İSTANBUL TECHNICAL UNIVERSITY ★ INSTITUTE OF SCIENCE AND TECHNOLOGY

SPIN^C STRUCTURES ON 8-MANIFOLDS

M.Sc. Thesis by
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Department : Mathematics

Programme: Mathematical Engineering

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CONTENTS

LIST OF SYMBOLS

M	**:** Manifold
$T_p M$	**:** Tangent Space at p $\in$ M
TM	**:** Tangent Bundle of M
E	**:** Vector Bundle
π	**:** Projection Map
γ^1_n	**:** Canonical Line Bundle
$P^n(R)$	**:** Real Projective Space
TM^*	**:** Cotangent Bundle of M
s	**:** Cross-section of E
$C^\infty(E)$	**:** Set of Differentiable Cross-sections of E
ω	**:** 1-form on M
∇	**:** Connection on E
A	: Lie Algebra Valued Connection 1-form
Ω	**:** Curvature of A
$M_n(R)$	**:** Real $n \times n$ Matrices
$M_n(C)$	**:** Complex $n \times n$ Matrices
TrA	**:** Trace of matrix A
P	**:** Invariant Polynominal
Cl(V)	: Clifford Algebra of V

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8-MANİFOLD ÜZERİNDEKİ $SPIN^C$ YAPILARI

öZET

Bu çalışmada, öncelikli olarak vektör demeti tanımı ve yapısı hakkında temel bilgiler verilmiş, vektör demetlerinin karakteristik sınıflarının eğrilik 2-formunun invariyant polinomları cinsinden ifadesi incelenmiş ve eğrilik 2-formunun çeşitli kuvvetlerinin izleri ile invariyant polinomlar arasındaki sayısal bağıntılar açık olarak hesaplanmıştır.

Daha sonra $spin^c$ yapısı ve tanımı hakkında temel bilgiler verilmiş, $spin^c$ yapılarının reel ve kompleks temsilleri incelenerek, $A^2 + \lambda^2 I = 0$ koşulunu sağlayan anti-hermitsel matrisler kümesi içindeki maksimal lineer alt uzayların boyutları incelenmiştir.

$SPIN^C$ STRUCTURES ON 8-MANİFOLDS

SUMMARY

In this study, basic information on the definition and structure of vector bundles are given, the expression of the characteristic classes of vector bundles in terms of the invariant polynomials of the curvature 2-form of a connection is reviewed, the numerical relations between the traces of powers of the curvature 2-form matrix and it's invariant polynomials are explicitly obtained.

Then we give basic information on $spin^c$ structures. The real and complex representations of $spin^c$ structures are reviewed and the dimension of maximal linear subspaces of the skew-hermitian matrices satisfying the condition, $A^2 + \lambda^2 I = 0$ is determined.

1. INTRODUCTION

1.1 Introduction and Aim of the Thesis

In this thesis we shall study $spin^c$ structures on even dimensional spaces and in particular $spin^c$ structures on 8-manifolds. We will concentrate on complex representation of a Clifford algebra and the main result of the thesis is the computation of the dimension of maximal linear subspaces lying in the set of $n \times n$ skew-hermitian matrices satisfying $A^2 + \lambda^2 I = 0$.

In Chapter 2, we present a very short overview of manifolds and vector bundles. Some well-known examples of manifolds and illustrative examples of vector bundles are also given in Chapter 2.

In Chapter 3, we will introduce certain notions related to the characteristic classes of a vector bundle. We will define the connection and curvature on a vector bundle as Lie algebra valued 1-forms and 2-forms respectively. Then we will define the invariant polynomials of the curvature 2-form matrix denoted by σ_n and we will obtain the relations between the σ_n's and traces of the powers of the curvature 2-form matrix.

In Chapter 4, we present Clifford algebras, giving their basic properties, their real and complex representations and we discuss $spin^c$ structures and the periodicity properties of the representation Clifford algebras. In particular if a k-dimensional Clifford algebra has a representation on skew-Hermitian or skew-symmetric matrices, the set of such $n \times n$ matrices satisfying $A^2 + \lambda^2 I = 0$ has a k-dimensional subspace. Here we study a related problem, namely we look for the maximal linear subspaces in the set of $n \times n$ skew-Hermitian or skew-symmetric matrices satisfying $A^2+\lambda^2 I=0$. For (real) skew-symmetric matrices the answer is known to be equal to the Radon-Hurwitz number, defined to be the number of linearly independent vector fields on the sphere S^{n-1}. We obtain here the corresponding numbers in the case (complex) skew-Hermitian matrices.

2. MANIFOLDS AND VECTOR BUNDLES

2.1 Manifolds and Vector Bundles: Basic Definitions

2.1.1 Manifolds

An n-dimensional manifold M is a topological space such that each point has a neighborhood homeomorphic to an open subset of the Euclidean space R^n. In addition, we assume that M is also a Hausdorff space. We note that, the Hausdorff condition is an essential part of the definition, because there are locally Euclidean spaces which are non-Hausdorff. **(Munkres, 2000)**

Let M be a manifold, a pair (U, ϕ) is called an n-dimensional chart or coordinate neigborhood of M if $U \subset M$ is an open set and ϕ is a homeomorphism of U to an open subset $\phi(U) \subset R^n$. Two charts (U_1, ϕ_1) and (U_2, ϕ_2) are called C^∞-compatible if whenever $U_1 \cap U_2$ is non-empty, the mapping

$$\phi_1 \circ \phi_2^{-1} : \phi_2(U_1 \cap U_2) \to \phi_1(U_1 \cap U_2) \tag{2.1}$$

is a diffeomorphism.

An atlas is a family of charts (U_α, ϕ_α) where any two are C^∞-compatible and $M = \bigcup_{\alpha \in I} U_\alpha$, where I is an index set. The manifold M with a smooth differentiable structure is called a differentiable manifold. **(Abraham, 1988)**. We give below some well-known examples of manifolds.

Example 1. (The Euclidean space R^n). Taking the open subset $U = R^n$, and using the identity mapping, $(R^n, \phi = I)$ gives an atlas for R^n. Hence the Euclidean space is an n-dimensional manifold.

Example 2. (The sphere S^n). For n=1, the circle S^1 can be thought of as the subset $\{(x, y) \in R^2 : x^2 + y^2 = 1\}$ of the Euclidean space R^2. By using stereographic projection it can be shown that it is a one dimensional manifold. Similarly the n-sphere can be regarded as the subset,

$$S^n = \{(x_1, x_2, ..., x_{n+1}) \in R^{n+1} : x_1^2 + x_2^2 + ... + x_{n+1}^2 = 1\}$$

of the Euclidean space R^{n+1}. By the stereographic projection, we can see that S^n is an n-dimensional manifold. Let us take an atlas for S^n such that

(U_1, ϕ_1) where $U_1 = S^n - \{t\}$, t=$(0, 0, ..., 1) \in R^{n+1}$,

(U_2, ϕ_2) where $U_1 = S^n - \{s\}$, s=$(0, 0, ..., -1) \in R^{n+1}$,

then $S^n = (U_1 \cup U_2)$, and

$$\phi_1(x_1, x_2, ..., x_{n+1}) = (\frac{x_1}{1 - x_{n+1}}, \frac{x_2}{1 - x_{n+1}}, ..., \frac{x_n}{1 - x_{n+1}}) \in R^n,$$

$$\phi_2(x_1, x_2, ..., x_{n+1}) = (\frac{x_1}{1 + x_{n+1}}, \frac{x_2}{1 + x_{n+1}}, ..., \frac{x_n}{1 + x_{n+1}}) \in R^n,$$

where $x_{n+1} = +\sqrt{1 - x_1^2 - x_2^2 - ... - x_n^2}$.

Example 3. (Torus T^n). For n=2, the torus is defined to be the Cartesian product $S^1 \times S^1$ of two circles. Since the product of manifolds is a manifold, It follows that the torus T^2 is a two dimensional manifold. Generally the n-torus;

$$T^n = \underbrace{S^1 \times \cdots \times S^1}_{n-times}$$

is an n-dimensional manifold.

Example 4. (The general linear group $GL(n, R)$). It is defined as the group of non-singular R-linear transformations of R^n into itself. Let $M = M(n, R)$ be the set of $n \times n$ matrices on R. As $M(n, R) \cong R^{n^2}$, it follows that $M(n, R)$ is an n^2 dimensional manifold. Let U=$GL(n, R)$,

$$U = GL(n, R) = \{\, A \in M(n, R) \mid detA \neq 0 \,\}$$

Define $f : U \mapsto R^{n^2}$

$$f(A) = (a_{11}, a_{12}, ..., a_{1n}, a_{21}, ..., a_{nn}) \in R^{n^2}$$

Note that as the det function is continuous, $det^{-1}(0)$ is a closed subset of M, hence its complement U is open in M. Thus $U = GL(n, R)$ as an open subset of R^{n^2} is an n^2 dimensional manifold.

