ISTANBUL TECHNICAL UNIVERSITY ★ INSTITUTE OF SCIENCE AND TECHNOLOGY

ON SYMMETRY PROPERTIES OF DAVEY-STEWARTSON AND GENERALIZED DAVEY-STEWARTSON EQUATIONS

MASTER OF SCIENCE THESIS

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Department : MATHEMATICS

Programme : MATHEMATICS ENGINEERING

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CONTENTS

ABBREVIATIONS

DS : Davey-Stewartson
GDS : Generalized Davey-Stewartson
NLS : Nonlinear Schrödinger
ODE : Ordinary Differential Equation
PDE : Partial Differential Equation
KP : Kadomtsev-Petviashvilli
KMV : Kac-Moody-Virasoro

ON SYMMETRY PROPERTIES OF DAVEY-STEWARTSON AND GENERALIZED DAVEY-STEWARTSON EQUATIONS

SUMMARY

In this thesis work, some symmetry properties of DS and GDS equations are investigated with the help of Lie group analysis of differential equations.

Analysis of differential equations through their symmetry groups is an effective tool especially for the investigation of solutions of nonlinear ODEs and PDEs. As the first step of analysis, the group of transformations leaving the equation studied invariant are found. Making use of this symmetry group, reducing the order of ODEs, lowering the number of independent variables of PDEs or even reducing PDEs to ODEs is possible.

Detailed studies are made for a work in literature related to symmetry group of DS equations. Lie symmetry algebra, point transformations and 1-dimensional subalgebras of DS system are found. These subalgebras are used to reduce DS equations to various equations involving two independent variables. Additionally, the Lie symmetry algebra of GDS equations is computed and it is shown that it is isomorphic to Lie algebra of DS equations with some conditions on parameters.

DAVEY-STEWARTSON VE GENELLEŞTİRİLMİŞ DAVEY-STEWARTSON DENKLEMLERİNİN SİMETRİ ÖZELLİKLERİ ÜZERİNE

ÖZET

Bu tez çalışmasında diferansiyel denklemlerin Lie grubu analizi yardımıyla DS ve GDS denklemlerinin bazı simetri özellikleri araştırılmıştır.

Diferansiyel denklemlerin simetri grupları yardımıyla incelenmesi, özellikle doğrusal olmayan adi diferansiyel denklemler ve kısmi türevli diferansiyel denklemlerin çözümlerinin araştırılması için etkin bir araçtır. Bu yöntemde önce, ele alınan denklemi değişmez bırakan grup dönüşümleri bulunmaktadır. Ardından bu dönüşümler kullanılarak adi diferansiyel denklemlerin mertebesinin düşürülmesi, kısmi türevli diferansiyel denklemlerin ise değişken sayısının azaltılması hatta adi diferansiyel denklemlere indirgenmesi mümkün olmaktadır.

DS denklemlerinin simetri grubuyla ilgili literatürdeki bir çalışma için kapsamlı çalışmalar yapılmıştır. DS sisteminin Lie simetri cebri, nokta dönüşümleri ve bir boyutlu alt cebirleri bulunmuştur. Bu alt cebirler DS denklemlerini iki değişken içeren çeşitli denkelemlere indirgemede kullanılmıştır. Ayrıca, GDS denklemlerinin Lie simetri cebri hesaplanmış ve parametrelerin bazı koşulları sağladığı takdirde bulunan cebrin DS denklemlerinin Lie cebrine izomorf olduğu gösterilmiştir.

1. INTRODUCTION

This thesis is devoted to the study of group-theoretical properties of Davey-Stewartson (DS) and generalized Davey-Stewartson (GDS) equations. These equations play an important role in modelling nonlinear waves. When we deal with waves propagating in a nonlinear media, one of the fundamental equations we encounter is the $(1+1)$ dimensional nonlinear Schrödinger (NLS) equation:

$$iA_t + pA_{xx} + q|A|^2 A = 0, \tag{1.1}$$

where t is time, x is spatial coordinate and A denotes the complex amplitude. NLS equation describes unidirectional wave modulation. (2+1) evolution equations are used if modulations transverse to the wave propagation direction are also allowed. When we replace the one dimensional dispersive term with a two dimensional dispersive term, we obtain

$$iA_t + pA_{xx} + sA_{xy} + rA_{yy} + q|A|^2 A = 0 \tag{1.2}$$

The $(2+1)$ dimensional form of the NLS equation (1.2) correctly describes $(2+1)$ dimensional wave motion when there is no resonance between the main quasi-harmonic wave and zero harmonics induce by nonlinear effects. The system that involves both short and long wave modes is called DS equations. DS was introduced in [1-3]

$$\begin{aligned} &i\psi_t + \psi_{xx} + \varepsilon_1 \psi_{yy} = \varepsilon_2 |\psi|^2 \psi + w\psi, \\ &w_{xx} + \delta_1 w_{yy} = \delta_2 (|\psi|^2)_{yy}, \end{aligned} \tag{1.3}$$

where ψ is the complex amplitude of the short wave, w is the long wave amplitude and $\delta_1, \delta_2, \varepsilon_1, \varepsilon_2$ are real constants with with $\varepsilon_1 = \pm 1$, $\varepsilon_2 = \pm 1$. DS equations describe the propagation of two-dimensional water waves moving under the force of gravity in water of finite depth. The DS equations belong to class of equations in more than $(1+1)$ dimensions that are exactly integrable by inverse scattering techniques and their generalizations [4]. A system of nonlinear partial differential equations in 2+1 dimensions as a model of wave propagation in a bulk medium composed of an elastic material with couple stresses has recently been derived in [5], namely

$$\begin{aligned} &i\psi_t + \delta\psi_{xx} + \psi_{yy} = \chi|\psi|^2\psi + \gamma(w_x + \phi_y)\psi \\ &w_{xx} + n\phi_{xy} + m_2 w_{yy} = (|\psi|^2)_x \\ &n w_{xy} + \lambda\phi_{xx} + m_1\phi_{yy} = (|\psi|^2)_y, \end{aligned} \tag{1.4}$$

with the condition $(\lambda - 1)(m_1 - m_2) = n^2$. Here $\psi(t, x, y)$ is a complex function, $w(t, x, y)$ and $\phi(t, x, y)$ are real functions and $\delta, n, m_1, m_2, \lambda, \chi, \gamma$ are real constants. The authors of [5] showed that if the parameters are related by

$$n = 1 - \lambda = m_1 - m_2, \tag{1.5}$$

then (1.4) can be reduced to the standard DS equations (in general not integrable) by a non-invertible point transformation of dependent variables. Therefore, they called (1.4) the GDS equations. Below, we justify this naming from a group-theoretical point of view. Also, in [5] some travelling type solutions of (1.4) in terms of elementary and elliptic functions are obtained. Based on some physically obvious Noetherian symmetries (time-space translations and constant change of phase), global existence and nonexistence results are given in [6]. In another recent work [7], under some constraints on the physical parameters, the so-called hyperbolic-elliptic-elliptic case of the system (1.4) (In [7] the system is classified into different types according to the signs of parameters $(\delta, m_1, m_2, \lambda)$) was shown to admit singular solutions that blow up in a finite time. To do this, inspired by the (pseudo) conformal invariance of DS system, they used the fact that time-dependent $\mathrm{SL}(2, \mathbb{R})$ invariant solutions can be generated from stationary radial solutions for an appropriate choice of coefficients.

In this thesis work, we show the detailed results for the DS symmetry algebra, the group transformations of DS and symmetry reductions for the DS equations in [8]. We compute the Lie symmetry algebra of the GDS and use this symmetry algebra to find new solutions for GDS system.

In second section, definitions and theorems about symmetry group of differential equations are given [9-12].

In third section, Lie symmetry algebra and group of point transformations of DS equations and symmetry reductions for the DS equations are given[5].

In fourth section, we compute Lie symmetry algebra of GDS equations and show that it is isomorphic to DS equations with some conditions on parameters.

2. GROUP-THEORETICAL APPROACH TO DIFFERENTIAL EQUATIONS

In this section we present a brief discussion for group-theoretical investigation of differential equations. In this approach there exist two main steps: First is to find the group of symmetry transformations, i.e. the transformations depending on some parameter(s) which leave the equation in study invariant. It is possible to classify differential equations through symmetry groups. What we are going to make use of is the second step we introduce: Once we have an equation with a symmetry algebra, it is possible reduce its order in case of an ODE, or it is possible to reduce the number of independent variables in case of a PDE.

2.1 Symmetry Groups of Differential Equations

We start with some basic definitions.

Definition 2.1. An **r-parameter Lie group** is a group G which also carries the structure of an r-dimensional smooth manifold in such a what that both the group operation,

$$m : G \times G \longrightarrow G, \qquad m(g, h) = g \cdot h, \qquad g, h \in G$$

and the inversion

$$i : G \longrightarrow G, \qquad i(g) = g^{-1}, \qquad g \in G$$

are smooth maps between manifolds.

Often we are not interested in the full Lie group, but only in group elements close to the identity element. In this case we can dispense with the abstract manifold theory and define a local Lie group solely in terms of local coordinate expressions for the group operations.

Definition 2.2. An **r-parameter local Lie group** consists of connected open subsets $V_0 \subset V \subset \mathbb{R}^r$ containing the origin 0, and smooth maps

$$m : V \times V \longrightarrow \mathbb{R}^r$$

defining the group operation, and

$$i : V_0 \longrightarrow V$$

defining the group inversion, with the following properties.

(a)Associativity. $x, y, z, m(x, y), m(y, z) \in V \ \Rightarrow \ m(x, m(y, z)) = m(m(x, y), z)$.

(b)Identity Element. $\forall x \in V, \ m(0, x) = x = m(x, 0)$.

(c)Inverses. $\forall x \in V_0, \ \exists i(x) : \ m(x, i(x)) = 0 = m(i(x), x)$.

Definition 2.3. Let M be a smooth manifold. A **local group of transformations** acting on M is given by a (local) Lie group G, an open subset $\mathcal{U}$, with

$$\{e\} \times M \subset \mathcal{U} \subset G \times M$$

which is the domain of definition of the group action, and a smooth map $\Psi : \mathcal{U} \longrightarrow M$ with the following properties:

(a) $g \cdot (h \cdot x) = (g \cdot h) \cdot x, \qquad g, h \in G, \quad x \in M,$

(b) $e \cdot x = x \qquad \forall x \in M,$

(c) $g^{-1} \cdot (g \cdot x) = x, \qquad g \in G, \quad x \in M.$

Definition 2.4. Symmetry Group of a Differential Equation. Let Δ be a system of differential equations. A symmetry group of the system Δ is a local group of transformations G acting on an open subset M of the space of independent and dependent variables for the system with the property that whenever $u = f(x)$ is a solution of Δ, and whenever $g \cdot f$ is defined for $g \in G$, then $\tilde{u} = g \cdot f(x)$ is also a solution of the system.

Let us consider a system of differential equations

$$\begin{aligned} &E^i(x, u, u^{(1)}, \ldots, u^{(n)}) = 0 \\ &x \in \mathbb{R}^p, \ u \in \mathbb{R}^q, \ i = 1, \ldots, m, \ p, q, m, n \in \mathbb{N} \end{aligned} \tag{2.1}$$

where $u^{(n)}$, denotes all partial derivatives of order k of all components u_α of u. We now wish to find all local point transformations of the form

$$\tilde{x} = \Lambda_g(x, u), \quad \tilde{u} = \Omega_g(x, u) \tag{2.2}$$

such that a solution $u = f(x)$ of the system (2.2) is transformed into a solution $\tilde{u} = \tilde{f}(\tilde{x})$. The transformations are called "point" ones because the new variables $(\tilde{x}, \tilde{u})$ depend only on the old ones (x, u), i.e., on a point in the space

$$M \subset X \times U, \quad X \sim \mathbb{R}^p, \quad U \sim \mathbb{R}^q$$

of independent and dependent variables. More general transformations, in which depend also on derivatives like u_x, u_{xx} or integral $D^{-1}u$ will not be considered.

The subscript g in equation (2.2) denotes a finite, or infinite number of group parameters and the transformations form a local Lie group. The word "local" in this context means that the transformations need only be defined and invertible for g close to the identity element of the group and for (x, u) close to the origin in $X \times U$ space.

In principle, one could use equation (2.2) directly to calculate derivatives like $\tilde{u}_{\tilde{x}}$. Substituting back into equation (2.1), one would get differential equations for the functions Λ and Ω. This approach is not fruitful; the equations determining Λ and Ω are at least as difficult to solve as the original system. Lie's outstanding contribution was that he showed that nearly all the relevant information can be obtained using an infinitesimal approach. Instead of equation (2.2) we consider transformation

$$\tilde{x}_i = x_i + \varepsilon\xi_i(x, u), \quad \tilde{u}_\alpha = u_\alpha + \varepsilon\phi_\alpha(x, u) \tag{2.3}$$

and obtain equations for ξ_i and ϕ_α, after putting (2.3) in (2.1) and ignoring all terms of order ε^p, $p \geq 2$. This provides us with a system of linear equations. Solving these determining equations we obtain the Lie algebra L of the symmetry group G, realized by vector fields

$$\hat{X} = \sum_{i=1}^{p} \xi_i(x, u)\partial_{x_i} + \sum_{\alpha=1}^{q} \phi_\alpha(x, u)\partial_{u_\alpha} \tag{2.4}$$

Vector field $\hat{X}$ contains arbitrary constants. Hence $\hat{X}$ can be written as a linear combination of vector fields $\hat{Y}$. Where

$$\hat{Y} = \sum_{i=1}^{p} \Xi_i(x, u)\partial_{x_i} + \sum_{\alpha=1}^{q} \Phi_\alpha(x, u)\partial_{u_\alpha} \tag{2.5}$$

Vector fields $\hat{Y}$ constitute a basis for the Lie algebra L.

Definition 2.5. A **Lie algebra** is a vector space L together with a bilinear operation

$$[\cdot, \cdot] : L \times L \to L$$

called the "Lie bracket" for L, satisfying the axioms

(a)Bilinearity. $[c\mathbf{v}+c'\mathbf{v}',\mathbf{w}] = c[\mathbf{v},\mathbf{w}]+c'[\mathbf{v}',\mathbf{w}]$, $[\mathbf{v},c\mathbf{w}+c'\mathbf{w}'] = c[\mathbf{v},\mathbf{w}]+c'[\mathbf{v},\mathbf{w}']$,

(b)Skew-Symmetry. $[\mathbf{v},\mathbf{w}] = -[\mathbf{w},\mathbf{v}]$,

(c)Jacobi Identity. $[\mathbf{u},[\mathbf{v},\mathbf{w}]]+[\mathbf{w},[\mathbf{u},\mathbf{v}]]+[\mathbf{v},[\mathbf{w},\mathbf{u}]] = 0$,

for every $\mathbf{u},\mathbf{v},\mathbf{v}',\mathbf{w},\mathbf{w}' \in L$ and $\forall c,c' \in \mathbb{R}$.

