, u_6^3u_4^3, u_6^2u_5^3, u_6^2u_5^2u_4^2, u_6^2u_5u_4^4, u_6^2u_4^6, \\
& u_6u_5^3u_4^3, u_6u_5^2u_4^5, u_6u_5^4u_4, u_6u_5u_4^7, u_6u_4^9, u_5^6, u_5^5u_4^2, u_5^4u_4^4, \\
& u_5^3u_4^6, u_5^2u_4^8, u_5u_4^{10}, u_4^{12}\rangle \\
M_3^{13} \quad = \quad & \langle u_{16}, u_{15}u_4, u_{14}u_5, u_{14}u_4^2, u_{13}u_6, u_{13}u_5u_4, u_{13}u_4^3, \\
& u_{12}u_7, u_{12}u_6u_4, u_{12}u_5^2, u_{12}u_5u_4^2, u_{12}u_4^4, \\
& u_{11}u_8, u_{11}u_7u_4, u_{11}u_6u_5, u_{11}u_6u_4^2, u_{11}u_5^2u_4, u_{11}u_5u_4^3, u_{11}u_4^5, \\
& u_{10}u_9, u_{10}u_8u_4, u_{10}u_7u_5, u_{10}u_7u_4^2, u_{10}u_6^2, u_{10}u_6u_5u_4, u_{10}u_6u_4^3, \\
& u_{10}u_5^3, u_{10}u_5^2u_4^2, u_{10}u_5u_4^4, u_{10}u_4^6, u_9^2u_4, u_9u_8u_5, u_9u_7u_6, u_9u_8u_4^2, \\
& u_9u_7u_5u_4, u_9u_6^2u_4, u_9u_6u_5^2, u_9u_7u_4^3, u_9u_6u_5u_4^2, u_9u_5^3u_4, u_9u_6u_4^4, \\
& u_9u_5^2u_4^3, u_9u_5u_4^5, u_9u_4^7, u_8^2u_6, u_8^2u_5u_4, u_8^2u_4^3, u_8u_7u_6u_4, u_8u_7u_5^2, \\
& u_8u_7u_5u_4^2, u_8u_7u_4^4, u_8u_6^2u_5, u_8u_6^2u_4^2, u_8u_6u_5^2u_4, u_8u_6u_5u_4^3, u_8u_6u_4^5, \\
& u_8u_5^4, u_8u_5^3u_4^2u_8u_5^2u_4^4, u_8u_5u_4^6, u_8u_7^2, u_8u_4^8, \\
& u_7^3u_4, u_7^2u_6u_5, u_7^2u_6u_4^2, u_7^2u_5^2u_4, u_7^2u_5u_4^3, u_7^2u_4^5, \\
& u_7u_6^3, u_7u_6^2u_5u_4, u_7u_6u_5^3, u_7u_6^2u_4^3, u_7u_6u_5^2u_4^2, u_7u_6u_5u_4^4, u_7u_6u_4^6, \\
& u_7u_5^4u_4, u_7u_5^3u_4^3, u_7u_5^2u_4^5, u_7u_5u_4^7, u_7u_4^9, \\
& u_6^4u_4, u_6^3u_5^2, u_6^3u_5u_4^2, u_6^3u_4^4, u_6^2u_5^3u_4, u_6^2u_5^2u_4^3, u_6^2u_5u_4^5, u_6^2u_4^7, \\
& u_6u_5^5, u_6u_5^4u_4^2, u_6u_5^3u_4^4, u_6u_5^2u_4^6, u_6u_5u_4^8, u_6u_4^{10}, \\
& u_5^6u_4, u_5^5u_4^3, u_5^4u_4^4, u_5^3u_4^7, u_5^2u_4^9, u_5u_4^{11}, u_4^{13}\rangle
\end{aligned}
$$

Appendix B

The quotient submodules $\overline{M_d^l}$ and their generating monomials (that are not total derivatives), where $d = 6, 5, 4, 3$ and $l = 1, 2, 3, ..., 11$ used in classification of $7th$ order evolution equations:

Quotient Submodules with base level 6

$$
\begin{aligned}
\overline{M_6^1} &= \langle \emptyset \rangle \\
\overline{M_6^2} &= \langle u_7^2 \rangle \\
\overline{M_6^3} &= \langle u_7^3 \rangle \\
\overline{M_6^4} &= \langle u_8^2, u_7^4 \rangle \\
\overline{M_6^5} &= \langle u_8^2 u_7, u_7^5 \rangle \\
\overline{M_6^6} &= \langle u_9^2, u_8^3, u_8^2 u_7^2, u_7^6 \rangle \\
\overline{M_6^7} &= \langle u_9^2 u_7, u_8^3 u_7, u_8^2 u_7^3, u_7^7 \rangle \\
\overline{M_6^8} &= \langle u_{10}^2, u_9^2 u_8, u_9^2 u_7^2, u_8^4, u_8^3 u_7^2, u_8^2 u_7^4, u_7^8 \rangle \\
\overline{M_6^9} &= \langle u_{10}^2 u_7, u_9^3, u_9^2 u_8 u_7, u_9^2 u_7^3, u_8^4 u_7, u_8^3 u_7^3, u_8^2 u_7^5, u_7^9 \rangle \\
\overline{M_6^{10}} &= \langle u_{11}^2, u_{10}^2 u_8, u_{10}^2 u_7^2, u_9^3 u_7, u_9^2 u_8^2, u_9^2 u_8 u_7^2, u_9^2 u_7^4, \\
&\quad u_8^5, u_8^4 u_7^2, u_8^3 u_7^4, u_8^2 u_7^6, u_7^{10} \rangle \\
\overline{M_6^{11}} &= \langle u_{11}^2 u_7, u_{10}^2 u_9, u_{10}^2 u_8 u_7, u_{10}^2 u_7^3, \\
&\quad u_9^3 u_8, u_9^3 u_7^2, u_9^2 u_8^2 u_7, u_9^2 u_8 u_7^3, u_8^5 u_7, u_9^2 u_7^5, \\
&\quad u_8^4 u_7^3, u_8^3 u_7^5, u_8^2 u_7^7, u_7^{11} \rangle \\
\overline{M_6^{12}} &= \langle u_{12}^2, u_{11}^2 u_8, u_{11}^2 u_7^2, u_{10}^3, u_{10}^2 u_9 u_7, u_{10}^2 u_8^2, u_{10}^2 u_8 u_7^2, u_{10}^2 u_7^4, \\
&\quad u_9^4, u_9^3 u_8 u_7, u_9^3 u_7^3, u_9^2 u_8^3, u_9^2 u_8^2 u_7^2, u_9^2 u_8 u_7^4, u_9^2 u_7^6, \\
&\quad u_8^6, u_8^5 u_7^2, u_8^4 u_7^4, u_8^3 u_7^6, u_8^2 u_7^8, u_7^{12} \rangle \\
\overline{M_6^{13}} &= \langle u_{12}^2 u_7, u_{11}^2 u_9, u_{11}^2 u_8 u_7, u_{11}^2 u_7^3, \\
&\quad u_{10}^3 u_7, u_{10}^2 u_9 u_8, u_{10}^2 u_9 u_7^2, u_{10}^2 u_8^2 u_7, u_{10}^2 u_8 u_7^3, \\
&\quad u_9^4 u_7, u_9^3 u_8^2, u_9^3 u_8 u_7^2, u_9^3 u_7^4, u_9^2 u_8^3 u_7, u_9^2 u_8^2 u_7^3, u_9^2 u_8 u_7^5, u_9^2 u_7^7, \\
&\quad u_8^6 u_7, u_8^5, u_7^3, u_8^4 u_7^5, u_8^3 u_7^7, u_8^2 u_7^9, u_7^{13} \rangle
\end{aligned}
$$

