İSTANBUL TECHNICAL UNIVERSITY ★ INSTITUTE OF SCIENCE AND TECHNOLOGY

GROEBNER BASES AND TORIC VARIETIES

M.Sc.Thesis by
Bahriye KARACA

Department : Mathematical Engineering

Programme : Mathematical Engineering

AUGUST 2011

İSTANBUL TECHNICAL UNIVERSITY ★ INSTITUTE OF SCIENCE AND TECHNOLOGY

GROEBNER BASES AND TORIC VARIETIES

M.Sc.Thesis by
Bahriye KARACA
(509081002)

Date of submission : 06 May 2011
Date of the defence examination : 09 August 2011

Thesis Supervisor (Chairman) : Prof. Dr. Ulviye BAŞER (ITU)
Members of Examining Committee : Prof. Dr. Ayşe KARA (YTU)
Assoc. Prof. Dr. Şeyda Canan TEK

AUGUST 2011

İSTANBUL TEKNİK ÜNİVERSİTESİ ★ FEN BİLİMLERİ ENSTİTÜSÜ

GROEBNER BAZLARI VE TORSAL VARYETELER

YÜKSEK LİSANS TEZİ

Bahriye KARACA
(509081002)

Tezin Enstitüye Verildiği Tarih : 06 Mayıs 2011
Tezin Savunulduğu Tarih : 09 Ağustos 2011

Tez Danışmanı : Prof. Dr. Ulviye BAŞER (İTÜ)
Diğer Jüri Üyeleri : Prof. Dr. Ayşe KARA (YTÜ)
Yar. Doç. Dr. Şeyda Canan TEKİN (İT

AĞUSTOS 2011

ACKNOWLEDGEMENT

I would like to express my deep appreciation and thanks for my advisor Assoc. Prof. Dr. Meral Tosun, providing her continuous support and patience throught the studies. I also would like to thank her for her unconditional support in some stage of my life. I have met the algebraic geometry through her, she provided me the opportunity to work about one of excellent subject and I am so grateful to study these subjects.

I would like to thank my supervisor Prof. Dr. Ulviye Başer for her valuable comments and guidance.

I would like to thank my friends Özgür Polat and Burçin Ünal for their feedbacks and encouragement.

My sincere thanks go to Dr. Kürşat Aker for his help.

Thankfulness goes to my family for always being with me in good and bad times since beginning of my life. They have been a great source of support during all of the step of this thesis. Special thanks to my brothers Ceyhun Karaca and Hikmet Karaca for their supports.

Finally, I would like to express my sincere appreciation to my mother for her unconditional love and encouragement while I was performing this study. Her affection and tenderness helped me a lot.

May 2011 Bahriye KARACA
(Mathematician)

This work is supported by TUBITAK 1001 Project 109T667, during April, 2010-April, 2011.

TABLE OF CONTENTS

Page

LIST OF NOTATIONS

σ	:	The strongly convex, rational, polyhedral cone
$\sigma^{\vee}$	:	The dual cone
τ	:	The face of the cone
LM(f)	:	The leading monomial of the polynomial f
LT(f)	:	The leading term of the polynomial f
N	:	The lattice
M	:	The dual lattice
e_i	:	unit vector
Δ	:	The fan
T_N	:	Affine Algebraic Torus
S_σ	:	The semigroup

LIST OF FIGURES

Page

GROEBNER BASES and TORIC VARIETIES

SUMMARY

In this thesis, our aim is to understand two different constructions of a toric variety: starting from a rational polyhedral cone and starting from a matrix, also connections between the Groebner bases and toric varieties. The detailed information can be found in [1] and [2].

First, we start by a brief introduction and give some basic definitions that we will use. In the third chapter, the Groebner bases that provides algorithmic solutions to problems in Commutative Algebra and Algebraic Geometry are introduced. Also the Buchberger's Algorithm is examined that allow us to compute the Groebner basis.

In the fourth chapter, a toric variety associated to a cone in n-dimensional space is constructed. In the fifth chapter, we construct a toric variety from a given set of convex cones which is called a fan. We glue of the affine toric varieties associated to the cones which have the common face.

In the sixth chapter, a toric variety is constructed from a matrix and the connections between the toric varieties and Groebner bases are examined.

GROEBNER BAZLARI ve TORSAL VARYETELER

ÖZET

Bu tezdeki amacımız, torsal varyetelerin rasyonel polihedral koniden başlayarak oluşturulan ve bir matristen başlayarak oluşturulan iki farklı inşasını ve Groebner bazları ile torsal varyeteler arasındaki ilişkiyi anlamaya çalışmaktır. Bu konu ile ilgili ayrıntılı bilgiler [1] ve [2] kitaplarında bulunabilir.

İlk olarak kısa bir giriş ve kullanacağımız temel kavramların tanımları ile başlayacağız. Üçüncü bölümde, değişmeli cebir ve cebirsel geometride bazı problemlere çözüm sağlayan Groebner bazları tanıtılacaktır. Ayrıca verilen bir idealin Groebner bazını hesaplamamızı sağlayan Buchberger'in algoritması incelenecektir.

Dördüncü bölümde, n-boyutlu bir uzayda verilen belirli özelliklere sahip bir koniye karşılık gelen afin torsal varyete oluşturulacaktır. Beşinci bölümde, birden fazla konveks koninin birleşimi olan fandan torsal varyete oluşturulacaktır. Ortak yüzleri bulunan konilere karşılık gelen torsal varyeteleri yapıştıracağız.

Altıncı bölümde, bir matristen torsal varyetenin oluşturulması ve torsal varyeteler ve Groebner bazları arasındaki ilişki incelenecektir.

1. INTRODUCTION

The theory of Groebner bases is an example of how the solution to one problem became the key to solving a great variety of other problems. This major step was taken by Buchberger in 1965 who formulated the concept as a solution to the Ideal Membership Problem, following a suggestion of his advisor Groebner, found an algorithm to compute such a basis and proved the correctness and the termination of the algorithm. Toric varieties are algebraic varieties, that contain $(\mathbb{C}^*)^n$ as a dense open subset, together with an action of $(\mathbb{C}^*)^n$ on it. Toric varieties were first defined in the 1970s and have connections with algebraic geometry and combinatorial geometry. Basic references for toric varieties are [1], [2], [3]. These references give complete proofs of the results and descriptions about toric varieties. Also, the toric varieties are defined as the zero set of ideals which are generated by binomials. By the properties of these ideals, it is noticed the relationship of toric varieties with the matrices associate to toric ideals [3]. We write the toric ideal associated to given a matrix and then calculate the Groebner basis of the toric ideal.

2. PRELIMINARIES

A *ring* is a set R with two binary operations, $+$ and . satisfying the following axioms:

(i) R is an additive group with respect to $+$,

(ii) $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ for all $a, b, c \in R$

(iii) $a \cdot (b + c) = a \cdot b + a \cdot c$

$(b + c) \cdot a = b \cdot a + c \cdot a$ for all $a, b, c \in R$

We denote it by $(R, +, .)$. If R has an identity element with respect to the multiplication then R is called *ring with identity.* If $a \cdot b = b \cdot a$ for all $a, b \in R$ then we say that R is a *commutative ring.*

Example 2.1.[4]$(\mathbb{Z}, +, .)$ is a commutative ring. The integers satisfy under addition and multiplication the axioms above. Also $(\mathbb{Q}, +, .)$ and $(\mathbb{R}, +, .)$ are commutative rings.

A commutative ring with identity is called *field* if every nonzero element has a multiplicative inverse in the ring.

Example 2.2. $(\mathbb{Q}, +, .)$ and $(\mathbb{R}, +, .)$ are fields. Since every nonzero element has not a multiplicative inverse, $(\mathbb{Z}, +, .)$ is not a field.

A monomial in a collection of variables in $x_1, \ldots, x_n$ is a product

$$x_1^{\alpha_1} \ldots x_n^{\alpha_n} \qquad \textbf{(2.1)}$$

where the α_i are non-negative integers. The exponent of any monomial is a vector $\alpha = (\alpha_1, ..., \alpha_n)$ in $\mathbb{Z}^n_{\geq 0}$. So there is a one-to-one correspondence between the monomials in n variables and vectors in $\mathbb{Z}^n_{\geq 0}$. The *total degree* of a monomial $x_1^{\alpha_1} \ldots x_n^{\alpha_n}$ is the sum $\alpha_1 + \ldots + \alpha_n$. We will denote the total degree of the monomial $x_1^{\alpha_1} \ldots x_n^{\alpha_n}$ by $|\alpha|$.

Example 2.3. $x_1^3 x_2^2 x_4$ is a monomial in the variables x_1, x_2, x_3, x_4, its exponent is $\alpha = (3, 2, 0, 1)$ and its total degree $|\alpha| = 6$.

If k is any field, we can form finite linear combinations of monomials with coefficients in k. The resulting objects are the *polynomials* in $x_1, \dots, x_n$.
A polynomial in the variables $x_1, \dots, x_n$ with coefficient in k has the form

$$f = \sum_{\alpha} a_\alpha x^\alpha \tag{2.2}$$

with $a_\alpha \in k$. The set of all polynomials in $x_1, \dots, x_n$ with coefficients in k is denoted by $k[x_1, \dots, x_n]$. We will also use the word *term* on occasion to refer to a product of a non-zero element of k and a monomial appearing in a polynomial. The sum and product of two polynomials is again a polynomial. Since under addition and multiplication, $k[x_1, \dots, x_n]$ satisfies the axioms above, it is called *polynomial ring.*
A *binary relation* on a set P is a subset of the Cartesian product $P \times P := \{(x, y) : x \in P \text{ and } y \in P\}$. An *ordered set* (or *partially ordered set*) is an ordered pair $(P, \leq)$ of a set P and a binary relation contained in $P \times P$, such that
(i) The relation $\leq$ is reflexive:

$$p \leq p \ \ \forall p \in P. \tag{2.3}$$

(ii) The relation $\leq$ is antisymmetric:

$$p \leq q \ \ and \ \ q \leq p \Rightarrow p = q \ \ \forall p, q \in P. \tag{2.4}$$

(iii) The relation $\leq$ is transitive:

$$p \leq q \ \ and \ \ q \leq r \Rightarrow p \leq r \ \ \forall p, q, r \in P \tag{2.5}$$

Natural numbers $\mathbb{N}$, Integers $\mathbb{Z}$, Rational numbers $\mathbb{Q}$ are all ordered sets with their usual order.
A *totally ordered set* (or *linearly ordered set*) has the property of comparability:

$$\forall p, q \in P, \ \ either \ \ p \leq q \ \ or \ \ q \leq p \tag{2.6}$$

in addition to partial order axioms. *Well ordering* on P is a total ordering on P such that every nonempty subset of P has a smallest element, i.e. for each nonempty $A \subseteq P$ there is some $s \in A$ such that $s \leq b$, for all $b \in A$.

Example 2.4. The set of real numbers ordered by the usual less than and equal

$\leq$ or greater than and equal $\geq$ relations is totally ordered.

Let R be a commutative ring with identity. The set of all prime ideals in the ring R is called *spectrum* of the ring and denoted by $Spec(R)$.

Example 2.5.[5] $Spec(\mathbb{Z}) = \{\langle 0\rangle\} \cup \{\langle p\rangle : p \text{ is prime }\}$.

The prime ideals in the one-variable polynomial ring $\mathbb{C}[x]$ are maximal ideals and they are of the form of $\langle x - a\rangle$. These maximal ideals correspond to the points in $\mathbb{C}$ so $Spec(\mathbb{C}[x]) = \mathbb{C}$.

Consider the polynomial ring $\mathbb{C}[x, y]$. For $a, b \in \mathbb{C}$, the maximal ideals in $\mathbb{C}[x, y]$ are of the form of $\langle x - a, y - b\rangle$. Also, the ideals generated by the irreducible polynomials $f(x, y) \in \mathbb{C}[x, y]$ and the ideal $\langle 0\rangle$ are prime ideals. The maximal ideals correspond to irreducible curves in $\mathbb{C}^2$. So $Spec(\mathbb{C}[x, y]) = \mathbb{C}^2$.

Remark 2.6.[6]In general, $Spec(\mathbb{C}[x_1, \ldots, x_n]) = \mathbb{C}^n$.

3. GROEBNER BASES

Finding Groebner basis is a technique that provides algorithmic solutions to a variety of problems in Commutative Algebra and Algebraic Geometry.

