THERMAL BUCKLING OF FUNCTIONALLY GRADED PLATES

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ABSTRACT

This paper presents thermal buckling analysis of rectangular and elliptical Kirchhoff plates with constant thickness made of functionally graded material (FGM). Metal-ceramic FGM plates are proposed for the use in thermal analyses where a metal-rich interior surface of the plate gradually transitions to a ceramic-rich exterior surface of the plate. In the entire analyses, the boundaries are assumed to be clamped. Rayleigh-Ritz method is employed to solve the partial differential equations and obtain the critical buckling loads. Material properties are assumed to be temperature dependent and graded in the thickness direction according to a power law distribution in terms of the volume fractions of the constituents. Thermal buckling analyses are carried out by taking into account the variations of the composition of the FGM and the aspect ratios of the plates. Results are presented in graphical and tabular forms and the effect of the FGM on the thermal buckling of rectangular and elliptical plates is discussed.

ÖZET

Bu çalışma, bir fonksiyona bağlı olarak değişen (FGM) sabit kalınlıklı dikdörtgen ve eliptik Kirchhoff plaklarının termal burkulması üzerine odaklanmıştır. Plağın metal yoğunluklu iç yüzeyinin aşamalı olarak seramik yoğunluklu dış yüzeye dönüştüğü metal-seramik FGM levhalar termal analizlerde kullanılmak üzere önerilmiştir. Tüm çalışmada levhalar basit mesnetli ve ankastre sınır koşullarına uygun olacak şekilde kabul edilmiştir. Enerji denklemlerinin çözümü ve kritik burkulma yüklerinin bulunması için Rayleigh-Ritz metodu kullanılmıştır. Malzeme özellikleri ısıya bağlı olup kuvvet dağılım kuralına göre, levha kalınlığı boyunca, bileşenlerin hacimsel oranına göre değiştiği varsayılmıştır. FGM bileşenlerinin ve plağın en/boy oranının değişimi esas alınarak termal burkulma analizleri gerçekleştirilmiştir. Sonuçlar grafik ve tablolar halinde sunulmuş, FGM malzemesinin dikdörtgen ve eliptik levhaların termal burkulması üzerindeki etkisi irdelenmiştir.

INTRODUCTION

Thin and thick various shapes of plates are extensively used in civil, mechanical, nuclear and aerospace engineering as basic structural elements. Because of their extensive application areas, the demand for the solution of plate problems has increased and numerous methods have been developed for analyzing both static and dynamic behavior of plates having a wide range of geometries and boundary conditions [1,2]. On the other hand, investigations about the thermal effect on the plates have received widespread attention in recent years. Since functionally graded materials (FGMs) have many advantages as a heat resistant material, FGMs have attracted much attention in high-temperature applications. Many different theories and solutions about the behavior of FGM under thermal environment are suggested in the literature [3,4,5,6,7]. Researches have been reporting thermal buckling analyses of functionally graded plates (FGM) subjected to mechanical or thermal loading. Among those studies, Shariyat [8] worked on the thermal buckling analysis of rectangular composite multilayered plates under uniform temperature rise by using layerwise plate theory. A finite element method algorithm is used to exactly incorporate the boundary conditions. Also, thermal and mechanical buckling of simply supported FGM rectangular plates is studied based on the classical and higher order shear deformation plate theories [9,10]. Na and Kim [11,12] used solid finite elements to calculate the buckling temperature of FGM plates with fully clamped edges. Thermal buckling and post-buckling behaviors due to uniform and non-uniform temperature rise are studied. In another study, critical buckling temperature of a thin rectangular FGM plate is investigated based on the higher order shear deformation theory and the results are compared with those obtained by finite element method [13]. Ganapathi and Prakash [14] presented the buckling loads for simply supported FGM skew plates subjected to in-plane mechanical loads and heat conduction by using the first-order shear deformation theory in conjunction with the finite element approach. Wu [15] studied the thermal buckling behavior of simply supported FGM rectangular plates under uniform temperature rise and gradient through the thickness based on the first-order shear deformation plate theory.

