ISTANBUL TECHNICAL UNIVERSITY ★ GRADUATE SCHOOL OF SCIENCE ENGINEERING AND TECHNOLOGY

COMMUTATIVE RINGS WHOSE PRIME IDEALS ARE RADICALLY PERFECT

Ph.D. THESIS

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Department of Mathematical Engineering

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İSTANBUL TEKNİK ÜNİVERSİTESİ ★ FEN BİLİMLERİ ENSTİTÜSÜ

ASAL İDELLERİ RADİKAL OLARAK MÜKEMMEL OLAN DEĞİŞMELİ HALKALAR

DOKTORA TEZİ

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Sevgi Harman, a **Ph.D.** student of ITU **Institute of Science and Technology** student ID 509042003, successfully defended the **thesis** entitled "**COMMUTATIVE RINGS WHOSE PRIME IDEALS ARE RADICALLY PERFECT**", which she prepared after fulfilling the requirements specified in the associated legislations, before the jury whose signatures are below.

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To my family and in particular to my dear son Yusuf Kerem HARMAN,

FOREWORD

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This thesis is dedicated to my dear son Yusuf Kerem Harman and to my esteemed mother Fatma Erdoğan who taught me get up after a fall and start again.

May 2012 Sevgi HARMAN
(Research Assistant)

TABLE OF CONTENTS

Page

ABBREVIATIONS

R[X]	**:** The polynomial ring over a ring R
dim(R)	**:** Krull dimension of a ring R
ht(I)	**:** The height of an ideal I
rad(I)	**:** The radical of an ideal I
j-rad(I)	**:** The intersection of all maximal ideals of a ring R that contains I
$\mathbf{A_f}$	**:** The content of a polynomial f
$\mathbf{P^{(n)}}$	**:** The symbolic power of a prime ideal P
Int(R)	**:** The ring of integer valued polynomials over a ring R
SpecR	**:** The set of all prime ideals in a ring R
MaxSpecR	**:** The set of all maximal ideals in a ring R
$\mathbf{K_0(R)}$	**:** Projective class group
PID	**:** Principal ideal domain
$\mathbf{H_I^i(M)}$	**:** i-th local cohomology of M with support in an ideal I

COMMUTATIVE RINGS WHOSE PRIME IDEALS ARE RADICALLY PERFECT

SUMMARY

The question of relating the number of generators of an ideal to the height of the ideal is one of the interesting research topics in commutative algebra which was first considered by Kronecker in late 19^{th} century. Since then an enormous amount of research has evolved around these types of questions. Among them still remains open is whether each height two ideal in $K[X,Y,Z]$ is a set theoretic complete intersection where K is a field. In other words, is every curve in 3-space the intersection of 2-hypersurfaces? When K is of positive characteristic this question has an affirmative answer, but not much is known when K is of characteristic zero.

In search of an answer to the characteristic zero case, Erdoğdu considered the following more general question: Under which circumstances on an integral domain R that contains a field of characteristic zero, can one conclude that all prime ideals of the polynomial ring $R[X]$ over R are radically perfect which is defined as follows: Call an ideal I of a ring R radically perfect if among the ideals of R whose radical is equal to the radical of I, the one with the least number of generators has this number of generators equal to the height of I. This is a generalization of the notion of set theoretic complete intersection of ideals in Noetherian rings to rings that need not be Noetherian.

The main objective of this thesis is to relate the height and the number of generators of ideals in rings that are not necessarily Noetherian. More precisely, the conditions on a ring R so that the prime ideals of R and also those of the polynomial ring $R[X]$ over R are radically perfect are determined. In many cases, it is shown that the condition of prime ideals of R or that of $R[X]$ being radically perfect is equivalent to a form of the class group of R being torsion. For instance, it is proved that over a Noetherian Hilbert domain R of finite character that contains a field of characteristic zero, each prime ideal of $R[X]$ is radically perfect if and only if R is a Dedekind domain with torsion ideal class group. It is also proved that in a Hilbert domain R in which each nonzero ideal is contained in finitely many maximal ideals, then each invertible maximal ideal is radically perfect if and only if $CopRad(R)$ is torsion, where $CopRad(R)$ denote the multiplicative semigroup generated by the set of pairwise coprime radical ideals of R that are invertible. Moreover, it is shown that in a two dimensional Krull domain R of finite character, each prime ideal of R is radically perfect if and only if the divisor class group of R is torsion. It is also shown that over a Prüfer domain R with coprimely

packed set of maximal ideals which has the property that each maximal ideal of it is finitely generated, then for each maximal ideal M of R the prime ideal $MR[X]$ of $R[X]$ is radically perfect if and only if R is a Dedekind domain with torsion ideal class group.

Finally, the relation between the notions of radically perfectness and coprime packedness of prime ideals of various rings is examined. Although these two notions coincide if R is an integral domain of dimension one, in general neither of these implies the other, and here, the conditions on a commutative ring R over which either of these notions implies the other are determined.

Also some basic facts on the analog generalization of complete intersection are outlined. An ideal I of a commutative ring R is perfect if the minimal number of generators of I is equal to the height of I, which turn out to be void in the case of R being a commutative ring with identity since such rings are necessarily Noetherian and so this notion coincides with the notion of complete intersection. In particular, it is shown that in an integral domain every height one prime ideal is perfect if and only if it is a factorial ring. It is also shown that, if R is a finite dimensional Prüfer domain then each prime ideal is perfect if and only if it is a principal ideal domain. It is also mentioned that if R is a finite dimensional Prüfer domain then each prime ideal of $R[X]$ which contracts to zero in R is perfect.

Throughout, R will always denote a commutative ring with identity, the dimension of a ring R will always mean its Krull dimension.

ASAL İDEALLERİ RADİKAL OLARAK MÜKEMMEL OLAN DEĞİŞMELİ HALKALAR

ÖZET

R bir Noether halkası ve I da R'nin kendisinden farklı, yüksekliği n olan bir asal ideali olsun. Eğer $rad(I) = rad(a_1, ..., a_n)$ olacak şekilde $a_1, ..., a_n \in I$ elemanları bulunabiliyorsa, I idealine kümesel tam arakesit denir. Bir idealin üreteç sayısı ile yüksekliği arasındaki ilişki değişmeli cebirin temel araştırma konularından biri olup, ilk defa 19. yüzyılda Kronecker tarafından ele alınmıştır. Kronecker, n-boyutlu bir Noether halkasında, her radikal idealin $n + 1$ eleman tarafından üretilen bir idealin radikali şeklinde yazılabileceğini göstermiştir. 1891 yılında, Vahlen Kronecker'in sonucunun mümkün olan en iyi sonuç olduğunu destekleyen bir örnek bulduğunu ileri sürmüş ve bu örnek vesilesi ile kompleks projektif üç boyutlu uzayda aldığı özel bir eğrinin üç hiperyüzeyin kesişimi şeklinde yazılamayacağını ifade etmiştir. Vahlen'in örneğinin doğru olmadığı yıllarca farkedilememiş olup, 1942 yılında Perron'un, kesişimleri Vahlen'in örneğindeki eğriyi veren üç hiperyüzey olduğunu göstermesi ile Vahlen'in bu savı çürütülmüştür. 1961 yılında Kneser, üç boyutlu uzaydaki her eğrinin üç hiperyüzeyin kesişimi şeklinde yazılabileceğini ispatlamak sureti ile, Perron'un bulduğu hiperyüzeylerin varlığının tesadüf olmadığını göstermiştir. 1973'te Eisenbud ve Evans, Kronecker'in ispatındaki üreteç sayısı olan $n + 1$'in n'ye düşürülebildiğini kanıtlamışlardır. Bu alanda kaydadeğer bir çok sonuç elde edilmiş olmasına rağmen, önemli problemlerden biri olan "karakteristiği sıfır olan bir K cismi üzerindeki $K[X, Y, Z]$ polinom halkasının, yüksekliği iki olan idealleri kümesel tam arakesit midir?", bir başka deyişle "üç boyutlu uzaydaki her eğri iki hiperyüzeyin kesişimi şeklinde yazılabilir mi?" sorusu halen çözülememiştir. K cisminin karakteristiğinin pozitif olduğu durumda, bu sorunun cevabının olumlu olduğu, 1978 yılında Cowsik ve Nori tarafından gösterilmiştir. Cowsik ve Nori, pozitif karakteristiğe sahip bir K cismi üzerindeki n değişkenli $K[X_1, ..., X_n]$ polinom halkasında, yüksekliği $n - 1$ olan ideallerin kümesel tam arakesit olduğunu göstermişlerdir.

1994 yılında Erdoğdu ve McAdam karakteristiğin pozitif olduğu durumun, karakteristiğin sıfır olduğu durumdan farklı davrandığını gösteren bir örnek vermişler ve bu vesile ile Erdoğdu daha sonraki çalışmalarında, bu örneğe paralel olarak, sanının karakteristiğin sıfır olduğu durumda doğru olmadığına dair olan kanaatini destekler mahiyette bazı önemli sonuçlar elde etmiştir. Bahsi geçen örnekte, K karakteristiği sıfır olan bir cisim, R de $K[[Y]]$'nin, Y terimlerinin katsayısı sıfır olan kuvvet serilerini içeren bir alt halkası olarak alınmıştır. Bu durumda R normal olmayan bir halkadır

ve $R[X]$ polinom halkasında, yüksekliği bir olan ve kümesel tam arakesit olmayan bir asal ideal vardır. Ancak, K'nın karakteristiği p ise, $R[X]$'in yüksekliği bir olan her asal ideali (aslında $R[X]$'in tüm asal idealleri), R normal olmaksızın kümesel tam arakesittir. Çünkü, bahsi geçen R halkasının boyutu birdir ve normalizasyonu $S = K[Y]$'dir ve dolayısı ile $S[X]$'in her asal ideali kümesel tam arakesittir. $R[X]$'in herhangi bir asal ideali P için, $rad(PS[X])$, $S[X]$'in içinde yüksekliği bir olan sonlu sayıda asal idealin kesişimi şeklinde yazılabilir. Ayrıca $S[X]$ bir çarpınım halkası olduğu için, $S[X]$ içinde yüksekliği bir olan asal idealler tek üreteçlidir ve sonlu sayıda tek üreteçli idealin kesişimi yine tek üreteçli olacağından, $rad(PS[X]) = rad(f)$ olacak şekilde bir f polinomu bulunabilir. K'nın karakteristiği p olduğu için, $S[X]$'in her elemanının p'ninci kuvveti $R[X]$'e dolayısı ile P'ye düşer ve sonuç olarak $R[X]$ içinde $P = rad(f^p)$ eşitliğinin sağlanması kaçınılmazdır. Bu gözlem ışığında, Erdoğdu, kümesel tam arakesit olma nosyonunun, R halkasının normal olmasıyla çok yakından ilintili olduğunu dile getirmiş ve aynı çalışmasında kümesel tam arakesit olma tanımını Noether olmayan halkalara da genişleterek, radikal olarak mükemmellik tanımını vermiştir. R halkasının bir I ideali için, radikali I idealinin radikaline eşit olan tüm idealler arasından, minimum üreteç sayısına sahip olanının üreteç sayısı, I'nın yüksekliğine eşit ise, I idealine radikal olarak mükemmel ideal denir. Bu bağlamda karakteristiği sıfır olan bir cisim içeren R tamlık bölgesi üzerindeki $R[X]$ polinom halkasında yüksekliği bir olan her asal idealin radikal olarak mükemmel olmasının R'nin normal olmasını gerektirdiği gösterilmiştir. Ayrıca, karakteristiği sıfır olan bir cisim içeren bir boyutlu bir R Noether tamlık bölgesi üzerindeki $R[X]$ polinom halkasının her asal idealinin radikal olarak mükemmel olması ile R'nin ideal sınıf grubu torsiyon olan bir Dedekind tamlık bölgesi olmasının eşdeğer olduğu kanıtlanmıştır. Bu sonuç daha önce 2004 yılında, R'nin normal olduğu kabulü altında ispatlanmış olup, normalliğin yerine R'nin karakteristiği sıfır olan bir cisim içermesi şartının koyulmasına sebep olan motivasyon ise yukarıda bahsi geçen örnektir. Bütün bu bilgiler ışığında, K karakteristiği sıfır olan bir cisim ve $R = K[X, Y]$ olarak alınırsa R, $K[X]$ temel ideal bölgesi üzerinde, tek değişkenli bir polinom halkası olarak düşünülebilir ve bu durumda Erdoğdu'nun elde ettiği yukarıda bahsi geçen sonuca göre, R'nin her asal ideali radikal olarak mükemmeldir. P^*, $K[X, Y, Z] = R[Z]$'de yüksekliği iki olan herhangi bir asal ideal ise, $P = P^* \cap R$ ideali R'de sıfırdan farklı bir asal ideal olur. Eğer P'nin yüksekliği iki ise, R'de $P = rad(f, g)$ olacak şekilde f, g polinomları bulunabilir ve dolayısı ile $K[X, Y, Z]$'de $P^* = PK[X, Y, Z] = rad(f, g)$ eşitliği sağlanır. Şayet P'nin yüksekliği bir ise; P asal ideali R'de indirgenemez bir h polinomu tarafından üretilir ve eğer $R/<h>$ ideal sınıf grubu torsiyon olmayan bir Dedekind tamlık bölgesi değil ise, o zaman yukarıdaki bilgiler ışığında $K[X, Y, Z]$'de yüksekliği iki olan, h'yi içeren ve radikal olarak mükemmel olmayan bir asal ideal bulunabilmesi söz konusu olabilir. Bu gözlem vesilesi ile " R (Noether olmak zorunda olmayan) karakteristiği sıfır olan bir cisim içeren bir tamlık bölgesi ve $R[X]$ de R üzerinde her asal ideali radikal olarak mükemmel olan bir polinom halkası ise R'nin boyutu bir midir?" sorusu gündeme gelmiştir. R'nin Bézout olduğu durumda bu sorunun cevabının pozitif olduğu kanıtlanmıştır. Sonlu boyutlu olan bir R Bezout tamlık bölgesi üzerindeki $R[X]$ polinom halkasının her asal idealinin radikal olarak mükemmel olmasının, R'nin boyutunun bir olmasına eşdeğer olduğu gösterilmiştir. Bunun yanı sıra, birtakım ekstra koşullar altında Prüfer tamlık bölgeleri için de sorunun cevabının olumlu olduğu ispatlanmıştır. R sonlu boyutlu ve maksimal idealler kümesi asal olarak kapalı olan bir Prüfer tamlık bölgesi ise, R' nin her maksimal ideali M için, $R[X]$'deki $MR[X]$ asal idealinin radikal olarak mükemmel olması için gerek ve yeter koşulun R' nin boyutunun bir olması ve her asal idealinin tek üreteçli bir idealin radikali şeklinde yazılması olduğu sonucu elde edilmiştir. Erdoğdu karakteristiğin sıfır olduğu duruma cevap ararken ayrıca daha genel olan "Karakteristiği sıfır olan bir cisim içeren R tamlık bölgesi üzerinde hangi şartlar altında, $R[X]$ polinom halkasının tüm asal idealleri radikal olarak mükemmeldir?" sorusunu sormuştur.

