ISTANBUL TECHNICAL UNIVERSITY ★ GRADUATE SCHOOL OF ARTS AND SOCIAL SCIENCES

COALITIONAL DEVIATION PROOF EQUILIBRIUM IN NONATOMIC GAMES

M.A. THESIS

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ATOMİK OLMAYAN OYUNLARDA KOALİSYONEL SAPMA İSPAT DENGESİ

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Kamil AHMADOV
(412141011)

Ekonomi Anabilim Dalı

Ekonomi Yüksek Lisans Programı

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Kamil Ahmadov, a M.A. student of ITU Graduate School of Arts and Social Sciences student ID 412141011, successfully defended the thesis entitled "COALITIONAL DEVIATION PROOF EQUILIBRIUM IN NONATOMIC GAMES", which he prepared after fulfilling the requirements specified in the associated legislations, before the jury whose signatures are below.

Thesis Advisor :	**Assist. Prof. Dr. Sinan ERTEMEL** Istanbul Technical University	....
Jury Members :	**Assist. Prof. Dr. Resul AYDEMIR** Istanbul Technical University	....
	Assist. Prof. Dr. Bora ERDAMAR Bahcesehir University	....

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To my family,

FOREWORD

I would like to dedicate this job to all my professors at school that taught me Economics. It was a great pleasure to spend time here, take the courses and write this thesis. I am grateful to Assist. Prof. Dr. David SEYMOUR for his support during the thesis period.

October 2019 Kamil AHMADOV

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ABBREVIATIONS

SE **:** Strategic Equilibrium

CDPE **:** Coalitional Deviation Proof Equilibrium

SYMBOLS

$\boldsymbol{T}$	: The set of all players
$\boldsymbol{\Sigma}$	: Collection of measurable set of players
$\boldsymbol{\nu}$	: Measure of players
l	: Single type of the atomless player
L	: Set of types of the atomless players
t	: The single player in a group
$\boldsymbol{a}$	: A single action
$\boldsymbol{A_l}$	: Action space for players of type *l*
$\boldsymbol{M}(\cdot)$	: Space of boral probability measures on a set
$\boldsymbol{Y}$	: Arbitrary seperable metric space
$\boldsymbol{\mathcal{M}}$	: Distribution of actions of all players
$\boldsymbol{\pi_l}$	**:** Distribution of the actions for players of type *l*
$\boldsymbol{U}$	**:** The space of real-valued continuous payoff functions
$\boldsymbol{T_l}$	: Set of players of type *l*
$\boldsymbol{u_l}$	**:** Payoff function for players of type *l*
$\boldsymbol{G}$	**:** Nonatomic game
$\boldsymbol{\mathcal{C}}$	**:** The set of Distributions of payoff funcitons and associated startegies
$\boldsymbol{\tau}$	**:** Distribution of payoff functions and associated startegies
$\boldsymbol{\tau_{l,U}}$	**:** The marginal of τ_l on U
$\boldsymbol{\tau_{l,A_l}}$	**:** The marginal of τ_l on A_l
$\boldsymbol{B}$	**:** Borel probability space
$\boldsymbol{T_l}$	**:** Players in group with low utility
ε	**:** Weight of the significance
$\boldsymbol{\delta_t}$	**:** Indicator function
$\boldsymbol{G_\varepsilon}$	**:** $\varepsilon - perturbed$ game
$\boldsymbol{u_\varepsilon}$	**:** Payoff function in the perturbed game
$\boldsymbol{\delta}$	**:** distance
$\boldsymbol{T_D}$	**:** New preferable group with high utility
$\boldsymbol{d_{A_l}}$	**:** Metric on the distribution of actions under sup norm

$\mathbf{T_d}$	**:** Coalition for players
$\boldsymbol{d_l^{TV}}$	**:** Total variation metric on l
$\boldsymbol{\nu_{l,t,\varepsilon}}$	**:** Measure of the player with weight ε
$\boldsymbol{\tau_l'}$	**:** Distribution of deviation with size ε

COALITIONAL DEVIATION PROOF EQUILIBRIUM IN NONATOMIC GAMES

SUMMARY

Nonatomic games are a useful way to model games with a large number players, which allows us to deal with a situation where each has a small influence on the outcomes by approximating a finite number of players by a continuum. However, in some instances, the results are problematic because there are implausible equilibria due to the strategic insignificance of the atomless players. Despite sometimes giving unrealistic results, nonatomic games are valuable because they may allow other parts of the game to use continuums. Therefore, it is important to develop an equilibrium concept for nonatomic games that gives players a small strategic influence on the outcome.

This study provides an equilibrium refinement for nonatomic games, that allows nonatomic players to have a small, but positive, impact on the outcome. Particularly, we develop an equilibrium concept where nonatomic players can deviate from the pure strategy Nash Equilibrium in arbitrarily small groups. We introduce the coalitional deviation proof equilibrium, an equilibrium that is robust to arbitrarily small deviations. Specifically, there is some positive small impact such that no coalitional deviations are profitable for almost all deviating players.