Each vector field $\hat{Y}$ generates a one-parameter subgroup of the symmetry group, obtained by integrating the vector field

$$\begin{aligned} \frac{d\tilde{x}_i}{d\varepsilon} &= \Xi_i(\tilde{x},\tilde{u}), \qquad \tilde{x}_i \mid_{\varepsilon=0}= x_i, \\ \frac{d\tilde{u}_\alpha}{d\varepsilon} &= \Phi_\alpha(\tilde{x},\tilde{u}), \qquad \tilde{u}_\alpha \mid_{\varepsilon=0}= u_\alpha. \end{aligned} \tag{2.6}$$

The general group transformation is obtained by composing the individual one-parameter transformations.

We shall present an algorithm for calculating the symmetry algebra, i.e., the vector fields $\hat{Y}$ of equation (2.4) that generate the symmetry group G. The basic tool is "prolongation theory".

The symmetry group G acts on the manifold M

$$G : \{x,u\} \in M \longrightarrow \{\tilde{x},\tilde{u}\} \in M. \tag{2.7}$$

Thus, it takes functions to function

$$u = f(x) \longrightarrow \tilde{u} = \tilde{f}(\tilde{x}) = g \cdot f(x). \tag{2.8}$$

The $n.th$ prolongation of G also takes derivatives of orders up to n into derivatives:

$$\mathrm{pr}^{(\mathrm{n})}G : \{x, f(x), f^{(1)}(x), \ldots, f^{(n)}(x)\} \longrightarrow \{\tilde{x}, \tilde{f}(\tilde{x}), \tilde{f}^{(1)}(\tilde{x}), \ldots, \tilde{f}^{(n)}(\tilde{x})\}. \tag{2.9}$$

The vector field $\hat{X}$ acts on functions of the variables x and u. Its n.th prolongation will act on functions of $x, u, u_x, \ldots, u_{nx}$. If we integrate $\mathrm{pr}^{(\mathrm{n})}\hat{X}$ we obtain the prolongation of the group action $\mathrm{pr}^{(\mathrm{n})}G$. The form of the prolongation of the vector field $\hat{X}$ is

$$\mathrm{pr}^{(\mathrm{n})}\,\hat{X} = \hat{X} + \sum_{\alpha=1}^{q}\sum_{k=1}^{n}\sum_{J} \phi_\alpha^J \frac{\partial}{\partial u_J^\alpha}. \tag{2.10}$$

Here J is a set of indices:

$$J \equiv J(k) = (j_1, \ldots, j_k), \qquad 1 \leq j_k \leq p, \quad k = j_1 + \cdots + j_k$$

The coefficients ϕ_α^J are expressed in terms of ξ, ϕ and their derivatives up to k. The formulas for these can be found in [10]. We find the recursion relations more useful and we shall reproduce those.

For the first prolongation we have

$$\mathrm{pr}^{(1)}\,\hat{X} \;=\; \hat{X} + \sum_{\alpha=1}^{q}\sum_{i=1}^{p}\, \phi_\alpha^i(x,u,u_x)\frac{\partial}{\partial u_{x_i}^\alpha}\;, \tag{2.11a}$$

$$\phi_\alpha^i \;=\; D_{x_i}\phi_\alpha - \sum_{j=1}^{p}\left(D_{x_i}\xi^j\right)u_{\alpha,x_j}, \tag{2.11b}$$

where D_{x_i} is the total derivative operator:

$$D_{x_i} = \frac{\partial}{\partial_{x_i}} + \sum_{\alpha=1}^{q}\frac{\partial u_\alpha}{\partial x_i}\frac{\partial}{\partial u_\alpha} + \sum_{\alpha=1}^{q}\sum_{j=1}^{p}\frac{\partial u_{\alpha,x_j}}{\partial x_i}\frac{\partial}{\partial u_{\alpha,x_j}} + \cdots\,. \tag{2.12}$$

If the $n.th$ prolongation is known, the $(n+1).th$ is given by

$$\mathrm{pr}^{(\mathrm{n+1})}\,\hat{X} \;=\; \mathrm{pr}^{(\mathrm{n})}\,\hat{X} + \sum_{\alpha=1}^{q}\;\sum_{i_1,\ldots,i_{n+1}=1}^{p}\;\phi_\alpha^{i_1\ldots i_{n+1}}\frac{\partial}{\partial u_{\alpha,x_{i_1},\ldots,x_{i_{n+1}}}}, \tag{2.13a}$$

$$\phi_\alpha^{i_1\ldots i_{n+1}} \;=\; D_{x_{i_{n+1}}}\phi_\alpha^{x_1,\ldots,i_n} - \sum_{j=1}^{p}\left(D_{x_{i_{n+1}}}\xi^j\right)u_{\alpha,x_{i_1}\ldots x_{i_n}x_j}. \tag{2.13b}$$

Theorem 2.1. Invariance of Differential Equations . Suppose

$$E^i(x,u,u^{(1)},\ldots,u^{(n)}) = 0, \qquad i = 1,\ldots,m$$

is a system of differential equations defined over $M \subset X \times U$. If G is a local group of transformations acting on M, and

$$\mathrm{pr}^{(\mathrm{n})}\,X\;\left(E^i\right)|_{E^k=0} = 0\,, \qquad i,k = 1,\ldots,m$$

for every infinitesimal generator X of G, then G is a symmetry group of the system. The algorithm for determining the symmetry algebra of the system (2.1) can now be stated quite simply. The $n.$ prolongation of the vector field must annihilate the equations on their solution set:

$$\mathrm{pr}^{(\mathrm{n})}\,\hat{X}\;\left(E^i\right)|_{E^k=0} = 0 \qquad i,k = 1,\ldots,m \tag{2.14}$$

The functions to be determined from equation (2.14) are the coefficients ξ_i and ϕ_α. They depend only x and u. Equation (2.14) will also involve the derivatives $x, u, u_x, u_{xx}, \ldots, u_{nx}$ explicitly. Hence, the coefficients multiplying each linearly independent expression in the derivatives must vanish identically. This provides us with a system of linear differential equations of order n. (or less), the so-called "determining equations" for the functions ξ_i and ϕ_α. These equations are linear, even if the system is nonlinear. If the algorithm is summed up:

(i) Calculate the $n.th$ prolongation (2.10) of the vector field (2.4). This does not depend on differential equation that is investigated, only on the number of independent and dependent variables of equation.

(ii) Solve system (2.1) for m of the highest derivatives in the system.

(iii) Implement equation (2.14) and substitute for all expressions calculated in (ii).

(iv) Identify all linearly independent expressions in the remaining derivatives and set the coefficients of these expressions equal to zero. This gives us determining equations.

(v) Solve the determining equations and obtain $\xi_i(x, u)$ and $\phi_\alpha(x, u)$.

The system of determining equations is nearly overdetermined. When the system is solved the following possibilities occur:

(a) The only solution is the trivial one: $\xi_i = \phi_\alpha = 0$. Here, no nontrivial symmetry group of (2.1) exists and the method is not applicable.

(b) The general solution of the determining equations depends on $N < \infty$ significant integration constants. The dimension of the symmetry algebra is equal to $N < \infty$.

(c) The general solution depends on arbitrary functions of some variables. The symmetry group is then infinite dimensional. That occurs for all linear PDEs but also for integrable nonlinear equations in 3 dimensions.

2.2 Symmetry Reduction

The most important application of the symmetry group G of Lie point transformations, leaving a system of PDEs invariant, is to perform symmetry reduction. In the case of PDEs that means a reduction of the number of independent variables in the equation. In particular, it may be possible and desirable to an ODE or even to an algebraic equation. The basic idea of symmetry reduction is to take some subgroup G_0 of the symmetry group G and look for solutions that are invariant under G_0. Requiring invariance is equivalent to imposing additional first-order linear equations on solutions. These can be solved and the result can be put into the original equations. This provides the reduced system to be solved.

The procedure for performing symmetry reduction can be outlined as an algorithm, consisting of the following steps.

(i) Find The symmetry group G, leaving the considered system (2.1) invariant, and obtain the corresponding Lie algebra L of vector fields (2.4).

(ii) Classify the subalgebras of L into conjugacy classes under the action of the Lie group G. Each subgroup $G_0 \subset G$, corresponding to a different conjugacy class of subalgebras L_0, will give a different type of invariant solution.

(iii) Consider a subalgebra $L_0 \subset L$, representing a class of subalgebras. The group G_0 acts on the space $M \sim X \times U$ of dependent and independent variables. Find the invariants of this action, i.e., the functionally independent solutions

$$\varphi^j = \varphi^j(x, u), \qquad j = 1, \ldots, N \tag{2.15}$$

of the set of first-order linear PDEs

$$X_i F(x, u) = 0, \qquad i = 1, \ldots, n_0 \tag{2.16}$$

where X_i form a basis of the Lie algebra L_0. The number of invariants N is equal to the codimension of the generic orbits of G_0 in M:

$$N = p + q - d$$

where d is the dimension of these orbits.

After solving the system (2.15), the following cases can arise.

(A) Among the invariants $\varphi^j(x, u)$ it is possible to choose q functions $\varphi^j(x, u)$ that provide an invertible mapping to the dependent variables. The Jacobian determinant then satisfies

$$J \equiv \left(\frac{\partial(\hat{\varphi}^1, \ldots, \hat{\varphi}^q)}{\partial(u_1, \ldots, u^q)} \right), \quad \det J \neq 0. \tag{2.17}$$

The remaining $k = N - q$ invariants can be chosen to depend only on the independent variables and we denote them

$$\zeta_1(x), \ldots, \zeta_k(x), \quad k < p \tag{2.18}$$

Now let us restrict to the solution set of equation (2.1). We consider u_j as functions of x and this can be imposed by setting

$$\hat{\varphi}^j = F_j(\zeta_1, \ldots, \zeta_k), \tag{2.19}$$

Using condition (2.17) we solve (2.19) for the dependent variables and obtain

$$u_j(x) = U_j(x, F_j(\zeta)). \tag{2.20}$$

Upon substitution into equation (2.1) we obtain a set of equations involving only the functions $F_j \quad (j = 1, \ldots, q)$ and the variables $\zeta_a \quad (a = 1, \ldots, k)$. Since the original equation (2.1) is G-invariant and equation (2.16) provides a complete set of G_0-invariants, the noninvariant quantities x in (2.20) must drop out. Since we have $k < p$, we have reduced the number of independent variables. If the reduced equations are solved for $F_j(\zeta)$, substitution into (2.20) provides solutions of the original system.

(B) Equation (2.17) is satisfied, but the complementary variables depend upon u. We proceed as above, however substitution of (2.20) into (2.1) is done as below

$$u_j(x) = U_j(x, F_j(\zeta)), \qquad \zeta = \zeta(x, u) \tag{2.21}$$

(2.21) yields implicit solutions. In some cases this can be solved and we obtain further explicit solutions.

(iv) Solve the reduced equations. The reduced equation may be integrable, even if the original one was not. Thus, it may be transformable into a linear equation, or solvable by inverse spectral transform techniques. If necessary, group theory can be applied once more to the reduced equation. Painlevé analysis, an analysis of the singularity structure of the solutions of the reduced PDE, or ODE is a fruitful approach.

3. THE SYMMETRY GROUP OF THE DS EQUATIONS AND STRUCTURE OF ITS LIE ALGEBRA

3.1 The Davey-Stewartson Symmetry Algebra

Firstly we will compute symmetry algebra of DS equations. We rewrite the corresponding system (1.3) in a real form by separating $\psi = u + iv$ into real and imaginary parts,

$$\begin{aligned} \Delta_1 &= u_t + v_{xx} + \varepsilon_1 v_{yy} - \varepsilon_2 v(u^2+v^2) - vw = 0, \\ \Delta_2 &= -v_t + u_{xx} + \varepsilon_1 u_{yy} - \varepsilon_2 u(u^2+v^2) - uw = 0, \\ \Delta_3 &= w_{xx} + \delta_1 w_{yy} - 2\delta_2(u_y^2 + uu_{yy} + v_y^2 + vv_{yy}) = 0. \end{aligned} \tag{3.1}$$

(3.1) is a system of three real partial differential equations. We apply the standard infinitesimal procedure [9] to find the symmetry algebra L and hence the symmetry group G of (3.1). We write the DS equations as a system $\Delta_i(t,x,y,u,v,w,\phi) = 0$, $i = 1,2,3$. A general element of the algebra is represented by a vector field

$$\mathbf{V} = \tau\partial_t + \xi\partial_x + \eta\partial_y + \varphi_1\partial_u + \varphi_2\partial_v + \varphi_3\partial_w, \tag{3.2}$$

where the coefficients $\tau, \xi, \eta, \varphi_i, \quad i = 1,2,3$ are functions of t,x,y,u,v,w. According to the general theory for symmetries of differential equations, to find these functions we prolong the vector field (3.2) to second order derivatives and require that the second prolonged vector field annihilates Δ_i on the solution manifold of the system, namely

$$\mathrm{pr}^{(2)}\mathbf{V}(\Delta_i(t,x,y,u,v,w,\phi))\Big|_{\Delta_i=0} = 0, \quad i = 1,2,3, \tag{3.3}$$

where $\mathrm{pr}^{(2)}\mathbf{V}$ is the second prolongation of the vector field $\mathbf{V}$. This condition provides us with a quite complicated system of determining equations (a system of linear partial differential equations) for the coefficients. This step is entirely algorithmic and is implemented on several computer algebra packages like REDUCE, MATHEMATICA, MAPLE (See [13] for a survey of symbolic softwares for symmetry). We find the determining equations by using MATHEMATICA