Bu çalışmanın ana amacı, Noether olma zorunluluğu olmayan halkalardaki ideallerin yükseklikleri ile üreteç sayıları arasındaki ilişkiyi ayrıntılı olarak incelemektir. Bu çalışmada, R halkasının ve R üzerindeki $R[X]$ polinom halkasının tüm asal ideallerinin radikal olarak mükemmel olması için R halkasının sahip olması gereken birtakım özellikler belirlenmeye çalışılmıştır. Birçok durumda, R ve $R[X]$'in asal ideallerinin radikal olarak mükemmel olması koşulunun, farklı formlardaki sınıf gruplarının torsiyon olması ile eşdeğer olduğu gösterilmiştir. Ayrıca sonlu karakterli halkalar üzerinde ayrıntılı çalışılmış olup bu bağlamda bazı önemli sonuçlar elde edilmiştir. Sonlu karakterli bir Hilbert tamlık bölgesinin boyutunun bir olduğu ve her maksimal idealinin iki üreteçli bir idealin radikali şeklinde yazılabildiği gösterilmiştir. Karakteristiği sıfır olan bir cisim içeren sonlu karakterli bir Noether Hilbert tamlık bölgesi R üzerindeki $R[X]$ polinom halkasının her asal idealinin radikal olarak mükemmel olması ile R'nin sınıf grubu torsiyon olan bir Dedekind tamlık bölgesi olmasının eşdeğer olduğu kanıtlanmıştır. Ayrıca, sonlu karakterli bir Hilbert tamlık bölgesinin tersinir maksimal ideallerinin radikal olarak mükemmel olması ile, ikili olarak aralarında asal radikal idealler kümesinin ürettiği çarpımsal yarıgrup olan $CopRad(R)$'ın torsiyon olmasının denk olduğu sonucu elde edilmiştir. Bunun yanında, boyutu iki olan sonlu karakterli bir Krull tamlık bölgesinin her asal idealinin radikal olarak mükemmel olmasını ile bölen sınıf grubunun torsiyon olmasının eşdeğer olduğu kanıtlanmıştır ve akabinde buna bağlı olarak iki boyutlu, sonlu karakterli, lokal olarak çarpınım halkası olan bir Krull tamlık bölgesi R'nin her asal idealin radikal olarak mükemmel olmasına eşdeğer bazı durumlar listelenmiştir. Bunlara ek olarak, sonlu karakterli bir R Prüfer tamlık bölgesinin her maksimal idealinin radikal olarak mükemmel olmasının R'nin boyutunun ikiden büyük olamayacağını gerektirdiği gösterilmiştir.

Daha sonra radikal olarak mükemmellik ve asal olarak kapalılık nosyonlarının arasındaki ilişki ayrıntılı olarak incelenmiştir. Genel olarak bu iki nosyon birbirini gerektirmez. Örneğin, R boyutu birden büyük bir valuasyon tamlık bölgesi olarak alındığında, R'nin idealler kümesinin asal olarak kapalı olduğu ama R'nin maksimal idealinin radikal olarak mükemmel olmadığı görülür. Öte yandan, R tamsayılar halkası üzerindeki $\mathbb{Z}[X]$ polinom halkası olarak alınırsa, R'nin her asal idealinin radikal olarak mükemmel olduğu ama maksimal idealler kümesinin asal olarak kapalı olmadığı görülür. Ancak, R'nin boyutu bir ise, iki nosyonun eşdeğer olduğu görülür. Bu noktada, R halkasının boyutunun birden büyük olduğu durumlarda bu iki nosyonun aralarındaki ilişkiyi analiz etmek, boyutunun bir olduğu durumlarda da $R[X]$ polinom halkasında paralel bir analizi yapmak önem arzetmektedir. Bu bağlamda, R bir boyutlu bir S-tamlık bölgesi olarak alınırsa, R üzerindeki $R[X]$ polinom halkasında, R'deki büzülmesi sıfırdan farklı olan asal ideallerin radikal olarak mükemmel olmasının, R'nin maksimal idealler kümesinin asal olarak kapalı olması ile eşdeğer olduğu sonucu elde edilmiştir. Ayrıca, R sonlu boyutlu bir valuasyon tamlık bölgesi olarak alındığında, R'nin her asal idealinin radikal olarak mükemmel olması ile R üzerindeki $R[X]$ polinom halkasının asal ideallerinin radikal olarak mükemmel olmasının denk olduğu önermesinin bağımsız ispatı verilmiştir. Yine aynı koşullar altında, R halkasının maksimal ideali M için, $MR[X]$ asal idealinin radikal olarak mükemmel olmasının $R[X]$'in asal ideallerinin kümesinin asal olarak kapalı olmasını gerektirdiği kanıtlanmıştır. Ek olarak, maksimal ideallerinin kümesi asal olarak kapalı olan ve maksimal idealleri sonlu üreteçli olan Prüfer tamlık bölgeleri üzerinde, R'nin her M maksimal ideali için, $R[X]$'de $MR[X]$ asal idelinin radikal olarak mükemmel olması ile R'nin sınıf grubu torsiyon olan bir Dedekind tamlık bölgesi olmasının muadil olduğu sonucu elde edilmiştir.

Ayrıca, literatürde Noether halkalar üzerinde tanımlanmış olan, "tam arakesit olma" nosyonunun radikal olarak mükemmellik nosyonundaki gibi, Noether olma kısıtı taşımayan halkalara genişletilmesinin mümkün olup olmadığı sorusundan yola çıkılarak, mükemmellik tanımı yapılmış ve bazı temel sonuçlar elde edilmiştir. Bu bağlamda bir halkanın tüm asal ideallerinin mükemmel olmasının, halkanın Noether olmasını gerektirdiği gözlemi altında, bu koşul altında tam arakesit olma ve mükemmellik nosyonlarının eşdeğer olduğu sonucuna varılmıştır. Buna ek olarak, sonlu boyutlu bir R tamlık bölgesinde, yüksekliği bir olan tüm asal ideallerinin mükemmel olması ile R'nin çarpınım halkası olmasının eşdeğer olduğu sonucuna varılmıştır. Ayrıca, sonlu boyutlu bir R Prüfer tamlık bölgesinde, her asal idealin mükemmel olması ile R'nin temel ideal bölgesi olmasının muadil olduğu ispatlanmıştır. Son olarak, R sonlu boyutlu bir Prüfer tamlık bölgesi ise, $R[X]$ polinom halkasının R'deki büzülmesi sıfır olan ideallerinin mükemmel olduğu gözleminden bahsedilmiştir.

Bu çalışmada tüm halkalar değişmeli ve birim elemana sahip halkalar olup, boyut ile de herzaman Krull boyutu kastedilmektedir.

1. INTRODUCTION

Here, we deal with the set theoretic number of generators of ideals in commutative rings. The problem of recognizing a "set theoretic complete intersection" is a rather delicate matter. There seems to exist a fairly large number of open questions in this area.

Before touching on the problem of "set theoretic complete intersections", it will be complementary to mention the related problem of “complete intersections" in algebraic geometry. Roughly speaking, this is the problem of trying to characterize ideals in polynomial rings which can be generated by the right number of elements. Geometrically, this translates into the problem of recognizing algebraic sets (in affine or projective spaces) which can be expressed as the intersection of the right number of hypersurfaces. In a broader sense, the "complete intersection problem" is also concerned with the determination of the minimal number of generators needed for a polynomial ideal or the determination of the minimal number of hypersurfaces needed to cut out a given algebraic variety.

On the other hand, if we are only interested in expressing a variety $V \subset \mathbf{A}^n_K$ as a set theoretic intersection of r hypersurfaces, then we do not need to know that the ideal I of V can be generated by r elements. Instead, it will be sufficient to know that I is the radical of some ideal $I^{'}$ that can be generated by r elements. This leads to the main definition: an ideal I of height r in a commutative Noetherian ring R is called a set theoretic complete intersection if $rad(I) = rad(a_1, ..., a_r)$ for suitable $a_i \in R$ $(1 \leq i \leq r)$. An algebraic set V of codimension r in $\mathbf{A}^n_K$ is called a set theoretic complete intersection if the ideal corresponding to V in $K[t_1, ..., t_n]$ is a set theoretic complete intersection.

The history of these subjects is rather interesting. As is outlined by Eisenbud and Evans in [1], these type of questions was first considered by Kronecker in late 19^{th}

century. In [2], Kronecker proved that any radical ideal in an n-dimensional Noetherian ring is the radical of an ideal generated by (n+1) elements. Since then an enormous amount of research has evolved around these types of questions. In 1891, nine years after Kronecker had announced his theorem, Vahlen produced an example which, he claimed, showed that Kronecker's result was the best possible. The example he gave is a curve in a complex projective 3-space which he showed is not the intersection of three hypersurfaces [3]. Vahlen's error seems to have gone undetected until 1941, when Perron [4] exhibited three hypersurfaces whose intersection is the curve in question. (The year before, Van der Waerden [5] had given the first modern proof of Kronecker's theorem.) In 1961, Kneser [6] showed that the existence of Perron's hypersurfaces was not an accident by proving that every curve in 3-space is an intersection of three hypersurfaces. In 1973, Eisenbud and Evans showed that a radical ideal in an n-dimensional Noetherian polynomial ring is always the radical of an ideal generated by n elements and they also proved a corresponding theorem for graded rings. Their proofs relied on a modification of Kneser's idea. In 1975, Murthy [7] showed that if K is a field, then in the ring $K[X,Y,Z]$, any ideal of height two which is locally a complete intersection can be generated by three elements. Murthy also gives an example to show that an ideal corresponding even to a nonsingular curve in 3-space need not be generated by two elements. In 1979, Szpiro [8] proved that a curve which is a local complete intersection in affine 3-space is a set theoretic complete intersection. This result was then extended by Mohan Kumar [9] to curves in spaces of arbitrary dimension. A consequence of Mohan Kumar's work is that any local complete intersection curve in any affine n-space is a set theoretic complete intersection. Boratynski [10] and Ferrand [11] independently showed that every locally complete intersection ideal of $K[X_1,...,X_n]$ of pure dimension one can be generated up to radical by n-1 elements. But the major question in the field remains open: Is every (irreducible) curve in 3-space the set theoretic intersection of two hypersurfaces? Although it is known that the answer to the corresponding question in 4-space is negative, this single old question still stands unsolved in the case of characteristic zero.

It seems worthwhile to mention this problem that remain unsolved in this area in detail. It is clear that each height one prime in $K[X,Y,Z]$ is principal, and so is a set theoretic complete intersection. On the other hand, by the result of Eisenbud and Evans [1], the maximal ideals of height three are also set theoretic complete intersections. The question is whether height two ideals in $K[X, Y, Z]$ are set theoretic complete intersections where K is of characteristic zero. The case of positive characteristic was resolved affirmatively by Cowsik and Nori [12] who proved that any curve in the affine n-space over a field K of positive characteristic p is a set theoretic complete intersection. In other words, they proved that if K is of positive characteristic then height (n-1) ideals in the polynomial rings in n variables over K are set theoretic complete intersection. Their proof uses Szpiro's and Mohan Kumar's results.