In one variable case, we first order the monomials in a polynomial in descending order with respect to their degree in x. But in several variables, there are many different ways to order the monomials. So, we will first introduce monomial orderings. Every pair of monomials $x_1^{\alpha_1}\dots x_n^{\alpha_n}$ and $x_1^{\beta_1}\dots x_n^{\beta_n}$ must be comparable, i.e. one of the followings must be held

- $x_1^{\alpha_1}\dots x_n^{\alpha_n} > x_1^{\beta_1}\dots x_n^{\beta_n}$
- $x_1^{\alpha_1}\dots x_n^{\alpha_n} = x_1^{\beta_1}\dots x_n^{\beta_n}$
- $x_1^{\alpha_1}\dots x_n^{\alpha_n} < x_1^{\beta_1}\dots x_n^{\beta_n}$

Next, the effect of multiplying and adding polynomials must be taken into account. Adding polynomials, after combining like terms, it may simply be rearranged into the appropriate order, so sums present no difficulties. We require the following property:

If $x_1^{\alpha_1}\dots x_n^{\alpha_n} > x_1^{\beta_1}\dots x_n^{\beta_n}$ and $x_1^{\gamma_1}\dots x_n^{\gamma_n}$ is any monomial, then

$$(x_1^{\alpha_1}\dots x_n^{\alpha_n})\cdot(x_1^{\gamma_1}\dots x_n^{\gamma_n}) > (x_1^{\beta_1}\dots x_n^{\beta_n})\cdot(x_1^{\gamma_1}\dots x_n^{\gamma_n}) \qquad \mathbf{(3.1)}$$

must be satisfied.

The formal definition of monomial ordering is:

Definition 3.1. [7] Let $x_1^{\alpha_1}\dots x_n^{\alpha_n} \in k[x_1,\dots,x_n]$ be any monomial with $\alpha = (\alpha_1,\dots,\alpha_n) \in \mathbb{Z}^n_{\geq 0}$. If a relation $>$ on the set of monomials in $k[x_1,\dots x_n]$ satisfies the following conditions, $>$ is called a **monomial ordering**.

(i) $>$ is a total ordering.

(ii) If $x_1^{\alpha_1}\dots x_n^{\alpha_n} > x_1^{\beta_1}\dots x_n^{\beta_n}$, then for any monomial $x_1^{\gamma_1}\dots x_n^{\gamma_n}$,

$$(x_1^{\alpha_1}\dots x_n^{\alpha_n})\cdot(x_1^{\gamma_1}\dots x_n^{\gamma_n}) = (x_1^{\alpha_1+\gamma_1}\dots x_n^{\alpha_n+\gamma_n}) > (x_1^{\beta_1+\gamma_1}\dots x_n^{\beta_n+\gamma_n})$$

(iii) $>$ is a well-ordering.

Example 3.2. Let $u = (u_1, \ldots, u_n)$ be a vector in $\mathbb{R}^n$ such that $u_1, \ldots, u_n$ are positives and $\mathbb{Q}$-linearly independent. For $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$, put $u \cdot \alpha = u_1\alpha_1 + \ldots + u_n\alpha_n$. Show that for $\alpha, \beta \in \mathbb{N}^n$,

$$x_1^{\alpha_1} \ldots x_n^{\alpha_n} <_u x_1^{\beta_1} \ldots x_n^{\beta_n} \Leftrightarrow (\alpha_1, \ldots, \alpha_n) <_u (\beta_1, \ldots, \beta_n) \qquad \mathbf{(3.2)}$$

$<_u$ is a monomial ordering.

Solution. (i) For all $\alpha, \beta \in \mathbb{N}^n$ and $\alpha \neq \beta \neq 0$, we have $u \cdot \alpha < u \cdot \beta$, $u \cdot \beta < u \cdot \alpha$ or $u \cdot \alpha = u \cdot \beta$

If $u \cdot \alpha < u \cdot \beta$ i.e. $u_1\alpha_1 + \ldots + u_n\alpha_n < u_1\beta_1 + \ldots + u_n\beta_n \Leftrightarrow$
$(\alpha_1, \ldots, \alpha_n) <_u (\beta_1, \ldots, \beta_n) \Leftrightarrow x_1^{\alpha_1} \ldots x_n^{\alpha_n} <_u x_1^{\beta_1} \ldots x_n^{\beta_n}$

If $u \cdot \beta < u \cdot \alpha$ i.e. $u_1\beta_1 + \ldots + u_n < u_1\alpha_1 + \ldots + u_n\alpha_n \Leftrightarrow$
$(\beta_1, \ldots, \beta_n) <_u (\alpha_1, \ldots, \alpha_n) \Leftrightarrow x_1^{\beta_1} \ldots x_n^{\beta_n} <_u x_1^{\alpha_1} \ldots x_n^{\alpha_n}$

If $u \cdot \alpha = u \cdot \beta$ i.e. $u_1\alpha_1 + \ldots + u_n\alpha_n = u_1\beta_1 + \ldots + u_n\beta_n$
$\Rightarrow u_1(\alpha_1 - \beta_1) + \ldots + u_n(\alpha_n - \beta_n) = 0$

Since $u = (u_1, \ldots, u_n)$ are linearly independent $\alpha_1 - \beta_1 = \ldots = \alpha_n - \beta_n = 0 \Rightarrow \alpha_1 = \beta_1, \ldots, \alpha_n = \beta_n \Rightarrow u \cdot \alpha = u \cdot \alpha \Rightarrow (\alpha_1, \ldots, \alpha_n) <_u (\alpha_1, \ldots, \alpha_n) \Leftrightarrow x_1^{\alpha_1} \ldots x_n^{\alpha_n} <_u x_1^{\alpha_1} \ldots x_n^{\alpha_n}$

(ii) Let $u \cdot \alpha < u \cdot \beta$ and $\gamma = (\gamma_1, \ldots, \gamma_n) \in \mathbb{N}^n$, since $\gamma \in \mathbb{N}^n$ and $u = (u_1, \ldots, u_n)$ are positives, we can add $u \cdot \gamma = u_1\gamma_1 + \ldots + u_n\gamma_n$ to both side of the inequality. Then

$u \cdot \alpha + u \cdot \gamma < u \cdot \beta + u \cdot \gamma \Rightarrow u \cdot (\alpha + \gamma) < u \cdot (\beta + \gamma)$
$\Leftrightarrow (\alpha_1 + \gamma_1, \ldots, \alpha_n + \gamma_n) <_u (\beta_1 + \gamma_1, \ldots, \beta_n + \gamma_n)$
$\Rightarrow (\alpha_1, \ldots, \alpha_n) + (\gamma_1, \ldots, \gamma_n) <_u (\beta_1, \ldots, \beta_n) + (\gamma_1, \ldots, \gamma_n)$
$\Leftrightarrow (x_1^{\alpha_1} \ldots x_n^{\alpha_n}) \cdot (x_1^{\gamma_1} \ldots x_n^{\gamma_n}) = (x_1^{\alpha_1+\gamma_1} \ldots x_n^{\alpha_n+\gamma_n}) <_u (x_1^{\beta_1} \ldots x_n^{\beta_n}) \cdot (x_1^{\gamma_1} \ldots x_n^{\gamma_n})$
$= (x_1^{\beta_1+\gamma_1} \ldots x_n^{\beta_n+\gamma_n})$

3.1. Examples Of Monomial Orderings

Here we will present 3 different monomial orderings. There are some other monomial orderings.

3.1.1. Lexicographic order

Let $x_1^{\alpha_1} \dots x_n^{\alpha_n}$ and $x_1^{\beta_1} \dots x_n^{\beta_n}$ be any two monomials. If in the vector $(\alpha_1 - \beta_1, \dots, \alpha_n - \beta_n) \in \mathbb{Z}^n_{\geq 0}$, the first nonzero entry is positive, we say that $x_1^{\alpha_1} \dots x_n^{\alpha_n}$ is larger than $x_1^{\beta_1} \dots x_n^{\beta_n}$ with respect to lexicographic order and we write $x_1^{\alpha_1} \dots x_n^{\alpha_n} >_{lex} x_1^{\beta_1} \dots x_n^{\beta_n}$.

Example 3.3. Let $x_1^2 x_2^3 x_3^4$ and $x_1 x_2^5 x_3^3$ be any two monomials. Since $\alpha - \beta = (2,3,4) - (1,5,3) = (1,-2,1)$ $x_1^2 x_2^3 x_3^4 >_{lex} x_1 x_2^5 x_3^3$.
One thing we might not like about lex order is that it doesn't respect degrees (e.g. $xy > y^3 z^4$).

Example 3.4. Considering the *lex* order for $\mathbb{R}[x_1, x_2, x_3]$ with $x_1 > x_2 > x_3$, we have

$$x_1^4 > x_1^3 x_2^2 x_3 > x_1^3 x_2 x_2^7 > x_1^3 x_2 x_3^4 > x_1^2 x_2 x_3^5 > x_1 x_2^3 x_3^2 > x_1 x_2 > x_1 x_3^2 > x_1 > x_2^6$$

3.1.2. Graded lexicographic order

Let $x_1^{\alpha_1} \dots x_n^{\alpha_n}$ and $x_1^{\beta_1} \dots x_n^{\beta_n}$ be any two monomials. If $|\alpha| = \sum_{i=1}^n \alpha_i > |\beta| = \sum_{i=1}^n \beta_i$, or $|\alpha| = |\beta|$ and $x_1^{\alpha_1} \dots x_n^{\alpha_n} >_{lex} x_1^{\beta_1} \dots x_n^{\beta_n}$, $x_1^{\alpha_1} \dots x_n^{\alpha_n}$ is larger than $x_1^{\beta_1} \dots x_n^{\beta_n}$ with respect to graded lex order and we write $x_1^{\alpha_1} \dots x_n^{\alpha_n} >_{grlex} x_1^{\beta_1} \dots x_n^{\beta_n}$.

Example 3.5. Let $x_1^3 x_2^3 x_3$ and $x_1^2 x_2^2 x_3^3$ be any two monomials. $\alpha = (3,3,1)$ and $\beta = (2,2,3)$. Since $|\alpha| = |3+3+1| = |\beta| = |2+2+3|$ and $x_1^3 x_2^3 x_3 >_{lex} x_1^2 x_2^2 x_3^3$ we say that $x_1^3 x_2^3 x_3 >_{grlex} x_1^2 x_2^2 x_3^3$.

3.1.3. Graded reverse lex order

Let $x_1^{\alpha_1} \dots x_n^{\alpha_n}$ and $x_1^{\beta_1} \dots x_n^{\beta_n}$ be any two monomials. If $|\alpha| = \sum_{i=1}^n \alpha_i > |\beta| = \sum_{i=1}^n \beta_i$, or in the vector $(\alpha_1 - \beta_1, \dots, \alpha_n - \beta_n) \in \mathbb{Z}^n_{\geq 0}$, the right-most nonzero entry is negative, $x_1^{\alpha_1}\dots x_n^{\alpha_n}$ is larger than $x_1^{\beta_1}\dots x_n^{\beta_n}$ with respect to graded reverse lex order and we write $x_1^{\alpha_1}\dots x_n^{\alpha_n} >_{grevlex} x_1^{\beta_1}\dots x_n^{\beta_n}$.

Graded reverse lexicographic order uses degree first, and uses "reverse lexicographic order" to break ties.

Example 3.6. Let $x_1^2x_2^3x_3$ and $x_1x_2^2x_3^2$ be any two monomials. $\alpha = (2,3,1)$ and $\beta = (1,2,2)$. Since $|\alpha| = |2+3+1| > |\beta| = |1+2+2|$, then $x_1^2x_2^3x_3 >_{grevlex} x_1x_2^2x_3^2$.

Example 3.7. For any two monomials $x_1x_2^3x_3^4$ and $x_1^4x_2^2x_3^2, \alpha = (1,3,4)$ and $\beta = (4,2,2)$, $|\alpha| = |1+3+4|, |\beta| = |4+2+2|$ and $(1,3,4)-(4,2,2) = (-3,1,2)$ then $x_1^4x_2^2x_3^2 >_{grevlex} x_1x_2^3x_3^4$.

Note that in the division of $h \in k[x]$ by a $g \in k[x]$, the remainder is unique. This fails with several variables. When we divide $f \in k[x_1,\dots,x_n]$ by $f_1,\dots,f_s \in k[x_1,\dots,x_n]$, the remainder may not be unique:

Example 3.8. Let $f = x^2y^2 + 3xy + y^2 - y \in \mathbb{R}[x,y]$.

If we divide f by $f_1 = xy + 3$ and $f_2 = y - 1$ using lex order with $x > y$ we get

$$f = xy.(xy+3) + y.(y-1) \tag{3.3}$$

where the remainder is zero.

Let us change the order of division. We divide first by f_2 then by f_1 with respect to lex order and we get

$$f = (x^2y + x^2 + 3x + y).(y-1) + 0.(xy+3) + x^2 + 3x \tag{3.4}$$

where the remainder is $x^2 + 3x$.