Quotient Submodules with base level 5

$$
\begin{aligned}
\overline{M_5^1} &= \langle \emptyset \rangle \\
\overline{M_5^2} &= \langle u_6^2 \rangle \\
\overline{M_5^3} &= \langle u_6^3 \rangle \\
\overline{M_5^4} &= \langle u_7^2, u_6^4 \rangle \\
\overline{M_5^5} &= \langle u_7^2 u_6, u_6^5 \rangle \\
\overline{M_5^6} &= \langle u_8^2, u_7^3, u_7^2 u_6^2, u_6^6 \rangle \\
\overline{M_5^7} &= \langle u_8^2 u_6, u_7^3 u_6, u_7^2 u_6^3, u_6^7 \rangle \\
\overline{M_5^8} &= \langle u_9^2, u_8^2 u_7, u_8^2 u_6^2, u_7^4, u_7^3 u_6^2, u_7^2 u_6^4, u_6^8 \rangle
\end{aligned}
$$

$$
\begin{aligned}
\overline{M_5^9} &= \langle u_9^2u_6, u_8^3, u_8^2u_7u_6, u_8^2u_6^3,\\
&\quad u_7^4u_6, u_7^3u_6^3, u_7^2u_6^5, u_6^9\rangle\\
\overline{M_5^{10}} &= \langle u_{10}^2, u_9^2u_7, u_9^2u_6^2, u_8^3u_6, u_8^2u_7^2, u_8^2u_7u_6^2, u_8^2u_6^4,\\
&\quad u_7^5, u_7^4u_6^2, u_7^3u_6^4, u_7^2u_6^6, u_6^{10}\rangle\\
\overline{M_5^{11}} &= \langle u_{10}^2u_6, u_9^2u_8, u_9^2u_7u_6, u_9^2u_6^3,\\
&\quad u_8^3u_7, u_8^3u_6^2, u_8^2u_7^2u_6, u_8^2u_7u_6^3, u_8^2u_6^5,\\
&\quad u_7^5u_6, u_7^4u_6^3, u_7^3u_6^5, u_7^2u_6^7, u_6^{11}\rangle\\
\overline{M_5^{12}} &= \langle u_{11}^2, u_{10}^2u_7, u_{10}^2u_6^2, u_9^3, u_9^2u_8u_6, u_9^2u_7^2, u_9^2u_7u_6^2, u_9^2u_6^4\\
&\quad u_8^4, u_8^3u_7u_6, u_8^3u_6^3, u_8^2u_7^3, u_8^2u_7^2u_6^2, u_8^2u_7u_6^4, u_8^2u_6^6,\\
&\quad u_7^6, u_7^5u_6^2, u_7^4u_6^4, u_7^3u_6^6, u_7^2u_6^8, u_6^{12}\rangle\\
\overline{M_5^{13}} &= \langle u_{10}^2u_8, u_{10}^2u_7u_6, u_{10}^2u_6^3, u_9^3u_6, u_9^2u_8u_7, u_9^2u_8u_6^2,\\
&\quad u_9^2u_7^2u_6, u_9^2u_7u_6^3, u_9^2u_6^5, u_8^4u_6, u_8^3u_7^2, u_8^3u_7u_6^2, u_8^3u_6^4\\
&\quad u_8^2u_7^3u_6, u_8^2u_7^2u_6^3, u_8^2u_7u_6^5, u_8^2u_6^7,\\
&\quad u_7^6u_6, u_7^5u_6^3, u_7^4u_6^5, u_7^3u_6^7, u_7^2u_6^9, u_6^{13}\rangle
\end{aligned}
$$

Quotient Submodules with base level 4

$$
\begin{aligned}
\overline{M_4^1} &= \langle\emptyset\rangle\\
\overline{M_4^2} &= \langle u_5^2\rangle\\
\overline{M_4^3} &= \langle u_5^3\rangle\\
\overline{M_4^4} &= \langle u_6^2, u_5^4\rangle\\
\overline{M_4^5} &= \langle u_6^2u_5, u_5^5\rangle\\
\overline{M_4^6} &= \langle u_7^2, u_6^3, u_6^2u_5^2, u_5^6\rangle\\
\overline{M_4^7} &= \langle u_7^2u_5, u_6^3u_5, u_6^2u_5^3, u_5^7\rangle\\
\overline{M_4^8} &= \langle u_8^2, u_7^2u_6, u_7^2u_5^2, u_6^4, u_6^3u_5^2, u_6^2u_5^4, u_5^8\rangle\\
\overline{M_4^9} &= \langle u_8^2u_5, u_7^3, u_7^2u_6u_5, u_7^2u_5^3, u_6^4u_5, u_6^3u_5^3, u_6^2u_5^5, u_5^9\rangle\\
\overline{M_4^{10}} &= \langle u_9^2, u_8^2u_6, u_8^2u_5^2, u_7^3u_5, u_7^2u_6^2, u_7^2u_6u_5^2, u_7^2u_5^4,\\
&\quad u_6^5, u_6^4u_5^2, u_6^3u_5^4, u_6^2u_5^6, u_5^{10}\rangle\\
\overline{M_4^{11}} &= \langle u_9^2u_5, u_8^2u_7, u_8^2u_6u_5, u_8^2u_5^3,\\
&\quad u_7^3u_6, u_7^3u_5^2, u_7^2u_6^2u_5, u_7^2u_6u_5^3, u_7^2u_5^5, u_6^5u_5, u_6^4u_5^3,\\
&\quad u_6^3u_5^5, u_6^2u_5^7, u_5^{11}\rangle\\
\overline{M_4^{12}} &= \langle u_{10}^2, u_9^2u_6, u_9^2u_5^2, u_8^3, u_8^2u_7u_5, u_8^2u_6^2, u_8^2u_6u_5^2, u_8^2u_5^4,\\
&\quad u_7^4, u_7^3u_6u_5, u_7^3u_5^3, u_7^2u_6^3, u_7^2u_6^2u_5^2, u_7^2u_6u_5^4, u_7^2u_5^6,\\
&\quad u_6^6, u_6^5u_5^2, u_6^4u_5^4, u_6^3u_5^6, u_6^2u_5^8, u_5^{12}\rangle\\
\overline{M_4^{13}} &= \langle u_{10}^2u_5, u_9^2u_7, u_9^2u_6u_5, u_9^2u_5^3, u_8^3u_5, u_8^2u_7u_6, u_8^2u_7u_5^2,\\
&\quad u_8^2u_6^2u_5, u_8^2u_6u_5^3, u_8^2u_5^5, u_7^4u_5, u_7^3u_6^2, u_7^3u_6u_5^2, u_7^3u_5^4\\
&\quad u_7^2u_6^3u_5, u_7^2u_6^2u_5^3, u_7^2u_6u_5^5, u_7^2u_5^7,\\
&\quad u_6^6u_5, u_6^5u_5^3, u_6^4u_5^5, u_6^3u_5^7, u_6^2u_5^9, u_5^{13}\rangle
\end{aligned}
$$

Quotient Submodules with base level 3

$$\begin{aligned}
\overline{M_3^1} &= \langle \emptyset \rangle \\
\overline{M_3^2} &= \langle u_4^2 \rangle \\
\overline{M_3^3} &= \langle u_4^3 \rangle \\
\overline{M_3^4} &= \langle u_5^2, u_4^4 \rangle \\
\overline{M_3^5} &= \langle u_5^2 u_4, u_4^5 \rangle \\
\overline{M_3^6} &= \langle u_6^2, u_5^3, u_5^2 u_4^2, u_4^6 \rangle \\
\overline{M_3^7} &= \langle u_6^2 u_4, u_5^3 u_4, u_5^2 u_4^3, u_4^7 \rangle \\
\overline{M_3^8} &= \langle u_7^2, u_6^2 u_5, u_6^2 u_4^2, u_5^4, u_5^3 u_4^2, u_5^2 u_4^4, u_4^8 \rangle \\
\overline{M_3^9} &= \langle u_7^2 u_4, u_6^3, u_6^2 u_5 u_4, u_6^2 u_4^3, u_5^4 u_4, u_5^3 u_4^3, u_5^2 u_4^5, u_4^9 \rangle \\
\overline{M_3^{10}} &= \langle u_8^2, u_7^2 u_5, u_7^2 u_4^2, u_6^3 u_4, u_6^2 u_5^2, u_6^2 u_5 u_4^2, u_6^2 u_4^4, \\
&\quad u_5^5, u_5^4 u_4^2, u_5^3 u_4^4, u_5^2 u_4^6, u_4^{10} \rangle \\
\overline{M_3^{11}} &= \langle u_8^2 u_4, u_7^2 u_6, u_7^2 u_5 u_4, u_7^2 u_4^3, u_6^3 u_5, u_6^3 u_4^2, u_6^2 u_5^2 u_4, u_6^2 u_5 u_4^3, u_6^2 u_4^5, \\
&\quad u_5^5 u_4, u_5^4 u_4^3, u_5^3 u_4^5, u_5^2 u_4^7, u_4^{11} \rangle \\
\overline{M_3^{12}} &= \langle u_9^2, u_8^2 u_5, u_8^2 u_4^2, u_7^3, u_7^2 u_6 u_4, u_7^2 u_5^2, u_7^2 u_5 u_4^2, u_7^2 u_4^4, \\
&\quad u_6^4, u_6^3 u_5 u_4, u_6^3 u_4^3, u_6^2 u_5^3, u_6^2 u_5^2 u_4^2, u_6^2 u_5 u_4^4, u_6^2 u_4^6, \\
&\quad u_5^6, u_5^5 u_4^2, u_5^4 u_4^4, u_5^3 u_4^6, u_5^2 u_4^8, u_4^{12} \rangle \\
\overline{M_3^{13}} &= \langle u_9^2 u_4, u_8^2 u_6, u_8^2 u_5 u_4, u_8^2 u_4^3, \\
&\quad u_7^3 u_4, u_7^2 u_6 u_5, u_7^2 u_6 u_4^2, u_7^2 u_5^2 u_4, u_7^2 u_5 u_4^3, u_7^2 u_4^5, \\
&\quad u_6^4 u_4, u_6^3 u_5^2, u_6^3 u_5 u_4^2, u_6^3 u_4^4, u_6^2 u_5^3 u_4, u_6^2 u_5^2 u_4^3, u_6^2 u_5 u_4^5, u_6^2 u_4^7, \\
&\quad u_5^6 u_4, u_5^5 u_4^3, u_5^4 u_4^4, u_5^3 u_4^7, u_5^2 u_4^9, u_4^{13} \rangle
\end{aligned}$$