In 1994, Erdoğdu and McAdam [13] showed that the radical of ideals in characteristic zero behaves differently than in characteristic positive case. For instance if K is of characteristic zero, $K[[Y]]$ is the power series ring in Y over K and R is the subring of K[[Y]] consisting of all those power series whose Y term has coefficient zero, then R is not normal and there is a height one prime ideal of $R[X]$ that is not a set theoretic complete intersection. However, if K is of positive characteristic then each height one prime ideal of $R[X]$ is a set theoretic complete intersection. With the help of this remark, Erdoğdu [14] observed that the notion of set theoretic complete intersection has more to do with the normality of the underlying ring R than anything else and realized that the notion of set theoretic complete intersection can be generalized to rings that need not be Noetherian. In [14], he called an ideal I of a ring R radically perfect if among the ideals of R whose radical is equal to the radical of I the one with the least number of generators has this number of generators equal to the height of I, and proved that if R is any integral domain (not necessarily Noetherian) containing a field of characteristic zero then each height one prime ideal in $R[X]$ is radically perfect implies that R is normal. He also proved that if R is a one dimensional Noetherian integral domain containing a field of characteristic zero, then each prime ideal of $R[X]$ is radically perfect if and only if R is a Dedekind domain with torsion ideal class group, and so if K is a field of characteristic zero and $R = K[X, Y]$, then each prime ideal of R is radically perfect. In the same work he also remarked that if P^* is any height two prime ideal of $K[X, Y, Z] = R[Z]$, then $P = P^* \cap R$ is a nonzero prime ideal of R, and if P is of height two, then P=rad(f, g) in R from which it follows that

$P^* = PK[X,Y,Z]$=rad(f,g) in $K[X,Y,Z]$. If on the other hand P is of height one in R, then it is generated by an irreducible polynomial p in $R = K[X,Y]$, and if $R/<p>$ is not a Dedekind domain with torsion ideal class group, then there might be a height two prime in $K[X,Y,Z]$ containing p and that is not radically perfect. Motivated by this remark he asked the following question: If R is an integral domain (not necessarily Noetherian) containing a field of characteristic zero and each prime in $R[X]$ is radically perfect, then does it follow that R is of dimension one?

If the answer to this question is yes, then one would know when and when not the prime ideals in $R[X]$ are radically perfect, which would then resolve the main conjecture by taking $R = K[Y,Z]$, where K is of characteristic zero.

In [14], [15] Erdoğdu answered the above question affirmatively in the cases when R is a finite dimensional Bézout domain, and R is a finite dimensional Prüfer domain with coprimely packed set of maximal ideals. In [15], he also observe that if in a ring R each element is contained in only finitely many maximal ideals, then each maximal ideal of R is the j-radical of an ideal generated by two elements. This observation was the starting point of this study that has the following layout:

This introduction constitute chapter 1.

Chapter 2 would contain the results (mostly known) that are used in the proofs of the main results of this thesis.

Chapter 3 constitute the main results of this thesis which consists of the following sections. In section 1, we recall some statements from [15] and also generalize Theorem 2.2 of [15]. In section 2, we examine radically perfectness of ideals in Hilbert domains of finite character and among other things, we prove that over a Noetherian Hilbert domain R of finite character that contains a field of characteristic zero, each prime ideal of $R[X]$ is radically perfect if and only if R is a Dedekind domain with torsion ideal class group. We also show that in a Hilbert domain R of finite character, each invertible maximal ideal is radically perfect if and only if *CopRad(R)* is torsion, where *CopRad(R)* denote the multiplicative semigroup generated by the set of pairwise coprime radical ideals of R that are invertible. In section 3, we examine radically perfectness of prime ideals in Krull domains, and our main result in this section states that in a two dimensional Krull domain R of finite character, each prime ideal of R is radically perfect if and only if the divisor class group of R is torsion. We also examine radically perfectness of maximal ideals in $Int(R)$, the ring of integer valued polynomials over R, and show that radically perfectness of maximal ideals of $Int(R)$ implies that each prime ideal P of R is the radical of an ideal generated by at most $ht(P) + 1$ elements. In section 4, we show that if each prime ideal of $R[X]$ over a finite dimensional Prüfer domain R of finite character is radically perfect, then R is of dimension at most two.

In chapter 4, we relate the notions of radically perfectness and coprime packedness of prime ideals of various rings. In particular, we show that in a finite dimensional Hilbert domain R, if $MaxSpecR$ is coprimely packed, then each maximal ideal M^* of $R[X]$ is the radical of an ideal generated by two elements. We also show that over a Prüfer domain R with coprimely packed set of maximal ideals which has the property that each maximal ideal of it is finitely generated, then for each maximal ideal M of R the prime ideal $MR[X]$ of $R[X]$ is radically perfect if and only if R is a Dedekind domain with torsion ideal class group.

In chapter 5, we briefly outline some basic facts on the analog generalization of complete intersection defined as: an ideal I of R is perfect if the minimal number of generators of I is equal to the height of I, which turn out to be void in the case of R being a commutative ring with identity since such rings are necessarily Noetherian and so this notion then coincides with the notion of complete intersection. We show that if R is a finite dimensional Prüfer domain, then each prime ideal of R is perfect if and only if it is a principal ideal domain. We also show that if R is a finite dimensional Prüfer domain whose set of maximal ideals is coprimely packed then each maximal ideal of R is perfect if and only if R is a Dedekind domain with torsion ideal class group.

Finally, chapter 6 is the conclusion chapter which contains a brief summary of the contributions of this thesis, along with some suggestions for future study.

The main results of this thesis (namely, the results of Chapter 3 through Chapter 5) will appear in the Journal of Commutative Algebra [16].

Throughout, R will always denote a commutative ring with identity, and $R[X]$ will denote the polynomial ring over R. Also by the dimension of R we will always mean the Krull dimension of R.

2. A GENERAL OUTLOOK ON RADICALLY PERFECT IDEALS

In this chapter we recall some basic definitions and collect the results that are used in the sequel. We begin by recalling that the height of a prime ideal P (in a ring R) is the number of strict inclusions in the longest chain of prime ideals contained in P, denoted by $ht(P)$; the height of an ideal I is the infimum of the heights of all prime ideals containing I, denoted by $ht(I)$ and the Krull dimension of a ring R is the number of strict inclusions in a maximal chain of prime ideals in R, denoted by $dim(R)$.

We recall that an integral domain R is a valuation ring if for every element x of its field of fractions K, at least one of x or x^{-1} belongs to R. We also recall that a Dedekind domain is an integral domain in which every nonzero proper ideal factors into a product of prime ideals and the ideal class group of a Dedekind domain R is defined to be the group of fractional ideals modulo the group of principal fractional ideals (i.e. the ideal class group measures how far a Dedekind domain is from being a principal ideal domain).

The content of this thesis is based on the following definition given in [14], [15].

Definition 2.1 An ideal I of a ring R is radically perfect if the height of I is equal to the infimum of the number of generators of ideals of R whose radical is equal to the radical of I.

This is a generalization of the definition of set theoretic complete intersection of ideals in Noetherian rings to rings that need not be Noetherian. According to Krull's Principal Ideal Theorem, the minimum number of generators of an ideal I of a Noetherian ring R must be greater than or equal to the height of I. Therefore, in the Noetherian case the optimal situation is attained when I can be generated by $ht(I)$ number of elements. This may not be the case when the ring is non-Noetherian, as an example take R to be a valuation ring of dimension greater than one, it is easy to see that the maximal ideal

M of R is the radical of a principal ideal but its height is greater than one. The fact that in a finite dimensional valuation ring every prime ideal is the radical of a principal ideal and so the only radically perfect ideal in a finite dimensional valuation ring is the height one prime ideal. In the non-Noetherian case to show that an ideal I is radically perfect is much more demanding than the Noetherian case, since in the non-Noeherian case one also needs to show that the radical of the ideal I can not be equal to the radical of an ideal J whose least number of generators is less than the height of I.

We now give some statements that are prelude to the main results of this thesis. In [14], Erdoğdu proved and gave the motivations for the following two statements.

Theorem 2.1 Let R be a Noetherian integral domain of dimension one which contains a field of characteristic zero. Then each prime ideal of $R[X]$ is radically perfect if and only if R is a Dedekind domain with torsion ideal class group.

The above statement is proved by Erdoğdu in Theorem 2.1 of [17] under the assumption that R is normal. The motivation for replacing the normality by the condition that R contains a field of characteristic zero is given in the following remark.

Remark 2.1 Let K be a field of characteristic zero and R be the subring of $K[[Y]]$ consisting of those power series whose Y term has coefficient zero. Then R is a local Noetherian integral domain of dimension one which is not normal and there is a height one maximal ideal in $R[X]$ that is not radically perfect. However, if K is of characteristic p (p a positive prime), then R is a Noetherian integral domain of dimension one whose normalization $S = K[Y]$ is a principal ideal domain (i.e. an integral domain in which every ideal is principal), and therefore by Theorem 2.1 of [17] each prime ideal of $S[X]$ is radically perfect. Let now P be any height one prime ideal of $R[X]$. Then $rad(PS[X])$ is the intersection of a finite number of height one primes in $S[X]$. Since $S[X]$ is a factorial ring (i.e. a UFD), height one primes in $S[X]$ are principal and that the intersections of a finite number of principal ideals in $S[X]$ is again a principal ideal, and so it follows that $rad(PS[X]) = rad(f)$ for some polynomial f in $S[X]$. Now K being of characteristic p, f^p is in $R[X]$ and therefore is in P (this is because the p^{th} power of every element of $S[X]$ is in $R[X]$). But then it is clear that $P = rad(f^p)$ in $R[X]$. Thus each height one prime in $R[X]$ (in fact each prime in $R[X]$) is radically perfect without R being normal.

These facts give rise to the following result:

Theorem 2.2 Let R be an integral domain (not necessarily Noetherian) containing a field of characteristic zero. If each height one prime ideal of R[X] is radically perfect, then R is normal.

Since Theorem 2.2 is used to prove Theorem Theorem 2.1, we first give the proof of Theorem 2.2 and then give the proof of Theorem 2.1.

Proof of Theorem 2.2. Let S be the integral closure of R in K, the field of fractions of R, and let a be any element of S. Then $(X - a)K[X]$ is a prime ideal of $K[X]$ and so $(X - a)K[X] \cap S[X] = (X - a)S[X]$ is a height one prime ideal of $S[X]$. Therefore $P = (X-a)S[X] \cap R[X]$ is a height one prime ideal of $R[X]$ and hence by the hypothesis $P = rad(f)$, for some polynomial f in P. Since $(X-a)S[X] \cap S = (0)$, there is a prime Q in $K[X]$ such that $Q \cap S[X] = (X-a)S[X]$ and hence it follows that $Q \cap R[X] = P$. Thus $(X-a)K[X]$ and Q are two primes in $K[X]$ lying over P. Since $P \cap R = (0)$, there can be only one prime in $K[X]$ lying over P. Therefore $PK[X] = rad(f)K[X] = (X - a)K[X]$. It is now clear that in the factorial ring $K[X]$, the only prime divisor of f is $(X - a)$. Hence it follows that $f = c(X - a)^n$ for some positive integer n and some constant c in K. To show that c must be a unit of R, we consider the ring homomorphism $\phi : R[X] \to R[a]$ given by $\phi(X) = a$. Since $R[a]$ is an integral domain, $Ker\phi$ is a prime ideal of $R[X]$. Moreover, $Ker\phi = P$ in $R[X]$. This is because if g is any element of P, then g is in $PS[X] \subseteq (X - a)S[X]$ and so $g(a) = 0$ which implies that $P \subseteq Ker\phi$. Now using the fact that $Ker\phi \cap R = (0)$ and the fact that there can not be a chain of three distinct primes in $R[X]$ contracting to 0 in R, we obtain $P = Ker\phi$. Since $P = rad(f) = Ker\phi$ contains the monic polynomial for a, f may be chosen monic and therefore $c = 1$ in R. Thus $f = (X - a)^n$ and so by comparing the coefficients one can see that $na \in R$. Now as n is invertible in R, we see that $a \in R$ and hence R is normal.□

Proof of Theorem 2.1. If each prime in $R[X]$ is radically perfect, then it follows from Theorem 2.2 that R is a Noetherian normal domain of dimension one and hence is a Dedekind domain. Let now P be any nonzero prime ideal of R. Then, as R

is Noetherian, $PR[X]$ is a height one prime ideal of $R[X]$ and so by the assumption $PR[X] = rad(f)$ for some polynomial f in $R[X]$. Let a be any element of P. Then a is in $PR[X] = rad(f)$ and hence there is a positive integer n such that $a^n \in (f)$. Thus $a^n = fg$ for some polynomial g in $R[X]$. Since $R[X]$ is an integral domain, by comparing the degrees of the polynomials on both sides, one can see that $f = b$ for some constant b in P. Thus $P = rad(b)$ in R and therefore $P^m = Rb$ for some positive integer m. But then it follows that the class group of R is torsion.
Conversely, if R is a Dedekind domain with torsion ideal class group, then each prime ideal of $R[X]$ is radically perfect by Theorem 2.1 of [17]. □

In light of the above results, Erdoğdu gave the following remark in [14].

