İSTANBUL TECHNICAL UNIVERSITY ★ INSTITUTE OF SCIENCE AND TECHNOLOGY

OPTIMAL SHAPE ANALYSIS OF ELASTIC BODIES BY USING DIFFERENTIAL TRANSFORM METHOD

M.Sc. Thesis by
Ülker ERDÖNMEZ

Department : **Aeronautical and Astronautical Engineering**

Programme : **Aeronautical and Astronautical Engineering**

Thesis Supervisor: Prof. Dr. İbrahim ÖZKOL

June 2009

İSTANBUL TECHNICAL UNIVERSITY ★ INSTITUTE OF SCIENCE AND TECHNOLOGY

OPTIMAL SHAPE ANALYSIS OF ELASTIC BODIES BY USING DIFFERENTIAL TRANSFORM METHOD

M.Sc. Thesis by
Ülker ERDÖNMEZ

(511071134)

Date of submission : **04 May 2009**
Date of defence examination: **03 June 2009**

Supervisor (Chairman) : **Prof. Dr. İbrahim ÖZKOL (ITU)**
Members of the Examining Committee : **Prof. Dr. Metin Orhan KAYA (ITU)**
Prof. Dr. Ata MUĞAN (ITU)

June 2009

İSTANBUL TEKNİK ÜNİVERSİTESİ ★ FEN BİLİMLERİ ENSTİTÜSÜ

DİFERANSİYEL DÖNÜŞÜM YÖNTEMİ İLE ELASTİK YAPILARIN OPTİMAL ŞEKİL ANALİZİ

YÜKSEK LİSANS TEZİ

Ülker ERDÖNMEZ

(511071134)

Tezin Enstitüye Verildiği Tarih : 04 Mayıs 2009
Tezin Savunulduğu Tarih : 03 Haziran 2009

Tez Danışmanı : Prof. Dr. İbrahim ÖZKOL (İTÜ)
Diğer Jüri Üyeleri : Prof. Dr. Metin Orhan KAYA (İTÜ)
Prof. Dr. Ata MUĞAN (İTÜ)

Haziran 2009

FOREWORD

I would like to express my deep appreciation and thanks for my advisor **Prof. Dr. İbrahim Özkol** due to the supports and contributions in preparing this thesis.

Also, I would like to express my thanks for **TUBITAK** – Türkiye Bilimsel ve Teknolojik Araştırma Kurumu - due to the financial supports during my M. Sc. Education.

Lastly, I would like to thanks to my big family that did not deprive their supports during preparation of this thesis.

May 2009 Ülker ERDÖNMEZ

Aeronautical Engineer

TABLE OF CONTENTS

Page

ABBREVIATIONS

DTM **:** Differential Transform Method

LIST OF TABLES

LIST OF FIGURES

Page

OPTIMAL SHAPE ANALYSIS OF ELASTIC BODIES BY USING DIFFERENTIAL TRANSFORM METHOD

SUMMARY

In this study optimal shape analysis of elastic bodies is carried out for different loading conditions. Simply supported rods and columns are used in the analysises as structural elements. Loading conditions examined in this study are axial compressive force, eccentrically placed compressive force –eccentricity at both ends and eccentricity at one end- and follower type of loading –uniformly distributed and exponentially varying-.

For each configuration, optimal distribution of cross-sectional area and volume of the structure with such cross-sectional area are determined. In addition, volume of uniform structure which is also subjected to same amount of loading is calculated and compared to the volume of the structure with optimal shape. This comparison gives the degree of success of the optimal shape analysis.

DİFERANSİYEL DÖNÜŞÜM YÖNTEMİ İLE ELASTİK YAPILARIN OPTİMAL ŞEKİL ANALİZİ

ÖZET

Bu çalışmada,farklı yükleme şartları altında elastik yapıların optimal şekil analizi gerçekleştirilmiştir. Analizlerde, basit mesnetli çubuk ve kolon tipi yapısal elemanlar göz önüne alınmıştır. Bu çalışmada ele alınan yükleme şartları eksenel uygulanan basma, eksantrik uygulanan basma –iki uçtan eksantrik ve tek uçtan eksantrik- ve follower tipi –düzgün dağılımlı ve eksponensiyel değişen- yüklemedir.

Her konfigürasyon için, yapının kesit alanının optimal dağılımı belirlenmiştir ve optimal kesit alanlı yapının hacmi bu dağılımdan yaralanılarak hesaplanmıştır. Bunlara ek olarak, optimal yapı ile aynı yüklemeye maruz kalan uniform yapının hacmi hesaplanmış ve optimal yapının hacmi ile kıyaslanmıştır. Bu kıyas yapılan optimizasyonun verimliliğini tayin etmektedir.

1. INTRODUCTION

1.1. Historical Development and Aim of the Study

Determination of the shape of elastic bodies was first investigated by Lagrange in 1773. Since then, the problem of maximizing the critical buckling force of a prismatic column of given length and volume is called as Lagrange problem ***(Cox, 1992)***. Optimization of shape of the columns was treated by Clausen. ***(Clausen, 1851).*** In the presented study, rods and columns under various loading conditions are examined and the effects of loading condition on the optimal distribution of the cross-sectional area are determined. These conditions are uniformly distributed follower type of loading, exponentially increasing follower type of loading and concentrated forces at both ends and eccentrically concentrated forces at both ends. In open literature, some studies exist, for example, Atanackovic and Simic obtained a solution for the optimal shape of a Pflüger column which is subjected to uniform follower type of loading in ***(Atanackovic, 1999).*** Optimization problem for compressed columns has been treated by Blasius ***(Blasius, 1914)*** Ratzerdorfer ***(Ratzdorfer, 1936)*** and Keller ***(Keller, 1960).***

In this study, for each case of loading, the optimal distribution of cross-sectional area of a simply-supported structural element is determined under the criteria of minimum volume and stability against buckling. The governing equation for columns is derived by considering an Euler-Bernoulli column and then Pontryagin's maximum principle with special treatment as in ***(Atanackovic, 1999)*** and ***(Galavardanov, 2001)*** is used to determine the optimal cross-sectional area. Pontryagin's maximum principle which is based on minimizing Hamiltonian function is used in optimal control theory to get the desired optimal solution ***(Vujanovic, 2004).*** The related Hamiltonian function is obtained via the governing equation and the volume of the structure can be minimized by regarding this Hamiltonian function.

In the rest of this study, analysis of nonlinear differential equations, which are obtained after the rearrangement of governing equations with the usage of Pontryagin's maximum principle, is carried out. In the analysis, Differential

Transform Method (DTM), which is a semi analytical-numerical computational technique used to solve ordinary and partial differential equations, is applied. This method which is also capable of solving fractional differential equations ***(Arıkoğlu, 2007),*** integral and integro-differential equations ***(Arıkoğlu, 2008)*** is introduced by Zhou in 1986 with the application to electrical circuits ***(Zhou, 1986).*** As results of this analysis, optimal shape and optimal volume of the column for three different conditions are determined and volume saving with such a cross-sectional area distribution is evaluated by considering the column with constant cross-section.

1.2. Content of the Study

In Section 2, general definitions and topics used in the analysis of elastic bodies are taken into consideration. As an introduction, general properties of elastic bodies are given under the topic of “general view to elastic bodies”. Also, beam theories used in engineering practice are introduced in general sense and buckling behavior of columns is analyzed.

In Section 3, basic variational principles used in mechanics are considered. These variational principles are classified under the topics of Lagrange- D’Alembert differential variational principle and Hamilton integral variational principle. Optimal control theory is introduced and especially Pontryagin’s maximum principle is tried to be introduced since this principle is used to carry out optimal shape analysis of structures.

In Section 4, differential transform method is mentioned. After the general description of differential transform is given; one-dimensional, two-dimensional and finally n-dimensional differential transform are presented. General properties of differential transform are also expressed.

In Section 5, optimal distribution of cross-sectional area of compressed rods is obtained under the criterion of minimum volume. After the volume of the rod with optimal cross-section is evaluated, it is compared to the volume of a uniform rod which is stable under the same load value to determine the efficiency of the rod with optimal cross-section. When the analysis is completed, it is noticed that the optimal compressed rod has zero cross-sectional areas at both ends. However, this cannot be an acceptable configuration in physical sense and for engineering applications.

Therefore, governing equations are rearranged and a minimum cross-sectional area is defined in order to overcome the problem of having zero cross-sectional area.

In Section 6, columns which are subjected to eccentrically concentrated forces are outlined and optimal distribution of cross-sectional area of such structures along the column length is determined. Two different configurations are considered, which are columns loaded eccentrically at both ends and columns loaded eccentrically at one end. Volume of the optimal column for each loading condition is determined. After the volume of the optimal column is evaluated, it is compared to the volume of a uniform column which is stable under the same load value to determine the efficiency of the optimal column.

In Section 7, columns of Euler-Bernoulli type are analyzed for the conditions uniformly distributed and exponentially varying follower type of loading. For each loading condition optimal distribution of cross-section along the column length and optimal volume of such column is determined. The efficiency of the columns with such cross-section by means of volume and load saving is determined for each loading condition by considering uniform column which is subjected to same amount of loading.

In Section 8, general discussions about the whole study are carried out and general summary of the analysis is outlined.

2. ELASTIC BODIES

In this section of this study, general definitions and topics used in the analysis of elastic bodies are taken into consideration. As an introduction, general properties of elastic bodies are given under the topic of “general view to elastic bodies”. Also, beam theories used in engineering practice are introduced in general sense and buckling behavior of columns is analyzed.

2.1. General View to Elastic Bodies

In this section, general definitions and topics used in the analysis of elastic bodies are taken into consideration. Elastic bodies utilized in engineering practice are commonly named as structural elements. Structural elements used in structural analysis simplify the structure by separating the structure into elements which can be defined as the simplest part of the whole. Structural elements can be linear, surfaces or volume. In other words, structural elements can be one-dimensional, two-dimensional or three-dimensional. Linear structural elements are rods (axial loading), beams (both bending and axial loading), columns (compressive loading), shafts (torsional loading) and beam-columns (both bending and buckling). Surface structural elements are membranes (in-plane loading only), shells (both in-plane loading and bending moments) and shear panels (shear loads only).

A structural element serves to transfer load from one place to another. When the dimensions and material properties of the structure, its support conditions, and loads applied to it are given, stress and deflection calculations can be performed. There are two main classes of methods which determine the behavior of a structure: mechanics of materials method and theory of elasticity method. With the mechanics of materials method, search begins by thinking about deformations. Using experiment, experience and intuition; it is decided that how the structure deforms. The selected deformation field may be exact. The deformation field yields to a strain field and with the use of an elastic law (i.e. Hooke’s Law) stress field can be determined. With the theory of elasticity method, there is no need to prescribe a deformation field to solve the problem. In this method, conditions of equilibrium at every point, continuity of the displacement field and loading and support conditions are to be satisfied simultaneously. Since the solution obtained with the use of theory of elasticity method meets all conditions required, this solution is called as “exact solution”. Similar to mechanics of materials method, this method also is based on some assumptions and is an approximation of nature. The elasticity solution is mathematically unique if the body has a linear load-displacement relation.

Theory of elasticity method is more complicated when compared to mechanics of materials method; however it is worth studying since it clarifies the shortcomings or range of applicability of approximate solutions.

2.1.1. Mechanics of Materials Method

Major function of mechanics of materials is to relate applied loads to stress by generating specific relations. The method of derivation, used in both elementary and advanced mechanics of materials is to prescribe the geometry of deformation by experiment, experience or intuition, to determine the strain field by analyzing the deformation field, to determine stress field by applying stress-strain relation and to relate stress to load.

In advanced mechanics of materials, relations derived in elementary mechanics of materials are used. Common definitions and relations in elementary mechanics of materials are as follows ***(Cook, 1985)***:

- Axial stress σ and elongation Δ of a bar which is under axial load ***P*** can be described as follows:

$$\sigma = \frac{P}{A} \qquad \Delta = \frac{PL}{AE} \tag{2.1}$$

where ***A*** is cross-sectional area and ***E*** is modulus of elasticity.

- Shear stress τ and angle of twist θ of a bar that carries a torque of T can be described as follows:

$$\tau = \frac{Tr}{J} \qquad \theta = \frac{TL}{JG} \tag{2.2}$$

where **r** is the radius to the point where shear stress is computed, ***J*** is polar moment of inertia and ***G*** is shear modulus.

- Flexural stress σ and curvature $d^2\upsilon / dx^2$ of a beam loaded by bending moment M can be described as follows:

$$\sigma = \frac{My}{I} \qquad \frac{d^2\upsilon}{dx^2} = \frac{M}{EI} \qquad \textbf{(2.3)}$$

where **y** is distance from the neutral surface to the point where flexural stress is computed and ***I*** is the second moment of inertia.

Definitions given above involve many assumptions and are only given to express general concept of mechanics of materials method. There are miscellaneous theorems which describe the behavior of materials after detailed analysis. These theorems are taken into consideration in the following sections.

2.1.2. Theory of Elasticity Method

Analysis of structures on the basis of theory of elasticity method concerns determining the structural deformation via the stress-strain behavior of the system. Classical elasticity theory is constructed under the assumptions of

- strains and deformations are small and
- material is homogeneous, isotropic and linearly elastic.

In determining the elastical behavior of the materials Hooke Law is used. In the simplest form, for a single-axial loading this behavior can be defined as follows:

$$\sigma_x = E\varepsilon_x \qquad \varepsilon_x = \frac{\sigma_x}{E} \qquad \textbf{(2.4)}$$

Moreover, in the case of single-axial loading there is elastic strain of the materials. Elastic strain in these directions can be expressed with the use of Poisson Ratio υ as follows:

$$\varepsilon_y = -\upsilon\varepsilon_x = -\upsilon\frac{\sigma_x}{E}$$

$$\varepsilon_z = -\upsilon\varepsilon_x = -\upsilon\frac{\sigma_x}{E} \qquad \textbf{(2.5)}$$

By generalizing the equations given above, "Elasticity Constitutive Equations" can be obtained for three-axial loading as follows:

$$\varepsilon_x = \frac{1}{E}[\sigma_x - \upsilon(\sigma_y + \sigma_z)]$$

$$\varepsilon_y = \frac{1}{E}[\sigma_y - \upsilon(\sigma_x + \sigma_z)]$$

$$\varepsilon_z = \frac{1}{E}[\sigma_z - \upsilon(\sigma_x + \sigma_y)] \qquad \textbf{(2.6)}$$

Shear strains can be expressed as follows:

$$\gamma_{xy} = \frac{\tau_{xy}}{G} \qquad \gamma_{yz} = \frac{\tau_{yz}}{G} \qquad \gamma_{zx} = \frac{\tau_{zx}}{G} \qquad \textbf{(2.7)}$$

For isotropic materials there is a relation between modulus of elasticity and shear modulus which can be given as follows:

$$G = \frac{E}{2(1+\upsilon)} \qquad \textbf{(2.8)}$$

2.2. Beam Theories used in Engineering Practice

There are four beam theories commonly used in engineering practice to formulate the partial differential equation of motion by considering the behavior of the structure under given loading, boundary conditions and initial conditions. Beam theories are Euler-Bernoulli beam theory, Rayleigh beam theory, Shear theory and Timoshenko beam theory.

The Euler-Bernoulli model, which is also known as classical beam theory, includes the strain energy due to bending and kinetic energy due to lateral displacement ***(Haym,1998)***. The effect of rotation of the cross-section is included by Rayleigh beam theory in addition to Euler-Bernoulli beam model (1877). The shear model adds shear distortion effect to the Euler-Bernoulli beam theory. Timoshenko beam theory is proposed in 1921, 1922 to add the effect of both shear and rotation. Namely, it can be said that Timoshenko beam theory covers the other beam theories.

2.2.1. Euler-Bernoulli model

2.2.1.1. Moment-curvature relation

Deflections due to bending are determined with the consideration of deformations taking place along a span ***(Popov,1990)***. Deflections due to shear are not considered in the Euler-Bernoulli model.

Radius of the elastic curve ρ, as it is shown in Fig. (2.1), is assumed as changing along the span. For positive y's, deformation of any fiber can be expressed as follows:

$$\Delta u = -y\Delta\theta \qquad \textbf{(2.9)}$$

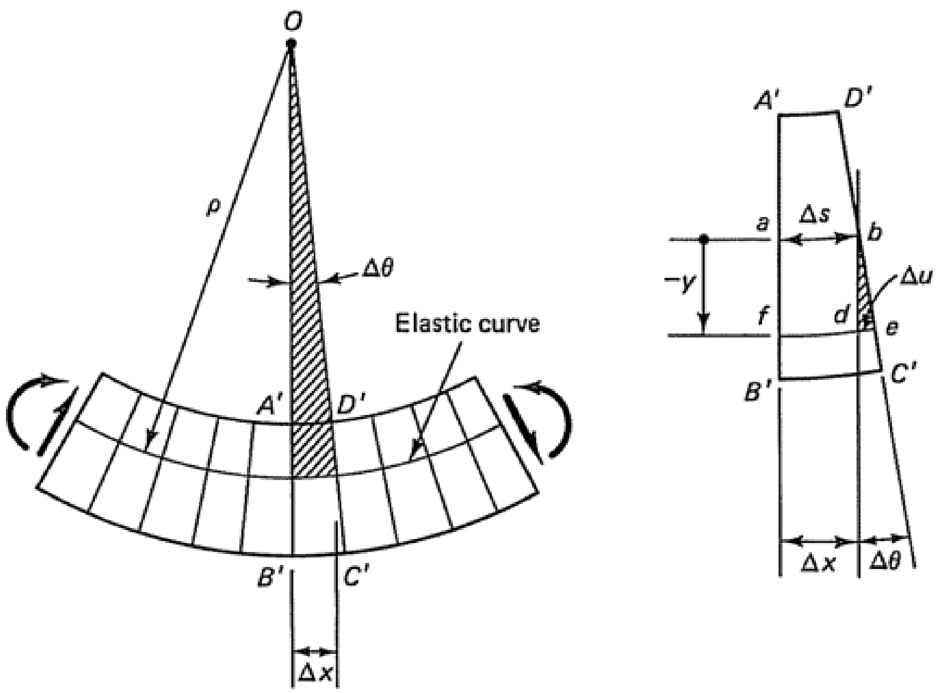


Figure 2.1 Deformation of a beam under bending

For negative y's, this yields elongation of fibers. By dividing the both sides of Eq. (2.9) by Δs and using the definition of normal strain (*du/ds*=*ε*), one can obtain

$$\frac{du}{ds} = -y\frac{d\theta}{ds} \qquad \textbf{(2.10)}$$

With the aid of Fig.(2.1), it can be seen that $\Delta s=\rho\Delta\theta$. On this basis, fundamental relation between curvature of the elastic curve and normal strain can be given as follows:

$$\frac{1}{\rho}=\kappa=-\frac{\varepsilon}{y} \qquad (2.11)$$

Note that, no material properties are used in the derivation of this relation. Therefore, this relation can also be used for inelastic problems as for elastic problems. By using elastic material relations given in Eq. (2.4) and longitudinal stress occurred by bending moment which is given in Eq. (2.3), relation between bending moment at a given section and curvature of elastic curve can be given as follows:

$$\frac{1}{\rho}=\frac{M}{EI} \qquad (2.12)$$

2.2.1.2. Governing differential equation

In Cartesian coordinates, the curvature of a line is defined as

$$\frac{1}{\rho}=\frac{\frac{d^2v}{dx^2}}{\left[1+\left(\frac{dv}{dx}\right)^2\right]^{\frac{3}{2}}}=\frac{v''}{[1+(v')^2]^{\frac{3}{2}}} \qquad (2.13)$$

where x and v are the coordinates of a point on a curve. By assuming that the term $(dv/dx)^2$ goes to zero, geometric nonlinearity is eliminated from the problem and Eq. (2.13) simplifies to

$$\frac{1}{\rho}\approx\frac{d^2v}{dx^2} \qquad (2.14)$$

By combining Eq. (2.12) and Eq. (2.14), constitutive relation can be obtained as follows:

$$\frac{d^2v}{dx^2}=\frac{M}{EI} \qquad (2.15)$$

2.2.1.3. Derivation of equation of motion

Equation of motion for an Euler-Bernoulli column or beam can be derived both by using Newton's equilibrium considerations and also applying energy method.

In the further sections of this study, Newton equilibrium equations are used to derive the equation of motion for the given systems. Therefore, it can be enough to analyze the system according to Newton approach.

Equation of motion of a beam or a column varies due to the loading condition. Therefore, derivation of equation of motion of a beam or column can be introduced by considering a structure under a certain loading condition.

2.2.1.4. An example- Beam loaded by distributed force

Consider a beam under given distributed force of ***f(x,t)*** which is the external force per unit length of the beam. A beam element under this type of loading can be shown as in Fig. (2.2).

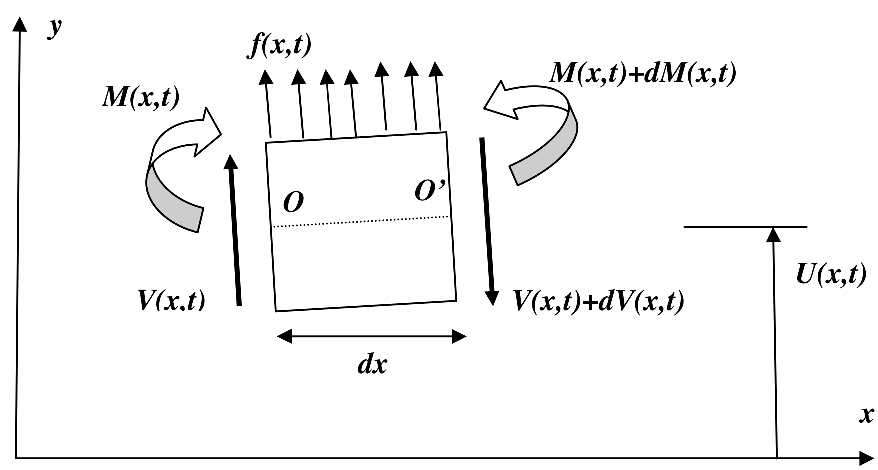


Figure 2.2 Configuration of an Euler-Bernoulli beam element after deformation

where ***M(x,t)*** is the bending moment, ***V(x,t)*** is the shear force. In addition ***dM*** and ***dV*** can be given in the following form

$$dM = \frac{\partial M}{\partial x}dx \qquad dV = \frac{\partial V}{\partial x}dx \qquad (\mathbf{2.16})$$

By accepting the counterclockwise direction as positive direction, moment equilibrium can be written (see Fig. (2.2)) as follows

$$M(x,t) + \frac{\partial M(x,t)}{\partial x}dx - M(x,t) - [V(x,t) + \frac{\partial V(x,t)}{\partial x}dx]dx + f(x,t)dx\frac{dx}{2} = 0 \qquad (\mathbf{2.17})$$

By eliminating the terms including the multiplication of $dx \cdot dx$, Eq. (2.16) goes

$$\frac{\partial M(x,t)}{\partial x} = V(x,t) \quad \mathbf{(2.18)}$$

Equilibrium along y-direction, by accepting y-direction is positive, can be written as

$$V(x,t) - \left(V(x,t) + \frac{\partial V(x,t)}{\partial x} dx \right) + f(x,t)dx = \rho A(x)dx \frac{\partial^2 U(x,t)}{\partial t^2} \quad \mathbf{(2.19)}$$

By eliminating the terms including the multiplication of $dx \cdot dx$, Eq. (2.16) goes

$$\frac{\partial V(x,t)}{\partial x} = f(x,t) - \rho A(x) \frac{\partial^2 U(x,t)}{\partial t^2} \quad \mathbf{(2.20)}$$

By taking the first derivative of Eq. (2.18) and substituting Eq. (2.20) into the obtained relation, one can obtain

$$\frac{\partial^2 M(x,t)}{\partial x^2} = f(x,t) - \rho A(x) \frac{\partial^2 U(x,t)}{\partial t^2} \quad \mathbf{(2.21)}$$

Constitutive relation for the column is as follows

$$M = EI(x) \frac{\partial^2 U(x,t)}{\partial x^2} \quad \mathbf{(2.22)}$$

By substituting Eq. (2.22) into Eq. (2.21) equation of motion can be obtained as follows

$$\frac{\partial^2}{\partial x^2} \left[EI(x) \frac{\partial^2 U(x,t)}{\partial x^2} \right] + \rho A(x) \frac{\partial^2 U(x,t)}{\partial t^2} = f(x,t) \quad \mathbf{(2.23)}$$

2.2.2. Timoshenko model

Timoshenko model is derived by adding the rotary inertia and shear deformation into the Euler-Bernoulli beam theory. Timoshenko beam theory is also known as the thick beam theory, since the cross-sectional dimensions are not small when compared to length of the beam.