Example 5. (Real Projective space $P^n(R)$). For n=2, the projective plane P^2 is the space obtained from S^2 by identifying each point x of S^2 with its antipodal point $-x$. We define an equivalence relation on S^2 by setting $x \sim (-x)$ then P^2 is the set of equivalence classes. If $p: S^2 \to P^2$ maps each point x to its equivalence class, we topologize P^2 by defining V be open in P^2 if and only if $p^{-1}(v)$ is open in S^2.

$$[x] = \pi(x)$$

$$\pi: x \to \pi(x) = [x] = \{ y \mid y \sim x \}$$

The Real Projective space $P^n(R)$ denotes the set of straight lines of R^{n+1} which pass through the origin $(0, 0, ..., 0) \in R^{n+1}$. Let $x, y \in R^{n+1}$, x and y are equivalent, $x \sim y$, if $y = \lambda x$, $\lambda \neq 0$.

$$[x] \in P^n(R), [x] = [x_1, x_2, ..., x_{n+1}]$$

$$P^n(R) = (R^{n+1} - \{0\})/\sim$$

We will now show that it is an n-dimensional manifold. **(Nakahara,1991)**

Let $x_i \neq 0 \Rightarrow [x_1, x_2, ..., x_{n+1}] = [\frac{x_1}{x_i}, \frac{x_2}{x_i}, ..., 1, ..., \frac{x_{n+1}}{x_i}]$, and take the open subsets of $P(R^n)$ as follows,

$$U_i = \{[x_1, x_2, ..., x_{n+1}] \mid x_i \neq 0\} \subset P(R^n)$$

If we define $y_1 = \frac{x_1}{x_i}, y_2 = \frac{x_2}{x_i}, ..., y_{i-1} = \frac{x_{i-1}}{x_i}, y_i = \frac{x_{i+1}}{x_i}, ..., y_n = \frac{x_{n+1}}{x_i}$, then the map

$$\varphi_i : U_i \to R^n, \qquad \varphi_i([\frac{x_1}{x_i}, \frac{x_2}{x_i}, ..., 1, ..., \frac{x_{n+1}}{x_i}]) = (y_1, y_2, ..., y_n)$$

is continuous and bijective. It can be shown that the φ_i's are smooth and have smooth inverses, hence (U_i, φ_i) constitute an atlas for $P(R^n)$.

2.1.2 Vector Bundles

A vector bundle on a manifold M locally looks like a product space $U \times R^n$, where U is an open subset of M. We now give the formal definition of a vector bundle.

Definition 1. A real vector bundle E over M consists of the following elements:

i) A topological space E called the total space,

ii) A topological space M called the base space,

iii) A continuous map $\pi : E \to M$ called projection map,

iv) The structure of a vector space over the real numbers is the set $\pi^{-1}(b)$ for each $b \in M$.

We often use a shorthand notation $E \xrightarrow{\pi} M$ or simply E to denote a real vector bundle (E, π, M). These elements must satisfy the following condition:

Condition of local triviality: For each $b \in M$ there must exist some neighborhood $U \subset M$ of b, an integer $n \geq 0$ and a homeomorphism

$$h : U \times R^n \to \pi^{-1}(U) \tag{2.2}$$

such that, for all $b \in U$, the correspondence $x \mapsto h(b, x)$ defines an isomorphism between the vector space R^n and the vector space $\pi^{-1}(b)$. **(Milnor, Stasheff, 1974)**. The product manifold $M \times R^n = E$ is the simplest example of a vector bundle, that is, if it is possible to choose U equal to the entire base space, then E is called a trivial bundle over M or product bundle.

For each $b \in M$, the vector space $\pi^{-1}(b)$ is called the fibre over b. This is denoted by F_b or $F_b(E)$. Because π is an onto map, F_b is never empty. If F_b has n-dimensional real vector space structure, then E is called n-dimensional real vector bundle over M.

The concept of a smooth vector bundle can be defined similarly. E and M are smooth manifolds, π is a smooth map, for all b of M there exists a neighborhood

$U \subset M$, and a diffeomorphism h such that (U, h) is a local coordinate system with $b \in U$. Changing R^n with C^n in the definition, we can obtain the definition of a complex vector bundle. Algebraic operations on vector spaces such as taking duals or tensor products can be carried over to vector bundles by applying these operations in each fiber and globalizing **(Milnor, Stasheff, 1974)**.

If we have two bundles E and G over the same base space M we can define their isomorphism as follows.

Definition 2. (Bundle isomorphism) Let E and G be two vector bundles over M. If there exist a homoemorphism

$$f : E \to G$$

between the total spaces which maps each vector space $F_b(E)$ isomorphically onto the corresponding vector space $F_b(G)$, then E is said to be *isomorphic* to G. This is denoted as $E \cong G$.

Definition 3. A cross-section of a vector bundle E with base space M is a continuous function

$$s : M \to E \tag{2.3}$$

which takes each $b \in M$ into the corresponding fiber $F_b(E)$. A cross-section is nowhere zero if $s(b)$ is a non-zero vector of $F_b(E)$ for all $b \in M$. For example if M is a smooth manifold, then a vector field on M is a cross-section of the tangent bundle.

Now consider a collection $\{s_1, \ldots, s_n\}$ of cross-section of a vector bundle E. The cross-sections $s_1, s_2, ..., s_n$ are nowhere dependent if, for all $b \in B$, the vectors $s_1(b), s_2(b), ..., s_n(b)$ are linearly independent.

2.2 Illustrative Examples

Example 1. Tangent Bundle:

Tangent vector: Let M be a manifold, $p \in M$, and $f, g \in C^\infty(M)$. A tangent vector v at p is a real valued function such that,

$$v : C^{\infty}(M) \to R$$

which satisfies the following conditions **(Kobayashi,v1,1996)**;

i)$v(af + bg) = av(f) + bv(g)$ (linearity)

ii)$v(fg) = v(f)g + v(g)f$ (Leibniz rule)

Tangent space: Let M be a manifold, $p \in M$, then the set of tangent vectors at p on M is called a tangent space at p.

$$T_pM = \{\, v_p \mid v_p : C^{\infty}(M) \to R \,\}. \tag{2.4}$$

The tangent space at p is a vector space. Let $v, w \in T_pM$, $\lambda \in R$ then

i) $(v + w)p = v(p) + w(p)$

ii) $(\lambda v)p = \lambda v(p)$

Now we will give the definition of the tangent bundle.

Tangent Bundle: The set of all tangent spaces of a manifold is called the tangent bundle. If M is an n-dimensional manifold, then it is known **(Warner,1983)** that the tangent bundle is a $2n$-dimensional manifold.

$$TM = \cup_{p\in M} T_pM = \{\, (p, v) \mid p \in M, v \in T_pM \,\} \tag{2.5}$$

Remarks.

1-) A vector bundle whose fibre is one-dimensional ($F = R$) is called a line bundle.

2-) A cylinder $S^1 \times R$ is a trivial R-line bundle. As we see below, the Möbius band can be viewed as a non-trivial line bundle over S^1:

i-) Total space E is obtained from $[0, 2\pi] \times R$ by identifying the left side boundary $[0] \times R$ with the right side boundary $[2\pi] \times R$ under the transformation $(0, t) \mapsto (2\pi, -t)$.

ii-) Base space M is a circle obtained from a line segment $(\theta, 0) \subset E$ by identifying its end points.

iii-) The projecton map, $\pi : E \to M$ which maps $(\theta, t) \mapsto (\theta, 0)$.

iv-) The fibre is $\pi^{-1}(b) \cong R, \quad b \in B$.