Remark 2.2 Let K be a field of characteristic zero and $R = K[X, Y]$. Then R is a polynomial ring in Y over the principal ideal domain $K[X]$ and hence it follows from Theorem 2.1 that each prime ideal of R is radically perfect. Let P^* be any height two prime ideal of $K[X, Y, Z] = R[Z]$, then $P = P^* \cap R$ is a nonzero prime ideal of R. If P is of height two, then $P = rad(f, g)$ in R. But then it follows that $P^* = PK[X, Y, Z] = rad(f, g)$ in $K[X, Y, Z]$. If on the other hand, P is of height one in R, then it is generated by an irreducible polynomial h in $R = K[X, Y]$, and if $R/ < h >$ is not a Dedekind domain with torsion ideal class group, then there might be a height two prime ideal in $K[X, Y, Z]$ containing h and that is not radically perfect. This observation leads Erdoğdu to ask the following question:

Question 2.1 If R is an integral domain (not necessarily Noetherian) containing a field of characteristic zero and such that each prime ideal of $R[X]$ is radically perfect, then does it follow that R of dimension one?

The next two sections contains results answering this question affirmatively in some special cases (see [14], [15]).

2.1 Radically Perfect Prime Ideals in Polynomial Rings over Bézout Domains

It is clear from Theorem 2.2 above that (in characteristic zero case) radically perfectness of prime ideals of $R[X]$ has more to do with the normality of R than anything else, and the following statement is a further justification of this fact. Recall that an integral domain R is a Bézout domain if each finitely generated ideal of R is principal, and it is a well known fact that Bézout domains are normal.

Theorem 2.3 Over a finite dimensional Bézout domain R every prime ideal of $R[X]$ is radically perfect if and only if R is of dimension one and each prime ideal of R is radically perfect.

Proof . Since R is Bézout, $dim(R[X]) = dim(R) + 1$, and that if P is a prime ideal of height n in R, then $PR[X]$ is of height n in $R[X]$. Let M be a maximal ideal of R of height n, then $MR[X]$ is of height n in $R[X]$, and hence by assumption, $MR[X] = rad(f_1, f_2, ..., f_n)$ for some polynomials $f_1, f_2, ..., f_n$ in R[X]. Let for each i $(1 \leq i \leq n)$, A_{f_i} be the content ideal of f_i in R, then as R is Bézout and each A_{f_i} is finitely generated, $A_{f_i} = Ra_i$ for some a_i in A_{f_i}. Since $(f_1, ..., f_n) \subseteq \sum_{i=1}^{n} A_{f_i}R[X] = \sum_{i=1}^{n} a_iR[X] \subseteq MR[X]$, it follows that the ideal $(a_1, a_2, ..., a_n)$ of R is contained only in the maximal ideal M of R but not in any other maximal ideal of R. Again using the fact that R is Bézout, we have $(a_1, a_2, ..., a_n) = Ra$ for some a in M and so M is the only maximal ideal of R containing a. If now P is any prime ideal of R containing a, then $(f_1, f_2, ..., f_n) \subseteq \sum_{i=1}^{n} a_iR[X] = aR[X] \subseteq PR[X] \subseteq MR[X]$. Taking the radical of both sides it is easily seen that $PR[X] = rad(f_1, f_2, ..., f_n) = MR[X]$ in $R[X]$ and it follows from the fact that the prime ideals of R contained in M containing the element a are linearly ordered, we obtain $P = M$ in R. Hence, $M = rad(a)$ in R and of course in $R[X]$, $MR[X] = rad(aR[X])$. Thus $n = 1$, and since $MR[X]$ is radically perfect, it is of height one. Therefore, it follows that height of each maximal ideal M of R is one, and hence R is of dimension one and each prime ideal of R is radically perfect.

Conversely, let P^* be a nonzero prime ideal of $R[X]$ and $P = P^* \cap R$. If P is different than zero in R, then we either have $PR[X] = P^*$ in which case $P^* = rad(aR[X])$ in $R[X]$, or else $PR[X]$ is properly contained in P^* and in that case P^* is of height two, and hence by Theorem 28 of [18], P^* is generated by P and some monic polynomial f. But then it is clear that $P^* = rad(a, f)$. Next we deal with the case when $P^* \cap R = (0)$ in R, in this case by Theorem 36 of [18], we have $P^*K[X] = gK[X]$ for some polynomial g in P^*, where K is the field of fractions of R. Let A_g be the content ideal of g in

R. Then $A_g = Ra$ for some a in A_g. Let $g = b_0 + b_1X + b_2X^2 + ... + b_kX^k$. Then $A_g = Rb_0 + Rb_1 + ... + Rb_k$, and hence $a = d_0b_0 + d_1b_1 + ... + d_kb_k$, for some $d_0, d_1, ..., d_k$ in R. On the other hand for each i ($0 \leq i \leq k$), we have $b_i = c_ia$ for some c_i in R. Thus, $a = ac_0d_0 + ac_1d_1 + ... + ac_kd_k$ and from this it follows that $1 = c_0d_0 + c_1d_1 + ... + c_kd_k$. Therefore if $f = c_0 + c_1X + c_2X^2 + ... + c_kX^k$, then $g = af$ and $A_f = R$. Hence $P^*K[X] = gK[X] = afK[X] = fK[X]$ and one can use the Dedekind-Mertens Lemma to show in fact that P^* is principal in $R[X]$ (see the last part of the proof of Proposition 4.3). Therefore it follows that each prime ideal of $R[X]$ is radically perfect. □

2.2 Radically Perfect Prime Ideals in Polynomial Rings over a Prüfer Domain with Coprimely Packed Set of Maximal Ideals

Recall that an integral domain R is Prüfer if every finitely generated ideal of R is invertible, equivalently if for each prime ideal P of R, the local ring R_P is a valuation ring and so Prüfer domains are normal. An almost Dedekind domain is also a Prüfer domain R with the property that each localization R_M at each maximal ideal M of R is a discrete valuation ring (i.e. a principal ideal domain with exactly one nonzero maximal ideal). We also recall that a ring R is said to be coprimely packed if whenever I is an ideal of R and S is a set of maximal ideals of R with $I \subseteq \bigcup\{M \in S\}$, then $I \subseteq M$ for some $M \in S$ and the set of maximal ideals of a ring R is coprimely packed if no maximal ideal of R is contained in the union of the remaining maximal ideals of R.

Theorem 2.4 Let R be a finite dimensional Prüfer domain with coprimely packed set of maximal ideals. Then for each maximal ideal M of R the prime ideal $MR[X]$ of $R[X]$ is radically perfect if and only if R is of dimension one and each prime ideal of R is the radical of a principal ideal.

Proof . Since R is a finite dimensional Prüfer domain, $dim(R[X]) = dim(R) + 1$ and that for each maximal ideal M of R, $ht(M) = ht(MR[X])$. Let now M be any maximal ideal of R of height n. Then as $MR[X]$ is a radically perfect ideal in $R[X]$ of height n, $MR[X] = rad(f_1, f_2, ..., f_n)$ for some polynomials $f_1, f_2, ..., f_n$ in $MR[X]$. Hence it follows that $M_MR_M[X] = rad(\hat{f}_1, \hat{f}_2, ..., \hat{f}_n)$, where $\hat{f}$ denotes the image of f in $R_M[X]$. Let for each i $(1 \leq i \leq n)$, $A_{\hat{f}_i}$ be the content ideal of $\hat{f}_i$ in R_M. Then since R_M is a valuation ring and each $A_{\hat{f}_i}$ is finitely generated and is obviously contained in M_M, and it follows that $A_{\hat{f}_i} = R_M\hat{a}_i$ for some $\hat{a}_i$ in M_M where $\hat{a}_i$ denotes the image of a_i in R_M. Hence in $R_M[X]$, as in the proof of Theorem 2.3 we have $(\hat{f}_1, \hat{f}_2, ..., \hat{f}_n) \subseteq \sum_{i=1}^{n} A_{\hat{f}_i} R_M[X] = \sum_{i=1}^{n} \hat{a}_i R_M[X] \subseteq M_M R_M[X]$. Now, again for the same reason as above, $\sum_{i=1}^{n} \hat{a}_i R_M = R_M\hat{a}$ for some $\hat{a}$ in M_M. Thus $M_M R_M[X] = rad(\hat{f}_1, \hat{f}_2, ..., \hat{f}_n) = rad(\hat{a}R_M[X])$. It is now clear that M_M is the only prime ideal of R_M containing the element $\hat{a}$, that is, no prime ideal of R_M properly contained in M_M contains $\hat{a}$. Now we show that if a is the element of M having $\hat{a}$ as its image in M_M, then $M = rad(a)$. Since the set of maximal ideals of R is coprimely packed, there is an element m in M that is not contained in any other maximal ideal of R. Consequently, each prime ideal of R containing m is contained only in the maximal ideal M and is not contained in any other maximal ideal of R. But then using the fact that in a Prüfer domain the set of prime ideals of R contained in a maximal ideal is linearly ordered, one obtains that $rad(m)$ is a prime ideal of R and that $M = j - rad(rad(m))$ (i.e. M is the intersection of all maximal ideals containing $rad(m)$). Hence it follows that $R/rad(m)$ is a valuation ring with maximal ideal $M/rad(m)$. Again using the one-to-one order preserving correspondence between the ideals of R contained in M which contain $rad(m)$ and the ideals of $R/rad(m)$ contained in $M/rad(m)$, together with the fact that the image of a in $M/rad(m)$ is nonzero give $M = rad(a)$ in R (of course here if it happens that $M = rad(m)$, then $m = a$). Hence it follows that for each maximal ideal M of R, $MR[X] = rad(aR[X])$, for some a in M. Since $MR[X]$ is radically perfect, $MR[X]$ is of height one, and so M is of height one. Therefore R is of dimension one and each maximal ideal of it is the radical of a principal ideal.

The converse is clear. □

Theorem 2.5 Let R be a finite dimensional Prüfer domain with coprimely packed set of maximal ideals. If for each maximal ideal M of R the prime ideal $MR[X]$ of $R[X]$ is radically perfect, then a power of each finitely generated maximal ideal M of R is principal.

Proof . By Theorem 2.4, R is of dimension one. Let now M be any finitely generated maximal ideal of R. Then since R is Prüfer of dimension one and since a finitely generated ideal in a Prüfer domain of dimension n is generated by $n + 1$ elements (which is even so for any invertible ideal I in a domain of dimension n, see, Theorem 3.1 of [19]), it can be seen that M is generated by two elements and is of height one. Again it follows from Theorem 2.4, that $M = rad(c)$, for some c in M. Therefore the principal ideal (c) is M-primary. Since M_M is generated by two elements and is the only nonzero prime ideal of R_M and is of height one, R_M is Notherian and so is a discrete valuation ring. If now $\hat{c}$ is the image of c in R_M, then the ideal $(\hat{c})$ is M_M-primary and so $(M_M)^n = (\hat{c})$, for some positive integer n. But then using the fact that there is a one to one correspondence between the primary ideals of R contained in M and those of R_M and the fact that a positive power of a maximal ideal is primary, we obtain $M^n = (c)$ in R. □

After proving Theorem 2.4 and Theorem 2.5 Erdoğdu asked the following question and answered it affirmatively in some special cases in Theorem 2.6 and its corollary below.

Question 2.2 Is the Picard group (i.e. the group of invertible fractional ideals modulo the subgroup of principal fractional ideals) of a Prüfer domain in which each maximal ideal is the radical of a principal ideal torsion?

Note that if R is an S-domain (i.e. for each height-one prime ideal P of R, the extension prime ideal $PR[X]$ in $R[X]$ is of height one) of dimension one and if for each maximal ideal M of R, the prime ideal $MR[X]$ is generated by $ht(MR[X])$ number of generators, then R is a principal ideal domain and therefore has trivial Picard group.

To relate radically perfectness of prime ideals in $R[X]$ to the Picard group of R being torsion when R is Prüfer, the following definition is needed.

Definition 2.2 An integral domain R is said to be of finite type if each finitely generated ideal I of R is a finite product of prime ideals that are minimal over I.

Theorem 2.6 Let R be a Prüfer domain of finite type with coprimely packed set of maximal ideals. Then the following statements are equivalent:

(1) For each maximal ideal M of R the prime ideal $MR[X]$ is radically perfect.

(2) R is of dimension one and each maximal ideal of R is the radical of a principal ideal.

(3) Each prime ideal of $R[X]$ is radically perfect.

(4) The Picard group of R is torsion and R is of dimension one, and each maximal ideal of R is the radical of a principal ideal.