Consider a beam element which has the configuration under deformation as it is shown in Fig. (2.3). As a result of shear deformation, the element undergoes distortion but no rotation ***(Rao, 2000)***.

Angle **γ** which is between the tangent to the deformed center line $O'T$ and the normal to the face $O'R'$ can be defined as follows:

$$\gamma = \phi - \frac{\partial U}{\partial x} \tag{2.24}$$

where ϕ is the slope of the deflection curve due to bending deformation.

Since $\phi = \partial U/\partial x$, constitutive relation can be written in terms of the slope of the deflection curve

$$M = EI\frac{\partial \phi}{\partial x} \tag{2.25}$$

$$V = kA(x)G\gamma = kA(x)G\left(\phi - \frac{\partial U}{\partial x}\right) \tag{2.26}$$

where G denotes the modulus of rigidity and k is Timoshenko's shear coefficient. Timoshenko's shear coefficient is a constant and depends on the geometry of the cross-section. For a rectangular cross-section k equals to 5/6 and equals to 9/10 for a circular cross-section ***(Cowper,1966).***

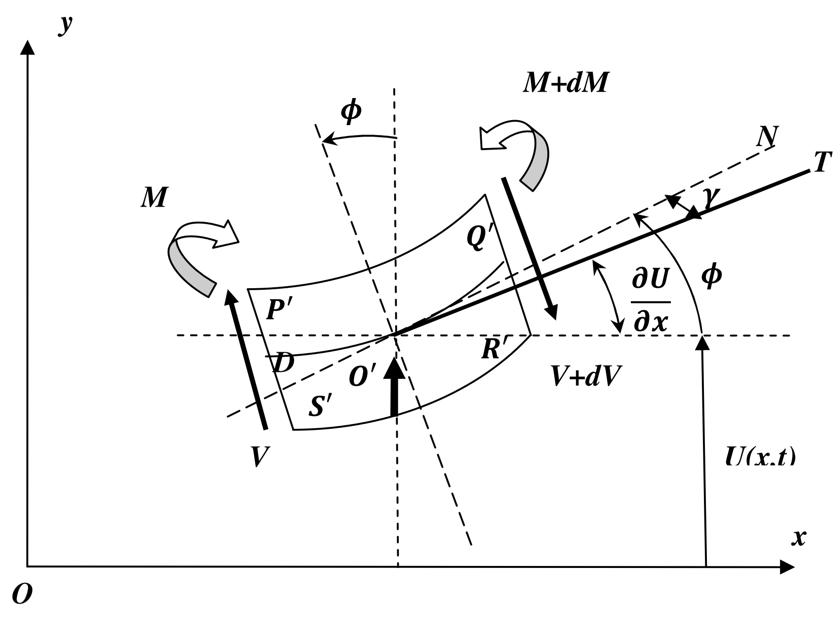


Figure 2.3 Configuration of a Timoshenko beam element after deformation

Translational inertia of the element equals to

$$\rho A(x)dx\frac{\partial^2 U(x,t)}{\partial t^2} \tag{2.27}$$

Rotational inertia of the element equals to

$$\rho I(x)dx\frac{\partial^2 \phi(x,t)}{\partial t^2} \tag{2.28}$$

By accepting the counterclockwise direction as positive direction, moment equilibrium about point ***D*** can be written (see Fig. (2.3)) as follows

$$M(x,t)+\frac{\partial M(x,t)}{\partial x}dx-M(x,t)-[V(x,t)+\frac{\partial V(x,t)}{\partial x}dx]dx$$
$$+f(x,t)dx\frac{dx}{2}=\rho I(x)dx\frac{\partial^2\phi(x,t)}{\partial t^2} \tag{2.29}$$

By eliminating the terms including the multiplication of $dx \cdot dx$, Eq. (2.29) goes

$$\frac{\partial M(x,t)}{\partial x}=V(x,t)+\rho I(x)\frac{\partial^2\phi(x,t)}{\partial t^2} \tag{2.30}$$

Equilibrium along y-direction, by accepting y-direction is positive, can be written as

$$V(x,t)-\left(V(x,t)+\frac{\partial V(x,t)}{\partial x}dx\right)$$
$$+f(x,t)dx=\rho A(x)dx\frac{\partial^2 U(x,t)}{\partial t^2} \tag{2.31}$$

By eliminating the terms including the multiplication of $dx \cdot dx$, Eq. (2.31) goes

$$\frac{\partial V(x,t)}{\partial x}=f(x,t)-\rho A(x)\frac{\partial^2 U(x,t)}{\partial t^2} \tag{2.32}$$

By substituting Eqs. (2.25) and (2.26), Eqs. (2.30) and (2.32) can be obtained in the form of

$$-kA(x)G\left(\frac{\partial\phi}{\partial x}-\frac{\partial^2 U}{\partial x^2}\right)+f(x,t)=\rho A(x)\frac{\partial^2 U(x,t)}{\partial t^2} \tag{2.33}$$

$$EI(x)\frac{\partial^2\phi}{\partial x^2}-kA(x)G\left(\phi-\frac{\partial U}{\partial x}\right)=\rho I(x)\frac{\partial^2\phi(x,t)}{\partial t^2} \tag{2.34}$$

Eqs. (2.33) and (2.34) describes the equations of motion of the given system.

2.3. Buckling of Columns

Structural instability as a result of compressive stresses is one of the basic engineering problems. Narrow beams, vacuum tanks, submarine hulls unless properly designed can collapse under an applied load ***(Popov,1990).***

2.3.1. Criteria for stability of equilibrium

Consider a rigid beam as it is shown in Fig. (2.4) with a torsional spring of stiffness k at the base point and is subjected to vertical force $\boldsymbol{P}$ and horizontal force $\boldsymbol{F}$. System has only one degree of freedom. For an assumed small rotation angle $\boldsymbol{\theta}$, stability limits are defined as follows:

$k\theta > PL\theta$ the system is stable **(2.35)**

$k\theta < PL\theta$ the system is stable **(2.36)**

$k\theta = PL\theta$ the system is in equilibrium **(2.37)**

This means that if restoring moment $k\theta$ which tends to upset the system is smaller than upsetting moment $PL\sin\theta$, system becomes unstable. Equality of this two moments yields to the equilibrium condition of the system. This condition is also known as the neutral point of the system. The force associated with this condition is called as the critical or buckling load.

$$P_{cr} = \frac{k}{L} \qquad \mathbf{(2.38)}$$

2.3.2. Euler load for columns

Euler load or critical buckling load is the least force value at which a buckled mode is possible. Euler load varies with the applied force.

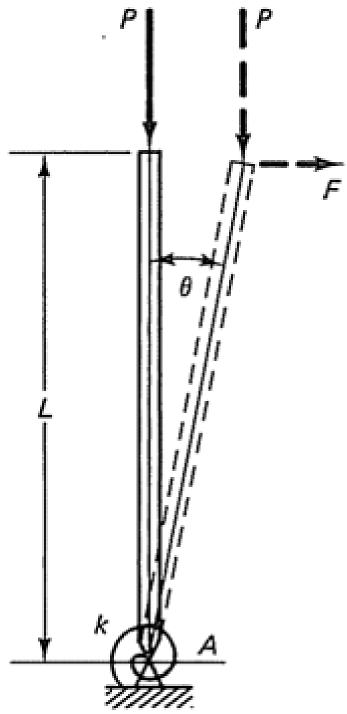


Figure 2.4 Buckling of a rigid bar

2.3.3. An example- Euler load of a simply supported column

Moment distribution of a simply supported column can be given as follows:

$$M(x) = Pu(x) \tag{2.39}$$

where ***u(x)*** is the displacement field of the column which is independent of time and ***P*** is the applied force as it is shown in Fig. (2.5).

By substituting Eq. (2.39) into Eq. (2.22) governing equation of the corresponding system can be obtained as follows:

$$\frac{d^2u}{dx^2} + \lambda^2 u = 0 \tag{2.40}$$

where $\lambda^2 = P/EI$ with the boundary conditions of

$$u(0) = u(L) = 0 \tag{2.41}$$

For a uniform column, solution of the differential equation given in Eq. (2.40) is in the form of

$$u(x) = A\, sin\, \lambda x + B \cos \lambda x \tag{2.42}$$

By applying the boundary conditions following relations are obtained as follows:

$B = 0$

$$A \sin \lambda L = 0 \tag{2.43}$$

Eq. (2.43) is satisfied for ***A=0***, but this gives trivial solution of the analyzed system. Non-trivial solution is obtained for

$$\sin \lambda L = 0 \qquad \lambda L = n\pi \qquad n = 1,2, \ldots \tag{2.44}$$

$$P_n = \frac{n^2 \pi^2 \mathrm{EI}}{L^2} \tag{2.45}$$

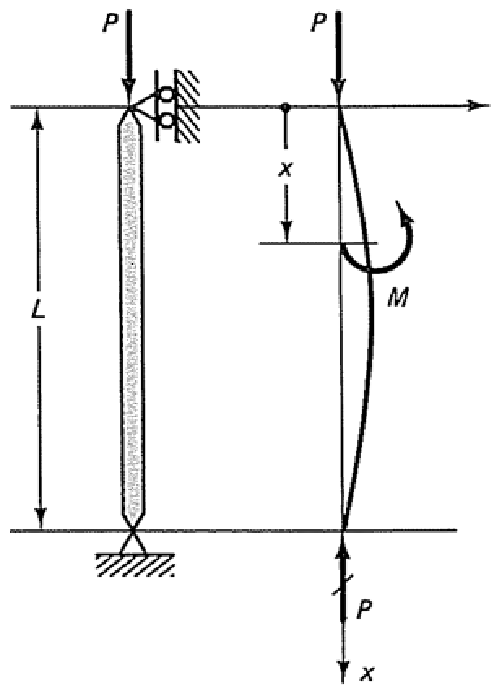


Figure 2.5 Buckling of a simply supported column

P_n is called as the eigenvalue of the problem. In buckling problems, smallest value of $\mathbf{P_n}$ is important, therefore critical buckling load or the Euler load for an initially perfectly straight elastic column can be given as follows:

$$P_{cr} = \frac{\pi^2 \mathrm{EI}}{L^2} \tag{2.46}$$

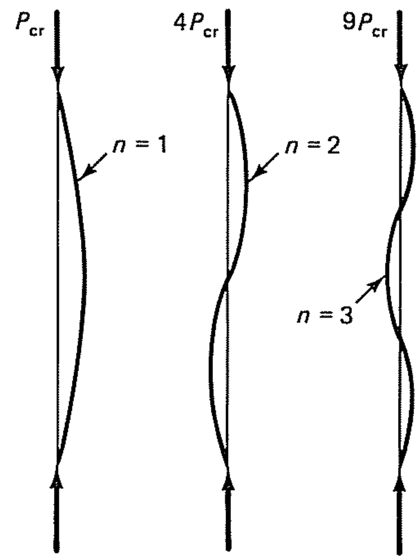


Figure 2.6 First three buckling modes for a simply supported column

3. VARIATIONAL METHODS IN MECHANICS

In this section, basic variational principles used in mechanics are considered. These variational principles can be classified under the topics of Lagrange- D'Alembert differential variational principle and Hamilton integral variational principle These two principles are the main subjects of analytical mechanics.

After the utility of variational principles are realized by physicians and engineers, optimal control theory is developed. In this section, optimal control theory is also taken into consideration and especially Pontryagin's maximum principle is tried to be introduced since this principle is used to carry out optimal shape analysis of structures.

3.1. Lagrange- D'Alembert Differential Variational Principle

This principle is based upon the local characteristics of motion; that is, the relations between its scalar and vector characteristics are considered simultaneously in one particular instant of time***(Vujanovic, 2004).***. The problem of describing the global characteristics of motion has been reduced to the integration of differential equation of motion.

Applications of Lagrange-D'Alembert differential variational principle include holonomic and non-holonomic dynamical systems and also conservative and non-conservative dynamical systems.

3.1.1. General definitions

As a simple approach, dynamical systems can be divided into two main groups as *free dynamical systems* and *constrained dynamical systems*. For the motion of a free dynamical system, Newton's second law determines the full configuration of the dynamical system. In fact, particles of a dynamical system are not completely free to move in the defined space of the motion. Namely, the motion of a dynamical system is commonly limited. Such limitations are called as constraints. Constraints of a dynamical system are specified by certain geometrical and kinematical relations.

For the case of constrained motion, dynamical systems are classified according to the structure of constraints they have. It can be possible to classify constraints in various ways. The most important classification of constraints named as holonomic dynamical systems and non-dynamical systems.

3.1.1.1. Holonomic dynamical systems

Holonomic dynamical systems are dynamical systems whose motion is restricted by holonomic constraints. It must be stated that all of the constraints of the dynamical system must be holonomic to define a dynamical system as holonomic dynamical system.

Holonomic constraints are of purely geometrical character and can be given as follows:

$$f_s(t, q_1, \dots, q_n) = 0 \qquad s = 1, \dots, k \qquad \textbf{(3.1)}$$

where $\boldsymbol{k}$ is the number of holonomic constraints given in the definition of the problem. In the case of free dynamical system without restrictions, such a dynamical system can be described as having $\boldsymbol{n}$ generalized coordinates. For constrained dynamical systems, number of degree of freedom is the difference between the number of generalized coordinates and number of constraints, namely for such a dynamical system degree of freedom equals to $\boldsymbol{n}$-$\boldsymbol{k}$. This is because the existence of constraints reduces the number of degree of freedom of the dynamical system.

3.1.1.1. Non-holonomic dynamical systems

Non-holonomic dynamical systems are dynamical systems whose motion is restricted by non-holonomic constraints. Non-holonomic constraints are of a kinematical character and can be given as follows:

$$A_{\alpha s}\dot{q}_s + B_\alpha = 0 \qquad \alpha = 1, \dots, r \qquad (\mathbf{3.2})$$

where $\boldsymbol{r}$ is the number of non-holonomic constraints of the dynamical system. $A_{\alpha s}$ and B_α depend on generalized coordinates and time $\boldsymbol{t}$. non-holonomic systems can also be defined as the non-linear functions of generalized velocities.

$$f_s(t, q_1, \dots, q_n, \dot{q}_1, \dots, \dot{q}_n) = 0 \qquad \alpha = 1, \dots, r \qquad \textbf{(3.3)}$$

The term non-holonomic is used to express the non-integrability of the differential equations given in Eqs. (3.1) and (3.2) and the impossibility of reducing them into the form of

$$\theta_\alpha(t, q_1, \dots, q_n) = C_\alpha = constant \qquad \textbf{(3.4)}$$

3.1.2. Euler-Lagrangian equations of motion

Euler-Lagrangian equations in classical mechanics basically can be expressed as follows:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_s} - \frac{\partial L}{\partial q_s} = Q_s \qquad s = 1, \dots, n \qquad \textbf{(3.5)}$$

$$L(t, q_1, \dots, q_n, \dot{q}_1, \dots, \dot{q}_n)$$
$$= T(t, q_1, \dots, q_n, \dot{q}_1, \dots, \dot{q}_n) - \pi(t, q_1, \dots, q_n) \qquad \textbf{(3.6)}$$

where $\boldsymbol{L}$ is the Lagrangian function, $\boldsymbol{T}$ is the kinetic energy function, $\boldsymbol{\pi}$ is potential energy function and $\boldsymbol{Q_s}$ is non-conservative forces. Note that potential forces do not depend on the generalized velocities.

In analytical mechanics, Lagrangian equation is used in the form of

$$\sum\left(\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_s} - \frac{\partial L}{\partial q_s} - Q_s\right)\delta q_s = 0 \qquad s = 1, \dots, n \qquad \textbf{(3.7)}$$

This form of Lagrangian equation has a great importance in the applications of analytical mechanics and is called as *central Lagrangian equation*.

3.1.3. Canonical differential equations of motion

Hamiltonian or canonical differential equations are of the first order with respect to the *generalized coordinates* q_i. Generalized momenta p_i and *generalized velocities* $\dot{q}_i$ can be defined by the following relations

$$p_i = \frac{\partial L}{\partial \dot{q}_i} \qquad i = 1, \dots, n \qquad \textbf{(3.8)}$$

$$\dot{q}_i = f(t, q_1, \dots, q_n, p_1, \dots, p_n) \qquad i = 1, \dots, n \qquad \textbf{(3.9)}$$

Canonical equations given above can be represented in different forms. A common representation can be given by means of Hamiltonian function. Hamiltonian function of dynamical system can be defined by transforming the Lagrangian position coordinates $q_1, \ldots, q_n$ into ***2n*** canonical variables $q_1, \ldots, q_n, p_1, \ldots, p_n$. This transformation is usually referred to as the Legendre transformation.

$$H(t, q_1, \ldots, q_n, p_1, \ldots, p_n) = p_i \dot{q}_i - L(t, q_1, \ldots, q_n, \dot{q}_1, \ldots, \dot{q}_n) \qquad \textbf{(3.10)}$$

By using Eq. (3.9) Lagrangian function can be obtained by means canonical variables as follows:

$$L(t, q_1, \ldots, q_n, \dot{q}_1, \ldots, \dot{q}_n) = p_i \dot{q}_i - H(t, q_1, \ldots, q_n, p_1, \ldots, p_n) \qquad \textbf{(3.11)}$$

By substituting Eq. (3.11) into central Lagrangian equation given in Eq. (3.7) one can obtain

$$\left(\dot{p}_i + \frac{\partial H}{\partial q_i} - Q_i\right) \delta q_i + \left(-\dot{q}_s + \frac{\partial H}{\partial p_s}\right) \delta p_s = 0 \qquad \textbf{(3.12)}$$

Since the generalized coordinates and generalized momenta are considered as mutually independent, Eq. (3.12) can be written in the form of

$$\dot{q}_s = \frac{\partial H}{\partial p_s} \qquad \textbf{(3.13)}$$

$$\dot{p}_i = -\frac{\partial H}{\partial q_i} + Q_i(t, q_1, \ldots, q_n, p_1, \ldots, p_n) \qquad \textbf{(3.14)}$$

These equations are called as Hamiltonian canonical differential equations of motion for non-conservative dynamical systems. For conservative systems for which Q_i equals to zero, Hamilton's canonical equations can be written as follows:

$$\dot{q}_s = \frac{\partial H}{\partial p_s} \qquad \textbf{(3.15)}$$

$$\dot{p}_i = -\frac{\partial H}{\partial q_i} \qquad \textbf{(3.16)}$$

3.2. Hamilton Integral Variational Principle

This principle is also known as Hamilton's principle of stationary action or Hamiltonian principle. Hamiltonian principle gives the central importance to global characteristics of motion. Hamiltonian principle is accepted as the cornerstone of the analytical mechanics. Variational calculus is the mathematical instrument used for the applications of Hamiltonian principle. Also many engineering problems which are formulated by variational calculus can be treated as characteristic formulations of the Hamiltonian principle.

3.2.1. General Definitions

Consider a dynamical system with a degree of freedom ***n*** ,whose position at any time ***t*** can be described by ***n*** independent generalized coordinates, $q_1(t),\ldots, q_2(t)$. It is considered that all physical behavior of the so-called Lagrangian systems is completely described by the *Lagrangian function* $L(t, q_1, \ldots, q_n, \dot{q}_1, \ldots, \dot{q}_n)$.

Configuration of such dynamical system at two instant time, t_o and t_1 is presented in Fig. (3.1). Actual trajectory describes the motion of a given dynamical system which joins configurations *A* and *B* and satisfies the differential equations of motion.

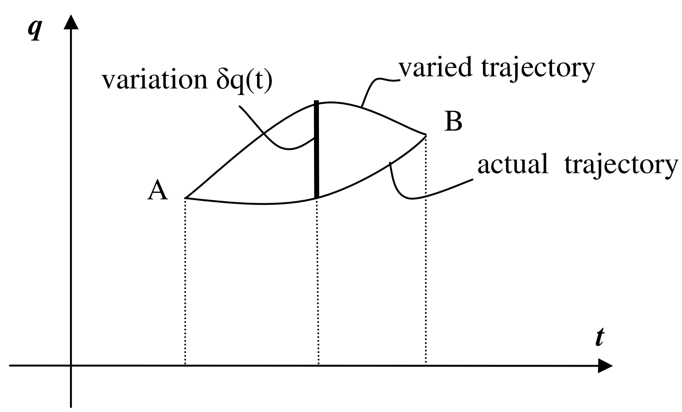


Figure 3.1 Definition of variation

Varied trajectory can be given as follows:

$$\bar{q}_i = q_i(t) + \delta q_i(t) = q_i(t) + \epsilon h_i(t) \qquad \textbf{(3.17)}$$

where $h_i(t)$ represents continuously differentiable arbitrary functions of time and $\delta q_i(t)$ represents variations or virtual displacements of the dynamical system. Variations at end points equal to zero as it is seen in Fig. (3.1) and given as follows:

$$\delta q_i(t_0) = \delta q_i(t_1) = 0, \qquad i = 1, \ldots, n \qquad \textbf{(3.18)}$$

Hamilton's action integral for the given holonomic dynamical system in the time interval of $[t_o, t_1]$ is given in Eq.(3.19).

Hamilton's principle states that the actual motion takes place on actual trajectory, makes the action integral ***I*** stationary.

$$I = \int_{t_o}^{t_1} L(t, q_1, \dots, q_n, \dot{q}_1, \dots, \dot{q}_n) dt \qquad (\mathbf{3.19})$$

$$\delta I = \delta \int_{t_o}^{t_1} L(t, q_1, \dots, q_n, \dot{q}_1, \dots, \dot{q}_n) dt = \int_{t_o}^{t_1} \delta L(t, \boldsymbol{q}, \dot{\boldsymbol{q}})\, dt = 0 \qquad (\mathbf{3.20})$$

By employing the commutativity property of variational calculus and integrating by parts, one can obtain

$$\delta I = \frac{\partial L}{\partial \dot{q}_i}\bigg|_{t_o}^{t_1} + \int_{t_o}^{t_1} \left(\frac{\partial L}{\partial q_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i}\right) \delta \dot{q}_i dt = 0 \qquad (\mathbf{3.21})$$

By applying the boundary conditions given in Eq.(3.18) , Eq. (3.21) yields to Euler-Lagrangian equations of motion.

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} = 0, \quad i = 1, \dots, n \qquad (\mathbf{3.22})$$

Hamiltonian principle is based on the selection of actual trajectory q_i (t) which satisfies the boundary conditions given in Eq. (3.18) and along which the functional affords an extreme value. It can also possible to express the action integral by means of canonical variables. By taking the Legendre transformation of Lagrangian ***L***, Hamiltonian action integral can be introduced in the form of

$$I_{can} = \int_{t_o}^{t_1} [p_i \dot{q}_i L(t, q_1, \dots, q_n, \dot{q}_1, \dots, \dot{q}_n)] dt \qquad (\mathbf{3.23})$$

It can be shown that Hamiltonian principle $\delta I_{can} = 0$ leads to the Hamilton's canonical equation of motion with the use of boundary conditions as follows:

$$\delta I_{can} = p_i \delta q_i|_{t_o}^{t_1} + \int_{t_o}^{t_1} \left[\left(\dot{q}_i - \frac{\partial H}{\partial p_i}\right) \delta p_i - \left(\dot{p}_i - \frac{\partial H}{\partial q_i}\right) \delta q_i\right] dt = 0 \qquad (\mathbf{3.24})$$

$$\dot{q}_i = \frac{\partial H}{\partial p_i} \qquad \dot{q}_i = -\frac{\partial H}{\partial q_i} \qquad i = 1, \dots, n \qquad (\mathbf{3.25})$$

3.2.2. Constrained problems

In constrained problems, dynamical system is described by means of independent, but restricted by auxiliary conditions, generalized coordinates. This restrictions defined on the generalized coordinates are called as constraints. Constraints are classified under three groups as isoperimetric constraints, algebraic constraints and differential equations constraints. Action integral which is constructed by using described constraints is called as augmented functional and represented as I_{aug}.

3.2.2.1. Isoperimetric constraints

Dynamical problem with isoperimetric constraints seeks the extremal of given action integral for the class of trajectories for which the auxiliary conditions occur as a set of definite integrals which must have specified constant values $\boldsymbol{C_k}$, $k = 1, \dots, r$ as follows

$$\int_{t_o}^{t_1} G_k(t, q_1, \dots, q_n, \dot{q}_1, \dots, \dot{q}_n) dt = C_k \qquad \mathbf{(3.26)}$$

where C_k are given constants. Extremal of isoperimetric variational problems can be found by using the method of Lagrangian undetermined multipliers.