$$\eta_x = 0, \quad \tau_x = 0, \quad \xi_y = 0, \quad \tau_y = 0,$$
$$\xi_u = 0, \quad \eta_u = 0, \quad \tau_u = 0, \quad (\varphi_3)_u = 0,$$
$$\xi_v = 0, \quad \eta_v = 0, \quad \tau_v = 0, \quad (\varphi_3)_v = 0,$$
$$\xi_w = 0, \quad \eta_w = 0, \quad \tau_w = 0, \quad (\varphi_1)_w = 0, \quad (\varphi_2)_w = 0 \tag{3.4a}$$
$$(\varphi_1)_{uu} = 0, \quad (\varphi_1)_{uv} = 0, \quad (\varphi_1)_{vv} = 0,$$
$$(\varphi_2)_{uu} = 0, \quad (\varphi_2)_{uv} = 0, \quad (\varphi_2)_{vv} = 0, \quad (\varphi_3)_{ww} = 0,$$
$$-\xi_{xx} + 2(\varphi_1)_{xu} = 0, \quad -\xi_{xx} + 2(\varphi_2)_{xv} = 0, \quad -\xi_{xx} + 2(\varphi_3)_{xw} = 0, \tag{3.4b}$$
$$-2u\delta_2(\varphi_1)_{yy} - 2v\delta_2(\varphi_2)_{yy} + (\varphi_3)_{xx} + \delta_(\varphi_3)_{yy} = 0, \tag{3.4c}$$
$$\begin{aligned}&2uv\varepsilon_2\varphi_1 + w\varphi_2 + u^2\varepsilon_2\varphi_2 + 3v^2\varepsilon_2\varphi_2 + v\varphi_3 + vw\tau_t + u^2v\varepsilon_2\tau_t + v^3\varepsilon_2\tau_t\\ &-(\varphi_1)_t - vw(\varphi_1)_u - u^2v\varepsilon_2(\varphi_1)_u - v^3\varepsilon_2(\varphi_1)_u\\ &+uw(\varphi_1)_v + u^3\varepsilon_2(\varphi_1)_v + uv^2(\varphi_1)_v - (\varphi_2)_{xx} - \varepsilon_1(\varphi_2)_{yy} = 0,\end{aligned} \tag{3.4d}$$
$$\begin{aligned}&-w\varphi_1 - 3u^2\varepsilon_2\varphi_1 - v^2\varepsilon_2\varphi_1 - 2uv\varepsilon_2\varphi_2 - u\varphi_3 - uw\tau_t - u^3\varepsilon_2\tau_t - uv^2\varepsilon_2\tau_t\\ &-(\varphi_2)_t - vw(\varphi_1)_u - u^2v\varepsilon_2(\varphi_1)_u - v^3\varepsilon_2(\varphi_1)_u\\ &+uw(\varphi_1)_v + u^3\varepsilon_2(\varphi_1)_v + uv^2(\varphi_1)_v + (\varphi_1)_{xx} - \varepsilon_1(\varphi_1)_{yy} = 0,\end{aligned} \tag{3.4e}$$
$$\xi_t - 2(\varphi_2)_{xu} = 0, \quad \xi_t + 2(\varphi_1)_{xv} = 0, \quad \xi_t - 2(\varphi_2)_{xu} = 0, \tag{3.4f}$$
$$-\xi_x + \eta_y = 0 \tag{3.4g}$$
$$\begin{aligned}&2\xi_x - \tau_t - (\varphi_1)_u + (\varphi_2)_v = 0,\\ &-2\xi_x + \tau_t - (\varphi_1)_u + (\varphi_2)_v = 0,\end{aligned} \tag{3.4h}$$
$$\begin{aligned}&2(\varphi_2)_y - v\eta_{yy} + 2u(\varphi_1)_{yv} + 2v(\varphi_2)_{yv} = 0,\\ &2(\varphi_1)_y - u\eta_{yy} + 2u(\varphi_1)_{yu} + 2v(\varphi_2)_{yu} = 0,\end{aligned} \tag{3.4i}$$
$$-\eta_{yy} + 2(\varphi_1)_{yu} = 0, \quad -\eta_{yy} + 2(\varphi_2)_{yv} = 0, \quad -\eta_{yy} + 2(\varphi_3)_{yw} = 0, \tag{3.4j}$$
$$\eta_t - 2\varepsilon_1(\varphi_2)_{yu} = 0, \quad \eta_t + 2\varepsilon_1(\varphi_1)_{yv} = 0, \tag{3.4k}$$
$$2\eta_y - \tau_t - (\varphi_1)_u + (\varphi_2)_v = 0, \quad -2\eta_y + \tau_t - (\varphi_1)_u + (\varphi_2)_v = 0, \tag{3.4l}$$
$$\begin{aligned}&\varphi_1 + u(\varphi_1)_u + v(\varphi_2)_u - u(\varphi_3)_w = 0,\\ &\varphi_2 + u(\varphi_1)_v + v(\varphi_2)_v - v(\varphi_3)_w = 0,\end{aligned} \tag{3.4m}$$
$$(\varphi_1)_v + (\varphi_2)_u = 0, \quad (\varphi_3)_w - 2(\varphi_1)_u = 0, \quad (\varphi_3)_w - 2(\varphi_2)_v = 0. \tag{3.4n}$$

When we solve the determining equations in (3.4), we will find the general element (3.2). Using (3.4a), we obtain

$$\begin{aligned}&\xi(x,y,t) = \xi(x,t), \quad \eta(x,y,t) = \eta(x,t), \quad \tau(x,y,t) = f(t),\\ &\varphi_1 = C_1(x,y,t)v + C_2(x,y,t)u + C_3(x,y,t),\\ &\varphi_2 = C_4(x,y,t)v + C_5(x,y,t)u + C_6(x,y,t),\\ &\varphi_3 = C_7(x,y,t)w + C_8(x,y,t).\end{aligned} \tag{3.5}$$

Using (3.4n), we obtain

$$\begin{aligned} C_1 + C_5 = 0 \quad &\Rightarrow \quad C_5 = -C_1, \\ C_7 - 2C_2 = 0 \quad &\Rightarrow \quad C_7 = 2C_2, \\ C_7 - 2C_4 = 0 \quad &\Rightarrow \quad C_7 = 2C_4, \end{aligned} \tag{3.6}$$

From (3.4m) and (3.6) , we get

$$\begin{aligned} vC_1 + 2uC_2 + C_3 + vC_5 - uC_7 = 0 \quad &\Rightarrow \quad C_3 = 0, \\ 2vC_4 + uC_5 + C_6 + uC_1 - vC_7 = 0 \quad &\Rightarrow \quad C_6 = 0. \end{aligned} \tag{3.7}$$

It is obtained from (3.4b) and (3.4j)

$$\begin{aligned} (\varphi_1)_{xu} = (\varphi_2)_{xv} = (\varphi_3)_{xw} \quad &\Rightarrow \quad (C_2)_x = (C_4)_x = (C_7)_x, \\ (\varphi_1)_{yu} = (\varphi_2)_{yv} = (\varphi_3)_{yw} \quad &\Rightarrow \quad (C_2)_y = (C_4)_y = (C_7)_y, \end{aligned} \tag{3.8}$$

Then from (3.7) and (3.8) it is easily seen that

$$C_2(x, y, t) = C_2(t). \tag{3.9}$$

Substituting (3.6) and (3.9) into (3.5), we obtain

$$\begin{aligned} &\xi(x, y, t) = \xi(x, t), \quad \eta(x, y, t) = \eta(x, t), \quad \tau(x, y, t) = f(t), \\ &\varphi_1 = C_1(x, y, t)v + C_2(t)u, \\ &\varphi_2 = C_2(t)v - C_1(x, y, t)u, \\ &\varphi_3 = 2C_2(t)w + C_8(x, y, t). \end{aligned} \tag{3.10}$$

From (3.4h) and (3.4l), it is obtained that

$$\begin{aligned} 2\xi_x - \tau_t - (\varphi_1)_u + (\varphi_2)_v = 2\xi_x - f'(t) - C_2 + C_2 = 0 \Rightarrow \xi = \frac{f'(t)}{2}x + g(t), \\ 2\eta_y - \tau_t - (\varphi_1)_u + (\varphi_2)_v = 2\eta_y - f'(t) - C_2 + C_2 = 0 \Rightarrow \eta = \frac{f'(t)}{2}y + h(t), \end{aligned} \tag{3.11}$$

where $g(t)$ and $h(t)$ are arbitrary functions. Solving (3.4f), we get

$$\xi_t - 2(\varphi_2)_{xu} = 0 \quad \Rightarrow (C_1)_x = -\frac{f''(t)}{4}x - \frac{g'(t)}{2}. \tag{3.12}$$

When we differentiate C_1 with respect to x in (3.12), it is found that

$$C_1(x, y, t) = -\frac{f''(t)}{8}x^2 - \frac{g'(t)}{2}x + C_9(y, t). \tag{3.13}$$

It is obtained by solving (3.4k) {remember that $(\varepsilon_1)^2 = 1$}

$$(\varphi_2)_{yu} = (C_1)_y = (C_9)_y = \frac{1}{2\varepsilon_1}\eta_t = -\frac{\varepsilon_1 f''(t)}{4}y - \frac{\varepsilon_1 h'(t)}{2}. \tag{3.14}$$

When we differentiate C_9 with respect to y in (3.14), it is found that

$$C_9(y,t) = -\frac{\varepsilon_1 f''(t)}{8} y^2 - \frac{\varepsilon_1 h'(t)}{2} y + m(t), \tag{3.15}$$

where $m(t)$ is an arbitrary function. If we solve (3.4d) by using

$$C_1(x,y,t) = -\frac{f''(t)}{8}(x^2 + \varepsilon_1 y^2) - \frac{g'(t)}{2} x - \frac{h'(t)}{2} y + m(t),$$

we obtain

$$\begin{aligned} &[\varepsilon_2 f'(t) - 2\varepsilon_2 C_2(t)]v^3 + [\varepsilon_2 f'(t) - 2\varepsilon_2 C_2(t)]u^2 v + [C_8 + (C_1)_t]v \\ &+ [C_2'(t) - f''(t)/2)]u + [f'(t) - 2C_2(t)]vw = 0. \end{aligned} \tag{3.16}$$

The all coefficients in (3.16) must be equal to zero. So we find that

$$C_2(t) = \frac{f'(t)}{2},\ C_8 = -\frac{f'''(t)}{8}(x^2 + \varepsilon_1 y^2) - \frac{g''(t)}{2} x - \frac{h''(t)}{2} y + m'(t). \tag{3.17}$$

Then the coefficients of the general element (3.2) are

$$\begin{aligned} \xi &= \frac{f'(t)}{2} x + g(t), \\ \eta &= \frac{f'(t)}{2} y + h(t), \\ \tau &= f(t), \\ \varphi_1 &= [-\frac{f''(t)}{8}(x^2 + \varepsilon_1 y^2) - \frac{g'(t)}{2} x - \frac{h'(t)}{2} y + m(t)]v + (\frac{f'(t)}{2})u, \\ \varphi_2 &= (\frac{f'(t)}{2})v - [-\frac{f''(t)}{8}(x^2 + \varepsilon_1 y^2) - \frac{g'(t)}{2} x - \frac{h'(t)}{2} y + m(t)]u, \\ \varphi_3 &= f'(t)w - \frac{f'''(t)}{8}(x^2 + \varepsilon_1 y^2) - \frac{g''(t)}{2} x - \frac{h''(t)}{2} y + m'(t). \end{aligned} \tag{3.18}$$

Substituting (3.18) in (3.4c), we obtain

$$[\frac{uv\delta_2\varepsilon_1}{2} - \frac{uv\delta_2\varepsilon_1}{2}]f''(t) - \frac{1}{4}[\delta_1\varepsilon_1 + 1]f'''(t) = 0 \tag{3.19}$$

Using (3.19) it is seen that $f(t)$ is arbitrary if

$$\delta_1 = -\varepsilon_1 = \pm 1, \tag{3.20}$$

otherwise $f(t) = c_2 t^2 + c_1 t + c_0$.

The functions $g(t)$, $h(t)$, and $m(t)$ are arbitrary functions of class $C^\infty(I)$, $I \subseteq \mathbb{R}$.

The general element can be written as

$$\mathbf{V} = T(f) + X(g) + Y(h) + W(m), \tag{3.21}$$

where

$$
\begin{aligned}
X(f) &= f(t)\partial_t + \frac{1}{2}f'(t)(x\partial_x + y\partial_y - u\partial_u - v\partial_v - 2w\partial_w) \\
&\quad - \frac{(x^2+\varepsilon_1 y^2)}{8}[f''(t)(v\partial_u - u\partial_v) + f'''(t)\partial_w], \\
Y(g) &= g(t)\partial_x - \frac{x}{2}[g'(t)(v\partial_u - u\partial_v) + g''(t)\partial_w], \\
Z(h) &= h(t)\partial_y - \frac{\varepsilon_1 y}{2}[h'(t)(v\partial_u - u\partial_v) + h''(t)\partial_w)], \\
W(m) &= m(t)(v\partial_u - u\partial_v) + m'(t)\partial_w.
\end{aligned}
\tag{3.22}
$$

The DS equations have been shown to be integrable precisely in the case when $f(t)$ is allowed to be arbitrary. We shall mainly concentrate on this case. The commutation relations for the DS algebra (3.21), (3.22) are easy to obtain, namely

$$
\begin{aligned}
&[X(f_1), X(f_2)] = X(f_1 f_2' - f_1' f_2) \\
&[X(f), Y(g)] = Y(fg' - \frac{1}{2}f'g) \\
&[X(f), Z(h)] = Z(fh' - \frac{1}{2}f'h) \\
&[X(f), W(m)] = W(fm') \\
&[Y(g_1), Y(g_2)] = -\frac{1}{2}W(g_1 g_2' - g_1' g_2) \\
&[Z(h_1), Z(h_2)] = -\frac{\varepsilon_1}{2}W(h_1 h_2' - h_1' h_2) \\
&[Y(g), Z(h)] = [Y(g), W(m)] = [Z(h), W(m)] = [W(m_1), W(m_2)] = 0.
\end{aligned}
\tag{3.23}
$$

We see that the DS Lie algebra L allows a Levi decomposition

$$
L = S \ni\!\!\!+ N, \tag{3.24}
$$

where $S = \{T(f)\}$ is a simple infinite dimensional Lie algebra and $N = \{X(g), Y(h), W(m)\}$ is a nilpotent ideal (nilradical). Here, $\ni\!\!\!+$ denotes the semi-direct sum. The DS equations are also invariant under a group of discrete transformations generated by

$$
\begin{aligned}
&t \to t, \quad x \to -x, \quad y \to y, \quad \psi \to \psi, \quad w \to w, \\
&t \to t, \quad x \to x, \quad y \to -y, \quad \psi \to \psi, \quad w \to w, \\
&t \to t, \quad x \to x, \quad y \to y, \quad \psi \to -\psi, \quad w \to w, \\
&t \to -t, \quad x \to x, \quad y \to y, \quad \psi \to \psi^*, \quad w \to w.
\end{aligned}
\tag{3.25}
$$

The obvious physical symmetries L_p of the DS equations are obtained by restricting all the functions f, g, h and m to be first order polynomials. Indeed, we have

$$
\begin{aligned}
&P_0 = X(1) = \partial_t, \quad P_1 = Y(1) = \partial_x, \quad P_2 = Z(1) = \partial_y,\\
&R_0 = W(1) = v\partial_u - u\partial_v,\\
&D = X(t) = t\partial_t + \frac{1}{2}(x\partial_x + y\partial_y - u\partial_u - v\partial_v - 2w\partial_w), \qquad (3.26)\\
&B_1 = Y(t) = t\partial_x - \frac{x}{2}(v\partial_u - u\partial_v), \quad B_2 = Z(t) = t\partial_y - \frac{\varepsilon_1 y}{2}(v\partial_u - u\partial_v),\\
&R_1 = W(t) = t(v\partial_u - u\partial_v) + R_0.
\end{aligned}
$$

We see that P_0, P_1, P_2 generate translations, D dilations, B_1 and B_2 Galilei boosts in the x and y directions, respectively. Finally, R_0 corresponds to a rotation in the (u, v) plane, i.e., a constant change of phase of ψ and R_1 to a change of phase of ψ, linear in t, accompanied by constant shift in w.