Let $I \subset k[x]$ be an ideal with $I = \langle g \rangle$ for some $g \in k[x]$. Given $f \in k[x]$, $f \in I$ if and only if the remainder is zero. In other words, we can solve the ideal membership problem for one variable ones by that way. But it doesn't solve the ideal membership problem for polynomials in several variables completely.

Now $I = \langle f_1,\dots,f_s \rangle \subset k[x_1,\dots,x_n]$, any $g \in k[x_1,\dots,x_n]$ can be written as

$$g = p_1f_1 + \dots + p_sf_s + r \tag{3.5}$$

where $p_1,\dots,p_s, r \in k[x_1,\dots,x_n]$.

We have $r = 0$ if $g \in I$. Thus $r = 0$ is a sufficient condition however, as the following example shows, $r = 0$ is not a necessary condition for being in the ideal.

Example 3.9. Let $I = \langle xy - 1, x - z \rangle \subset \mathbb{C}[x, y, z]$ and $g = xy^2z + xy + x - yz - z - 1$. If we divide g by $f_1 = xy - 1$ and $f_2 = x - z$ with respect to $z < y < x$ and

$$g = (yz + 1).f_1 + 1.f_2 \tag{3.6}$$

with $r = 0$, we have $g \in I$.

On the other hand, if we divide g first by $f_2 = x - z$ then by $f_1 = xy - 1$, then g can be written as

$$g = (y^2z + y + 1).f_2 + 0.f_1 + y^2z^2 - 1 \tag{3.7}$$

where $y^2z^2 - 1$ is the remainder and not equal to zero.

This example shows that even if $g \in \langle f_1, f_2 \rangle$, it is possible to obtain a nonzero remainder on division by (f_1, f_2).

These problems such that the remainder r is not uniquely determined and $r = 0$ is not a necessary condition for being in the ideal occur because of choosing any arbitrary basis for the ideal I. In dealing with a collection of polynomials $f_1, \ldots, f_s \in k[x_1, \ldots, x_n]$, it is frequently desirable to pass to the ideal I they generate. This allows the possibility of going from $f_1, \ldots, f_s$ to a different generating set for I. For such a generating set, we would want the remainder on division by this generating set to be uniquely determined and that condition $r = 0$ should be necessary condition for being in the ideal.

Hence, we will introduce a new generating set for the ideal I called Groebner bases which first introduced by H.Hironaka (circa 1964 whilst) in Harvard working in the field of algebraic geometry where he called standard bases.

Definition 3.10. Let $f = \sum_\alpha a_\alpha x^\alpha$ be a nonzero polynomial in $k[x_1, \ldots, x_n]$ and let $>$ be a monomial order. The **multidegree** of f with respect to $>$ is

$$multideg(f) = max(\alpha \in \mathbb{Z}^n_{\geq 0} : a_\alpha \neq 0) \tag{3.8}$$

The **leading monomial** of f, $LM(f)$ is the largest of the monomials occurring in the expression of f with respect to $>$. The **leading term** of f, $LT(f)$ is the largest of the monomial terms in f with respect to $>$.

Example 3.11. Let $f = 2x^3y^2z + 3y^2z^5 + 3x^3y^4z - xy^5 - 2y^7z + x^2y^6$
With respect to lex order, we get $f = 3x^3y^4z + 2x^3y^2z + x^2y^6 - xy^5 - 2y^7z + 3y^2z^5$ and $LT(f) = 3x^3y^4z,\ LM(f) = x^3y^4z$
With respect to grlex order, we get $f = 3x^3y^4z + x^2y^6 - 2y^7z + 3y^2z^5 + 2x^3y^2z - xy^5$ and $LT(f) = 3x^3y^4z,\ LM(f) = x^3y^4z$
With respect to grevlex order, we get $f = x^2y^6 + 3x^3y^4z - 2y^7z + 3y^2z^5 - xy^5 + 2x^3y^2z$ and $LT(f) = x^2y^6,\ LM(f) = x^2y^6$

Definition 3.12. Let $f,\ g \in k[x_1, \ldots, x_n]$ be nonzero polynomials. Let $multideg(f) = (\alpha_1, \ldots, \alpha_n)$ and $multideg(g) = (\beta_1, \ldots, \beta_n)$ and $\gamma_i = max(\alpha_i, \beta_i)$ for each i. We call $x_1^{\gamma_1} \ldots x_n^{\gamma_n}$ **the least common multiple** of $LM(f)$ and $LM(g)$, written $x_1^{\gamma_1} \ldots x_n^{\gamma_n} = LCM(LM(f), LM(g))$.

Example 3.13. Let I be a nonzero ideal in $k[x_1, \ldots, x_n]$. Fix a monomial ordering.

$$LT(I) = \{LT(f) \mid for\ \ all\ f \in I\} \tag{3.9}$$

The ideal generated by the elements of $LT(I)$ is $\langle LT(I) \rangle$. Note that for any ideal $I = \langle f_1, \ldots, f_s \rangle$, the ideals $\langle LT(f_1), \ldots LT(f_s) \rangle$ and $\langle LT(I) \rangle$ may be different ideals. By definition, $LT(f_i) \in LT(I)$ for each $i = 1, \ldots, s$ and $\langle LT(f_1), \ldots LT(f_s) \rangle \subset \langle LT(I) \rangle$.

Example 3.14. Let $I = \langle f_1, f_2, f_3 \rangle \subset k[x, y]$ be an ideal with $f_1 = x^2y - 2x$, $f_2 = x^3 - y$, $f_3 = xy^3 - y$. We obtain

$$\langle LT(f_1), LT(f_2), LT(f_3) \rangle = \left\langle x^2y, x^3, xy^3 \right\rangle \tag{3.10}$$

It is clear that $\langle LT(f_1), LT(f_2), LT(f_3) \rangle \subset \langle LT(I) \rangle$ since $f_1, f_2, f_3 \in I$ with respect to lex order. However, if we consider the polynomial

$$f = -x.(x^2y - 2x) + y.(x^3 - y) = 2x^2 - y^2 \tag{3.11}$$

$f \in I$ and $f \in \langle LT(I) \rangle$ but $LT(2x^2 - y^2) = x^2 \notin \langle LT(f_1), LT(f_2), LT(f_3) \rangle$. Because x^2 is not divisible by $LT(f_1), LT(f_2)$ or $LT(f_3)$.
So $\langle LT(f_1), LT(f_2), LT(f_3) \rangle \subset \langle LT(I) \rangle$.
The equality holds for a special set of generators for I.

Definition 3.15. Let $I \subset k[x_1, \ldots, x_n]$ be an ideal. Let $>$ be a monomial ordering and $G = \{g_1, \ldots, g_s\} \subset I$ be a finite subset. If the leading term of every nonzero polynomial $f \in I$ is divisible by at least one of the $LT(g_i)$, then G is called a **Groebner basis** of I.

Definition 3.16. [8] Let $f, g \in k[x_1, \ldots, x_n]$. Let $LT(f) = ax_1^{\alpha_1} \ldots x_n^{\alpha_n}$ and $LT(f) = bx_1^{\beta_1} \ldots x_n^{\beta_n}$ with respect to a monomial ordering($a, b \in k^*$). Let $x_1^{\gamma_1} \ldots x_n^{\gamma_n} = LCM(x_1^{\alpha_1} \ldots x_n^{\alpha_n}, x_1^{\beta_1} \ldots x_n^{\beta_n})$. The **S-polynomial** of f and g is defined as

$$S(f, g) = \frac{x_1^{\gamma_1} \ldots x_n^{\gamma_n}}{LT(f)} \cdot f - \frac{x_1^{\gamma_1} \ldots x_n^{\gamma_n}}{LT(g)} \cdot g \tag{3.12}$$

Since $\frac{x_1^{\gamma_1} \ldots x_n^{\gamma_n}}{LT(f)}$ and $\frac{x_1^{\gamma_1} \ldots x_n^{\gamma_n}}{LT(g)}$ are monomials, then the $S(f, g)$ is a linear combination with polynomial coefficient of f and g, belongs to the same ideal generated by f and g.

Example 3.17. Let $F = \{f_1, f_2\}$, where $f_1 = xy^2z - xyz$ and $f_2 = x^2yz - z^2$. These polynomials are ordered with respect to lex order. $LM(f_1) = xy^2z$ and $LM(f_2) = x^2yz$. Let $LCM(LM(f_1), LM(f_2) = x^2y^2z$. Then

$$S(f_1, f_2) = \frac{x^2y^2z}{xy^2z} \cdot f_1 - \frac{x^2y^2z}{x^2yz} \cdot f_2 \tag{3.13}$$

$$= x \cdot f_1 - y \cdot f_2 \tag{3.14}$$

$$= -x^2yz + yz^2 \tag{3.15}$$

Example 3.18. Let $F = \{f_1, f_2\}$, where $f_1 = xy + z^3$ and $f_2 = z^2 - 3z$ ordered with respect to lex order. $LM(f_1) = xy$ and $LM(f_2) = z^2$. Then $LCM(LM(f_1), LM(f_2) = xyz^2$. So

$$S(f_1, f_2) = \frac{xyz^2}{xy} \cdot f_1 - \frac{xyz^2}{z^2} \cdot f_2 \tag{3.16}$$

$$= -3xyz + z^5 \tag{3.17}$$

The following theorem allows us to form an algorithm to compute the Groebner basis.

3.2. Buchberger's Algorithm

Theorem 3.19. (Buchberger Criterion) Let us consider the subset $G =$

$\{g_1, \ldots, g_s\}$ of an ideal $I \subset k[x_1, \ldots, x_n]$. The set G is a Groebner basis for I if and only if for all pairs g_i and g_j with $i \neq j$, $(i, j = 1, \ldots, s)$ the remainder of division of $S(g_i, g_j)$ by G is zero.

Algorithm 3.20. (Buchberger's Algorithm)
Let $G = \{g_1, \ldots, g_s\}$ be a generating set for $I \subset k[x_1, \ldots, x_n]$.
Step 1: Calculate the polynomial $S(g_1, g_2)$.
Step 2: See whether $S(g_1, g_2)$ is divided by each $g_1, \ldots, g_s$.

- If the remainder $r \neq 0$, then r will be added to the set G.
- If the remainder $r = 0$, then the set G is a Groebner basis for I.

Step 3: Repeat the algorithm for each g_i and g_j for $i \neq j$ and $i, j = 1, \ldots, s$.

Example 3.21. Let $G = \{x - 2y, x^2 - 1\}$ be a generating set for $I \subset \mathbb{C}[x, y]$ with respect to lex order $x > y$. We have $LM(f_1) = x$ and $LM(f_2) = x^2$ This gives $LCM(x, x^2) = x^2$. We obtain

$$S(f_1, f_2) = \frac{x^2}{x} \cdot (x - 2y) - \frac{x^2}{x^2} \cdot (x^2 - 1) \tag{3.18}$$

$$= 2xy - 1 \tag{3.19}$$

Let us divide $S(f_1, f_2)$ by f_1 and f_2. We get

$$2xy - 1 = 2y.(x - 2y) + 0.(x^2 - 1) + 4y^2 - 1 \tag{3.20}$$

Since $r = 4y^2 - 1 \neq 0$, $\{f_1, f_2\}$ is not a Groebner basis for I. By the algorithm we add a new generator $f_3 = 4y^2 - 1$ to the set G. Put $G_1 = \{f_1, f_2, f_3\}$. Consider $LM(f_1) = x$ and $LM(f_3) = y^2$ then $LCM(x, y^2) = xy^2$. So

$$S(f_1, f_3) = \frac{xy^2}{x} \cdot (x - 2y) - \frac{xy^2}{4y^2} \cdot (4y^2 - 1) \tag{3.21}$$

$$= x - 8y^3 \tag{3.22}$$

Dividing $S(f_1, f_3)$ by f_1, f_2 and f_3, we obtain

$$x - 8y^3 = 1.(x - 2y) + 0.(x^2 - 1) + 2y.(4y^2 - 1) \tag{3.23}$$

so $r = 0$. $LM(f_2) = x^2$ and $LM(f_3) = y^2$ consider $LCM(x^2, y^2) = x^2y^2$, then

$$S(f_2, f_3) = \frac{x^2y^2}{x^2} \cdot (x^2 - 1) - \frac{x^2y^2}{4y^2} \cdot (4y^2 - 1) \tag{3.24}$$

$$= x^2 - 4y^2 \tag{3.25}$$

If we divide $S(f_2, f_3)$ by f_1, f_2 and f_3, we get

$$x^2 - 4y^2 = 0.(x - 2y) + 1.(x^2 - 1) + 1.(4y^2 - 1) \tag{3.26}$$

We check for each pair (f_i, f_j) and we obtain $r = 0$ in each case. So the set $\{f_1, f_2, f_3\}$ is a Groebner basis for I.