Appendix C

The submodules M_d^l and their generating monomials with level $l(m_i)$, where $d = 8, 7$ and $l = 1, 2, 3, ..., 14$ used in classification of $9th$ order evolution equations:

Submodules with base level 8

$$
\begin{aligned}
M_8^1 &= \langle u_9 \rangle \\
M_8^2 &= \langle u_{10}, u_9^2 \rangle \\
M_8^3 &= \langle u_{11}, u_{10}u_9, u_9^3 \rangle \\
M_8^4 &= \langle u_{12}, u_{11}u_9, u_{10}^2, u_{10}u_9^2, u_9^4 \rangle \\
M_8^5 &= \langle u_{13}, u_{12}u_9, u_{11}u_{10}, u_{11}u_9^2, u_{10}^2u_9, u_{10}u_9^3, u_9^5 \rangle \\
M_8^6 &= \langle u_{14}, u_{13}u_9, u_{12}u_{10}, u_{12}u_9^2, u_{11}^2, u_{11}u_{10}u_9, u_{11}u_9^3, \\
&\quad u_{10}^3, u_{10}^2u_9^2, u_{10}u_9^4, u_9^6 \rangle \\
M_8^7 &= \langle u_{15}, u_{14}u_9, u_{13}u_{10}, u_{13}u_9^2, u_{12}u_{11}, u_{12}u_{10}u_9, u_{12}u_9^3, \\
&\quad u_{11}^2u_9, u_{11}u_{10}^2, u_{11}u_{10}u_9^2, u_{11}u_9^4, \\
&\quad u_{10}^3u_9, u_{10}^2u_9^3, u_{10}u_9^5, u_9^7 \rangle \\
M_8^8 &= \langle u_{16}, u_{15}u_9, u_{14}u_{10}, u_{14}u_9^2, u_{13}u_{11}, u_{13}u_{10}u_9, u_{13}u_9^3, \\
&\quad u_{12}^2, u_{12}u_{11}u_9, u_{12}u_{10}^2, u_{12}u_{10}u_9^2, u_{12}u_9^4, \\
&\quad u_{11}^2u_{10}, u_{11}^2u_9^2, u_{11}u_{10}^2u_9, u_{11}u_{10}u_9^3, u_{11}u_9^5, \\
&\quad u_{10}^4, u_{10}^3u_9^2, u_{10}^2u_9^4, u_{10}u_9^6, u_9^8 \rangle \\
M_8^9 &= \langle u_{17}, u_{16}u_9, u_{15}u_{10}, u_{15}u_9^2, \\
&\quad u_{14}u_{11}, u_{14}u_{10}u_9, u_{14}u_9^3, \\
&\quad u_{13}u_{12}, u_{13}u_{11}u_9, u_{13}u_{10}^2, u_{13}u_{10}u_9^2, u_{13}u_9^4, \\
&\quad u_{12}^2u_9, u_{12}u_{11}u_{10}, u_{12}u_{11}u_9^2, u_{12}u_{10}^2u_9, u_{12}u_{10}u_9^3, u_{12}u_9^5, \\
&\quad u_{11}^3, u_{11}^2u_{10}u_9, u_{11}^2u_9^3, u_{11}u_{10}^3, u_{11}u_{10}^2u_9^2, u_{11}u_{10}u_9^4, u_{11}u_9^6, \\
&\quad u_{10}^4u_9, u_{10}^3u_9^3, u_{10}^2u_9^5, u_{10}u_9^7, u_9^9 \rangle \\
M_8^{10} &= \langle u_{18}, u_{17}u_9, u_{16}u_{10}, u_{16}u_9^2, u_{15}u_{11}, u_{15}u_{10}u_9, u_{15}u_9^3, \\
&\quad u_{14}u_{12}, u_{14}u_{11}u_9, u_{14}u_{10}u_9^2, u_{14}u_{10}^5u_9^4, u_{14}u_{10}^2, \\
&\quad u_{13}^2, u_{13}u_{12}u_9, u_{13}u_{11}u_{10}, u_{13}u_{11}u_9^2, u_{13}u_{10}^2u_9, u_{13}u_{10}u_9^3, u_{13}u_9^5, \\
&\quad u_{12}^2u_{10}, u_{12}^2u_9^2, u_{12}u_{11}^2, u_{12}u_{11}u_{10}u_9, u_{12}u_{11}u_9^3, \\
&\quad u_{12}u_{10}^3, u_{12}u_{10}^2u_9^2, u_{12}u_{10}u_9^4, u_{12}u_9^6, \\
&\quad u_{11}^3u_9, u_{11}^2u_{10}^2, u_{11}^2u_{10}u_9^2, u_{11}^2u_9^4, u_{11}u_{10}^3u_9, u_{11}u_{10}^2u_9^3, \\
&\quad u_{11}u_{10}u_9^5, u_{11}u_9^7, \\
&\quad u_{10}^5, u_{10}^4u_9^2, u_{10}^3u_9^4, u_{10}^2u_9^6, u_{10}u_9^8, u_9^{10} \rangle \\
M_8^{11} &= \langle u_{19}, u_{18}u_9, u_{17}u_{10}, u_{17}u_9^2, u_{16}u_{11}, u_{16}u_{10}u_9, u_{16}u_9^3, \\
&\quad u_{15}u_{12}, u_{15}u_{11}u_9, u_{15}u_{10}^2, u_{15}u_{10}u_9^2, u_{15}u_9^4, \\
&\quad u_{14}u_{13}, u_{14}u_{12}u_9, u_{14}u_{11}u_{10}, u_{14}u_{11}u_9^2, u_{14}u_{10}^2u_9, u_{14}u_{10}u_9^3, u_{14}u_9^5, \\
&\quad u_{13}^2u_9, u_{13}u_{12}u_{10}, u_{13}u_{12}u_9^2, u_{13}u_{11}^2, u_{13}u_{11}u_{10}u_9, u_{13}u_{11}u_9^3, \\
&\quad u_{13}u_{10}^3, u_{13}u_{10}^2u_9^2, u_{13}u_{10}u_9^4, u_{13}u_9^6, \\
&\quad u_{12}^2u_{11}, u_{12}^2u_{10}u_9, u_{12}^2u_9^3, u_{12}u_{11}^2u_9,
\end{aligned}
$$