Augmented functional can be defined by accounting for the constraints by introducing $\boldsymbol{r}$ unknown constant Lagrangian multipliers λ_k ,$k=1,\dots,r$ as follows:

$$I_{aug} = \int_{t_o}^{t_1} L_{aug}(t, \boldsymbol{q}, \dot{\boldsymbol{q}}, \boldsymbol{\lambda}) dt$$

$$= \int_{t_o}^{t_1} [L(t, \boldsymbol{q}, \dot{\boldsymbol{q}}) + \lambda_k G_k(t, \boldsymbol{q}, \dot{\boldsymbol{q}})] dt \qquad \mathbf{(3.27)}$$

3.2.2.2. Algebraic constraints

Dynamical problem with algebraic constraints seeks the extremal of given action integral in the presence of such constraints given below

$$f_s(t, q_1, \dots, q_n) = 0 \quad s = 1, \dots, k \quad k < n \qquad \mathbf{(3.28)}$$

where $\boldsymbol{n}$ is the number of degree of freedom of the dynamical system and $\boldsymbol{k}$ is the number of algebraic constraints given in the definition of the problem. It can be said that there are k dependent and *(n-k)* independent generalized coordinates which describes the configuration of the dynamical system.

Extremal of algebraic variational problems can be found by using the method of Lagrangian undetermined multipliers by considering the augmented functional in the form of

$$I_{aug} = \int_{t_o}^{t_1} [L(t, \boldsymbol{q}, \dot{\boldsymbol{q}}) + \lambda_s(t) f_s(t, \boldsymbol{q}, \dot{\boldsymbol{q}})] dt \qquad \mathbf{(3.29)}$$

3.2.2.3. Differential equations constraints

Dynamical problem with algebraic constraints seeks the extremal of given action integral subject to constraints in the form of differential equation as it can be given as follows:

$$h_s(t, q_1, \dots, q_n, \dot{q}_1, \dots, \dot{q}_n) = 0 \quad s = 1, \dots, k \quad k < n \qquad \mathbf{(3.30)}$$

where $\boldsymbol{n}$ is the number of degree of freedom of the dynamical system and $\boldsymbol{k}$ is the number of differential equation constraints given in the definition of the problem. It can be said that there are k dependent and *(n-k)* independent generalized coordinates which describes the configuration of the dynamical system.

Extremal of algebraic variational problems can be found by using the method of Lagrangian undetermined multipliers by considering the augmented functional in the form of

$$I_{aug} = \int_{t_o}^{t_1} [L(t, \boldsymbol{q}, \dot{\boldsymbol{q}}) + \lambda_s(t) h_s(t, \boldsymbol{q}, \dot{\boldsymbol{q}})] dt \qquad \mathbf{(3.31)}$$

3.3. Optimal Control Theory

Optimal control theory depends of Hamilton's variational principle and can be defined as a mathematical optimization method used for controlling the given system and obtaining the extremal solution. This method is mostly constructed by Lev Pontryagin and his collaborators and Richard Bellman.

3.3.1. General definitions

The objective of the optimal control theory is to obtain desired extremal solution of a given constrained system by determining the control signals that satisfies the physical constraints. A problem can be formulated by means of optimal control theory with the requirements of ***(Dirk, 2004)***

- A mathematical description of the process to be controlled
- A statement of the physical constraints
- Specification of a performance criterion

Optimal control problems are generally has a non-linear characteristic and therefore cannot be solved analytically. Numerical methods are employed to optimal control problems to obtain the solution. There are indirect and direct methods used for solving optimal control problems.

3.3.1.1. Indirect methods

In the early years of optimal control (between nearly 1950 and 1980) indirect methods are commonly used to solve the optimal control problems. An indirect method uses calculus of variations to obtain first order optimality conditions, which results in a two-point boundary-value problem. More clearly, by taking the derivative of Hamiltonian, a boundary value problem which concerns the optimality conditions of the given optimal control problem is obtained.

There is a great disadvantage of indirect methods, since it is generally difficult to solve the obtained boundary value problem. Specially for problems with span large time intervals and problems with interior point constraints, solution is more difficult. A well-known software program which can be used for obtaining the desired solution is BNDSCO ***(Oberle, 1989).***

3.3.1.2. Direct methods

With the increasing interest on numerical optimal control over the past two decades (after 1980s), direct methods are developed to solve the optimal control problems. A direct method controls the given system by using an appropriate function approximation, e.g. polynomial approximation. Coefficients of the corresponding function approximations are considered as optimization variables and the problem is adapted to a non-linear optimization problem.

Size of the non-linear optimization problem depends on the type of direct methods. For instance, for a direct shooting or quasi-linearization method size of the non-linear optimization problem is quite small and for a direct collocation method size is quite large. There are a lot of well-known software programs which can be used for obtaining the desired solution, such as SNOPT ***(Gill, 2007).*** Many such programs written in FORTRAN and MATLAB exists and commonly used for solving such problems.

3.3.2. Pontryagin's maximum principle

Pontryagin's maximum principle can also be called as Pontryagin's minimum principle. This principle is used in optimal control theory to find best possible solution for a given dynamical system in the presence of constraints ***(Vujanovic, 2004)***. Here, best solution means the optimal solution of the system. The content of optimal solution varies in accordance with the aim of the problem. For instance, a benefit function is tried to be maximized but a cost function is tried to be minimized.

In the previous steps of this study, Pontryagin's maximum principle is used to determine the extremals of an action integral. Therefore, a special case of Pontryagin's maximum principle is considered in this section which is compatible with the optimal control problem considered in this study. Note that, in the previous section optimal distribution of cross-sectional area is tried to be determined with the criterion of minimum volume. Since the problem depends on minimization phenomena, this principle must be referred as Pontryagin's minimum principle.

Consider an action integral which is in the form of

$$I = \int_{t_o}^{t_1} L(t, q_1, \dots, q_n, \dot{q}_1, \dots, \dot{q}_n) dt \qquad \mathbf{(3.32)}$$

where the time interval is specified as $[t_o, t_1]$. Assume that the initial and terminal configurations A and B are defined as follows:

$$q_i(t = t_o) = A_i \qquad q_i(t = t_1) = B_i \qquad \mathbf{(3.33)}$$

where A_i and B_i are constants. Hamilton's principle $\delta I = 0$ can be reformulated by considering the functional

$$\bar{I} = \int_{t_o}^{t_1} L(t, q_1, \dots, q_n, u_1, \dots, u_n) dt \qquad \textbf{(3.34)}$$

subject to the differential constraints

$$\dot{q}_1 = u_1\,, \quad \dot{q}_2 = u_2\,, \dots, \quad \dot{q}_n = u_n \qquad \textbf{(3.35)}$$

Differential constraints of the system can be written in accordance with Eq. (3.30) as follows:

$$h_i = u_i - \dot{q}_i = 0 \qquad \textbf{(3.36)}$$

This functional can be called as criterion of optimality, objective functional or performance measure. According to optimal control theory generalized coordinates q_i are called as *state variables* and u_i are called as *control variables* or *costate variables*. By substituting Eq. (3.36) into Eq. (3.31), augmented functional can be obtained as follows

$$\bar{I}_{aug} = \int_{t_o}^{t_1} [L(t, q_1, \dots, q_2, u_1, \dots, u_n) + \lambda_i(u_i - \dot{q}_i)] dt \qquad \textbf{(3.37)}$$

By taking the first variation of Eq. (3.37) , using commutativity rule of calculus of variations and applying the partial integration one can obtain

$$\delta\bar{I}_{aug} = -\lambda_i(t)\delta q_i|_{t_o}^{t_1} + \int_{t_o}^{t_1} [\left(\frac{\partial L}{\partial u_i} + \lambda_i\right)\delta u_i$$

$$+\left(\frac{\partial L}{\partial u_i} + \dot{\lambda}_i\right)\delta q_i + \delta\lambda_i(u_i - \dot{q}_i)] dt = 0 \qquad \textbf{(3.38)}$$

By considering that the variations at the end points equal to zero as it is given in Eq. (3.18) and by applying the extremality condition $\delta I_{aug} = 0$, the following system of equations are obtained as follows:

$$\lambda_i = -\frac{\partial L}{\partial u_i} \qquad \dot{\lambda}_i = -\frac{\partial L}{\partial q_i} \qquad \textbf{(3.39)}$$

Augmented action integral can be transformed appropriately to generate Hamiltonian canonical differential equations.

Hamiltonian function can be introduced as follows:

$$\overline{H}(t, \boldsymbol{q}, \boldsymbol{\lambda}, \boldsymbol{u}) = L(t, \boldsymbol{q}, \boldsymbol{u}) + \lambda_i u_i \quad \textbf{(3.40)}$$

Augmented action integral can be constructed by substituting Eq. (3.40) into Eq. (3.34) as follows:

$$\bar{I}_{aug} = \int_{t_o}^{t_1} [\overline{H}(t, \boldsymbol{q}, \boldsymbol{\lambda}, \boldsymbol{u}) - \lambda_i \dot{q}_i] dt \quad \textbf{(3.41)}$$

By taking the first variation of Eq. (3.41) and applying the extremality condition $\delta I_{aug} = 0$, canonical equations are obtained in the form of system of equations as follows:

$$\dot{\lambda}_i = -\frac{\partial \overline{H}}{\partial q_i} \qquad \dot{q}_i = \frac{\partial \overline{H}}{\partial \lambda_i} \quad \textbf{(3.42)}$$

4. DIFFERENTIAL TRANSFORM METHOD

In this section, differential transform method is mentioned. After the general description of differential transform is given; one-dimensional, two-dimensional and finally n-dimensional differential transform are presented. General properties of differential transform are also expressed.

4.1. Description of Differential Transform

Differential transform method is a semi analytical-numerical computational technique which is used to solve ordinary and partial differential equations. This method uses the form of polynomials which are sufficiently differentiable in approximating to the exact solution. Differential transform method provides iterative procedures to obtain the high-order Taylor series. In contrast, the traditional Taylor series method requires symbolic computation of the necessary derivatives and is expensive for large orders. Differential transform method, which is capable of solving fractional differential equations, integral and integro-differential equations is introduced by Zhou in 1986 with the application to electrical circuits. ***(Zhou,1986)***

4.2. One-Dimensional Differential Transform

If function of $u(x)$ can be differentiated continuously in the domain X with respect to variable x, it can be said that $u(x)$ is an analytic function.

$$\frac{\partial^k u(x)}{\partial x^k} = \varphi(x,k) \qquad \forall k \in K \tag{4.1}$$

For $x = x_i$ then $\varphi(x,k) = \varphi(x_i,k)$, where k belongs to the set of non-negative integers, denoted as K domain. Therefore Eq. (4.1) can be written as

$$U(k) = \varphi(x_i k) = \left[\frac{\partial^k u(x)}{\partial x^k}\right]_{x=x_i} \qquad \forall k \in K \tag{4.2}$$

where $U(k)$ is called the spectrum of $u(x)$.

If $u(x)$ can be expressed by Taylor series then $u(x)$ can be represented as

$$u(x) = \sum_{k=0}^{\infty} \frac{(x - x_i)^k}{k!} U(k) \quad \textbf{(4.3)}$$

In Eq. (4.3), $u(x)$ is the inverse transformation of $U(k)$. If $U(k)$ is defined as

$$U(k) = M(k) \left[\frac{\partial^k q(x) u(x)}{\partial x^k} \right]_{x=x_0} \quad , \text{ where } \quad k = 0, 1, 2, ..., \infty \quad \textbf{(4.4)}$$

then the function $u(x)$ can be described as

$$u(x) = \frac{1}{q(x)} \sum_{k=0}^{\infty} \frac{(x - x_0)^k}{k!} \frac{U(k)}{M(k)} \quad \textbf{(4.5)}$$

where $M(k)$ is called the weighting factor ($M(k) \neq 0$) and $q(x)$ is regardless as a kernel corresponding to $u(x)$ ($q(x) \neq 0$). If $M(k) = 1$ and $q(x) = 1$ then Eqs. (4.2) and (4.4) are equivalent. As a result of this, Eq. (4.3) can be treated as a special case of Eq. (4.5).

With the use of definitions above, one dimensional differential transform of the k^{th} derivative of a function $u(x)$ around x_o and the differential inverse transformation can be defined as ***(Abdel, 2002)***

$$U(k) = \frac{1}{k!} \left[\frac{d^k u(x)}{dx^k} \right]_{x=x_o} \quad \textbf{(4.6)}$$

$$u(x) = \sum_{k=0}^{n} U(k)(x - x_o)^k \quad \textbf{(4.7)}$$

where $u(x)$ is the original function and $U(k)$ is the transformed function. Here d^k / dx^k means the k^{th} derivative of the function with respect to x.

By combining Eqs. (4.6) and (4.7), original function can be expressed by a finite series as follows:

$$u(x) = \sum_{k=0}^{n} \frac{(x - x_o)^k}{k!} \left[\frac{d^k u(x)}{dx^k} \right] \Bigg|_{x=x_0} \quad \textbf{(4.8)}$$

which means that $\sum_{k=n+1}^{\infty}((x-x_o)^k / k!)(d^k u(x)/dx^k)_{x=x_o}$ is negligibly small. Number of terms which are taken into consideration in the solution process (n in Eq. (4.8)) depends on the convergence decision of the solution.

These definitions show that the Differential Transform Method is derived from Taylor series expansion. By using Eqs. (4.6) and (4.7), some basic properties of one-dimensional differential transform can be represented as follows:

$$u(x) = f(x) \mp g(x) \qquad U(k) = F(k) \mp G(k) \qquad \textbf{(4.9)}$$

$$u(x) = \beta f(x) \qquad U(k) = \beta F(k) \qquad \textbf{(4.10)}$$

$$u(x) = f(x)g(x) \qquad U(k) = \sum_{k_1=0}^{k} F(k_1)G(k-k_1) \qquad \textbf{(4.11)}$$

$$u(x) = \frac{d^m f(t)}{dx^m} \qquad U(k) = \frac{(m+k)!}{k!}F(k+m) \qquad \textbf{(4.12)}$$

$$u(x) = x^m \qquad U(k) = \delta(k-m) = \begin{cases} 1 & k = m \\ 0 & k \neq m \end{cases} \qquad \textbf{(4.13)}$$

where F(k) and G(k) are differential transforms of functions u(x) and v(x), respectively and β is a constant.

4.2.1. Differential transform of boundary conditions

As the original functions described in Eqs. (4.9)-(4.14), boundary conditions of the problem which is desired to be solved can also be transformed as follows:

At point $x = 0$,

$$u(0) = 0 \qquad U(0) = 0 \qquad \textbf{(4.14)}$$

$$\left.\frac{d^m u(x)}{dx^m}\right|_{x=0} = 0 \qquad U(m) = 0 \qquad \textbf{(4.15)}$$

At point x=1,

$$u(1) = 0 \qquad \sum_{k=0}^{\infty} U(k) = 0 \qquad \textbf{(4.16)}$$

$$\left.\frac{du(x)}{dx}\right|_{x=1}=0 \qquad \sum_{k=0}^{\infty}kU(k)=0 \tag{4.17}$$

$$\left.\frac{d^2u(x)}{dx^2}\right|_{x=1}=0 \qquad \sum_{k=0}^{\infty}k(k-1)U(k)=0 \tag{4.18}$$

$$\left.\frac{d^mu(x)}{dx^m}\right|_{x=1}=0 \qquad \sum_{k=0}^{\infty}k(k-1)\cdots(k-(m-1))U(k)=0 \tag{4.19}$$

4.3. Two-Dimensional Differential Transform

Two-dimensional differential transform is used to obtain solutions of partial differential equations. By using two-dimensional differential transform technique, a closed form series solution or an approximate solution of partial differential equations can be obtained.

Two-dimensional differential transform of function $w(x, y)$ around x_o and y_o can be defined as follows:

$$W(r,s)=\frac{1}{r!s!}\left[\frac{\partial^{r+s}}{\partial x^r\partial y^s}w(x,y)\right]_{\substack{x=x_o\\ y=y_o}} \tag{4.20}$$

where $w(x, y)$ is the original function and $W(r,s)$ is the transformed function.

Differential inverse transform of $W(r,s)$ is as follows:

$$w(x,y)=\sum_{k=0}^{\infty}\sum_{k=0}^{\infty}W(r,s)(x-x_o)^r(y-y_o)^s \tag{4.21}$$

By combining Eqs. (4.20) and (4.21), original function can be expressed as

$$w(x,y)=\sum_{k=0}^{\infty}\sum_{k=0}^{\infty}\frac{(x-x_o)^r(y-y_o)^s}{r!s!}\left[\frac{\partial^{r+s}}{\partial x^r\partial y^s}w(x,y)\right]_{\substack{x=x_o\\ y=y_o}} \tag{4.22}$$

Eq. (4.22) signifies that the concept of two-dimensional differential transform is also derived from two-dimensional Taylor series expansion. In real applications, original function $w(x, y)$ is expressed by a finite series and can be written as follows:

$$w(x,y)=\sum_{k=0}^{m}\sum_{k=0}^{n}\frac{(x-x_o)^r(y-y_o)^s}{r!s!}\left[\frac{\partial^{r+s}}{\partial x^r\partial y^s}w(x,y)\right]_{\substack{x=x_o\\ y=y_o}} \tag{4.23}$$

This means

$$w(x,y)=\sum_{k=m+1}^{\infty}\sum_{k=n+1}^{\infty}\frac{(x-x_o)^r(y-y_o)^s}{r!s!}\left[\frac{\partial^{r+s}}{\partial x^r\partial y^s}w(x,y)\right]_{\substack{x=x_o\\ y=y_o}} \qquad \textbf{(4.24)}$$

is negligibly small. By using Eqs. (4.20) and (4.21), some basic properties of two-dimensional differential transform can be represented as follows:

$$w(x,y)=f(x,y)\mp g(x,y) \qquad W(r,s)=F(r,s)\mp G(r,s) \qquad \textbf{(4.25)}$$

$$w(x,y)=\beta f(x,y) \qquad W(r,s)=\beta F(r,s) \qquad \textbf{(4.26)}$$

$$w(x,y)=f(x,y)g(x,y) \qquad W(r,s)=\sum_{r_1=0}^{r}\sum_{s_1=0}^{s}F(r_1,s-s_1)G(r-r_1,s_1) \qquad \textbf{(4.27)}$$

$$w(x,y)=\frac{\partial^{h+\ell}f(x,y)}{\partial x^h\partial y^\ell} \qquad W(r,s)=(r+1)(r+2)\cdots(r+h)$$

$$\times(s+1)(s+2)\cdots(s+\ell)W(r+h,s+\ell) \qquad \textbf{(4.28)}$$

$$w(x,y)=x^m y^n \qquad W(r,s)=\delta(r-m,s-n)=\delta(r-m)\delta(s-n) \qquad \textbf{(4.29)}$$

where

$$\delta(r-m)=\begin{cases}1 & r=m\\ 0 & r\neq m\end{cases} \quad \text{and} \quad \delta(s-n)=\begin{cases}1 & s=n\\ 0 & s\neq n\end{cases} \qquad \textbf{(4.30)}$$

4.4. n-Dimensional Differential Transformation

Let $x=(x_1,x_2,\ldots,x_n)$ be a vector of n variable and $k=(k_1,k_2,\ldots,k_n)$ be a vector of n nonnegative integers then n-dimensional transform of function $w(x_1,x_2,\ldots,x_n)$ about point $x=0$ is as follows:

$$W(k_1,k_2,\ldots,k_n)=\frac{1}{k_1!k_2!\cdots k_n!}\left[\frac{\partial^{k_1+k_2+\cdots+k_n}}{\partial x_1^{k_1}\partial x_2^{k_2}\cdots\partial x_n^{k_n}}w(x_1,x_2,\ldots,x_n)\right]_{(0,0,\ldots,0)} \qquad \textbf{(4.31)}$$

where $w(x)=w(x_1,x_2,\ldots,x_n)$ is the original function and $W(k)=W(k_1,k_2,\ldots,k_n)$ is the transformed function.

Differential inverse transform of $W(k)$ can be defined as follows:

$$w(x_1, x_2, \ldots, x_n) = \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} \cdots \sum_{k_n=0}^{\infty} W(k_1, k_2, \ldots, k_n) \prod_{i=1}^{n} x_i^{k_i} \tag{4.32}$$

By substituting Eq. (4.31) into Eq. (4.32), original function can be obtained as

$$w(x_1, x_2, \ldots, x_n) = \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} \cdots \sum_{k_n=0}^{\infty} \frac{\prod_{i=1}^{n} x_i^{k_i}}{k_1!k_2!\cdots k_n!} \times \left[\frac{\partial^{k_1+k_2+\cdots+k_n}}{\partial x_1^{k_1} \partial x_2^{k_2} \cdots \partial x_n^{k_n}} w(x_1, x_2, \ldots, x_n) \right]_{(0,0,\ldots,0)} \tag{4.33}$$

In real applications, original function $w(x) = w(x_1, x_2, \ldots, x_n)$ is expressed by a finite series and can be written as follows:

$$w(x_1, x_2, \ldots, x_n) = \sum_{k_1=0}^{m_1} \sum_{k_2=0}^{m_2} \cdots \sum_{k_n=0}^{m_n} \frac{\prod_{i=1}^{n} x_i^{k_i}}{k_1!k_2!\cdots k_n!} \times \left[\frac{\partial^{k_1+k_2+\cdots+k_n}}{\partial x_1^{k_1} \partial x_2^{k_2} \cdots \partial x_n^{k_n}} w(x_1, x_2, \ldots, x_n) \right]_{(0,0,\ldots,0)} \tag{4.34}$$

By using Eqs. (4.31) and (4.32), some basic properties of n-dimensional differential transform can be represented as follows:

$$w(x_1, x_2, \ldots, x_n) = u(x_1, x_2, \ldots, x_n) \pm v(x_1, x_2, \ldots, x_n)$$

$$W(k_1, k_2, \ldots, k_n) = U(k_1, k_2, \ldots, k_n) \pm V(k_1, k_2, \ldots, k_n) \tag{4.35}$$

$$w(x_1, x_2, \ldots, x_n) = \beta \cdot u(x_1, x_2, \ldots, x_n)$$

$$W(k_1, k_2, \ldots, k_n) = \beta \cdot U(k_1, k_2, \ldots, k_n) \tag{4.36}$$

$$w(x_1, x_2, \ldots, x_n) = \frac{\partial^{r_1+\cdots+r_n} u(x_1, x_2, \ldots, x_n)}{\partial x_1^{r_1} \cdots \partial x_n^{r_n}}$$

$$W(k_1, k_2, \ldots, k_n) = \frac{(k_1+r_1)!\cdots(k_n+r_n)!}{k_1!\cdots k_n!} U(k_1+r_1, \ldots, k_n+r_n) \tag{4.37}$$

$$w(x_1, x_2, \ldots, x_n) = x_1^{m_1} x_2^{m_2} \cdots x_n^{m_n}$$

$$W(k_1, k_2, \ldots, k_n) = \delta(k_1 - m_1, k_2 - m_2, \ldots, k_n - m_n) = \prod_{i=1}^{n} \delta(k_i - m_i) \qquad \textbf{(4.38)}$$

5. OPTIMAL SHAPE OF COMPRESSED RODS

In this section, optimal distribution of cross-sectional area of compressed rods is obtained under the criterion of minimum volume. After the volume of the rod with optimal cross-section is evaluated, it is compared to the volume of a uniform rod which is stable under the same load value to determine the efficiency of the rod with optimal cross-section.

When the analysis is completed, it is noticed that the optimal compressed rod has zero cross-sectional areas at both ends. However, this cannot be an acceptable configuration in physical sense and for engineering applications. Therefore, governing equations are rearranged and a minimum cross-sectional area is defined in order to overcome the problem of having zero cross-sectional area.

Note that, the problem of determining the optimal shape of compressed rods is also known as Lagrange's problem in open literature.

5.1. Governing Equation for the Problem

Consider a rod with length ***L*** and subjected to concentrated forces at both ends. Then for such a centrally compressed rod with inextensible axis, equilibrium equation without influence of shear is as follows ***(Rao, 2004)***

$$\frac{\partial^2 u}{\partial x^2} + \frac{P}{EI} u = 0 \qquad \textbf{(5.1)}$$

where **E** is modulus of elasticity, ***I*** is area moment of inertia, **u(x)** is displacement field.