This work is motivated by the necessity to understand the buckling behavior of FGM members under thermal effects. It is intended to determine the critical buckling temperature of rectangular and elliptical FGM plates accurately. The governing equations are solved for clamped boundary conditions by using Rayleigh-Ritz method. The results are examined with respect to a/b ratios and n which are the plate aspect ratio and the power law index, respectively. The effects of material and geometric properties are studied.

METHODOLOGY

Material Properties of FGMs:

The functionally graded plates are typically made from a controlled mixture of ceramics and metal or a combination of different metals. Here, the material on the top surface (z=t/2) and on the bottom surface (z=-t/2) is ceramic and metal-rich, respectively. The ceramic constituent of the material provides high-temperature resistance due to its low thermal conductivity. The ductile metal constituent prevents failure caused by stresses due to the high-temperature gradient in a very short period of time. The material properties change in the thickness direction. For the validity of classical thin plate theory, the transverse deflections are assumed to be small compared to plate dimensions.

Since FGM plates are composed of more than one material; effective material properties of the mixture govern the plate behavior. Through-the-thickness composition of the material is assumed to be governed by a volume fraction rule. The volume fractions of ceramic, Vc, and metal, Vm, corresponding to the power law are expressed as

$$V_c(z) = \left(\frac{2z+t}{2t}\right)^n ; \qquad V_c = \int_{-t/2}^{t/2} V_c(z)\,dz$$
$$V_m = 1 - V_c \tag{1}$$

where, z is the thickness coordinate variable; and $-t/2 \le z \le t/2$, where t is the thickness of the plate and n is the power law index that takes values greater than or equal to zero ($0 \le n \le \infty$). The variation of the composition of ceramics and metal is linear for n=1 and as it is seen from Eq. (1), the value of n equal to zero represents a fully ceramic plate. Also, for the high values of n, the dominant constituent material first exhibit changes in small increments and then rapidly changes at the opposite side. On the other hand, for the low values of n, the material properties change quickly near the surface. The variations of the volume fractions through the thickness are illustrated in Figure 1.

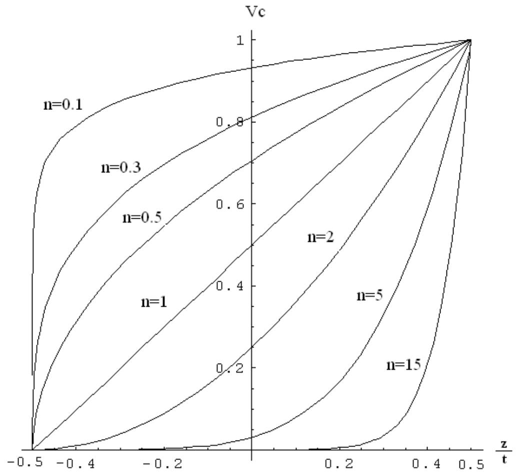

Figure 1. Through the thickness distribution of the volume fraction of ceramic

It is assumed that the non-homogenous material properties such as the Young's modulus E and the coefficient of thermal expansion α change in the thickness direction z, whereas Poisson's ratio ν is assumed to be constant.

$$\begin{aligned} E(z) &= E_c V_c + E_m(1-V_c) \\ \alpha(z) &= \alpha_c V_c + \alpha_m(1-V_c) \\ \nu(z) &= \nu_0 \end{aligned} \tag{2}$$

where, subscripts m and c refer to the metal and ceramic constituents, respectively. When Eq. (1) is substituted into Eq. (2), material properties of the FGM plate are determined. Thus,

$$\begin{aligned} E(z) &= (E_c - E_m)(\frac{2z+t}{2t})^n + E_m \\ \alpha(z) &= (\alpha_c - \alpha_m)(\frac{2z+t}{2t})^n + \alpha_m \\ \nu(z) &= \nu_0 \end{aligned} \tag{3}$$

Solution by Rayleigh-Ritz Method:

Rayleigh-Ritz Method is used in conjunction with the potential energy U which can be written as

$$U = \frac{D}{2}\iint_A \left\{ \left(\nabla^2 w\right)^2 + 2(1-\nu)\left[\left(\frac{\partial^2 w}{\partial x \partial y}\right)^2 - \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}\right]\right\} dx\,dy \tag{4}$$

