İSTANBUL TECHNICAL UNIVERSITY ★ INSTITUTE OF SCIENCE AND TECHNOLOGY

COPRIMELY PACKED POLYNOMIAL RINGS

T.C. YÜKSEKÖĞRETİM KURULU
DOKÜMANTASYON MERKEZİ

Master Thesis by
Esra ŞENGELEN, M.Sc.

509951050

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Supervisor: Prof. Dr. Vahap ERDOĞDU

Members of the Examining Committee: Prof.Dr. Ulviye BAŞER

Prof.Dr. Mahmut HORTAÇSU

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T.C. YÜKSEKÖĞRETİM KURULU
DOKÜMANTASYON MERKEZİ

YÜKSEK LLİSANS TEZİ

Araş. Gör. Esra ŞENGELEN
509951050

101356

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Diğer Jüri Üyeleri: Prof.Dr. Ulviye BAŞER
Prof.Dr. Mahmut HORTAÇSU

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CONTENTS

LIST OF SYMBOLS

$\cup$	: Union (of sets)
$\cap$	: Intersection (of sets)
$<$	: Less than
$\leq$	: Less or equal
$>$	: Grater than
$\geq$	: Grater or equal
$=$	: Is equal to
$\neq$	: Is not equal to
$\supset$	: Is a proper superset of
$\supseteq$	: Is a superset of
$\subset$	: Is a proper subset of
$\subseteq$	: Is a subset of
$\not\subset$	: Is not a subset of
$\in$	: Is an element of
$\notin$	: Is not an element of
$\cong$	: Is isomorphic to
=>	: Is necessary condition
$\Leftarrow$	: Is sufficient condition
$\Leftrightarrow$	: Necessary and sufficient condition
$\otimes_R$	: Tensor product over R
$\prod$	: Product
$\sum$	: Summation

T.C. YÜKSEKÖĞRETİM KURULU
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ABSTRACT

COPRIMELY PACKED POLYNOMIAL RINGS

In this work we prove some new results in line with the work of V. Erdoğdu about the inheritance of coprime packedness of R by a polynomial ring extension R[x] of R. We also examine the coprime packedness of some overrings of R[x] when R is coprimely packed.

YÖK

ASAL İDELLARİ TARAFINDAN KAPALI POLİNOM HALKALARI

ÖZET

R birimli değişmeli bir halka ve I R'nin bir ideali olsun. Eğer I, R'nin herhangi bir asal idealler kümesinin her elemanı ile aralarında asal ise ve I söz konusu asal idealler kümesinin birleşimi tarafından kapsanmıyorsa, o zaman I idealine, R'nin asal idealler tarafından kapalıdır denir. Eğer R'nin her ideali R'nin asal idealleri tarafından kapalı ise o zaman R'ye asal idealleri tarafından kapalı bir halka denir. Bir R Noetherian halkası üzerinde R[x] polinom halkasının asal idealler tarafından kapalı olması durumunda R'nin sonlu sayıda asal ideallere sahip olduğu [7] de gösterilmiştir. Şayet R bir Dedekind tamlık bölgesi ise, o zaman R[x]' in asal idealler tarafından kapalı olması için gerek ve yeter koşul R'nin semilocal temel ideal bölgesi olmasıdır.

R'nin asal idealler tarafından kapalı olması her zaman R üzerindeki polinom halkası R[x]'in de aynı özelliği göstermesi gerekmediğini R = Z (Z tam sayılar halkası olmak üzere) olması durumunda göstermektedir. Çünkü Z temel ideal bölgesi ve her temel ideal bölgesi asal idealleri tarafından kapalıdır ancak Z[x] asal idealleri tarafından kapalı değildir.

Bu çalışmada R Noetherian ve sonlu sayıda maksimal ideale sahip olmadığı durumlarda R[x] polinom halkasının asal idealleri tarafından kapalı olduğu durumlar incelenmiştir. R asal idealleri tarafından kapalı, Krull boyutu bir olan tam kapalı bir tamlık bölgesi olsun. O zaman R üzerinde R[x] polinom halkasının monik bir polinom içeren Q* asal ideali ile bölümünden elde edilen R[x] / Q* bölüm halkasının da asal idealleri tarafından kapalı olduğu gösterilmiştir. Bunun dışında R üzerindeki çeşitli polinom halka genişlemelerinin asal idealleri tarafından kapalı olma özellikleri de incelenmiştir.

COPRIMELY PACKED POLYNOMIAL RINGS

SUMMARY

An ideal I of a commutative ring R with identity is said to be coprimely packed by prime ideals of R if whenever I is coprime to each element of a family of prime ideals of R, I is not contained in the union of prime ideals of the family. We say that R is coprimely packed if every ideal of R is coprimely packed.