The method provides a rational basis for providing players a small strategic influence as an alternative to Barlo and Carmona (2015). They provide a method for dealing with small strategic significance by allowing nonatomic players to believe they have a small influence on the outcome. However, our model provides an alternative method of giving players small strategic significance that does not rely on players having irrational expectations about other players behaviors. As our main result, we find that some coalitional deviation proof equilibrium in our environment are not a strategic equilibrium in Barlo and Carmona (2015) model. We conjecture that assuming players any equilibrium outcome under a strategic equilibrium is also a coalitional deviation proof equilibrium for the environment with atomless players.

Modeling economic situation is easier with a continuum of agents because then we can consider agents as players with no impact on the average. Because of this method we were able to use continuum in many areas of economics. The continuum allows us to focus on the aspects of the problem and use different mathematical methods that give us a notion of the equilibrium. Specifically, we can deal with a large number of groups where each player's impact goes to zero.

In this thesis, we examine a situation where each individual is atomless and has no impact on the average. However, there is strategic significance that we can assume for the players which give us a more realistic outcome. With this method that allows us to give players a small amount of weight, we give back the strategic significance that was removed by assuming continuum.

We give each player an epsilon amount of weight over an outcome because in a finite model players would have strategic significance with coalition formation. With this formation, we can get a more realistic approximation where players have irrational beliefs about the behaviors of the other players.

ATOMİK OLMAYAN OYUNLARDA KOALİSYONEL SAPMA İSBAT DENGESİ

ÖZET

Bu tezde, süreklilik kavramının yer aldığı oyunlarda daha gerçekçi sonuçlar elde etmek için oyunculara çok küçük stratejik etki vererek denge kuramını araştırıyoruz. Spesifik olarak, her oyuncunun etkisiz olduğu ve stratejik öneme sahip olmadığı bir durumu inceliyoruz. Dengelerin keyfi olarak küçük koalisyon sapmalarına dayanıklı olmasını istiyoruz. Bunun oyunculara küçük bir ağırlık vermenin doğal bir yolu olduğunu düşünüyoruz, çünkü süreklilik kavramı oyuncuların sahip oldukları etkiyi ellerinden alarak onları etkisiz bir oyuncu olarak varsaymamıza neden oluyor, fakat koalisyon oyunculara ortalama üzerindeki stratejik etkiyi geri veriyor.

Atomik olmayan oyunlar sonlu sayıda oyunculardan ibaret büyük oyunların modellenmesi için kullanışlı yöntemdir, öyle ki bu oyunlar bize bireylerin sonuçlar üzerinde etkilerinin olmadığı durumları yaklaştırma yöntemi ile incelememize yardımcı oluyor. Ancak, bazı durumlarda, elde ettiğiğmiz sonuçlar makul olmayan denge kuramından dolayı problemli sonuçlara yol açabiliyor. Bazen gerçekçi olmayan sonuçlar vermesine rağmen, bu sayede biz süreklilik kavramını oyunların diğer bölümlerinde de kullanabiliyoruz. Bu nedenle, oyuncuların ortalama üzerinde küçük etkisinin olduğu atomik olmayan oyunlarda denge kavramının geliştirilmesi önemlidir.

Oyun teorisinde süreklilik kavramının kullanılması, ekonomide devrim yarattı, her bireyin önemsiz olmasına izin vermek, çok sayıda oyuncunun yer aldığı oyunlar ile ekonomik durumların modellenmesini kolaylaştırdı. Son altmış yılda, bireylerin süreklilik kavramı ile modellenmesi ekonominin birçok alanına uygulanmıştır. Örneğin, rekabetçi firmalar, dengeyi belirlemeyi kolaylaştırmak için mikroekonomide süreklilik kavramını kullanılarak modellemiştir. Makroekonomide, temsilci modelinde tüketiciler, tüketiciler arasındaki stratejik etkileşimi analiz etmekten kaçınmak için süreklilik kavramı altında modellenmiştir. Temsilcilerin sonuç üzerinde göz ardı edilebilir bir etkiye sahip olduğu ekonomik durumları analiz etmek için sürekliliği kullanmak, sorunun önemli olduğu konulara odaklanmamızı ve analizi basitleştirmek için gerçek analiz araçlarını kullanmamızı sağlar. Ayrıca, bu modelleme küçük stratejik etkiye sahip çok fakat sınırlı sayıda oyuncunun bulunduğu durumlara iyi bir yaklaşım sağlar. Ancak, oyun teorisinde ajanların sürekliliği daha az yaygındır. Oyuncular önemsiz olduklarında, sonuç üzerinde stratejik bir etkisi olmaz ve dengesiz sonuçlara yol açarlar.

Bu çalışma, oyunculara ortalama üzerinde çok küçük ama pozitif etki vererek, atomik olmayan oyunlar için denge arınması sağlar. Özellikle, biz burada atomik olmayan oyuncuların kendi istekleri dahilinde saf Nash dengesi stratejisinden gruplar halinde sapabilme denge kavramını geliştiriyoruz. Biz burada, küçük sapmalar için dirençli

olan, koalisyonel sapma denge isbatı kavramını sunuyoruz. Spesifik olarak, çok küçük pozitif etken vardır, öyle ki neredeyse tüm dengeden sapan oyuncular için hiç bir koalisyon sapması karlı değildir.