Example 3.22. Let $F = \{x-1, x^2-y, z-1\}$ be a generating set for $I \subset \mathbb{C}[x, y, z]$ with respect to lex order $x > y > z$. The leading monomial of $f_1 = x - 1$ is x, the leading monomial of $f_2 = x^2 - y$ then $LCM(x, x^2) = x^2$. Therefore

$$S(f_1, f_2) = \frac{x^2}{x} \cdot (x - 1) - \frac{x^2}{x^2} \cdot (x^2 - y) \tag{3.27}$$

$$= x - y \tag{3.28}$$

If we divide $S(f_1, f_2)$ by f_1, f_2 and f_3, we get

$$x - y = 1.(x - 1) + 0.(x^2 - y) + 0.(z - 1) + y - 1 \tag{3.29}$$

$y - 1 \neq 0$, hence $\{f_1, f_2, f_3\}$ is not a Groebner basis for I. We add that remainder to F as a new generator $f_4 = y - 1$. Put $F_1 = \{f_1, f_2, f_3, f_4\}$. $LM(f_1) = x$ and $LM(f_3) = z$ then $LCM(x, z) = xz$. So

$$S(f_1, f_3) = \frac{xz}{x} \cdot (x - 1) - \frac{xz}{z} \cdot (z - 1) \tag{3.30}$$

$$= x - z \tag{3.31}$$

Dividing $S(f_1, f_3)$ by f_1, f_2, f_3 and f_4, we obtain

$$x - z = 1.(x - 1) + 0.(x^2 - y) + 1.(z - 1) + 0.(y - 1) \tag{3.32}$$

The remainder is zero, so there is nothing to add to the set F. Consider $LM(f_1) = x$, $LM(y) = y$ and $LCM(x, y) = xy$. Then

$$S(f_1, f_4) = \frac{xy}{x} \cdot (x - 1) - \frac{xy}{y} \cdot (y - 1) \tag{3.33}$$

$$= x - y \tag{3.34}$$

Dividing $S(f_1, f_4)$ by f_1, f_2, f_3 and f_4, we get

$$x - y = 1.(x - 1) + 0.(x^2 - y) + 0.(z - 1) + 1.(y - 1) \tag{3.35}$$

So $r = 0$. $LM(f_2) = x^2$, $LM(f_3) = z$ and $LCM(x^2, z) = x^2z$.

$$S(f_2, f_3) = \frac{x^2z}{x^2} \cdot (x^2 - y) - \frac{x^2z}{z} \cdot (z - 1) \quad \textbf{(3.36)}$$

$$= x^2 - yz \quad \textbf{(3.37)}$$

If we divide $S(f_2, f_3)$ by f_1, f_2, f_3 and f_4, we get

$$x^2 - yz = 0.(x - 1) + 1.(x^2 - y) + y.(z - 1) + 0.(y - 1) \quad \textbf{(3.38)}$$

Since $r = 0$, we don't add it to F. $LM(f_2) = x^2$, $LM(f_4) = y$ and $LCM(x^2, y) = x^2y$.

$$S(f_2, f_4) = \frac{x^2y}{x^2} \cdot (x^2 - y) - \frac{x^2y}{y} \cdot (y - 1) \quad \textbf{(3.39)}$$

$$= x^2 - y^2 \quad \textbf{(3.40)}$$

Dividing $S(f_2, f_4)$ by f_1, f_2, f_3 and f_4, we obtain

$$x^2 - y^2 = 0.(x - 1) + 1.(x^2 - y) + 0.(z - 1) + y.(y - 1) \quad \textbf{(3.41)}$$

The remainder is zero. Consider $LM(f_3) = z$, $LM(f_4) = y$ and $LCM(z, y) = yz$.

$$S(f_3, f_4) = \frac{yz}{z} \cdot (z - 1) - \frac{yz}{y} \cdot (y - 1) \quad \textbf{(3.42)}$$

$$= y - z \quad \textbf{(3.43)}$$

Since the remainder on division of $S(f_3, f_4)$ by f_1, f_2, f_3 and f_4 is zero, the set $\{f_1, f_2, f_3, f_4\}$ is a Groebner basis for I.

Definition 3.23. A reduced Groebner basis for a polynomial ideal I is a Groebner basis G such that

(i) All $g \in G$ are monic,

(ii) For all $g \in G$, no monomial of g lies in $\langle LM(G - \{g\})\rangle$.

Example 3.24. The reduced Groebner bases of the sets in example 3.20 and 3.21 are the sets $\{x - 2y, y^2 - \frac{1}{4}\}$ and $\{x - 1, z - 1, y - 1\}$ respectively.

4. TORIC VARIETIES

Toric varieties are important class of algebraic varieties. They provide an alternative way to see many phenomena in algebraic geometry.

In this section, we will define the toric varieties and give some of their properties. Then we will relate with Groebner basis.

A discrete subgroup N of $\mathbb{Z}^n$ is called *lattice*. In a lattice, for every point x in N, there is a neighbourhood U of x such that $N \cap U = \{x\}$. For example $\mathbb{Z}^2 \subset \mathbb{R}^2$ is a lattice.

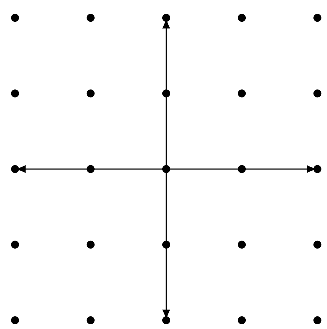

Figure 4.1: The lattice $N \cong \mathbb{Z}^2 \subset \mathbb{R}^2$.

Throughout this section, $N \cong \mathbb{Z}^n$ and $N = \mathbb{Z}.e_1 \oplus ... \oplus \mathbb{Z}.e_n$. To define vectors in our lattice N, consider the real vector space

$$N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R} = \mathbb{R}.e_1 \oplus ... \oplus \mathbb{R}.e_n \cong \mathbb{R}^n \qquad \textbf{(4.1)}$$

Let $U = \{u_1, ..., u_n\}$ be a finite set of vectors in $\mathbb{R}^n$, the set

$$\sigma = \langle u_1, ..., u_n \rangle = \{u \in N_{\mathbb{R}} : u = a_1u_1 + ... + a_nu_n,\ a_i \in \mathbb{R},\ a_i \geq 0\} \qquad \textbf{(4.2)}$$

is called a *polyhedral cone*. The vectors $u_1, ..., u_n$ are generators of the cone σ.

If $U = \emptyset$ then $\sigma = \{0\}$ is called *zero cone*. A cone spanned by a single non-zero vector u is called *ray*.

A cone σ is a *lattice* (or *rational*) cone if all the generators u_i of σ belong to N.

Example 4.1. The cone $\sigma = \left\langle \frac{\sqrt{2}}{3}e_1, \sqrt{7}e_2 \right\rangle$ is not a lattice cone since the generators doesn't belong to the lattice $N \cong \mathbb{Z}^2$.

A vector $u \in \mathbb{Z}^n$ is *primitive* if its coordinates are coprime. A cone is *regular* if the vectors $(u_1, ..., u_r)$ spanning the cone are primitive and there exists primitive vectors $(u_{r+1}, ..., u_n)$ such that $det(u_1, ..., u_n) = \pm 1$. In another words, the vectors $(u_1, ..., u_r)$ can be completed in a basis of the lattice N.

Remark 4.2. The dimension of σ is the dimension of the smallest vector space containing σ.

Example 4.3. $\sigma = \langle (1,0), (0,1) \rangle \subset \mathbb{R}^2$ is a rational polyhedral cone of dimension 2. It looks like:

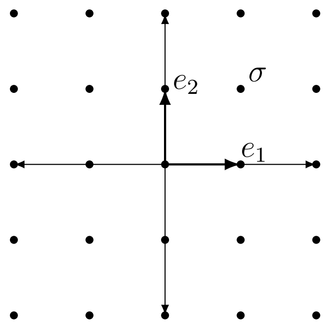

Figure 4.2: The polyhedral lattice cone in the lattice $N \cong \mathbb{Z}^2 \subset \mathbb{R}^2$.

Definition 4.4. A cone σ is *strongly convex* if it does not contain any straight line going through the origin, i.e. $\sigma \cap (-\sigma) = \{0\}$.

The dual cone is defined in the dual vector space of N. The dual of the vector space N is defined as

$$N^* = M = Hom(N, \mathbb{Z}) = \{v : N \longrightarrow \mathbb{Z} : v(u) = \langle v, u \rangle \, , \, \forall u \in N\} \qquad \textbf{(4.3)}$$

We will deal with the real vector space

$$M_{\mathbb{R}} = M \otimes_{\mathbb{Z}} \mathbb{R} = \mathbb{R}.e_1 \oplus ... \oplus \mathbb{R}.e_n \cong \mathbb{R}^n \qquad \textbf{(4.4)}$$

If we assume that M is generated by $\mp e_1^*, ..., \mp e_n^*$, then the condition

$$\langle e_i^*, e_j \rangle = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{if } i \neq j \end{cases} \qquad \textbf{(4.5)}$$

Example 4.5. Let $u = e_1 + 2e_2 \in N_{\mathbb{R}}$ and $v = e_1^* + 3e_2^* \in M_{\mathbb{R}}$,

$$\langle u, v \rangle = (e_1 + 2e_2).(e_1^* + 3e_2^*) = (e_1, e_1^*) + 6(e_2, e_2^*) = 7$$

To each cone $\sigma \subset N_{\mathbb{R}}$, we associate the dual cone $\sigma^\vee \subseteq M_{\mathbb{R}}$ such that

$$\sigma^\vee = \{v \in M_{\mathbb{R}} : \langle u, v\rangle \geq 0 \ \ \forall u \in \sigma\} \tag{4.6}$$

Example 4.6. The dual of the cone $\sigma = \langle(1,0),(0,1)\rangle$ in example 4.5 is $\sigma^\vee = \langle(1,0),(0,1)\rangle \subset (\mathbb{R}^2)^*$.

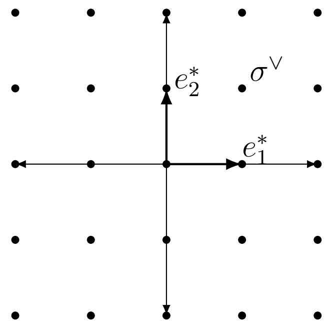

Figure 4.3: The dual cone $\sigma^\vee = \langle e_1^*, e_2^*\rangle$.

Proposition 4.7 [9]If σ is a rational polyhedral cone, then $\sigma^\vee$ is rational polyhedral cone.

Proof. Let $u_1, ..., u_n$ be the generators of σ. Then

$$\sigma^\vee = \cap_{i=1}^{n}(u_i)^\vee \tag{4.7}$$

So $\sigma^\vee$ consists of all solutions v_i of the following linear inequality

$$(u_i, v_i) \geq 0 \tag{4.8}$$

and all non-negative linear combination of $v_i's$. So $\sigma^\vee$ is a rational polyhedral cone.

■

Butif σ is a strongly convex cone, then $\sigma^\vee$ is not necessarily a strongly convex cone.

Proposition 4.8. Let $\sigma = \{0\}$, then $\sigma^\vee = M_{\mathbb{R}}$ and $M_{\mathbb{R}}$ is not a strongly convex cone.

Proposition 4.9. For every $\sigma \subset N_{\mathbb{R}}$, $(\sigma^\vee)^\vee = \sigma$.

Proof. Let $u \in (\sigma^\vee)^\vee$. From the definition, for every $v \in \sigma^\vee$ $\langle u, v\rangle \geq 0$. This means that $u \in \sigma$. Then $(\sigma^\vee)^\vee \subset \sigma$. Now, let us take $u \notin (\sigma^\vee)^\vee$. From the definition there is at least $v \in \sigma^\vee$ such that $\langle u, v\rangle < 0$. So $u \notin \sigma$ and $\sigma \subseteq (\sigma^\vee)^\vee$. By both inclusion we get $(\sigma^\vee)^\vee = \sigma$. ■

Remark 4.10. $\sigma_1 \subset \sigma_2 \Leftrightarrow \sigma_2^\vee \subset \sigma_1^\vee$.