$$
\begin{aligned}
& u_{12}u_{11}u_{10}^2, u_{12}u_{11}u_{10}u_9^2, u_{12}u_{11}u_9^4, \\
& u_{12}u_{10}^3u_9, u_{12}u_{10}^2u_9^3, u_{12}u_{10}u_9^5, u_{12}u_9^7, \\
& u_{11}^3u_{10}, u_{11}^3u_9^2, u_{11}^2u_{10}^2u_9, u_{11}^2u_{10}u_9^3, u_{11}^2u_9^5, \\
& u_{11}u_{10}^4, u_{11}u_{10}^3u_9^2, u_{11}u_{10}^2u_9^4, u_{11}u_{10}u_9^6, u_{11}u_9^8, \\
& u_{10}^5u_9, u_{10}^4u_9^3, u_{10}^3u_9^5, u_{10}^2u_9^7, u_{10}u_9^9, u_9^{11}\rangle \\
M_8^{12} \quad = \quad & \langle u_{20}, u_{19}u_9, u_{18}u_{10}, u_{18}u_9^2, u_{17}u_{11}, u_{17}u_{10}u_9, u_{17}u_9^3, \\
& u_{16}u_{12}, u_{16}u_{11}u_9, u_{16}u_{10}^2, u_{16}u_{10}u_9^2, u_{16}u_9^4, \\
& u_{15}u_{13}, u_{15}u_{12}u_9, u_{15}u_{11}u_{10}, u_{15}u_{11}u_9^2, u_{15}u_{10}^2u_9, u_{15}u_{10}u_9^3, u_{15}u_9^5, \\
& u_{14}^2, u_{14}u_{13}u_9, u_{14}u_{12}u_{10}, u_{14}u_{12}u_9^2, u_{14}u_{11}^2, u_{14}u_{11}u_{10}u_9, u_{14}u_{11}u_9^3, \\
& u_{14}u_{10}^3, u_{14}u_{10}^2u_9^2, u_{14}u_{10}u_9^4, u_{14}u_9^6, \\
& u_{13}^2u_{10}, u_{13}^2u_9^2, u_{13}u_{12}u_{11}, u_{13}u_{12}u_{10}u_9, u_{13}u_{12}u_9^3, \\
& u_{13}u_{11}^2u_9, u_{13}u_{11}u_{10}^2, u_{13}u_{11}u_{10}u_9^2, u_{13}u_{11}u_9^4, \\
& u_{13}u_{10}^3u_9, u_{13}u_{10}^2u_9^3, u_{13}u_{10}u_9^5, u_{13}u_9^7, \\
& u_{12}^3, u_{12}^2u_{11}u_9, u_{12}^2u_{10}^2, u_{12}^2u_{10}u_9^2, u_{12}^2u_9^4, \\
& u_{12}u_{11}^2u_9^2, u_{12}u_{11}^2u_{10}, u_{12}u_{11}u_{10}^2u_9, u_{12}u_{11}u_{10}u_9^3, u_{12}u_{11}u_9^5, \\
& u_{12}u_{10}^4, u_{12}u_{10}^3u_9^2, u_{12}u_{10}^2u_9^4, u_{12}u_{10}u_9^6, u_{12}u_9^8, \\
& u_{11}^4, u_{11}^3u_{10}u_9, u_{11}^3u_9^3, u_{11}^2u_{10}^3, u_{11}^2u_{10}^2u_9^2, u_{11}^2u_{10}u_9^4, u_{11}^2u_9^6, \\
& u_{11}u_{10}^4u_9, u_{11}u_{10}^3u_9^3, u_{11}u_{10}^2u_9^5, u_{11}u_{10}u_9^7, u_{11}u_9^9, \\
& u_{10}^6, u_{10}^5u_9^2, u_{10}^4u_9^4, u_{10}^3u_9^6, u_{10}^2u_9^8, u_{10}u_9^{10}, u_9^{12}\rangle \\
M_8^{13} \quad = \quad & \langle u_{21}, u_{20}u_9, u_{19}u_{10}, u_{19}u_9^2, u_{18}u_{11}, u_{18}u_{10}u_9, u_{18}u_9^3, \\
& u_{17}u_{12}, u_{17}u_{11}u_9, u_{17}u_{10}^2, u_{17}u_{10}u_9^2, u_{17}u_9^4, \\
& u_{16}u_{13}, u_{16}u_{12}u_9, u_{16}u_{11}u_{10}, u_{16}u_{11}u_9^2, u_{16}u_{10}^2u_9, u_{16}u_{10}u_9^3, u_{16}u_9^5, \\
& u_{15}u_{14}, u_{15}u_{13}u_9, u_{15}u_{12}u_{10}, u_{15}u_{11}^2, u_{15}u_{12}u_9^2, \\
& u_{15}u_{11}u_{10}u_9, u_{15}u_{11}u_9^3, u_{15}u_{10}^3, u_{15}u_{10}^2u_9^2, u_{15}u_{10}u_9^4, u_{15}u_9^6, u_{14}^2u_9, \\
& u_{14}u_{13}u_{10}, u_{14}u_{13}u_9^2, u_{14}u_{12}u_{11}, u_{14}u_{12}u_{10}u_9, u_{14}u_{12}u_9^3, \\
& u_{14}u_{11}^2u_9, u_{14}u_{11}u_{10}^2, u_{14}u_{11}u_{10}u_9^2, u_{14}u_{11}u_9^4, u_{14}u_{10}^3u_9, \\
& u_{14}u_{10}^2u_9^3, u_{14}u_{10}u_9^5, u_{14}u_9^7, \\
& u_{13}^2u_{11}, u_{13}^2u_{10}u_9, u_{13}^2u_9^3, u_{13}u_{12}^2, \\
& u_{13}u_{12}u_{11}u_9, u_{13}u_{12}u_{10}^2, u_{13}u_{12}u_{10}u_9^2, u_{13}u_{12}u_9^4, \\
& u_{13}u_{11}^2u_{10}, u_{13}u_{11}^2u_9^2, u_{13}u_{11}u_{10}^2u_9, u_{13}u_{11}u_{10}u_9^3, u_{13}u_{11}u_9^5, \\
& u_{13}u_{10}^4, u_{13}u_{10}^3u_9^2, u_{13}u_{10}^2u_9^4, u_{13}u_{10}u_9^6, u_{13}u_9^8, \\
& u_{12}^3u_9, u_{12}^2u_{11}u_{10}, u_{12}^2u_{11}u_9^2, u_{12}^2u_{10}^2u_9, u_{12}^2u_{10}u_9^3, u_{12}^2u_9^5, \\
& u_{12}u_{11}^3, u_{12}u_{11}^2u_{10}u_9, u_{12}u_{11}^2u_9^3, \\
& u_{12}u_{11}u_{10}^3, u_{12}u_{11}u_{10}^2u_9^2, u_{12}u_{11}u_{10}u_9^4, u_{12}u_{11}u_9^6, \\
& u_{12}u_{10}^4u_9, u_{12}u_{10}^3u_9^3, u_{12}u_{10}^2u_9^5, u_{12}u_{10}u_9^7, u_{12}u_9^9, \\
& u_{11}^4u_9, u_{11}^3u_{10}^2, u_{11}^3u_{10}u_9^2, u_{11}^3u_9^4, \\
& u_{11}^2u_{10}^3u_9, u_{11}^2u_{10}^2u_9^3, u_{11}^2u_{10}u_9^5, u_{11}^2u_9^7, \\
& u_{11}u_{10}^5, u_{11}u_{10}^4u_9^2, u_{11}u_{10}^3u_9^4, u_{11}u_{10}^2u_9^6, u_{11}u_{10}u_9^8, u_{11}u_9^{10}, \\
& u_{10}^6u_9, u_{10}^5u_9^3, u_{10}^4u_9^5, u_{10}^3u_9^7, u_{10}^2u_9^9, u_{10}u_9^{11}, u_9^{13}\rangle \\
M_8^{14} \quad = \quad & \langle u_{22}, u_{21}u_9, u_{20}u_{10}, u_{20}u_9^2, u_{19}u_{11}, u_{19}u_{10}u_9, u_{19}u_9^3, \\
& u_{18}u_{12}, u_{18}u_{11}u_9, u_{18}u_{10}^2, u_{18}u_{10}u_9^2, u_{18}u_9^4, \\
& u_{17}u_{13}, u_{17}u_{12}u_9, u_{17}u_{11}u_{10}, u_{17}u_{11}u_9^2, u_{17}u_{10}^2u_9, u_{17}u_{10}u_9^3, u_{17}u_9^5,
\end{aligned}
$$