To check the local triviality condition, let us take two open subsets of base space such that $U = (\pi/4, 7\pi/4)$, $V = (-3\pi/4, 3\pi/4)$. $h : U \times R \to \pi^{-1}(U)$ and $k : V \times R \to \pi^{-1}(v)$ are two transformations.

h: $\quad (t, x) \to [t, x]$. $E = [t, x]$, $[t, x] = (t, x)$ if $0 < t < 2\pi$. $[t, x] = \{(0, x), (2\pi, -x)\}$.

k: $\quad (s, y) \to [s, y]$. $E = [s, y]$, $[s, y] \to (s, y)$, if $s \geq 0$, $[s, y] \to (s + 2\pi, -y)$ if $s \leq 0$.

h and k are homeomorphism which $h(b, x)$ and $k(b, y)$ define an isomorphism between the vector space R and the vector space $\pi^{-1}(b), \quad b \in M$. Hence the condition of local triviality is satisfied. We shall see in the next example that the canonical line bundle γ_n^1 is a non-trivial vector bundle. In fact for n=1, this bundle is a Möbius band **(Milnor, Stasheff, 1974)** hence the Möbius band is a non-trivial vector bundle.

Example 2. The Canonical Real Line Bundle

The canonical real line bundle γ_n or γ_n^1 is a one-dimensional real vector bundle over the projective space $P^n(R)$ with total space

$$E(\gamma_n^1) = \{([x], v) \in P^n(R) \times R^{n+1} : v = \lambda x \quad for \;\; some \quad \lambda \in R\} \qquad (2.6)$$

where $[x]$ denotes the line that passes through $x \in R^{n+1}$. The projection map $\pi : E(\gamma_n^1) \to P^n(R)$ is defined by $\pi([x], v) = [x]$. Thus each fiber $\pi^{-1}([x])$ is the line in R^{n+1} that passes through x and $-x$. Each such a line is to be given by its usual vector space structure.

Let us show the condition of local triviality. If $U \subset S^n$ is any open set that is sufficiently small, and this set contains no pair of antipodal points, and also if

U' is corresponding set $\pi(U)$ in $P^n(R)$, then a local homeomorphism,

$$h : U' \times R \to \pi^{-1}(U')$$

can be defined by

$$h([x], \lambda) = ([x], \lambda x)$$

for all $(x, \lambda) \in U \times R$. By this homeomorphism, the condition of local triviality is satified.

Now we want to show that the canonical real line bundle γ_n^1 is not trivial. To prove this, we show that γ_n^1 has no nowhere zero section. Let us take a cross-section from base space to total space

$$s : P^n(R) \to E(\gamma_n^1)$$

and think about the composition from S^n to total space $E(\gamma_n^1)$

$$S^n \to P^n \xrightarrow{s} E(\gamma_n^1).$$

In this composition, the image of every $x \in S^n$ in total space is

$$([x], f(x)x) \in E(\gamma_n^1)$$

where $f(x)$ is a continuous real valued funtion which satifies

$$f(-x) = -f(x).$$

S^n is connected space and by the intermediate value theorem that $f(x_0) = 0$ for some $f(x_0)$. Hence $s([x_0]) = ([x_0], 0)$. This shows that there is no nowhere zero cross-section for the canonical real line bundle. Hence γ_n^1 is not a trivial bundle. **(Milnor, Stasheff, 1974)**.

Now we give an example for sections of a vector bundle.

Example 3. We will show that the unit sphere S^n admits a vector field which is nowhere zero provided that n is odd.

For n=1, the tangent bundle of the circle $S^1 \subset R^2$ admits one nowhere zero cross-section. The arrows lead from $x \in S^1$ to $x + v$, where

$$s(x) = (x, v) = ((x_1, x_2), (-x_2, x_1)).$$

For n=3, the 3-sphere $S^3 \subset R^4$ admits three nowhere dependent vector fields $s_i(x) = (x, v_i(x))$ where

$$v_1(x) = (-x_2, x_1, -x_4, x_3),$$

$$v_2(x) = (-x_3, x_4, x_1, -x_2),$$

$$v_3(x) = (-x_4, -x_3, x_2, x_1).$$

For n=2m-1, then

$$v(x) = (-x_2, x_1, -x_4, x_3, \ldots, -x_{2m}, x_{2m-1})$$

is a nowhere zero vector field on S^n.

3. CHARACTERISTIC CLASSES OF A VECTOR BUNDLE

3.1 Connections and Curvature on Vector Bundle

We consider E as a smooth complex vector bundle with smooth base space M. We know that a cross-section of the smooth complex bundle E with the smooth base space M is a continuous function $s : M \rightarrow E$. Let S be the set of cross-sections $s : M \rightarrow E$, then S has the structure of a vector space over C with addition and scalar multiplication defined by

i) $(s_1 + s_2)(x) = s_1(x) + s_2(x), \qquad s_1, s_2 \in S, \qquad x \in M,$

ii) $(\lambda s)(x) = \lambda(s(x)), \qquad s \in S, \lambda \in C, \qquad x \in M.$

The vector space of smooth sections of E is denoted by $C^{\infty}(E)$. As noted before, algebraic operations among vector spaces can be used to define new vector bundles from old ones, as described for example in **(Milnor, Statsheff, 1974)**. In particular, we have the dual of the tangent bundle TM

$$TM^{*} = Hom_R(TM, R), \tag{3.1}$$

which is also called the cotangent bundle. Its sections are 1-forms. Then one can obtain new bundles by taking tensor products and k-th exterior products. The sections of these latter bundles will be called k-forms.

Definition 1. The *complexification* of the cotangent bundle is defined as

$$Hom_R(TM, R) \otimes_R C \cong Hom_R(TM, C). \tag{3.2}$$

It is known that this complexification of the cotangent bundle of M is also vector bundle. **(Milnor, Stasheff, 1974)** that we denote as

$$TM_C^{*} = Hom_R(TM, C).$$

Since TM_C^* and E are complex vector bundles over the same base space M, the tensor product of $TM_C^* \otimes E$ is also a complex vector bundle over M. Similarly, the cross-sections of the complex vector bundle $TM_C^* \otimes E$ form a vector space. The vector space of smooth sections of $TM_C^* \otimes E$ is denoted by $C^\infty(TM_C^* \otimes E)$.

Definition 2. A *connection* on the vector bundle E is a complex-linear transformation

$$\nabla : C^\infty(E) \to C^\infty(TM_C^* \otimes E)$$

such that

$$\nabla(fs) = df \otimes s + f\nabla(s), \tag{3.3}$$

for any $s \in C^\infty(E)$ and for all $f \in C^\infty$.

It is well known that if the base manifold is paracompact, the connection can be defined by

$$\nabla(s_i) = \sum w_{ij} \otimes s_j \tag{3.4}$$

where $\{s_i\}$ is a local basis of sections and $[w_{ij}]$ is an arbitrary $n \times n$ matrix of 1-forms on U. When the base space is paracompact, these local connections can be patched by a partition of unity, leading to a globally defined connection on the vector bundle.

The image of the sections under ∇ is called covariant derivative of s $\in C^\infty(E)$, that is, $\nabla(s)$. Since the connection ∇ is a C-linear mapping, the following properties are satisfied;

i) $\nabla(s_i + s_j) = \nabla(s_i) + \nabla(s_j)$, $\qquad s_i, s_j \in C^\infty(E)$,

ii) $\nabla(ks_i) = k\nabla(s_i) \qquad k \in C$.

We defined above a connection ∇ on the complex vector bundle E. Similarly, we will define another connection $\hat{\nabla}$ on the complex vector bundle $TM_C^* \otimes E$ by using ∇ and the exterior differentiation operator.

Definition 3. For a given connection ∇, the complex-linear transformation $\hat{\nabla}$;

$$\hat{\nabla} : C^\infty(TM_C^* \otimes E) \to C^\infty(\Lambda^2 TM_C^* \otimes E)$$

is defined by

$$\hat{\nabla}(\varphi \otimes s) = d\varphi \otimes s - \varphi \wedge \nabla(s) \tag{3.5}$$

for all 1-forms φ and for all sections $s \in C^{\infty}(E)$. Therefore $\hat{\nabla}$ satisfies the equality, $\hat{\nabla}(f(\varphi \otimes s)) = df \wedge (\varphi \otimes s) + f\hat{\nabla}(\varphi \otimes s)$.

Let us consider the composition of two complex-linear transformation ∇ and $\hat{\nabla}$ on complex vector bundle and use K to be the composition $K = \hat{\nabla} \circ \nabla$ such that **(Milnor, Stasheff, 1974)**

$$C^{\infty}(E) \xrightarrow{\nabla} C^{\infty}(TM_C^* \otimes E) \xrightarrow{\hat{\nabla}} C^{\infty}(\Lambda^2 TM_C^* \otimes E). \tag{3.6}$$

Before giving the definition of the curvature tensor, the value of $K(s)$ is defined $K(s) = \hat{\nabla}(\nabla(s))$ for any $s \in C^{\infty}(E)$. Here $s(x) \to K(s)(x)$ defines a smooth section of the complex vector bundle $Hom(E, \Lambda^2 TM_C^* \otimes E)$.