In [7] it is prove that if a polynomial ring R[x] over a Noetherian ring R is coprimly packed, then R has only finitely many prime ideals and that over a Dedekind domain R, R[x] is coprimely packed if and only if R is a semilocal principal ideal domain. If R is coprimely packed then it does not follow that the polynomial ring extension R[x] of R is coprimely packed (e.g. if R is the ring of integers Z, then Z being a principal ideal domain is coprimely packed but Z[x] is not coprimely packed). The quest we pursue here is that what form of a polynomial ring extension is coprimely packed when the underlying ring R is, in the case R may have infinitely many maximal ideals and may not be Noetherian. In Chapter 5 we show that if R is a coprimely packed integrally closed domain of Krull dimension one and Q^* is a prime ideal of R[x] contaning a monic polynomial then R[x] / Q^* contains R as a subring and it is coprimely packed. We then investigate the other forms of polynomial ring extensions which inherits the coprimely packedness property from that of R.

1.INTRODUCTION

The ring R is said to be coprimely packed if whenever I is an ideal of R and S is a set of maximal ideals of R with $I \subseteq \cup \{M \in S\}$, then $I \subseteq M$ for some $M \in S$. This concept was introduced in [3]. [4] raised the question of which rings R have the property that R[x] is coprimely packed, and showed taht for R Noetherian and Hilbert, R[x] is coprimely packed if and only if R is Artinian. In [7] it is shown that for an arbitrary Noetherian ring R, if R[x] is coprimely packed, then R has at most finitely many prime ideals. Our aim here is to find out what form of a polynomial ring extension of R is coprimely packed when R has infinitely many prime ideals and may not be Noetherian.

The outline of this work is as follows :

In Chapter 2, we deal with integral extensions and examine the properties of such extensions. In particular we examine the integral extentions of polynomial rings.

In Chapter 3, we outline certain properties of prime ideals.

In Chapter 4, we gather the information related to the notion of coprime packedness.

Finally, in Chapter 5, we prove our main results.

Throughout this work R will denote a commutative ring with identity.

2.INTEGRAL DEPENDENCE

This chapter contains an exposition of integral dependence. The material of this chapter can be found in [1] and in [9]. For the sake of completeness we reproduce the proofs.

Definition. Let B be a ring, A a subring of B. An element x of B is said to be integral over A if x is a root of a monic polynomial with coefficients in A, that is if x satisfies an equation of the form

$$x^n + a_1x^{n-1} + \ldots + a_n = 0 \quad (1)$$

where the a_i are elements of A. Clearly every element of A is integral over A.

Lemma 2.1. Let M be a finitely generated A-module, let I be an ideal of A, and let ϕ be an A-module endomorphism of M such that $\phi(M) \subseteq IM$. Then ϕ satisfies an equation of the form

$$\phi^n + a_1\phi^{n-1} + \ldots + a_n = 0$$

where the a_i in I.

Proof. Let $x_1, \ldots, x_n$ be a set of generators of M. Then each $\phi(x_i) \in IM$, so that we have say $\phi(x_i) = \sum_{j=1}^{n} a_{ij}x_j$ $(1 \le i \le n\ ;\ a_{ij} \in I)$,

$$\sum_{j=1}^{n} (\delta_{ij}\phi - a_{ij})x_j = 0$$

where δ_{ij} is the Kronecker delta. By multiplying on the left by the adjoint of the matrix $(\delta_{ij}\phi - a_{ij})$ it follows that det $(\delta_{ij}\phi - a_{ij})$ annihilates each x_i, hence is the zero endomorphism of M .Expanding out the determinant, we have an equation of the required form.

Proposition 2.2. The following are equivalent :

i) $x \in B$ is integral over A.

ii) A[x] is finitely generated A-module.

iii) A[x] is contained in a subring C of B such that C is a finitely generated A-module.

iv) There exists a faithful A[x]-module M which is finitely generated A-module.

Proof. i) $\Rightarrow$ ii). From equation (1) we have

$$x^{n+r} = -(a_1 x^{n+r-1} + \ldots + a_n x^r)$$

for all $r \geq 0$; hence, by induction,all positive powers of x lie in the A-module generated by $1, x, \ldots, x^{n-1}$. Hence A[x] is generated (as an A-module) by $1, x, \ldots, x^{n-1}$.

ii) $\Rightarrow$ iii). Take C=A[x]

iii) $\Rightarrow$ iv). Take M=C, which is faithful A[x]-module (since $yC = 0 \Rightarrow y.1 = 0$).

iv) $\Rightarrow$ i). This follows from (2.1): take ϕ to be multiplication by x, and I = A (we have $xM \subseteq M$ since M is an A[x]-module) ; since M is faithful, we have

$$x^n + a_1 x^{n-1} + \ldots + a_n = 0 \text{ for suitable } a_i \in A.$$

Corollary 2.3. Let x_i $(1 \leq i \leq n)$ be elements of B, each integral over A. Then the ring $A[x_1, \ldots x_n]$ is a finitely –generated A-module.