Proof. (1) $\Leftrightarrow$ (2). This follows from Theorem 2.4.

(2) $\Rightarrow$ (3). Since R is a Prüfer domain of dimension one, $dim(R[X]) = 2$. Let P^* be a nonzero prime ideal of $R[X]$. Then either $P^* \cap R = P$ is nonzero or $P^* \cap R = (0)$ in R. If $P^* \cap R = P$ is nonzero, then either $P^* = PR[X]$ in which case P^* is the radical of the principal ideal in $R[X]$ generated by the principal ideal of R whose radical is P in R, or else P^* properly contains $PR[X]$ in this case P^* is of height two and so is maximal in $R[X]$. But then $P^* = (P, f)$ for some monic polynomial f, and since $P = rad(a)$ for some element a in R, one can see that $P^* = rad(a, f)$ in $R[X]$. In the case when $P^* \cap R = (0)$ in R (clearly such P^* is of height one), as in the proof of Theorem 2.3 we have $P^*K[X] = gK[X]$, for some polynomial g in P^* where K is the field of fractions of R. Since the content ideal A_g of g in R is finitely generated, A_g is a finite product of its minimal primes. Thus $A_g = M_1M_2...M_n$, where for each i $(1 \leq i \leq n)$, M_i is a maximal ideal of R. Since R is Prüfer and A_g is finitely generated, it is invertible. Therefore it follows that each M_i is invertible. But then by Theorem 2.5, a power of each M_i is principal and from this we see that a positive power of A_g is principal, say $(A_g)^k = (c)$. Again using the fact that polynomials over Prüfer rings are Gaussian, we obtain $(A_g)^k = A_{g^k} = (c)$. Next one may use an argument similar to the last part of Theorem 2.3 and obtain $g^k = bh$ with b in A_g and $P^* = rad(h)$. Therefore $P^*$is a radically perfect ideal of $R[X]$.

(3) $\Rightarrow$ (4). By Theorem 2.4, each prime ideal of R is the radical of a principal ideal. Now Theorem 2.5 together with the above proof of (2) $\Rightarrow$ (3) imply that each finitely generated ideal of R has a positive power is principal, and so the Picard group of R is torsion.

(4) $\Rightarrow$ (1). This follows from Theorem 2.4. □

Corollary 2.1 Let R be an almost Dedekind domain with coprimely packed set of maximal ideals. Then the following statements are equivalent.

(1) For each maximal ideal M of R, the prime ideal $MR[X]$ is radically perfect.

(2) Each maximal ideal of R is the radical of a principal ideal.

(3) Each prime ideal of $R[X]$ is radically perfect.

(4) R is a Dedekind domain with torsion ideal class group.

Proof. The proof of (1) ⇒ (2) is the same as the proof of (1) ⇒ (2) of Theorem 2.6.

(2) ⇔ (3) ⇔ (4) ⇒ (1). This follows from Theorem 2.1 of [17] and Theorem 2.1 of [20]. □

3. RADICALLY PERFECT PRIME IDEALS IN DOMAINS OF FINITE CHARACTER

3.1 Finite Generation of Ideals in Rings of Finite Character up to j-radical

We begin this section by recalling the following simple but useful proposition from [15] that motivated the main results of this thesis. We recall that a ring R is of *finite character* if each nonzero element of it is contained in only finitely many maximal ideals. We denote by $j - rad(I)$ the intersection of all maximal ideals of R containing the ideal I, and by $j_{max} - rad(I)$, the intersection of all maximal ideals of R of maximal height that contains I.

Proposition 3.1 ([15], Proposition 2.1) Let R be a ring of finite character. Then each maximal ideal of R is the j-radical of an ideal generated by two elements.

Proof. Let M be any maximal ideal of R and u a nonzero element of M. Then u is contained in finitely many maximal ideals and let $M = M_1, M_2, ..., M_n$ be the only maximal ideals of R containing u. It is clear that there is an element v in $M = M_1$ and an element a in $\bigcap_{k=2}^{n} M_k$ such that $v + a = 1$ in R and that $M = M_1 = j - rad(u, v)$. □

Before giving the proof of the main result of this section which is a slight generalization of Theorem 2.2 of [15], we need the following proposition.

Proposition 3.2 Let M^* be a maximal ideal of $R[X]$. Then M^* contains a monic polynomial if and only if $M^* \cap R$ is maximal ideal of R.

Proof. If $M^* \cap R = M$ is maximal then $M^* = (M, f)$ for some monic polynomial over R.

Conversely, if M^* contains a monic polynomial then one can see that $R/M^* \cap R \subseteq R[X]/M^*$ is an integral extension and therefore $M^* \cap R$ is a maximal ideal of R since M^* is a maximal ideal of $R[X]$. □

Theorem 3.1 Let R be a finite dimensional integral domain of finite character. Then each maximal ideal M^* of $R[X]$ of maximal height is the $j-$radical of an ideal generated by three elements and j_{max}-radical of an ideal generated by two elements.

Proof. Let M^* be a maximal ideal of $R[X]$ of maximal height such that $M^* \cap R = M$. We first show that M is maximal in R. To do so, we note that there is a chain of prime ideals of $R[X]$ contained in M^* passing through $MR[X]$ of length equal to the dimension of $R[X]$. Since there can not be a chain of three distinct prime ideals of $R[X]$ having the same contraction in R, it follows that there can not be a prime ideal between $MR[X]$ and M^* in $R[X]$. If M is not maximal, then M would be contained in a maximal ideal N of R and that would give a longer chain of prime ideals in $R[X]$ of length equal to $dimR[X] + 1$. This contradiction shows that M is maximal in R and therefore $M^* = (M, f)$ for some monic polynomial f in M^*. Now use the proof of Proposition 3.1 to see that M^* contains $j - rad(u, v, f)$, where $M = j - rad(u, v)$ for some elements u and v in M. Let Q^* be any maximal ideal of $R[X]$ containing the ideal (u, v, f). Then since Q^* is maximal and contains a monic polynomial it follows that $Q^* \cap R$ is a maximal ideal of R containing u and v and so has to be the maximal ideal M. Therefore $M^* = Q^*$ in $R[X]$ and so $M^* = j - rad(u, v, f)$. Next we show that M^* is the j_{max}-radical of an ideal generated by two elements. For this let $M = M_1, M_2, ..., M_n$ be the only maximal ideals of R containing u and let a be in $\bigcap_{k=2}^{n} M_k$ and v be in M such that $v + a = 1$ in R put $g = v + af$. Then clearly the ideal (u, g) of $R[X]$ is contained in M^*. Suppose now that N^* is any maximal ideal of $R[X]$ of maximal height containing (u, g). Then $N = N^* \cap R$ is a maximal ideal of R and it contains u and so is one of $M = M_1, M_2, ..., M_n$. If $N = M_i$ for some $i \neq 1$, then N^* contains a and therefore it contains $v = g - af$. But then it follows that N^* contains 1, a contradiction. Therefore $N = M$ in R and $N^* = M^*$ in $R[X]$, consequently $M^* = j_{max} - rad(u, g)$. □

In the remaining part of this chapter we present results on radically perfectness of ideals in certain classes of rings of finite character.

3.2 Radically Perfect Prime Ideals in Hilbert Domains of Finite Character

Recall that a Hilbert domain is an integral domain in which each prime ideal is an intersection of maximal ideals. In this section, we show that the condition of finite character on a Hilbert domain R implies that each maximal ideal M^* of $R[X]$ is the radical of an ideal generated by two elements and that radically perfectness of maximal ideals of R is equivalent to the condition that the multiplicative semigroup *CopRad(R)* (generated by the set of pairwise coprime radical ideals of R that are invertible) is torsion. Observe that if R is a Hilbert domain of finite character then R is finite dimensional.

Proposition 3.3 A Hilbert domain R of finite character is of dimension one and each maximal ideal of it is the radical of an ideal generated by two elements.

Proof. Let P be a nonzero prime ideal of R and a be a nonzero element of P. Then since R is a Hilbert domain of finite character, $rad(a) = M_1 \cap ... \cap M_n$ for some maximal ideals $M_1, ..., M_n$, $(1 \leq i \leq n)$. But then it follows that P contains at least one of the maximal ideals M_i for some i, and so P is maximal. Therefore R is of dimension one. If now M is a maximal ideal of R, then by Proposition 3.1, there are elements u and v in M such that $M = j - rad(u, v)$. Since R is a Hilbert domain, for any ideal I of R, $j - rad(I) = rad(I)$, and so $M = rad(u, v)$. □

Recall that the contraction of a maximal ideal of the polynomial ring $R[X]$ to R is not necessarily a maximal ideal in general, but when R is a Hilbert domain then the the contraction of a maximal ideal of $R[X]$ to R is always maximal [21] (e.g. if we take R to be $\mathbb{Z}_p = \{\frac{x}{y} : p \nmid y\}$ where p is a prime number, then it is well-known that R is not Hilbert and $(px - 1)$ is a maximal ideal of $R[X]$ whose contraction is not maximal in R).

Proposition 3.4 Let R be a Hilbert domain of finite character and $R[X]$ be the polynomial ring over R and let M^* be any maximal ideal of $R[X]$. Then the least number of generators among the ideals in $R[X]$ with radical M^* is two.

Proof. Let M^* be any maximal ideal of $R[X]$. Then by Theorem 5 of [21], $M = M^* \cap R$ is a maximal ideal in R. Hence $M^* = (M, f)$ for some monic polynomial f in M^*. Now by adapting the proof of the last part of Theorem 3.1, we obtain $M^* = rad(u, g)$ for some element u in M and some polynomial g in M^*. Next we show that M^* can not be the radical of a single element. For if $M^* = rad(h)$ for some h in M^*, then it is clear that h must be a constant c. Since the monic polynomial f is in M^*, f^k is in (c) for some positive integer k. But then c divides all the coefficients of f^k and hence it divides the highest coefficient 1. This is a contradiction to the fact that M^* is a maximal ideal. Therefore the least number of generators among the ideals in $R[X]$ with radical M^* is two. □

Corollary 3.1 Let R be a Hilbert domain of finite character and $R[X]$ be the polynomial ring over R. Then each maximal ideal of $R[X]$ of maximal height is radically perfect if and only if $R[X]$ is of dimension two.

Proof. This follows from Proposition 3.4. It is worth noting that without radically perfectness assumption, one can only conclude from Proposition 3.3 that $dim(R[X]) \geq 2$. □

Proposition 3.5 Let R be a Noetherian Hilbert domain of finite character which contains a field of characteristic zero. Then each prime ideal of $R[X]$ is radically perfect if and only if R is a Dedekind domain with torsion ideal class group.

Proof. By Proposition 3.3, R is of dimension one, and by Theorem 2.2 R is normal but then the statement follows from Theorem 2.1 of [17]. □

Proposition 3.6 For a ring R the following statements are equivalent:
(a) R is a Noetherian Hilbert ring of finite character with zero divisors.
(b) R is Artinian.
(c) R is Noetherian and each ideal of $R[X]$ is the radical of a principal ideal.

Proof.
$(a) \Rightarrow (b)$. By the proof of Proposition 3.3, we have each prime ideal of R is maximal, and thus R is a Noetherian ring of dimension zero, and so is Artinian.
$(b) \Rightarrow (a)$. This is obvious.
$(b) \Rightarrow (c)$. Let NR be the nil-radical of R (the set of all nilpotent elements in the ring, or equivalently the radical of the zero ideal), then $NR[X]$ is contained in every prime

ideal of $R[X]$. But $R[X]/NR[X] \cong (R/NR)[X]$ is a principal ideal ring. Therefore, if I^* is any ideal of $R[X]$, then the image of I^* in $(R/NR)[X]$ is principal. Let f be the element of I^* whose image in $(R/NR)[X]$ generates the image of I^* in $(R/NR)[X]$. But then it follows that $rad(I^*) = rad(f)$ [17].
$(c) \Rightarrow (b)$. The condition implies that R is a Noetherian ring of dimension zero, and so is Artinian. □

Theorem 3.2 Let R be a Hilbert domain of finite character. Then each invertible maximal ideal of R being radically perfect implies that a positive power of each invertible radical ideal is principal.

Proof. By Proposition 3.3, we have R is of dimension one. Let M be an invertible maximal ideal of R. Then M_M is principal in R_M and M_M being the only nonzero prime ideal of R_M implies that R_M is Noetherian, and so is a discrete valuation ring. Since M is radically perfect, $M = rad(c)$ for some $c \in M$, and so (c) is M-primary. Note that as (c) is M-primary, $(c) = cR_M \cap R$ and as M^r is primary for all natural numbers r we have $M^r R_M \cap R = M^r$. Now, as R_M is a rank one discrete valuation ring, $M^n R_M = cR_M$ for some natural number n, and as in the proof of Theorem 2.5 we obtain that $M^n = (c)$ in R. Let now I be an invertible radical ideal of R. Then $I = rad(I) = M_1 ... M_n$. Since I is invertible, it follows that each M_i is invertible. But then a power of each M_i is principal and from this we see that a positive power of I is principal. □

Let R be a ring and let *CopRad(R)* denote the multiplicative semigroup generated by the set of pairwise coprime radical ideals of R that are invertible. Then each generator of *CopRad(R)* represents a class of finitely generated projective $R-$modules of rank one. Note that for any two generators I, J of $CopRad(R)$, $rad(IJ) = rad(I \cap J) = rad(I) \cap rad(J) = I \cap J = IJ$ and this semigroup has an identity, and the identity being the ring R itself. Clearly, *CopRad(R)* is a multiplicative subsemigroup of the Picard group of R.