3.2 The Group Transformations for the Davey-Stewartson Equations

The elements of the connected part of the symmetry group of the DS equations are obtained by integrating the general element of the DS Lie algebra (3.3). We consider separately the cases $f(t) = 0$ and $f(t) \neq 0$. Note that sometimes it is more convenient to use the polar decomposition $u + iv = Re^{i\sigma}$ so that in (3.22) we can write

$$u\partial_u + v\partial_v = R\partial_R, \quad -(v\partial_u - u\partial_v) = \partial_\sigma.$$

Now we will integrate the equations

$$\frac{d\tilde{x}}{d\lambda} = \frac{f(\tilde{t})}{2}\tilde{x} + g(\tilde{t}), \qquad (3.27a)$$

$$\frac{d\tilde{y}}{d\lambda} = \frac{f(\tilde{t})}{2}\tilde{y} + h(\tilde{t}), \qquad (3.27b)$$

$$\frac{d\tilde{t}}{d\lambda} = f(\tilde{t}), \qquad (3.27c)$$

$$\frac{d\tilde{R}}{d\lambda} = -\frac{f'(\tilde{t})}{2}\tilde{R}, \qquad (3.27d)$$

$$\frac{d\tilde{\sigma}}{d\lambda} = \frac{(\tilde{x}^2 + \varepsilon_1\tilde{y}^2)}{8}f''(\tilde{t}) + \frac{\tilde{x}}{2}g'(\tilde{t}) + \frac{\varepsilon_1\tilde{y}}{2}h'(\tilde{t}) - m(\tilde{t}), \qquad (3.27e)$$

$$\frac{d\tilde{w}}{d\lambda} = -f'(\tilde{t})\tilde{w} - \frac{(\tilde{x}^2 + \varepsilon_1\tilde{y}^2)}{8}f'''(\tilde{t}) - \frac{\tilde{x}}{2}g''(\tilde{t}) - \frac{\varepsilon_1\tilde{y}}{2}h''(\tilde{t}) + m'(\tilde{t}) \qquad (3.27f)$$

with boundary conditions $\tilde{x}(0) = x$, $\tilde{y}(0) = y$, $\tilde{t}(0) = t$, $\tilde{R}(0) = R$, $\tilde{\sigma}(0) = \sigma$ and $\tilde{w}(0) = w$.

(i) Case $f(t) = 0$,

It is easily seen that

$$\tilde{t}(\lambda) = t, \quad \tilde{x}(\lambda) = \lambda g(t) + x, \quad \tilde{y}(\lambda) = \lambda h(t) + y. \tag{3.28}$$

We will find $\tilde{\Psi} = \tilde{R}\exp[i\tilde{\sigma}]$ by integrating (3.27d) and (3.27e). It is obvious that $\tilde{R} = R$ from (3.27d).

Integrating (3.27e)and by using (3.28) we get

$$\tilde{\sigma} = \frac{\lambda}{2}g'(\tilde{t})(\tilde{x} - \frac{\lambda}{2}g(\tilde{t})) + \frac{\varepsilon_1\lambda}{2}h'(\tilde{t})(\tilde{y} - \frac{\lambda}{2}h(\tilde{t})) - \lambda m(\tilde{t}) + c. \tag{3.29}$$

Using initial condition $\tilde{\sigma} = \sigma$ when $\lambda = 0$ is, we obtain

$$c = \sigma - \frac{\lambda}{2}g'(t)(x - \frac{\lambda}{2}g(t)) - \frac{\varepsilon_1\lambda}{2}h'(t)(y - \frac{\lambda}{2}h(t)) + \lambda m(t). \tag{3.30}$$

Substituting (3.30) into (3.29), it is obtained

$$\tilde{\Psi}(\tilde{x}, \tilde{y}, \tilde{t}) = \Psi(x, y, t)\exp i\left\{\frac{\lambda}{2}g'(\tilde{t})(\tilde{x} - \frac{\lambda}{2}g(\tilde{t})) + \frac{\varepsilon_1\lambda}{2}h'(\tilde{t})(\tilde{y} - \frac{\lambda}{2}h(\tilde{t})) - \lambda m(\tilde{t})\right\}. \tag{3.31}$$

When we integrate (3.27e) and by using (3.28), we get

$$\tilde{w}(\tilde{x}, \tilde{y}, \tilde{t}) = w(x, y, t) - \frac{\lambda}{2}g''(\tilde{t})\left(\tilde{x} - \frac{\lambda}{2}g(\tilde{t})\right) - \frac{\varepsilon_1\lambda}{2}h''(\tilde{t})\left(\tilde{y} - \frac{\lambda}{2}h(\tilde{t})\right) + \lambda m(\tilde{t}). \tag{3.32}$$

(ii) Case $f(t) \neq 0$,

If we integrate (3.27d) with boundary condition, we obtain $\frac{d\tilde{t}}{f(\tilde{t})} = \lambda + \int \frac{dt}{f(t)}$. If we assume that

$$\int \frac{dt}{f(t)} = \phi(t),$$

we obtain $\phi(\tilde{t}) = \lambda + \phi(t)$ and so it is seen that $\tilde{t} = \phi^{-1}(\lambda + \phi(t))$. When we integrate $\phi(\tilde{t})$ with respect to λ, we get $d\lambda = \frac{d\tilde{t}}{f(\tilde{t})}$.

(3.27a) is a first order linear differential equation. We solve it to find

$$\tilde{x} = f^{1/2}(\tilde{t})\left(\int f^{-3/2}(\tilde{t})g(\tilde{t})d\tilde{t} + c\right).$$

With the initial conditions, we have

$$\tilde{x} = \frac{f^{1/2}(\tilde{t})}{f^{1/2}(t)}[x + G(t, \tilde{t})], \tag{3.33}$$

where

$$G(t,\tilde{t}) = f^{1/2}(t)\int_t^{\tilde{t}} f^{-3/2}(s)g(s)ds.$$

(3.27b) is solved like (3.27a) and it is found that

$$\tilde{y} = \frac{f^{1/2}(\tilde{t})}{f^{1/2}(t)}[y + H(t,\tilde{t})], \tag{3.34}$$

where

$$H(t,\tilde{t}) = f^{1/2}(t)\int_t^{\tilde{t}} f^{-3/2}(s)h(s)ds.$$

If we solve (3.27d), we see that

$$\tilde{R} = R[\frac{f(t)}{f(\tilde{t})}]^{1/2}. \tag{3.35}$$

When we integrate (3.27e), we obtain

$$\tilde{\sigma} = \int \left(\frac{(\tilde{x}^2 + \varepsilon_1\tilde{y}^2)}{8f(\tilde{t})} f''(\tilde{t}) + \frac{\tilde{x}}{2f(\tilde{t})} g'(\tilde{t}) + \frac{\varepsilon_1\tilde{y}}{2f(\tilde{t})} h'(\tilde{t}) - \frac{m(\tilde{t})}{f(\tilde{t})} \right) d\tilde{t} + c. \tag{3.36}$$

For the first three integrals of (3.36), we let

$$\begin{aligned} \frac{(\tilde{x}^2 + \varepsilon_1\tilde{y}^2)}{f(\tilde{t})} &= u, \quad f''(\tilde{t})d\tilde{t} = dv, \\ \frac{\tilde{x}}{f(\tilde{t})} &= u, \quad g'(\tilde{t})d\tilde{t} = dv, \\ \frac{\tilde{y}}{f(\tilde{t})} &= u, \quad h'(\tilde{t})d\tilde{t} = dv, \end{aligned} \tag{3.37}$$

respectively. The derivatives of $\tilde{x}$ and $\tilde{y}$ according to $\tilde{t}$ can easily be computed as

$$\tilde{x}' = \frac{d\tilde{x}}{d\tilde{t}} = \left(\frac{f'(\tilde{t})}{2f(\tilde{t})}\tilde{x} + \frac{g(\tilde{t})}{f(\tilde{t})} \right), \quad \tilde{y}' = \frac{d\tilde{y}}{d\tilde{t}} = \left(\frac{f'(\tilde{t})}{2f(\tilde{t})}\tilde{y} + \frac{h(\tilde{t})}{f(\tilde{t})} \right). \tag{3.38}$$

By using integration by parts for the integrals in (3.36), we obtain

$$\tilde{\sigma} = \frac{1}{8}\frac{f'(\tilde{t})}{f(\tilde{t})}(\tilde{x}^2 + \varepsilon_1\tilde{y}^2) + \frac{1}{2}\frac{g(\tilde{t})}{f(\tilde{t})}\tilde{x} + \frac{\varepsilon_1}{2}\frac{h(\tilde{t})}{f(\tilde{t})}\tilde{y} - \frac{1}{2}\int \frac{g^2(\tilde{t}) + h^2(\tilde{t}) + 2m(\tilde{t})f(\tilde{t})}{f(\tilde{t})} d\tilde{t} + c. \tag{3.39}$$

Using $\tilde{\sigma} = \sigma$ when $\lambda = 0$ is, we find

$$\begin{aligned} \tilde{\sigma} =& \frac{1}{8}\frac{f'(\tilde{t})}{f(\tilde{t})}(\tilde{x}^2 + \varepsilon_1\tilde{y}^2) + \frac{1}{2}\frac{g(\tilde{t})}{f(\tilde{t})}\tilde{x} + \frac{\varepsilon_1}{2}\frac{h(\tilde{t})}{f(\tilde{t})}\tilde{y} - \frac{1}{2}\int \frac{g^2(\tilde{t}) + h^2(\tilde{t}) + 2m(\tilde{t})f(\tilde{t})}{f(\tilde{t})} d\tilde{t} \\ &- \frac{1}{8}\frac{f'(t)}{f(t)}(x^2 + \varepsilon_1 y^2) + \frac{1}{2}\frac{g(t)}{f(t)}x + \frac{\varepsilon_1}{2}\frac{h(t)}{f(t)}y - \frac{1}{2}\int \frac{g^2(t) + h^2(t) + 2m(t)f(t)}{f(t)} dt + \sigma. \end{aligned} \tag{3.40}$$

It is easily calculated that

$$\begin{aligned} G(\tilde{t},t) &= -\frac{f^{1/2}(\tilde{t})}{f^{1/2}(t)}G(t,\tilde{t}), \quad H(\tilde{t},t) = -\frac{f^{1/2}(\tilde{t})}{f^{1/2}(t)}H(t,\tilde{t}), \\ G^2(\tilde{t},t) &= \frac{f(\tilde{t})}{f(t)}G^2(t,\tilde{t}), \quad H^2(\tilde{t},t) = \frac{f(\tilde{t})}{f(t)}H^2(t,\tilde{t}). \end{aligned} \tag{3.41}$$

From (3.33) and (3.34), we see that

$$\begin{aligned} x &= \frac{f^{1/2}(t)}{f^{1/2}(\tilde{t})}(\tilde{x} + G(\tilde{t},t)) = \frac{f^{1/2}(t)}{f^{1/2}(\tilde{t})}\left(\tilde{x} - \frac{f^{1/2}(\tilde{t})}{f^{1/2}(t)}G(t,\tilde{t})\right), \\ y &= \frac{f^{1/2}(t)}{f^{1/2}(\tilde{t})}(\tilde{y} + H(\tilde{t},t)) = \frac{f^{1/2}(t)}{f^{1/2}(\tilde{t})}\left(\tilde{y} - \frac{f^{1/2}(\tilde{t})}{f^{1/2}(t)}H(t,\tilde{t})\right). \end{aligned} \tag{3.42}$$

Substituting (3.42) in (3.40), we get

$$\begin{aligned} \tilde{\psi}(\tilde{x},\tilde{y},\tilde{t}) =& [\frac{f(t}{f(\tilde{t})}]^{1/2}\psi(x,y,t)\exp i(\tilde{x}^2 + \varepsilon_1\tilde{y}^2)(\frac{f'(\tilde{t}) - f'(t)}{8f(\tilde{t})}) \\ &+ \tilde{x}\frac{1}{2f^{1/2}(\tilde{t})f^{1/2}(t)}\left[g(\tilde{t})\left[\frac{f(t)}{f(\tilde{t})}\right]^{1/2} - g(t) + \frac{1}{2}f'(t)G(t,\tilde{t})\right] \\ &+ \tilde{y}\frac{\varepsilon_1}{2f^{1/2}(\tilde{t})f^{1/2}(t)}\left[h(\tilde{t})\left[\frac{f(t)}{f(\tilde{t})}\right]^{1/2} - h(t) + \frac{1}{2}f'(t)H(t,\tilde{t})\right] \\ &+ \frac{1}{2f(t)}[g(t)G(t,\tilde{t}) + h(t)H(t,\tilde{t})] - \frac{f'(t)}{8f(t)}[G^2(t,\tilde{t}) + H^2(t,\tilde{t})] \\ &- \int_t^{\tilde{t}} \frac{g^2(s) + h^2(s) + 2m(s)f(s)}{2f(s)}ds. \end{aligned} \tag{3.43}$$

(3.27f) is again a linear first order differential equation, then we find that integrating factor is

$$\mu = \exp(\int f'(\tilde{t})d\lambda) = \exp(\int f'(\tilde{t})\frac{d\tilde{t}}{f(\tilde{t})}) = f(\tilde{t}). \tag{3.44}$$

By using (3.44), we obtain

$$\begin{aligned} \tilde{w} =& -\frac{1}{f(\tilde{t})}\int \frac{(\tilde{x}^2 + \varepsilon_1\tilde{y}^2)}{8}f'''(\tilde{t})d\tilde{t} - \frac{1}{f(\tilde{t})}\int \frac{\tilde{x}}{2}g''(\tilde{t})d\tilde{t} - \frac{1}{f(\tilde{t})}\int \frac{\varepsilon_1\tilde{y}}{2}h''(\tilde{t})d\tilde{t} \\ &+ \frac{m(\tilde{t})}{f(\tilde{t})} + \frac{c}{f(\tilde{t})}. \end{aligned} \tag{3.45}$$