The total energy function, F, is expressed in Eq. (5) where U is the strain energy due to bending, T is the kinetic energy of the plate and V is the potential energy of the in-plane uniform pressure, N. Here, the scalar indicators, γ_1 and γ_2, are chosen according to the type of the plate problem. If the problem is a vibration problem, then, γ_1=1 and γ_2=0. If the problem is a buckling problem, then, γ_1=0 and γ_2=1.

$$F = U - \gamma_1 T + \gamma_2 V \tag{5}$$

T and V in the open form are given in Eq. (6) as

$$T = \frac{1}{2}\rho t w^2 \iint_A w^2 dx\,dy$$
$$V = -\frac{1}{2}\iint_A N\left\{\left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2\right\}dx\,dy \tag{6}$$

where ρ is the mass density, N is the in-plane normal force per unit length.

In applying Rayleigh-Ritz method, which is based on the energy principle, first of all, an appropriate deflection shape must be selected for the system. In a plate problem, the deflected middle surface may be represented in the form of a series:

$$w(x,y) = \alpha_1 f_1(x,y) + \alpha_2 f_2(x,y) + ...+ \alpha_n f_n(x,y) = \sum_{i=1}^{n}\alpha_i f_i(x,y) \tag{7}$$

where $f_i(x,y),(i=1,2,.........,n)$ are continuous functions that satisfy individually at least the geometrical boundary conditions and are capable of representing the deflected plate surface. The unknown constants $\alpha_1,\alpha_2,\alpha_3.....,\alpha_n$ are determined from the minimum potential energy principle as

$$\frac{\partial F}{\partial \alpha_1} = 0, \frac{\partial F}{\partial \alpha_2} = 0, ..., \frac{\partial F}{\partial \alpha_n} = 0 \tag{8}$$

With this minimization procedure, n simultaneous algebraic equations, in terms of the unknown coefficients $\alpha_1,\alpha_2,\alpha_3.....,\alpha_n$ will be obtained. The number of equations is equal to the number of unknown parameters, so $\alpha_1,\alpha_2,\alpha_3.....,\alpha_n$ can be calculated.

Then, the strain and the potential energy equations are computed into the total energy equation. According to this, for the buckling problem, the total energy equation becomes

$$F = \frac{D}{2}\iint_A\left\{\left(\nabla^2 w\right)^2 + 2(1-\nu)\left[\left(\frac{\partial^2 w}{\partial x \partial y}\right)^2 - \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}\right]\right\}dx\,dy - \frac{1}{2}\iint_A\left[N_x\left(\frac{\partial w}{\partial x}\right)^2 + N_y\left(\frac{\partial w}{\partial y}\right)^2\right]dx\,dy \tag{9}$$

Eq. (9) is minimized with respect to the unknown coefficients α_i. This procedure yields a set of homogeneous linear simultaneous equations in terms of α_i and in this way the problem is reduced to an eigenvalue problem.

Basic Assumptions and Equations:

The results are obtained for rectangular and elliptical plates which are compressed in their plane, in the directions of x and y, by Nx and Ny, respectively. The configurations of the analyzed rectangular and elliptical plates are given in Figure 2.

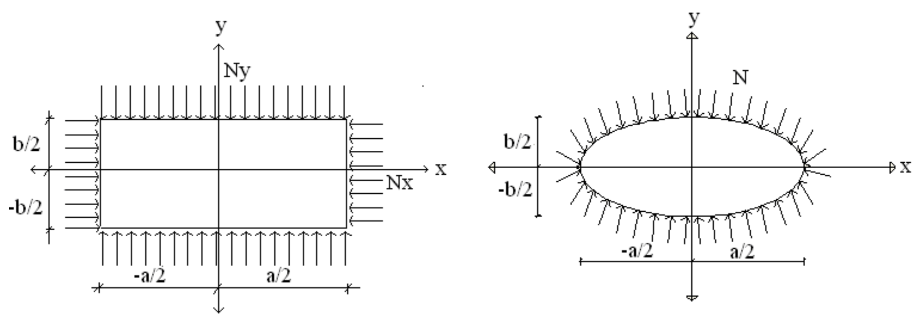

Figure 2. Configuration of the analyzed plates

The boundary shape equation of these rectangular and elliptical plates can be represented by Eq. (10) and Eq. (11), respectively. If the external boundary condition is clamped, in the equations Ω=2.