Kullandığımız yöntem, oyunculara stratejik değer vererek, Barlo ve Carmonaya (2015) alternatif olan rasyonel temel sağlar. Onlar, atomik olmayan oyuncuların sonuç üzerinde küçük bir etkiye sahip olduğuna inanmalarını sağlayarak, sonuç üzerinde çok küçük stratejik etkiye sahip oyuncular için bir yöntem sunarlar. Bununla birlikte, modelimiz, diğer oyuncular hakkında irrasyönel beklentide bulunmayan davranışa sahip oyunculara stratejik ağırlık vererek alternatif yöntem sunar. Biz, çalışmanın esas sonucu olarak, bizim ekonomik ortamımız için bazı koalisyonel sapma isbat dengelerinin, Barlo ve Carmonanın (2015) sunduğu stratejik denge ile aynı olmadığı gösteriyoruz. Aynı zamanda, stratejik dengeye sahip oyuncular için, ortalama üzerinde pozitif etkilerinin olduğunu varsayarak, atomik olmayan oyunculardan ibaret oyunlarda koalisyonel sapma isbat dengesine de ulaşabileceğimizi tahmin ediyoruz.

Süreklilik kavramı ile ekonomik durumun modellenmesi daha kolaydır çünkü bu zaman ajanları ortalamaya etkisi olmayan oyuncular olarak değerlendirebiliriz. Bu yöntem sayesinde ekonominin birçok alanında süreklilik kavramını kullanabildik. Süreklilik, sorunun yönlerine odaklanmamızı ve bize denge hakkında bir fikir veren farklı matematiksel yöntemleri kullanmamızı sağlıyor. Özellikle, her oyuncunun etkisinin sıfıra gittiği çok sayıda grubu araştırabiliriz.

Bu tezde, her bireyin atomsuz olduğu ve ortalamaya etkisi olmadığı bir durumu inceliyoruz. Ancak, oyuncuların stratejik öneme sahip olmaları bize daha gerçekçi bir sonuç vrecektir. Oyunculara az miktarda ağırlık vermemizi sağlayan bu yöntem ile, sürekliliği varsaymak suretiyle ortadan kaldırılan stratejik önemi oyunculara geri veriyoruz.

Her oyuncuya bir sonuç üzerinden epsilon ağırlık veriyoruz çünkü sonlu bir modelde oyuncular koalisyon oluşturmakla stratejik öneme sahip olacaklar. Bu oluşumla, oyuncuların diğer oyuncuların davranışları hakkında irrasyönel inançlara sahip olması halinde daha gerçekçi bir yaklaşım elde edeceğiz.

1. INTRODUCTION

Using a continuum of agents has revolutionized economics, allowing each individual to be insignificant has made modeling economic situations with a large number of agents easier. Over the last sixty years, continuums of agents have been applied to many areas of economics. For instance, monopolistically competitive firms are modeled using a continuum in microeconomics, to make the equilibrium easier to determine. In macroeconomics, consumers in representative agent models are represented by continuums to avoid analyzing the strategic interaction between consumers. Using continuums to analyze economic situations where agents have a negligible influence on the outcome allows us to focus on the aspects of the problem that are important and allows us to use real analysis tools to simplify the analysis. Further, it often provides a good approximation to situations where there are a large but finite number of players with small strategic influence. However, continuums of agents are less common in game theory. When players are insignificant, they have no strategic influence on the outcome, leading to implausible equilibria.

In this thesis, we provide an equilibrium concept that maintains a continuum of players but gives each player a small amount of strategic significance, leading to a more realistic outcome for games with a continuum of players. Specifically, we examine a situation where each player is atomless and has no strategic significance. We require that equilibria are robust to arbitrarily small coalitional deviations. We think this is a natural way to give players a small amount of weight because the coalition gives players back the strategic significance that was removed by assuming a continuum.

Although continuums of agents are unrealistic assumptions in most situations where they are used, continuums often provide a useful idealization for economic theory. They often give a valid approximation of a finite agent model while being more straightforward to solve. In particular, it is often easier to determine if an equilibrium exists and solve for the equilibrium conditions under a continuum of players. Unfortunately, modeling a large but finite number of agents using a continuum may lead to implausible outcomes in strategic situations. Because each player is insignificant, their actions do not affect the distribution of actions of the entire group of players.

The lack of strategic influence is best shown in an example from Barlo and Carmona (2015). They consider a game where each player has an action set {0,1}, and the player's payoff is the average choice of all of the players. If there are a finite number of players, all players have a dominant strategy of choosing 1. However, if the players are on the continuum [0,1] each player is atomless and has no influence the average choice. Even though the game has a dominant strategy, when there are a finite number of players, any outcome is an equilibrium when a continuum of players is used. In this instance, the atomless game is a poor approximation of the atomic game because players lose their strategic significance.

One approach that can be used to give players in a continuum of agents strategic significance is to allow the players to believe that they have a greater influence on the outcome than they actually do. Barlo and Carmona (2015) considered a perturbed environment where the players believe they have a small impact on the outcome. Given these beliefs, an equilibrium is the limit point where almost every player maximizes their utility as the impact that each player believes they have goes to zero. In other words, it chooses an equilibrium distribution from the unperturbed game that is the limit point of a series of equilibrium where each player believes he has a small influence on the outcome. In this model, players do not behave rationally because they believe that they have a small impact on the outcome but do not think others have a small impact. Therefore, the player's model of how people behave is different from how they actually behave.