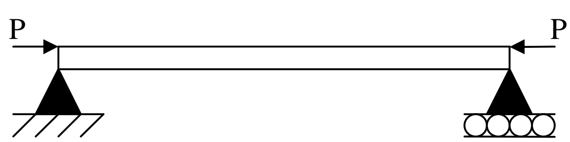
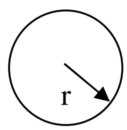
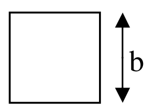
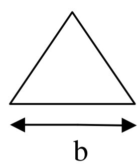
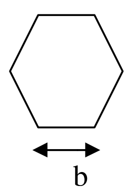


Figure 5.1 Simply Supported Compressed Rod

Since the optimal shape analysis concerns the determination of the cross-sectional area distribution, rod is taken as having variable cross-section. As a result of this assumption, second moment of inertia ***I*** and volume ***W*** can be expressed in terms of cross-sectional area ***A(x)*** as follows:

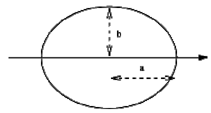
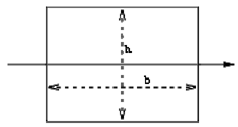
$$I(x) = \alpha A^2(x) \qquad W = \int_0^L A(x)dx \qquad \textbf{(5.2)}$$

Table 5.1 : Shape factors for cross-sections with regular geometries.

Description	Figure	Area moment of inertia	Area	α
Circular		$I = \frac{\pi}{4} r^4$	$A = \pi r^2$	$\frac{1}{4\pi}$
Square		$I = \frac{1}{12} b^4$	$A = b^2$	$\frac{1}{12}$
Triangular		$I = \frac{\sqrt{3}}{32} b^4$	$A = \frac{\sqrt{3}}{4} b^2$	$\frac{\sqrt{3}}{6}$
Hexagonal		$I = \frac{5\sqrt{3}}{16} b^4$	$A = \frac{3\sqrt{3}}{2} b^2$	$\frac{5\sqrt{3}}{108}$

α is a constant for regular geometries such as circular, triangular, etc., since cross-sectional area and second moment of inertia depend on only one parameter for this type of geometries. For regular geometries see Table 5.1. For irregular geometries, **α** also depends on the parameters on which cross-sectional area and second moment of inertia depend. For regular geometries see Table 5.2. Note that, **α** is a dimensionless quantity for either regular or irregular geometries.

Table 5.2 : Shape factors for cross-sections with irregular geometries.

Description	Figure	Area moment of inertia	Area	α
Elliptical		$I = \frac{\pi}{4} ab^3$	$A = \pi ab$	$\frac{1}{4\pi}\frac{b}{a}$
Rectangular		$I = \frac{1}{12} bh^3$	$A = bh$	$\frac{1}{12}\frac{h}{b}$

By substituting (5.2) into (5.1),

$$u'' + \frac{P}{E\alpha A^2} u = 0 \quad \textbf{(5.3)}$$

where $(\cdot)' = d(\cdot)/dx$. Boundary conditions for the simply supported rod can be expressed as:

$$u(0) = 0 \qquad u(L) = 0 \quad \textbf{(5.4)}$$

For the sake of simplicity, some dimensionless quantities can be introduced as:

$$\bar{u} = \frac{u}{L} \qquad \xi = \frac{x}{L} \qquad a = \frac{A}{L^2} \qquad \lambda = \frac{P}{E\alpha L^2} \qquad w = \frac{W}{L^3} \quad \textbf{(5.5)}$$

With the use of preceding dimensionless quantities, (5.3) can be written in the form of:

$$\ddot{\bar{u}} + \frac{\lambda}{a^2(\xi)} \bar{u} = 0 \quad \textbf{(5.6)}$$

$$\bar{u}(0) = 0 \qquad \bar{u}(1) = 0 \quad \textbf{(5.7)}$$

where $\dot{(\cdot)} = d(\cdot)/d\xi$.

5.2. Optimization Problem

Optimization problem can be defined as the determination of a(ξ) in Eq. (5.6) which minimizes volume of the rod by using Pontryagin's maximum principle. For this problem; this principle is based on minimization of Hamiltonian function, since the aim is to minimize volume within the stability limits. To construct Hamiltonian function it is required to define state and costate variables. State variables are determined by separating the governing equations into a system of differential equations.

The volume which is aimed to be minimized can be given in the dimensionless form as follows:

$$w = \int_0^1 a(\xi) d\xi \tag{5.8}$$

Eqs.(5.6) and (5.7) can be written by means of state variables q_1 and q_2 as follows:

$$\dot{q}_1 = q_2 \qquad \dot{q}_2 = -\frac{\lambda}{a^2(\xi)} q_1 \tag{5.9}$$

$$q_1(0) = q_1(1) = 0 \tag{5.10}$$

where $q_1(\xi)$ denotes $\overline{u}(\xi)$ in the original differential equation. Costate variables p_1 and p_2 have to satisfy the following system of differential equations

$$\dot{p}_1 = -\frac{\partial H(a, p_1, p_2, q_1, q_2, \xi)}{\partial q_1}$$

$$\dot{p}_2 = -\frac{\partial H(a, p_1, p_2, q_1, q_2, \xi)}{\partial q_2} \tag{5.11}$$

where $\boldsymbol{H(a,p_1,p_2,q_1,q_2,\xi)}$ represents Hamiltonian function which can be constructed by using Eq.(5.8) and corresponding variables defined above (See Section 3.3.2).

$$H = a(\xi) + p_1 q_2 - p_2 \frac{\lambda}{a^2(\xi)} q_1 \tag{5.12}$$

After constructing Hamiltonian function, it is necessary to control whether Eq.(5.11) is satisfied.

By substituting Eq.(5.12) into Eq.(5.11), relation between state and costate variables can be obtained as

$$p_1(\xi) = q_2(\xi) \qquad p_2(\xi) = -q_1(\xi) \tag{5.13}$$

It is possible to obtain different relations between state and costate variables, but this does not change the conclusion, only affects the solution procedure.

Optimal cross-sectional area function is such ***a(ξ)*** which minimizes Hamiltonian function. Therefore, Eq.(5.12) is differentiated with respect to a(ξ) to determine the optimal shape where *H* is minimum.

$$\frac{\partial H}{\partial a} = 1 + p_2 \frac{2\lambda}{a^3} q_1 = 0 \tag{5.14}$$

By solving Eqs. (5.13) and (5.14), a(ξ) is obtained in terms of $q_1(\xi)$. By replacing $q_1(\xi)$ to $\bar{u}(\xi)$, optimal cross-sectional area function is found in the form of

$$a(\xi) = (2\lambda \bar{u}^2(\xi))^{1/3} \tag{5.15}$$

Substituting Eq.(5.15) into Eqs.(5.6) and (5.7) gives the nonlinear differential equation system as

$$\ddot{\bar{u}} + (\frac{\lambda}{4})^{1/3} \bar{u}^{-1/3} = 0 \tag{5.16}$$

$$\bar{u}(0) = \bar{u}(1) = 0 \tag{5.17}$$

5.3. Solution of Governing System

In the solution step, Differential Transform Method (**DTM**) is applied to the system. To simplify the application of differential transform method, governing equation can be written in the form of

$$(\ddot{\bar{u}})^3 \bar{u} = \bar{\lambda}^3 \tag{5.18}$$

where $\bar{\lambda} = (\lambda / 4)^{1/3}$.

Differential transform of Eq. (5.18) around ξ_0=0 is

$$\sum_{k_3=0}^{k} \sum_{k_2=0}^{k_3} \sum_{k_1=0}^{k_2} U(k_1)(k_2+1-k_1)(k_2+2-k_1)U(k_2+2-k_1)(k_3+1-k_2)(k_3+2-k_2)U(k_3+2-k_2) \times (k+1-k_3)(k+2-k_3)U(k+2-k_3) = -\bar{\lambda}^3 \delta(k) \tag{5.19}$$

where U(k) represents the differential transform of $\bar{u}(\xi)$.

Differential transform of boundary conditions (5.17) are respectively leads to

$$U(0)=0 \qquad \sum_{k=0}^{n} U(k)=0 \tag{5.20}$$

Insert $U(1)=b$, where b is a constant which will be evaluated at the end of solution with the use of Eq.(5.20).

At k=1 with substituting U(0) and U(1) into Eq.(5.19)

$$U(2)=0 \tag{5.21}$$

At k=2 with substituting (5.21) into (5.22), the result of $U(3)=0$ is obtained. Following the same procedure, n terms of the series is calculated as

$$U(3)=U(4)=\cdots=U(n)=0$$

By using the differential transform of second boundary condition, which is given in (3.19), missed boundary condition ***b*** is obtained as *b=0.*

Therefore, it can be said that Differential Transform Method gives the trivial solution of the system. To obtain the non-trivial solution of the system (5.16)-(5.17), it can be useful to analyze the non-linear differential equation analytically.

Differential equation which is aimed to be analyzed can be written in the form of

$$\ddot{x}+\bar{\lambda}x^{-1/3}=0 \tag{5.22}$$

with the replacement of $\bar{u}(\xi)=x(\xi)$. Eq. (5.22) can be written in the form of

$$\frac{1}{2}\frac{d}{dx}\left(\frac{dx}{d\xi}\right)^2+\bar{\lambda}x^{-1/3}=0 \tag{5.23}$$

By integrating Eq. (5.23)

$$\left(\frac{dx}{d\xi}\right)^2=-\frac{2\bar{\lambda}}{2/3}x^{2/3}+C_1 \tag{5.24}$$

$$\frac{dx}{d\xi}=\sqrt{3\bar{\lambda}}\sqrt{\frac{C_1}{3\bar{\lambda}}-x^{2/3}} \tag{5.25}$$

To get the solution of Eq.(5.25), a transformation, which is in the following form, can be used.

$$x^{1/3} = \sqrt{\frac{C_1}{3\bar{\lambda}}}\sin\theta \quad \textbf{(5.26)}$$

$$\frac{dx}{d\xi} = 3\sqrt{\frac{C_1}{3\bar{\lambda}}}x^{2/3}\cos\theta\frac{d\theta}{d\xi} \quad \textbf{(5.27)}$$

By substituting Eqs.(5.26) and (5.27) into Eq.(5.28),

$$\frac{C_1}{\bar{\lambda}}\sin^2\theta d\theta = \sqrt{3\bar{\lambda}}d\xi \quad \textbf{(5.28)}$$

Eq.(5.28) is integrated by using the trigonometric relation of $\sin^2\theta = (1-\cos 2\theta)/2$ to obtain a solution to the system. Therefore, the solution is in the form of

$$\frac{C_1}{2\bar{\lambda}}(\theta - \frac{\sin 2\theta}{2}) = \sqrt{3\bar{\lambda}}\xi + C_2 \quad \textbf{(5.29)}$$

5.4. Obtaining the Integration Constants

Boundary conditions are applied to Eq. (5.29) to get the constants of integration C_1 and C_2.

$$x(\xi = 0) = 0 \quad \sin\theta = 0 \quad \theta = n\pi \quad n = 0,1,\ldots \quad \textbf{(5.30)}$$

$$x(\xi = 1) = 0 \quad \sin\theta = 0 \quad \theta = n\pi \quad n = 0,1,\ldots \quad \textbf{(5.31)}$$

Applying first boundary condition (5.30) to Eq. (5.29) gives

$$C_2 = \frac{n\pi}{2\bar{\lambda}}C_1 \qquad n = 0,1,\ldots \quad \textbf{(5.32)}$$

Applying second boundary condition (5.31) to Eq. (5.29) gives

$$C_2 = \frac{n\pi}{2\bar{\lambda}}C_1 - \sqrt{3\bar{\lambda}} \qquad n = 0,1,\ldots \quad \textbf{(5.33)}$$

The condition of ***n=0*** gives trivial solution to the problem.

Nontrivial solution can be obtained for the condition of ***n=0*** and ***C_2=0*** as follows:

$$C_1 = 2\sqrt{3}\frac{\bar{\lambda}^{3/2}}{\pi} \quad \textbf{(5.34)}$$

5.5. Results for Centrally Compressed Rod

5.5.1. Optimal distribution of cross-sectional area

Taking the inverse transform of Eq. (5.35) by using Eq.(5.26) gives the solution of the function **x(ξ).**

$$\arcsin\left(\sqrt{\pi\sqrt{\frac{3}{4\bar{\lambda}}}x^{1/3}}\right) - \sqrt{\pi\sqrt{\frac{3}{4\bar{\lambda}}}x^{1/3}}\sqrt{1-\pi\sqrt{\frac{3}{4\bar{\lambda}}}x^{2/3}} = \pi\xi \quad \textbf{(5.35)}$$

For a given critical load $\bar{\lambda}$, distribution of ***x(ξ)*** can be obtained from Eq. (5.35). By changing x(ξ) to $\bar{u}(\xi)$, optimal distribution of cross-sectional area which is a function of $\bar{u}(\xi)$, as it is described in Eq.(3.15), can be obtained for any given λ. Eq. (5.36) gives optimal distribution of cross-sectional area.

$$\arcsin\left(\sqrt{\frac{\pi}{4\bar{\lambda}}\sqrt{\frac{3}{\bar{\lambda}}}a^{1/2}}\right) - \sqrt{\frac{\pi}{4\bar{\lambda}}\sqrt{\frac{3}{\bar{\lambda}}}a^{1/2}}\sqrt{1-\frac{\pi}{4\bar{\lambda}}\sqrt{\frac{3}{\bar{\lambda}}}a} = \pi\xi \quad \textbf{(5.36)}$$

Fig.(5.2) shows the optimal distribution of cross-sectional area, corresponding moment distribution and optimal distribution of radius for circular cross-sectional rods for $\lambda = \pi^2$

For the rod with circular cross-sectional area with the representation of $\bar{r}(\xi) = r(\xi)/L = \sqrt{a(\xi)/\pi}$. $\bar{r}(\xi)$ in Fig.(5.2) denotes the dimensionless radius of the rod.

Cross-sectional area distribution cannot be obtained as a smooth function of rod-length. At some selected points; dimensionless displacement, dimensionless cross-sectional area and dimensionless radius of the rod is evaluated and a curve is fitted as it is seen in Fig. (5.2).

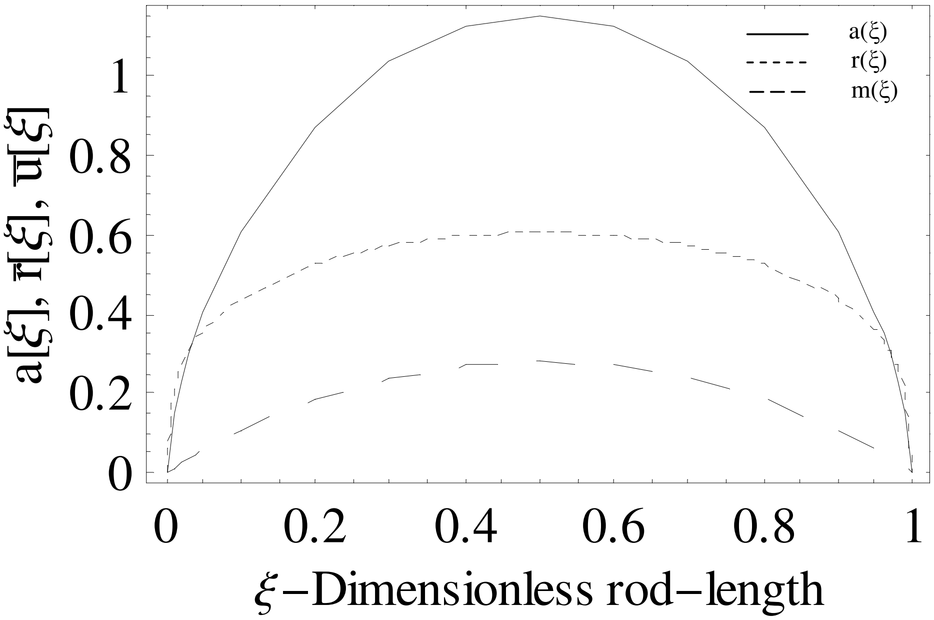


Figure 5.2 Optimal shape distribution of a centrally compressed rod

5.5.2. Volume of optimal rod

By using the obtained cross-sectional area distribution volume of the optimal compressed rod can be evaluated by using Eq. (5.8) as $w_{optimal} =$

Alternate to this evaluation, volume of the optimal rod can be calculated by using the fact that the optimal distribution of the cross-sectional area has a symmetrical attribute and therefore has a maximum at mid-point. This leads to the conclusion of that first derivative of the obtained cross-sectional area distribution must be equal to zero at mid-point. This conclusion can be represented as follows:

$$\left.\frac{da}{d\xi}\right|_{\xi=0.5} = 0 \qquad \textbf{(5.37)}$$

By differentiating Eq. (5.36) with respect to ξ gives

$$\dot{a}(\xi) = \frac{\pi}{\gamma^3 a(\xi)^{1/2}}\sqrt{1-\gamma^2 a(\xi)} \qquad \text{where} \qquad \gamma = \sqrt{\frac{\pi}{4\bar{\lambda}}\sqrt{\frac{3}{\bar{\lambda}}}} \qquad \textbf{(5.38)}$$

Applying the condition given in Eq. (5.37) leads to

$$1-\gamma^2 a(\xi = 0.5) = 0 \qquad \textbf{(5.39)}$$

$$a_o^2 = \frac{16\bar{\lambda}^3}{3\pi^2} \quad \textbf{(5.40)}$$

where a_o is the mid-point cross-sectional area. By replacing $\bar{\lambda} = (\lambda / 4)^{1/3}$

$$a_o = \frac{1}{\pi}\sqrt{\frac{4\lambda}{3}} \quad \textbf{(5.41)}$$

Eq. (5.8) can be written with the property of symmetrical distribution of cross-sectional area as follows:

$$w = 2\int_0^{1/2} a(\xi)d\xi \quad \textbf{(5.42)}$$

Eq. (5.38) can be written in the form of

$$d\xi = \frac{\gamma^3 a(\xi)^{1/2}}{\pi\sqrt{1-\gamma^2 a(\xi)}} da \quad \textbf{(5.43)}$$

Substituting Eq.(5.43) into Eq. (5.42) gives

$$w = \frac{2\gamma^3}{\pi}\int_0^{a_o} \frac{a^{3/2}}{\sqrt{1-\gamma^2 a}} da \quad \textbf{(5.44)}$$

After integrating Eq. (5.44) and using Eq. (5.41) gives the volume of the optimal rod as follows:

$$w_{optimal} = \frac{1}{\pi}\sqrt{\frac{3\lambda}{4}} \quad \textbf{(5.45)}$$

For given λ volume of the optimal rod can be found by using Eq. (5.45). For $\lambda = \pi^2$ volume of the rod with optimal shape leads to $w_{optimal} = 0.866025$.

5.6. Comparison of results with a uniform rod

To determine the efficiency of the optimal rod, deformation analysis of the rod with constant cross-sectional area must be carried out.

By considering cross-sectional area ***a*** is a constant, solution of the governing system given in Eqs. (5.6) and (5.7) can be obtained as follows:

$$\bar{u}(\xi) = A\sin(\sqrt{\frac{\lambda}{a^2}}\xi) + B\cos(\sqrt{\frac{\lambda}{a^2}}\xi) \qquad \textbf{(5.46)}$$

By applying the boundary conditions, ***B=0*** is obtained and the eigenvalues of the system are as follows:

$$\lambda_n = n^2\pi^2a^2 \qquad n = 1,2,.. \qquad \textbf{(5.47)}$$

Critical load for centrally compressed rod is the least eigenvalue of the system. Therefore, by taking ***n=1*** in the Eq. (5.47), critical load of the system can be given as follows

$$\lambda_{cr} = \pi^2a^2 \qquad n = 1,2,.. \qquad \textbf{(5.48)}$$

5.6.1. Comparison of volume

For uniform rod $w = a$ from Eq. (5.8), therefore volume of a centrally compressed rod which is stable under the given critical load can be obtained as follows:

$$w_{constant} = \frac{1}{\pi}\sqrt{\lambda} \qquad \textbf{(5.49)}$$

For a uniform rod, volume of the column is evaluated as $w_{constant} = 1$ for the given load $\lambda = \pi^2$. This means volume saving of optimal rod with respect to uniform rod leads to ***13.4%*** for the given load.

For given load, relation between volumes of optimal and uniform rods is

$$w_{optimal} = \sqrt{\frac{3}{4}}w_{constant} \qquad \textbf{(5.50)}$$

From Eq. (5.50) it can be seen that the ratio between the volumes of optimal rod and uniform rod does not depend on the given critical load value. It can be said that, for an arbitrary critical load, volume saving of optimal rod with respect to uniform load always equals to ***13.4%.***

5.6.2. Comparison of critical loads

Another comparison between optimal rod and uniform rod can be made through the dimensionless load which will be carried by each rod. For a given volume, amount of critical loads are evaluated for each rod.

For this comparison, Eq. (5.45) can be rearranged as follows to obtain the critical load which simply supported compressed rod with optimal distribution of cross-sectional area can carry for a given volume.

$$\lambda_{optimal} = \frac{4}{3}\pi^2 w^2 \qquad \textbf{(5.51)}$$

Eq. (5.49) can also be written in the form of

$$\lambda_{cons\tan t} = \pi^2 w^2 \qquad \textbf{(5.52)}$$

For a given volume of $w = 1$, optimal rod can carry a load of $\lambda_{optimal} = 13.1595$ from Eq. (5.51) and uniform can carry a load of $\lambda_{cons\tan t} = 9.8696$ from Eq. (5.52). This means optimal rod carries ***33.3%*** mucher than that of carried by constant cross-sectional area.

For given volume, relation between critical load which can be carried by optimal and constant cross-sectional rods can be given as follows:

$$\lambda_{optimal} = \frac{4}{3}\lambda_{cons\tan t} \qquad \textbf{(5.53)}$$

From Eq. (5.53) it can be seen that the ratio between the critical loads of optimal rod and uniform rod does not depend on the given volume. It can be said that, for an arbitrary volume, critical load saving of optimal rod with respect to uniform load always equals to ***33.3%.***

5.7. Further Considerations – Rearranged Compressed Rod Problem

It can be seen from Fig. (5.2) that the optimal compressed rod has zero cross-sectional area at end points. But this is not an acceptable configuration for engineering practice. Therefore, problem can be modeled by using material limits of the rod. By using limits of given material of which compressed rod is composed, the cross-sectional area at end points can be defined as a condition of the problem.

5.7.1. Rearranging governing equations

Eq. (5.16) can be constructed by means of optimal distribution of cross-sectional area ***a(ξ)*** which is given in Eq. (5.15). Eq. (5.15) can be written as follows:

$$u(\xi) = \frac{a^{3/2}(\xi)}{\sqrt{2\lambda}} \tag{5.54}$$

By differentiating Eq. (5.54) two times and substituting into Eq. (5.16), governing equation can be obtained as follows:

$$\ddot{a}a + \frac{1}{2}\dot{a}^2 + \frac{2}{3}\lambda = 0 \tag{5.55}$$

5.7.2. Defining end point cross-sectional areas

Boundary conditions of the problem can be defined by using the minimal cross-sectional area that is allowed by material limits. In other words, for every rod-length ξ_i, optimal cross-section of the column must satisfy the condition of

$$a_i = a(\xi = \xi_i) \geq a_o \tag{5.56}$$

where a_o is a constant given in the definition of the problem and denotes the allowable minimum cross-sectional area.

End point cross-sectional area, which is decided to be defined as a condition of the problem (See Eq. (5.56)), is the minimum cross-sectional area allowed by material limits. For a compressive loading, by ignoring the bending effect, compressive stress can be found as follows:

$$\sigma = \frac{P}{A} \tag{5.57}$$

In the dimensionless form it can be written as follows:

$$\bar{\sigma} = \frac{\lambda}{a} \tag{5.58}$$

where $\bar{\sigma} = \sigma/(E\alpha)$ is the dimensionless compressive stress. For a material, compressive strength is known and by using this limit minimum cross-sectional area can be determined by using the condition of $\bar{\sigma} \leq \bar{\sigma}_{\lim it}$.