$$\phi(x,y)=\left[\left(x^2-\frac{a^2}{4}\right)\left(y^2-\frac{b^2}{4}\right)\right]^{\Omega} \tag{10}$$

$$\phi(x,y)=\left[\frac{x^2}{(a/2)^2}+\frac{y^2}{(b/2)^2}-1\right]^{\Omega} \tag{11}$$

In order to write the displacement surface, w, a polynomial deflection function is assumed as

$$w(x,y)=\sum_{i}^{r}\sum_{j}^{r}\alpha_{ij}\phi(x,y)\,x^i y^j \tag{12}$$

where, $i+j\le r$, so r is the degree of the polynomial trial function and α_{ij} are the coefficients to be determined. The existence of $\phi(x,y)$ in the equation guarantees that every element of these trial functions satisfies the boundary conditions of the problem. Knowing that the deflection function of the chosen system is an even function, the elements of the trial function which has odd powers of x and y are eliminated. It should be noted that selection of the trial functions has crucial importance in approximation and time-consuming.

In order to deal with the plate bending problem, a complete set of independent, continuous functions which are capable of representing the plate deflections are obtained by substituting Eq. (10) and Eq. (11) into Eq. (12). Here, Eq. (13) represents the function of the rectangular plate whereas Eq. (14) represents the function of the elliptical plate.

$$w(x,y)=\sum_{i}^{r}\sum_{j}^{r}\alpha_{ij}\left[\left(x^2-\frac{a^2}{4}\right)\left(y^2-\frac{b^2}{4}\right)\right]^{\Omega}x^i y^j \tag{13}$$

$$w(x,y)=\sum_{i}^{r}\sum_{j}^{r}\alpha_{ij}\left[\frac{x^2}{(a/2)^2}+\frac{y^2}{(b/2)^2}-1\right]^{\Omega}x^i y^j \tag{14}$$

According to this, for the buckling problem of the given rectangular and elliptical plates, the total energy equation, Eq. (9), is solved where the integration areas of the rectangular and elliptical plate are expressed as

$$A_{rectangular}=\int_{-a/2}^{a/2}\int_{-b/2}^{b/2}dx\,dy \tag{15}$$

$$A_{elliptical}=\int_{-a/2}^{a/2}\int_{-b/2\sqrt{1-(2x/a)^2}}^{b/2\sqrt{1-(2x/a)^2}}dx\,dy \tag{16}$$

NUMERICAL RESULTS

Thermal buckling anaysis of clamped rectangular and elliptical FGM plate with constant thickness has been carried out to validate Rayleigh-Ritz solution. In order to find the critical temperatures for the plates, the buckling load factors of the FGM plates should be obtained. In the analysis, three different shape functions are used which are namely the expansions of Eq.

(12) for r=2, 4, and 6. It is seen that as the number of the unknown terms are increased in the shape functions, the results get more convenient. Therefore, in the static analysis while applying Rayleigh-Ritz method, shape functions at the length of r=6 are used. Also, for the entire study, a is kept constant as 1, and b is chosen for 8 different numbers from 1 to 3 in order to obtain results for various a/b ratios.

Also, for the solution of FGM plates, the volume fractions of the ceramic and metal are found due to the variation of the power law index, n. Since functionally graded structures are most commonly used in high-temperature environment where significant changes in mechanical properties of the constituent materials are to be expected, it is essential to take into consideration this temperature-dependency for accurate prediction of the mechanical response. Thus, the effective Young's modulus, E_{eff}, and thermal expansion coefficient, α_{eff}, are assumed to be temperature dependent. According to these, the effective elastic moduli, E_{eff}, and effective thermal expansion coefficients, α_{eff}, are obtained again due to the variation of the power law index, n. The results are presented in Table 1. On the other hand, Poisson's ratio depends weakly on temperature change and is assumed to be constant as 0,3. Here, for the numerical calculations of FGM plates, the folowing properties of ceramic and metal materials are used in the analyses. Elastic modulus and the coefficient of thermal expansion for metal are E_m=70 GPa and α_m=23x10^{-6} 1/ºC and for the ceramic E_c=380 GPa and α_c=7.4x10^{-6} 1/ºC, respectively.