Definition 4. The curvature tensor of the connection ∇ is the section of K_{∇} of the complex vector bundle $Hom(E, \Lambda^2 TM_C^* \otimes E) \cong \Lambda^2 TM_C^* \otimes Hom(E, E)$.

Lemma 1. K is a $C^{\infty}(M, C)$-linear operator.

Proof. We must show that $K(fs) = fK(s)$, for every $f \in C^{\infty}(M, C)$. Let us compute $K(fs) = \hat{\nabla}(\nabla(fs))$. Using definition of the connection, we show this as follows,
$K(fs) = \hat{\nabla}(\nabla(fs)) = \hat{\nabla}(df \otimes s + f\nabla(s)) = d^2 f \otimes s - df \wedge \nabla(s) + df \wedge \nabla(s) + f\hat{\nabla}(\nabla(s)) = f\hat{\nabla}(\nabla(s)) = fK(s)$.

Let $\{s_1, s_2, ..., s_n\}$ be a local basis of the section of the vector bundle, and let $\nabla(s_i) = \sum w_{ij} \otimes s_j$. Then $K(s_i) = \hat{\nabla}(\nabla(s_i)$,

$$s_i \xrightarrow{\nabla} \sum w_{ij} \otimes s_j \xrightarrow{\hat{\nabla}} \sum [dw_{ij} \otimes s_j - (w_{ij} \wedge w_{jk}) \otimes s_k].$$

Hence

$$K(s_i) = \sum dw_{ij} \otimes s_j - (w_{ij} \wedge w_{jk}) \otimes s_k$$

$$= \sum dw_{il} \otimes s_l - (w_{ij} \wedge w_{jl}) \otimes s_l$$

$$= \sum (dw_{il} - w_{ij} \wedge w_{jl}) \otimes s_l$$

$$= \sum \Omega_{il} \otimes s_l$$

where $\Omega_{il} = dw_{il} - w_{ij} \wedge w_{jl}$ is the $n \times n$ matrix of 2-forms. That is,

$$K(s_i) = \sum \Omega_{il} \otimes s_l. \tag{3.7}$$

Similiarly, ∇ is described by the matrix $w = [w_{ij}]$ of 1-forms. Finally, in matrix notation

$$\Omega = dw - w \wedge w. \tag{3.8}$$

3.2 Invariant Polynomials and Characteristic Classes

An invariant polynomial on $n \times n$ complex matrices $M_n(C)$ is a function

$$P : M_n(C) \to C$$

which can be expressed as a complex polynomial in the entries of the matrix, and which satisfies

$$P(XY) = P(YX), \tag{3.9}$$

where X,Y $\in M_n(C)$. **(Milnor, Stasheff, 1974)**

Example 1. The trace function $[x_{ij}] \mapsto \sum x_{ii}$, and the determinant function are well known examples of invariant polynomials on $M_n(C)$.

Theorem 1. For any invariant polynomial P, the exterior form $P(K)$ is closed, that is dP(K)=0, where K is the curvature matrix.

Proof. Let $P(A) = P([a_{ij}])$ where the a_{ij}'s are indeterminates, be an invariant polynomial of the matrix A. We form the matrix of the first derivatives as

$$[\partial P / \partial a_{ij}].$$

The exterior derivative $dP(A)$ is equal to the expression,

$$dP(A) = \sum(\partial P/\partial a_{ij})da_{ij}. \tag{3.10}$$

We write these expressions in matrix form as

$$[\partial P/\partial a_{ij}] = \begin{pmatrix} P_{11} & P_{12} & \dots & P_{1n} \\ P_{21} & P_{22} & \dots & P_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ P_{n1} & P_{n2} & \dots & P_{nn} \end{pmatrix}, d[a_{ij}] = \begin{pmatrix} da_{11} & da_{12} & \dots & da_{1n} \\ da_{21} & da_{22} & \dots & da_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ da_{n1} & da_{n2} & \dots & da_{nn} \end{pmatrix},$$

$$dP(A) = \sum(\partial P/\partial a_{ij})da_{ij} = P_{11}da_{11} + P_{12}da_{12} + ... + P_{nn}da_{nn}$$

If we denote the transpose of the first derivative matrix by the symbol $P'(A)$ then ;

$$dP(A) = P_{11}da_{11} + P_{12}da_{12} + ... + P_{nn}da_{nn} = tr(P'(A)da_{ij}).$$

Now let $\Omega = [\Omega_{ij}]$ be the curvature matrix (3.8), then the exterior derivative $dP(\Omega)$ is

$$dP(\Omega) = \sum(\partial P/\partial\Omega_{ij})d\Omega_{ij},$$

which can be written in matrix form as

$$dP(\Omega) = tr(P'(\Omega)d\Omega).$$

As the curvature matrix is $\Omega = d\omega - \omega\wedge\omega$, taking the exterior derivative of Ω, we compute the 3-form $d\Omega$ as $d\Omega = d(d\omega - \omega\wedge\omega) = 0 - d(\omega\wedge\omega) = -d\omega\wedge\omega + w\wedge d\omega = w\wedge(\Omega + \omega\wedge\omega) - (\Omega + \omega\wedge\omega)\wedge\omega$, which implies

$$d\Omega = \omega\wedge\Omega - \Omega\wedge\omega. \tag{3.11}$$

This equality is called the Bianchi identity. We now need to prove the following claim.

Claim: For any invariant polynominal P, the transposed matrix of first derivatives $P'(A)$ commutes with A.

Proof of the Claim: Let E_{ji} denote the matrix with entry 1 in the (j,i)-th place and zero elsewhere. As P is an invariant polynomial, $P(BA) = P(AB)$, and taking $B = (I + tE_{ji})$ we obtain,

$$P((I + tE_{ji})A) = P(A(I + tEji))$$

Let $C = ((I + tE_{ji})A)$, $D = (A(I + tEji))$. Their components are respectively

$$C_{\alpha\beta} = [I + tE_{ji}]_{\alpha\gamma}[A]_{\gamma\beta} = [\delta_{\alpha\gamma} + t\delta_{\alpha j}\delta_{\gamma i}]A_{\gamma\beta} = A_{\alpha\beta} + t\delta_{\alpha j}A_{i\beta}$$

$$D_{\alpha\beta} = A_{\alpha\gamma}[I + tE_{ji}]_{\gamma\beta} = A_{\alpha\gamma}[\delta_{\gamma\beta} + t\delta_{\gamma j}\delta_{\beta i}] = A_{\alpha\beta} + tA_{\alpha j}\delta_{\beta i}$$

Differentiating the equality $P(C) = P(D)$ with respect to t we obtain

$$\frac{dP}{dt} = \frac{\partial P}{\partial C_{\alpha\beta}}\frac{dC_{\alpha\beta}}{dt} = \frac{\partial P}{\partial D_{\alpha\beta}}\frac{dD_{\alpha\beta}}{dt}$$
$$\frac{\partial P}{\partial C_{\alpha\beta}}\delta_{\alpha j}A_{i\beta} = \frac{\partial P}{\partial D_{\alpha\beta}}A_{\alpha j}\delta_{\beta i}$$
$$\frac{\partial P}{\partial C_{j\beta}}A_{i\beta} = \frac{\partial P}{\partial D_{\alpha i}}A_{\alpha j}$$

Finally if we set $t = 0$, then $A = C = D$, hence the last equality gives

$$\frac{\partial P}{\partial A_{j\beta}}A_{i\beta} = \frac{\partial P}{\partial A_{\alpha i}}A_{\alpha j}$$

or

$$A_{i\beta}[\frac{\partial P}{\partial A_{j\beta}}] = [\frac{\partial P}{\partial A_{\alpha i}}]A_{\alpha j}.$$

Thus the transposed matrix of the first derivatives $[\frac{\partial P}{\partial A_{ij}}]$ commutes with A. Now replacing A with Ω, it follows that;

$$\Omega \wedge P'(\Omega) = P'(\Omega) \wedge \Omega$$

$$dP(\Omega) = trace(P'(\Omega)d\Omega)$$

Using the Bianchi identity we have

$$dP(\Omega) = trace(P'(\Omega) \wedge (\omega \wedge \Omega - \Omega \wedge \omega))$$

As $tr(AB) = tr(BA)$ for any A, B, we can write

$$dP(\Omega) = trace(\omega \wedge \Omega \wedge P'(\Omega) - \omega \wedge \Omega \wedge P'(\Omega))$$

Now using the claim above we obtain

$$dP(\Omega) = trace(\omega \wedge \Omega \wedge P'(\Omega) - \omega \wedge \Omega \wedge P'(\Omega)) = 0,$$

which proves the theorem.