By substituting this condition into the condition given in Eq. (5.58), allowable minimum cross-sectional area can be obtained as follows:

$$a_o = \frac{\lambda}{\bar{\sigma}_{limit}} \qquad (5.59)$$

By substituting Eq. (5.59) into (5.56), optimal distribution of cross-sectional are is limited with the use of material limits and applied force as follows:

$$a \geq \frac{\lambda}{\bar{\sigma}_{\lim it}} \qquad \textbf{(5.60)}$$

By using given limitation on the minimum cross-sectional area, boundary conditions of the problem can be defined as follows:

$$a(0) = a(1) = a_o \qquad \textbf{(5.61)}$$

5.7.3. Solution of governing system

In the solution step, Differential Transform Method (DTM) is applied to the governing system given in Eqs. (5.55) and (5.61).

Differential transform of Eq. (5.55) around ξ_0=0 is

$$\sum_{k_1=0}^{k}(k_1+1)(k_1+2)A(k_1+2)A(k-k_1)$$

$$+\frac{1}{2}\sum_{k_1=0}^{k}(k_1+1)A(k_1+1)(k-k_1+1)A(k-k_1+1)+\frac{2}{3}\lambda\delta(k)=0 \qquad \textbf{(5.62)}$$

where A(k) represents the differential transform of $a(\xi)$. Differential transform of boundary conditions (5.61) are respectively leads to

$$A(0) = a_o \qquad \sum_{k=0}^{n} A(k) = a_o \qquad \textbf{(5.63)}$$

Insert $A(1) = b$, where b is a constant which will be evaluated at the end of solution with the use of Eq.(5.63).

At k=1 with substituting A(0) and A(1) into Eq.(5.62)

$$A(2) = -\frac{3b^2 + 4\lambda}{12a_o} \qquad \textbf{(5.64)}$$

At k=2 with additionally substituting (5.64) into (5.62)

$$A(3)=\frac{3b^3+4b\lambda)}{18a_o^2} \tag{5.65}$$

By following the same procedure, the other terms of the series can be evaluated as follows:

$$A(4)=-\frac{63b^4+96b^2\lambda+16\lambda^2}{432a_o^3} \tag{5.66}$$

$$A(5)=\frac{315b^5+552b^3\lambda+176b\lambda^2}{2160a_o^4} \tag{5.67}$$

$$A(6)=-\frac{12285b^6+24516b^4\lambda+11376b^2\lambda^2+704\lambda^3}{77760a_o^5} \tag{5.68}$$

⋮

⋮

By substituting the terms A(0) to A(n) into second boundary condition given in Eq. (5.63), **b** can be found for given end point cross-sectional area $\boldsymbol{a_o}$ and critical load value **λ**.

According to one dimensional differential transform method (See Section 4.2), inverse transform can be obtained as having the following form

$$a(\xi)=\sum_{k=0}^{n}A(k)\xi^k \tag{5.69}$$

After missed boundary condition ***b*** is found, all terms of the series can be obtained for a given load and optimal distribution of cross-sectional area can be determined by using Eq. (5.69).

5.7.4. Results for rearranged compressed rod problem

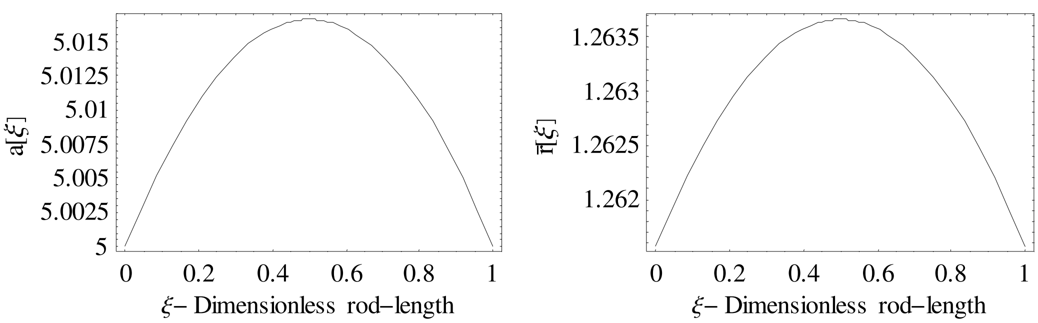
Results for this problem is given for $\lambda=1$ and for a material with a dimensionless compressive strength of $\bar{\sigma}_{\lim it}=0.2$. For these given values of dimensionless critical load and compressive strength of the material, minimum cross-sectionl area that is allowed can be found by using Eq. (5.59) as $a_o=5$.

5.7.4.1. Optimal distribution of cross-sectional area

For $\lambda = 1$, $\bar{\sigma}_{\lim it} = 0.2$ and $a_o = 5$, the terms of the series given in Eqs. (5.64)- (5.68) are obtained and substituted into Eq. (5.69) to obtain optimal distribution of cross-sectional area. Optimal shape of the compressed rod is presented in Fig. (5.3). Fig. (5.4) also presents the distribution of dimensionless radius of the column which has circular cross-section ($\bar{r}(\xi) = r(\xi)/L = \sqrt{a(\xi)/\pi}$).

5.7.4.2. Volume of optimal rod

Dimensionless volume of the optimal column is calculated from Eq.(5.8) as $w_{optimal} = 5.0119$.



(a) (b)

Figure 5.3 (a) Optimal distribution of cross-sectional area of compressed rod with end point constraint

(b) Optimal distribution of radius with end point cross-section along rod-length for circular cross-sectional rod

6. OPTIMAL SHAPE OF ECCENTRICALLY COMPRESSED COLUMNS

In this section, columns which are subjected to eccentrically concentrated forces are outlined and optimal distribution of cross-sectional area of such structures along the column length is determined. Two different configurations are considered, which are columns loaded eccentrically at both ends and columns loaded eccentrically at one end. Volume of the optimal column for each loading condition is determined. After the volume of the optimal column is evaluated, it is compared to the volume of a uniform column which is stable under the same load value to determine the efficiency of the optimal column.

6.1. Eccentrically Concentrated Forces at Both Ends

6.1.1. Governing equation for the problem

Governing equation which describes the dynamical behavior of a column which is subjected to concentrated forces at both ends as it is seen in Fig. (6.1) can be obtained by considering an Euler-Bernoulli column (See Section 2.2.1).

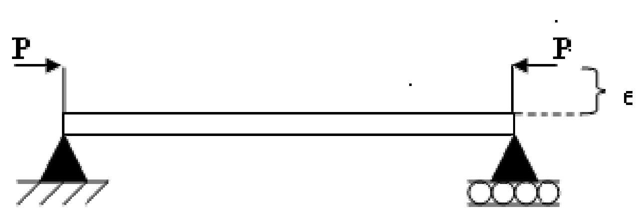


Figure 6.1 Eccentrically Compressed Column at Both Ends

By accepting the counterclockwise direction as positive direction, moment equilibrium can be written (see Fig. (6.2)) as follows;

$$M(x,t)+\frac{\partial M(x,t)}{\partial x}dx-M(x,t)+[V(x,t)$$

$$+\frac{\partial V(x,t)}{\partial x}dx]dx-H(x,t)\frac{\partial U(x,t)}{\partial x}dx=0 \qquad \textbf{(6.1)}$$

By eliminating the terms including the multiplication of $dx \cdot dx$, Eq. (6.1) goes to

$$\frac{\partial M(x,t)}{\partial x} = H(x,t)\frac{\partial U(x,t)}{\partial x} - V(x,t) \qquad \textbf{(6.2)}$$

Equilibrium along y-direction, by accepting y-direction is positive, can be written as

$$V(x,t) + \frac{\partial V(x,t)}{\partial x}dx - V(x,t) = \rho A(x) dx \frac{\partial^2 U(x,t)}{\partial t^2} \qquad \textbf{(6.3)}$$

$$\frac{\partial V(x,t)}{\partial x} = \rho A(x)\frac{\partial^2 U(x,t)}{\partial t^2} \qquad \textbf{(6.4)}$$

Equilibrium along x-direction, by accepting x-direction is positive, can be written as

$$H(x,t) + \frac{\partial H(x,t)}{\partial x}dx - H(x,t) = 0 \qquad \textbf{(6.5)}$$

$$\frac{\partial H(x,t)}{\partial x} = 0 \qquad \textbf{(6.6)}$$

$$H(x,t) = cons. = P \qquad \textbf{(6.7)}$$

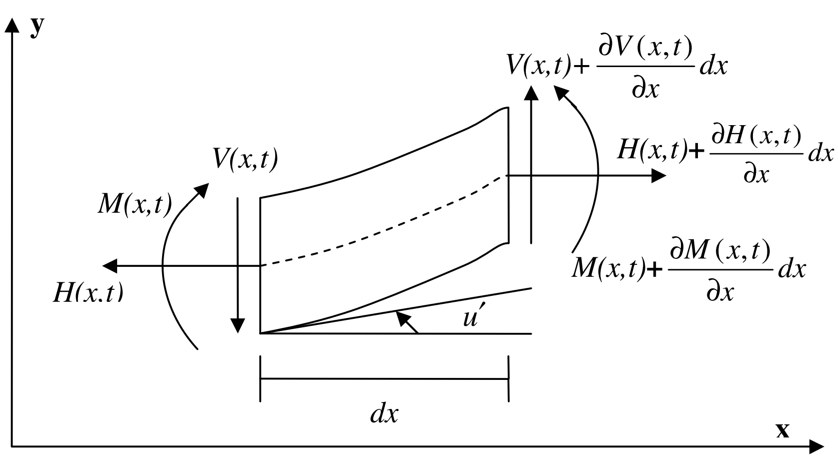


Figure 6.2 Configuration of the column after deformation

Constitutive relation for the column is as follows

$$M = EI(x)\frac{\partial^2 U(x,t)}{\partial x^2} \qquad \textbf{(6.8)}$$

By substituting Eqs.(6.4) and (6.7) into Eq.(6.2) gives the equation of motion for the compressed columns. Equation of motion for this type of loading can be represented as follows:

$$\frac{\partial^2 M(x,t)}{\partial x^2} + \rho A(x)\frac{\partial^2 U(x,t)}{\partial t^2} + P\frac{\partial^2 U(x,t)}{\partial x^2} = 0 \qquad \textbf{(6.9)}$$

where ***E*** is modulus of elasticity, ***I*** is area moment of inertia of the column, ρ is mass density, ***A*** is cross-sectional area of the column, ***P*** is the applied concentrated force and ***U(x,t)*** is displacement field.

For the steady-state solution where $U(x,t) = u(x)$, then the governing equation reduces to the following form

$$\frac{d^2 M(x)}{dx^2} + P\frac{d^2 u}{dx^2} = 0 \qquad \textbf{(6.10)}$$

By substituting Eq. (6.8) into Eq. (6.10), governing equation can be written by means of moment distribution as follows:

$$M'' + \frac{P}{EI(x)}M = 0 \qquad \textbf{(6.11)}$$

where $(\cdot)' = d(\cdot)/dx$. By using Eq.(5.2) which gives the second moment of inertia distribution of the column with variable cross-section, Eq. (6.11) can be written in the following form

$$M'' + \frac{P}{E\alpha A^2(x)}M = 0 \qquad \textbf{(6.12)}$$

Boundary conditions for simply supported columns loaded by eccentrically concentrated forces at both ends are as follows:

$$M(0) = Pe \qquad M(L) = Pe \qquad \textbf{(6.13)}$$

where **e** is eccentricity (see Fig. 6.1).

Dimensionless quantities for this problem can be defined as

$$m = \frac{M}{E\alpha L^3} \qquad \lambda = \frac{P}{E\alpha L^2} \qquad \xi = \frac{x}{L} \qquad a = \frac{A}{L^2} \qquad \overline{e} = \frac{e}{L} \qquad \textbf{(6.14)}$$

Governing equation for eccentrically compressed column is obtained in terms of preceeding dimensionless quantities as follows

$$\ddot{m}+\frac{\lambda}{a^2(\xi)}m=0 \quad \textbf{(6.15)}$$

$$m(0)=m(1)=\lambda\overline{e} \quad \textbf{(6.16)}$$

where $\dot{(\cdot)}=d(\cdot)/d\xi$

6.1.2. Optimization problem

By following the same procedure described in Section (5.2), Hamiltonian function can be obtained by separating Eq.(6.15) by means of state and costate variables as follows (See Section 3.3.2)

$$\dot{q}_1=q_2 \qquad \dot{q}_2=-\frac{\lambda}{a^2(\xi)}q_1 \quad \textbf{(6.17)}$$

Hamiltonian function can be represented as follows:

$$H=a(\xi)+p_1q_2-p_2\frac{\lambda}{a^2(\xi)}q_1 \quad \textbf{(6.18)}$$

By using Eq.(5.11),

$$\dot{p}_1=\frac{\lambda}{a^2}p_2 \qquad \dot{p}_2=-p_1 \quad \textbf{(6.19)}$$

By using Eq (6.19), the relation between state and costate variables can be obtained as

$$p_1(\xi)=q_2(\xi) \qquad p_2(\xi)=-q_1(\xi) \quad \textbf{(6.20)}$$

Optimal cross-sectional area function is such a(ξ) which minimizes Hamiltonian function. Therefore, Eq.(6.18) is differentiated with respect to a(ξ) to determine the optimal shape where *H* is minimum.

$$\frac{\partial H}{\partial a}=1+\frac{2\lambda}{a^3}p_2q_1=0 \quad \textbf{(6.21)}$$

As a result of solving Eq. (6.21) for a(ξ) and replacing $q_1(\xi)$ to m(ξ), optimal shape of the eccentrically compressed column can be represented by means of the related moment distribution of the column as follows

$$a(\xi) = (2\lambda(m(\xi))^2)^{1/3} \quad \textbf{(6.22)}$$

By substituting Eq.(6.22) into Eq.(6.15), the system reduces to a second-order nonlinear differential equation as follows

$$\ddot{m} + (\frac{\lambda}{4})^{1/3} m^{-1/3} = 0 \quad \textbf{(6.23)}$$

$$m(0) = m(1) = \lambda\overline{e} \quad \textbf{(6.24)}$$

6.1.3. Solution of governing system

In the solution step, Differential Transform Method (**DTM**) is applied to the system. To simplify the application of differential transform method, governing equation can be written in the form of

$$(\ddot{m})^3 m = \frac{\lambda}{4} \quad \textbf{(6.25)}$$

Differential transform of Eq. (6.23) around ξ_0=0 is

$$\sum_{k_3=0}^{k}\sum_{k_2=0}^{k_3}\sum_{k_1=0}^{k_2} M(k_1)(k_2+1-k_1)(k_2+2-k_1)M(k_2+2-k_1)(k_3+1-k_2)(k_3+2-k_2)M(k_3+2-k_2) \times(k+1-k_3)(k+2-k_3)M(k+2-k_3) = -\frac{\lambda}{4}\delta(k) \quad \textbf{(6.26)}$$

where M(k) represents the differential transform of m(ξ).

Differential transforms of boundary conditions (6.24) are respectively

$$M(0) = \lambda\overline{e} \qquad \sum_{k=0}^{n} M(k) = \lambda\overline{e} \quad \textbf{(6.27)}$$

Insert $M(0) = \lambda\overline{e}$ and $M(1) = b$, where b is a constant which will be evaluated at the end of solution with the use of Eq.(6.27). At k=1 with substituting ***M(0)*** and ***M(1)*** into Eq.(6.26)

$$M(2) = -\frac{1}{2^{5/3}\overline{e}^{1/3}} \quad \textbf{(6.28)}$$

At k=2 with additionally substituting (7.30) into (7.28)

$$M(3)=\frac{1}{2^{5/3}\overline{e}^{1/3}}\frac{b}{9\lambda\overline{e}} \quad \textbf{(6.29)}$$

By following the same procedure, the other terms of the series can be evaluated as follows:

$$M(4)=-\frac{1}{2^{2/3}\overline{e}^{1/3}}\frac{1}{54}\left(\frac{b}{\lambda\overline{e}}\right)^2-\frac{1}{2^{1/3}\overline{e}^{2/3}}\frac{1}{144}\frac{1}{\lambda\overline{e}} \quad \textbf{(6.30)}$$

$$M(5)=\frac{1}{2^{2/3}\overline{e}^{1/3}}\frac{7}{810}\left(\frac{b}{\lambda\overline{e}}\right)^3+\frac{1}{2^{1/3}\overline{e}^{2/3}}\frac{13}{2160}\left(\frac{1}{\lambda\overline{e}}\right)^2 \quad \textbf{(6.31)}$$

$$M(6)=-\frac{1}{2^{2/3}\overline{e}^{1/3}}\frac{7}{1458}\left(\frac{b}{\lambda\overline{e}}\right)^4-\frac{1}{2^{1/3}\overline{e}^{2/3}}\frac{47}{9720}\left(\frac{1}{\lambda\overline{e}}\right)\left(\frac{b}{\lambda\overline{e}}\right)^2-\frac{1}{\overline{e}}\frac{13}{25920}\left(\frac{1}{\lambda\overline{e}}\right)^2 \quad \textbf{(6.32)}$$

$$\vdots$$

$$\vdots$$

Following the same procedure, n terms of the series are calculated. Terms of the corresponding series can be represented by new parameters which can be described as follows:

$$\beta=\frac{b}{\lambda\overline{e}} \quad \text{and} \quad \mu=\frac{1}{\lambda\overline{e}} \quad \textbf{(6.33)}$$

All of the terms evaluated are as follows:

$$M(0)=\frac{1}{\mu}$$

$$M(1)=\frac{1}{\mu}\beta$$

$$M(2)=-\frac{1}{2^{5/3}\overline{e}^{1/3}}$$

$$M(3)=\frac{1}{2^{5/3}9\overline{e}^{1/3}}\beta$$

$$M(4) = -\left(\frac{1}{2^{2/3} 54 \overline{e}^{1/3}} \beta^2 + \frac{\mu}{2^{1/3} 144 \overline{e}^{2/3}} \right)$$

$$M(5) = \frac{7}{2^{2/3} 810 \overline{e}^{1/3}} \beta^3 + \frac{13\mu^2}{2^{1/3} 2160 \overline{e}^{2/3}}$$

$$M(6) = -\left(\frac{7}{2^{2/3} 1458 \overline{e}^{1/3}} \beta^4 + \frac{47\mu}{2^{1/3} 9720 \overline{e}^{2/3}} \beta^2 + \frac{13\mu^2}{25920 \overline{e}} \right)$$

$$\vdots$$

$$\vdots \qquad \textbf{(6.34)}$$

By substituting the terms ***M(0)*** to ***M(n)*** into second boundary condition given in Eq.(6.27), one can obtain

$$\frac{1}{\mu} + \frac{1}{\mu}\beta - \frac{1}{2^{5/3} \overline{e}^{1/3}} + \frac{1}{2^{5/3} 9 \overline{e}^{1/3}} \beta - \left(\frac{1}{2^{2/3} 54 \overline{e}^{1/3}} \beta^2 + \frac{\mu}{2^{1/3} 144 \overline{e}^{2/3}} \right)$$
$$+ \left(\frac{7}{2^{2/3} 810 \overline{e}^{1/3}} \beta^3 + \frac{13\mu^2}{2^{1/3} 2160 \overline{e}^{2/3}} \right) - \cdots = \frac{1}{\mu} \qquad \textbf{(6.35)}$$

It can be useful to re-arrange Eq. (6.35) as

$$\frac{1}{\mu}\beta - \frac{1}{2^{5/3} \overline{e}^{1/3}} + \frac{1}{2^{5/3} 9 \overline{e}^{1/3}} \beta - \left(\frac{1}{2^{2/3} 54 \overline{e}^{1/3}} \beta^2 + \frac{\mu}{2^{1/3} 144 \overline{e}^{2/3}} \right) + \cdots = 0 \qquad \textbf{(6.36)}$$

to get the following form

$$f^{(n)}(\beta) = 0 \qquad \textbf{(6.37)}$$

In the case of known λ and $\overline{e}$, b can be found by solving Eq.(6.37) for β. By considering n terms of the series, and solving Eq.(6.37) gives the eigenvalue $\beta_i^{(n)}$ corresponding to the n-term estimation.

Computation can be stopped for such *n* which satisfies the following inequality

$$\left| \beta_i^{(n)} - \beta_i^{(n-1)} \right| \le \varepsilon \qquad \textbf{(6.38)}$$

where **ε** is the value of acceptable truncation error.

For this problem Eq. (6.37) can be constructed as follows:

$$f^{(n)}(\beta) = -\frac{1}{2^{5/3} \overline{e}^{1/3}} + \left(\frac{1}{\mu} + \frac{1}{2^{5/3} 9 \overline{e}^{1/3}} \right)\beta - \frac{\mu}{2^{1/3} 144 \overline{e}^{2/3}} - \frac{1}{2^{2/3} 54 \overline{e}^{1/3}} \beta^2 + \cdots = 0 \quad \textbf{(6.39)}$$

Since λ and $\bar{e}$ are given, μ is a known parameter, therefore missed boundary condition **b** can be obtained by solving Eq.(6.39) for β. Fig. (6.3) presents change in **b** with respect to critical load for given eccentricity. All terms of the series can be determined by substituting missed boundary condition into Eq. (6.34).

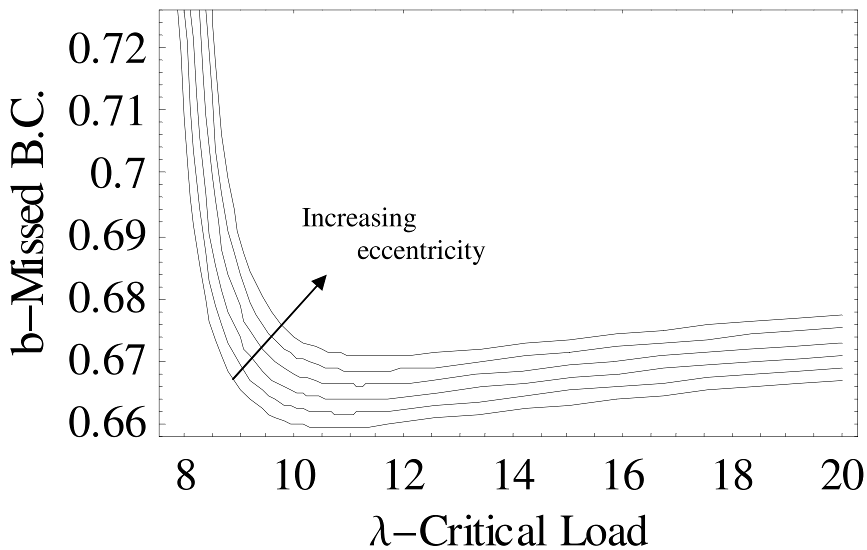


Figure 6.3 Effect of critical load on the solution for column subjected to eccentrically concentrate forces at both ends

According to one dimensional differential transform method (See Section 4.2), inverse transform can be obtained as having the following form

$$m(\xi) = \sum_{k=0}^{n} M(k)\xi^k \tag{6.40}$$

6.1.4. Results for eccentrically compressed column at both ends

Since there are two parameters which have to be given in the description of problem (load and eccentricity), first of all optimal shape of the column for $\lambda = 15$ and for different values of eccentricity is represented in Fig. (6.4). Optimal distribution of cross-sectional area is obtained from Eq.(6.22) by determining the moment distribution along column-length for given λ and $\bar{e}$.

6.1.4.1. Optimal distribution of cross-sectional area for $\lambda = 15$ and $\bar{e} = 0.07$

For $\lambda = 15$ and $\bar{e} = 0.07$, missed boundary condition can be evaluated as $\beta = 0.6883$ and $b = 0.7227$. Moment distribution corresponding to this problem can be obtained as follows:

$$m(\xi) = 1.05 + 0.7227\xi - 0.7643\xi^2 + 0.0584\xi^3 - 0.0443\xi^4 + 0.0227\xi^5 - \cdots \quad \textbf{(6.41)}$$

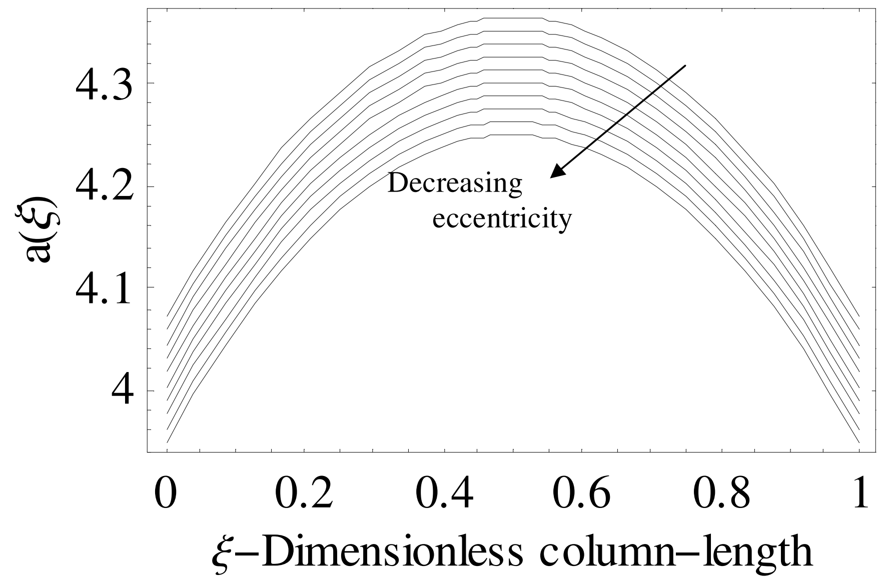


Figure 6.4 Optimal shape of column subjected to eccentrically concentrated forces at both ends for different values of eccentricity

By using obtained m(ξ), optimal distribution of cross-sectional area is determined for eccentrically compressed columns at both ends from Eq.(6.22) as it is shown in Fig. (6.5). Fig. (6.5) also presents the distribution of dimensionless radius of the column which has circular cross-section ($\bar{r}(\xi) = r(\xi)/L = \sqrt{a(\xi)/\pi}$).