Proof. By induction on n. The case n=1 is part of (2.2). Assume $n > 1$, let

T.C. YÜKSEKÖĞRETİM KURULU DOKÜMANTASYON MERKEZİ

$A_r = A[x_1, \ldots, x_r]$; then by the inductive hypothesis A_{n-1} is finitely generated A-module. $A_n = A_{n-1}[x_n]$ is a finitely generated A_{n-1}-module (by the case n=1,since x_n is integral over A_{n-1}). But then A_n is a finitely generated A-module.

Corollary 2.4 The set C of elements of B which are integral over A is a subring of B containing A.

Proof. If $x,y \in C$ then A[x,y] is a finitely generated A-module by (2.3). Hence $x \pm y$ and x.y are integral over A, by iii) of (2.2).

The ring C in (2.4) is called the integral closuer of A in B. If C=A, then A is said to be integrally closed in B, the ring B is said to be integral over A.

Corollary 2.5. If $A \subseteq B \subseteq C$ are rings and if B is integral over A, and C is integral over B, then C is integral over A (transitivity of integral dependence)

Proof. Let $x \in C$, then we have an equation

$$x^n + b_1 x^{n-1} + \ldots + b_n = 0 \quad (b_i \in B)$$

The ring $B' = A[b_1, \ldots, b_n]$ is a finitely generated A-module by (2.3) and $B'[x]$ is a finitely generated B' - module (since x is integral over B'). Hence $B'[x]$ is a finitely generated A-module and therefore x is integral over A by iii) of (2.2)

Corollary 2.6. Let $A \subseteq B$ be rings and let C be the integral closure of A in B. Then C is integrally closed in B.

Proof. Let $x \in B$ be integral over C. By (2.5) x is integral over A, hence $x \in C$.

Proposition 2.7. Let $A \subseteq B$ be rings, B integral over A.

i) If J is an ideal of B and $I = J^c = A \cap J$, then B/J is integral over A/I

ii) If S is a multiplicatively closed subset of A, then $S^{-1}B$ is integral over $S^{-1}A$.

Proof. i) If $x \in B$ we have, say, $x^n + a_1 x^{n-1} + \ldots + a_n = 0$ with $a_i \in A$. Reduce this equation mod J.

ii) Let $x/s \in S^{-1}B$ ($x \in B$, $s \in S$). Then the equation above gives

$$(x/s)^n + (a_1/s)(x/s)^{n-1} + \ldots + a_n/s = 0$$

which shows that x/s is integral over $S^{-1}A$.

Proposition2.8. Let $A \subseteq B$ be integral domains, B integral over A. Then Bis a field if and only if A is a field.

Proof. Suppose A is a field ; let $y \in B$, $y \neq 0$. Let

$$y^n + a_1 y^{n-1} + \ldots + a_n = 0 \quad (a_i \in A)$$

be an equation of integral dependence for y of smallest possible degree. Since B is an integral domain we have $a_n \neq 0$, hence

$$y(y^{n-1} + a_1 y^{n-2} + \ldots + a^{n-1}) = -a_n$$

$$y(y^{n-1} + a_1 y^{n-2} + \ldots + a^{n-1})(-a_n^{-1}) = 1$$

It follows that,

$$y^{-1} = -a_n^{-1}(y^{n-1} + a_1 y^{n-2} + \ldots + a^{n-1}) \in B$$

Hence Bis a field.

Conversly, suppose B is a field ; let $x \in A$, $x \neq 0$. Then $x^{-1} \in B$, hence is integral over A, so that we have an equation

$$x^{-m} + a_1 x^{-m+1} + \ldots + a_m = 0 \ (a_i \in A)$$

It follows that,

$$x^{-1} = -(a_1 + a_2 x + \ldots + a_m x^{m-1}) \in A,$$

hence Ais a field.

Corollary 2.9. Let $A \subseteq B$ be rings, B integral over A ; let q be a prime ideal of B and let $p = q^c = q \cap A$. Then q is maximal if and only if p is maximal.