For the first three integral of (3.45), we let

$$\begin{aligned} \tilde{x}^2 + \varepsilon_1\tilde{y}^2 = u, &\quad f'''(\tilde{t})d\tilde{t} = dv, \\ \tilde{x} = u, &\quad g''(\tilde{t})d\tilde{t} = dv, \\ \tilde{y} = u, &\quad h''(\tilde{t})d\tilde{t} = dv, \end{aligned} \tag{3.46}$$

respectively. By using integration by parts in (3.45), we get

$$\begin{aligned}\tilde{w} &= -\frac{f''(\tilde{t})}{8f(\tilde{t})}(\tilde{x}^2+\varepsilon_1\tilde{y}^2) - \frac{1}{4f(\tilde{t})}\int\frac{g(\tilde{t})f''(\tilde{t})}{f(\tilde{t})}\tilde{x}d\tilde{t} - \frac{1}{4f(\tilde{t})}\int\frac{\varepsilon_1 h(\tilde{t})f''(\tilde{t})}{f(\tilde{t})}\tilde{y}d\tilde{t}\\ &+ \frac{1}{8f(\tilde{t})}\int\frac{\tilde{x}^2+\varepsilon_1\tilde{y}^2}{f(\tilde{t})}f'(\tilde{t})f''(\tilde{t})d\tilde{t} - \frac{g'(\tilde{t})}{2f(\tilde{t})}\tilde{x} - \frac{\varepsilon_1 h'(\tilde{t})}{2f(\tilde{t})}\tilde{y}\\ &+ \frac{1}{4f(\tilde{t})}\int\frac{g'(\tilde{t})f'(\tilde{t})}{f(\tilde{t})}\tilde{x}d\tilde{t} + \frac{1}{2f(\tilde{t})}\int\frac{g'(\tilde{t})g(\tilde{t})}{f(\tilde{t})}d\tilde{t} + \frac{\varepsilon_1}{4f(\tilde{t})}\int\frac{h'(\tilde{t})f'(\tilde{t})}{f(\tilde{t})}\tilde{y}d\tilde{t}\\ &+ \frac{\varepsilon_1}{2f(\tilde{t})}\int\frac{h'(\tilde{t})h(\tilde{t})}{f(\tilde{t})}d\tilde{t} + \frac{m(\tilde{t})}{f(\tilde{t})} + \frac{c}{f(\tilde{t})}.\end{aligned} \tag{3.47}$$

$$\begin{aligned}\int\frac{\tilde{x}^2+\varepsilon_1\tilde{y}^2}{f(\tilde{t})}f'(\tilde{t})f''(\tilde{t})d\tilde{t} \quad &\Rightarrow \quad \frac{\tilde{x}^2+\varepsilon_1\tilde{y}^2}{f(\tilde{t})} = u(\tilde{t}),\ f''(\tilde{t})f'(\tilde{t})d\tilde{t} = dv.\\ \int\frac{g'(\tilde{t})g(\tilde{t})}{f(\tilde{t})}d\tilde{t} \quad &\Rightarrow \quad \frac{1}{f(\tilde{t})} = u, \qquad g'(\tilde{t})g(\tilde{t})d\tilde{t} = dv.\\ \int\frac{h'(\tilde{t})h(\tilde{t})}{f(\tilde{t})}d\tilde{t} \quad &\Rightarrow \quad \text{we let}\frac{1}{f(\tilde{t})} = u, \quad h'(\tilde{t})h(\tilde{t})d\tilde{t} = dv.\end{aligned} \tag{3.48}$$

Applying the above substitution, we obtain

$$\begin{aligned}\tilde{w} &= -\frac{f''(\tilde{t})}{8f(\tilde{t})}(\tilde{x}^2+\varepsilon_1\tilde{y}^2) + \frac{1}{4f(\tilde{t})}\int\frac{g(\tilde{t})f''(\tilde{t})}{f(\tilde{t})}\tilde{x}d\tilde{t} + \frac{1}{4f(\tilde{t})}\int\frac{\varepsilon_1 h(\tilde{t})f''(\tilde{t})}{f(\tilde{t})}\tilde{y}d\tilde{t}\\ &\frac{[f'(\tilde{t})]^2}{16f^2(\tilde{t})}(\tilde{x}^2+\varepsilon_1\tilde{y}^2) - \frac{1}{8f(\tilde{t})}\int\frac{g(\tilde{t})f'(\tilde{t})^2}{f^2(\tilde{t})}\tilde{x}d\tilde{t} - \frac{1}{8f(\tilde{t})}\int\frac{h(\tilde{t})f'(\tilde{t})^2}{f^2(\tilde{t})}\tilde{y}d\tilde{t}\\ &\frac{-g'(\tilde{t})}{2f(\tilde{t})}\tilde{x} - \frac{\varepsilon_1 h'(\tilde{t})}{2f(\tilde{t})}\tilde{y} + \frac{1}{4f(\tilde{t})}\int\frac{g'(\tilde{t})f'(\tilde{t})}{f(\tilde{t})}\tilde{x}d\tilde{t} + \frac{\varepsilon_1}{4f(\tilde{t})}\int\frac{h'(\tilde{t})f'(\tilde{t})}{f(\tilde{t})}\tilde{y}d\tilde{t}\\ &\frac{g^2(\tilde{t})}{4f^2(\tilde{t})} + \frac{1}{4f(\tilde{t})}\int\frac{g^2(\tilde{t})f'(\tilde{t})}{f^2(\tilde{t})}d\tilde{t} + \frac{\varepsilon_1 h^2(\tilde{t})}{4f^2(\tilde{t})} + \frac{\varepsilon_1}{4f(\tilde{t})}\int\frac{h^2(\tilde{t})f'(\tilde{t})}{f^2(\tilde{t})}d\tilde{t}\\ &+ \frac{m(\tilde{t})}{f(\tilde{t})} + \frac{c}{f(\tilde{t})}.\end{aligned} \tag{3.49}$$

We rearrange (3.49) and we get

$$\begin{aligned}\tilde{w} &= (\tilde{x}^2+\varepsilon_1\tilde{y}^2)(\frac{[f'(\tilde{t})]^2}{16f^2(\tilde{t})} - \frac{f''(\tilde{t})}{8f(\tilde{t})}) - \frac{g'(\tilde{t})}{2f(\tilde{t})}\tilde{x} - \frac{\varepsilon_1 h'(\tilde{t})}{2f(\tilde{t})}\tilde{y} + \frac{g^2(\tilde{t})}{4f^2(\tilde{t})} + \frac{\varepsilon_1 h^2(\tilde{t})}{4f^2(\tilde{t})}\\ &+ \frac{1}{4f(\tilde{t})}\int[\frac{g(\tilde{t})f''(\tilde{t})}{f(\tilde{t})}\tilde{x} - \frac{g(\tilde{t})f'(\tilde{t})^2}{2f^2(\tilde{t})}\tilde{x} + \frac{g'(\tilde{t})f'(\tilde{t})}{f(\tilde{t})}\tilde{x} + \frac{g^2(\tilde{t})f'(\tilde{t})}{f^2(\tilde{t})}]d\tilde{t}\\ &+ \frac{\varepsilon_1}{4f(\tilde{t})}\int[\frac{h(\tilde{t})f''(\tilde{t})}{f(\tilde{t})}\tilde{y} - \frac{h(\tilde{t})f'(\tilde{t})^2}{2f^2(\tilde{t})}\tilde{y} + \frac{h'(\tilde{t})f'(\tilde{t})}{f(\tilde{t})}\tilde{y} + \frac{h^2(\tilde{t})f'(\tilde{t})}{f^2(\tilde{t})}]d\tilde{t}\\ &+ \frac{m(\tilde{t})}{f(\tilde{t})} + \frac{c}{f(\tilde{t})}.\end{aligned} \tag{3.50}$$

It can be computed that

$$\begin{aligned}\frac{f'(\tilde{t})g(\tilde{t})}{f(\tilde{t})}\tilde{x} &= \int[\frac{g(\tilde{t})f''(\tilde{t})}{f(\tilde{t})}\tilde{x} - \frac{g(\tilde{t})f'(\tilde{t})^2}{2f^2(\tilde{t})}\tilde{x} + \frac{g'(\tilde{t})f'(\tilde{t})}{f(\tilde{t})}\tilde{x} + \frac{g^2(\tilde{t})f'(\tilde{t})}{f^2(\tilde{t})}]d\tilde{t},\\ \frac{f'(\tilde{t})h(\tilde{t})}{f(\tilde{t})}\tilde{y} &= \int[\frac{h(\tilde{t})f''(\tilde{t})}{f(\tilde{t})}\tilde{y} - \frac{h(\tilde{t})f'(\tilde{t})^2}{2f^2(\tilde{t})}\tilde{y} + \frac{h'(\tilde{t})f'(\tilde{t})}{f(\tilde{t})}\tilde{y} + \frac{h^2(\tilde{t})f'(\tilde{t})}{f^2(\tilde{t})}]d\tilde{t}.\end{aligned} \tag{3.51}$$

If we substitute (3.51) in (3.50), we obtain

$$\begin{aligned}\tilde{w} =& \frac{(\tilde{x}^2+\varepsilon_1\tilde{y}^2)}{8f^2(\tilde{t})}\left(\frac{[f'(\tilde{t})]^2}{2} - f''(\tilde{t})f(\tilde{t})\right) + \frac{\tilde{x}}{4f^2(\tilde{t})}[f'(\tilde{t})g(\tilde{t}) - 2g'(\tilde{t})f(\tilde{t})] \\ &+ \frac{\varepsilon_1\tilde{y}}{4f^2(\tilde{t})}[f'(\tilde{t})h(\tilde{t}) - 2h'(\tilde{t})f(\tilde{t})] + \frac{g^2(\tilde{t})}{4f^2(\tilde{t})} + \frac{\varepsilon_1 h^2(\tilde{t})}{4f^2(\tilde{t})} + \frac{m(\tilde{t})}{f(\tilde{t})} + \frac{c}{f(\tilde{t})}.\end{aligned} \tag{3.52}$$

Using the initial conditions of (3.27f) we find that

$$\begin{aligned}c =& wf(t) + \frac{(x^2+\varepsilon_1 y^2)}{8f(t)}[f''(t) - \frac{[f'(t)]^2}{2}] + \frac{x}{4f^2(t)}[f'(t)]g(t) - 2g'(t)f(t)] \\ &+ \frac{\varepsilon_1 y}{4f^2(t)}[f'(t)]h(t) - 2h'(t)f(t)] - \frac{g^2(t)}{4f^2(t)} - \frac{\varepsilon_1 h^2(t)}{4f^2(t)} - m'(t).\end{aligned} \tag{3.53}$$

Substituting (3.42) and (3.53) in (3.52), we obtain

$$\begin{aligned}\tilde{w}(\tilde{x},\tilde{y},\tilde{t}) =& [f(t)/f(\tilde{t})]w(x,y,t) \\ &- \frac{1}{8f^2(\tilde{t})}(\tilde{x}^2+\varepsilon_1\tilde{y}^2)\left[f''(\tilde{t})f(\tilde{t}) - \frac{[f'(\tilde{t})]^2}{2} - f''(t)f(t) + \frac{[f'(t)]^2}{2}\right] \\ &- \frac{1}{4f^{1/2}(t)f^{3/2}(\tilde{t})}\tilde{x}([2g'(\tilde{t})f(\tilde{t}) - f'(\tilde{t})g(\tilde{t})]\left[\frac{f(t)}{f(\tilde{t})}\right]^{1/2} \\ &- [2g'(t)f(t) - f'(t)g(t)] + [f''(t)f(t) - \frac{[f'(t)]^2}{2}]G(t,\tilde{t})) \\ &- \frac{\varepsilon_1}{4f^{1/2}(t)f^{3/2}(\tilde{t})}\tilde{y}([2h'(\tilde{t})f(\tilde{t}) - f'(\tilde{t})h(\tilde{t})]\left[\frac{f(t)}{f(\tilde{t})}\right]^{1/2} \\ &- [2h'(t)f(t) - f'(t)h(t)] + [f''(t)f(t) - \frac{[f'(t)]^2}{2}]H(t,\tilde{t})) \\ &- \frac{1}{4f(t)f(\tilde{t})}\{[2g'(t)f(t) - f'(t)g(t)]G(t,\tilde{t}) \\ &+ \varepsilon_1[2h'(t)f(t) - f'(t)h(t)]H(t,\tilde{t})\} \\ &+ \frac{1}{4f(t)f(\tilde{t})}[f''(t)f(t) - \frac{[f'(t)]^2}{2}][G^2(t,\tilde{t}) + H^2(t,\tilde{t})] \\ &+ \frac{g^2(\tilde{t}) + \varepsilon_1 h^2(\tilde{t}) + 4m(\tilde{t})f(\tilde{t})}{4f^2(\tilde{t})} - \frac{g^2(t) + \varepsilon_1 h^2(t) + 4m(t)f(t)}{4f(\tilde{t})f(t)}.\end{aligned} \tag{3.54}$$

3.3 Symmetry Reduction for the Davey-Stewartson Equations

We shall now use the results of the previous sections to reduce the DS equations to a system of equations involving two independent variables only. To do this we make use of the one dimensional subalgebras of the DS algebra. Depending on which of the functions $g(t)$, $h(t)$ and $m(t)$ are nonzero, precisely six conjugacy classes of one-dimensional subalgebras exist:

$$\begin{aligned}&L_{1,1} = \{X(1)\}, \quad L_{1,2}^a = \{Y(1) + aZ(1)\}, \quad L_{1,3}(h) = \{Y(1) + Z(h)\}, \\ &L_{1,4} = \{Z(1)\}, \quad L_{1,5} = \{W(t)\}, \quad L_{1,6} = \{W(1)\}\end{aligned} \tag{3.55}$$

with $a \geq 0$. The method is standard and quite simple. We consider an auxiliary function $F(x, y, t, u, v, w)$ and request that it be annihilated by the elements of the one-dimensional subalgebra X:

$$XF = 0. \tag{3.56}$$

(3.56) implies that F is a function of five variables only, namely the invariants of the Lie group generated by X. Two invariants ξ and η can be chosen to depend on x, y and t only, these are the new symmetry variables. The remaining invariants yield the dependence of u, v and w on the symmetry variables.

We used one-dimensional subalgebras $L_{1,1}$, $L^a_{1,2}$, $L_{1,3}(h)$ and $L_{1,4}$ which generate actions on the coordinate space (t, x, y) to reduce the DS equations to integrable one in two variables. We shall perform the reduction using the standard basis elements of (3.55).

The Algebra $L_{1,1}$

The equation $X(1)F(x, y, t, u, v, w) = \partial_t F(x, y, t, u, v, w) = 0$ tells us that the invariants of $\exp X(1)$ are x,y,u,v and w. The reduction is hence obtained by setting

$$\Psi(x, y, t) = \phi(\xi, \eta), \quad w(x, y, t) = Q(\xi, \eta), \quad \xi = x, \quad \eta = y. \tag{3.57}$$

Substituting (3.57) into the DS equations (1.3), we obtain the reduced system

$$\phi_{\xi\xi} + \varepsilon_1 \phi_{\eta\eta} = \varepsilon_2 |\phi|^2 \phi + \phi Q, \tag{3.58a}$$

$$Q_{\xi\xi} + \delta_1 Q_{\eta\eta} = \delta_2 (|\phi|^2)_{\eta\eta}. \tag{3.58b}$$

Applying a general DS group transformation to a solution of (3.58a) and (3.58b) we obtain a class of solutions of the DS equations, depending on four arbitrary functions $f(t), g(t), h(t)$ and $m(t)$. Thus assuming $f(t) \neq 0$, we obtain

$$\begin{aligned}
&\xi = x f^{-1/2} - \int_0^t g(s)[f(s)]^{-3/2} ds, \\
&\eta = y f^{-1/2} - \int_0^t h(s)[f(s)]^{-3/2} ds, \\
&\Psi = \phi(\xi, \eta) f^{-1/2} \exp i \left[\frac{f'}{8f}(x^2 + \varepsilon_1 y^2) + \frac{1}{2f}(xg + \varepsilon_1 yh) - \frac{1}{2} \int \frac{\varepsilon_1 h^2 + g^2 + 2mf}{f^2} dt \right], \\
&w = \frac{1}{f} Q(\xi, \eta) - \frac{1}{8f^2}(ff'' - \frac{[f']^2}{2})(x^2 + \varepsilon_1 y^2) - \frac{x}{4f^2}(2g'f - gf') - \frac{\varepsilon_1 y}{4f^2}(2h'f - hf') \\
&+ \frac{g^2 + \varepsilon_1 h^2 + 4mf}{4f^2}.
\end{aligned} \tag{3.59}$$

Substituting (3.59) into the DS equations (1.3) we find that $\phi(\xi,\eta)$ and $Q(\xi,\eta)$ must satisfy (3.58a) and

$$8[Q_{\xi\xi} + \delta_1 Q_{\eta\eta} - \delta_2(|\phi|^2)_{\eta\eta}] = (\delta_1\varepsilon_1 + 1)[2ff_{tt} - (f_t)^2], \tag{3.60}$$

which reduces to (3.58b) if $\delta_1 = -\varepsilon_1$ or if $f(t) = (a+bt)^2$.