6.1.4.2. Volume of optimal column $\lambda = 15$ and $\bar{e} = 0.07$

Dimensionless volume of the optimal column is calculated from Eq.(5.8) as $w_{optimal} = 3.4398$. The values of optimal volume corresponding to each eccentricity value are given in Fig. (6.5).

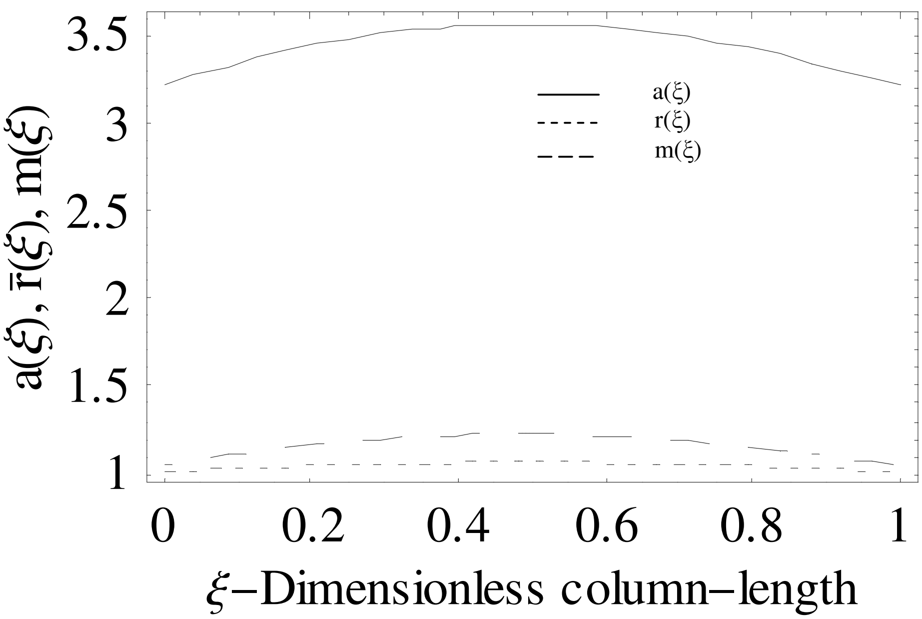


Figure 6.5 Optimal shape of column subjected to eccentrically concentrated forces at both ends for $\lambda = 15$ and $\bar{e} = 0.07$

6.1.5. Comparison of results with uniform column

To determine the efficiency of the optimal column it can be useful to analyze the behavior of the column which has constant cross-section.

For a column with constant cross-section, solution of Eq. (6.15) has the form of

$$m(\xi) = A\sin\sqrt{\frac{\lambda}{a^2}}\xi + B\cos\sqrt{\frac{\lambda}{a^2}}\xi \qquad \textbf{(6.42)}$$

by accepting that ***a*** is a constant.

By applying boundary conditions given in Eq. (6.16), moment distribution along the column length can be determined as follows:

$$m(\xi) = \lambda\bar{e}(\tan\frac{1}{2}\sqrt{\frac{\lambda}{a^2}}\sin\sqrt{\frac{\lambda}{a^2}}\xi + \cos\sqrt{\frac{\lambda}{a^2}}\xi) \qquad \textbf{(6.43)}$$

Dimensionless moment at the midpoint $\bar{M}$ is as follows:

$$\bar{M} = \lambda\bar{e}\sec\frac{1}{2}\sqrt{\frac{\lambda}{a^2}} \qquad \textbf{(6.44)}$$

where $\bar{M} = M/(E\alpha L^3)$ and M is the midpoint moment ***(Chilver, 1993)***.

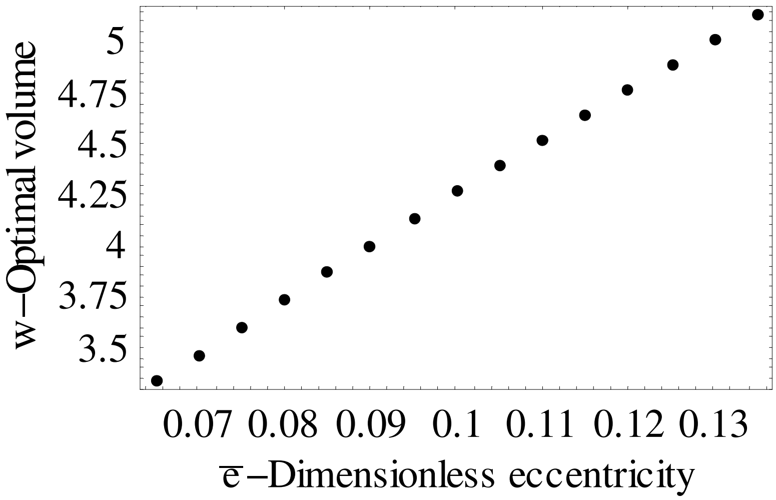


Figure 6.6 Effect of eccentricity on the optimal volume

As it is seen in Fig. (6.7), moment at the midpoint goes to zero as $\lambda / a^2 \rightarrow \pi^2$. Hence, it can be said that the critical load for the eccentrically compressed column can be taken as

$$\lambda = \pi^2 a^2 \quad \textbf{(6.45)}$$

From Eq. (5.8), it can be seen that the dimensionless volume of constant cross-sectional column equals to dimensionless cross-sectional area, namely $w_{constant} = a$. Therefore volume of constant cross-sectional column can be determined by using Eq. (6.45) as follows:

$$w_{constant} = \frac{\sqrt{\lambda}}{\pi} \quad \textbf{(6.46)}$$

However, it cannot be accepted as the critical value of the applied force since this would imply an infinitely large value of midpoint deflection and material breakdown would occur at some smaller value of λ. Therefore, it is needed to use material limits to evaluate the volume of the column which has constant cross-sectional area. Since the largest lateral deflection and also greatest bending moment occurs at mid-length of the column, it can be useful to determine the longitudinal stresses at this section.

6.1.5.1. Maximum compressive stress for uniform column

The bending moment at mid-length of the column is determined before and given in Eq. (6.44). By using this bending moment, longitudinal stress at mid-length can be determined as

$$\sigma_1 = \frac{Mc}{\alpha A^2} \qquad \textbf{(6.47)}$$

where **c** is the distance from centroidal axis. The average longitudinal compressive stress can be determined as follows:

$$\sigma_2 = \frac{P}{A} \qquad \textbf{(6.48)}$$

In the dimensionless form, maximum longitudinal compressive stress given above can be described as follows:

$$\bar{\sigma}_{\max} = \frac{\bar{M}\bar{r}}{a^2} + \frac{\lambda\alpha}{a} \quad \text{where} \quad \bar{\sigma}_{\max} = \frac{(\sigma_1 + \sigma_2)_{c=r}}{E} \qquad \textbf{(6.49)}$$

where $\bar{r} = r / L$ is the dimensionless radius of the column with constant cross-section. By substituting Eq. (6.44) into Eq. (6.49), maximum longitudinal stress for the compressed column which has constant cross-sectional area can be obtained as

$$\bar{\sigma}_{\max} = \lambda(\frac{\bar{e}\bar{r}}{a^2} \sec\frac{1}{2}\sqrt{\frac{\lambda}{a^2}} + \frac{\alpha}{a}) \qquad \textbf{(6.50)}$$

For the columns with circular cross-section, maximum longitudinal stress

$$\bar{\sigma}_{\max} = \lambda(\frac{\bar{e}}{\pi^2\bar{r}^3} \sec\frac{1}{2}\frac{\sqrt{\lambda}}{\pi\bar{r}^2} + \frac{1}{4\pi^2\bar{r}^2}) \qquad \textbf{(6.51)}$$

6.1.5.2. Volume of uniform column

Consider a material with a dimensionless compressive stress limit of $\bar{\sigma}_{\lim it} = 0.01$, then the results for $e = 0.07$ and $\lambda = 15$ can be represented as $\bar{r}_{cons\tan t} = 2.0777$ and $w_{cons\tan t} = 13.5618$.

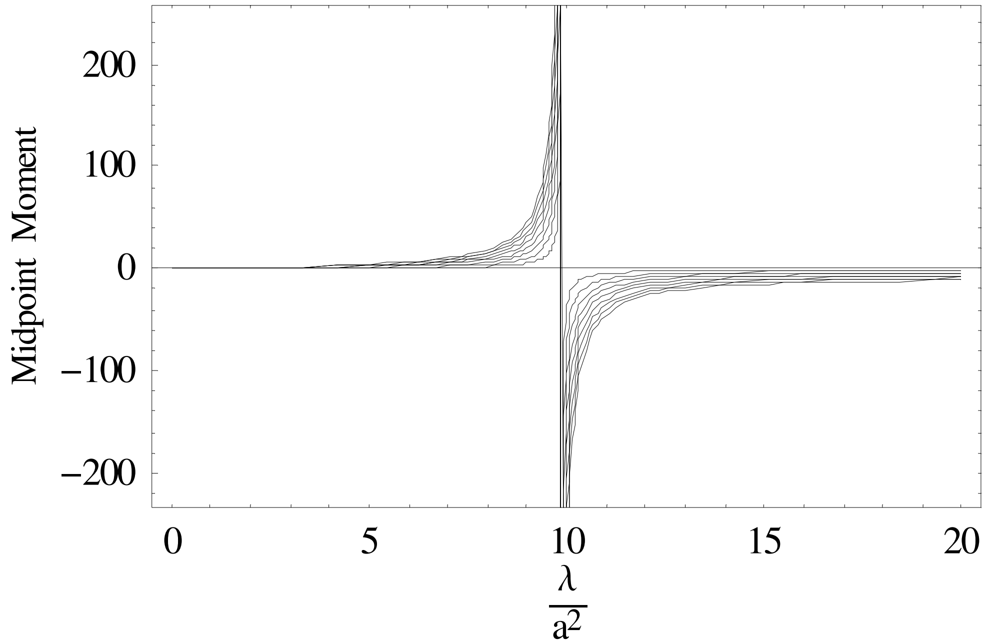


Figure 6.7 Effect of eccentricity on the optimal volume

6.2. Eccentrically Concentrated Forces at One End

Optimal shape analysis of columns which are loaded by eccentrically concentrated forces at one end differs only in the solution step, since the only difference is sourced by the boundary conditions.

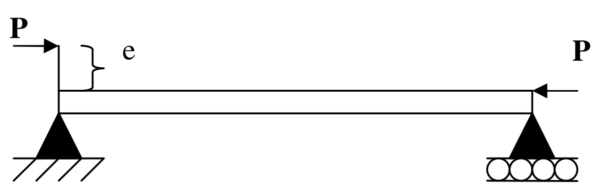


Figure 6.8 Eccentrically Compressed Column at One End

6.2.1. Governing system for the problem

Governing system has the same form as it is in the problem of eccentrically concentrated force at both ends. Therefore, there is no need to define a new optimization problem.

Governing system for the optimal shape analysis of columns subjected to eccentric loading at one end can be given as follows:

$$\ddot{m}+(\frac{\lambda}{4})^{1/3}m^{-1/3}=0 \quad \textbf{(6.52)}$$

$$m(0)=\lambda\overline{e} \qquad m(1)=0 \quad \textbf{(6.53)}$$

6.2.2. Solution of governing system

Since the governing system is same, differential transform of the system does not differ. Only difference comes from the differential transform of boundary conditions.

Differential transforms of boundary conditions (6.53) are respectively

$$M(0)=\lambda\overline{e} \qquad \sum_{k=0}^{n}M(k)=0 \quad \textbf{(6.54)}$$

The terms of the series are same since the first boundary condition is same with the problem defined for the columns loaded by eccentrically concentrated forces at both ends.

By substituting the terms M(0) to M(n) (refer to Eq. (6.34)) into second boundary condition given in Eq.(6.54), one can obtain

$$\frac{1}{\mu}+\frac{1}{\mu}\beta-\frac{1}{2^{5/3}\overline{e}^{1/3}}+\frac{1}{2^{5/3}9\overline{e}^{1/3}}\beta-\left(\frac{1}{2^{2/3}54\overline{e}^{1/3}}\beta^{2}+\frac{\mu}{2^{1/3}144\overline{e}^{2/3}}\right)$$
$$+\left(\frac{7}{2^{2/3}810\overline{e}^{1/3}}\beta^{3}+\frac{13\mu^{2}}{2^{1/3}2160\overline{e}^{2/3}}\right)-\cdots=0 \quad \textbf{(6.55)}$$

In the case of known λ and $\overline{e}$, b can be found by solving Eq.(6.55) for β. Since λ and $\overline{e}$ are given, μ is a known parameter, therefore missed boundary condition **b** can be obtained by solving Eq.(6.55) for β. Fig. (6.2) presents change in **b** with respect to different values of critical load for given eccentricity. All terms of the series can be determined by substituting missed boundary condition into Eq. (6.34).

According to one dimensional differential transform method (See Section 4.2), inverse transform can be obtained as having the following form

$$m(\xi)=\sum_{k=0}^{n}M(k)\xi^{k} \quad \textbf{(6.56)}$$

6.2.3. Results for eccentrically compressed column at one end

For $\lambda = 15$ and $\bar{e} = 0.07$, missed boundary condition can be evaluated as $\beta = -0.2014$ and $b = -0.2114$ by solving Eq. (6.55).

6.2.3.1. Optimal distribution of cross-sectional area for $\lambda = 15$ and $\bar{e} = 0.07$

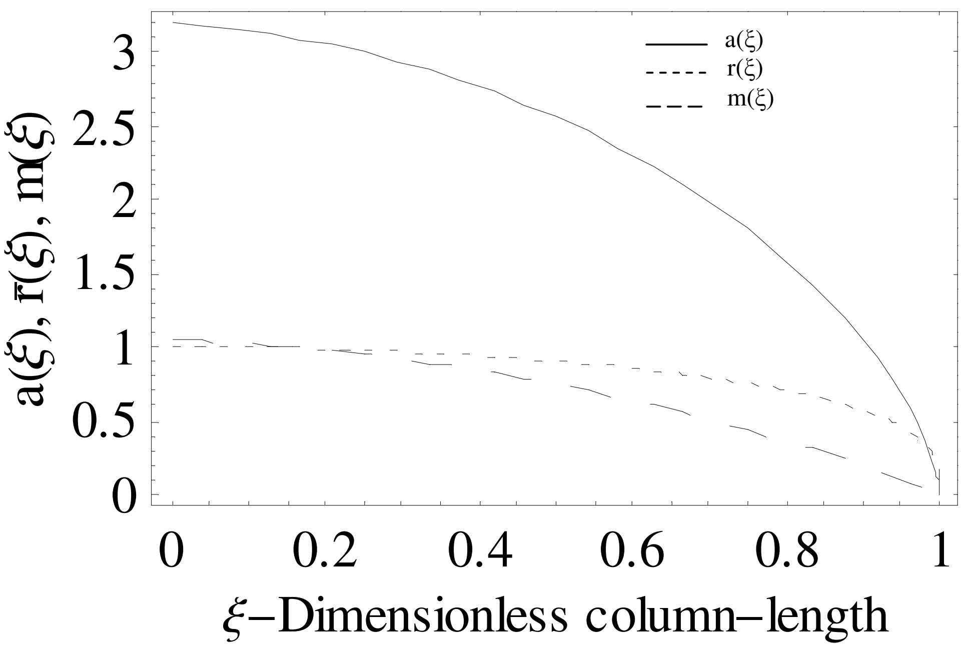


Figure 6.9 Optimal shape of column subjected to eccentrically concentrated forces at both ends for $\lambda = 15$ and $\bar{e} = 0.07$

Moment distribution corresponding to this problem can be obtained as follows:

$$m(\xi) = 1.05 - 0.2114\xi - 0.7643\xi^2 - 0.0171\xi^3 - 0.0321\xi^4 - 0.0055\xi^5 - \cdots \quad \textbf{(6.57)}$$

By using obtained m(ξ), optimal distribution of cross-sectional area is determined for eccentrically compressed columns at one end from Eq.(6.22) as it is shown in Fig. (6.9). Fig. (6.9) also presents the distribution of dimensionless radius of the column which has circular cross-section ($\bar{r}(\xi) = r(\xi)/L = \sqrt{a(\xi)/\pi}$).

6.2.3.2. Volume of optimal column $\lambda = 15$ and $\bar{e} = 0.07$

Dimensionless volume of the optimal column is calculated from Eq.(5.8) as $w_{optimal} = 2.3172$.

6.2.4. Comparison of results with uniform column

To make a comparison and determine the efficiency of the optimal column, column with constant cross-section can be taken into consideration. For a column with constant cross-section, solution of Eq. (6.15) has the form of

$$m(\xi) = A\sin\sqrt{\frac{\lambda}{a^2}}\xi + B\cos\sqrt{\frac{\lambda}{a^2}}\xi \tag{6.58}$$

by accepting that ***a*** is a constant. By applying boundary conditions given in Eq. (6.53), moment distribution along the column length can be determined as follows:

$$m(\xi) = \lambda\overline{e}(-\cot\frac{1}{2}\sqrt{\frac{\lambda}{a^2}}\sin\sqrt{\frac{\lambda}{a^2}}\xi + \cos\sqrt{\frac{\lambda}{a^2}}\xi) \tag{6.59}$$

Dimensionless moment at the midpoint $\overline{M}$ is as follows:

$$\overline{M} = \frac{\lambda\overline{e}}{2}\sec\frac{1}{2}\sqrt{\frac{\lambda}{a^2}} \tag{6.60}$$

As it is seen in Fig. (7.6), moment at the midpoint goes to zero as $\lambda / a^2 \to \pi^2$. Hence, it can be said that the critical load for the eccentrically compressed column can be taken as $\lambda = \pi^2 a^2$. However, as it is explained above, it cannot be accepted as the critical value of the applied force since this would imply an infinitely large value of midpoint deflection and material breakdown would occur at some smaller value of λ. Therefore, it is needed to use material limits to evaluate the volume of the column which has constant cross-sectional area. Since the greatest bending moment occurs at one end of column on which eccentric loading applied (ξ=0) , it can be useful to determine the longitudinal stresses at this cross-section and make a comparison.

6.2.4.1 Maximum compressive stress for uniform column

At ξ=0, moment is $m(0) = \lambda\overline{e}$. Maximum longitudinal stress for the compressed column which has constant cross-sectional area can be obtained by using Eqs. (6.47) and (6.48) as follows:

$$\overline{\sigma}_{\max} = \lambda(\frac{\overline{er}}{a^2} + \frac{\alpha}{a}) \tag{6.61}$$

For the columns with circular cross-section, maximum longitudinal stress

$$\bar{\sigma}_{\max} = \lambda\left(\frac{\bar{e}}{\pi^2 \bar{r}^3} + \frac{1}{4\pi^2 \bar{r}^2}\right) \quad \textbf{(6.62)}$$

6.2.4.2. Volume of uniform column

Consider a material with a dimensionless compressive stress limit of $\bar{\sigma}_{\lim it} = 0.01$, then the results for $e = 0.07$ and $\lambda = 15$ can be represented as $\bar{r}_{cons\tan t} = 2.0765$ and $w_{cons\tan t} = 13.5462$.

7. OPTIMAL SHAPE OF COLUMNS UNDER FOLLOWER TYPE LOADING

In this section, columns of Euler-Bernoulli type are analyzed for the conditions uniformly distributed and exponentially varying follower type of loading. For each loading condition optimal distribution of cross-section along the column length and optimal volume of such column is determined. The efficiency of the columns with such cross-section by means of volume and load saving is determined for each loading condition by considering uniform column which is subjected to same amount of loading.

7.1. Uniformly Distributed Follower Type of Loading

7.1.1. Governing equation for the problem

For the optimal shape analysis of the column subjected to uniformly distributed follower type of loading, it is considered that the loading has a constant intensity of $\mathbf{f_o}$.

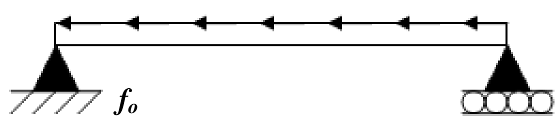


Figure 7.1 Simply supported column subjected to uniformly distributed follower loading

By considering the differential element shown in Fig. (6.2), moment equilibrium can be written with the acceptance that counterclockwise direction is positive direction.

$$M(x,t) + \frac{\partial M(x,t)}{\partial x}dx - M(x,t) + [V(x,t) + \frac{\partial V(x,t)}{\partial x}dx]dx - H(x,t)\frac{\partial U(x,t)}{\partial x}dx = 0 \qquad \textbf{(7.1)}$$

By eliminating the terms including the multiplication of $dx \cdot dx$, Eq. (7.1) goes

$$\frac{\partial M(x,t)}{\partial x} = H(x,t)\frac{\partial U(x,t)}{\partial x} - V(x,t) \quad \textbf{(7.2)}$$

Equilibrium along y-direction, by accepting y-direction is positive, can be written as

$$V(x,t) + \frac{\partial V(x,t)}{\partial x}dx - V(x,t) = \rho A(x)dx\frac{\partial^2 W(x,t)}{\partial t^2} \quad \textbf{(7.3)}$$

$$\frac{\partial V(x,t)}{\partial x} = \rho A(x)\frac{\partial^2 W(x,t)}{\partial t^2} \quad \textbf{(7.4)}$$

Equilibrium along x-direction, by accepting x-direction is positive, can be written as

$$H(x,t) + \frac{\partial H(x,t)}{\partial x}dx - H(x,t) - f_o dx = 0 \quad \textbf{(7.5)}$$

$$\frac{\partial H(x,t)}{\partial x} = f_o \quad \textbf{(7.6)}$$

Constitutive relation for the column is as follows

$$M = EI(x)\frac{\partial^2 U(x,t)}{\partial x^2} \quad \textbf{(7.7)}$$

where ***E*** is modulus of elasticity, ***I*** is area moment of inertia of the column, ρ is mass density, ***A*** is cross-sectional area of the column, ***P*** is the applied concentrated force and ***U(x,t)*** is displacement field.

For the steady-state solution where $U(x,t) = u(x)$, Eqs. (7.2), (7.4) and (7.6) leads to

$$\frac{dM(x)}{dx} = H(x)\frac{du(x)}{dx} - V(x) \quad \textbf{(7.8)}$$

$$V(x) = 0 \quad \textbf{(7.9)}$$

$$H(x) = -f_o(L - x) \quad \textbf{(7.10)}$$

respectively, by using that H(L)=0 at the movable end.

Substituting Eqs. (7.9) and (7.10) into Eq. (7.8) and taking the first derivative of released equation gives

$$\frac{d^2 M(x)}{dx^2} = -f_0(L - x)\frac{d^2 u(x)}{dx^2} \quad \textbf{(7.11)}$$

By using constitutive relation given in Eq. (6.8), governing equation can be written by means of moment distribution as follows:

$$M'' + \frac{f_0}{EI(x)}(L-x)M = 0 \tag{7.12}$$

where $(\cdot)' = d(\cdot)/dx$. By using Eq.(5.2) which gives the second moment of inertia distribution of the column with variable cross-section, Eq. (7.12) can be written in the following form

$$M'' + \frac{f_0}{E\alpha A^2(x)}(L-x)M = 0 \tag{7.13}$$

Boundary conditions for simply supported columns loaded by follower type loading are as follows:

$$M(0) = 0 \qquad M(L) = 0 \tag{7.14}$$

To simplify and generalize Eq.(7.13), it can be useful to define some dimensionless quantities as follows:

$$m = \frac{M}{E\alpha L^3} \qquad \lambda = \frac{f_0}{E\alpha L} \qquad \xi = \frac{x}{L} \qquad a = \frac{A}{L^2} \qquad w = \frac{W}{L^3} \tag{7.15}$$

By using Eq.(7.15), the governing system can be obtained in terms of dimensionless quantities as follows:

$$\ddot{m} + \frac{\lambda}{a^2(\xi)}(1-\xi)m = 0 \tag{7.16}$$

$$m(0) = m(1) = 0 \tag{7.17}$$

where $\dot{(\cdot)} = d(\cdot)/d\xi$.