Proof. By (2.7), B/q is integral over A/p, and both these rings are integral domain. Now use (2.8)

Proposition2.10 Let A be a subring of an integral domain B, and let C be the integral closure of A in B. Let f,g be monic poliynomials in B[x] such that fg $\in$ C[x]. Then f,g are in C[x].

Proof. $A \subseteq C \subseteq B$, f,g $\in$ B[x] and f,g are monic polynomials such that f.g $\in$ C[x]. Let K be quotient field of B, then f,g $\in$ K[x]. Let L be an extention field of K such that the polynomial f, g split into lineer factors ; say $f = \prod(x-\xi_i)$ and $g = \prod(x-\eta_i)$. Each ξ_i and η_i is a root of f.g, hence is integral over C. Hence the coefficients of f and g are integral over C.

Proposition 2.11. Let A be a subring of a ring B and C be the integral closure of A in B. Than C[x] is the integral closure of A[x] in B[x].

Proof. If f $\in$ B[x] is integral over A[x], then

$$f^m + g_1 f^{m-1} + \ldots + g_m = 0 \ (g_t \in A[x])$$

Let r > (m, $\deg g_1, \ldots, \deg g_m$) and let $f_1 = f - x^r$, so that

$$(f_1 + x^r)^m + g_1(f_1 + x^r)^{m-1} + \ldots + g_m = 0$$

or say $f_1^m + h_1 f_1^{m-1} + \ldots + h_m = 0$ where $h_m = (x^r)^m + g_1(x^r)^{m-1} + \ldots + g_m \in A[x]$ Now apply (2.10) to the polynomials $h_m = -f_1^m - h_1 f_1^{m-1} - \ldots - h_{m-1} f_1 \in A[x]$ $\Rightarrow h_m = -f_1(f_1^{m-1} + h_1 f_1^{m-2} + \ldots + h_{m-1}) \Rightarrow -f_1$ and $f_1^{m-1} + h_1 f_1^{m-2} + \ldots + h_{m-1} \in C[x]$ $\Rightarrow f - x^r \in C[x] \Rightarrow f \in C[x]$.

Theorm 2.12. Let $A \subseteq B$ be rings, B integral over A, and let p be a prime ideal of A. Then there exists a prime ideal q of B such that $q \cap A = p$

Proof. By (2.7), B_p is integral over A_p, and the diagram

T.C. YÜKSEKÖĞRETİM KURULU DOKÜMANTASYON MERKEZİ

$$A \to B$$

$$\alpha\downarrow \quad \downarrow\beta$$

$$A_p \to B_p$$

(in which the horizontal arrows are injections) is commutative. Let n be a maximal ideal of B_p; then $m = n \cap A_p$ is maximal by (2.9), hence is the unique maximal ideal of the local ring A_p. If $q = \beta^{-1}(n)$, then q is prime and we have $q \cap A = \alpha^{-1}(m) = p$

Theorm 2.13.("Going-up theorem") Let $A \subseteq B$ be rings, B integral over A ; let $p_1 \subseteq \ldots \subseteq p_n$ be a chain of prime ideals of A and $q_1 \subseteq \ldots \subseteq q_m$ $(m < n)$ a chain of prime ideals of B such that $q_i \cap A = p_i$ $(1 \le i \le m)$. The chain $q_1 \subseteq \ldots \subseteq q_m$ can be extended to a chain $q_1 \subseteq \ldots \subseteq q_n$ in B such that $q_i \cap A = p_i$ for $(1 \le i \le n)$.

Proof. By induction we reduce immediately to the case $m = 1$, $n = 2$. Let $A^* = A / p_1$, $B^* = B / q_1$; then $A^* \subseteq B^*$, and B^* is integral over A^* by (2.7). Hence, by (2.12), there exists a prime ideal q_2^* of B^* such that $q_2^* \cap A^* = p_2^*$ the image of p_2 in A^*. Lift back q_2^* to B and we have a prime ideal q_2 with the required properties.

3.SOME PROPERTIES OF PRIME IDEALS

The aim of this short chapter is to show that no three distinct prime ideals in a polynomial ring R[x] contract to the same ideal in R, and as well as state and demonstrate the prime avoidance theorem. The material in here can be found in [1] and [8].

Theorem 3.1. (The prime avoidance theorem) i) Let R be a ring $p_1, \ldots, p_n$ be prime ideals of R and let I an ideal of R contained in $\bigcup_1^n p_i$. Then $I \subseteq p_i$ for some i.

ii) Let $I_1, \ldots, I_n$ be ideals and let p be a prime ideal containing $\bigcap_1^n I_i$. Then $p_i \supseteq I_i$ for some i . If $p = \bigcap_1^n I_i$, then $p = I_i$ for some i .