Theorem 3.3 Let R be a Hilbert domain in which each nonzero ideal is contained in finitely many maximal ideals. Then each invertible maximal ideal of R is radically perfect if and only if *CopRad(R)* is torsion.

Proof. The only if part follows from Theorem 3.2. For the if part take M to be any invertible maximal ideal of R. Then M is in *CopRad(R)*, and hence $M^k = (c)$ for some c in R and some positive integer k. But then it is clear that $M = rad(c)$ and by Proposition 3.3, R is of dimension one. Therefore, M is radically perfect. □

We remark that Theorems 3.2 and 3.3 are true for any integral domain of dimension one. Since in their proofs, the condition "Hilbert" is used to ensure that the dimension of the ring is one.

3.3 Radically Perfect Prime Ideals in Krull Domains of Finite Character

In this section, we investigate radically perfectness of prime ideals in Krull domains of finite character.

An integral domain R is a Krull domain if it satisfies the following three conditions:
(i) For each height one prime P in R, R_P is a discrete valuation ring.
(ii) $R = \bigcap_{P \in X^{(1)}} R_P$ where $X^{(1)}$ is the set of prime ideals of height one in R.
(iii) Each nonzero element f in R is contained in at most a finite number of prime ideals of height one.

Recall that a fractional ideal of an integral domain R is an R-submodule of its quotient field K and a fractional ideal of the form Rx where $x \in K$, $x \neq 0$ is called a principal fractional ideal. An ideal I is called divisorial if $I = (I^{-1})^{-1} = (R : (R : I))$ (or equivalently if it is the intersection of principal fractional ideals). The free group generated by the fractional divisorial ideals of an integral domain R is denoted by $D(R)$. In the case of Krull domain this group is the same as the free group $Div(R)$ generated by the set of minimal nonzero prime ideals. The set of principal divisorial fractional ideals form a subgroup $Prin(R)$ of $D(R)$ and the divisor class group of the Krull domain R is the quotient group $Div(R)/Prin(R)$. We also recall that a fractional ideal I of R is invertible if $II^{-1} = R$ (note that $II^{-1} \subset R$ is automatic) where the fractional ideal $I^{-1} := (R : I)$ denotes the inverse of I which is the set of all x in K with $xI \subset R$.

An integral domain R is a factorial ring (or a UFD) provided every element in R is uniquely (up to multiplication by a unit) a finite product of irreducible (or prime) elements and a factorial ring is a Krull domain whose divisor class group is trivial.

The following theorem is one of our main results of this section.

Theorem 3.4 Let R be a two dimensional Krull domain of finite character. Then each prime ideal P of R is radically perfect if and only if the divisor class group of R is torsion.

Proof. The only if part follows from the fact that in a Krull domain R each height one prime ideal is the radical of a principal ideal if and only if the divisor class group of R is torsion (Theorem 3.2 of [22]).

The proof of the if part is as follows. Since we are assuming that the divisor class group of R is torsion, each height one prime ideal of R is the radical of a principal ideal and hence is radically perfect. So, we are left to consider the case of the prime ideals of height two. Let M be a height two prime ideal of R. Then by Proposition 3.1, there are elements u, v in M such that $M = j{-}rad(u, v)$. Clearly, $rad(u, v)$ is contained in M, and so any prime ideal of R that contains u and v is contained in M. Therefore, if no prime ideal of R contained in M contains u and v, then $M = rad(u, v)$. So, suppose that there is a prime ideal N of R containing (u, v). Then necessarily N is properly contained in M and so is of height one. Therefore $N = rad(c)$ for some $c \in N$. Since N contains u and v and $N = rad(c)$, some powers of u, v are in (c). Let x be an element in M which is not in N. Then clearly the ideal (c, x) is contained only in M and not contained in any other prime ideal of R. For if P is any other prime ideal of R containing c and x, then N must be strictly contained in P, as x is not in N, which implies that P is of height two and it contains u and v and therefore P must be nothing other than M. Hence $M = rad(c, x)$. In order to conclude that M is radically perfect we need to show that M is not the radical of a principal ideal. Suppose that $M = rad(b)$ for some $b \in M$. Then since R is a Krull domain, it follows from the corollary to Theorem 12.3 of [23] that $Rb = P_1^{(n_1)} \cap P_2^{(n_2)} \cap ... \cap P_k^{(n_k)}$ for some finite number of height one prime ideals of R where $P_i^{(n_i)}$ are the symbolic powers of P_i $(1 \leq i \leq n)$. But then clearly $M = rad(b) = rad(P_1^{(n_1)} \cap P_2^{(n_2)} \cap ... \cap P_k^{(n_k)}) = rad(P_1^{(n_1)}) \cap rad(P_2^{(n_2)}) \cap ... \cap rad(P_k^{(n_k)}) =$

$P_1 \cap P_2 \cap ... \cap P_n \subseteq P_i$ for each $(1 \leq i \leq n)$. This is a contradiction to the fact that M is of height two. Therefore the least number among the generators of ideals with radical M in R is two, and so M is radically perfect. □

Corollary 3.2 Let R be a two dimensional factorial ring of finite character; then each prime ideal of R is radically perfect.

Proof. Follows from the above proof by noting that every factorial ring is a Krull domain and that in a factorial ring height one primes are principal. □

Corollary 3.3 Let R be a two dimensional locally factorial Krull domain of finite character. Then the following statements are equivalent:
(a) Each height one prime ideal of $R[X]$ is radically perfect.
(b) The divisor class group of R and that of $R[X]$ are torsion.
(c) A positive power of each height one prime ideal of R is principal.
(d) A positive power of each divisorial ideal of R is principal.
(e) Each prime ideal of R is radically perfect.

Proof. $(a) \Leftrightarrow (b)$. Each height one prime ideal of $R[X]$ is radically perfect if and only if the divisor class group of $R[X]$ is torsion which is so if and only if the class group of R is torsion.

$(b) \Rightarrow (c)$. Let P be a height one prime ideal of R. Since R is locally factorial Krull domain P is invertible by Theorem 3.1 of [22], and since the divisor class group of R is torsion, by Theorem 3.2 of [22], $P^{(n)}$ is principal for some $n > 0$. Since for such prime P, we have $P^{(n)} = P^n$, and so P^n is principal.

$(c) \Rightarrow (d)$. Let I be a divisorial ideal in R, then R being Krull, $I = P^{(n_1)} \cap ... \cap P^{(n_k)}$ where $P_1, ..., P_k$ are height one primes and since by assumption a power of each P_i is principal, it follows that a power of $I = P^{(n_1)} \cap ... \cap P^{(n_k)} = P^{n_1}...P^{n_k}$ is principal.

$(d) \Rightarrow (e)$ Since in a Krull domain each height one prime ideal is divisorial, it follows that a power of each height one prime ideal is principal and so is radically perfect. The case of height two primes being radically perfect follows from Theorem 3.1.

$(e) \Rightarrow (b)$. This is nothing but Theorem 3.1. □

Corollary 3.4 Let R be a two dimensional locally factorial Krull domain of finite character. Then each prime ideal of $R[X]$ is radically perfect implies that the ring $R(X)$, the localization of $R[X]$ at the set of polynomials with unit content, is a factorial ring.

Proof. This follows from Corollary 3.3 and Theorem 3.1 of [22]. □

Proposition 3.7 Let R be a two dimensional Krull domain of finite character. Suppose that the divisor class group of R is torsion. Then each maximal ideal of $R[X]$ of maximal height is the radical of an ideal generated by at most three elements.

Proof. Let M^* be a maximal ideal of $R[X]$ of maximal height such that $M^* \cap R = M$. Then it follows from the proof of Theorem 3.1 that M is a maximal ideal of R. Hence $M^* = (M, f)$ for some monic polynomial f in M^*. Now by using the proof of Theorem 3.1 one can easily see that M^* is the radical of an ideal generated by at most three elements. □

Next we make the following observation:
Let R be a two dimensional Noetherian domain of finite character that contains a field of characteristic zero. Suppose that each prime ideal of the polynomial ring $R[X]$ over R is radically perfect then it follows from Theorem 2.2 that R is normal. Since Noetherian normal domains are Krull, R is a Krull domain and therefore $R[X]$ is a Krull domain having the property that each prime ideal of it is radically perfect. Hence each height one prime ideal of $R[X]$ is the radical of a principal ideal and therefore by Theorem 3.2 of [22] the divisor class group of $R[X]$ is torsion. Knowing that the divisor class group of R and that of $R[X]$ are isomorphic, we see that the divisor class group of R is torsion, and hence by Theorem 3.1 each prime ideal of R is radically perfect.

Conversely, suppose that R is a two dimensional Noetherian Krull domain of finite character whose divisor class group is torsion. Then for any nonzero prime ideal P^* of $R[X]$ we have the following three cases.

Case 1. $ht(P^*) = 3$. Then Proposition 3.7 together with the fact that R is Noetherian imply P^* is radically perfect.

Case 2. $ht(P^*) = 1$. Since the divisor class group of R and hence that of $R[X]$ are torsion, it follows from Theorem 3.2 of [22] that P^* is radically perfect.

Case 3. $ht(P^*) = 2$. Then either $ht(P^* \cap R) = 2$ in which case P^* is radically perfect, or $ht(P^* \cap R) = 1$ in that case $R/P^* \cap R$ is a Noetherian semilocal domain of dimension one. Hence by Proposition 3 of [13], each prime ideal of $(R/P^* \cap R)[X]$ except perhaps those primes contracting to zero in $R/P^* \cap R$ are radically perfect and each maximal ideal of $(R/P^* \cap R)[X]$ is the j-radical of a single element.

The following result is related to the above Propositon 3.6 which correlate radically perfectness of the prime ideals of R to radically perfectness of maximal ideals of $Int(R)$, the ring of integer-valued polynomials over R.

Recall that $Int(R) := \{f \in K[X] : f(R) \subseteq R\}$ and for any α in R and any prime ideal P of R, the set $\mathcal{B}(\alpha, P) = \{f \in Int(R) : f(\alpha) \in P\}$ is a prime ideal of $Int(R)$ and if P is a maximal ideal of R then $\mathcal{B}(\alpha, P)$ is a maximal ideal of $Int(R)$ [24]. We also recall the following definition which is used in the proof of the next statement. An integral domain R is said to be essential if R is an intersection of valuation rings that are localizations of R. As this notion does not carry up to localizations, R is said to be locally essential if R_P is essential for each prime ideal P of R.

Note that in the following statement R is not assumed to be of finite character.

Proposition 3.8 Let R be a two dimensional Noetherian Krull domain whose divisor class group is torsion. Then each maximal ideal of $Int(R)$ is radically perfect implies that each prime ideal P of R is either radically perfect or is the radical of $ht(P) + 1$ elements.

Proof. Let P be a prime ideal of R. Then either P is of height one or it is of height two. If the height of P is one, then since the divisor class group of R is torsion, $P = rad(a)$ for some $a \in P$. Therefore P is radically perfect. If on the other hand, the height of P is two, then there exist a chain $(0) \subsetneq P_1 \subsetneq P$ of prime ideals of R of length three which give rise to a chain $(0) \subsetneq \mathcal{B}(\alpha, (0)) \subsetneq \mathcal{B}(\alpha, P_1) \subsetneq \mathcal{B}(\alpha, P)$ of prime ideals in $Int(R)$. From this it follows that $ht(\mathcal{B}(\alpha, P)) \geq 3$. Now since Krull domains are locally essential domains, it follows from Theorem 2.1 of [25] that $dim(IntR) = dimR[X]$. But then R being Noetherian gives that $dim(Int(R)) = dimR[X] = 3$, and so $\mathcal{B}(\alpha, P)$ is a maximal ideal of maximal height in $Int(R)$. Hence, by the hypothesis we have $\mathcal{B}(\alpha, P) = rad(f_1, f_2, f_3)$ for some $f_1, f_2, f_3 \in \mathcal{B}(\alpha, P)$ from which it follows that $(f_1(\alpha), f_2(\alpha), f_3(\alpha)) \subseteq P$. Let Q be any other prime ideal of R containing $(f_1(\alpha), f_2(\alpha), f_3(\alpha))$. Then in $Int(R)$, $\mathcal{B}(\alpha, Q)$ contains the ideal (f_1, f_2, f_3) and so it contains $\mathcal{B}(\alpha, P)$. Since $\mathcal{B}(\alpha, P)$ is maximal, we have $\mathcal{B}(\alpha, Q) = \mathcal{B}(\alpha, P)$, and so $P = Q$ (here we use the fact that $P \subseteq \mathcal{B}(\alpha, P)$ for any prime ideal P of R and that if Q is a prime ideal of R with $Q \subseteq \mathcal{B}(\alpha, P)$, then $Q \subseteq P$). Therefore it follows that $P = rad(f_1(\alpha), f_2(\alpha), f_3(\alpha))$, and hence P is the radical of an ideal generated by $ht(P) + 1$ elements. □

3.4 Radically Perfect Prime Ideals in Prüfer Domains of Finite Character

Here we show that each prime ideal of a Prüfer domain R or of the polynomial ring $R[X]$ over R being radically perfect implies that R is of dimension at most two and in some special cases is of dimension one.