7.1.2. Optimization problem

By following the same procedure described in Section (5.2), state and costate variables of the optimization problem can be defined as follows:

$$\dot{q}_1 = q_2 \qquad \dot{q}_2 = -\frac{\lambda}{a^2(\xi)}(1-\xi)q_1 \tag{7.18}$$

$$q_1(0) = q_1(1) = 0 \tag{7.19}$$

where $q_1(\xi)$ denotes $m(\xi)$ in the original differential equation.

Hamiltonian function can be constructed by using Eq.(5.8) which gives the volume of the rod that is aimed to be minimized and corresponding variables defined above.

$$H = a(\xi) + p_1 q_2 - p_2 \frac{\lambda}{a^2(\xi)}(1-\xi)q_1 \qquad \textbf{(7.20)}$$

By using Eq.(5.11),

$$\dot{p}_1 = \frac{\lambda}{a^2}(1-\xi)p_2 \qquad \dot{p}_2 = -p_1 \qquad \textbf{(7.21)}$$

By using Eq (6.19), the relation between state and costate variables can be obtained as

$$p_1(\xi) = q_2(\xi) \qquad p_2(\xi) = -q_1(\xi) \qquad \textbf{(7.22)}$$

Optimal cross-sectional area function is such $a(\xi)$ which minimizes Hamiltonian function. Therefore, Eq.(7.20) is differentiated with respect to $a(\xi)$ to determine the optimal shape where *H* is minimum.

$$\frac{\partial H}{\partial a} = 1 + p_2 \frac{2\lambda}{a^3}(1-\xi)q_1 = 0 \qquad \textbf{(7.23)}$$

By solving Eqs. (7.23) with the substitution of Eq.(7.22), $a(\xi)$ is obtained in terms of $q_1(\xi)$. By replacing $q_1(\xi)$ to $m(\xi)$, optimal cross-sectional area function is found in the form of

$$a(\xi) = (2\lambda(1-\xi)m^2(\xi))^{1/3} \qquad \textbf{(7.24)}$$

Substituting Eq.(7.24) into Eqs.(7.16) gives the nonlinear differential equation system as

$$\ddot{m} + (\frac{\lambda}{4})^{1/3}(1-\xi)^{1/3} m^{-1/3} = 0 \qquad \textbf{(7.25)}$$

$$m(0) = m(1) = 0 \qquad \textbf{(7.26)}$$

7.1.3. Solution of governing system

In the solution step, Differential Transform Method (**DTM**) is applied to the system. To simplify the application of differential transform method, governing equation can be written in the form of

$$(\ddot{m})^3 m = \bar{\lambda}^3 (1-\xi) \qquad \textbf{(7.27)}$$

where $\bar{\lambda} = (\lambda/4)^{1/3}$. Differential transform of Eq. (7.27) around ξ_0=0 is

$$\sum_{k_3=0}^{k}\sum_{k_2=0}^{k_3}\sum_{k_1=0}^{k_2} M(k_1)(k_2+1-k_1)(k_2+2-k_1)M(k_2+2-k_1)(k_3+1-k_2)(k_3+2-k_2)M(k_3+2-k_2)$$

$$\times(k+1-k_3)(k+2-k_3)M(k+2-k_3) = -\bar{\lambda}^3(\delta(k)-\delta(k-1)) \quad \textbf{(7.28)}$$

Differential transform of boundary conditions (7.26) are respectively leads to

$$M(0)=0 \qquad \sum_{k=0}^{n} M(k)=0 \qquad \textbf{(7.29)}$$

Insert $M(0)=0$ and $M(1)=b$, where b is a constant which will be calculated at the end of solution with the use of Eq.(7.29).

At k=1 with substituting M(0) and M(1) into Eq.(7.28)

$$M(2) = \frac{1}{2b^{1/3}}\bar{\lambda} \qquad \textbf{(7.30)}$$

At k=2 with additionally substituting (7.30) into (7.28)

$$M(3) = -\frac{1}{36b^{5/3}}\bar{\lambda}^2 \qquad \textbf{(7.31)}$$

By following the same procedure, the other terms of the series can be evaluated as follows:

$$M(4) = \frac{7}{1296b^3}\bar{\lambda}^3 \qquad \textbf{(7.32)}$$

$$M(5) = -\frac{23}{15552b^{13/3}}\bar{\lambda}^4 \qquad \textbf{(7.33)}$$

$$M(6) = \frac{5}{10368b^{17/3}}\bar{\lambda}^5 \qquad \textbf{(7.34)}$$

$$M(7) = -\frac{6179}{35271936 b^{17/3}} \bar{\lambda}^6 \tag{7.35}$$

$\vdots$

$\vdots$

Following the same procedure, n terms of the series are calculated. Terms of the corresponding series can be represented by a new parameter which can be described as follows:

$$\beta = \frac{\bar{\lambda}}{b^{4/3}} \tag{7.36}$$

All of the terms evaluated are as follows:

$$M(0) = 0$$

$$M(1) = b$$

$$M(2) = \frac{\beta}{2} b$$

$$M(3) = -\frac{1}{36 b^{5/3}} \bar{\lambda}$$

$$M(4) = \frac{7\beta^2}{1296} b$$

$$M(5) = -\frac{23\beta^3}{15552} b$$

$$M(6) = \frac{5\beta^4}{10368} b$$

$$M(7) = -\frac{6179\beta^5}{35271936} b$$

$\vdots$

$\vdots$ **(7.37)**

By substituting the terms M(0) to M(n) into second boundary condition given in Eq. (7.29), the following form can be obtained.

$$b f^{(n)}(\beta) = 0 \tag{7.38}$$

In the case of known $\bar{\lambda}$, b can be found by solving Eq.(7.38) for β, since $b \neq 0$.

$$f^{(n)}(\beta) = 0 \quad \textbf{(7.39)}$$

By considering n terms of the series, and solving Eq.(7.39) gives the eigenvalue $\beta_i^{(n)}$ corresponding to the n-term estimation. Computation can be stopped for such n which satisfies the following inequality

$$\left|\beta_i^{(n)} - \beta_i^{(n-1)}\right| \leq \varepsilon \quad \textbf{(7.40)}$$

where **ε** is the value of acceptable truncation error.

For this problem Eq. (7.39) can be constructed as follows:

$$f^{(n)}(\beta) = 1 + \frac{\beta}{2} - \frac{\beta^2}{36} + \frac{7\beta^3}{1296} - \frac{23\beta^4}{15552} + \frac{5\beta^5}{10368} - \cdots = 0 \quad \textbf{(7.41)}$$

By taking n=50 terms, $\beta_i^{(50)} = 1.70872$ and for error estimation $\beta_i^{(49)} = 1.70874$. Substituting these values into Eq. (7.40) gives $\left|\beta_i^{(50)} - \beta_i^{(49)}\right| = 0.00002$. This error value is in the acceptable region.

According to one dimensional differential transform method (See Section 4.2), inverse transform can be obtained as having the following form

$$m(\xi) = \sum_{k=0}^{n} M(k)\xi^k \quad \textbf{(7.42)}$$

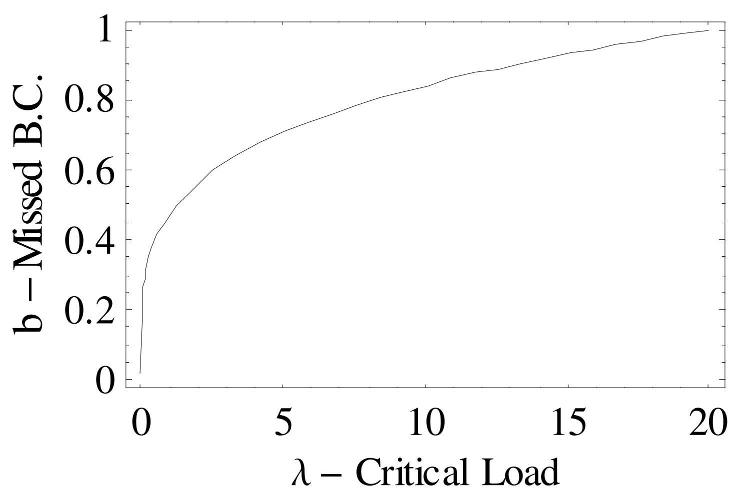


Figure 7.2 Effect of critical load on the solution for column subjected to uniformly distributed follower type of loading

Therefore, moment distribution along the length of the column **m(ξ)** which is governed by Eq. (7.25) can be obtained by using (7.42) with n=50 and $\beta = 1.70872$.

$$m(\xi) = b(\xi + 0.85456\xi^2 - 0.0811\xi^3 + 0.0270\xi^4 - 0.0126\xi^5 + 0.0070\xi^6 - \cdots) \quad \textbf{(7.43)}$$

where ***b*** for different values of critical load is shown in Fig. (7.2).

7.1.4. Results for uniformly distributed follower type of loading

The results for the column which is subjected to uniformly distributed follower type of loading are presented for the case of $\lambda = 15$.

7.1.4.1. Optimal distribution of cross-sectional area for $\lambda = 15$

Optimal shape of the column is determined by calculating the missed boundary condition **b** for the given critical load from Fig. (7.2) to obtain moment distribution given in Eq. (7.43). By using obtained m(ξ), optimal distribution of cross-sectional area is determined for uniformly distributed follower type of loading from Eq.(7.24) as it is shown in Fig. (7.3). Fig. (7.3) also presents the distribution of dimensionless radius of the column which has circular cross-section ($\bar{r}(\xi) = r(\xi)/L = \sqrt{a(\xi)/\pi}$).

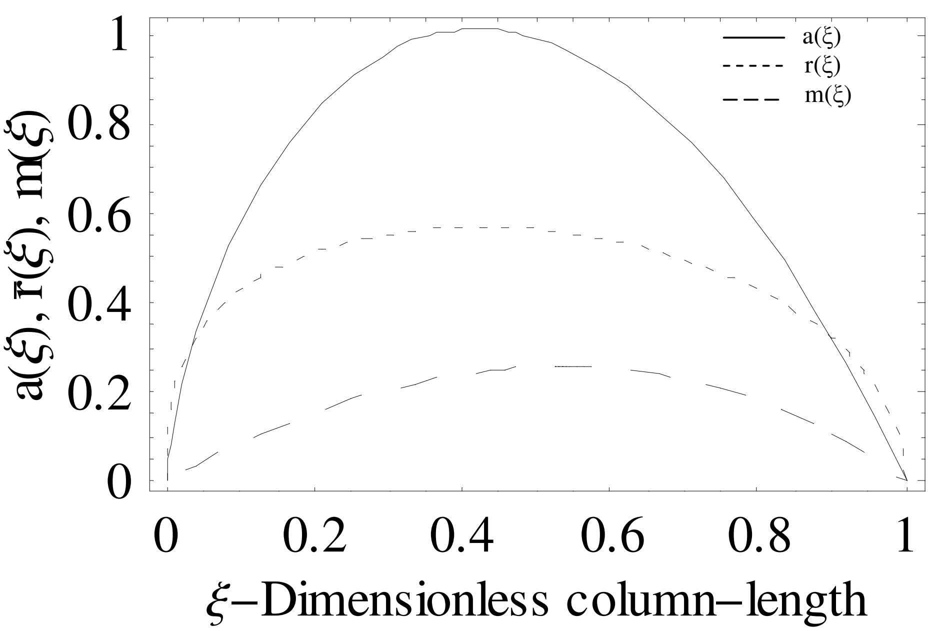


Figure 7.3 Optimal shape of column subjected to uniformly distributed follower type of loading

7.1.4.2. Volume of optimal column for $\lambda = 15$

Dimensionless volume of the optimal column is calculated from Eq.(5.8) as $w_{optimal} = 0.7083$.

7.1.5. Comparison of results with uniform column

For a comparison, volume of the column, which has constant cross-sectional area and is stable under the case of $\lambda = 15$, is evaluated. To determine the efficiency of the optimal column it can be useful to analyze the behavior of the column with constant cross-section. For a column with constant cross-section, differential transforms of Eqs. (7.16) and (7.17) are

$$(k+1)(k+2)M(k+2) + \frac{\lambda}{a^2}\sum_{k_1=0}^{k}(\delta(k_1) - \delta(k_1 - 1))M(k - k_1) = 0 \qquad \textbf{(7.44)}$$

$$M(0) = 0 \qquad \sum_{k=0}^{n} M(k) = 0 \qquad \textbf{(7.45)}$$

After calculation of few terms of the series and application of boundary conditions, the general form as in Eq.(7.38) is obtained with $f^{(n)}(\beta)$ of

$$f^{(n)}(\beta) = 1 - \frac{\beta}{12} + \frac{\beta^2}{504} - \frac{\beta^3}{45360} + \frac{\beta^4}{798336} - \cdots = 0 \qquad \textbf{(7.46)}$$

where $\beta = \lambda / a^2$.

Solving Eq.(7.46) for β gives the result of $\beta = 18.956266$ and therefore $w_{constant} = 0.8895$ for $\lambda = 15$. This means volume saving of optimal column with respect to uniform column leads to ***20.4%*** for the given load.

We also compared our result with the literature. We obtained the result of $w_{optimal} = 0.7961$ by considering the moment distribution corresponding to the case of $\lambda = 18.956266$. The same result was obtained by Atanackovic and Simic ***(Atanackovic, 1999)*** as $w_{optimal} = 0.8105$ for $\lambda = 18.956266$.

7.2. Exponentially Varying Follower Type of Loading

7.2.1. Governing equation for the problem

For the optimal shape analysis of the column subjected to exponentially increasing follower type of loading, it is considered that loading has the form of $f(x) = f_0 e^{x/L}$.

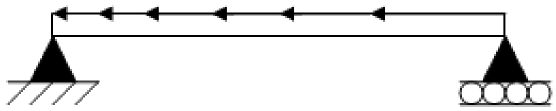


Figure 7.4 Simply supported column subjected to exponentially varying follower type of loading

By considering the differential element shown in Fig. (6.2), moment equilibrium can be written with the acceptance that counterclockwise direction is positive direction (See Section 2.2.1).

$$M(x,t) + \frac{\partial M(x,t)}{\partial x} dx - M(x,t) + [V(x,t) + \frac{\partial V(x,t)}{\partial x} dx] dx - H(x,t) \frac{\partial U(x,t)}{\partial x} dx = 0 \quad \textbf{(7.47)}$$

By eliminating the terms including the multiplication of $dx \cdot dx$, Eq. (7.47) goes

$$\frac{\partial M(x,t)}{\partial x} = H(x,t) \frac{\partial U(x,t)}{\partial x} - V(x,t) \quad \textbf{(7.48)}$$

Equilibrium along y-direction, by accepting y-direction is positive, can be written as

$$V(x,t) + \frac{\partial V(x,t)}{\partial x} dx - V(x,t) = \rho A(x) dx \frac{\partial^2 W(x,t)}{\partial t^2} \quad \textbf{(7.49)}$$

$$\frac{\partial V(x,t)}{\partial x} = \rho A(x) \frac{\partial^2 W(x,t)}{\partial t^2} \quad \textbf{(7.50)}$$

Equilibrium along x-direction, by accepting x-direction is positive, is

$$H(x,t) + \frac{\partial H(x,t)}{\partial x} dx - H(x,t) - f_o e^{x/L} dx = 0 \quad \textbf{(7.51)}$$

$$\frac{\partial H(x,t)}{\partial x} = f_o e^{x/L} \quad \textbf{(7.52)}$$

For the steady-state solution where $U(x,t)=u(x)$, Eqs. (7.48), (7.50) and (7.52) leads to

$$\frac{dM(x)}{dx}=H(x)\frac{du(x)}{dx}-V(x) \tag{7.53}$$

$$V(x)=0 \tag{7.54}$$

$$H(x)=f_o L(e-e^{x/L}) \tag{7.55}$$

respectively, by using that H(L)=0 at the movable end. Substituting Eqs. (7.54) and (7.55) into Eq. (7.53) and taking the first derivative of released equation gives

$$\frac{d^2M(x)}{dx^2}=-f_0 L(e-e^{x/L})\frac{d^2u(x)}{dx^2} \tag{7.56}$$

By using constitutive relation given in Eq. (6.8), governing equation can be written by means of moment distribution as follows:

$$M''+\frac{f_0 L}{EI(x)}(e-e^{x/L})M=0 \tag{7.57}$$

where $(\cdot)'=d(\cdot)/dx$. By using Eq.(5.2) which gives the second moment of inertia distribution of the column with variable cross-section, Eq. (7.57) can be written in the following form

$$M''+\frac{f_0 L}{E\alpha A^2(x)}(e-e^{x/L})M=0 \tag{7.58}$$

$$M(0)=0 \qquad M(L)=0 \tag{7.59}$$

By using Eq.(7.15), the governing system can be obtained in terms of dimensionless quantities as follows:

$$\ddot{m}+\frac{\lambda}{a^2(\xi)}(e-e^{\xi})m=0 \tag{7.60}$$

$$m(0)=m(1)=0 \tag{7.61}$$

where $\dot{(\cdot)}=d(\cdot)/d\xi$.

7.2.2. Optimization problem

By following the same procedure described in Section (5.2), state and costate variables of the optimization problem can be defined as follows (See Section 3.3.2):

$$\dot{q}_1 = q_2 \qquad \dot{q}_2 = -\frac{\lambda}{a^2(\xi)}(e - e^{\xi})q_1 \tag{7.62}$$

$$q_1(0) = q_1(1) = 0 \tag{7.63}$$

where $q_1(\xi)$ denotes $m(\xi)$ in the original differential equation.

Hamiltonian function can be constructed by using Eq.(5.8) which gives the volume of the rod that is aimed to be minimized and corresponding variables defined above.

$$H = a(\xi) + p_1 q_2 - p_2 \frac{\lambda}{a^2(\xi)}(e - e^{\xi})q_1 \tag{7.64}$$

By using Eq.(5.11),

$$\dot{p}_1 = \frac{\lambda}{a^2}(e - e^{\xi})p_2 \qquad \dot{p}_2 = -p_1 \tag{7.65}$$

By using Eq (6.19), the relation between state and costate variables can be obtained as

$$p_1(\xi) = q_2(\xi) \qquad p_2(\xi) = -q_1(\xi) \tag{7.66}$$

Optimal cross-sectional area function is such $a(\xi)$ which minimizes Hamiltonian function. Therefore, Eq.(7.64) is differentiated with respect to $a(\xi)$ to determine the optimal shape where H is minimum.

$$\frac{\partial H}{\partial a} = 1 + p_2 \frac{2\lambda}{a^3}(e - e^{\xi})q_1 = 0 \tag{7.67}$$

By solving Eqs. (7.67) with the substitution of Eq.(7.66), $a(\xi)$ is obtained in terms of $q_1(\xi)$. By replacing $q_1(\xi)$ to $m(\xi)$, optimal cross-sectional area function is found in the form of

$$a(\xi) = (2\lambda(e - e^{\xi})m^2(\xi))^{1/3} \tag{7.68}$$

Substituting Eq.(7.24) into Eqs.(7.16) gives the nonlinear differential equation system as

$$\ddot{m}+(\frac{\lambda}{4})^{1/3}(e-e^{\xi})^{1/3}m^{-1/3}=0 \qquad \textbf{(7.69)}$$

$$m(0)=m(1)=0 \qquad \textbf{(7.70)}$$

7.2.3. Solution of governing system

In the solution step, Differential Transform Method (**DTM**) is applied to the system. To simplify the application of differential transform method, governing equation can be written in the form of

$$(\ddot{m})^3 m=\bar{\lambda}^3(e-e^{\xi}) \qquad \textbf{(7.71)}$$

where $\bar{\lambda}=(\lambda/4)^{1/3}$.

Differential transform of Eq. (7.69) around ξ_0=0 is

$$\sum_{k_3=0}^{k}\sum_{k_2=0}^{k_3}\sum_{k_1=0}^{k_2}M(k_1)(k_2+1-k_1)(k_2+2-k_1)M(k_2+2-k_1)(k_3+1-k_2)(k_3+2-k_2)M(k_3+2-k_2)$$
$$\times(k+1-k_3)(k+2-k_3)M(k+2-k_3)=-\bar{\lambda}^3(1/k!-e\delta(k)) \qquad \textbf{(7.72)}$$

Differential transform of boundary conditions (7.70) are respectively leads to

$$M(0)=0 \qquad \sum_{k=0}^{n}M(k)=0 \qquad \textbf{(7.73)}$$

Insert $M(1)=b$, where b is a constant which will be calculated at the end of solution with the use of Eq.(7.73).

At k=1 with substituting M(0) and M(1) into Eq.(7.72)

$$M(2)=\frac{1}{2b^{1/3}}\bar{\lambda} \qquad \textbf{(7.74)}$$

At k=2 with additionally substituting (7.30) into (7.28)

$$M(3)=-\frac{\bar{\lambda}(b^{4/3}+\bar{\lambda})}{36b^{5/3}} \qquad \textbf{(7.75)}$$

By following the same procedure, the other terms of the series can be evaluated as follows:

$$M(4)=\frac{\overline{\lambda}(-9b^{8/3}+4b^{4/3}\overline{\lambda}+7\overline{\lambda}^2)}{1296b^3} \qquad \textbf{(7.76)}$$

$$M(5)=-\frac{\overline{\lambda}(156b^4-57b^{8/3}\overline{\lambda}+70b^{4/3}\overline{\lambda}^2+115\overline{\lambda}^3)}{77760b^{13/3}} \qquad \textbf{(7.77)}$$

$$M(6)=-\frac{\overline{\lambda}(4848b^{16/3}-1446b^4\overline{\lambda}+1357b^{8/3}\overline{\lambda}^2-2300b^{4/3}\overline{\lambda}^3-3375\overline{\lambda}^4)}{6998400b^{17/3}} \qquad \textbf{(7.78)}$$

$\vdots$

$\vdots$

Following the same procedure, n terms of the series are calculated. Terms of the corresponding series can be represented a new parameter which can be described as follows:

$$\beta=\frac{\overline{\lambda}}{b^{4/3}} \qquad \textbf{(7.79)}$$

All of the terms evaluated are as follows:

$$M(0)=0$$

$$M(1)=b$$

$$M(2)=\frac{\beta}{2}b$$

$$M(3)=-(\frac{\beta}{36}+\frac{\beta^2}{36})b$$

$$M(4)=(-\frac{9}{1296}\beta+\frac{4}{1296}\beta^2+\frac{7}{1296}\beta^3)b$$

$$M(5)=-(\frac{156}{77760}\beta-\frac{57}{77760}\beta^2+\frac{70}{77760}\beta^3+\frac{115}{77760}\beta^4)b$$

$$M(6)=-(\frac{4848}{6998400}\beta-\frac{1446}{6998400}\beta^2+\frac{1357}{6998400}\beta^3-\frac{2300}{6998400}\beta^4-\frac{3375}{6998400}\beta^5)b$$

$\vdots$

$\vdots$ **(7.80)**

By substituting the terms ***M(0)*** to ***M(n)*** into second boundary condition given in Eq. (7.73), the general form as in Eq. (7.38) is obtained with $f^{(n)}(\beta)$ which is in the form of

$$f^{(n)}(\beta)=1+\frac{\beta}{2}-\frac{\beta^2}{36}+\frac{7\beta^3}{1296}-\frac{23\beta^4}{15552}+\frac{5\beta^5}{10368}-\cdots=0 \quad \textbf{(7.81)}$$

By taking n=20 terms, $\beta_i^{(20)}=1.61374$ and for error estimation $\beta_i^{(19)}=1.61404$. Substituting these values into (7.40) gives $\left|\beta_i^{(20)}-\beta_i^{(19)}\right|=0.0003$. This error value is in the acceptable region.