Lemma 4.11. Let $\sigma_1, \sigma_2 \in N_{\mathbb{R}}$ then $(\sigma_1 + \sigma_2)^\vee = \sigma_1^\vee \cap \sigma_2^\vee$

Proof. Let $v \in (\sigma_1 + \sigma_2)^\vee$ then $\langle u, v\rangle \geq 0$ for every $u \in (\sigma_1 + \sigma_2)$ i.e. $\langle u, v\rangle \geq 0 \quad \forall u \in \sigma_1, \forall u \in \sigma_2$. So $v \in \sigma_1^\vee \cap \sigma_2^\vee$. Now let $v \in \sigma_1^\vee \cap \sigma_2^\vee$, then $v \in \sigma_1^\vee$ and $v \in \sigma_2^\vee$. There is $u \in \sigma_1$ and σ_2 such that $\langle u, v\rangle \geq 0$ so $v \in (\sigma_1 + \sigma_2)^\vee$.

■

Lemma 4.12. A cone is strongly convex ifandonlyif $\sigma^\vee$ has dimension n.

Proof. Let $\sigma^\vee$ has dimension less than n, then it is contained in a hyperplane in $M_{\mathbb{R}}$. This happens if and only if there exists $0 \neq u \in \sigma$ such that $\langle u, v\rangle = 0$ for all $v \in \sigma^\vee$. Then there exists $0 \neq u \in \sigma$ such that $-u \in \sigma$. This completes the proof.

■

Example 4.13. *Let* N=Z^2 and $\sigma = \langle e_1, 2e_1 + e_2\rangle$ in $N_{\mathbb{R}}$. Then this cone is

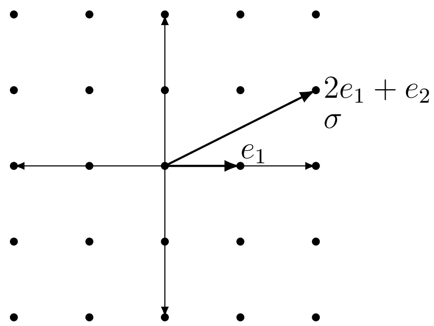

Figure 4.4: The cone $\sigma = \langle e_1,\ 2e_1 + e_2\rangle$.

Let us assume that $\sigma^\vee \subset M_{\mathbb{R}}$ is generated by the vectors $v_1 = a_1 e_1^* + a_2 e_2^*$ and $v_2 = b_1 e_1^* + b_2 e_2^*$, for $a_1, a_2, b_1, b_2 \in \mathbb{R}$.

From the definition if $i = j$ then $\langle e_i, e_j\rangle = 0$, otherwise $\langle e_i, e_j\rangle > 0$, so

$$\langle e_1, a_1 e_1^* + a_2 e_2^*\rangle = 0, \qquad \langle 2e_1 + e_2, a_1 e_1^* + a_2 e_2^*\rangle > 0 \quad \text{and}$$
$$\langle e_1, b_1 e_1^* + b_2 e_2^*\rangle > 0, \qquad \langle 2e_1 + e_2, b_1 e_1^* + b_2 e_2^*\rangle = 0$$

Then we obtain

$$2x_1 + x_2 = 0 \qquad \text{and} \qquad 2y_1 + y_2 > 0$$
$$x_1 > 0 \qquad\qquad y_1 = 0$$

Solving these equations, we get $v_1 = a_1e_1^* - 2a_1e_2^*$ and $v_2 = b_2e_2^*$.
Therefore $\sigma^\vee = \langle e_1^* - 2e_2^*, e_2^* \rangle$

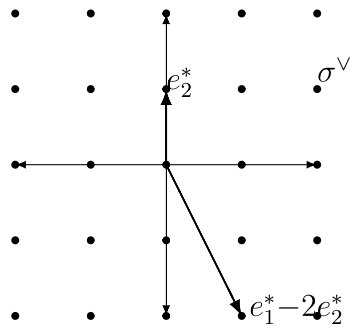

Figure 4.5: The dual cone $\sigma^\vee = \langle e_1^* - 2e_2^*, e_2^* \rangle$.

Definition 4.14. Let σ be a cone and $v \in \sigma^\vee$. The set

$$\tau = \sigma \cap v^\perp = \{u \in \sigma : \langle u, v \rangle = 0\} \quad \textbf{(4.9)}$$

is called a *face* of σ, denoted by $\tau \prec \sigma$.

Example 4.15. The faces of the cone $\sigma = \langle e_1, 2e_1 + e_2 \rangle$ in $N_\mathbb{R}$ are $\tau_1 = \{0\}$, $\tau_2 = \langle e_1 \rangle$, $\tau_3 = \langle 2e_1 + e_2 \rangle$ and $\tau_4 = \sigma$.
A cone is a face of itself, the other faces are called *proper faces.* An 1-dimensional face is called an *edge.* If the face has $(n-1)-$ dimension then it is called as *facet.*

Remark 4.16. In the n-dimensional cone, the number of faces with dimension k is given by the formula

$$C(n, k) = \frac{n!}{k!(n-k)!} \quad \textbf{(4.10)}$$

So the number of faces is

$$C(n, 0) + C(n, 1) + ... + C(n, n-1) + C(n, n) = 2^n \quad \textbf{(4.11)}$$

Proposition 4.17. Every face of polyhedral convex cone is a polyhedral convex cone.

Proof. For any $v \in \sigma^\vee$, $\tau = \sigma \cap v^\perp$ implies that τ is a subset of σ. Therefore τ is a convex cone. All u_i that satisfies the condition $(v, u_i) = 0$ generate the face τ. But we know that there are only finitely many $u_i \in \sigma$ that satisfies the above equality. Therefore σ is also polyhedral cone. ■

Remark 4.18. Every face of a face is a face of the cone.

Proposition 4.19. If τ is a proper face of σ, then $\sigma^\vee \subset \tau^\vee$.

Proof. Let $v \in \sigma^\vee$, from the definition $\langle u, v\rangle \geq 0$ for all $u \in \sigma$. Since $\tau \prec \sigma$, $\langle \lambda, v\rangle \geq 0$ holds for every $\lambda \in \tau$. Then $v \in \sigma^\vee$ and we obtain $\sigma^\vee \subset \tau^\vee$. ■

Proposition 4.20. Every intersection of faces of σ is a face of σ.

Proposition 4.21. [1] If $\tau \prec \sigma$, then $\sigma^\vee \cap v^\perp$ is a face of $\sigma^\vee$ with $\dim(\sigma)+\dim(\sigma^\vee \cap v^\perp)= n$. This provides a one-to-one correspondence (with reverse order) between faces of σ and faces of $\sigma^\vee$.

Proposition 4.22. The set

$$S_\sigma = \sigma^\vee \cap M = \{v \in M : \langle u, v\rangle \geq 0 \; \forall u \in \sigma\} \quad (\mathbf{4.12})$$

is a semigroup.

Proof. The dual lattice $M_\mathbb{R}$ is an additive group so $\sigma^\vee$ and S_σ are also additive groups. For every $v_1, v_2, v_3 \in S_\sigma$, $v_1 + v_2 \in S_\sigma$ and $v_1 + (v_2 + v_3) = (v_1 + v_2) + v_3$. ■

Theorem 4.23.$(Gordan's Lemma) If$ σ is a polyhedral lattice cone, then $S_\sigma = \sigma^\vee \cap M$ is finitely generated semigroup.

Proof. Let σ be a polyhedral cone. So $\sigma^\vee$ is a polyhedral cone. Let us assume that σ is generated by the vectors $v_1, ..., v_s \in M_\mathbb{R}$. Let

$$F_\sigma = \{t_1 v_1 + ... + t_s v_s \mid \; 0 \leq t_i \leq 1\}$$

F_σ is compact and M is discrete, hence $F_\sigma \cap M$ is a finite set. We claim that $F_\sigma \cap M$ generates $\sigma^\vee \cap M$ as a semigroup: Every $v \in \sigma^\vee \cap M$ can be written as,

$$v = a_1 v_1 + ... + a_s v_s \text{ where } a_i \in \mathbb{Q}_{\geq 0} \; \forall i.$$

If we write $a_i = \lfloor a_i \rfloor + t_i$, then

$$v = \sum_{i=1}^{s} \lfloor a_i \rfloor v_i + \sum_{i=1}^{s} t_i v_i$$

where $\lfloor a_i \rfloor$ denotes the greatest integer less than or equal to a_i. This shows that $F_\sigma \cap M$ generates $\sigma^\vee \cap M$ as a semigroup. ■

Example 4.24. Consider $\sigma = \langle 2e_1 - e_2, e_2\rangle \subset \mathbb{R}^2$. Let us find $S_\sigma = \sigma^\vee \cap M$. The dual cone is $\sigma^\vee = \langle e_1^*, e_1^* + 2e_2^*\rangle$. $\sigma^\vee \cap M$ contains the vectors with integer coefficients in $\sigma^\vee$. But $\sigma^\vee \cap M$ is not generated by the vectors $v_1 = e_1^*$ and $v_2 = e_1^* + 2e_2^*$. Because, every element of S_σ can not be written as a linear combinations of v_1 and v_2 with integer coefficients. For example, $e_1^* + e_2^*$ can be written only such that $v_3 = \frac{1}{2}v_1 + \frac{1}{2}v_2$ and since the coefficients are not integer, we must take the vector v_3 as a generator of S_σ.
Hence S_σ is generated by $v_1 = e_1^*$, $v_2 = e_1^* + 2e_2^*$ and $v_3 = e_1^* + e_2^*$.

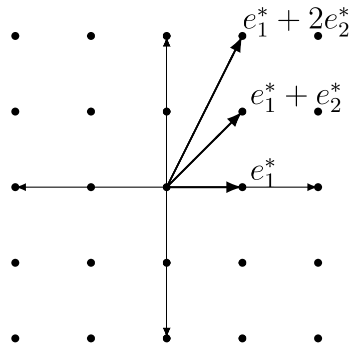

Figure 4.6: The semigroup $S_\sigma = \langle e_1^*, e_1^* + e_2^*, e_1^* + 2e_2^*\rangle$.

Remark 4.25. If $\sigma = \{0\}$, $S_\sigma = \sigma^\vee \cap M$ is generated by the vectors $(e_1^*, -e_1^*, e_2^*, -e_2^*)$ and also generated by the vectors $(e_1^*, e_2^*, -e_1^* - e_2^*)$. Then S_σ is all of the dual space M. In general, for any $\sigma' \subset N_\mathbb{R}$, $S_{\sigma'} \subset S_{\{0\}} = M$.

Remark 4.26. For any rational polyhedral convex cone σ and $\tau = \sigma \cap v^\perp$; which is a face of σ, with $v \in S_\sigma = \sigma^\vee \cap M$, then

$$S_\tau = S_\sigma + \mathbb{Z}_{\geq 0}(-v) \tag{4.13}$$

Example 4.27. In the cases considered in example 4.25, we obtain for the face $\tau = \langle e_2\rangle$ of σ, the vector $v = e_1^*$ satisfies $\tau^\vee = \sigma^\vee + \mathbb{R}_{\geq 0}(-v)$ and one has

$$S_\tau = S_\sigma + \mathbb{Z}_{\geq 0}(-v) \tag{4.14}$$

A Laurent polynomial with coefficient in the field $\mathbb{C}$, expressed in the form

$$... + b_{-n}t^{-n} + b_{-(n-1)}t^{-(n-1)} + ... + b_{-1}t^{-1} + b_0 + b_1t^1 + ... + b_nt^n + ... \tag{4.15}$$

where the b_i are elements of $\mathbb{C}$ and only finitely many of the b_i are nonzero. We denote by $\mathbb{C}[t, t^{-1}] = \mathbb{C}[t_1, ..., t_n, t_1^{-1}, ..., t_n^{-1}]$ the ring of Laurent polynomials.

Consider

$$\theta : \mathbb{Z}^n \longrightarrow \mathbb{C}[t_1, ..., t_n, t_1^{-1}, ..., t_n^{-1}] \quad (\mathbf{4.16})$$

$$b = (b_1, ..., b_n) \longmapsto t^b = t_1^{b_1} ... t_n^{b_n} \quad (\mathbf{4.17})$$

the isomorphism between the additive group $\mathbb{Z}^n$ and the multiplicative group of monic Laurent polynomials. This says that we have

$$\chi : M \longrightarrow \mathbb{C}[M] = \mathbb{C}[X_1, ..., X_n, X_1^{-1}, ..., X_n^{-1}] \quad (\mathbf{4.18})$$

by $\chi^{e_i^*} = X_i$ and $\chi^{-e_i^*} = X_i^{-1}$
where $\mp e_i^*$ is a generator of the dual space M.
Thus, for every $v \in S_\sigma$, there is a generator $\chi^v \in \mathbb{C}[S_\sigma]$ corresponds to v. Since S_σ is finitely generated by Gordan's Lemma, $\mathbb{C}[S_\sigma]$ has finite number of generators. Also for every $v_1, v_2 \in S_\sigma$, we have

$$\chi^{v_1} . \chi^{v_2} = \chi^{v_1 + v_2} \quad (\mathbf{4.19})$$

Then

$$\mathbb{C}[S_\sigma] = \{\sum b_v \chi^v \mid v \in S_\sigma,\ b_v \in \mathbb{C}\} \quad (\mathbf{4.20})$$

Here $\chi^0 = 1$ is the constant polynomial corresponding to $0 \in S_\sigma = \sigma^\vee \cap M$.