Proposition 3.9 Let R be a finite dimensional Prüfer domain of finite character. Then each maximal ideal of R is radically perfect implies that R is of dimension at most two.

Proof. Since R is of finite character, for each maximal ideal M of R, $M = j - rad(u, v)$. Now R being Prüfer implies that $rad(u, v)$ is a prime ideal of R with $M = j - rad(rad(u, v))$. Hence $R/rad(u, v)$ is a valuation ring with maximal ideal $M/rad(u, v)$. If $R/rad(u, v)$ is a field, then $M = rad(u, v)$. If however $R/rad(u, v)$ is not a field, then it follows that $M/rad(u, v)$ is nonzero and in which case $M = rad(x)$ for some $x \in M - rad(u, v)$. From this it follows that R is of dimension at most two. □

The next result is on a special type of Prüfer domain whose definition is as follows: An integral domain R is a QR-domain if each overring of it is a quotient ring. Recall that a Prüfer domain is a QR-domain if and only if the radical of a finitely generated ideal is the radical of a principal ideal [26].

Proposition 3.10 Let R be a finite dimensional QR-domain of finite character. Then each maximal ideal of R is the radical of a principal ideal.

Proof. Let M be any maximal ideal of R. Since R is a QR-domain of finite character, it follows from the above proof that $M = j - rad(rad(u, v))$ for some $u, v \in M$ and as $rad(u, v) = rad(c)$ for some $c \in M$, $R/rad(c)$ is a valuation ring and from this it follows that M is the radical of a principal ideal. □

Note that without the assumption of finite character if each maximal ideal of a finite dimensional QR-domain is radically perfect then it is of dimension one.

In order to appreciate the content of the next statement we remark that if $R = K[Y, Z]$ where K is a field of characteristic zero and P^* is a prime ideal of $R[X]$ of height two with $P^* \cap R = P$ a height one prime ideal of R, then R/P is a normal domain of dimension one. If R/P is not normal, then not every prime ideal of $R/P[X]$ is radically perfect. As an example, take $P^* = (Y^2 - Z^3, X)$, then $P^* \cap R = P = (Y^2 - Z^3)$ is a prime ideal of height one in R and R/P is not normal. Therefore not every prime ideal of $R/P[X]$ is radically perfect (see Theorem 2.2). □

Proposition 3.11 Let R be a Prüfer domain of finite character. Suppose that each prime ideal of $R[X]$ is radically perfect. Then either R is of dimension one and each prime ideal of R is the radical of a principal ideal or R is of dimension two and for each height one prime ideal P of R, each prime ideal of $(R/P)[X]$ is radically perfect.

Proof. Let M be a maximal ideal of maximal height in R. Let P be a prime ideal of height one which is contained in M. Then as $PR[X]$ is radically perfect, $P = rad(c)$ for some $c \in P$. Since R is of finite character, c is contained in only finitely many maximal ideals and therefore R/P is a semilocal Prüfer domain and hence is Bézout. Hence by Corollary 2 of [20], each maximal ideal of R/P is the radical of a principal ideal, and so $M/P = rad(\bar{d})$ for some $\bar{d} \in M/P$, but then $M = rad(c, d)$ where d is the preimage of $\bar{d}$ in R, from which we obtain $MR[X] = rad(cR[X] + dR[X])$. Therefore $ht(M) = ht(MR[X]) \leq 2$ and hence $dim(R) \leq 2$. But then R/P is of dimension at most one. If R/P is a field then R is of dimension one and each prime ideal of R is the radical of a principal ideal. If on the other hand R/P is of dimension one then R is of dimension two and by Theorem 2.5 of [17], each prime ideal of $(R/P)[X]$ is radically perfect.

We failed, in our attempt to see whether or not height one prime ideals in the polynomial ring $R[X]$ over a Prüfer domain R contracting to zero in R are radically perfect. This primes are known to be invertible. This is a consequence of the fact that every Prüfer domain is a generalized GCD domain (i.e. an integral domain in which the intersection of two invertible ideals is invertible) and the primes of $R[X]$ contracting to zero in a generalized GCD domain R is invertible (see, Theorem 15 of [27]).

4. COPRIMELY PACKED IDEALS VERSUS RADICALLY PERFECT IDEALS

An ideal I of a ring R is coprimely packed by prime ideals of R if whenever I is coprime to each element of a family of prime ideals of R, I is not contained in the union of the primes in the family. We say that R is coprimely packed if every ideal of R is coprimely packed [28]. The following definition provides several conditions equivalent to this definition [13].

Definition 4.1 ([13], Lemma 2) A ring R is said to be coprimely packed if it satisfies one of the following equivalent conditions:
(i) If I is an ideal in R, and if S is a set of prime ideals in R with $I \subseteq \bigcup\{P \in S\}$, then $I + P \neq R$ for some $P \in S$.
(ii) If I is an ideal in R, and if S is a set of maximal ideals in R with $I \subseteq \bigcup\{M \in S\}$, then $I \subseteq M$ for some $M \in S$.
(iii) If I is an ideal in R, then $I \not\subset \bigcup\{M \mid \text{M is a maximal ideal with } I \not\subset M\}$.
(iv) If P is a prime ideal in R, and if S is a set of maximal ideals in R, with $P \subseteq \bigcup\{M \in S\}$, then $P \subseteq M$ for some $M \in S$.
(v) For each prime ideal P of R, there is an element $b \in P$ with $j{-}rad(Rb) = j{-}rad(P)$.

Proposition 4.1 Let R be an S-domain of dimension one. Then each prime ideal of $R[X]$ that does not contract to zero ideal of R is radically perfect if and only if $MaxSpecR$ is coprimely packed.

Proof. Let M be a maximal ideal of R. Then the prime ideal $MR[X]$ of $R[X]$ is radically perfect, and so $MR[X]$ is the radical of a constant c in $R[X]$, which clearly implies that $M = rad(c)$ in R. Therefore $MaxSpecR$ is coprimely packed.

For the converse let P^* be a prime ideal of $R[X]$ such that $P^* \cap R = P$ is a nonzero prime ideal in R. Since $MaxSpecR$ is coprimely packed, $P = rad(aR[X])$ for some

a in P. Now we either have $P^* = PR[X]$ or $PR[X]$ is strictly contained in P^*. If $P^* = PR[X]$, then $P^* = rad(a)$ in R[X]. If on the other hand P^* strictly contains $PR[X]$, then P^* is a maximal ideal of height two. Since $P*$ is maximal ideal of maximal height, $P^* = (M, f)$ in $R[X]$ for some monic polynomial $f \in P^*$ and therefore $P* = rad(a, f)$. Now use the argument of the proof of Proposition 3.4 to see that P^* can not be the radical of a principal ideal, and so is radically perfect. □

Proposition 4.2 Let R be a finite dimensional integral domain. If *MaxSpecR* is coprimely packed, then each maximal ideal M^* of $R[X]$ of maximal height is the j-radical of an ideal generated by two elements. Moreover, if in addition R is Hilbert, then the condition implies that each maximal ideal of $R[X]$ is the radical of an ideal generated by two elements.

Proof. Let M^* be a maximal ideal of $R[X]$ of maximal height. Then use the proof of Theorem 3.1 to see that $M^* \cap R = M$ is a maximal ideal of R. Hence $M^* = (M, f)$ for some monic polynomial $f \in M^*$. Since *MaxSpecR* is coprimely packed, $M = j - rad(m)$ for some $m \in M$ and hence $j - rad(m, f)$ is contained in M^*. If now Q^* is any maximal ideal of $R[X]$ containing the ideal (m, f), then we see that $Q^* \cap R = M$, and hence $Q^* = M^*$. Therefore, $M^* = j - rad(m, f)$. In the case of R being Hilbert, all we need to note that if M^* is any maximal ideal (not just a maximal ideal of maximal height) of $R[X]$, $M^* \cap R$ is a maximal ideal of R and that $j - rad(I) = rad(I)$ for any ideal I in R. □

Corollary 4.1 Let R be a finite dimensional Hilbert domain with coprimely packed set of maximal ideals. Then each maximal ideal of $R[X]$ is radically perfect if and only if the height of each maximal ideal of $R[X]$ is two.

Proof. Let M^* be a maximal ideal of $R[X]$. Then by the proof of Proposition 4.2, it follows that $M^* = rad(m, f)$ where m is a constant and f is a monic polynomial. Now using the proof of Proposition 3.4, we see that M^* can not be the radical of a principal ideal. Since M^* is radically perfect and is the radical of an ideal generated by two elements, it is of height two.

The converse follows from the proof of Proposition 4.2. □

Although the following result may have been a corollary to Theorem 2.3, we state it as new and give an alternative proof of it.

Proposition 4.3 Let R be a valuation ring of finite dimension. Then each prime ideal of R is radically perfect if and only if the same is so for the polynomial ring $R[X]$ over R.

Proof. Suppose that each prime ideal of R is radically perfect. Let M be the maximal ideal of R. Then as R is a valuation ring of finite dimension, say of dimension n, there is a chain of prime ideals $M = P_n \supseteq ... \supseteq P_0 = (0)$ in R. Let $a \in M = P_n - P_{n-1}$. Then clearly $M = rad(a)$. Since each prime ideal of R is radically perfect, and $M = rad(a)$, M is of height one and hence $dim(R) = 1$.

Let P^* be nonzero prime ideal of $R[X]$. Then either $P^* \cap R = (0)$ or $P^* \cap R = M$ is satisfied. If $P^* \cap R = (0)$ then it is clear that P^* is of height one and that $P^*K[X] = gK[X]$ in $K[X]$ where K is the field of fractions of R and $g \in P^*$. Since R is a valuation ring, there exists a coefficient c of g such that $g = cf$ for some polynomial f in P^*. If now A_f denotes the content ideal of f in R, then $A_f = R$, and hence $P^*K[X] = gK[X] = fK[X]$. Our aim is to show that in fact $P^* = (f)$ in $R[X]$. For this we take h to be any element of P^*, then $h = h_1 f$ for some polynomial $h_1 \in K[X]$. Now there is an element r in R such that rh_1 is in $R[X]$ and by Dedekind-Mertens Lemma there is a positive integer n such that $A_{rh}(A_f)^n = A_{rh_1}(A_f)^{n+1}$,and since $A_f = R$, we obtain $A_{rh_1 f} = A_{rh} = A_{rh_1}$ which implies that $A_h = A_{h_1}$ and so $h_1 \in R[X]$. Therefore, $P^* = (f)$ in $R[X]$ and so P^* is perfect (see Chapter 5).
If $P^* \cap R = M$, then either $P^* = MR[X]$ or $MR[X]$ is strictly contained in P^*. If $P^* = MR[X]$, then it follows that $P^* = rad(aR[X])$ in $R[X]$ for some a in M. If on the other hand P^* strictly contains $MR[X]$, then $P^* = (M, f)$ for some monic polynomial f in P^*, and so $P^* = rad(a, f)$. In order to conclude that P^* is radically perfect we we adapt the last part of the proof of Proposition 3.4 and see that P^* can not be the radical of a principal ideal in $R[X]$. Therefore in any case it follows that P^* is radically perfect.

Conversely, suppose that each prime ideal of $R[X]$ is radically perfect. Then use the proof of Theorem 2.3 to see that R is of dimension one and the maximal ideal M of R is radically perfect. □

Corollary 4.2 Let R be a valuation ring of finite dimension. Then each maximal ideal of $R[X]$ of maximal height is the j-radical of a single element.

This corollary is true in a more general manner as is shown in the following proposition.

Proposition 4.4 If R is a finite dimensional local integral domain, then each maximal ideal of maximal height of $R[X]$ is the j-radical of a principal ideal.

Proof. Let M^* be any maximal ideal of $R[X]$ of maximal height. Then $M^* \cap R$ is the maximal ideal M of R , and so $M^* = (M, f)$ for some monic polynomial $f \in M^*$. It follows from Proposition 3.2 that any maximal ideal N^* of $R[X]$ containing the monic polynomial f has to contain M and therefore $M^* = j - rad(f)$. □

Proposition 4.5 Let R be a finite dimensional valuation ring with maximal ideal M. If $MR[X]$ is radically perfect then $SpecR[X]$ is coprimely packed.

Proof. Following the proof of Proposition 4.3, we see that R is of dimension one and the maximal ideal M of R is the radical of a principal ideal.