The Algebra $L^a_{1,2}$

We have

$$[Y(1) + aZ(1)]F = [\partial_x + a\partial_y]F = 0. \tag{3.61}$$

The characteristic system for (3.61) is

$$\frac{dx}{1} = \frac{dy}{a} = \frac{dt}{0} = \frac{du}{0} = \frac{dv}{0} = \frac{dw}{0}. \tag{3.62}$$

By solving (3.62) and remembering $\psi = u + iv$ we obtain

$$\psi(x,y,t) = \Omega(\xi,\zeta), \quad w(x,y,t) = \Theta(\xi,\zeta) \quad ,\xi = t, \quad \zeta = y - ax. \tag{3.63}$$

By substituting into the DS equations we obtain the reduced system

$$\begin{aligned} &i\Omega_\xi + (a^2+\varepsilon_1)\Omega_{\zeta\zeta} = \varepsilon_2|\Omega|^2\Omega + \Theta\Omega, \\ &(a^2+\varepsilon_1)\Theta_{\zeta\zeta} = \delta_2(|\Omega|^2)_{\zeta\zeta}. \end{aligned} \tag{3.64}$$

We solve the second equation choosing $a^2 \neq -\delta_1$:

$$\Theta(\xi,\zeta) = \frac{\delta_2}{(a^2+\delta_1)}|\Omega|^2 + \alpha(\xi)\zeta + \beta(\xi). \tag{3.65}$$

where $\alpha(\xi)$ and $\beta(\xi)$ are arbitrary functions. Expression (3.65) can be substituted back into (3.64) and we obtain an equation for $\Omega(\xi,\zeta)$ alone

$$i\Omega_\xi + (a^2+\varepsilon_1)\Omega_{\zeta\zeta} = (\varepsilon_2 + \frac{\delta_2}{(a^2+\delta_1)})|\Omega|^2\Omega + [\alpha(\xi)\zeta + \beta(\xi)]\Omega. \tag{3.66}$$

(3.66) can be reduced to a NLS equation with variable coefficients. Substituting

$$\Omega(\xi,\zeta) = K\phi(\xi,\eta)\exp i(\zeta F(\xi) + G(\xi)), \quad \zeta = y - ax, \quad \eta = c_1\zeta + H(\xi),$$

in (3.65), we get

$$\begin{aligned} &K\exp i[\zeta F(\xi) + G(\xi)]\{[-F'(\xi) - \alpha(\xi)]\zeta\phi - [G'(\xi) + F^2(\xi) - \beta(\xi)]\phi \\ &+ i\phi_\xi + [iH'(\xi) + 2ic_1F(\xi)]\phi_\eta + (a^2+\delta_1)c_1^2\phi_{\eta\eta}\} \\ &= K\exp i[\zeta F(\xi) + G(\xi)](\varepsilon_2 + \frac{\delta_2}{a^2+\delta_1})K^2|\phi|^2\phi. \end{aligned} \tag{3.67}$$

If we normalize the coefficients of $\phi_{\eta\eta}$, $|\phi|^2\phi$ in (3.67), we find

$$
\begin{aligned}
&(a^2+\delta_1)c_1^2 = 1 \quad \Rightarrow \quad c_1 = [\varepsilon_3(a^2+\delta_1)]^{-1/2}, \qquad \varepsilon_3 = \mathrm{sgn}(a^2+\delta_1),\\
&(\varepsilon_2 + \frac{\delta_2}{a^2+\delta_1})K^2 = 1 \Rightarrow \quad K = \frac{a^2+\delta_1}{\varepsilon_2(a^2+\delta_1)+\delta_2}\varepsilon_4, \quad \varepsilon_4 = \mathrm{sgn}(\frac{a^2+\delta_1}{\varepsilon_2(a^2+\delta_1)+\delta_2}).
\end{aligned} \tag{3.68}
$$

In (3.67), we choose the coefficients of ϕ and ϕ_η so as to be zero. Hence we get

$$
\begin{aligned}
-F'(\xi) - \alpha(\xi) = 0 \quad &\Rightarrow \quad F(t) = -\int \alpha(t)dt,\\
G'(\xi) + F^2(\xi) - \beta(\xi) = 0 \quad &\Rightarrow \quad G(t) = -\int (a^2+\delta_1)F^2(t) + \beta(t)dt,\\
iH'(\xi) + 2ic_1F(\xi) = 0 \quad &\Rightarrow \quad H(t) = -2[\varepsilon_3(a^2+\delta_1)]^{-1/2}\int F(t)dt.
\end{aligned} \tag{3.69}
$$

Then (3.66) is

$$
\varepsilon_3 i\phi_\xi + \phi_{\eta\eta} = \varepsilon_3\varepsilon_4|\phi|^2\phi. \tag{3.70}
$$

By a rescaling of ξ, we see that (3.70) satisfies the NLS equation

$$
i\phi_\xi + \phi_{\eta\eta} = \varepsilon_3\varepsilon_4|\phi|^2\phi.
$$

The Algebra $L_{1,3}(h)$

We have

$$
[Y(1) + Z(h)F] = \{\partial_x + h\partial_y - (\varepsilon_1/2)y[h'(v\partial_u - u\partial_v) + h''\partial_w]\}F = 0. \tag{3.71}
$$

The characteristic system for (3.71) is

$$
\frac{dx}{1} = \frac{dy}{h} = \frac{dt}{0} = -\frac{2du}{\varepsilon_1 yh'u} = \frac{2dv}{\varepsilon_1 yh'u} = \frac{-2dw}{\varepsilon_1 yh''}. \tag{3.72}
$$

By solving (3.72)we obtain

$$
\begin{aligned}
\Psi &= \phi(\xi,\eta)\exp i(\frac{\varepsilon_1 h'}{4h}y^2), \quad \xi = t,\\
w &= \Theta(\xi,\eta) - \frac{\varepsilon_1 h''}{4h}y^2, \qquad \eta = y - h(t)x.
\end{aligned} \tag{3.73}
$$

Substituting (3.73) in DS equations, we get

$$
i\phi_\xi + (\varepsilon_1 + h^2)\phi_{\eta\eta} + \frac{ih'}{h}\eta\phi_\eta + \frac{ih'}{2h}\phi = \varepsilon_2|\phi|^2\phi + \phi\Theta, \tag{3.74a}
$$

$$
(h^2+\delta_1)\Theta_{\eta\eta} - \frac{\varepsilon_1\delta_1 h''}{2h} = \delta_2(|\phi|^2)_{\eta\eta}. \tag{3.74b}
$$

Solving (3.74b) and substituting into (3.74a), we find

$$
\begin{aligned}
&i\phi_\xi + (\varepsilon_1 + h^2)\phi_{\eta\eta} + \frac{ih'}{h}\eta\phi_\eta + \{\frac{ih'}{2h} - \frac{\varepsilon_1\delta_1 h''}{4h(h^2+\delta_1)}\eta^2 - \alpha(t)\eta - \beta(t))\}\phi\\
&= (\varepsilon_2 + \frac{\delta_2}{h^2+\delta_1})|\phi|^2\phi.
\end{aligned} \tag{3.75}
$$

(3.75) can be reduced to a NLS equation with variable coefficients. To see this, set

$$\begin{aligned} &\phi(\xi,\eta) = A(t)\Omega(\xi,\zeta)\exp[i(\eta^2 H(t) + \eta F(t) + G(t))],\\ &\zeta(\xi,\eta) = \gamma(t)\eta + K(t). \end{aligned} \tag{3.76}$$

Then (3.75) becomes

$$\begin{aligned} i\phi_\xi =& \exp[i(\eta^2 H(t) + \eta F(t) + G(t))]\{iA'\Omega + A[i\Omega_\xi + i\gamma'\eta\Omega_\zeta + iK'\Omega_\zeta\\ &- (\eta^2 H' + \eta F' + G')\Omega]\},\\ (\varepsilon_1 + h^2)\phi_{\eta\eta} =& \exp[i(\eta^2 H(t) + \eta F(t) + G(t))]A(\varepsilon_1 + h^2)\{2i(2\eta H + F)\gamma\Omega_\zeta\\ &- (2\eta H + F)^2\Omega + \gamma^2\Omega_{\zeta\zeta} + 2iH\Omega\},\\ \frac{ih'}{h}\eta\phi_\eta =& \exp[i(\eta^2 H(t) + \eta F(t) + G(t))]A\{\frac{ih'}{h}\eta\gamma\Omega_\zeta - \frac{h'}{h}\eta(2\eta H + F)\Omega\} \end{aligned} \tag{3.77}$$

Substituting (3.77) in (3.75), we get

$$\begin{aligned} &A\{H' + \frac{2h'}{h}H + 4(\varepsilon_1 + h^2)H^2 + \frac{\varepsilon_1\delta_1 h''}{4h(h^2+\delta_1)}\}\eta^2\Omega\\ &- A\{F' + \frac{h'}{h}F + 4(\varepsilon_1 + h^2)HF + \alpha\}\eta\Omega\\ &- A\{G' + (\varepsilon_1 + h^2)F^2 + \beta\}\Omega + \{A' + [\frac{h'}{2h} + 2(\varepsilon_1 + h^2)H]A\}i\Omega\\ &+ \{\gamma' + [\frac{h'}{h} + 4(\varepsilon_1 + h^2)H]\gamma\}\eta\Omega_\zeta + \{K' + 2(\varepsilon_1 + h^2)F\}\Omega_\zeta + (\varepsilon_1 + h^2)\gamma^2\Omega_{\zeta\zeta}\\ &= A^3(\varepsilon_2 + \frac{\delta_2}{h^2+\delta_1})|\Omega|^2\Omega. \end{aligned} \tag{3.78}$$

In (3.77) we choose the coefficients of Ω and Ω_ζ so as to be zero. Hence we get

$$\begin{aligned} &H' + 4(\varepsilon_1 + h^2)H^2 + \frac{\varepsilon_1\delta_1 h''}{4h(h^2+\delta_1)} + \frac{2h'}{h}H = 0,\\ &F' + 4(\varepsilon_1 + h^2)HF + \frac{h'}{h}F + \alpha = 0,\\ &G' + (\varepsilon_1 + h^2)F^2 + \beta = 0,\\ &A' + [\frac{h'}{2h} + 2(\varepsilon_1 + h^2)H]A = 0,\\ &\gamma' + [\frac{h'}{h} + 4(\varepsilon_1 + h^2)H]\gamma = 0,\\ &K' + 2(\varepsilon_1 + h^2)F = 0. \end{aligned} \tag{3.79}$$

From (3.79), we find that

$$\begin{aligned} &K = -2\int(\varepsilon_1 + h^2)\gamma F dt,\\ &A = h^{-1/2}\exp[-2\int(\varepsilon_1 + h^2)H dt],\\ &\gamma = h^{-1}\exp[-4\int(\varepsilon_1 + h^2)H dt] = A^2 \end{aligned} \tag{3.80}$$

and the function Ω in (3.76) then satisfies 1-dimensional NLS equation

$$i\Omega_\xi + (\varepsilon_1 + h^2)A^4\Omega_{\zeta\zeta} = (\varepsilon_2 + \frac{\delta_2}{h^2+\delta_1})A^2|\Omega|^2\Omega. \quad (3.81)$$

For $\delta_1 = -\varepsilon_1$ a particular solution of the Riccati equation in the first equation of (3.79) is

$$H = \frac{h'}{4h(h^2-\varepsilon_1)}.$$

The Algebra $L_{1,4}$

The algebra generated by $Z(1) = \partial_y$ leads in a simple manner to NLS equation. Indeed straightforward reduction with $\Psi = \Omega(x,t), \quad w = Q(x,t)$ yields

$$i\Omega_t + \Omega_{xx} = \varepsilon_2|\Omega|^2\Omega + Q\Omega, \quad Q_{xx} = 0. \quad (3.82)$$

Then $Q = \alpha(t)x + \beta(t)$. Substituting this into (3.82), it is obtained that

$$i\Omega_t + \Omega_{xx} = \varepsilon_2|\Omega|^2\Omega + \Omega[\alpha(t)x + \beta(t)]. \quad (3.83)$$

(3.83) can be reduced to a NLS with variable coefficients. Substituting

$$\Omega = \phi(t,\xi)\exp i[F(t)x + G(t)], \quad \xi = x + H(t) \quad (3.84)$$

into (3.83), we get

$$\begin{aligned} &\exp i[F(t)x + G(t)]\{[-F'(t)x - \alpha(t)x - G'(t) - F^2(t) - \beta(t)]\phi + i\phi_t \\ &+ [iH'(t) + 2iF(t)]\phi_\xi + \phi_{\xi\xi}\} = \exp i[F(t)x + G(t)]\varepsilon_2|\phi|^2\phi. \end{aligned} \quad (3.85)$$

The coefficients of $x\phi$, ϕ, ϕ_ξ in (3.85) must be equal to zero

$$\begin{aligned} -F'(t)x - \alpha(t)x = 0 \quad &\Rightarrow \quad F(t) = -\int \alpha(t)dt, \\ -G'(t) - F^2(t) - \beta(t) = 0 \quad &\Rightarrow \quad G(t) = -\int (F^2(t) + \beta(t))dt, \\ iH'(t) + 2iF(t) = 0 \quad &\Rightarrow \quad H(t) = -2\int F(t)dt. \end{aligned} \quad (3.86)$$

and $\phi(t,\xi)$ satisfies the 1-dimensional NLS equation.

$$i\phi_\xi + \phi_{\eta\eta} = |\phi|^2\phi.$$

4. THE SYMMETRY GROUP OF THE GDS EQUATIONS AND STRUCTURE OF ITS LIE ALGEBRA

For our purposes, we find it more convenient to consider the differentiated form of (1.4). Thus, differentiating the last two equations of (1.4) with respect to x and y, respectively and then making the substitution $w_x \to w$, $\phi_y \to \phi$ and rewriting the corresponding system in a real form by separating $\psi = u + iv$ into real and imaginary parts, we obtain a system of four real partial differential equations

$$\begin{aligned}
\Delta_1 &= u_t + \delta v_{xx} + v_{yy} - \chi v(u^2 + v^2) - \gamma v(w + \phi) = 0, \\
\Delta_2 &= -v_t + \delta u_{xx} + u_{yy} - \chi u(u^2 + v^2) - \gamma u(w + \phi) = 0, \\
\Delta_3 &= w_{xx} + n\phi_{xx} + m_2 w_{yy} - 2(u_x^2 + uu_{xx} + v_x^2 + vv_{xx}) = 0, \\
\Delta_4 &= nw_{yy} + \lambda\phi_{xx} + m_1\phi_{yy} - 2(u_y^2 + uu_{yy} + v_y^2 + vv_{yy}) = 0.
\end{aligned} \tag{4.1}$$

In the sequel, we shall call (4.1) the GDS equations.