In deRham cohomology theory **(Warner, 1983)**, closed forms on a manifold define cohomology classes. Thus invariant polynomials of the curvature 2-form of the bundle define deRham cohomology classes of the base manifold. It is known that the deRham cohomology classes defined above are related to Chern classes and Pontrjagin classes as follows **(Milnor and Stasheff, 1974)**.

$[\sigma_i] \cong c_i$, c_i: Chern class of the bundle,

$[\sigma_{2i}] \cong c_{2i} \cong p_i$, p_i: Pontrjagin class of the bundle.

We will now concentrate on the computation of the invariant polynomials of the curvature 2-form matrix. For any $n \times n$ matrix A, we can compute the characteristic polynomial equation for A as follows;

$$det(A + \lambda I) = \lambda^n + \sigma_1 \lambda^{n-1} + ... + \sigma_j \lambda^{n-j} + ... + \sigma_n.$$

Note that whenever the entries of a matrix are even forms, it makes sense to take the determinant. Hence taking A=Ω (3.8), we obtain the characteristic polynomial of the curvature matrix Ω. The σ_n's are representatives of the characteristic classes that we are interested in. In the next subsection we shall obtain the expression of the σ_n's above in terms of the traces of the powers of Ω.

3.3 Relations Between σ_n and $tr(A^n)$

Let us take any $n \times n$ matrix A and write it's Jordan canonical form.

$$A = PJP^{-1}, J = \begin{pmatrix} J_1 & & & \\ & J_2 & & \\ & & \ddots & \\ & & & J_k \end{pmatrix},$$

where J_i's are the Jordan blocks of J. Note that as each Jordan block is lower triangular, det(A+λI) is independent of the canonical form but depends only on the eigenvalues. We compute det(A+λI) as follows;

$$det(A + \lambda I) = det(PJP^{-1} + \lambda PP^{-1}) = det(P(J + \lambda I)P^{-1})$$

$$= (detP)det(J + \lambda I)(detP^{-1}) = det(J + \lambda I).$$

But det(J+ λI)= $\prod det(J_i + \lambda I)$ =$\prod(\lambda_i + \lambda)^{k_i}$, where k_i is the size of the Jordan block J_i. Hence we get the equality below.

$$det(A + \lambda I) = \lambda^n + \sigma_1 \lambda^{n-1} + ... + \sigma_j \lambda^{n-j} + ... + \sigma_n = \prod_i (\lambda + \lambda_i)^{k_i}. \quad (3.12)$$

It is well known that the σ_n's can be calculated either in terms of eigenvalues of A (**Hungerford, 1987**);

$$\sigma_j = \textstyle\sum_{1 \leq i_1 < i_2 ... < i_j \leq n} \lambda_{i_1} \lambda_{i_2} ... \lambda_{i_j}, \qquad j! \sigma_j = \textstyle\sum_{i_1 \neq i_2 ... \neq i_j} \lambda_{i_1} \lambda_{i_2} ... \lambda_{i_j},$$

or in terms of the entries of A as

$\sigma_j = \sum$ j'th principal minors. **(Gantmacher,1960)**

Let us compute the σ_n's for a 3×3 matrix.

Let

$$A = \begin{pmatrix} a & b & c \\ d & e & f \\ m & n & p \end{pmatrix} \quad (3.13)$$

A has one of the following Jordan forms,

$$J_1 = \begin{pmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_3 \end{pmatrix}, J_2 = \begin{pmatrix} \lambda_1 & & \\ & \lambda_1 & \\ & & \lambda_2 \end{pmatrix}, J_3 = \begin{pmatrix} \lambda_1 & & \\ 1 & \lambda_1 & \\ & & \lambda_2 \end{pmatrix}$$

$$J_4 = \begin{pmatrix} \lambda & & \\ & \lambda & \\ & & \lambda \end{pmatrix}, J_5 = \begin{pmatrix} \lambda & & \\ 1 & \lambda & \\ & & \lambda \end{pmatrix}, J_6 = \begin{pmatrix} \lambda & & \\ 1 & \lambda & \\ & 1 & \lambda \end{pmatrix}$$

and in all of the 6 cases, σ_n's are

$\sigma_1{=}\sum_{i=1}^{3} \lambda_i{=}\lambda_1 + \lambda_2 + \lambda_3,$

$\sigma_2{=}\sum_{1\leq i<j\leq 3}^{3} \lambda_i\lambda_j{=}\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3,$

$\sigma_3{=}\sum_{1\leq i<j<k\leq 3}^{3} \lambda_i\lambda_j\lambda_k{=}\lambda_1\lambda_2\lambda_3,$

where the eigenvalues may coincide.

On the other hand, we can find σ_n's by using n-th principal minors as follows;

$\sigma_1{=}\ |a| + |e| + |p|{=}\ a + e + p\ ,$

$$\sigma_2{=}\begin{vmatrix} a & b \\ d & e \end{vmatrix} + \begin{vmatrix} a & c \\ m & p \end{vmatrix} + \begin{vmatrix} e & f \\ n & p \end{vmatrix} {=}ae - bd + ap - cm + ep - nf,$$

$$\sigma_3{=}\begin{vmatrix} a & b & c \\ d & e & f \\ m & n & p \end{vmatrix} {=}\ aep - afn + bdp - bfm + cdn - cem.$$

Now let us find the relation between σ_n and $\mathrm{tr}(A^n)$.

For n=1, $\sigma_1{=}a + e + p\ {=}\lambda_1 + \lambda_2 + \lambda_3{=}$tr J= tr A.

For n=2, $\sigma_2{=}\ ae - bd + ap - cm + ep - nf{=}\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3.$

$trJ^2{=}\lambda_1^2 + \lambda_2^2 + \lambda_3^2$, tr J= $\lambda_1 + \lambda_2 + \lambda_3$, on the other hand we can see that

$$2(\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3) = -(\lambda_1^2 + \lambda_2^2 + \lambda_3^2) + (\lambda_1 + \lambda_2 + \lambda_3)^2.$$

Hence,

$$2!\sigma_2 = -(trA^2) + (trA)^2. \tag{3.14}$$

For n=3, $\sigma_3 = det(A) = \lambda_1\lambda_2\lambda_3$. We will express this in terms of $trA^3 = trJ^3 = \lambda_1^3 + \lambda_2^3 + \lambda_3^3$, trA^2 and $tr(A)$. We write

$\lambda_1\lambda_2\lambda_3 = \alpha_1(\lambda_1^3 + \lambda_2^3 + \lambda_3^3) + \alpha_2(\lambda_1^2 + \lambda_2^2 + \lambda_3^2)(\lambda_1 + \lambda_2 + \lambda_3) + \alpha_3(\lambda_1 + \lambda_2 + \lambda_3)^3$

Equating the coefficients of various terms in the λ_i's, we can see that $\alpha_1 + \alpha_2 + \alpha_3 = 0, \ \ \alpha_2 + 3\alpha_3 = 0, \ \ 6\alpha_3 = 1.$

Hence

$$3!\sigma_3 = 2(trA^3) - 3(trA^2)(trA) + (trA)^3. \tag{3.15}$$

We will now give a general method for the computation of the σ_n in terms of $tr(A^k)$'s. For this let us first rewrite σ_n, $n = 1, \ldots, 4$ as below.

$$1!\sigma_1 = C^1_{1,1} trA,$$

$$2!\sigma_2 = C^1_{2,1}(trA^2) + C^2_{2,1}(trA)^2,$$

$$3!\sigma_3 = C^1_{3,1}(trA^3) + C^2_{3,1}(trA^2)(trA) + C^3_{3,1}(trA)^3,$$

$$4!\sigma_4 = C^1_{4,1}(trA^4) + C^2_{4,1}(trA^3)(trA) + C^2_{4,2}(trA^2)^2 + C^3_{4,1}(trA^2)(trA)^2 + C^4_{4,1}(trA)^4,$$

where $C^\gamma_{n,\alpha}$ are constants. We will determine these constants by assigning specific values to the eigenvalues. At the first stage we take $\lambda_1 = 1$, all others zero, then $\lambda_1 = \lambda_2 = 1$, the rest zero, and so on. We will see that this procedure will be insufficient to determine $C^\gamma_{n,\alpha}$, whenever $\alpha \geq 2$.