Alternatively, we solve the problem that agents in a continuum lack strategic significance by requiring that equilibria are robust to arbitrarily small coalitional deviations. An equilibrium has a coalitional deviation of size ε if the set of players that deviate has a measure of at most ε. A group of players has a profitable coalitional deviation where each member chooses strategies that make them better off then they are under the equilibrium. An equilibrium is robust too coalitional deviations if, for every δ, there is some ε for which there are no profitable deviations of size ε where the players are deviating by more than δ. By assuming the measure of the coalition goes to zero, the coalition's strategic impact also approaches zero but does not become zero. In the limit, any coalition of players is insignificant relative to the players, making coalition atomless within the whole group.

Intuitively, we use coalition formation as a way to give players back the strategic significance that they would have in a finite model. That is why we give each player an epsilon amount of weight over an outcome. In some sense, each finite player can be modeled as a coalition of players of zero measure. This assumption gives a more realistic approximation of a finite person game without relying on players having irrational beliefs about the behaviors of other players.

Although modeling a finite number of players using a continuum of agents is an approximation that does not realistically describe an environment, returning strategic significance to the players makes this model more realistic. Further, using a continuum instead of a finite number of players allows for other parts of the game to be based on a continuum. Customers in an economic system that choose between different platforms is a good example. We know that agents can easily think that they have a small but positive impact on a societal choice. The example shows that when there is a continuum of players in nonatomic games, small players cannot affect outcomes with their decisions but they know that the aggregate choice could be effective.

In section 2, we discuss the relevant literature. Section 3 introduces our framework and assumptions. In the 4th section, we discuss the main theoretical results. The 5th section is a conclusion.

2. LITERATURE REVIEW

The use of continuums began in economics with Aumann (1964). In his seminal paper, he considered an economy with a continuum of agents to represent a large group where each individual's impact is negligible. Since then, most areas of economics use a continuum of agents analyze situations where each agent in a market with has little individual influence on the economy as a whole. Although his paper is related to the general equilibrium theory and demonstrates an effective solution to the core allocation problem for the pure exchange economies, his continuum of traders model played a revolutionary role for many areas of economics. The idea behind this was adapted to other areas of economics and is the typical way to model a large number of agents, goods, or agent characteristics.[1]

A decade later, Schmeidler (1973) extended the Nash equilibrium Nash (1950) to incorporate a continuum of players. He established a nonatomic game with atomless players with measurable utility functions by adapting the continuum of agents model to game theory. Under the assumption that each player has a finite action space, he found that there is a mixed strategy equilibrium just as in Nash (1950). When the payoff functions are further restricted, so they depend on the average response of the other players, an equilibrium when almost every player chooses a pure strategy. In his environment, the pure strategy equilibrium exists because players have a finite action space.

Dubey et al. (1980) showed that in an exchange economy where there are a large but finite number of traders, the limit of the equilibrium when there are a large but finite number of traders are equilibria in the associated nonatomic games. Green (1984) extended this characterization to nonatomic games. He discussed that, in small games, decision-makers have a significant impact on each other and can affect their welfare. However, when a continuum of players is considered, no player can individually affect

[1]Dornbush, Fisher, Samuelson (1977) used a continuum model to extend the Ricardian two commodity model into a continuum of goods model. Economists use similar models for modeling voting, auctions, and bargaining.

the payoffs of other players where the presence of each player creates the aggregate choice that they face. The limiting equilibrium when there are a large but finite number of players is equilibria in the associated nonatomic games.

Following Schmeidler, the literature has mostly focused on two approaches; the distributional approach and the average response approach. Under the distribution approach, the distribution of strategies played by the players is important, but the identity of the players playing the strategies is not. Under the average response approach, the players' payoffs are conditional on their own strategies and the average action that is chosen by all players. Under both models, players do not care about who chooses which strategy; instead, payoff only depends on relative proportions of the actions that players play. Therefore, both frameworks are anonymous.

The distribution approach was formulated by Mas-Colell (1984). It reformulated Schmeidler's approach by considering an equilibrium in distributions rather than in strategies. He proposed a type of anonymous games with a continuum of players where each player's payoff depends on their action and distribution of strategies chosen by the other players. Using distributions rather than strategies makes it easier to show that equilibria exist. When payoff functions are conditional on distributions of choices of the other players, there is a pure strategy equilibrium. Additionally, by using distributions rather than strategies, the players' payoff does not depend on the player playing the action.

Khan (1986) generalized the action space to include separable Banach spaces and considered preferences over outcomes rather than payoff functions. Using set-valued mapping as Schmeidler but changing the strategy space from Euclidean-n space into separable Banach space with Radon-Nikodym property we can find the same pure strategy Nash Equilibrium but compare with the Schmeidler model we ignore the available strategies for each player and accept convex hull which is an extreme points as the pure strategy set.

Rath (1992) reformulated Schmeidler's model to allow players payoff to depend on the average response of the other players, allowing for a simpler proof for the existence of the pure strategy Nash Equilibria in games with a continuum of players. Rath does not show mixed strategies, only considered the pure strategy equilibrium of the game and also extended the previous model where the action set of each player is a compact subset of Euclidean-n space. He argued that in his model Schmeidler allowed for mixed strategies and a broader class of payoff functions where he first showed mixed strategy equilibrium and then under restrictions for the payoff functions proved the existence of the pure strategy equilibrium.