If we apply the same procedure to find symmetry algebra which we applied for finding symmetry algebra of DS above, we find that the general element can be written as

$$\begin{aligned}
&\xi_y = 0, \quad \xi_u = 0, \quad \xi_v = 0, \quad \xi_w = 0, \quad \xi_\phi = 0, \\
&\eta_x = 0, \quad \eta_u = 0, \quad \eta_v = 0, \quad \eta_w = 0, \quad \eta_\phi = 0, \\
&\tau_x = 0, \quad \tau_y = 0, \quad \tau_u = 0, \quad \tau_v = 0, \quad \tau_w = 0, \quad \tau_\phi = 0, \\
&(\varphi_1)_w = 0, \quad (\varphi_1)_\phi = 0, \quad (\varphi_1)_{uu} = 0, \quad (\varphi_1)_{uv} = 0, \quad (\varphi_1)_{vv} = 0, \\
&(\varphi_2)_w = 0, \quad (\varphi_2)_\phi = 0, \quad (\varphi_2)_{uu} = 0, \quad (\varphi_2)_{uv} = 0, \quad (\varphi_2)_{vv} = 0, \\
&(\varphi_3)_u = 0, \quad (\varphi_3)_v = 0, \quad (\varphi_3)_\phi = 0, \quad (\varphi_3)_{ww} = 0, \\
&(\varphi_4)_u = 0, \quad (\varphi_4)_v = 0, \quad (\varphi_4)_w = 0, \quad (\varphi_4)_{\phi\phi} = 0.
\end{aligned} \tag{4.2}$$

From (4.2) , we get

$$\begin{aligned}
&\xi(x, y, t) = \xi(x, t), \quad \eta(x, y, t) = \eta(x, t), \quad \tau(x, y, t) = f(t), \\
&\varphi_1 = P_1(x, y, t)u - P_2(x, y, t)v + S_1(x, y, t), \\
&\varphi_2 = P_2(x, y, t)u + P_1(x, y, t)v + S_2(x, y, t), \\
&\varphi_3 = 2P_1(x, y, t)w + S_3(x, y, t), \\
&\varphi_4 = 2P_1(x, y, t)\phi + S_4(x, y, t).
\end{aligned} \tag{4.3}$$

If we substitute (4.3) in prolongation formulas, we get

$$\begin{aligned}
\mathrm{pr}^{(2)}\mathbf{V}(\Delta_1) &= -u^2v\chi P_1 + v^3\chi P_1 + \gamma vwP_1 + \gamma v\phi P_1 - u^3\chi P_2 - 3uv^2\chi P_2 \\
&- \gamma uwP_2 - \gamma uwP_2 - \gamma u\phi P_2 - \gamma vwP_3 - \gamma v\phi P_4 - u^3\chi R_1 - 3uv^2\chi R_1 - \gamma uwR_1 \\
&- \gamma u\phi R_1 - u^2v\chi R_2 - 3v^3\chi R_2 - \gamma vwR_2 - \gamma v\phi R_2 - 2uv\chi S_1 - u^2\chi S_2 \\
&- 3v^2\chi S_2 - \gamma wS_2 - b\phi S_2 - \gamma vS_3 - \gamma vS_4 + v(R_2)_{yy} + (S_2)_{yy} + \delta u(P_2)_{xx} \\
&+ \delta v(R_2)_{xx} + \delta(S_2)_{xx} - u^2v\chi f'(t) - v^3\chi f'(t) - \gamma vwf'(t) - \gamma v\phi f'(t) \\
&+ u(P_1)_t + v(R_1)_t + (S_1)_t + [2\delta(P_2)_x - \xi_t]u_x + [2\delta(R_2)_x - \delta\xi_{xx}]v_x \\
&+ [2(P_2)_y - \eta_t]u_y + [2(R_2)_y - \eta_{yy}]v_y + [P_2 + R_1]u_{yy} + [R_2 - P_1 + f'(t) - 2\eta_y]v_{yy} \\
&+ [\delta P_2 + \delta R_1]u_{xx} + [-\delta P_1 + \delta R_2 + \delta f'(t) - 2\delta\xi_x]v_{xx} = 0
\end{aligned} \tag{4.4}$$

$$\begin{aligned}
\mathrm{pr}^{(2)}\mathbf{V}(\Delta_2) &= -3u^3\chi P_1 - uv^2\chi P_1 - \gamma uwP_1 - \gamma u\phi P_1 - 3u^2v\chi P_2 - v^3\chi P_2 \\
&- \gamma w\phi P_2 - \gamma uwP_3 - \gamma u\phi P_4 - 3u^2v\chi R_1 - v^3\chi R_1 - bvwR_1 - \gamma v\phi R_1 + u^3\chi R_2 \\
&- v^2u\chi R_2 + \gamma uwR_2 + \gamma u\phi R_2 - 3u^2\chi S_1 - v^2\chi S_1 - \gamma vS_1 - \gamma\phi S_1 - 2uv\chi S_2 \\
&- \gamma uS_3 - \gamma uS_4u(P_1)_{yy} + v(R_2)_{yy} + (S_2)_{yy} + \delta u(P_1)_{xx} + \delta v(R_1)_{xx} + \delta(S_1)_{xx} \\
&- \gamma vwP_2 - u^3\chi f'(t) - uv^2\chi f'(t) - \gamma uwf'(t) - \gamma u\phi f'(t) - u(P_2)_t - v(R_2)_t \\
&- (S_2)_t + [2(P_1)_y - \eta_{yy}]u_y + [2(R_1)_y + \eta_t]v_y + [2\delta(R_1)_x + \xi_t]v_x \\
&+ [2\delta(P_1)_x - \delta\xi_{xx}]u_x + [P_1 - R_2 + f'(t) - 2\eta_y]u_{yy} + [P_2 + R_1]v_{yy} \\
&+ [\delta P_2 + \delta R_1]v_{xx} + [-\delta R_2 + \delta P_1 + \delta f'(t) - 2\delta\xi_x\}v_{yy} = 0.
\end{aligned} \tag{4.5}$$

$$\begin{aligned}
\mathrm{pr}^{(2)}\mathbf{V}(\Delta_3) &= wm_2(P_3)_{yy} + m_2(S_3)_{yy} - 2u^2(P_1)_{xx} - 2uv(P_2)_{xx} + w(P_3)_{xx} \\
&+ n\phi(P_4)_{xx} - 2uv(R_1)_{xx} - 2v^2(R_2)_{xx} - 2u(S_1)_{xx} - 2v(S_2)_{xx} + (S_3)_{xx} \\
&+ n(S_4)_{xx} + [2u\xi_{xx} - 8u(P_1)_x - 4v(P_2)_x - 4v(R_1)_x - 4(S_1)_x]u_x \\
&+ [2(P_3)_x - \xi_{xx}]w_x + [2v\xi_{xx} - 8v(R_2)_x - 4u(P_2)_x - 4u(R_1)_x - 4(S_2)_x]v_x \\
&+ [2m_2(P_3)_y - m_2\eta_{yy}]w_y + [-4uP_1 - 2vP_2 + 2uP_3 - 2vR_1 - 2S_1]u_{xx} \\
&+ [-2uP_2 + 2vP_3 - 2uR_1 - 4vR_2 - 2S_2]v_{xx} + [2n(P_4)_x - n\xi_{xx}]\phi_x \\
&- [4P_2 + 4R_1]u_xv_x + [2P_3 - 4R_2](v_x)^2 + [2P_3 - 4P_1](u_x)^2 \\
&+ [2m_2\xi_x - 2m_2\eta_y]w_{yy} + [nP_4 - nP_3]\phi_{xx} = 0
\end{aligned} \tag{4.6}$$

$$
\begin{aligned}
&\mathrm{pr}^{(2)}\mathbf{V}(\Delta_4) = nw(P_3)_{yy} + n(S_3)_{yy} - 2u^2(P_1)_{yy} - 2uv(P_2)_{yy} + m_1\phi(P_4)_{yy} \\
&- 2uv(R_1)_{yy} - 2v^2(R_2)_{yy} - 2u(S_1)_{yy} - 2v(S_2)_{yy} + m_1(S_4)_{yy} + \lambda(S_4)_{xx} \\
&+ \lambda\phi(P_4)_{xx} + [2u\eta_{yy} - 8u(P_1)_y - 4v(P_2)_y - 4v(R_1)_y - 4(S_1)_y]u_y \\
&+ [2v\eta_{yy} - 8v(R_2)_y - 4u(P_2)_y - 4u(R_1)_y - 4(S_2)_y]v_y + [2n(P_3)_y - n\eta_{yy}]w_y \\
&+ [-4uP_1 - 2vP_2 + 2uP_4 - 2vR_1 - 2S_1]u_{yy} + [2P_4 - 4P_1](u_y)^2 \\
&+ [-2uP_2 + 2vP_4 - 2uR_1 - 4vR_2 - 2S_2]v_{yy} + [2P_4 - 4R_2](v_y)^2 \\
&+ [2\lambda(P_4)_x - \lambda\xi_{xx}]\phi_x + [-4P_2 - 4R_1]u_yv_y + [2m_1(P_4)_y - m_1\eta_{yy}]\phi_y \\
&+ [nP_3 - nP_4]w_{yy} + [2\lambda\eta_y - 2\lambda\xi_x]\phi_{xx} = 0
\end{aligned}
\tag{4.7}
$$

The coefficients of v_{yy} in (4.5), $(v_x)^2$ in (4.6), ϕ_{xx} in (4.6), $(u_y)^2$ in (4.7) and w_{yy} in (4.6) must be equal to zero. So find that

$$
\begin{aligned}
P_2 + R_1 = 0, &\quad \Rightarrow \quad P_2 = -R_1, \\
2P_3 - 4R_2 = 0, &\quad \Rightarrow \quad P_3 = 2R_2, \\
nP_4 - nP_3 = 0, &\quad \Rightarrow \quad P_4 = P_3, \\
2P_4 - 4P_1 = 0, &\quad \Rightarrow \quad P_4 = 2P_1, \\
2m_2\xi_x - 2m_2\eta_y = 0, &\quad \Rightarrow \quad \eta_y = \xi_x,
\end{aligned}
\tag{4.8}
$$

Using (4.8), it is easily seen that

$$
P_2 = -R_1, \quad P_4 = P_3 = 2R_2 = 2P_1 \quad \text{and} \quad \xi_{xx} = \eta_{yy} = 0. \tag{4.9}
$$

When we substitute (4.9) in the coefficients of u_{xx} in (4.6), v_{xx} in (4.6), u_y in (4.5), u_x in (4.5) and v_{yy} in (4.4), we get

$$
\begin{aligned}
-4uP_1 - 2vP_2 + 2uP_3 - 2vR_1 - 2S_1 = 0 &\quad \Rightarrow \quad S_1 = 0, \\
-2uP_2 + 2vP_3 - 2uR_1 - 4vR_2 - 2S_2 = 0 &\quad \Rightarrow \quad S_2 = 0, \\
2(P_1)_y - \eta_{yy} = 0 &\quad \Rightarrow \quad (P_1)_y = 0, \\
2\delta(P_1)_x - \delta\xi_{xx} = 0 &\quad \Rightarrow \quad (P_1)_x = 0, \\
R_2 - P_1 + f'(t) - 2\eta_y = 0 &\quad \Rightarrow \quad \eta_y = f'(t)/2.
\end{aligned}
\tag{4.10}
$$

Then we find

$$
P_1(x, y, t) = P_1(t), \tag{4.11a}
$$

$$
\eta(y, t) = \frac{f'(t)}{2}y + h(t). \tag{4.11b}
$$

By using $\eta_y = \xi_x$, it is easily seen

$$
\xi(x, t) = \frac{f'(t)}{2}x + g(t), \tag{4.12}
$$

where $g(t)$ and $h(t)$ are arbitrary functions. When we substitute (4.11b) and (4.12) in the coefficients of u_y in (4.4) and u_x in (4.4), we get

$$\begin{aligned} 2(P_2)_y - \eta_t = 0 \quad &\Rightarrow \quad (P_2)_y = \frac{1}{2}\eta_t = \frac{y}{4}f''(t) + \frac{1}{2}h'(t), \\ 2\delta(P_2)_x - \xi_t = 0 \quad &\Rightarrow \quad (P_2)_x = \frac{1}{2\delta}\xi_t = \frac{x}{4\delta}f''(t) + \frac{1}{2\delta}g'(t), \end{aligned} \tag{4.13}$$

From (4.13), it is easily found that

$$P_2 = \frac{f''(t)}{8\delta}(x^2 + \delta y^2) + \frac{g'(t)}{2\delta}x + \frac{h'(t)}{2}y - m(t), \tag{4.14}$$

where $m(t)$ is an arbitrary function. By using all of these equations, we get

$$\begin{aligned} &\mathrm{pr}^{(2)}\mathbf{V}(\Delta_1) = -2u^2 v\chi P_1 - 2v^3\chi P_1 - 2\gamma vwP_1 - 2\gamma\phi P_1 - \gamma vS_3 \\ &- \gamma vS_4 + uC_1' - u^2 v\chi f'(t) - v^3\chi f'(t) - \gamma vwf'(t) - \gamma v\phi f'(t) + \frac{1}{2}uf''(t) \\ &- v[\frac{f'''(t)}{8\delta}(x^2 + \delta y^2) + \frac{g''(t)}{2\delta}x + \frac{h''(t)}{2}y - m'(t)] = 0. \end{aligned} \tag{4.15}$$

(4.15) is a polynomial. So the coefficient of v^3 must be equal to zero. Then we find

$$-2\chi P_1 - \chi f'(t) = 0 \quad \Rightarrow \quad P_1 = -\frac{f'(t)}{2} \tag{4.16}$$

The coefficients of general element is

$$\begin{aligned} \xi &= \frac{f'(t)}{2}x + g(t), \\ \eta &= \frac{f'(t)}{2}y + h(t), \\ \tau &= f(t), \\ \varphi_1 &= [-\frac{f''(t)}{8\delta}(x^2 + \delta\varepsilon_1 y^2) - \frac{g'(t)}{2\delta}x - \frac{h'(t)}{2}y + m(t)]v - (\frac{f'(t)}{2})u, \\ \varphi_2 &= -(\frac{f'(t)}{2})v + [\frac{f''(t)}{8\delta}(x^2 + \delta\varepsilon_1 y^2) + \frac{g'(t)}{2\delta}x + \frac{h'(t)}{2}y - m(t)]u, \\ \varphi_3 &= -f'(t)w + S_3(x, y, t), \\ \varphi_4 &= -f'(t)\phi + S_4(x, y, t). \end{aligned} \tag{4.17}$$