Table 1. Effective values of the elastic modulus and the thermal expansion coefficient

n	*Vc*	*Vm*	*Eeff* (GPa)	*αeff* (1/ºC)
0,1	0,909	0,091	351,79	8,82E-06
0,3	0,769	0,231	308,39	1,1E-05
0,5	0,667	0,333	276,77	1,26E-05
1	0,5	0,5	225	1,52E-05
2	0,333	0,667	173,23	1,78E-05
5	0,167	0,833	121,77	2,04E-05
15	0,062	0,938	89,22	2,2E-05

After finding the effective values of elastic moduli and the thermal expansion coefficients, it is possible to obtain the buckling load factors for FGM plates which are shown as λ' and presented as

$$\lambda' = Na^2 \frac{12(1-\nu^2)}{t^3} \tag{17}$$

The buckling load factors of the clamped rectangular and elliptical FGM plates are given in Table 2 and Table 3, respectively. In order to see the influence of the aspect ratio, the results are obtained for a wide range of a/b ratios.

Table 2. Buckling load factors for FGM rectangular plates

				n			
	0,1	0,3	0,5	1	2	5	15
a/b				**λ'**			
1	18416,21	16144,22	14488,91	11778,75	9068,59	6374,66	4670,67
1,125	21008,90	18417,05	16528,70	13437,00	10345,30	7272,10	5328,22
1,25	24178,53	21195,64	19022,40	15464,25	11906,10	8369,25	6132,09
1,375	27914,54	24470,75	21961,70	17853,75	13745,80	9662,45	7079,61

1,5	32195,82	28223,85	25329,99	20592,00	15854,01	11144,39	8165,41
2	54488,75	47766,53	42868,91	34850,25	26831,59	18860,96	13819,29
2,5	84482,37	74059,86	66466,32	54033,75	41601,18	29243,07	21426,18
3	121592,70	106591,92	95662,78	77769,00	59875,22	42088,58	30838,00

Table 3. Buckling load factors for FGM elliptical plates

				n			
	0,1	0,3	0,5	1	2	5	15
a/b				**λ'**			
1	20660,63	18111,74	16254,70	13214,25	10173,80	7151,55	5239,89
1,125	23538,27	20634,37	18518,68	15054,75	11590,82	8147,63	5969,71
1,25	26996,36	23665,85	21239,33	17266,50	13293,67	9344,63	6846,74
1,375	31017,32	27190,75	24402,81	19838,25	15273,69	10736,46	7866,53
1,5	35580,04	31190,56	27992,52	22756,50	17520,48	12315,82	9023,71
2	58741,89	51494,96	46215,05	37570,50	28925,95	20333,15	14897,96
2,5	89168,21	78167,61	70152,89	57030,75	43908,61	30865,04	22614,59
3	126672,54	111045,07	99659,34	81018,00	62376,66	43846,94	32126,34

After obtaining the buckling load factors for each case, the critical buckling temperatures of FGM plates can be found. By the help of Eq. (6) and Eq. (7), the methodology for the thermal buckling of plates is applied by using the thermal stress resultants and the buckling load factors. Thermal expansion coefficient and the elastic modulus change in the thermal analysis. It is evident that E_{eff} and α_{eff} are both temperature and position dependent. According to this, the critical temperature results under the uniform temperature rise for the clamped rectangular and elliptical FGM plates are obtained and the variation of the critical temperatures vs. dimensionless geometrical parameters a/b are plotted for each case. Figures 3 and 4 show the effect of the aspect ratio and seven arbitrary values of the power law index, n= 0.1, 0.3, 0.5, 1, 2, 5, 15, on the critical buckling temperature of the rectangular and elliptical FGM plates, respectively.