$u_{16}u_{14}, u_{16}u_{13}u_9, u_{16}u_{12}u_{10}, u_{16}u_{12}u_9^2, u_{16}u_{11}^2,$
$u_{16}u_{11}u_{10}u_9, u_{16}u_{11}u_9^3, u_{16}u_{10}^3, u_{16}u_{10}^2u_9^2, u_{16}u_{10}u_9^4, u_{16}u_9^6,$
$u_{15}^2, u_{15}u_{14}u_9, u_{15}u_{13}u_{10}, u_{15}u_{13}u_9^2,$
$u_{15}u_{12}u_{11}, u_{15}u_{12}u_{10}u_9, u_{15}u_{12}u_9^3,$
$u_{15}u_{11}^2u_9, u_{15}u_{11}u_{10}^2, u_{15}u_{11}u_{10}u_9^2, u_{15}u_{11}u_9^4,$
$u_{15}u_{10}^3u_9, u_{15}u_{10}^2u_9^3, u_{15}u_{10}u_9^5, u_{15}u_9^7,$
$u_{14}^2u_{10}, u_{14}^2u_9^2, u_{14}u_{13}u_{11}, u_{14}u_{13}u_{10}u_9, u_{14}u_{13}u_9^3,$
$u_{14}u_{12}^2, u_{14}u_{12}u_{11}u_9, u_{14}u_{12}u_{10}^2, u_{14}u_{12}u_{10}u_9^2, u_{14}u_{12}u_9^4,$
$u_{14}u_{11}^2u_{10}, u_{14}u_{11}^2u_9^2, u_{14}u_{11}u_{10}^2u_9, u_{14}u_{11}u_{10}u_9^3, u_{14}u_{11}u_9^5,$
$u_{14}u_{10}^4, u_{14}u_{10}^3u_9^2, u_{14}u_{10}^2u_9^4, u_{14}u_{10}u_9^6, u_{14}u_9^8,$
$u_{13}^2u_{12}, u_{13}^2u_{11}u_9, u_{13}^2u_{10}^2, u_{13}^2u_{10}u_9^2, u_{13}^2u_9^4,$
$u_{13}u_{12}^2u_9, u_{13}u_{12}u_{11}u_{10}, u_{13}u_{12}u_{11}u_9^2,$
$u_{13}u_{12}u_{10}^2u_9, u_{13}u_{12}u_{10}u_9^3, u_{13}u_{12}u_9^5,$
$u_{13}u_{11}^3, u_{13}u_{11}^2u_{10}u_9, u_{13}u_{11}^2u_9^3,$
$u_{13}u_{11}u_{10}^3, u_{13}u_{11}u_{10}^2u_9^2, u_{13}u_{11}u_{10}u_9^4, u_{13}u_{11}u_9^6,$
$u_{13}u_{10}^4u_9, u_{13}u_{10}^3u_9^3, u_{13}u_{10}^2u_9^5, u_{13}u_{10}u_9^7, u_{13}u_9^9,$
$u_{12}^3u_{10}, u_{12}^3u_9^2, u_{12}^2u_{11}^2, u_{12}^2u_{11}u_{10}u_9, u_{12}^2u_{11}u_9^3,$
$u_{12}^2u_{10}^3, u_{12}^2u_{10}^2u_9^2, u_{12}^2u_{10}u_9^4, u_{12}^2u_9^6,$
$u_{12}u_{11}^3u_9, u_{12}u_{11}^2u_{10}^2, u_{12}u_{11}^2u_{10}u_9^2, u_{12}u_{11}^2u_9^4,$
$u_{12}u_{11}u_{10}^3u_9, u_{12}u_{11}u_{10}^2u_9^3, u_{12}u_{11}u_{10}u_9^5, u_{12}u_{11}u_9^7,$
$u_{12}u_{10}^5, u_{12}u_{10}^4u_9^2, u_{12}u_{10}^3u_9^4, u_{12}u_{10}^2u_9^6, u_{12}u_{10}u_9^8, u_{12}u_9^{10},$
$u_{11}^4u_{10}, u_{11}^4u_9^2, u_{11}^3u_{10}^2u_9, u_{11}^3u_{10}u_9^3, u_{11}^3u_9^5,$
$u_{11}^2u_{10}^4, u_{11}^2u_{10}^3u_9^2, u_{11}^2u_{10}^2u_9^4, u_{11}^2u_{10}u_9^6, u_{11}^2u_9^8,$
$u_{11}u_{10}^5u_9, u_{11}u_{10}^4u_9^3, u_{11}u_{10}^3u_9^5, u_{11}u_{10}^2u_9^7, u_{11}u_{10}u_9^9, u_{11}u_9^{11},$
$u_{10}^7, u_{10}^6u_9^2, u_{10}^5u_9^4, u_{10}^4u_9^6, u_{10}^3u_9^8, u_{10}^2u_9^{10}, u_{10}u_9^{12}, u_9^{14}\rangle$

Submodules with base level 7

$$
\begin{aligned}
M_7^1 &= \langle u_8\rangle \\
M_7^2 &= \langle u_9, u_8^2\rangle \\
M_7^3 &= \langle u_{10}, u_9u_8, u_8^3\rangle \\
M_7^4 &= \langle u_{11}, u_{10}u_8, u_9^2, u_9u_8^2, u_8^4\rangle \\
M_7^5 &= \langle u_{12}, u_{11}u_8, u_{10}u_9, u_{10}u_8^2, u_9^2u_8, u_9u_8^3, u_8^5\rangle \\
M_7^6 &= \langle u_{13}, u_{12}u_8, u_{11}u_9, u_{11}u_8^2, u_{10}u_9u_8, u_{10}u_8^3, \\
&\quad u_9^3, u_9^2u_8^2, u_9u_8^4, u_8^6\rangle \\
M_7^7 &= \langle u_{14}, u_{13}u_8, u_{12}u_9, u_{12}u_8^2, u_{11}u_{10}, u_{11}u_9u_8, u_{11}u_8^3, \\
&\quad u_{10}^2u_8, u_{10}u_9^2, u_{10}u_9u_8^2, u_{10}u_8^4, u_9^3u_8, u_9^2u_8^3, u_9u_8^5, u_8^7\rangle \\
M_7^8 &= \langle u_{15}, u_{14}u_8, u_{13}u_9, u_{13}u_8^2, u_{12}u_{10}, u_{12}u_9u_8, u_{12}u_8^3, \\
&\quad u_{11}^2, u_{11}u_{10}u_8, u_{11}u_9^2, u_{11}u_9u_8^2, u_{11}u_8^4, \\
&\quad u_{10}^2u_9, u_{10}^2u_8^2, u_{10}u_9^2u_8, u_{10}u_9u_8^3, u_{10}u_8^5, \\
&\quad u_9^4, u_9^3u_8^2, u_9^2u_8^4, u_9u_8^6, u_8^8\rangle \\
M_7^9 &= \langle u_{16}, u_{15}u_8, u_{14}u_9, u_{14}u_8^2, u_{13}u_{10}, u_{13}u_9u_8, u_{13}u_8^3,
\end{aligned}
$$