Remark 4.28. If $\sigma = \{0\}$ then $\mathbb{C}[S_\sigma] = \mathbb{C}[M]$.
In general for $\sigma' \subset N_\mathbb{R}$, $\mathbb{C}[S_{\sigma'}]$ is a subset of $\mathbb{C}[M]$.

Example 4.29.[10] Let $\sigma = \langle e_1, e_2 \rangle \subset \mathbb{R}^2$, then the dual cone $\sigma^\vee$ is generated by the vectors $v_1 = e_1^*$ and $v_2 = e_2^*$. The semigroup S_σ is also generated by the vectors $v_1 = e_1^*$ and $v_2 = e_2^*$. So the generators of $\mathbb{C}[S_\sigma]$ are $\chi^{e_1^*} = X_1$ and $\chi^{e_2^*} = X_2$. Hence $\mathbb{C}[S_\sigma] = \mathbb{C}[X_1, X_2]$

Example 4.30. Let σ be a cone generated by $u_1 = 2e_1 - e_2$ and $u_2 = e_2$. The semigroup S_σ is generated by $v_1 = e_1^*$, $v_2 = e_1^* + 2e_2^*$ and $v_3 = e_1^* + e_2^*$.
Then $\mathbb{C}[S_\sigma]$ is generated by the vectors $\chi^{e_1^*} = X_1$, $\chi^{e_1^* + e_2^*} = X_1 X_2$ and $\chi^{e_1^* + 2e_2^*} = X_1 X_2^2$. Therefore $\mathbb{C}[S_\sigma] = \mathbb{C}[X_1, X_1X_2, X_1X_2^2]$.
Let us define the map

$$f : \mathbb{C}[Y_1, ..., Y_m] \longrightarrow \mathbb{C}[S_\sigma] \quad (\mathbf{4.21})$$

by $\chi^{v_i} = Y_i$, where v_i is a generator of S_σ. The kernel of the map f is an ideal of the polynomial ring $\mathbb{C}[Y_1, ..., Y_m]$ see [4]. If we denote this ideal by I, we get

$$\mathbb{C}[S_\sigma] \cong \mathbb{C}[Y_1, ..., Y_m]/I \quad \textbf{(4.22)}$$

Since the maps χ and θ are defined in the same way, the ideal I is also kernel of the θ isomorphism. This means that the ideal I is defined by relations between the generators of the semigroup S_σ. Those relations are in the form of

$$\alpha_1 v_1 + ... + \alpha_m v_m = \beta_1 v_1 + ... + \beta_m v_m \quad \textbf{(4.23)}$$

where $\alpha_i, \beta_i \in \mathbb{Z}_{\geq 0}$ Those correspond to the relations

$$(\chi^{v_1})^{\alpha_1} ... (\chi^{v_m})^{\alpha_m} = (\chi^{v_1})^{\beta_1} ... (\chi^{v_m})^{\beta_m} \quad \textbf{(4.24)}$$

in the ring $\mathbb{C}[S_\sigma]$. We obtain the relations

$$Y_1^{\alpha_1} ... Y_m^{\alpha_m} = Y_1^{\beta_1} ... Y_m^{\beta_m} \quad \textbf{(4.25)}$$

in the ring $\mathbb{C}[Y_1, ..., Y_m]$.

Proposition 4.31.[2] The ideal I, defined above is generated by finite number of binomials such that

$$Y_1^{\alpha_1} ... Y_m^{\alpha_m} - Y_1^{\beta_1} ... Y_m^{\beta_m} \quad \textbf{(4.26)}$$

Now we can define the affine toric variety associated with a cone σ.

Definition 4.32 [2] The affine toric variety corresponding to a rational polyhedral strongly convex cone $\sigma \subset N_\mathbb{R} \cong \mathbb{R}^n$ is $V_\sigma \cong Spec(\mathbb{C}[S_\sigma])$. The dimension of V_σ is n.

Theorem 4.33.[3] If $\sigma \subset N_\mathbb{R} \cong \mathbb{R}^n$ and I is generated by the relations between the generators of the ring $\mathbb{C}[S_\sigma]$, then V_σ is the variety $V(I)$ in the space $\mathbb{C}^m$.

Example 4.34. Let $\sigma = \langle e_1, e_2 \rangle \subset \mathbb{R}^2$. Since $\sigma^\vee$ is generated by the vectors e_1^* and e_2^*, the semi-group $S_\sigma = \sigma^\vee \cap M$ is also generated by e_1^* and e_2^*. In that case the polynomial ring $\mathbb{C}[S_\sigma]$ is defined such that

$$\mathbb{C}[S_\sigma] = \mathbb{C}[\chi^{e_1^*}, \chi^{e_2^*}] = \mathbb{C}[X_1, X_2]$$

with writing Y_1, Y_2 into X_1, X_2 we obtain

$$\mathbb{C}[S_\sigma] \cong \mathbb{C}[Y_1, Y_2]/I$$

Since $dim(V_\sigma) = 2$ and $dim(\mathbb{C}[Y_1, Y_2]) = 2$ there is no relation between the generators of S_σ and $I = \langle 0 \rangle$. Hence

$$V_\sigma = Spec(\mathbb{C}[S_\sigma]) \cong V(\langle 0 \rangle) = \mathbb{C}^2$$

Definition 4.35. The set $T_N \cong \mathbb{C}^* \times ... \times \mathbb{C} = (\mathbb{C}^*)^n$ is called *n-dimensional affine algebraic torus.*

Let us see that for every $\sigma \subset N_\mathbb{R}$, the affine toric variety V_σ contains the torus $T_N = (\mathbb{C}^*)^n$ as a Zariski open dense subset:

For $\sigma = \langle 0 \rangle$ the semigroup S_σ is the dual space M and $\mathbb{C}[S_\sigma] = \mathbb{C}[M]$. So $Spec(\mathbb{C}[M])$ is $(\mathbb{C}^*)^n$, $\mathbb{C}^* = \mathbb{C} - \{0\}$.

The affine toric variety associated with the cone $\sigma = \langle 0 \rangle \subset \mathbb{R}^2$ is $(\mathbb{C}^*)^2$. $\sigma^\vee$ is the dual space $M_\mathbb{R}$. Therefore the semi-group S_σ is generated by the vectors $\mp e_1^*$ and $\mp e_2^*$. With the map χ, we obtain $\mathbb{C}[S_\sigma] = \mathbb{C}[\chi^{e_1^*}, \chi^{e_2^*}, \chi^{-e_1^*}, \chi^{-e_2^*}] = \mathbb{C}[X_1, X_2, X_1^{-1}, X_2^{-1}]$.

If we show the variables $X_1, X_2, X_1^{-1}, X_2^{-1}$ with Y_1, Y_2, Y_3, Y_4 respectively, in this case, we obtain the natural isomorphism

$$\mathbb{C}[S_\sigma] \cong \mathbb{C}[Y_1, Y_2, Y_3, Y_4]/I$$

Here the ideal I is generated by the relations $Y_1.Y_3 = 1$ and $Y_2.Y_4 = 1$

It follows that

$$\mathbb{C}[S_\sigma] \cong \mathbb{C}[Y_1, Y_2, Y_3, Y_4]/\langle Y_1.Y_3 - 1, Y_2.Y_4 - 1 \rangle$$

By the definition, the affine toric variety equals to Spec($\mathbb{C}[S_\sigma]$) i.e.

$$V_\sigma = Spec(\mathbb{C}[S_\sigma]) \cong V(Y_1.Y_3 - 1, Y_2.Y_4 - 1)$$

In other words,

$$V_\sigma = \{(u_1, u_2, u_3, u_4) \in \mathbb{C}^4 \mid u_1.u_3 = 1, u_2.u_4 = 1\}$$

Here with the restriction $u_1 \neq 0$, $u_2 \neq 0$ and with the projection map $(u_1, u_2, u_3, u_4) \mapsto (u_1, u_2)$ we obtain the affine toric variety as

$$V_\sigma = \{(u_1, u_2) \in \mathbb{C}^2 \mid u_1 \neq 0,\ u_2 \neq 0\} = (\mathbb{C}^*)^2$$

More generally: Since $\sigma = \langle 0 \rangle$ is a face of every cone, for any cone $\sigma' \subset N_{\mathbb{R}}$ the condition $\mathbb{C}[S_{\sigma'}] \subset \mathbb{C}[S_\sigma]$ holds. If we generalise this for $\tau \prec \sigma'$, then $S_{\sigma'} \subset S_\tau$ and $\mathbb{C}[S_{\sigma'}] \subset \mathbb{C}[S_\tau]$.

$\mathbb{C}[S_\sigma] = \mathbb{C}[\chi^{e_1^*}, \chi^{e_2^*}, ..., \chi^{e_n^*}, \chi^{-e_1^*}, ..., \chi^{-e_n^*}] = \mathbb{C}[X_1, ..., X_n, X_1^{-1}, ..., X_n^{-1}]$

If we denote the variables $X_1, ..., X_n, X_1^{-1}, ..., X_n^{-1}$ by $Y_1, ..., Y_n, Y_{n+1}, ..., Y_{2n}$ respectively, and $\mathbb{C}[S_\sigma]$ is isomorphic to $\mathbb{C}[Y_1, ..., Y_n, Y_{n+1}, ..., Y_{2n}]/I$.

$$\mathbb{C}[S_\sigma] \cong \mathbb{C}[Y_1, ..., Y_n, Y_{n+1}, ..., Y_{2n}]/I$$

Since $dim(V_\sigma) = n$ the ideal I is generated by $2n - n = n$ number of generators. This means that there are n number of relations between the variables $Y_1, ..., Y_n, Y_{n+1}, ..., Y_{2n}$. Those relations are

$$Y_1.Y_{n+1} = X_1.X_1^{-1} = 1,\ Y_2.Y_{n+2} = X_2.X_2^{-1} = 1, ...,\ Y_n.Y_{2n} = X_n.X_n^{-1} = 1$$

By the theorem 4.25, $V_\sigma \cong Spec(\mathbb{C}[S_\sigma]) \cong V(Y_1.Y_{n+1} - 1, ..., Y_n.Y_{2n} - 1)$

$$V_\sigma = \{(u_1, ..., u_n, u_{n+1}, ..., u_{2n}) \in \mathbb{C}^{2n} \mid u_1 u_{n+1} = 1, ..., u_n u_{2n} = 1\}$$

With the projection map $(u_1, ..., u_n, u_{n+1}, ..., u_{2n}) \mapsto (u_1, ..., u_n)$ and restriction $u_1 \neq 0, ..., u_n \neq 0$, we obtain the affine toric variety as

$$V_\sigma = \{(u_1, ..., u_n) \in \mathbb{C}^n \mid u_1 \neq 0, ..., u_n \neq 0\}$$

and it is equal to $(\mathbb{C}^*)^n$.

Remark 4.36. If the cone $\sigma \subset N_{\mathbb{R}} \cong \mathbb{R}^n$ is generated by n number of vectors which are generators of $N_{\mathbb{R}}$, then $V_\sigma \cong \mathbb{C}^n$.

Example 4.37. Let $\sigma = \langle e_2 \rangle \subset \mathbb{R}^2$. Show that $V_\sigma \cong \mathbb{C}^* \times \mathbb{C}$.

Proof. The dual cone $\sigma^\vee$ is $\{v \in \sigma^\vee : \langle v, e_2 \rangle \geq 0,\ e_2 \in \sigma\}$. $\sigma^\vee = \langle e_1^*, e_2^*, -e_1^* \rangle$ and the semi-group $S_\sigma = \langle e_1^*, e_2^*, -e_1^* \rangle$. The polynomial ring $\mathbb{C}[S_\sigma]$ is

$$\mathbb{C}[S_\sigma] = \mathbb{C}[X_1, X_2, X_1^{-1}]$$

Hence the kernel of the map $\mathbb{C}[S_\sigma] \longrightarrow \mathbb{C}[Y_1, Y_2, Y_3]$ is $I = \langle Y_1 Y_3 - 1 \rangle$ and

$$\mathbb{C}[S_\sigma] \cong \mathbb{C}[Y_1, Y_2, Y_3]/I$$

The affine toric variety is $Spec\, \mathbb{C}[S_\sigma]$ which is

$$V_\sigma \cong V(Y_1 Y_3 - 1) \subset \mathbb{C}^3$$

$$= \{(u_1, u_2, u_3) \in \mathbb{C}^3 \mid u_1.u_3 = 1\}$$

Taking $u_1 \neq 0$ with the projection map $(u_1, u_2, u_3) \mapsto (u_1, u_2)$ we obtain

$$V_\sigma \cong \mathbb{C}^* \times \mathbb{C}$$

■

5. GLUING OF AFFINE TORIC VARIETIES

We glue together the affine toric varieties. This brings us to the concept of a fan.