Proof. i) Is proved by induction on n in the form

$$I \not\subset p_i \,(1 \le i \le n) \Rightarrow I \not\subset \bigcup_1^n p_i$$

It is certainly true for n = 1. If n > 1 and the result is true for n-1, then for each i there exists $x_i \in I$ such that $x_i \notin p_j$, whenever $j \neq i$. If for some i we have $x_i \notin p_i$, we are through. If not, then $x_i \in p_i$ for all i.

Consider the element

$$y = \sum_1^n x_1 x_2 \ldots x_{i-1} x_{i+1} x_{i+2} \ldots x_n$$

we have $y \in I$ and $y \notin p_i \,(1 \le i \le n)$. Hence $I \not\subset \bigcup_1^n p_i$

ii) Suppose $p \not\subset I_i$ for all i. Then there exists $x_i \in I_i$, $x_i \notin p$ $(1 \leq i \leq n)$ and threfore $\prod_1^n x_i \in \bigcap_1^n I_i$; but $\prod_1^n x_i \notin p$ (since p is prime). Hence $\bigcap_1^n I_i \not\subset p$. Finally, if $p = \bigcap_1^n I_i$, then $p \subseteq I_i$, and hence $p = I_i$ for some i.

Theorem3.2. Suppose taht R is an integral domain. If P is a nonzero prime ideal of R[x] such that $P \cap R = 0$, then for every $f(x) \in R[x] \setminus P$ we have $(P + f(x)) \cap R \neq 0$.

Proof. Let K be the quotient field of R. If $S = R\setminus\{0\}$, then $S^{-1}R[x] = K[x]$. Since $P \cap R = 0$, $P \cap S$ is empty, and consequently $S^{-1}P$ is a nonzero proper prime ideal of K[x]. As such, it is a maximal ideal of K[x]. Furthermore $S^{-1}P \cap R[x] = P$. Therefore, if $f(x) \in R[x] / P$, then $f(x) \notin S^{-1}P$; hence f(x) and $S^{-1}P$ generate K[x]. Now K[x] is a principal ideal domain, so $S^{-1}P$ is generated by some g(x) / s, where $g(x) \in P$ and $s \in S$. Hence there are elements $h(x), k(x) \in R[x]$ and elements $u, v \in S$ such that $1 = (h(x) / u)(g(x) / s) + (k(x) / v)f(x)$. Then

$$uvs = vh(x)g(x) + usk(x)f(x) \in (P +(f(x))) \cap R \text{ and } uvs \neq 0$$

Corollary3.3. Suppose that R is an integral domain. Let P and P′ be nonzero ideals of R[x] such taht $P \subset P'$. If P is a prime ideal, then $P' \cap R \neq 0$.

Proof. If $P \cap R \neq 0$, then $P' \cap R \neq 0$. If $P \cap R = 0$ and if $f(x)\in P' \setminus P$, then

$$0 \neq (P + (f(x))) \cap \subseteq P' \cap R.$$

Corollary3.4. Let P_1,P_2,P_3 be ideals of R[x],with P_2 prime, such that $P_1 \subset P_2 \subset P_3$. Then $P_1 \cap R \subset P_3 \cap R$.

Proof. If $P_1 \cap R \subset P_2 \cap R$, we have finished. Suppose that $P_1 \cap R = P_2 \cap R = P$, which is a prime ideal of R. Then R[x] / PR[X] may be viewed as the polynomial ring in one indeterminate over the integral domain R / P. Since $PR[x] \subset P_1 \subset P_2$ we have $0 \neq P_2 / PR[x] \subset P_3 / PR[x]$. Hence, by (3.3), $P_3 / PR[x] \cap R / P \neq 0$. Thus there is an element $a \in R \setminus P$ and a polynomial $f(x) \in P_3$ such that $a - f(x) \in PR[x]$. If b is the constant term of f(x), then $b \notin P$. However $b \in P_3 \cap R$, so $P_1 \cap R \subset P_3 \cap R$.

4.COPRIMELY PACKEDNESS

Here we recall the notion of coprimely packedness of a ring which was introduced and stuided in [3], [4], [5], [6], [7].

An ideal I of a ring R is said to be coprimely packed if $I + P_s = R$ where P_s ($s \in S$) are prime idals of R, then $I \not\subset \bigcup_{s \in S} P_s$.

A non-empty subset X of the set of prime ideals of R is said to be coprimely packed if whenever an element P of X is coprime to each element of a subset Y of X, then P is not contained in the union of prime idals in Y.

If every ideal of R is coprimeley packed then R is a coprimely packed ring.