$$
\begin{aligned}
& u_{12}u_{11}, u_{12}u_{10}u_8, u_{12}u_9^2, u_{12}u_9u_8^2, u_{12}u_8^4,\\
& u_{11}^2u_8, u_{11}u_{10}u_9, u_{11}u_{10}u_8^2, u_{11}u_9^2u_8, u_{11}u_9u_8^3, u_{11}u_8^5,\\
& u_{10}^3, u_{10}^2u_9u_8, u_{10}^2u_8^3, u_{10}u_9^3, u_{10}u_9^2u_8^2, u_{10}u_9u_8^4, u_{10}u_8^6,\\
& u_9^4u_8, u_9^3u_8^3, u_9^2u_8^5, u_9u_8^7, u_8^9\rangle\\
M_7^{10} \quad = \quad & \langle u_{17}, u_{16}u_8, u_{15}u_9, u_{15}u_8^2, u_{14}u_{10}, u_{14}u_9u_8, u_{14}u_8^3,\\
& u_{13}u_{11}, u_{13}u_{10}u_8, u_{13}u_9u_8^2, u_{13}u_9^2, u_{13}u_8^4,\\
& u_{12}^2, u_{12}u_{11}u_8, u_{12}u_{10}u_9, u_{12}u_{10}u_8^2, u_{12}u_9^2u_8, u_{12}u_9u_8^3, u_{12}u_8^5,\\
& u_{11}^2u_9, u_{11}^2u_8^2, u_{11}u_{10}^2, u_{11}u_{10}u_9u_8, u_{11}u_{10}u_8^3,\\
& u_{11}u_9^3, u_{11}u_9^2u_8^2, u_{11}u_9u_8^4, u_{11}u_8^6,\\
& u_{10}^3u_8, u_{10}^2u_9^2, u_{10}^2u_9u_8^2, u_{10}^2u_8^4,\\
& u_{10}u_9^3u_8, u_{10}u_9^2u_8^3, u_{10}u_9u_8^5, u_{10}u_8^7,\\
& u_9^5, u_9^4u_8^2, u_9^3u_8^4, u_9^2u_8^6, u_9u_8^8, u_8^{10}\rangle\\
M_7^{11} \quad = \quad & \langle u_{18}, u_{17}u_8, u_{16}u_9, u_{16}u_8^2, u_{15}u_{10}, u_{15}u_9u_8, u_{15}u_8^3,\\
& u_{14}u_{11}, u_{14}u_{10}u_8, u_{14}u_9^2, u_{14}u_9u_8^2, u_{14}u_8^4,\\
& u_{13}u_{12}, u_{13}u_{11}u_8, u_{13}u_{10}u_9, u_{13}u_{10}u_8^2, u_{13}u_9^2u_8, u_{13}u_9u_8^3, u_{13}u_8^5,\\
& u_{12}^2u_8, u_{12}u_{11}u_9, u_{12}u_{11}u_8^2, u_{12}u_{10}^2, u_{12}u_{10}u_9u_8, u_{12}u_{10}u_8^3,\\
& u_{12}u_9^3, u_{12}u_9^2u_8^2, u_{12}u_9u_8^4, u_{12}u_8^6,\\
& u_{11}^2u_{10}, u_{11}^2u_9u_8, u_{11}^2u_8^3, u_{11}u_{10}^2u_8,\\
& u_{11}u_{10}u_9^2, u_{11}u_{10}u_9u_8^2, u_{11}u_{10}u_8^4, u_{11}u_9^3u_8, u_{11}u_9^2u_8^3,\\
& u_{11}u_9u_8^5, u_{11}u_8^7, u_{10}^3u_9, u_{10}^3u_8^2,\\
& u_{10}^2u_9^2u_8, u_{10}^2u_9u_8^3, u_{10}^2u_8^5,\\
& u_{10}u_9^4, u_{10}u_9^3u_8^2, u_{10}u_9^2u_8^4, u_{10}u_9u_8^6, u_{10}u_8^8,\\
& u_9^5u_8, u_9^4u_8^3, u_9^3u_8^5, u_9^2u_8^7, u_9u_8^9, u_8^{11}\rangle\\
M_7^{12} \quad = \quad & \langle u_{19}, u_{18}u_8, u_{17}u_9, u_{17}u_8^2, u_{16}u_{10}, u_{16}u_9u_8, u_{16}u_8^3,\\
& u_{15}u_{11}, u_{15}u_{10}u_8, u_{15}u_9^2, u_{15}u_9u_8^2, u_{15}u_8^4,\\
& u_{14}u_{12}, u_{14}u_{11}u_8, u_{14}u_{10}u_9, u_{14}u_{10}u_8^2, u_{14}u_9^2u_8, u_{14}u_9u_8^3, u_{14}u_8^5,\\
& u_{13}u_{12}u_8, u_{13}u_{11}u_9, u_{13}u_{11}u_8^2, u_{13}u_{10}^2, u_{13}u_{10}u_9u_8, u_{13}u_{10}u_8^3,\\
& u_{13}u_9^3, u_{13}u_9^2u_8^2, u_{13}u_9u_8^4, u_{13}u_8^6,\\
& u_{12}^2u_9, u_{12}^2u_8^2, u_{12}u_{11}u_{10}, u_{12}u_{11}u_9u_8, u_{12}u_{11}u_8^3,\\
& u_{12}u_{10}^2u_8, u_{12}u_{10}u_9^2, u_{12}u_{10}u_9u_8^2, u_{12}u_{10}u_8^4,\\
& u_{12}u_9^3u_8, u_{12}u_9^2u_8^3, u_{12}u_9u_8^5, u_{12}u_8^7,\\
& u_{11}^3, u_{11}^2u_{10}u_8, u_{11}^2u_9^2, u_{11}^2u_9u_8^2, u_{11}^2u_8^4,\\
& u_{11}u_{10}^2u_9, u_{11}u_{10}^2u_8^2, u_{11}u_{10}u_9^2u_8, u_{11}u_{10}u_9u_8^3, u_{11}u_{10}u_8^5,\\
& u_{11}u_9^4, u_{11}u_9^3u_8^2, u_{11}u_9^2u_8^4, u_{11}u_9u_8^6, u_{11}u_8^8,\\
& u_{10}^4, u_{10}^3u_9u_8, u_{10}^3u_8^3, u_{10}^2u_9^3, u_{10}^2u_9^2u_8^2, u_{10}^2u_9u_8^4, u_{10}^2u_8^6,\\
& u_{10}u_9^4u_8, u_{10}u_9^3u_8^3, u_{10}u_9^2u_8^5, u_{10}u_9u_8^7, u_{10}u_8^9,\\
& u_9^6, u_9^5u_8^2, u_9^4u_8^4, u_9^3u_8^6, u_9^2u_8^8, u_9u_8^{10}, u_8^{12}\rangle\\
M_7^{13} \quad = \quad & \langle u_{20}, u_{19}u_8, u_{18}u_9, u_{18}u_8^2, u_{17}u_{10}, u_{17}u_9u_8, u_{17}u_8^3,\\
& u_{16}u_{11}, u_{16}u_{10}u_8, u_{16}u_9^2, u_{16}u_9u_8^2, u_{16}u_8^4,\\
& u_{15}u_{12}, u_{15}u_{11}u_8, u_{15}u_{10}u_9, u_{15}u_{10}u_8^2, u_{15}u_9^2u_8, u_{15}u_9u_8^3, u_{15}u_8^5,\\
& u_{14}u_{13}, u_{14}u_{12}u_8, u_{14}u_{11}u_9, u_{14}u_{11}u_8^2, u_{14}u_{10}^2,\\
& u_{14}u_{10}u_9u_8, u_{14}u_{10}u_8^3, u_{14}u_9^3, u_{14}u_9^2u_8^2, u_{14}u_9u_8^4, u_{14}u_8^6,
\end{aligned}
$$