Both the cardinality of the action space and the equilibrium concept are relevant for ensuring the existence of the pure strategy Nash Equilibrium Khan et al. (1997). For both the average response and the distributional models, there exists a pure strategy equilibrium under a finite action space and an n-dimensional Euclidean space. The average response model further ensures an equilibrium for the infinite-dimensional space for the average responses model. If the action set is countable, a pure strategy Nash Equilibriums exist for the average responses approach and the distributional approach model in a paper Khan and Sun (1995).

Although Schmeidler's model parallels Nash's finite normal form game where players are divided along the unit interval, we cannot extend Schmeidler's model to the general games. To ensure an equilibrium, either action space or payoff functions must be denumerable Khan and Sun (1995). However, when we assume our set of action is uncountable the existence of the pure strategies fails for the distributional approach model Rath et al., (1995). For the average responses model we can achieve in $\mathbb{R}^n$; however, there won't be an equilibrium in the infinite-dimensional setting Rath et al., (1995).

First, Carmona (2009) proved the equivalence of the measurability assumptions. Then, Carmona and Podczeck (2009) showed that all the previous models that have been used to show equilibrium in nonatomic games are equally strong; therefore, choosing a model is just a point of preference. They show that games with a continuum of players are just an idealization of the games with a large but finite number of players. The result they obtain indicates that there is a tradeoff between strengthened some assumptions while weakening others to generalize the existence of an equilibrium. Thus, from their equivalence result, they conclude that the assumptions that were made to show an equilibrium for different formalizations are just compensations of each other.

The space of player types plays a role in determining whether equilibria exist in non-anonymous games Noguchi (2009). He showed that when a game has many player types and there is no asymptotic information, we do have an equilibrium even when every player has an identical strategy space. He shows that the “many players of almost every type,” as in Podczeck (1997) variant, is a strong assumption. It does not make these non-anonymous games with many players rich enough; however, by applying Lebesgue measure, we can move to “many players every type” variant which is sufficient to achieve a pure strategy Nash equilibrium. Although Podczeck variant is also sufficient, he defined uncountable compact action space which is not rich enough when non-anonymous games defined on atomless probability space.

Barlo and Cormona (2015) showed that, in nonatomic games, players do not have any strategic influence on other players. To give players a small strategic significance, they developed the Strategic Equilibrium, a refinement of Nash Equilibrium. Under a Strategic Equilibrium, nonatomic players believe that they have a positive influence on the outcome, even though they cannot affect an outcome alone. Although the players are atomless, they have a small strategic influence on outcomes because they believe that they have a positive but small effect on an average choice. A shortcoming of Barlo and Carmona (2015), is that they assume that players are not rational. Therefore, the Strategic Equilibrium does not satisfy the rationality as in Selten (1975).

Aumann and Brandenburger (1995) also discuss the common knowledge and rationality between players when we have a large group of players. They showed that to achieve Nash equilibrium players need common knowledge, not rationality.

He et al. (2017) examined modeling infinitely many agents to determine which measure space is more suitable for modeling many economic agents. They proposed nowhere equivalence theorem and showed many examples of game theory, for a principal-agent model with equilibria in distributions, for the general equilibrium theory. They criticized some formalizations and failure of the equilibriums. They examined determinateness and equilibria existence problems especially for Khan and Sun (1999) and Khan et al. (2013) and proved that nowhere equivalence theorem gives us an equilibrium which does not exist in those models.

Khan et al. (2017) investigate two: large individualized games and large distributionalized games with a biosocial topology and show that they are both are quasi equal. They prove that the equilibrium condition for the large individualized games induces a Nash equilibrium distribution of the large distributionalized games. However, the converse is not always true; some equilibrium condition for the large distributionalized games are not induced by the Nash Equilibrium of the large individualized games.

In our work, we consider an alternative way to give players a small strategic influence. To give each player some strategic weight, we allow small coalitions to deviate collectively through choosing collective actions. Intuitively, an outcome is not an equilibrium if players can form arbitrarily small coalitions and change their actions such that almost all players in the coalition are better off.

We use the distributional approach from Mas-Colell (1984), where individuals' payoffs depend on their own action and the distribution on all players actions. This equilibrium is appropriate for atomless games because they approximate a situation where there are a large number of players, each with a small influence on the outcome.

Our model differs from Barlo and Carmona (2015). We provide a different method for giving players strategic significance. Additionally, they assumed that in addition to nonatomic players, there are also players that are atomic and can influence the outcome individually. We intentionally disregard atomic players because adding atomic players increases the complexity of the model, but keeps the same aspects as a standard game and does not add any new theoretical issues.

3. MODEL

3.1 Setup

The set of atomless players is denoted by T and given by a probability space (T, Σ, ν) where each player has zero meaure. A probablility space is used without loss of generality because every finite measure space can be normalized to a probability space. We partition T into a finite number of player types $l \in L$, where T_l is the set of players of type l. We assume that players in T_l are on the interval $[0,1]$ for all $l \in L$.

The players who are the same type have the same nonempty, compact metric action space A_l. Then $a \in A_l$ is a single action for a player in group l. Given a seperable metric space Y, $M(Y)$ is a space of a Borel probability measures on Y endowed with the weak convergence of probability measures. Then, $M(A_l)$ is the set of distributions for the players of type l and we can denote $\mathcal{M} = M(A_1) \times \cdots \times M(A_L)$. A distribution of actions for atomless players is $(\pi_1, \ldots, \pi_L) \in \mathcal{M}$ where π_l is the distribution of actions of players of type l.