Theorem4.1. Every semilocal ring is coprimely packed.

Proof. Let R be a semilocal ring and let I be any non-zero ideal of R. Suppose that $I + P = R$ for all $P \in X$, where X is any non-empty subset of SpecR. We want to show that $I \not\subset \bigcup_{P \in X} P$. Now because MaxSpecR is finite, we may pick a subset $\{M_1, M_2, \dots, M_n\}$ of MaxSpecR in such a way that for each M_i in $\{M_1, M_2, \dots, M_n\}$ there exists an element P in X such that $P \subseteq M_i$ and for each $P \in X$ there exists an element M_i in $\{M_1, M_2, \dots, M_n\}$ such that $P \subseteq M_i$. Since $I + P = R$ for all $P \in X$, it follows that $I + M_i = R$ for all $i = 1, 2, \dots, n$. Hence by (3.1) $I \not\subset \bigcup_{1}^{n} M_i$. But $\bigcup_{P \in X} P \subseteq \bigcup_{1}^{n} M_i$, and so $I \not\subset \bigcup_{P \in X} P$. Therefore it follows that R is coprimely packed.

Theorem 4.2. Let R be a ring. Suppose that every prime ideal of R is coprimely packed by the set of maximal ideals of R. Then R is coprimely packed.

Proof. Let $\Delta = \{ J \mid J$ is ideal of R, and there is a set S(J) of maximal ideals such that $J \subseteq \cup \{M \in S(J)\}$ but $J \not\subset M$ for all $M \in S(J)\}$. We must show Δ is empty. If not, use Zorn's Lemma to show that Δ contains a maximal member, say H. Clearly H is not

prime. Thus there are ideals K and L, both properly containing H, such that $KL \subseteq H$. As $K \notin \Delta$, $K \not\subset \cup\{M \in S(H)\}$ (since otherwise, $H \subset K \subseteq M$ for some $M \in S(H)$). Therefore, there is an $x \in K - \cup\{M \in S(H)\}$. Similary, there is a $y \in L - \cup \{M \in S(H)\}$. However, $xy \in KL \subseteq H \subseteq \cup \{M \in S(H)\}$, which gives a contradiction. Therefore R is coprimely packed.

Definitions. 1) Let R be a ring and consider a chain $P_0 \subseteq P_1 \subseteq \dots \subseteq P_r$ of $r+1$ proper prime ideals of R . The lenght of such a chain is the integer r . Its first term is P_0 and its last term is P_r.
The Krull dimension of R is the supremum of the lengths of all chains of distinct proper prime ideals of R.

2) The heigth of a prime ideal P is the Krull dimension of R_p.

3) A Dedekind domain is a Noetherian integrally closed domain of Krull dimension one.

Remarks :

1) If R is coprimely packed,then clearly every quotient ring R/J is coprimely packed. In particular if $\prod_1^n R_i$ is coprimely packed, then each factor is coprimely packed.

2) Let $\prod_1^n R_i$ be a product of domains where each factor is of dimension 1. If $\prod_1^n R_i$ coprimely packed, so is each R_i. This is because each maximal ideal of a coprimely packed integral domain of Krull dimension one is the radical of a principal ideal.

Because of its use we state (without proof, as it is lengthy) the following result taken from [5].

Theorem4.3. Let R be a Dedekind domain. Then the following statement are equivalent.

i) MaxSpecR is coprimely packed.

ii) The ideal class group of R is torsion.

iii) For each non-empty subset X of MaxSpecR and for each Q$\in$ (MaxSpecR) – X, $R_Q \otimes_R S_X^{-1}R = K$ where K is the field of fractions of R and $S_X = R - \bigcup_{P\in X} P$.

Definition. A ring R is Hilbert if every prime ideal of R is an intersection of maximal ideals.

Theorem4.4. Let R be a Hilbert ring. Then MaxSpecR is coprimely packed if and only if every maximal ideal of R is the radical of principal ideal.

Proof. Let M be any maximal ideal of R. Then $M \not\subset \bigcup_{N\neq M} N$ (N ranging in the set of maximal ideals of R different than M). Let x $\in$ M which is not in any other maximal ideal of R. Then $\sqrt{x}$ is the intersection of all prime ideals of containing x. Since each prime containing x is contained only in M but not contained in any other maximal of R, and each such prime is nothing other than M itself, $M = \sqrt{x}$

Conversly ; if $M = \sqrt{x}$, then $x \notin \bigcup_{N\neq M} N$ (N is maximal ideal of R and N $\neq$ M). Hence MaxSpecR is coprimely packed.