i. Let $\lambda_1 = 1, \quad \lambda_2 = \lambda_3 = ... = \lambda_n = 0, \quad trA^k = 1, \quad \sigma_1 = 1, \quad \sigma_k = 0 \ \ k > 1.$

$$C^1_{1,1} = 1$$

$$0 = C^1_{2,1} + C^2_{2,1}$$

$$0 = C^1_{3,1} + C^2_{3,1} + C^3_{3,1}$$

$$0 = C^1_{4,1} + C^2_{4,1} + C^2_{4,2} + C^3_{4,1} + C^4_{4,1}$$

ii. Let $\lambda_1 = \lambda_2 = 1, \ \lambda_3 = \lambda_4 = ... = \lambda_n = 0, \ trA^k = 2, \ \sigma_2 = 1, \ \sigma_k = 0 \ \ k > 2.$

$$2! = C^1_{2,1}2 + C^2_{2,1}(2)^2$$

$$0 = C^1_{3,1}2 + C^2_{3,1}(2)(2) + C^3_{3,1}(2)^3$$

$$0 = C^1_{4,1}2 + C^2_{4,1}(2)(2) + C^2_{4,2}(2)^2 + C^3_{4,1}(2^2)(2) + C^4_{4,1}(2)^4$$

iii. Let $\lambda_1 = \lambda_2 = \lambda_3 = 1, \ \lambda_4 = ... = \lambda_n = 0, \ trA^k = 3, \ \sigma_3 = 1, \ \sigma_k = 0 \ \ k > 3.$

$$3! = C^1_{3,1}(3) + C^2_{3,1}(3)(3) + C^3_{3,1}(3)^3$$

$$0 = C^1_{4,1}3 + C^2_{4,1}(3)(3) + C^2_{4,2}(3)^2 + C^3_{4,1}(3^2)(3) + C^4_{4,1}(3)^4$$

iv. Let $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = 1,\ \ \lambda_5 = ... = \lambda_n = 0,\ \ trA^k = 4,\ \ \sigma_4 = 1,\ \ \sigma_k = 0\ \ k > 4.$

$$4! = C^1_{3,1}(4) + C^2_{3,1}(4)(4) + C^3_{3,1}(4)^3$$

$$0 = C^1_{4,1}4 + C^2_{4,1}(4)(4) + C^2_{4,2}(4)^2 + C^3_{4,1}(4^2)(4) + C^4_{4,1}(4)^4$$

Note that for $n = 1, 2, 3$ we can determine the $C^\gamma_{n,\alpha}$'s from the equations above, but this procedure gives only the sum $C^2_{4,1} + C^2_{4,2}$. To determine the coefficients $C^\gamma_{n,\alpha}$ for same γ but different α, we could take for example $\lambda_1 = \lambda_2 = ... = \lambda_{n-1} = \lambda, \lambda_n = \beta$ then,

$4!(\ \binom{n}{4}\ \text{-}\binom{n-1}{3})\ \lambda^4 + 4!\binom{n-1}{3}\lambda^3\beta = C^1_{4,1}(trJ^4) + C^2_{4,1}(trJ^3)(trA) + C^2_{4,2}(trJ^2)^2 + C^3_{4,1}(trJ^2)(trJ)^2 + C^4_{4,1}(trJ)^4, \qquad \forall n \in N,$

$4!(\binom{n}{4}\text{-}\binom{n-1}{3})\ \lambda^4 + 4!\binom{n-1}{3}\lambda^3\beta = C^1_{4,1}((n-1)\lambda^4 + \beta^4) + C^2_{4,1}((n-1)\lambda^3 + \beta^3)((n-1)\lambda + \beta) + C^2_{4,2}((n-1)\lambda^2 + \beta^2)^2 + C^3_{4,1}((n-1)\lambda^2 + \beta^2)((n-1)\lambda + \beta)^2 + C^4_{4,1}((n-1)\lambda + \beta)^4, \qquad \forall n \in N,$

Equating the coefficients of λ^4 and λ^3 in the both sides of this equation we obtain

$4!\ (\binom{n}{4}\text{-}\binom{n-1}{3})\ \lambda^4 = [C^1_{4,1}(n-1) + (C^2_{4,1} + C^2_{4,2})(n-1)^2 + C^3_{4,1}(n-1)^3 + C^4_{4,1}(n-1)^4]\lambda^4,$

$4!\ (\binom{n-1}{3})\ \lambda^3\beta = (C^2_{4,1} + 2C^3_{4,1} + 4C^4_{4,1})(n-1)^3\lambda^3\beta.$

Combining these equations with the previous ones we can determine the coefficients of $C^\gamma_{4,\alpha}$ as $C^1_{4,1} = -6, C^2_{4,1} = 8, C^2_{4,2} = 3, C^3_{4,1} = -6, C^4_{4,1} = 1$. Hence

$$4!\sigma_4 = -6(trA^4) + 8(trA^3)(trA) + 3(trA^2)^2 - 6(trA^2)(trA)^2 + (trA)^4 \quad (3.16)$$

The expressions of σ_k, $k = 5, 6, 7$ are obtained similarly by using Mathematica. We give these explicit expressions below.

$5!\sigma_5 = 24(trA^5) - 30(trA^4)(trA) - 20(trA^3)(trA^2) + 20(trA^3)(trA)^2 + 15(trA^2)^2(trA) - 10(trA^2)(trA)^3 + (trA)^5.$

$6!\sigma_6 = -120(trA^6) + 144(trA^5)(trA) + 90(trA^4)(trA^2) + 40(trA^3)^2 - 90(trA^4)(trA)^2 - 120(trA^3)(trA^2)(trA) - 15(trA^2)^3 + 40(trA^3)(trA)^3 + 45(trA^2)^2(trA)^2 - 15(trA^2)(trA)^4 + (trA)^6.$

$7!\sigma_7 = 720(trA^7) - 735(trA^6)(trA) - 189(trA^5)(trA^2) - 840(trA^4)(trA^3) + 189(trA^5)(trA)^2 + 1050(trA^4)(trA^2)(trA) + 385(trA^3)^2(trA) - 210(trA^4)(trA)^3 - 420(trA^3)(trA^2)(trA)^2 - 105(trA^2)^3(trA) + 70(trA^3)(trA)^4 + 105(trA^2)^2(trA)^3 - 21(trA^2)(trA)^5 + (trA)^7.$

We will now write the expression of σ_n in terms of the partitions of the integer n. In the expression given by Eq. (3.17), the coefficients are to be determined by assigning specific values to the λ_i's as above.

Definition 4. A partition of an integer $n \geq 0$ is an unordered sequence $r_1 r_2 ... r_s$ of positive integers with sum n. The number of partitions of n is denoted by p(n). Let i_α be the number of distinct partition of n into α summands. Let us give below partition of integers, n=1,2,..,5.

p(1)=1, $n = 1 \Rightarrow \underbrace{\{1\}}_{i_1=1},$

p(2)=2, $n = 2 \Rightarrow \underbrace{\{2\}}_{i_1=1},\ \underbrace{\{1+1\}}_{i_2=1},$

p(3)=3, $n = 3 \Rightarrow \underbrace{\{3\}}_{i_1=1},\ \underbrace{\{2+1\}}_{i_2=1}, \underbrace{\{1+1+1\}}_{i_3=1},$

p(4)=5, $n = 4 \Rightarrow \underbrace{\{4\}}_{i_1=1},\ \underbrace{\{3+1, 2+2\}}_{i_2=2}, \underbrace{\{2+1+1\}}_{i_3=1}, \underbrace{\{1+1+1+1\}}_{i_4=1},$

p(5)=7, $n = 5 \Rightarrow \underbrace{\{5\}}_{i_1=1},\ \underbrace{\{4+1, 3+2\}}_{i_2=2}, \underbrace{\{3+1+1, 2+2+1\}}_{i_3=2},$
$\underbrace{\{2+1+1+1\}}_{i_4=1}, \underbrace{\{1+1+1+1+1\}}_{i_5=1},$

Note that p(n)=$\sum_{\alpha=1}^{n} i_\alpha$. The numbers in the braces will be denoted by $J^\gamma_{\alpha,\beta}$. For simplicity of notation we will not indicate the dependency of J on n. Comparing with the table above we can write the first few $J^\gamma_{\alpha,\beta}$'s as below, with $\alpha = 1, ..., n$,

i_α as in the table above and $\beta = 1, ..., \alpha \quad \gamma = 1, ..., i_\alpha$.