Definition 5.1.[1] A *fan* $\triangle$ in the space $N_{\mathbb{R}} \cong \mathbb{R}^n$ is a finite union of cones such that :

(i) every cone of $\triangle$ is a strongly convex rational polyhedral cone,

(ii) every face of a cone of $\triangle$ is a cone of $\triangle$,

(iii) if σ_1 and σ_2 are cones of $\triangle$, then $\sigma_1 \cap \sigma_2$ is a common face of σ_1 and σ_2.

Example 5.2. In $N_{\mathbb{R}} \cong \mathbb{R}^2$, the following is a fan with four cones.

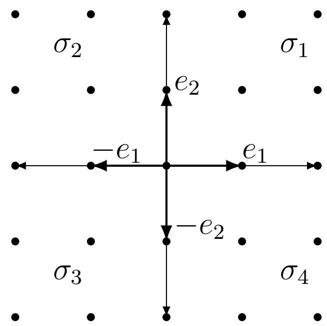

Figure 5.1: The fan $\triangle = \{\sigma_1, \sigma_2, \sigma_3, \sigma_4\}$.

5.1. If The Fan Contains 2 Cones

Let $\triangle = \{\sigma_1, \sigma_2\} \subset N_{\mathbb{R}} \cong \mathbb{R}^n$. Let τ be the common face of the cones σ_1 and σ_2, i.e. $\tau = \sigma_1 \cap \sigma_2$. Since $\tau \prec \sigma_1$ and $\tau \prec \sigma_2$ V_{σ_1} and V_{σ_2} contains V_τ as an open subset. That follows that

$V_{\sigma_1} \cap V_\tau = V_{\sigma_1 \cap \tau}$ and $V_{\sigma_2} \cap V_\tau = V_{\sigma_2 \cap \tau}$

Let us see the maps, $V_\tau \longrightarrow V_{\sigma_1}$ and $V_\tau \longrightarrow V_{\sigma_2}$:

Let $\lambda_1 \in \sigma_1^\vee$ and $\tau = \sigma_1 \cap \lambda_1^\perp$. The semi-group S_τ is obtained from the semi-group S_{σ_1} by adding one generator $-\lambda_1$. As λ_1 can be choosen as an element of a system of generators $(a_1, ..., a_k)$ for S_{σ_1}, we may assume that $\lambda_1 = a_k$ is the last vector in the system of generators of S_{σ_1}, and we denote $a_{k+1} = -\lambda_1$. In order to obtain the relationship between the generators of S_τ, we have to consider previous relationships between the generators $(a_1, ..., a_k)$ of S_{σ_1} and the supplementary relationship $a_k + a_{k+1} = 0$. This relationship corresponds to the multiplicative one $X_k X_{k+1} = 1$ in the polynomial ring $\mathbb{C}[S_\tau]$, and that is the only supplementary relationship we need in order to obtain $\mathbb{C}[S_\tau]$ from $\mathbb{C}[S_\sigma]$. So we obtain the affine toric varietis V_τ and V_{σ_1} as

$$V_{\sigma_1} = \{(u_1, ..., u_k) \in \mathbb{C}^k\}$$

$$V_\tau = \{(u_1, ..., u_k, u_{k+1}) \in \mathbb{C}^{k+1} : u_k u_{k+1} = 1\}$$

With the projection map $(u_1, ..., u_k, u_{k+1}) \overset{\phi_1}{\longmapsto} (u_1, ..., u_k)$ and with $(u_k \neq 0)$, we obtain the isomorphism:

$$V_\tau \cong V_{\sigma_1} - \{u_k = 0\}$$

Now, let us take the face τ as a face of σ_2 and for $\lambda_2 \in \sigma_2^\vee$, it is defined as $\sigma_2 \cap \lambda_2^\perp$. The semi-group S_τ is obtained from the semi-group S_{σ_2} by adding one generator $-\lambda_2$. As λ_2 can be choosen as an element of a system of generators $(b_1, ..., b_j)$ for S_{σ_2}, we may assume that $\lambda_2 = b_j$ is the last vector in the system of generators of S_{σ_2}, and we denote $b_{j+1} = -\lambda_2$. In order to obtain the relationship between the generators of S_τ, we have to consider previous relationships between the generators $(b_1, ..., b_j)$ of σ_2 and the supplementary relationship $b_j + b_{j+1} = 0$. This relationship corresponds to the multiplicative one $X_j X_{j+1} = 1$ in the polynomial ring $\mathbb{C}[S_\tau]$. In this case, the affine toric varieties V_τ and V_{σ_2} are in the form of

$$V_{\sigma_2} = \{(v_1, ..., v_j) \in \mathbb{C}^j\}$$

$$V_\tau = \{(v_1, ..., v_j, v_{j+1}) \in \mathbb{C}^{j+1} : v_j v_{j+1} = 1\}$$

Therefore, with the projection map $(v_1, ..., v_j, v_{j+1}) \overset{\phi_2}{\longmapsto} (v_1, ..., v_j)$ and with $(v_j \neq 0)$ we get the following isomorphism:

$$V_\tau \cong V_{\sigma_2} - \{v_j = 0\}$$

By the composition of the maps ϕ_1 and ϕ_2 we obtain a gluing map as

$$\psi = \phi_2 \circ \phi_1^{-1} = V_{\sigma_1} - \{u_k = 0\} \longrightarrow V_\tau \longrightarrow V_{\sigma_2} - \{v_j = 0\}$$

This map allows us to glue together V_{σ_1} and V_{σ_2} along their common part V_τ. Hence, from a fan $\triangle$, the toric variety $X(\triangle)$ is constructed by taking disjoint union of V_{σ_1} and V_{σ_2} i.e.

$$X(\triangle) = V_{\sigma_1} \sqcup V_{\sigma_2}$$

Example 5.3. Let $\triangle = \{\sigma_1, \sigma_2\}$ in $N_\mathbb{R} \cong \mathbb{R}$ with $\sigma_1 = \langle e_1 \rangle$ and $\sigma_2 = \langle -e_1 \rangle$.

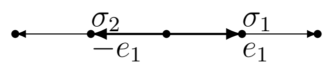

Figure 5.2: The lattice fan $\triangle = \{\sigma_1, \sigma_2\}$.

By the remark 4.42. $V_{\sigma_1} \cong \mathbb{C}$ and $V_{\sigma_2} \cong \mathbb{C}$. The common face of the cones σ_1 and σ_2 is $\tau = \{0\}$.

The face $\tau = \{0\}$ is defined as $\tau = \sigma_1 \cap (e_1^*)^\perp$ in the cone σ_1 and it is defined as $\tau = \sigma_2 \cap (-e_1^*)^\perp$ in the cone σ_2.The semi-group S_τ is

$$S_\tau = \langle e_1^*, -e_1^* \rangle$$

and the polynomial ring as

$$\mathbb{C}[S_\tau] = \mathbb{C}[X, X^{-1}]$$

Hence, the associated affine toric variety is
$V_\tau = Spec(\mathbb{C}[X, X^{-1}]) = \{(u_1, u_2) \in \mathbb{C}^2 : u_1.u_2) = 1\}$. With $(u_1 \neq 0)$ and the projection map $(u_1, u_2) \longmapsto u_1$, we get

$$V_\tau \cong \mathbb{C}^*$$

It follows that $V_\tau \cong V_{\sigma_1} - \{u_1 = 0\}$ and $V_\tau \cong V_{\sigma_2} - \{u_2 = 0\}$

Therefore, the gluing map between the V_{σ_1} and V_{σ_2} is defined by the change of coordinate $X \mapsto X^{-1}$, with the maps $\kappa_i : \mathbb{C} \longrightarrow \mathbb{P}^1$ for $i = 0, 1$ and writing $\frac{t_1}{t_0}$ into X, we get the homogeneous coordinates on $\mathbb{P}^1$.

$$X = \frac{t_1}{t_0} \overset{\kappa_0}{\longmapsto} (1 : \frac{t_1}{t_0})$$

$$X^{-1} = \frac{t_0}{t_1} \stackrel{\kappa_1}{\longmapsto} (\frac{t_0}{t_1} : 1)$$

This means that the affine toric varieties V_{σ_1} and V_{σ_2} corresponds to the two open subsets of $\mathbb{P}^1$ defined with $t_0 \neq 0$ and $t_1 \neq 0$.

We can glue together V_{σ_1} and V_{σ_2} along V_τ i.e. we take the union of the open subsets of $\mathbb{P}^1$. We come by the toric variety as

$$X(\triangle) = \mathbb{P}^1$$

Example 5.4. Let $\triangle = \{\sigma_1 = \langle e_1, e_2 \rangle, \sigma_2 = \langle e_1, -e_2 \rangle\}$ in $N_\mathbb{R} \cong \mathbb{R}^2$. Then $X(\triangle)$ is $\mathbb{C} \times \mathbb{P}^1$. This fan is such that

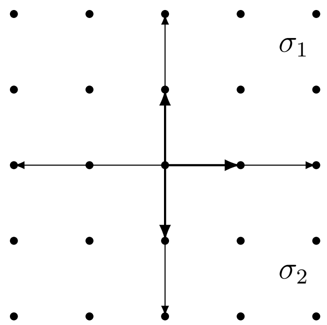

Figure 5.3: The fan $\triangle = \{\sigma_1, \sigma_2\}$

By the remark 4.42, we find the affine toric variety correspond to the cones as $V_{\sigma_1} = \mathbb{C}^2_{(X,Y)}$ and $V_{\sigma_2} = \mathbb{C}^2_{(X,Y^{-1})}$.

The common face of σ_1 and σ_2 is $\tau = \sigma_1 \cap \sigma_2 = \langle e_1 \rangle$. The affine toric variety associated to this face is $V_\tau = \mathbb{C} \times \mathbb{C}^*$. Then we get the following isomorphism: $V_\tau \cong V_{\sigma_1} - \{u_2 = 0\}$and $V_\tau \cong V_{\sigma_2} - \{u_2^{-1} = 0\}$

Writing $\frac{t_1}{t_0}$ into Y and with the maps $h_i : \mathbb{C} \longrightarrow \mathbb{P}^1, (i = 0, 1)$, we obtain $1-1$ correspondence between the affine toric varieties V_{σ_1} and V_{σ_2} and open subsets of projective space.

$$(X, Y) = (X, \frac{t_1}{t_0}) \stackrel{h_0}{\longmapsto} (X, (1 : \frac{t_1}{t_0})$$

$$(\mathrm{X,Y}^{-1}) = (X, \frac{t_0}{t_1}) \stackrel{h_1}{\longmapsto} (X, (\frac{t_0}{t_1} : 1)$$

Then by the gluing map $(X, Y) \longmapsto (X, Y^{-1})$ we get $\mathbb{C} \times \mathbb{P}^1$ with the coordinates $(u_1, (t_0 : t_1))$.