Corollary4.5. A coprimely packed Noetherian Hilbert ring is of Krull dimension at most 1.

Proof. Let M be a maximal ideal of R then by the above theorem $M = \sqrt{x}$. But then the fact that R is of Krull dimension at most 1 follows from Krull's principal ideal theorem.

Proposition4.6. There exists a ring R and a polynamial ring R[x] in x over R such taht MaxSpecR is coprimely packed without MaxSpecR[x] being coprimely packed.

Proof. Take R to be the ring of integers Z and let Z[x] be the polynamial ring in x over Z. Then clearly MaxSpecZ is coprimely packed. Now Z and Z[x] are both Hilbert rings. If MaxSpecZ[x] where coprimely packed,then by (4.4), every maximal ideal of Z[x] would be the radical of a principal ideal, which would then imply that

Z[x] is of Krull dimension one, which is not the case, since it is of Krull dimension two. Therefore MaxSpecZ[x] is not coprimely packed.

5.COPRIMELY PACKED POLYNOMIAL RINGS

In this Chapter we state and prove our main results. We begin with the following well known fact.

Proposition5.1. If P^* is a prime ideal and contains a monic polynomial in R[x] such taht $P^* \cap R = 0$, then $R \subseteq R[x] / P^*$ is an integral extention.

Proof. Let $f(x) = x^m + a_{m-1}x^{m-1} + \ldots + a_0$ be the monic polynomial of R[x] contained in P^*. Let $\theta = x + P^*$, then $R[x] / P^*$ is generated by $1, \theta, \ldots, \theta^{m-1}$. This is because R[x] as an R-module is generated by powers of x and so $R[x] / P^*$ as an R-module is generated by powers of θ. Since $f + P^* = P^*$ in $R[x] / P^*$, it follows that $\theta^m + a_{m-1}\theta^{m-1} + \ldots + a_1\theta + a_0 = 0$ in $R[x] / P^*$. But then $\theta^m = -(a_{m-1}\theta^{m-1} + \ldots + a_1\theta + a_0)$. Hence $R[x] / P^*$ as an R-module is generated by $1, \theta, \ldots, \theta^{m-1.}$ Therfore by (2.2) $R[x] / P^*$ is integral over R.

Proposition 5.2. Let R be a domain and let M^* be a maximal ideal of R[x] containing a monic polynomial. Then $M^* \cap R$ is maximal.

Proof. Since $R / M^* \cap R \subseteq R[x] / M^*$ and since M^* contains a monic polynomial $R / M^* \cap R \subseteq R[x] / M^*$ is an integral extention. As M^* is maximal, $R[x] / M^*$ is a field and hence $R / M^* \cap R$ is a field by (2.9). Therefore $M^* \cap R$ is a maximal ideal in R.

Theorem5.3. Let R be an integrally closed domain and Q^* a prime ideal of R[x] containing a monic poliynomial. Then Q^* contains a height one prime ideal P^* such that P^* is generated by an irreducible monic poliynomial.

Proof. Let f be a monic polynomial in Q^* and suppose that $f = g_1h_1$ for some polynomials g_1, h_1 in R[x]. If now a is the leading coefficient of g_1 and b is the leading coefficient of h_1, then 1 = ab is the leading coefficient of f and thus a and b are units in R. Since Q^* is a prime ideal, it follows that either g_1 or h_1 in Q^*, say $g_1 \in Q^*$. But then $g_2 = (1/a)g_1 \in Q^*$ and g_2 is a monic polynomial. If g_2 is reducible, then using the above argument we obtain a monic polynomial g_3 in Q^* such that $g_3 \mid g_2 \mid g_1 \mid f$. Continuing in this way and taking in account that degree cannot decrease indefinitely, we arrive at a monic polynomial g in Q^*.

Since g is irreducible in R[x] it is irreducible in K[x], where K is the field of fractions of R. This is because if $g = g_1^* g_2^*$ in K[x], then the coeeficients of g_1^* and g_2^* are integral over R. But R is in integrally closed imply, $g_1^*, g_2^* \in R[x]$, a contradiction.

Thus gK[x] = P is a prime ideal in K[x], as K[x] is a U.F.D.

If now $\theta = x + <g>$ and g is of degree n then R[x] / <g> is a free R-module generated by $1, \theta, \ldots, \theta^{n-1}$. This is because if $\sum_{i=0}^{n-1} a_i \theta^i = 0$ then $\sum_{i=0}^{n-1} a_i x^i \in <g>$, a contradiction as g is of degree n, unless $a_i = 0$ for $i = 0, 1, \ldots, n-1$.