Moment distribution along the length of the column **m(ξ)** which is governed by Eq. (7.60) can be calculated by using (7.42) with n=20 and $\beta=1.61374$.

$$m(\xi)=b(\xi-0.8070\xi^2-0.1172\xi^3-0.0345\xi^4-0.01498\xi^5-0.0081\xi^6-\cdots) \quad \textbf{(7.82)}$$

where *b* for different values of critical load is shown in Fig. (7.3).

7.2.4. Results for exponentially varying follower type of loading

The results for the column which is subjected to exponentially varying follower type of loading are presented for the case of $\lambda=15$.

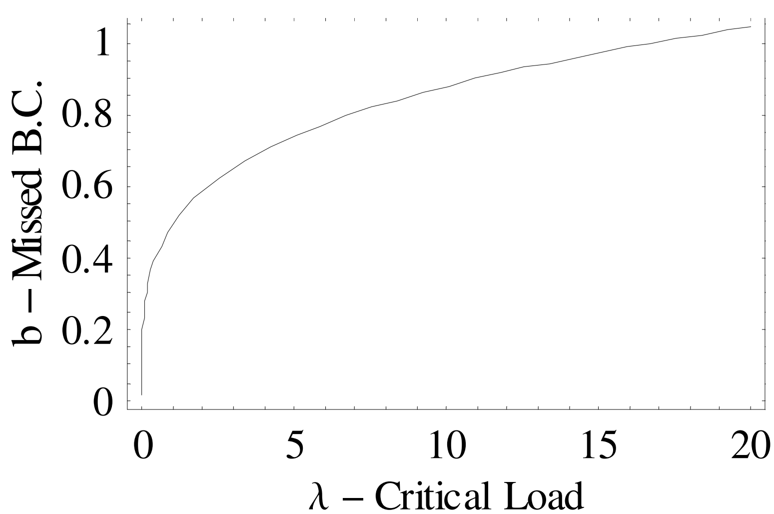


Figure 7.5 Effect of critical load on the solution for column subjected to exponentially varying follower type of loading

7.2.4.1. Optimal distribution of column for $\lambda = 15$

Optimal shape of the column is determined by calculating the missed boundary condition **b** for the given critical load from Fig. (7.5) to obtain moment distribution given in Eq. (7.82). By using obtained *m(ξ),* optimal distribution of cross-sectional area is determined for exponentially varying follower type of loading from Eq.(7.68) as it is shown in Fig. (7.6). Fig. (7.6) also presents the distribution of dimensionless radius of the column which has circular cross-section ($\bar{r}(\xi) = r(\xi)/L = \sqrt{a(\xi)/\pi}$).

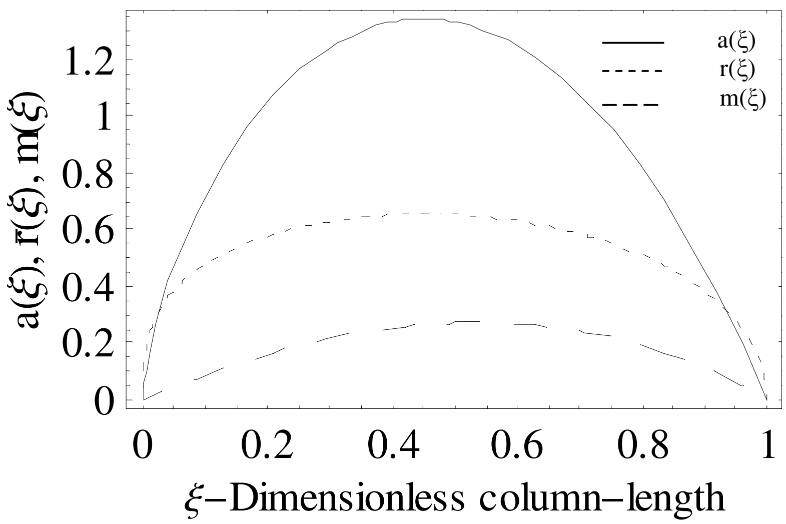


Figure 7.6 Optimal shape of column subjected to exponentially varying follower type of loading

7.2.4.2. Volume of the column for $\lambda = 15$

Dimensionless volume of the optimal column is calculated from Eq.(5.8) as $w_{optimal} = 0.9476$.

7.2.5. Comparison of results with uniform column

For a comparison, volume of the column, which has constant cross-sectional area and is stable under the case of $\lambda = 15$, is evaluated. To determine the efficiency of the optimal column it can be useful to analyze the behavior of the column with constant cross-section.

For a uniform column, differential transforms of Eqs. (7.60) and (7.61) can be written as

$$(k+1)(k+2)M(k+2)+\frac{\lambda}{a^2}\sum_{k_1=0}^{k}(\frac{1}{k_1!}-e^1\delta(k_1))M(k-k_1)=0 \qquad \textbf{(7.83)}$$

$$M(0)=0 \qquad \sum_{k=0}^{n}M(k)=0 \qquad \textbf{(7.84)}$$

Carrying the calculation of few terms and applying the boundary conditions, the general form as in Eq.(7.38) is obtained with $f^{(n)}(\beta)$ of

$$f^{(n)}(\beta)=1-0.171329\beta+0.008390\beta^2-0.000192\beta^3-\cdots=0 \qquad \textbf{(7.85)}$$

where $\beta=\lambda/a^2$. By solving Eq.(7.85) for β, then the result becomes $\beta=9.222766$. Volume of the column, which has constant cross-sectional area and is stable under the case of $\lambda=15$, is $w_{cons\tan t}=1.2753$.

This means volume saving of optimal column with respect to uniform column leads to ***25.7%*** for the given load.

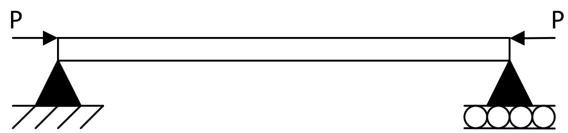
8. DISCUSSIONS AND CONCLUSION

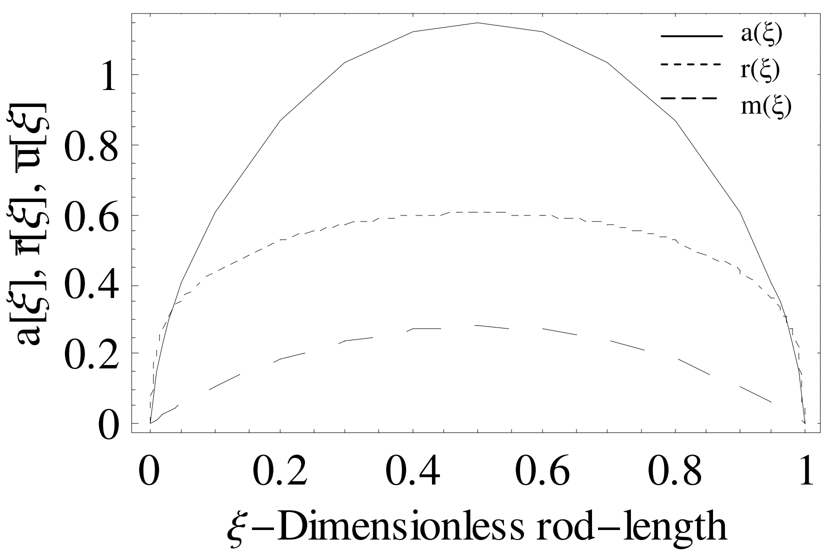
In this study optimal shape analysis of elastic bodies is carried out for different loading conditions. Rods are the structural elements which generally carries axial loading, bending effects of the applied load is neglected in the derivation of the equation of motion. Columns are the structural elements which are subjected to compressive loading and for this type of elements bending effect is not negligible since the ratio between the column length and column thickness is bigger than ***10*** for columns.

Loading conditions examined in this study are axial compressive force, eccentrically placed compressive force –eccentricity at both ends and eccentricity at one end- and follower type of loading –uniformly distributed and exponentially varying-. For each configuration, optimal distribution of cross-sectional area and volume of the structure with such cross-sectional area are determined. In addition, volume of uniform structure which is also subjected to same amount of loading is calculated and compared to the volume of the structure with optimal shape. This comparison gives the degree of success of the optimal shape analysis. Percent of volume saving (s_{volume}) is computed at the end of the study to determine the efficiency of the optimal structure.

In Sections 6&7, the effect of loading condition on the optimal shape of an Euler-Bernoulli column is analyzed by using Pontryagin's maximum principle and applying Differential Transform Method. Two types of loading, follower type loading and eccentric loading are considered in this section. It is seen that for the same critical load value, volume of the column which is subjected to uniformly distributed follower type of loading is smaller than that of exponentially increasing follower type of loading. Additionally, effect of eccentricity on the volume is analyzed and it is also observed that the volume of the column which is stable under the given load is increasing with increasing eccentricity.

<u>CONFIGURATION 1</u>: CENTRALLY COMPRESSED ROD





Volume for given critical load $\lambda = \pi^2$;

Optimal Rod: $w_{optimal} = 0.866025$

Uniform Rod: $w_{constant} = 1$

Volume Saving: s_{volume}=13.4%

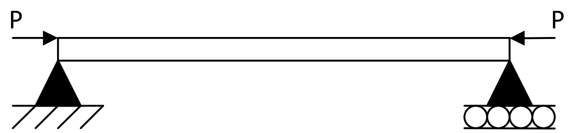
Critical load for given volume $w = 1$;

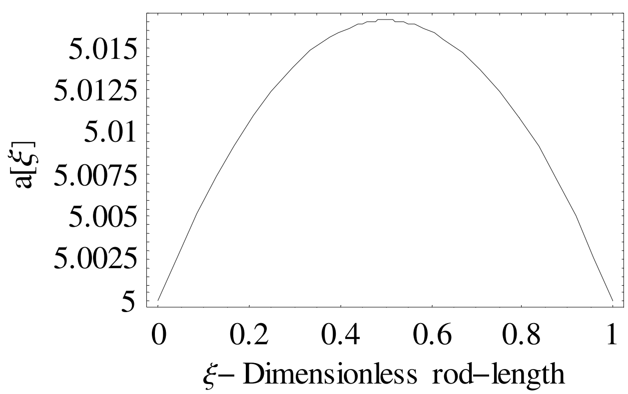
Optimal Rod: $\lambda_{optimal} = 13.1595$

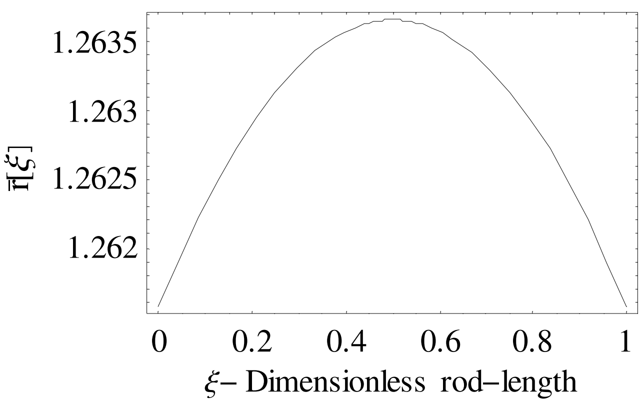
Uniform Rod: $\lambda_{constant} = 9.8696$

Critical Load Saving: s_{volume}=33.3%

<u>CONFIGURATION 2</u>: CENTRALLY COMPRESSED ROD WITH END POINT CONSTRAINT







Material limits: $\bar{\sigma}_{\lim it} = 0.2$

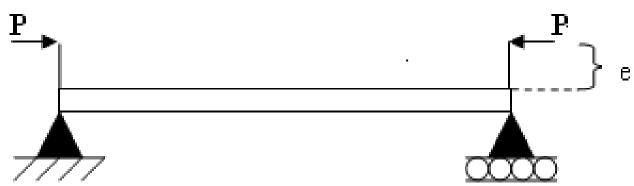
End-point cross-sectional area for $\lambda = 1$:

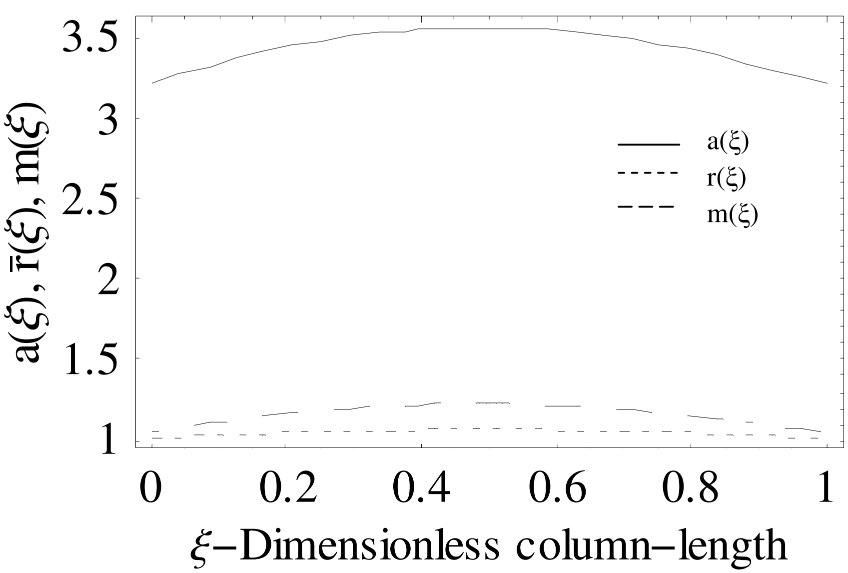
Allowable minimum cross-sectional area: $a_o = 5$

Volume for $\lambda = 1$:

Optimal Rod: $w_{optimal} = 5.0119$

<u>CONFIGURATION 3</u>: ECCENTRICALLY COMPRESSED COLUMN AT BOTH ENDS



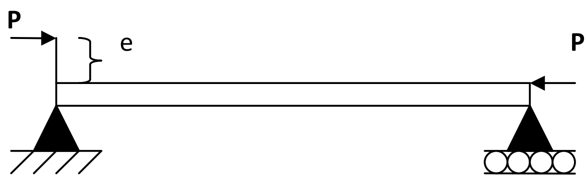


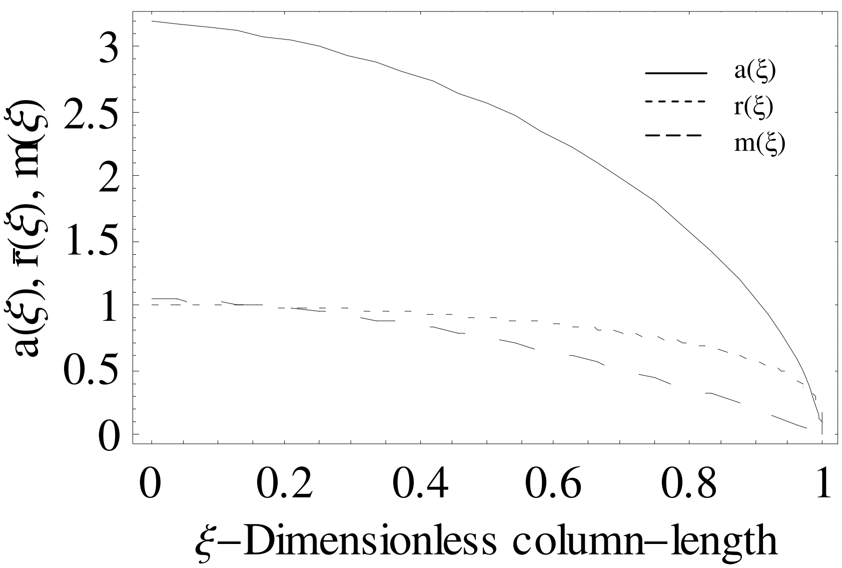
Volume for given critical load $\lambda = 15$ and $\bar{e} = 0.07$;

Optimal Column: $w_{optimal} = 3.4398$

Uniform Column: $w_{constant} = 13.5618$

CONFIGURATION 4: ECCENTRICALLY COMPRESSED COLUMN AT ONE END



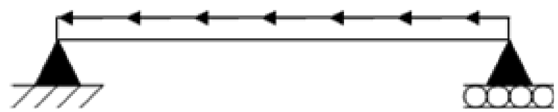


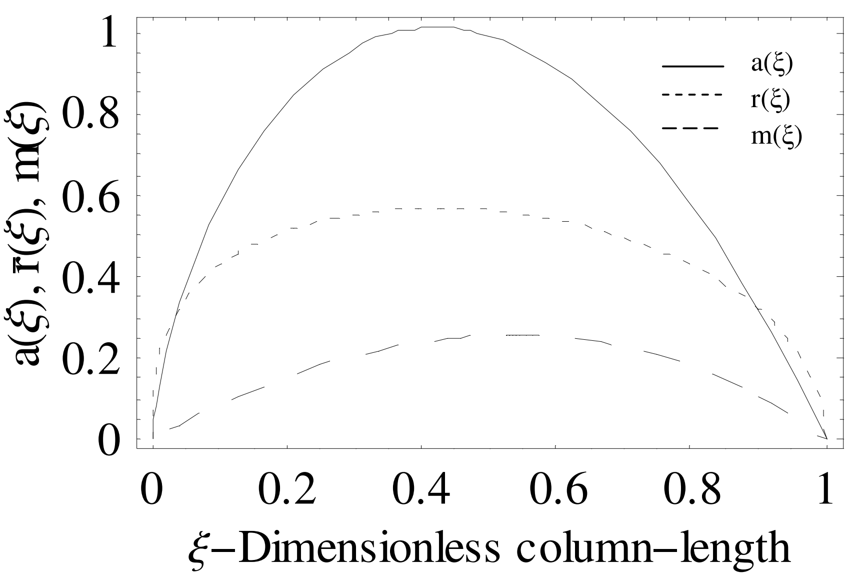
Volume for given critical load $\lambda = 15$ and $\overline{e} = 0.07$;

Optimal Column: $w_{optimal} = 2.3172$

Uniform Column: $w_{constant} = 13.5462$

<u>CONFIGURATION 5</u>: COLUMN SUBJECTED TO UNIFORM FOLLOWER TYPE OF LOADING





Volume for given critical load $\lambda = 15$;

Optimal Column: $w_{optimal} = 0.7083$

Uniform Column: $w_{constant} = 0.8895$

Volume Saving: s_{volume}=20.4%

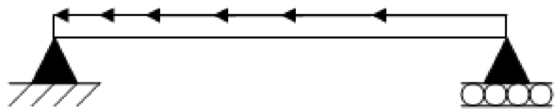
Volume for given critical load $\lambda = 18.956266$;

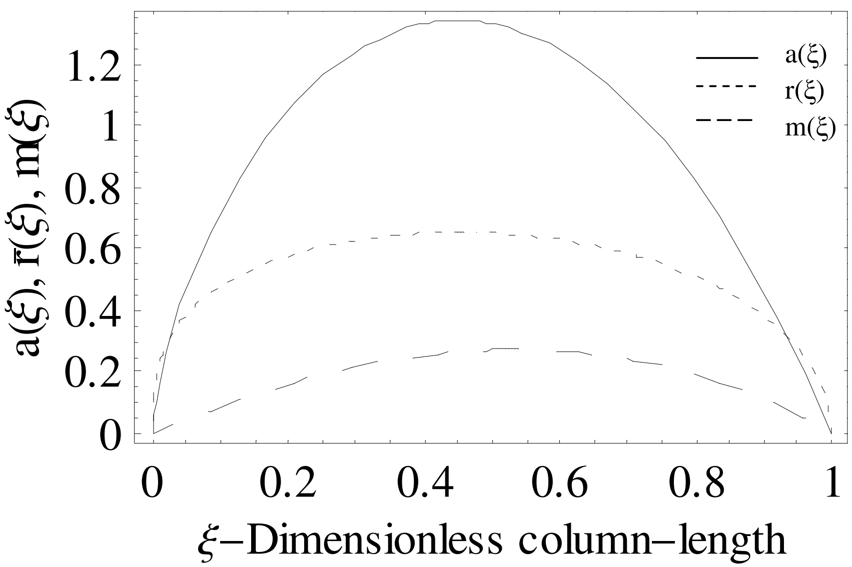
Optimal Column: $w_{optimal} = 0.7961$

Uniform Column: $w_{constant} = 1$

Volume Saving: s_{volume}=20.4%

CONFIGURATION 6: COLUMN SUBJECTED TO EXPONENTIALLY VARYING FOLLOWER TYPE OF LOADING





Volume for given critical load $\lambda = 15$;

Optimal Column: $w_{optimal} = 0.9476$

Uniform Column: $w_{constant} = 1.2753$

Volume Saving: s_{volume}=25.7%

REFERENCES

Abdel, I. H., Hassan, H., 2002: Different applications for the differential transformation in .the differential equations, Applied Mathematics and Computation 2002; 129; 183-201

Arıkoğlu, A., and Özkol, İ., 2008: Solutions of integral and integro-differential equation systems by using differential transform method, Computers and Mathematics with Appl. 2008; 56; 2411-17.

Arıkoğlu, A., and Özkol, İ., 2007: Solution of fractional differential equations by using differential transform method, Chaos Soliton. Fract. 2007; 34; 1473-81.

Atanackovic, T. M., and Simic, S. S., 1999: On the optimal shape of a Pflüger column, European Journal of Mechanics A-Solids 1999; **18**; 903-13.

Blasius, H., 1914. Träger kleinster Durchbiegung und Stäbe großter Knickfestigkeit bei gegebenem Materialverbrauch, Z. Math. Physik 1914; 62; 182-97.

Case, J., Chilver, A. H., Ross, C. T. F., 1993: Strength of Materials and Structures with an introduction to finite element methods.

Clausen, T., 1851: Über die Form architektonischer Säulen Bull. Cl. Phys. Math. Acad. Imp. Sci. St.Petersbourg; 9; 369-80.

Cook, R. D., and Young, W. C., 1985: Advanced Mechanics of Materials, Macmillan Publishing Company, New York.

Cowper, G. R., 1966: The shear coefficient in Timoshenko's beam theory, *Journal of Applied Mechanics*, Vol. **33**, pp 335-340.

Cox, S. J., 1992: The shape of the ideal column, *Mathematical Intelligencer* 1992; 14; 16-24.

Gill, P. E., Murray, W. M., and Saunders, M. A., 2007 User's Manual for SNOPT Version 7: Software for Large-Scale Nonlinear Programming, University of California, San Diego Report.

Glavardanov, B. V., Atanackovic T. M., 2001: Optimal shape of a twisted and compressed rod, European. *Journal of Mechanics A-Solids* 2001; 20; 795-809.

Han, S. H., Benaroya, H., and Wei, T., 1999: Dynamics of Transversely Vibrating Beams by Using Four Engineering Theories. *Journal of Sound and Vibration.* Vol. **225**, no. 5, pp. 935-988.

Keller, J., 1960: The shape of the strongest column, *Arch. Rational Mech.Anal.*; Vol. **5**; pp. 275-85

Popov, E. P., 1990: Engineering Mechanics of Solids, Prentice Hall, New Jersey

Rao, S. S., 2000: Mechanical Vibrations, Addison-Wesley, California.

Ratzerdorfer , J., 1936: Die Knickfestigkeit von Stäben und Stabwerken, Springer, Wien, 1936

Vujanovic, B. D., Atanackovic, T.M., 2004: An Introduction to Modern Variational Techniques in Mechanics and Engineering, Birkhäuser, Boston, 2004

Zhou, J. K., 1986: Differential Transformation and Its Application for Electrical Circuits, Huazhong University Press, Wuhan.

CURRICULUM VITAE

Candidate’s full name: **Ülker ERDÖNMEZ**

Place and date of birth: **İskenderun 15.05.1986**

Permanent Address: **Sancaktar Hayrettin Mah. İzci Türk Sok. No:55**

Birlik Palas Apt. D.7 Fatih/İSTANBUL

ulkererdonmez@yahoo.com

Universities and Colleges attended:

2004 –2007 **İstanbul Technical University(İstanbul)**
Aeronautical Engineering Bachelor Degree

1997 –2004 **İbni Sina Anatolian High School (İskenderun)**
High School

1992 –1997 **Mithat Paşa Primary School (İskenderun)**
Primary School

Publications:

Erdönmez Ü., Özkol İ., An Application to Peridynamic Approach as Reformulation of Elasticity Theory, 4. Ankara International Aerospace Conference, 2007, METU, Ankara

Erdönmez Ü., Özkol İ., Optimal Shape Analysis of a Column Structure under Various Loading Conditions by Using Differential Transform Method, Applied Mathematics and Computation (with editor)

Erdönmez Ü., Özkol İ., Optimal Shape Analysis of an Eccentrically Compressed Column by using Differential Transform Method, 5. Ankara International Aerospace Conference, 2009, METU, Ankara