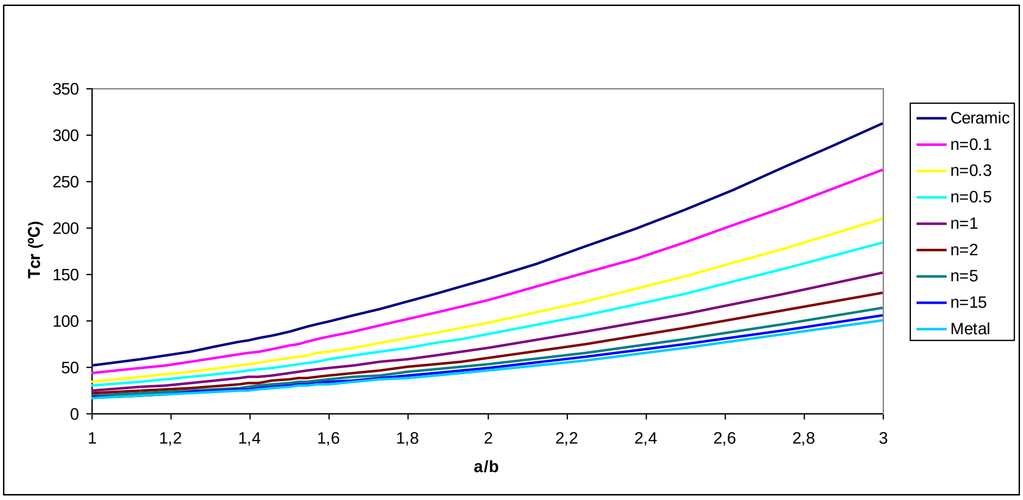

Figure 3. Critical buckling temperature of rectangular FGM plates vs. a/b

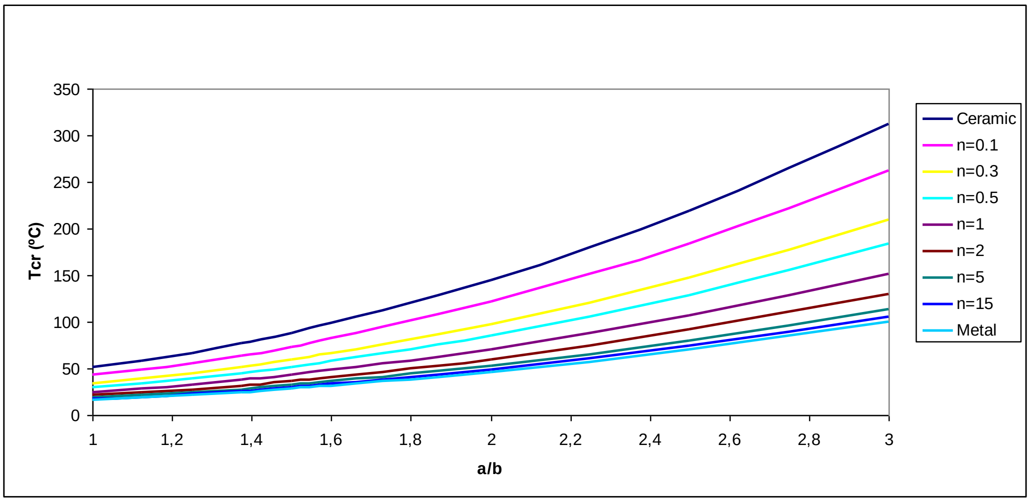

Figure 4. Critical buckling temperature of elliptical FGM plates vs. a/b

CONCLUSIONS

This study deals with the thermal buckling analysis of rectangular and elliptical FGM plates having constant thickness with fully clamped edges. According to the obtained results, the critical buckling temperature increases by the increase of the aspect ratio a/b and decreases by the increase of the power law index n from 0 to 15. The critical buckling temperature difference between ceramic and metal increases with the increase of a/b ratio which means the buckling temperature increment of ceramic dominant plates is higher than the buckling temperature increment of the metal dominant plates. It is also seen that after the value of n=5, for the higher values of n, there is no significant change in the temperatures for all plates. Also, the critical buckling temperatures for homogeneous plates, n=0, are considerably higher than those for the functionally graded plates, n>0, especially for the comparatively longer and thicker plates.

Further studies about FGM plates with different boundary conditions and variable thickness are also conducted and it is seen that critical buckling temperatures obtained for the clamped case are higher than those obtained for the simply supported case for both rectangular and elliptical plates [16].

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