Thus R[x] / <g> is a free R-module and since $R \to K$ is a monomorphism, we have $R[x] / <g> = R[x] / <g> \otimes_R R \to K \otimes_R R[x] / <g> = K[x] / P$ is a monomorphism, and hence R[x] / <g> is an integral domain as K[x] / P is a field. Therefore <g> is a prime ideal of R[x].

Finally, we show that <g> is a height one prime ideal of R[x]. If not then there is a non-zero prime ideal P^* of R[x] such that $0 \subset P^* \subset <g>$. Since $<g> \cap R = 0$ and by (3.4) there can not be a chain of three distinct prime ideals in R[x] contracting to the same ideal in R, <g> is of height one.

Theorem5.4. Let R be a coprimeley packed integrally closed domain of Krull dimension one and Q^* a prime ideal of R[x] containing a monic polynomial. Then $R[x] / Q^*$ is coprimely packed.

Proof. Let $P = Q^* \cap R$. Then as Q^* contains a monic polynomial, $R / P \subseteq R[x] / Q^*$ is an integral extention. There are two possibilities to be examined.

Case 1. $P \neq 0$ in R in which case R / P is a field and hence by (2.8) $R[x] / Q^*$ is a field being an integral extention of R / P. Therefore R[x] / Q* is coprimely packed.

Case2. P = 0. Then it follows that Q^* is an upper to zero, and so by going-up theorem there is a prime ideal M^* of R[x] properly containing Q^*. Thus Q* is a non–zero prime ideal of R[x] of height one containing a monic polynomial (since otherwise there would be three distinct prime ideals contracting to zero in R). Hence by (5.3), Q^* is generated by an irreducible monic polynomial g in R[x]. Again by the proof of (5.3), we have $R[x] / Q^* = \prod_1^n R$. But then the coprimely packedness of $R[x] / Q^*$ follows from Remark 2 of Chapter 4.

In fact $R[x] / Q^* \cong R$. This is because $R[x] / Q^*$ and $\prod_1^n R$ as R-modules and as rings are one and the same. Since

$$r(f + Q^*) = rf + Q^* = (r + Q^*)(f + Q^*),$$

also for $r \in R$, $(r_1, \ldots, r_n) \in \prod_1^n R$

$$r(r_1, \ldots, r_n) = (rr_1, \ldots, rr_n) = (r, \ldots, r)(r_1, \ldots, r_n)$$

Now $R[x] / Q^*$ being an integral domain and $R[x] / Q^* = \prod_1^n R$ implies n=1.

By virtue of (4.3) we have,

Corollary5.5. Let R be a Dedekind domain of torsion ideal class group and Q^* be a prime ideal of R[x] containing a monic polynomial. Then the class group of R[x] / Q* is torsion.

Proposition.5.6. Let R be principal ideal domain and S be the set of monic polynomials in R[x], T be the set of polynomials in R[x] with unit content.Let $R\langle x\rangle = S^{-1}R[x]$, $R(x) = T^{-1}R[x]$. Then in the following chain of rings $R \subset R[x] \subset R\langle x\rangle \subset R(x)$ all except possibly R[x] are coprimely packed.

Proof. We note at the outset that all rings in the chain $R \subset R[x] \subset R<x> \subset R(x)$ are unique factorization domains. Let M be a prime ideal of R[x] of height 2. Then $M \cap R = P \neq 0$ in R, and so $P[x] \subset M$ and $M / P[x]$ is a maximal ideal in $R[x] / P[x] \cong (R / P)[x]$. Since R / P is a field, $M/ P[x]$ is a principal ideal in $(R / P)[x]$ and hence $M = < f, P[x] >$ for some monic polynomial f in M. But then $M \cap S$ and $M \cap T$ are both non empty. Thus the prime ideals in R<x> and R(x) are of height one and so are principal. Therefore R<x> and R(x) are coprimely packed. If R is not semilocal, then R[x] is not coprimely packed as shown in (4.6).

Finally we remark that in general the coprimely packedness of R always implies the same for R(x) (see [6]), however it does not imply the same for R<x>. For if K is a field and Y_1, Y_2 are two indeterminates then $R = K[[Y_1, Y_2]]$ being a local ring is coprimely packed by (4.1). But R<x> is not coprimely packed. Because by [2], R<x> is a Noetherian Hilbert ring of Krull dimension 2 and so by (4.5) it is not coprimely packed.

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CURRICULUM VITAE

I Esra Şengelen was born in Istanbul in 1971, and have completed my B.Sc. degree at Istanbul University in 1995. In 2000, I have started working as a research assistant in the Department of Mathematics at Istanbul Technical University and still hold the same position.

MERKEZI