$$n = 1, \quad i_1 = 1, \quad J^1_{1,1} = 1$$

$$n = 2, \quad i_1 = 1, \quad J^1_{1,1} = 2$$

$$i_2 = 1, \quad J^1_{2,1} = 1, J^1_{2,2} = 1$$

$$n = 3, \quad i_1 = 1, \quad J^1_{1,1} = 3$$

$$i_2 = 1, \quad J^1_{2,1} = 2, J^1_{2,2} = 1$$

$$i_3 = 1, \quad J^1_{3,1} = 1, J^1_{3,2} = 1, J^1_{3,3} = 1$$

$$n = 4, \quad i_1 = 1, \quad J^1_{1,1} = 4$$

$$i_2 = 2, \quad J^1_{2,1} = 3, J^1_{2,2} = 1$$

$$J^2_{2,1} = 2, J^2_{2,2} = 2$$

$$i_3 = 1, \quad J^1_{3,1} = 2, J^1_{3,2} = 1, J^1_{3,3} = 1$$

$$i_4 = 1, \quad J^1_{4,1} = 1, J^1_{4,2} = 1, J^1_{4,3} = 1, J^1_{4,4} = 1$$

With the notations above, the general expression of σ_n in terms of trA^n is as follows,

$$n!\sigma_n = \sum_{\alpha=1}^{n} \sum_{\gamma=1}^{i_\alpha} C^\gamma_{n,\alpha} \left(\prod_{\beta=1}^{\alpha} trA^{J^\gamma_{\alpha,\beta}} \right) \tag{3.17}$$

We note that a method for iterative computation of the σ_n, known as the method of Faddeev, is given in **(Gantmacher, 1960)**.

3.4 Topological Invariants and The Yang-Mills Action

Let E be a vector bundle over an $2n$-dimensional smooth manifold without boundary M and let σ_k's be the invariant polynomials the curvature 2-form of E. If P is a polynomial in the σ_k's such that $P(\sigma_k)$ is a homogeneous $2n$-form, then the integral of $P(\sigma_k)$ over M will be a topological invariant of the bundle.

For example in 4-dim, $P(\sigma_i) = a\sigma_1^2 + b\sigma_2$ is a 4-form and the integral

$$\int_{M^4} (a\sigma_1^2 + b\sigma_2)$$

is a topological invariant.

In 8-dim, one can choose $P(\sigma_i) = a\sigma_1^4 + b\sigma_1^2\sigma_2 + c\sigma_1\sigma_3 + d\sigma_2^2 + e\sigma_4$ and obtain the topological invariant,

$$\int_{M^8} (a\sigma_1^4 + b\sigma_1^2\sigma_2 + c\sigma_1\sigma_3 + d\sigma_2^2 + e\sigma_4)$$

If the connection on the bundle E is compatible with an inner product, then the connection and the curvature takes their values in the Lie algebra $o(n)$, i.e. $A = [\omega_{ij}]$ is a skew-symmetric matrix of 1-forms. This can be seen easily as follows. If the connection ∇ is compatible with an inner product, then

$$\nabla < Y, Z > = < \nabla Y, Z > + < Y, \nabla Z >$$

where $< Y, Z >$ is the inner product of X and Y. For $X = s_i$ and $Y = s_j$, we have $\nabla s_i = \sum \omega_{ik} \otimes s_k, \quad < s_i, s_j > = \delta_{ij}$.

$$\begin{aligned} \nabla < s_i, s_j > = \nabla(\delta_{ij}) = 0 &= < \nabla s_i, s_j > + < s_i, \nabla s_j > \\ &= < \omega_{ik} s_k, s_j > + < s_i, \omega_{jl} s_l > \\ &= \omega_{ij} + \omega_{ji} \end{aligned}$$

which implies $A = [\omega_{ij}]$ is a skew-symmetric matrix. Similarly the curvature 2-form matrix $[\Omega_{ij}]$ is also skew-symmetric, since

$$\begin{aligned} [\Omega_{ik}]^t &= (d\omega_{ik} - \omega_{ij} \wedge \omega_{jk})^t \\ &= d\omega_{ki} - \omega_{kj} \wedge \omega_{ji} \\ &= -d\omega_{ik} - \omega_{jk} \wedge \omega_{ij} \\ &= -d\omega_{ik} + \omega_{ij} \wedge \omega_{jk} \\ &= -\Omega_{ik} \end{aligned}$$

Thus, if the connection is compatible with a metric, then $Tr\Omega = \sigma_1 = 0$. Then the topological invariants of a bundle with a metric connection on four and eight manifolds reduce respectively to,

$$\int_{M^4} (\sigma_2),$$

$$\int_{M^8} (a\sigma_2^2 + b\sigma_4).$$

In gauge theories, vector bundles may be used to represent potentials for physical forces. In this context, one is interested in determining the connection on the vector bundle in such a way that certain quantities called the "action integrals" are minimized. A widely used action integral is the expression

$$\int_{M_4} < F, F > dV \tag{3.18}$$

known as the Yang-Mills action. It can be seen that the Yang-Mills action satisfies the inequality

$$\int_{M_4} TrF^2 \leq \int_{M_4} < F, F > dV$$

As the integral at the left hand side of the inequality is a topological invariant, it can be altered by changing the connection, hence it is a topological lower bound for the Yang-Mills action.

In eight dimensions, the search of suitable actions with topological lower bounds is an ongoing research problem. **(Bilge, Dereli, Koçak, 1999)**. In particular on eight manifolds admitting a $spin^c$ structure, it is possible to write down a topological lower bound for the Yang-Mills action. In the next section we shall study $spin^c$ structures in detail.

4.CLIFFORD ALGEBRAS AND $SPIN^C$ STRUCTURES

4.1 Clifford Algebras: Basic Definitions

Let V be an n-dimensional real vector space with inner product $< .,. >$ and choose an orthonormal basis $\{e_1, e_2, ..., e_n\}$. A Clifford algebra of V, $Cl(V)$ is a 2^n-dimensional real vector space and an associative algebra with unit element 1. $Cl(V)$ is generated by the basis elements $\{e_1, e_2, ..., e_n\}$ with multiplication rules

i-) $e_i^2 = -1$,
ii-) $e_ie_j + e_je_i = 0 \qquad$ for $i \neq j$.

A basis of $Cl(V)$ as a real vector space is given by the elements

$$1, e_1, e_2, ..., e_n, e_1e_2, ..., e_{n-1}e_n, e_1e_2e_3, ..., e_1e_2e_3...e_n \tag{4.1}$$

or

$$e_0 = 1, \quad e_I = e_{i_1}e_{i_2}...e_{i_k}$$

for $I = \{i_1, i_2, ..., i_k\} \subset \{1, ..., n\}$ with $1 \leq i_1 < i_2 < ... < i_k \leq n$. We denote k= $|I|= deg(e_I)$. Any element of $Cl(V)$ can be written $x = \sum_I x_I e_I$.

The Clifford algebras have been studied in the literature and the following isomorphism are well known. (**Salamon, 1996**).

$Cl(R^1) \cong C$,
$Cl(R^2) \cong H$,
$Cl(R^3) \cong H + H$,
$Cl(R^4) \cong H^{2\times 2}$,
$Cl(R^5) \cong C^{2\times 2}$,
$Cl(R^6) \cong R^{8\times 8}$,

$Cl(R^7) \cong R^{8\times 8} + R^{8\times 8}$,
$Cl(R^8) \cong R^{16\times 16}$.

After 8-dimension, there is a periodicity situation **(Porteous, 1995)** such that;

$$Cl(n+8) = Cl(n) \otimes R^{16\times 16}. \tag{4.2}$$

4.2 $Spin^c$ Structures: Basic Definitions

Let V be a $2n$-dimensional real vector space, and W be a 2^n-dimensional Hermitian vector space. We can define an algebra homomorphism from $Cl(V)$ to $End(W)$ such that;

i-) $\Gamma(v_1 + v_2) = \Gamma(v_1) + \Gamma(v_2)$,
ii-) $\Gamma(v_1 v_2) = \Gamma(v_1)\Gamma(v_2)$,
iii-) $\Gamma(\tilde{v}) = \Gamma(v)^*$,

where $\tilde{v}$ is an involution of the $v \in Cl(V)$. An involution of $Cl(V)$ is a one-to-one transformation within $Cl(V)$ $\quad v \mapsto \tilde{v} \in Cl(V)$ and whose square is unity: $\tilde{v} \mapsto \tilde{\tilde{v}} = v$. An involution of $Cl(V)$ is defined by

$$Cl(V) \longrightarrow Cl(V): \qquad v \mapsto \tilde{v}$$

$$\tilde{v} = \sum_I \epsilon_I v_I e_I, \quad \epsilon_I = (-1)^k, \quad k = |I|.$$

For example; let us take $v \in Cl(V)$ as follows,

$$v = a_0 + a_1 e_1 + a_2 e_2 + a_{23} e_{23} + a_{123} e_{123}$$

then the involution of v is

$$\tilde{v} = a_0 - a_1 e_1 - a_2 e_2 + a_{23} e_{23} - a_{123} e_{123}$$

This algebra homomorphism is determined by the restriction of Γ to V. Now we can define this $spin^c$-structure as follows.