We denote the space of real-valued continuous payoff functions U defined on $\mathcal{M} \times A_l$ which is endowed with the sup norm. Each of the players in T_l is assigned a utility function using a measurable function $u_l: T_l \to U$ and describes the set of payoff functions for all players in group l. Given the utility function u_l, a players utility is denoted by $u_l(t)$, where each players utility can depend on his choice of actions $a \in A_l$ and on the distributions of actions $(\pi_l, \ldots, \pi_L)$ of the players in $A_1, \ldots, A_L$.

We allow the strategies played and the utility functions to vary by the group because players may have different types with different strategies available in their types. Assuming players have different types is a more realistic assumption, as it allows the strategies of different types of player to have different strategies. When looking at coalitional deviations, we are trying to recreate a Nash equilibrium in a finite game.

Under a Nash equilibrium, a single player deviates; therefore, when considering coalitional deviations, players with different types are not allowed to deviate together.

A game $G = \left((T, \Sigma, \nu), (A_l, u_l)_{l=1}^{L}\right)$ consist of players, strategies and payoff functions of the players. In our environment, G is nonatomic as (T, Σ, ν) is atomless.

We define $\mathcal{C} = \prod_{l=1}^{L} M(U \times A_l)$ as the set of distribution of payoff functions and associated strategies, and $(\tau_1, \dots, \tau_L) \in \mathcal{C}$ as the distribution of payoff functions and associated actions of the players. $\tau_{l,U}$ is the marginal of τ_l on U which is the distribution of utility function of the players in *l*. Similarly, $\tau_{l,\, A_l}$ is a the marginal of τ_l on A_l and is the distribution of actions chosen by players in *l*.

3.2 Equilibrium

The equilibrium concept is the distributional approach Mass-Collel (1984). In equilibrium, almost every atomless player maximize their payoff for the given distribution of actions of the players.

Definition: A distribution $(\tau_1, \dots, \tau_l) \in \mathcal{C}$ is a equilibrium of the game G if for every *l*

1) $\nu(T_l)\tau_{l,U}(B) = \nu(\{t \in T_l\}: u_l(t) \in B)$ for each Borel measurable $B \subseteq U$, and
2) $\tau_l\left(\left\{(u, a) \in U \times A_l : u\left(a, \tau_{1,A_1}, \dots, \tau_{L,A_L}\right) \geq u\left(a', \tau_{1,A_1}, \dots, \tau_{L,A_L}\right) \text{ for each } a' \in A\right\}\right) = 1$

The first condition ensures that utility functions are consistent with the distribution of players in T_l. For each group, the measure of any subset of utility functions in the distribution τ_l is equal to the measure of available utility functions. The second condition tells us that almost all players in each group maximizes their utility given the distribution of actions chosen by other players. In the equilibrium, players do not have strategic significance, because they are not able to affect the distribution of actions; therefore, their individual actions do not affect the distribution of choices by their group.

3.3 Strategic Equilibrium

We summarize the strategic equilibrium of Barlo and Carmona (2015). A strategic equilibrium of a nonatomic game is an equilibrium refinement that looks at the limiting equilibrium of games where each player believes that he and he alone is an atom that can influence the distribution of actions chosen by his group of players. In this perturbed game, the players all think they can influence the distribution of actions even though each player $t \in T$ has zero measure.

The perturbed game where each player thinks that he has an ε impact on average is constructed by adjusting the player's payoffs to incorporate the false beliefs. A player t in a group l places a weight $\varepsilon > 0$ on the effect of her action on the distribution. Letting δ_t indicator function associated with the action of player t, the belief of player t of the distribution of actions chosen by her group of players is $\nu_{l,t,\varepsilon} = \varepsilon\delta_t + (1-\varepsilon)\nu_l$. Using the measure $\nu_{l,t,\varepsilon}$, the payoff function of each payer is adjusted to coincide with the belief of player t that she has an ε impact on the distribution of strategies played by her group.

For the game $G = \left((T, \Sigma, \nu), (A_l, u)_{l=1}^{L}\right)$ and for $\varepsilon > 0$, an ε-perturbed game is G_ε a nonatomic game with the same players and action space, but with each player's utility function adjusted to account for the belief that they have ε influence on the outcome. Given a game G and an $\varepsilon > 0$, the perturbed game is $G_\varepsilon = \left((T, \Sigma, \nu), (A_l, u_\varepsilon)_{l=1}^{L}\right)$, the payoff function is defined as

$$u_\varepsilon(t)(a, \pi) = u_l(t)\big(a, (\varepsilon\delta_a + (1-\delta)\pi_l, \pi_{-l})\big) \tag{1}$$

For all $1 \le l \le L$, $t \in T_l$, $\alpha \in A_l$, and $(\pi_l, \dots, \pi_L) \in \mathcal{M}$, Here, $u_\varepsilon(t)$ is continuous for each player $t \in T_l$ and $u_\varepsilon\colon T_l \to U$ is measurable; therefore, the game G_ε is nonatomic and has the same action spaces and players but with different payoff functions.