Let P^* be any nonzero prime ideal of $R[X]$. Then either $P^* \cap R$ is zero, or is the maximal ideal M of R. If $P^* \cap R = (0)$, then as in the proof of Theorem 2.3, $P^* = f$ for some polynomial f in $R[X]$. But then it is clear that P^* is coprimely packed. If however $P^* \cap R = M$, then again using the proof of Proposition 4.3 we either have $P^* = MR[X]$ in which case $P^* = rad(c)$ in $R[X]$, or $P^* = (M, f)$ for some monic polynomial f. The case $P^* = rad(cR[X])$ clearly implies that P^* is coprimely packed. If on the other hand $P^* = (M, f)$, then by Proposition 4.4, $P^* = j - rad(f)$ and so is coprimely packed. □

From the above information it easily follows that if R is a one dimensional valuation ring, then $SpecR[X]$ is coprimely packed.

Theorem 4.1 For a finite dimensional QR-domain R the following statements are equivalent:

(1) For each maximal ideal M of R, the prime ideal $MR[X]$ of $R[X]$ is radically perfect.

(2) R is of dimension one and $MaxSpecR$ is coprimely packed.

(3) Each prime ideal of $R[X]$ is radically perfect.

Proof. (1)⇒(2). Let M be any maximal ideal of R and $ht(MR[X]) = ht(M) = n$. Then $MR[X] = rad(f_1, ..., f_n)$ for some $f_1, ..., f_n \in MR[X]$. But then it is easy to see that $MR[X] = rad(A_{f_i}, ..., A_{f_n})$ in $R[X]$ where A_{f_i} is the content ideal of f_i. Since $rad(A_{f_i}, ..., A_{f_n}) = rad(a)$ in R for some $a \in R$, it follows that $MR[X] = rad(a)$ in $R[X]$ and that $M = rad(a)$ in R. Thus $ht(MR[X]) = ht(M) = 1$. Hence R is of dimension one and *MaxSpecR* is coprimely packed.

(2)⇒(3). Let P^* be any prime ideal of $R[X]$ and $P^* \cap R = P$. Then we have three cases to consider:

Case 1. $ht(P^*) = 2$ in which case, $P \neq 0$ and so $P^* = (P, f) = rad(c, f)$ where $P = rad(c)$ and f is a monic polynomial in P^*. As in the proof of Proposition 3.4, P^* can not be written as the radical of a principal ideal and therefore is radically perfect.

Case 2. $ht(P^*) = 1$ and $P \neq 0$, in which case $P^* = PR[X] = rad(c)$ in $R[X]$ where $P = rad(c)$ in R, and so P^* is radically perfect.

Case 3. $ht(P^*) = 1$ and $P = 0$ in this case by Theorem 15 of [27], P^* is invertible, and therefore is finitely generated. Let $P^* = (f_1, ..., f_n)$ and A_{f_i} be the content ideal of f_i. Then using the proof of (1)⇒(2) above, we see that $P^* = rad(a)$ for some $a \in R$ and therefore is radically perfect.

(3)⇒(1). Is clear. □

Theorem 4.2 Let R be a Prüfer domain in which each maximal ideal is finitely generated and whose set of maximal ideals is coprimely packed. Then for each maximal ideal M of R, the prime ideal $MR[X]$ of $R[X]$ is radically perfect if and only if R is a Dedekind domain with torsion ideal class group.

Proof. By Theorem 2.4, R is of dimension one and so R is a Dedekind domain. But then the result follows from Theorem 2.1 of [17]. □

We conclude this chapter by noting that if R is an integral domain of dimension one in which each nonzero ideal is contained in finitely many maximal ideals, then the condition that *SpecR* is coprimely packed implies that *CopRad(R)* is torsion. The justification of this fact is the same as the one given in the proof of Theorem 3.3. We also note that, in general, the property that *MaxSpecR* is coprimely packed does not imply that each prime ideal of R is radically perfect, as an example take R to be a valuation ring of dimension greater than one. It is also not the case that each prime ideal of R being radically perfect implies that *MaxSpecR* is coprimely packed, as an example take R to be the polynomial ring $\mathbb{Z}[X]$ over the ring of integers $\mathbb{Z}$ (Theorem 1.3 of [29] and Theorem 2.1 [17]). However, if R is an integral domain of dimension one, then *MaxSpecR* is coprimely packed if and only if each prime ideal of R is radically perfect. Moreover, either of these conditions imply that $G(R)$, the subsemigroup of *CopRad(R)* generated by the set of invertible pairwise coprime radical ideals that are contained in finitely many maximal ideals, is torsion. Semilocal integral domains of dimension one are examples of such domains.

5. PERFECT IDEALS

This short chapter is the outcome of the obvious generalization of complete intersection of ideals to rings that may not be Noetherian. The analog generalization of radically perfectness in this case would be that an ideal I is perfect if the minimum number of generators of I is equal to the height of I. We note that with this definition each prime ideal of a finite dimensional ring being perfect implies that the ring is Noetherian, and so the notion of perfectness coincides with the notion of complete intersection. Of course, there are Noetherian rings whose prime ideals are not all perfect for as an example take R to be a Dedekind domain whose class group is not torsion.

Proposition 5.1 If R is a finite dimensional ring in which each prime ideal is perfect, then R is Noetherian.

Proof. Let P be a prime ideal of R of height n. Then since P is perfect, $P = (x_1, ..., x_n)$ and so R is Noetherian by Cohen's Theorem. □

Proposition 5.2 For an integral domain R of dimension one, the following statements are equivalent:

(i) Each maximal ideal of R is perfect.

(ii) R is a principal ideal domain.

(iii) R is a factorial ring.

(iv) The divisor class group of R is trivial.

(v) R is a Dedekind domain with trivial ideal class group.

Proof. (i)⇔(ii). Clear.

(ii)⇔(iii). Clear.

(iii)⇔(iv). Clear.

(iii)⇒(v). It is well known that in a factorial ring, the only ideals of R which are invertible as fractional ideals are the principal ideals, and hence the ideal class group of R is trivial.

(v)⇒(ii). Let R be a Dedekind domain with trivial ideal class group and I be a nonzero ideal of R. Then I is clearly invertible and since the class group is trivial, I is principal. Therefore, R is a principal ideal domain. □

Theorem 5.1 In an integral domain every height one prime ideal is perfect if and only if it is a factorial ring.

Proof. This follows from the fact that an integral domain R is a factorial ring if and only if every nonzero prime ideal in R contains a principal prime [18].

Proposition 5.3 Let R be one dimensional integral domain. If R satisfies any one of the conditions in Proposition 5.2, then the projective class group, $K_0(R)$ of R is isomorphic to the additive group of integers $\mathbb{Z}$.

Proof. Since the conditions in Proposition 5.2 are equivalent, one may suppose that R satisfies the second condition. Since any projective module over a principal ideal domain is free, a finitely generated projective R-module can be written as a finite direct sum of cyclic submodules isomorphic to R. Therefore, the projective class group $K_0(R)$ is generated by the symbols $[R \oplus R \oplus R \oplus ... \oplus R]$, and so for any finitely generated projective R-module P, the class represented by it is of the form $[P] = [R \oplus R \oplus R \oplus ... \oplus R] = n_P[R]$ where n_P is the number of copies of R in the direct sum. But then, it is clear that $K_0(R) \cong \mathbb{Z}$. Another way of seeing this is to use the well known fact that $K_0(R) = \mathbb{Z} \oplus Cl(R)$ and the fact that in a principal ideal domain $Cl(R) = (0)$ and obtain $K_0(R) \cong \mathbb{Z}$. □

Remark 5.1 In general $K_0(R)$ being isomorphic to $\mathbb{Z}$ does not imply that R is a principal ideal domain. It is well known that if R is a principal ideal domain, then all finitely generated projective $R[X_1, ..., X_n]$-modules are free (Serre's conjecture) and so it follows from the above proof of Proposition 5.3 that $K_0(R') \cong \mathbb{Z}$ where $R' = R[X_1, ..., X_n]$ and R' is not a principal ideal domain.

Proposition 5.4 In a Prüfer domain, any finitely generated nonzero prime ideal is maximal.

Proof. Let R be a Prüfer domain and P be a finitely generated nonzero prime ideal of R. Assume that P is not maximal, then there exists a maximal ideal M containing P. Let $P' = P + Rx$ for some $x \in M - P$ and since P' is finitely generated and R is a Prüfer domain, there exists an ideal I of R such that $P = P'I = (P + Rx)I$. It is clear that $P \neq I$ since otherwise $P = I$ implies that $P = P'P$ from which one obtains $P' = R$ (as P is invertible), a contradiction. Therefore $P \neq I$ and so there exists $i \in I - P$, and $r \in P' - P$ such that $r'i \in P'I = P$, which is not possible, and hence P is maximal. □

Proposition 5.5 Let R be a finite dimensional Prüfer domain. Then each nonzero minimal prime ideal of R is perfect if and only if it is a principal ideal domain.

Proof. Let P be a nonzero minimal prime ideal of R. Then since P is perfect, $P = (x)$ for some $x \in P$ and so P is maximal by the above proposition. Therefore R is of dimension one, and hence R is a principal ideal domain. Alternatively, one can argue as follows. Let M be a maximal ideal of R, then R_M is a valuation ring and its nonzero minimal prime ideal P_M is principal. Suppose that there is a height two prime ideal Q_M of R_M, then P_M is properly contained in Q_M and so there exist a $\bar{y} \in Q_M$ which is not in P_M. Since R_M is a valuation ring, $Rx \subset Ry$ and so $\bar{x} = \bar{a}\bar{y}$ for some $\bar{a} \in P_M$ but then $\bar{a} = \bar{b}\bar{x}$ for some $\bar{b} \in R_M$. Hence $\bar{x} = \bar{b}\bar{x}\bar{y}$ in R_M which implies $\bar{b}\bar{y} = 1$ in Q_M. This contradiction shows that for each maximal ideal M of R, the dimension of R_M is one. Therefore R is of dimension one and each nonzero minimal prime is maximal and so R is a principal ideal domain.

The converse is clear. □

Note that if R is a finite dimensional Prüfer domain whose prime ideals are perfect, then R satisfies all conditions of Proposition 5.2.

When the statement of Proposition 5.5 is weakened from all prime ideals to maximal ideals one obtains the following result with an extra condition imposed on R.

Proposition 5.6 Let R be a finite dimensional Prüfer domain whose set of maximal ideals is coprimely packed. Then each maximal ideal of R is perfect if and only if R is a principal ideal domain.

Proof. By Theorem 4.1, R is of dimension one and therefore is a principal ideal domain by Proposition 5.5. $\square$

Corollary 5.2 In a finite dimensional valuation ring, the maximal ideal is perfect if and only if it is a discrete valuation ring.

We remark that there are non-Noetherian valuation rings whose maximal ideal is principal that is not perfect. As an example, take the valuation ring $R = \bigcup_{k\geq 1} K[x, y/x^k]_{m_k}$ where K is a field and m_k is the maximal ideal $(x, y/x^k)$. The maximal ideal m of R is generated by x and the intersection $\bigcap_{n\geq 1} m^n$ is $p = (y, y/x, y/x^2, ..., y/x^i, ...)R$ for which there is no minimal system of generators. Also there are non-Noetherian non-local two dimensional Hilbert domains whose maximal ideals of height two are not perfect (see [30]).

6. CONCLUSIONS

In this thesis, we explored the newly defined notion of radically perfect ideals in commutative rings, and related this notion to the well established notions of Algebra such as class groups and the notion of being normal. For if R is an integral domain containing a field of characteristic zero, then each height one prime ideal of $R[X]$ is radically perfect implies that R is normal. This then evidently suggested that one should consider the notion of radically perfect ideals on rings that are normal and we did so, and examined it on Krull domains, factorial rings, Bézout domains, QR-domains and Prüfer domains, and in each of these cases it was shown that this newly defined notion is equivalent to a form of the class group of the underlying ring being torsion. We also observed that in many cases each prime ideal of the polynomial ring $R[X]$ over a commutative ring R being radically perfect implied that R is of dimension at most two, and when R contains a field of characteristic zero, we think this could be improved to R being of dimension at most one. If one can establish that this is always the case, it will then follow that not every height two ideal of the ring $K[X,Y,Z]$ is a set theoretic complete intersection (or radically perfect), and this would then completely resolve the long standing conjecture that each height two ideal of $K[X,Y,Z]$ is a set theoretic complete intersection where K is a field of characteristic zero.

As it is evident from the above information that the next obvious question one would like to answer is whether each prime ideal of $R[X]$ being radically perfect implies R is of Krull dimension at most one. This would be one of the question that we would like to answer in our future work.

Another related question that we would like to consider is to what extend an ideal being radically perfect is related to the vanishing of its local cohomology which is suggested by the well known fact that if I is an ideal of a ring R of height n, and if there exists some R-module M such that $H_I^i(M) \neq 0$ for $i > n$, then I is not a set theoretic complete intersection.

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