Definition: A $spin^c$-structure on 2n(even)-dimensional real vector space V is a pair (W, Γ), where W is a 2^n-dimensional Hermitian vector space and Γ is a linear map from V to $End(W)$ satisfying following conditions;

i-) $\Gamma(v)^* + \Gamma(v) = 0,$

ii-) $\Gamma(v)^*\Gamma(v) = \|v\|^2 \quad$ for $v \in V.$

4.3 Real and Complex Representations of Clifford Algebras

Let V be k-dimensional real vector space spanned by $\{e_1, e_2, ..., e_k\}$, and $\Gamma(v)$ be $N \times N$ matrix such that $\Gamma(e_i) = A_i$;

$$A_i^2 = -1, \;\; and \;\; A_iA_j + A_jA_i = 0, \;\; for \;\; i = 1, 2..., k \tag{4.3}$$

A_i's are either skew-symmetric matrix over real numbers, or skew-hermitian matrix over complex numbers. If we choose real representation of $Cl(V_k)$, then A_i's are $d(k) \times d(k)$ skew-symmetric matrices, when we choose complex representation of $Cl(V_k)$, then A_i are $d_c(k) \times d_c(k)$ skew-hermitian matrices. The dimension of the representation is given as follows.

k	N=d(k)	N $=d_c(k)$
1	2	1
2	4	2
3	4	2
4	8	4
5	8	4
6	8	8
7	8	8
8	16	16

We also know how to compute the $d(k)$ and $d_c(k)$ in any other dimension. This is given as follows **(Lawson, Michelsohn, 1989)**.

$$d(m+8k) = 2^{4k}d(m), \qquad d_c(m+2k) = 2^k d_c(m). \tag{4.4}$$

We know that k is the number of vector fields lying on $(d(k)-1)$-dimensional sphere $S^{d(k)-1}$. This number k is called Radon-Hurwitz number in the literature. On the sphere S^{N-1} there exist k linearly independent vector fields where k is computed as follows.

$$N = 2^{4d+c}(2a+1), \;\; 0 \leq c \leq 3 \qquad then \;\; k = 8d + 2^c - 1$$

(Lawson, Michelsohn, 1989). Note that this construction gives three vector fields on S^3, seven on S^7, and eight on S^{15}.

4.4 Dimension of Maximal Linear Subspaces of Matrices Satisfying $A^2 + \lambda^2 I = 0$

We are interested in the following two problems.

Problem(A): Over the real numbers, what is the dimension of maximal linear subspaces L_R of the set $S_R = \{A : A^T + A = 0, A^2 + \lambda^2 I = 0\}$. **(Bilge, Dereli, Kocak, 1997)**

Problem(B): Over the complex numbers what is the dimension of maximal linear subspaces L_C of the set $S_C = \{A : A^* + A = 0, A^2 + \lambda^2 I = 0\}$.

Solution of Problem(A): A is a skew-symmetric matrix that satisfies $A^2 + \lambda^2 I = 0$. This is known from **(Bilge, Dereli, Kocak, 1997)** that S_R has a manifold structure. We work locally to determine its dimension. We know that dimension of manifold is equal to dimension of its tangent space at any point on the manifold. Pick a point $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Study the dimension of tangent space at J. This number is the number skew-symmetric matrices anti-commute with J. i.e., $BJ + JB = 0$. Then we can see that $B = \begin{pmatrix} B_1 & B_2 \\ B_2 & -B_1 \end{pmatrix}$,

$$dimB = dimB_1 + dimB_2,$$

where B_1, B_2 are skew-symmetric matrices. The dimension of B_1, B_2 are as follows, $dimB_1 = dimB_2 = n(n-1)/2$. Hence we find dim B$= n^2 - n$. Finally, the number of skew-symmetric matrices which anti-commute with J and the skew-symmetric matrix J implies the dimension of S_R such that $dimS_R = n^2 - n + 1$.

Solution of Problem(B): A is a skew-hermitian matrix which satisfies $A^2 + \lambda^2 I = 0$. Similarly, we will find the dimension of tangent space at any point to determine the dimension of the manifold. Pick a point J, i.e., $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. The dimension of the tangent space at J is the number of skew-hermitian matrices which anti-commute with J. i.e. $AJ + JA = 0$. This condition implies that $A = \begin{pmatrix} A_1 & A_2 \\ A_2 & -A_1 \end{pmatrix}$, where A_1 and A_2 are skew-hermitian matrices. We can write

$$A_1 = A_{1R} + iA_{1I}$$

where A_{1R} is a skew-symmetric matrix and A_{1I} is a symmetric matrix. A_{1R} and A_{1I} show that $dimA_{1R} = n(n-1)/2$, $dimA_{1I} = n(n+1)/2$. Hence $dimA_1 = n^2$. Similarly,

$$A_2 = A_{2R} + iA_{2I}$$

where A_{2R} is a skew-symmetric matrix and A_{2I} is a symmetric matrix. A_{2R} and A_{2I} show that $dimA_{2R} = n(n-1)/2$, $dimA_{2I} = n(n+1)/2$. Hence $dimA_2 = n^2$.

$$dimA = dimA_1 + dimA_2 = 2n^2$$

The number of skew-symmetric matrices which anti-commute with J is $2n^2$, and adding with the skew-symmetric matrix J implies that $dimS_C = 2n^2 + 1$.

We know from **(Bilge, Dereli, Kocak, 1997)** that $dimL_R =$ Radon-Hurwitz number. In particular, if

$$N = 2(2a+1), \text{ then } dimL_R(N) = 1.$$

We will give a lemma before calculating the dimension of L_C.

Lemma. Let A_1, A_2 be skew-hermitian matrices. If $A_3 = A_1.A_2$, $A_i^2 = -1$, $A_iA_j + A_jA_i = 0$, $BA_i + A_iB = 0$ for i=1,2,3. then B=0.

Proof.

$BA_2 + A_2B = 0$

$A_1(BA_2 + A_2B) = 0$

$A_3B - BA_3 = 0$ with $BA_3 + A_3B = 0$, so this shows that B=0.

If we take $A_3 = A_1.A_2$ then the linear subspaces spanned by $\{A_1, A_2, A_1A_2\}$ can not be extended, hence we take $A_3 \neq A_1.A_2$. Now pick two points, J_1, J_2, i.e., $J_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $J_2 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$.

Any matrix B which anti-commutes J_1 and J_2 is such that $B = \begin{pmatrix} 0 & \beta \\ \beta & 0 \end{pmatrix}$, where β is a skew-hermitian matrix. This shows that,

$$L_C(2n) = L_C(n) + 2. \tag{4.5}$$

The dimensions of maximal linear subspaces of complex-hermitian matrices satisfying $A^2 + \lambda^2 I = 0$ are listed below.

2n	dim $L_C(2n)$
2	$3 \longrightarrow \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$
4	3+2=5
6	1+2=3
8	5+2=7
10	1+2=3
12	3+2=5
14	1+2=3
16	7+2=9
18	1+2=3
20	3+2=5

The general formula can be given as follows.

$k = 2(2a+1) \longrightarrow L_C(k) = 3,$

$k = 2^2(2a+1) \longrightarrow L_C(k) = 5,$

$k = 2^{l-1}(2a+1) \longrightarrow L_C(k) = 2l-1.$

5. RESULTS AND DISCUSSIONS

In this study, the numerical relations between the traces of powers of the curvature 2-form matrix Ω and its invariant polynomials σ_n are obtained.

We present Clifford algebras with their real and complex representations. For any dimensional Clifford algebras, they have representations on skew-symmetric and skew-hermitian matrices. These representations are given.

Finally, the dimensions of maximal linear subspaces of the skew-hermitian matrices satisfying $A^2 + \lambda^2 I = 0$ are determined.

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