A strategic equilibrium is the limit of ε-perturbed games where the players beliefs about their influence on the outcome approach 0. It is the set of limits of Nash equilibrium distributions of ε-perturbed game where ε tends to 0. This structure gives the players an

arbitrarily small amount of strategic significance. The following definition of a strategic equilibrium is from Barlo and Carmona (2015).

Definition: A distribution $(\tau_1^*, \dots, \tau_L^*) \in \mathcal{C}$ is a strategic equilibrium of the game G if there exists a sequence $\{\varepsilon_k\}_{k=1}^{\infty} \subseteq (0,1)$ decreasing to zero and a sequence $\{\tau_1^k, \dots, \tau_L^*\}_{k=1}^{\infty} \subseteq \mathcal{C}$ converging to $(\tau_1^*, \dots, \tau_L^*)$ such that $(\tau_1^k, \dots, \tau_L^k)$ is an equilibrium of G_{ε_k} for every $\in \mathbb{N}$.

Although a strategic game is built upon incorrect beliefs about the payoff function, it has appealing properties. A strategic equilibrium is the limit points of equilibrium of a large finite game, allowing it to be used to approximate to large finite games. Further, under the assumptions of the model, the equilibrium exists.

However, calculating strategic equilibria might be difficult because we have to determine the payoff functions for all players in the ε-perturbed game. It is necessary to calculate the limits of the sequence of the Nash Equilibriums. One of the advantages of using continuums in economics is that it is not necessary to calculate complicated limits to determine the equilibrium outcome.

3.4 Coalitional deviation proof equilibrium

As an alternative method to give players a small amount of strategic significance, we allow players to deviate in small coalitions. An ε-coalition deviation is profitable if given a set of strategies, a set of players of size at most ε can benefit by deviating from the equilibrium. By allowing players to deviate in small groups, they can affect the outcome even though they are players with $\nu(\{t\}) = 0$.

We imagine a single player in a large, but finite, game deviating from the equilibrium. In this case, the players' strategy would have a small effect on the total distribution of actions taken by the group. As the group size gets arbitrarily large, the total influence of the player decreases but is not eliminated. In a nonatomic game, the players are separate decision makers each with zero measure. To give them the positive influence they have in large finite games, we allow them to form small groups which behaves

collectively. When we give them positive weight, we allow each player to deviate from equilibrium as a small group of players of the same type.

To model this behavior in an atomless game, we allow players to deviate in small coalitions of players of the same type. An ε-coalitional deviation is profitable if given a distribution of actions, a set of players with a positive measure of at most ε can benefit by deviating from the outcome. By allowing players to deviate in small groups, we give these players small strategic significance in equilibrium even though each player has zero measure under the modeling assumptions.

Definition: Given a game G and a distribution $(\tau_1, \dots, \tau_L) \in \mathcal{C}$, a distribution τ_l' is a deviation of size ε if

1. $d_l^{TV}(\tau, \tau') < \epsilon$, where $d_l^{TV}(\cdot,\cdot)$ is the total variation metric on l, and
2. $\tau_{l,U}(B) = \tau'_{l,U}(B)$ for each Borel measurable $B \subseteq U$

The deviating set is the smallest set $T_D \subseteq T_l$ such that $\tau'(T_D) - \tau(T_D) = d_l^{TV}(\tau, \tau')$.

A deviation of size ε requires that the set of players that are deviating is sufficiently small. Additionally, the distribution of payoff functions cannot change as a result of the action space changing. For a given deviation of size ε, a deviation is profitable if almost every player in the deviating set benefits from deviating.

Definition: A coalition of size $\boldsymbol{\varepsilon}$ has a profitable deviation if almost all players in $t \in T_D$ choose the same action and the payoff from deviating is strictly higher for almost all $t \in T_D$.

Definition: Let d_{A_l} be a metric on the distribution of actions under sup norm. A Nash equilibrium $\tau = (\tau_1, \dots, \tau_L)$ is Coalitional Deviation Proof Equilibrium (CDPE) if for any $\delta > 0$, there exists $\varepsilon > 0$ such that any profitable deviation τ_l' of size ε has $d_l(\pi_l, \pi_l') < \delta$.

An outcome is a coalitional deviation proof equilibrium if players do not have an incentive to deviate too much from the equilibrium strategies. The deviation is profitable for players if there is a chance to better off by joining a coalition that can provide a

higher payoff. For an outcome to be a coalitional deviation proof equilibrium, large deviations cannot be profitable as the coalition size becomes small. The following example shows that the equilibrium concept gives a plausible outcome in our motivating example Barlo and Carmona (2015).

Example 1: Let $T = [0,1]$, $A = \{0,1\}$, and $u(t) = \int_0^1 a(t)\,dt$. When almost all players choose 1, $u(t) = 1$ for all $t \in T$. If any coalition T_d of size $0 < T_d < \varepsilon$ deviated, then $a(t) = 0$ for all $t \in T_d$. The payoff from players that are deviating is $u(t) = 1 - \nu(T_D)$ for all $t \in T_d$. As the payoff from deviating is lower, there is no profitable coalitional deviation of any $\varepsilon > 0$. So, the outcome $a(t) = 1$ for almost all $t \in T$ is a coalitional deviation proof equilibrium.