5.2. If The Fan $\triangle$ Contains Three Cones

Let $\triangle = \{\sigma_1, \sigma_2, \sigma_3\} \subset N_{\mathbb{R}} \cong \mathbb{R}^n$. Let τ_1 be the common face of the cones σ_1 and σ_2, i.e. $\tau_1 = \sigma_1 \cap \sigma_2$. For the vectors $\lambda_1 \in \sigma_1^\vee$ and $\lambda_2 \in \sigma_2^\vee$, let us define $\tau_1 = \sigma_1 \cap \lambda_1^\perp$ and $\tau_1 = \sigma_2 \cap \lambda_2^\perp$. Then we obtain the following isomorphisms between the affine varieties $V_{\tau_1}, V_{\sigma_1}, V_{\sigma_2}$.

$$\phi_1 : V_{\tau_1} \longrightarrow V_{\sigma_1} - \{u_k = 0\}$$
$$\phi_2 : V_{\tau_1} \longrightarrow V_{\sigma_2} - \{v_j = 0\}$$

Hence, the gluing map is defined as

$$\vartheta_1 = \phi_2 \circ \phi_1^{-1} : V_{\sigma_1} \longrightarrow V_{\sigma_2}$$
$$(\mathrm{u}_1, ..., u_k, u_{k+1}) \longmapsto (v_1, ..., v_l, v_{l+1})$$

Now, let us consider the cones σ_2 and σ_3, and τ_2 be the common face of σ_2 and σ_3. For the vector $\xi_1 \in \sigma_2^\vee$, let us define $\tau_2 = \sigma_1 \cap \xi_1^\perp$ and the vector ξ_1 corresponds to the monomial which is defined as $X^{\xi_1} = X_r$ in the affine toric variety V_{σ_2}. Then we can define the following map

$$\delta_1 : V_{\tau_2} \longrightarrow V_{\sigma_2} - \{v_r = 0\}$$

Now, for the vector $\xi_2 \in \sigma_3^\vee$, let us define $\tau_2 = \sigma_3 \cap \xi_2^\perp$ and the vector ξ_2 corresponds to the monomial which is defined as $X^{\xi_2} = X_p$. Hence, we get the following map

$$\delta_2 : V_{\tau_2} \longrightarrow V_{\sigma_3} - \{w_p = 0\}$$

By the composition of the maps δ_1 and δ_2, we obtain the affine toric varieties V_{σ_2} and V_{σ_3}.

$$\vartheta_2 = \delta_2 \circ \delta_1^{-1} : V_{\sigma_2} \longrightarrow V_{\sigma_3}$$

Using the gluing maps ϑ_1 and ϑ_2 we can glue together the affine toric varieties V_{σ_1}, V_{σ_2} and V_{σ_3}. In other words, the map is such that:

$$\vartheta_2 \circ \vartheta_1 : V_{\sigma_1} \xrightarrow{\phi_1^{-1}} V_{\tau_1} \xrightarrow{\phi_2} V_{\sigma_2} \xrightarrow{\delta_1^{-1}} V_{\tau_2} \xrightarrow{\delta_2} V_{\sigma_3}$$

So we obtain the toric variety corresponds to the fan as

$$X(\triangle) = V_{\sigma_1} \sqcup V_{\sigma_2} \sqcup V_{\sigma_3}$$

Example 5.5. Let $\triangle = \{\sigma_1 = \langle e_1, e_2, e_3, e_4\rangle, \sigma_2 = \langle e_1, e_2, -e_3 - e_4, e_4\rangle,$
$\sigma_3 = \langle e_1, e_2, e_3, -e_3 - e_4\rangle$ in $N_{\mathbb{R}} \cong \mathbb{R}^4$.
We find the dual of those cones as $\sigma_1^\vee = \langle e_1^*, e_2^*, e_3^*, e_4^*\rangle$,
$\sigma_2^\vee = \langle e_1^*, e_2^*, -e_3^*, -e_3^* + e_4^*\rangle, \sigma_3^\vee = \langle e_1^*, e_2^*, -e_4^*, e_3^* - e_4^*\rangle$ Then the affine toric varieties correspond to those cones are

$$V_{\sigma_1} = \mathbb{C}^4_{(X,Y,Z,W)},$$

$$\mathrm{V}_{\sigma_2} = \mathbb{C}^4_{(X,Y,Z^{-1},Z^{-1}W)},$$

$$\mathrm{V}_{\sigma_3} = \mathbb{C}^4_{(X,Y,W^{-1},ZW^{-1})}$$

Writing Z$= \dfrac{t_1}{t_0}$ and $W = \dfrac{t_2}{t_0}$ with the maps $h_i : \mathbb{C}^2 \longrightarrow \mathbb{P}^2$, $(i = 0, 1, 2)$, we obtain the $1-1$ correspondence between the affine toric varieties V_{σ_1}, V_{σ_2}, V_{σ_3} and open subsets of the projective space.

$$(X, Y, Z, W) \overset{h_0}{\longmapsto} (X, Y, (1 : \frac{t_1}{t_0} : \frac{t_2}{t_0}))$$

$$(X, Y, Z^{-1}, Z^{-1}W) \overset{h_1}{\longmapsto} (X, Y, (\frac{t_0}{t_1} : 1 : \frac{t_2}{t_1}))$$

$$(X, Y, W^{-1}, ZW^{-1}) \overset{h_2}{\longmapsto} (X, Y, (\frac{t_0}{t_2} : \frac{t_1}{t_2} : 1))$$

We glue together V_{σ_1} and V_{σ_2} with the map $(X, Y, Z, W) \longmapsto (X, Y, Z^{-1}, Z^{-1}W)$ and then glue together V_{σ_2} and V_{σ_3} with the map $(X, Y, Z^{-1}, Z^{-1}W) \longmapsto (X, Y, W^{-1}, ZW^{-1})$. We obtain the toric variety $X(\triangle) = \mathbb{C}^2 \times \mathbb{P}^2$ with the coordinates $(u_1, u_2, (t_0 : t_1 : t_2))$.

5.3.The Fan Contains n Number Of Cones

Let $\triangle = \sigma_1, \sigma_2, \ldots, \sigma_n \subset N_{\mathbb{R}} \cong \mathbb{R}^n$. First, we find the affine toric varieties associated to the cone $\sigma_i \in \triangle$, for each i. Then we glue together the affine toric varieties associated to the cones which have the common face. And going on the process in this way, we obtain the toric variety correspond to the fan as

$$X(\triangle) = \cup_{\sigma \in \triangle} V_\sigma$$

6. TORIC VARIETIES ATTACHED INTEGRAL MATRICES

Let k be any field and $A = (a_{ij})$ a fixed $m \times n$ matrix with non-negative integer entries a_{ij} and with non-zero columns. Let $k[Z_1, \ldots, Z_n]$ and $k[X_1, \ldots, X_m]$ be two polynomial rings over k and ϕ the graded homomorphism of $k-$algebras, $\phi : k[Z_1, \ldots, Z_n] \longrightarrow k[X_1, \ldots, X_m]$, induced by $\phi(Z_j) = X^{a_j}$, where $a_j = (a_{1j}, \ldots, a_{mj})$ is the jth column of A and $X^{a_j} = X_1^{a_{1j}}, \ldots, X_m^{a_{mj}}$[11].
The kernel of ϕ is called the *toric ideal* associated with A. If $\alpha = (\alpha_i) \in N^n$, we set $Z^\alpha = \prod_{i=1}^n Z_i^{\alpha_i}$ for the corresponding monomial in $k[Z_1, \ldots, Z_n]$. Note that the map ϕ is closely related to the homomorphism $\psi : Z^n \longrightarrow Z^m$, determined by the matrix A in the standard bases of Z^n and Z^m: A binomial $Z^\alpha - Z^\beta$ belongs to $ker(\phi)$ if and only if $\alpha - \beta$ belongs to $ker(\psi)$[12], [13].
We can write every vector $u \in Z^n$ uniquely as $u_+ - u_-$, where u_+ and u_- are non-negative and have disjoint support. For instance, $(2, 0, -1, -4, 3) = (2, 0, 0, 0, 3) - (0, 0, 1, 4, 0)$.

Lemma 6.1. [14] The toric ideal I_A is generated by the binomials $Z^{u_+} - Z^{u_-}$, where u runs over all integer vectors in the nullspace of A.

Definition 6.2. The toric ideal of X_A is the zero set of the ideal I_A, i.e. $V(I) \subset \mathbb{C}^n$.

We define the all of toric ideals in the ring $k[Z_1, \ldots, Z_n]$ such that:

Theorem 6.3. [14] The ideal in the ring $k[Z_1, \ldots, Z_n]$ is toric ideal if and only if it is prime and binomial.
When any matrix A is given, to find the toric ideal associated to it, we have to find the vectors in the nullspace of A. In other words, we look for the solutions of the equation of $Au = 0$.

Example 6.4. [15]Let $A = \begin{pmatrix} 3 & 2 & 1 & 0 \\ 0 & 1 & 2 & 3 \end{pmatrix}$ *be a* $2{\times}4$ matrix.
Consider the following vectors from the solution set of equation $Au = 0$

$\{(1,-2,1,0)^t,(0,1,-2,1)^t,(1,-1,-1,1)^t\}$ in $\mathbb{Z}^4$.

Then we can write $u_1 = (1,0,1,0)-(0,2,0,0)$, $u_2 = (0,1,0,1)-(0,0,2,0)$, $u_3 = (1,0,0,1)-(0,1,1,0)$ respectively.

We obtain the toric ideal such that

$$I_A = \left\langle Z_1Z_3 - Z_2^2, Z_2Z_4 - Z_3^2, Z_1Z_4 - Z_2Z_3\right\rangle \qquad (\mathbf{6.1})$$

and $X_A = V(I)$. Also, we can find the vectors $(2,-3,0,1)^t$ and $(1,-2,1,0)^t$ in the nullspace of A. Then we obtain the toric ideal as $I'_A = \langle Z_1^2Z_4 - Z_2^3, Z_1Z_3 - Z_2^2\rangle$. The zero sets of I_A and I'_A define the same toric variety; because if we find the Groebner bases of both of them, we obtain the same toric ideal. Their Groebner Basis is $I_G = \langle Z_1Z_3 - Z_2^2, Z_2Z_4 - Z_3^2, Z_1Z_4 - Z_2Z_3\rangle$. This means that, we solve this problem by considering the common generating set of these ideals.

Example 6.5. Let $I = \langle X_1^2 - X_2, X_1^3 - X_3\rangle$ be an ideal in $\mathbb{C}[X_1,X_2,X_3]$. By the Buchberger's Algorithm, we find the Groebner basis of I as

$$I_G = \langle X_1^2 - X_2, X_1X_2 - X_3, X_1X_3 - X_2^2, X_2^3 - X_3^2\rangle$$

and $X_A = V(I_G)$.

7. CONCLUSION

In conclusion, we first find the nullspace of any given matrix, then we write the toric ideal associated to it and calculate the Groebner basis of that ideal. After that we find the zero set of that ideal as a toric variety.

REFERENCES

[1] **Fulton, W.,** 1969. Introduction to Toric Varieties, *Princeton University Press*, New Jersey.

[2] **Oda, T.,** 1985. Convex Bodies and Algebraic Geometry, *Springer-Verlag*, New York.

[3] **Ewald, G.,** 1976. Combinatorial Convexity and Algebraic Geometry, *Springer-Verlag*, Bochum.

[4] **Fraleigh, J. B. and Katz, V. J.,** 1994, A First Course in Abstract Algebra, Addison-Wesley.

[5] **Reid, M.,** 2001. Undergraduate Algebraic Geometry, *Cambridge University Press*, Warwick.

[6] **Atiyah M. F.,** 1969, Introduction to Commutative Algebra, Addison-Wesley, Massachusetts.

[7] **Cox, D.,** 1991. Ideals Varieties and Algorithms, *Springer-Verlag*, New York.

[8] **Miller, E., Sturmfels, B.,** 2004 Combinatorial Commutative Algebra, *Springer-Verlag*

[9] **Danilov, V.,** 1967. The Geometry of Toric Varieties, *Russian Math. Surveys.* **33:2**:97-154

[10] **Barthel, G., Kaup, L.,** Introduction To Basic Toric Geometry

[11] **Reyes, E.,Villarreal, R.,Zárate L.,** 2000. A Note on Affine Toric Varieties, *Linear Algebra and Its Applications*, **318** (2000)173-179.

[12] **Katsabekis, A.,Thoma, A.,** 2006. Parametrizations Of Toric Varieties Over Any Field, University Of Ionnina 45110, Greece.

[13] **Katsabekis, A.,Thoma, A.,** 2003. Toric Sets and Orbits On Toric Varieties, *J.Pure Appl. Algebra.* **181** (2003)75-83.

[14] **Sturmfels, B.** 1995. Groebner Bases and Convex Polytopes, *American Mathematical Society, University Lecture Series*, **8**

[15] **Tosun, M.** Ade Surface Singularities, Chambers and Toric Varieties, Seminaires Congress **10**, 2005 p.341-350

[16] **Kirwan, F.,** 1995. Complex Algebraic Curves, *Cambridge University Press*, Cambridge.

[17] **Sharp, R.Y.,** 2000. Steps in Commutative Algebra, *Cambridge University Press*, Cambridge.

[18] **Dimitrov, P.,** 1994, Toric Varieties: A Short Introduction

CURRICULUM VITAE

Bahriye Karaca was born in Izmir in 1987. She graduated from Izmir 60th Year Anatolian High School in 2004. She took a bachelor's degree of mathematics at Yıldız Technical University in 2008. She began her graduate education at Istanbul Technical University Mathematics Programme in 2008 and she continues this education.