When we calculate all of the prolongations, we get

$$\begin{aligned} \mathrm{pr}^{(2)}\mathbf{V}(\Delta_1) &= -v[\gamma S_3 + \gamma S_4 + \frac{f'''(t)}{8\delta}(x^2 + \delta y^2) + \frac{g''(t)}{2\delta}x + \frac{h''(t)}{2}y - m'(t)] = 0, \\ \mathrm{pr}^{(2)}\mathbf{V}(\Delta_2) &= -u[\gamma S_3 + \gamma S_4 + \frac{f'''(t)}{8\delta}(x^2 + \delta y^2) + \frac{g''(t)}{2\delta}x + \frac{h''(t)}{2}y - m'(t)] = 0, \\ \mathrm{pr}^{(2)}\mathbf{V}(\Delta_3) &= m_2(S_3)_{yy} + (S_3)_{xx} + n(S_4)_{xx} = 0, \\ \mathrm{pr}^{(2)}\mathbf{V}(\Delta_4) &= n(S_3)_{yy} + m_1(S_4)_{yy} + \lambda(S_4)_{xx} = 0. \end{aligned} \tag{4.18}$$

A compatibility of (4.18) gives

$$S_3 = S_4 = -\frac{f'''(t)}{16\delta\gamma}(x^2+\delta y^2) - \frac{g''(t)}{4\delta\gamma}x - \frac{h''(t)}{4\gamma}y + \frac{m'(t)}{2\gamma}, \tag{4.19a}$$

$$-\frac{f'''(t)}{8\delta\gamma}(\delta m_2 + n + 1) = 0, \tag{4.19b}$$

$$-\frac{f'''(t)}{8\delta\gamma}(m_1\delta + n\delta + \lambda) = 0. \tag{4.19c}$$

Thus, we obtain

$$\begin{aligned}
\xi &= \frac{f'(t)}{2}x + g(t),\\
\eta &= \frac{f'(t)}{2}y + h(t),\\
\tau &= f(t),\\
\varphi_1 &= [-\frac{f''(t)}{8\delta}(x^2+\delta y^2) - \frac{g'(t)}{2\delta}x - \frac{h'(t)}{2}y + m(t)]v + (-\frac{f'(t)}{2})u,\\
\varphi_2 &= (-\frac{f'(t)}{2})v + [\frac{f''(t)}{8\delta}(x^2+\delta y^2) + \frac{g'(t)}{2\delta}x + \frac{h'(t)}{2}y - m(t)]u,\\
\varphi_3 &= -f'(t)w - \frac{f'''(t)}{16\delta\gamma}(x^2+\delta y^2) - \frac{g''(t)}{4\delta\gamma}x - \frac{h''(t)}{4\gamma}y + \frac{m'(t)}{2\gamma},\\
\varphi_4 &= -f'(t)\phi - \frac{f'''(t)}{16\delta\gamma}(x^2+\delta y^2) - \frac{g''(t)}{4\delta\gamma}x - \frac{h''(t)}{4\gamma}y + \frac{m'(t)}{2\gamma}.
\end{aligned} \tag{4.20}$$

The general element can be written as

$$\mathbf{V} = T(f) + X(g) + Y(h) + W(m), \tag{4.21}$$

where

$$\begin{aligned}
T(f) &= f(t)\partial_t + \frac{1}{2}f'(t)(x\partial_x + y\partial_y - u\partial_u - v\partial_v - 2w\partial_w - 2\phi\partial_\phi)\\
&\quad - \frac{(x^2+\delta y^2)}{8\delta}[f''(t)(v\partial_u - u\partial_v) + \frac{f'''(t)}{2\gamma}(\partial_w + \partial_\phi)],\\
X(g) &= g(t)\partial_x - \frac{x}{2\delta}[g'(t)(v\partial_u - u\partial_v) + \frac{g''(t)}{2\gamma}(\partial_w + \partial_\phi)],\\
Y(h) &= h(t)\partial_y - \frac{y}{2}[h'(t)(v\partial_u - u\partial_v) + \frac{h''(t)}{2\gamma}(\partial_w + \partial_\phi)],\\
W(m) &= m(t)(v\partial_u - u\partial_v) + \frac{m'(t)}{2\gamma}(\partial_w + \partial_\phi).
\end{aligned} \tag{4.22}$$

The functions $g(t)$, $h(t)$, and $m(t)$ are arbitrary functions of class $C^\infty(I)$, $I \subseteq \mathbb{R}$. The function $f(t)$ is arbitrary if

$$m_2\delta + n + 1 = 0, \quad m_1\delta + n\delta + \lambda = 0, \tag{4.23}$$

otherwise $f(t) = c_2t^2 + c_1t + c_0$. We mention that these conditions come from the fact that two of the determining equations are

$$(m_2\delta + n + 1)f'''(t) = 0, \quad (m_1\delta + n\delta + \lambda)f'''(t) = 0,$$

whereas the remaining ones are solved without any constraints on g, h and m.

We mainly focus on the case when $f(t)$ is allowed to be arbitrary. The symmetry algebra realized by the vector fields (4.21) and (4.22) is then infinite-dimensional and more important has the structure of a Kac-Moody-Virasoro (KMV) algebra as we shall see below. More interestingly, it is generic among the symmetry algebras of a few 2+1-dimensional integrable partial differential equations (the Kadomtsev-Petviashvilli (KP) equation, the modified KP equation, the potential KP equation, the integrable three-wave resonant equations and the integrable DS equations).

The commutation relations for the GDS algebra are easily obtained as follows:

$$\begin{aligned}
&[T(f_1), T(f_2)] = T(f_1 f_2' - f_1' f_2) \\
&[T(f), X(g)] = X(f g' - \frac{1}{2} f' g) \\
&[T(f), Y(h)] = Y(f h' - \frac{1}{2} f' h) \\
&[T(f), W(m)] = W(f m') \\
&[X(g_1), X(g_2)] = -\frac{1}{2\delta} W(g_1 g_2' - g_1' g_2) \\
&[Y(h_1), X(h_2)] = -\frac{1}{2} W(h_1 h_2' - h_1' h_2) \\
&[X(g), Y(h)] = [X(g), W(m)] = [Y(h), W(m)] = [W(m_1), W(m_2)] = 0.
\end{aligned} \tag{4.24}$$

From (4.24) we see that the GDS system has a Lie symmetry algebra L isomorphic to that of the DS symmetry algebra [8]. Indeed, it allows a Levi decomposition

$$L = S \oplus N, \tag{4.25}$$

where $S = \{T(f)\}$ is a simple infinite dimensional Lie algebra and

$$N = \{X(g), Y(h), W(m)\}$$

is a nilpotent ideal (nilradical). Here, $\oplus$ denotes the semi-direct sum. The algebra $\{T(f)\}$ is isomorphic to the Lie algebra corresponding to the Lie group of diffeomorphisms of a real line.

Expanding the arbitrary functions f, g, h and m into Laurent polynomials and considering each monomial t^n (n not necessarily positive integer) separately, we obtain a realization of a KMV algebra without central extension. Here the factor subalgebra S is the Virasoro part, the nilpotent subalgebra N is the Kac-Moody part of the GDS algebra [14]. Furthermore, just as the DS algebra [8] it can be shown that the GDS algebra with (4.23) can be imbedded into a Kac-Moody-type loop algebra.

Theorem 4.1. The system (4.1) is invariant under an infinite-dimensional Lie point symmetry group, the Lie algebra of which has a Kac-Moody-Virasoro structure isomorphic to the DS algebra if and only if the conditions (4.23) hold.

The GDS equations are also invariant under a group of discrete transformations generated by

$$\begin{aligned} &t \to t, \quad x \to -x, \quad y \to y, \quad \psi \to \psi, \quad w \to w, \quad \phi \to \phi \\ &t \to t, \quad x \to x, \quad y \to -y, \quad \psi \to \psi, \quad w \to w, \quad \phi \to \phi \\ &t \to t, \quad x \to x, \quad y \to y, \quad \psi \to -\psi, \quad w \to w, \quad \phi \to \phi \\ &t \to -t, \quad x \to x, \quad y \to y, \quad \psi \to \psi^*, \quad w \to w, \quad \phi \to \phi. \end{aligned} \tag{4.26}$$

The obvious physical symmetries L_p of the GDS equations are obtained by restricting all the functions f, g, h and m to be first order polynomials. Indeed, we have

$$\begin{aligned} &T = T(1) = \partial_t, \quad P_1 = X(1) = \partial_x, \quad P_2 = Y(1) = \partial_y \\ &W_0 = W(1) = v\partial_u - u\partial_v, \\ &D = T(t) = t\partial_t + \frac{1}{2}(x\partial_x + y\partial_y - u\partial_u - v\partial_v - 2w\partial_w - 2\phi\partial_\phi) \\ &B_1 = X(t) = t\partial_x - \frac{x}{2\delta}(v\partial_u - u\partial_v), \quad B_2 = Y(t) = t\partial_y - \frac{y}{2}(v\partial_u - u\partial_v) \\ &W_1 = W(t) = t(v\partial_u - u\partial_v) + \frac{1}{2\gamma}(\partial_w + \partial_\phi). \end{aligned} \tag{4.27}$$

We see that T, P_1, P_2 generate translations, D dilations, B_1 and B_2 Galilei boosts in the x and y directions, respectively. Finally, W_0 and W_1 generate a constant change of phase of ψ and a change of phase of ψ, linear in t, plus constant shifts in w and ϕ, respectively.

The generators (4.27) form a basis of a eight-dimensional solvable Lie algebra $L_p = \{D, T, P_1, P_2, B_1, B_2, W_0, W_1\}$. It has a seven-dimensional nilpotent ideal (the nilradical) $N = \{T, P_1, P_2, B_1, B_2, W_0, W_1\}$.

Another finite-dimensional algebra, not contained in L_p is obtained by restricting $f(t)$ to quadratic polynomials. We obtain $T = T(1)$, $D = T(t)$ as in (4.27), and in addition

$$C = T(t^2) = t^2\partial_t + tD - \frac{(x^2 + \delta y^2)}{4\delta}(v\partial_u - u\partial_v). \tag{4.28}$$

The commutation relations are

$$[T, D] = T, \quad [T, C] = 2D, \quad [D, C] = C,$$

so that we have obtained the algebra $\mathrm{sl}(2, \mathbb{R})$ with C generating conformal type of transformations

$$\begin{aligned}
&\tilde{t} = \frac{t}{1 - pt}, \quad \tilde{x} = \frac{x}{1 - pt}, \quad \tilde{y} = \frac{y}{1 - pt}, \\
&\tilde{R} = (1 - pt)R, \quad \tilde{\sigma} = \frac{p(x^2 + \delta y^2)}{4\delta(1 - pt)} + \sigma, \\
&\tilde{w} = (1 - pt)^2 w, \quad \tilde{\phi} = (1 - pt)^2 \phi,
\end{aligned} \tag{4.29}$$

where p is the group parameter. Further, composing (4.29) with time translations generated by T and dilations generated by D we obtain the $\mathrm{SL}(2, \mathbb{R})$ group generated by actions on the space of independent and depend variables. It should be mentioned that any finite dimensional subalgebra of the Virasoro algebra of 2+1 dimensional integrable equations is isomorphic to $\mathrm{sl}(2, \mathbb{R})$ or one of its subalgebras. The transformed variables and the new solution in terms of the original ones are given by the formulas

$$\begin{aligned}
&\tilde{t} = \frac{c + dt}{a + bt}, \quad \tilde{x} = \frac{x}{a + bt}, \quad \tilde{y} = \frac{y}{a + bt}, \quad ad - bc = 1 \\
&\tilde{\psi} = (a + bt)^{-1} \exp\{\frac{ib(x^2 + \delta y^2)}{4\delta(a + bt)}\}\psi(\tilde{t}, \tilde{x}, \tilde{y}) \\
&\tilde{w} = (a + bt)^{-2} w(\tilde{t}, \tilde{x}, \tilde{y}) \\
&\tilde{\phi} = (a + bt)^{-2} \phi(\tilde{t}, \tilde{x}, \tilde{y}).
\end{aligned} \tag{4.30}$$

Here a, b, c are the group parameters of $\mathrm{SL}(2, \mathbb{R})$. These are exactly the formulas which played an essential role in the construction of analytic blow-up profiles [7] in which the authors made use of stationary radial solutions (ψ, w, ϕ) to generate new solutions (time dependent) $(\tilde{\psi}, \tilde{w}, \tilde{\phi})$ of the GDS equations. More generally, the elements of the connected part of the full symmetry group of the GDS equations can be obtained by integrating the vector fields (3.21), (3.22).

Let us now return to the isomorphic GDS and DS symmetry algebras, and transform the GDS vector fields (3.2) by the point transformation $q = w + \phi - |\psi|^2$. It is easy to see that the component $\frac{1}{2}(\partial_w + \partial_\phi)$ transforms to ∂_q, and $D \to x\partial_x + y\partial_y - u\partial_u - v\partial_v - 2q\partial_q$, and the rest remains unaltered, namely the DS symmetry algebra is obtained. This means that the functions (ψ, q) satisfy the DS equations whenever (ψ, w, ϕ) satisfy the GDS equations, but not vice versa. At this time, it remains open whether it is possible to construct an invertible point transformation relating these two systems.

CONCLUSIONS

In this thesis we worked out the group-theoretical properties of DS and GDS equations. Symmetry group analysis is an effective tool for studying nonlinear ordinary and partial differential equations because no techniques exist for finding explicit solutions. When we consider nonlinear ODEs or PDEs, the determining system of equations for the symmetry transformations always becomes a linear system of PDEs, and hence is easier to cope with, compared to the equation in question.

Our study found its motivation from a work done by Champagne and Winternitz. They treated a complete analysis of symmetry properties of the DS equations. In the third section of the thesis, based on this work we showed that DS equations has an infinite-dimensional symmetry algebra. We calculated the group transformations for the cases $f(t) = 0$ and $f(t) \neq 0$. Champagne and Winternitz [8] found the one-dimensional subalgebras

$$L_{1,1} = \{X(1)\}, \quad L^{a}_{1,2} = \{Y(1) + aZ(1)\}, \quad L_{1,3}(h) = \{Y(1) + Z(h)\}, \quad a \geq 0,$$

$$L_{1,4} = \{Z(1)\}, \quad L_{1,5} = \{W(t)\}, \quad L_{1,6} = \{W(1)\}.$$

We used one-dimensional subalgebras $L_{1,1}, L^{a}_{1,2}, L_{1,3}(h)$ and $L_{1,4}$ which generate actions on the coordinate space (t, x, y) to reduce the DS equations to integrable ones in two variables. This section includes computational details of this paper.

In the fourth section, we showed that GDS equations have an infinite-dimensional symmetry algebra with a Virasoro structure. The GDS algebra will have the same conjugacy classes of subalgebras as the DS algebra.

As an open problem, we can introduce the use of subalgebras to reduce the integrable GDS system to integrable one in two variables and thus to obtain subgroup invariant solutions.

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