Let $T_0 = \{t \in T : a(t) = 0\}$. Assume there is an equilibrium with $\nu(T_0) > 0$, then for any $\varepsilon > 0$, there is a set of players $T_d \in T_0$ such that $0 < \nu(T_d) < \varepsilon$. The payoff under the equilibrium is $u(t) = 1 - \nu(T_d)$. If the players in T_d deviate to $a(t) = 1$, then their payoff is $u(t) = 1 - \nu(T_0) + \nu(T_d)$. Therefore, the outcome $a(t) = 1$ for almost all $t \in T$ is the only coalitional deviation proof equilibrium.

4. RESULTS

CDPE give behavior that seems consistent strategic equilibria; however, there are equilbria that are CDPE but are not strategic equilibria. As the strategic equlibria and the limiting equilbria for large finite games are the same, this means some CDPE that are not limit points for finite games. We introduce the result in the proposition below:

Proposition : Some coalitional deviation proof equilibrium are not strategic equilibrium.

In Example 2, below, we have an equilibrium that has multiple coalitional deviation proof equilibria but only a single strategic equilibrium. Under the strategic equilibrium in any perturbed game, the players each think they have an ε influence on the outcome. Therefore, they each have an incentive to choose a slightly higher action than the average for any average less than 1. So the equilibrium of the perturbed game is each player choosing 1.

Example 2: Let $T = [0,1]$, $A = [0,1]$, and $u(t) = \int_0^1 a(t)\,dt - c\big(a(t)\big)$, where $c\big(a(t)\big) = \min\left\{0, \left(a(t) - \int_0^1 a(t)\,dt\right)^2\right\}$. Then the outcome $a(t) = 1$ for all $t \in T$ is the only strategic equilibrium, but for any $r \in [0,1]$, $a(t) = r$ for all $t \in T$ is an equilibrium.

Proof of proposition:

Given the environment in example 2, we show that $a(t) = 1$ for almost all $t \in T$ is a strategic equilibrium. Let $\varepsilon > 0$ be the weight each player places on their own strategy. Then the payoff to player t is $u(t) = (1-\varepsilon)\int_0^1 a(t)\,dt + \varepsilon a(t) - c(a(t))$. The payoff maximizing choice of a is $a = \min\left\{1, 2/(2-\varepsilon)\int_0^1 a(t)\,dt\right\}$; therefore, the only equilibrium has $\int_0^1 a(t) = 1$.

Let $a(t) = \bar{a}$ for almost all $t \in T$, and let T_d be a deviating set of size ε. If the players in T_d all change their strategy to $a(t)$, then $u(t) = (1 - \varepsilon) \int_0^1 a(t)\, dt + \varepsilon a(t) - c(a(t))$; therefore, is $a = \min\{1, 2/(2 - \varepsilon)\, \bar{a}\}$; so, $a - \bar{a} \leq \varepsilon/(2 - \varepsilon)\, \bar{a} < \varepsilon$. *QED.*

In this example, every outcome where almost all players play the average is an equilibrium under the coalitional deviation proof equilibrium. Under a coalitional deviation of size ε, each coalition has an incentive to deviate from the equilibrium. However, as the measure of the deviating set decreases, the players' deviation from the average gets arbitrarily small. Therefore, every player playing the average is a coalitional deviation proof equilibrium.

In each ε-perterbed game of the strategic equilibrium, each player has an incentive to choose a slightly higher a than the average. Therefore, in every ε-perterbed game, the equilbrium has $a = 1$. Thefore, the only strategic equlibrium is $a = 1$.

5. CONCLUSION

To get more plausible equilibria in nonatomic games, we allow small groups of players to deviate from the outcome. Those coalitions can block the equilibrium when small coalitions have an incentive to make large deviations. This equilibrium provides a way to give nonatomic players a small but positive strategic influence without relying on irrational assumptions about the players' expectations. We find that some equilibria are not strategic equilibria. Given the results in Barlo and Carmona (2015), this shows that although the coalitional deviation proof equilibria eliminate implausible equilibria, they leave some equilibria that are not the limit of arbitrarily large finite games.

We have an environment where players with a positive impact by forming small coalitions with players of the same type. As calculating the limits of the sequence of the equilibriums to determine strategic equilibrium is hard, calculating coalitional deviation proof equilibrium is easier because we do not have to compute the limit of a sequence of games. Therefore, although the equilibria in our method are not always the limit of finite games, our method provides a simple method for determining equilibria while eliminating some implausible equilibria.

We conjecture that any strategic equilibrium is a coalitional deviation proof equilibrium in our environment. If this conjecture is correct, the result provides an equilibrium concept such that every equilibrium is a strategic equilibrium in the sense of Barlo and Carmona (2015). For future research, we suggest determining whether every strategic equilibrium is also a coalitional deviation proof equilibrium. Then, this would show that for any game, there exists a coalitional deviation proof equilibrium. It is also worth determining under what assumptions these equilibrium concepts agree.

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CURRICULUM VITAE

Name Surname : Kamil Ahmadov

Place and Date of Birth : Azerbaijan, Qax, İlisu.
April 17, 1990.

E-Mail : kamilahmadov@hotmail.com

EDUCATION :

High School Group:	Zengi High School Biochemistry	2003-2007
Undergraduate Faculty:	Azerbaijan State University of Oil and Industry Engineering Economy	2007-2011