$$
\begin{aligned}
& u_{13}^2u_8, u_{13}u_{12}u_9, u_{13}u_{12}u_8^2,\\
& u_{13}u_{11}u_{10}, u_{13}u_{11}u_9u_8, u_{13}u_{11}u_8^3,\\
& u_{13}u_{10}^2u_8, u_{13}u_{10}u_9^2, u_{13}u_{10}u_9u_8^2, u_{13}u_{10}u_8^4,\\
& u_{13}u_9^3u_8, u_{13}u_9^2u_8^3, u_{13}u_9u_8^5, u_{13}u_8^7,\\
& u_{12}^2u_{10}, u_{12}^2u_9u_8, u_{12}^2u_8^3, u_{12}u_{11}^2,\\
& u_{12}u_{11}u_{10}u_8, u_{12}u_{11}u_9^2, u_{12}u_{11}u_9u_8^2, u_{12}u_{11}u_8^4,\\
& u_{12}u_{10}^2u_9, u_{12}u_{10}^2u_8^2, u_{12}u_{10}u_9^2u_8, u_{12}u_{10}u_9u_8^3, u_{12}u_{10}u_8^5,\\
& u_{12}u_9^4, u_{12}u_9^3u_8^2, u_{12}u_9^2u_8^4, u_{12}u_9u_8^6, u_{12}u_8^8,\\
& u_{11}^3u_8, u_{11}^2u_{10}u_9, u_{11}^2u_{10}u_8^2, u_{11}^2u_9^2u_8, u_{11}^2u_9u_8^3, u_{11}^2u_8^5,\\
& u_{11}u_{10}^3, u_{11}u_{10}^2u_9u_8, u_{11}u_{10}^2u_8^3,\\
& u_{11}u_{10}u_9^3, u_{11}u_{10}u_9^2u_8^2, u_{11}u_{10}u_9u_8^4, u_{11}u_{10}u_8^6,\\
& u_{11}u_9^4u_8, u_{11}u_9^3u_8^3, u_{11}u_9^2u_8^5, u_{11}u_9u_8^7, u_{11}u_8^9,\\
& u_{10}^4u_8, u_{10}^3u_9^2, u_{10}^3u_9u_8^2, u_{10}^3u_8^4,\\
& u_{10}^2u_9^3u_8, u_{10}^2u_9^2u_8^3, u_{10}^2u_9u_8^5, u_{10}^2u_8^7,\\
& u_{10}u_9^5, u_{10}u_9^4u_8^2, u_{10}u_9^3u_8^4, u_{10}u_9^2u_8^6, u_{10}u_9u_8^8, u_{10}u_8^{10},\\
& u_9^6u_8, u_9^5u_8^3, u_9^4u_8^5, u_9^3u_8^7, u_9^2u_8^9, u_9u_8^{11}, u_8^{13}\rangle\\
M_7^{14} \quad = \quad & \langle u_{21}, u_{20}u_8, u_{19}u_9, u_{19}u_8^2, u_{18}u_{10}, u_{18}u_9u_8, u_{18}u_8^3,\\
& u_{17}u_{11}, u_{17}u_{10}u_8, u_{17}u_9^2, u_{17}u_9u_8^2, u_{17}u_8^4,\\
& u_{16}u_{12}, u_{16}u_{11}u_8, u_{16}u_{10}u_9, u_{16}u_{10}u_8^2,\\
& u_{16}u_9^2u_8, u_{16}u_9u_8^3, u_{16}u_8^5,\\
& u_{15}u_{13}, u_{15}u_{12}u_8, u_{15}u_{11}u_9, u_{15}u_{11}u_8^2, u_{15}u_{10}^2,\\
& u_{15}u_{10}u_9u_8, u_{15}u_{10}u_8^3, u_{15}u_9^3, u_{15}u_9^2u_8^2, u_{15}u_9u_8^4, u_{15}u_8^6,\\
& u_{14}^2, u_{14}u_{13}u_8, u_{14}u_{12}u_9, u_{14}u_{12}u_8^2,\\
& u_{14}u_{11}u_{10}, u_{14}u_{11}u_9u_8, u_{14}u_{11}u_8^3,\\
& u_{14}u_{10}^2u_8, u_{14}u_{10}u_9^2, u_{14}u_{10}u_9u_8^2, u_{14}u_{10}u_8^4,\\
& u_{14}u_9^3u_8, u_{14}u_9^2u_8^3, u_{14}u_9u_8^5, u_{14}u_8^7,\\
& u_{13}^2u_9, u_{13}^2u_8^2, u_{13}u_{12}u_{10}, u_{13}u_{12}u_9u_8, u_{13}u_{12}u_8^3,\\
& u_{13}u_{11}^2, u_{13}u_{11}u_{10}u_8, u_{13}u_{11}u_9^2, u_{13}u_{11}u_9u_8^2, u_{13}u_{11}u_8^4,\\
& u_{13}u_{10}^2u_9, u_{13}u_{10}^2u_8^2, u_{13}u_{10}u_9^2u_8, u_{13}u_{10}u_9u_8^3, u_{13}u_{10}u_8^5,\\
& u_{13}u_9^4, u_{13}u_9^3u_8^2, u_{13}u_9^2u_8^4, u_{13}u_9u_8^6, u_{13}u_8^8,\\
& u_{12}^2u_{11}, u_{12}^2u_{10}u_8, u_{12}^2u_9^2, u_{12}^2u_9u_8^2, u_{12}^2u_8^4,\\
& u_{12}u_{11}^2u_8, u_{12}u_{11}u_{10}u_9, u_{12}u_{11}u_{10}u_8^2,\\
& u_{12}u_{11}u_9^2u_8, u_{12}u_{11}u_9u_8^3, u_{12}u_{11}u_8^5,\\
& u_{12}u_{10}^3, u_{12}u_{10}^2u_9u_8, u_{12}u_{10}^2u_8^3,\\
& u_{12}u_{10}u_9^3, u_{12}u_{10}u_9^2u_8^2, u_{12}u_{10}u_9u_8^4, u_{12}u_{10}u_8^6,\\
& u_{12}u_9^4u_8, u_{12}u_9^3u_8^3, u_{12}u_9^2u_8^5, u_{12}u_9u_8^7, u_{12}u_8^9,\\
& u_{11}^3u_9, u_{11}^3u_8^2, u_{11}^2u_{10}^2, u_{11}^2u_{10}u_9u_8, u_{11}^2u_{10}u_8^3,\\
& u_{11}^2u_9^3, u_{11}^2u_9^2u_8^2, u_{11}^2u_9u_8^4, u_{11}^2u_8^6,\\
& u_{11}u_{10}^3u_8, u_{11}u_{10}^2u_9^2, u_{11}u_{10}^2u_9u_8^2, u_{11}u_{10}^2u_8^4,\\
& u_{11}u_{10}u_9^3u_8, u_{11}u_{10}u_9^2u_8^3, u_{11}u_{10}u_9u_8^5, u_{11}u_{10}u_8^7,\\
& u_{11}u_9^5, u_{11}u_9^4u_8^2, u_{11}u_9^3u_8^4, u_{11}u_9^2u_8^6, u_{11}u_9u_8^8, u_{11}u_8^{10},\\
& u_{10}^4u_9, u_{10}^4u_8^2, u_{10}^3u_9^2u_8, u_{10}^3u_9u_8^3, u_{10}^3u_8^5,
\end{aligned}
$$

$$u_{10}^2u_9^4, u_{10}^2u_9^3u_8^2, u_{10}^2u_9^2u_8^4, u_{10}^2u_9u_8^6, u_{10}^2u_8^8,$$
$$u_{10}u_9^5u_8, u_{10}u_9^4u_8^3, u_{10}u_9^3u_8^5, u_{10}u_9^2u_8^7, u_{10}u_9u_8^9, u_{10}u_8^{11},$$
$$u_9^7, u_9^6u_8^2, u_9^5u_8^4, u_9^4u_8^6, u_9^3u_8^8, u_9^2u_8^{10}, u_9u_8^{12}, u_8^{14}\rangle$$

Appendix D

The quotient submodules $\overline{M_d^l}$ and their generating monomials (that are not total derivatives), where $d = 8, 7$ and $l = 1, 2, 3, ..., 14$ used in classification of $9th$ order evolution equations:

Quotient Submodules with base level 8

$$
\begin{aligned}
\overline{M_8^1} &= \langle \emptyset \rangle \\
\overline{M_8^2} &= \langle u_9^2 \rangle \\
\overline{M_8^3} &= \langle u_9^3 \rangle \\
\overline{M_8^4} &= \langle u_{10}^2, u_9^4 \rangle \\
\overline{M_8^5} &= \langle u_{10}^2 u_9, u_9^5 \rangle \\
\overline{M_8^6} &= \langle u_{11}^2, u_{10}^3, u_{10}^2 u_9^2, u_9^6 \rangle \\
\overline{M_8^7} &= \langle u_{11}^2 u_9, u_{10}^3 u_9, u_{10}^2 u_9^3, u_9^7 \rangle \\
\overline{M_8^8} &= \langle u_{12}^2, u_{11}^2 u_{10}, u_{11}^2 u_9^2, \\
&\quad u_{10}^4, u_{10}^3 u_9^2, u_{10}^2 u_9^4, u_9^8 \rangle \\
\overline{M_8^9} &= \langle u_{12}^2 u_9, u_{11}^3, u_{11}^2 u_{10} u_9, u_{11}^2 u_9^3, \\
&\quad u_{10}^4 u_9, u_{10}^3 u_9^3, u_{10}^2 u_9^5, u_9^9 \rangle \\
\overline{M_8^{10}} &= \langle u_{13}^2, u_{12}^2 u_{10}, u_{12}^2 u_9^2, u_{11}^3 u_9, u_{11}^2 u_{10}^2, u_{11}^2 u_{10} u_9^2, u_{11}^2 u_9^4, \\
&\quad u_{10}^5, u_{10}^4 u_9^2, u_{10}^3 u_9^4, u_{10}^2 u_9^6, u_9^{10} \rangle \\
\overline{M_8^{11}} &= \langle u_{13}^2 u_9, u_{12}^2 u_{11}, u_{12}^2 u_{10} u_9, u_{12}^2 u_9^3, \\
&\quad u_{11}^3 u_{10}, u_{11}^3 u_9^2, u_{11}^2 u_{10}^2 u_9, u_{11}^2 u_{10} u_9^3, u_{11}^2 u_9^5, \\
&\quad u_{10}^5 u_9, u_{10}^4 u_9^3, u_{10}^3 u_9^5, u_{10}^2 u_9^7, u_9^{11} \rangle \\
\overline{M_8^{12}} &= \langle u_{14}^2, u_{13}^2 u_{10}, u_{13}^2 u_9^2, \\
&\quad u_{12}^3, u_{12}^2 u_{11} u_9, u_{12}^2 u_{10}^2, u_{12}^2 u_{10} u_9^2, u_{12}^2 u_9^4, \\
&\quad u_{11}^4, u_{11}^3 u_{10} u_9, u_{11}^3 u_9^3, \\
&\quad u_{11}^2 u_{10}^3, u_{11}^2 u_{10}^2 u_9^2, u_{11}^2 u_{10} u_9^4, u_{11}^2 u_9^6, \\
&\quad u_{10}^6, u_{10}^5 u_9^2, u_{10}^4 u_9^4, u_{10}^3 u_9^6, u_{10}^2 u_9^8, u_9^{12} \rangle \\
\overline{M_8^{13}} &= \langle u_{14}^2 u_9, u_{13}^2 u_{11}, u_{13}^2 u_{10} u_9, u_{13}^2 u_9^3, \\
&\quad u_{12}^3 u_9, u_{12}^2 u_{11} u_{10}, u_{12}^2 u_{11} u_9^2, u_{12}^2 u_{10}^2 u_9, u_{12}^2 u_{10} u_9^3, u_{12}^2 u_9^5, \\
&\quad u_{11}^4 u_9, u_{11}^3 u_{10}^2, u_{11}^3 u_{10} u_9^2, u_{11}^3 u_9^4, \\
&\quad u_{11}^2 u_{10}^3 u_9, u_{11}^2 u_{10}^2 u_9^3, u_{11}^2 u_{10} u_9^5, u_{11}^2 u_9^7, \\
&\quad u_{10}^6 u_9, u_{10}^5 u_9^3, u_{10}^4 u_9^5, u_{10}^3 u_9^7, u_{10}^2 u_9^9, u_9^{13} \rangle \\
\overline{M_8^{14}} &= \langle u_{15}^2, u_{14}^2 u_{10}, u_{14}^2 u_9^2, \\
&\quad u_{13}^3 u_{12}, u_{13}^2 u_{11} u_9, u_{13}^2 u_{10}^2, u_{13}^2 u_{10} u_9^2, u_{13}^2 u_9^4, \\
&\quad u_{12}^3 u_{10}, u_{12}^3 u_9^2, u_{12}^2 u_{11}^2, u_{12}^2 u_{11} u_{10} u_9, u_{12}^2 u_{11} u_9^3, \\
&\quad u_{12}^2 u_{10}^3, u_{12}^2 u_{10}^2 u_9^2, u_{12}^2 u_{10} u_9^4, u_{12}^2 u_9^6, \\
&\quad u_{11}^4 u_{10}, u_{11}^4 u_9^2, u_{11}^3 u_{10}^2 u_9, u_{11}^3 u_{10} u_9^3,
\end{aligned}
$$

$u_{11}^2u_{10}^4, u_{11}^2u_{10}^3u_9^2, u_{11}^2u_{10}^2u_9^4, u_{11}^2u_{10}u_9^6, u_{11}^2u_9^8,$
$u_{10}^7, u_{10}^6u_9^2, u_{10}^5u_9^4, u_{10}^4u_9^6, u_{10}^3u_9^8, u_{10}^2u_9^{10}, u_9^{14}\rangle$

Quotient Submodules with base level 7

$$
\begin{aligned}
\overline{M_7^1} &= \langle\emptyset\rangle \\
\overline{M_7^2} &= \langle u_8^2\rangle \\
\overline{M_7^3} &= \langle u_8^3\rangle \\
\overline{M_7^4} &= \langle u_9^2, u_8^4\rangle \\
\overline{M_7^5} &= \langle u_9^2u_8, u_8^5\rangle \\
\overline{M_7^6} &= \langle u_{10}^2, u_9^3, u_9^2u_8^2, u_8^6\rangle \\
\overline{M_7^7} &= \langle u_{10}^2u_8, u_9^3u_8, u_9^2u_8^3, u_8^7\rangle \\
\overline{M_7^8} &= \langle u_{11}^2, u_{10}^2u_9, u_{10}^2u_8^2, u_9^4, u_9^3u_8^2, u_9^2u_8^4, u_8^8\rangle \\
\overline{M_7^9} &= \langle u_{11}^2u_8, u_{10}^3, u_{10}^2u_9u_8, u_{10}^2u_8^3, \\
&\quad u_9^4u_8, u_9^3u_8^3, u_9^2u_8^5, u_8^9\rangle \\
\overline{M_7^{10}} &= \langle u_{12}^2, u_{11}^2u_9, u_{11}^2u_8^2, u_{10}^3u_8, \\
&\quad u_{10}^2u_9^2, u_{10}^2u_9u_8^2, u_{10}^2u_8^4, u_9^5, u_9^4u_8^2, u_9^3u_8^4, u_9^2u_8^6, u_8^{10}\rangle \\
\overline{M_7^{11}} &= \langle u_{12}^2u_8, u_{11}^2u_{10}, u_{11}^2u_9u_8, u_{11}^2u_8^3, u_{10}^3u_9, u_{10}^3u_8^2, \\
&\quad u_{10}^2u_9^2u_8, u_{10}^2u_9u_8^3, u_{10}^2u_8^5, \\
&\quad u_9^5u_8, u_9^4u_8^3, u_9^3u_8^5, u_9^2u_8^7, u_8^{11}\rangle \\
\overline{M_7^{12}} &= \langle u_{13}^2, u_{12}^2u_9, u_{12}^2u_8^2, u_{11}^3, \\
&\quad u_{11}^2u_{10}u_8, u_{11}^2u_9^2, u_{11}^2u_9u_8^2, u_{11}^2u_8^4, \\
&\quad u_{10}^4, u_{10}^3u_9u_8, u_{10}^3u_8^3, \\
&\quad u_{10}^2u_9^3, u_{10}^2u_9^2u_8^2, u_{10}^2u_9u_8^4, u_{10}^2u_8^6, \\
&\quad u_9^6, u_9^5u_8^2, u_9^4u_8^4, u_9^3u_8^6, u_9^2u_8^8, u_8^{12}\rangle \\
\overline{M_7^{13}} &= \langle u_{13}^2u_8, u_{12}^2u_{10}, u_{12}^2u_9u_8, u_{12}^2u_8^3, \\
&\quad u_{11}^3u_8, u_{11}^2u_{10}u_9, u_{11}^2u_{10}u_8^2, u_{11}^2u_9^2u_8, u_{11}^2u_9u_8^3, u_{11}^2u_8^5, \\
&\quad u_{10}^4u_8, u_{10}^3u_9^2, u_{10}^3u_9u_8^2, u_{10}^3u_8^4, \\
&\quad u_{10}^2u_9^3u_8, u_{10}^2u_9^2u_8^3, u_{10}^2u_9u_8^5, u_{10}^2u_8^7, \\
&\quad u_9^6u_8, u_9^5u_8^3, u_9^4u_8^5, u_9^3u_8^7, u_9^2u_8^9, u_8^{13}\rangle \\
\overline{M_7^{14}} &= \langle u_{14}^2, u_{13}^2u_9, u_{13}^2u_8^2, \\
&\quad u_{12}^2u_{11}, u_{12}^2u_{10}u_8, u_{12}^2u_9^2, u_{12}^2u_9u_8^2, u_{12}^2u_8^4, \\
&\quad u_{11}^3u_9, u_{11}^3u_8^2, u_{11}^2u_{10}^2, u_{11}^2u_{10}u_9u_8, u_{11}^2u_{10}u_8^3, \\
&\quad u_{11}^2u_9^3, u_{11}^2u_9^2u_8^2, u_{11}^2u_9u_8^4, u_{11}^2u_8^6, \\
&\quad u_{10}^4u_9, u_{10}^3u_9^2u_8, u_{10}^3u_9u_8^3, u_{10}^3u_8^5, \\
&\quad u_{10}^2u_9^4, u_{10}^2u_9^3u_8^2, u_{10}^2u_9^2u_8^4, u_{10}^2u_9u_8^6, u_{10}^2u_8^8, \\
&\quad u_9^7, u_9^6u_8^2, u_9^5u_8^4, u_9^4u_8^6, u_9^3u_8^8, u_9^2u_8^{10}, u_8^{14}\rangle
\end{aligned}
$$

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