

Relativistic Stars in Starobinsky Model of Gravity

M.Sc. THESIS

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Starobinsky Gravitasyon Modelinde Relativistik Yıldızlar

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FOREWORD

I would like to express my deep gratitude to my thesis advisor Prof. Dr. Yavuz Ekşi for guidance, patience, helps, motivations and sparing his precious time.

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ABBREVIATIONS

PPN	: Parametrized Post-Newtonian
EoS	: Equation of State
STT	: Scalar-Tensor Theory
MAE	: Matched Asymptotic Expansions
GR	: General Relativity
TOV	: Tolman-Oppenheimer-Volkof

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LIST OF SYMBOLS

R	: Ricci scalar
S	: Action
S_{matter}	: Action of matter
g	: Determinant of metric tensor
$G_{\mu\nu}$: Einstein tensor
$R_{\mu\nu}$: Ricci tensor
$T_{\mu\nu}$: Energy-momentum tensor
ρ	: Density
P	: Pressure
∇	: Covariant derivative
\square	: D'Alembertian
m	: Mass
\bar{m}	: Dimensionless mass
\bar{P}	: Dimensionless pressure
\bar{R}	: Dimensionless Ricci scalar
$\bar{\rho}$: Dimensionless density
r	: Radial coordinate
r_s	: Radial distance from center to the surface of the star
x	: Dimensionless radial coordinate
ξ	: Coordinate variable of inner region

Relativistic Stars in Starobinsky Model of Gravity

SUMMARY

The strong gravity regime of neutron stars provide a good opportunity to test general relativity and constrain modified models of gravity. In this thesis the hydrostatic equilibrium of neutron stars in Starobinsky model of gravity is studied. The field equations and hydrostatic equilibrium equations are obtained for a spherically symmetric metric in the Starobinsky model. The trace of the field equations contains second order derivative of Ricci scalar and poses a singular perturbation problem. Accordingly, the *matched asymptotic expansion* (MAE) method is used to solve the set of differential equations for the Ricci scalar, mass within radial coordinate and pressure distribution for a uniform density object. It is found that the solution for the Ricci scalar can not match the Schwarzschild solution at the surface. This leads to the conclusion that uniform density neutron star solutions matching the Schwarzschild solution do not exist in the Starobinsky model of gravity. This does not exclude the possible existence of solutions that match an appropriate vacuum solution in this gravity model.

Starobinsky Gravitasyon Modelinde Relativistik Yıldızlar

ÖZET

Einstein alan denklemleri iki şekilde modifiye edilebilir; (i) Einstein alan denklemlerinin sağ tarafına enerji-momentum bileşeni eklemek, (ii) Einstein-Hilbert Lagrajyen'ine bazı terimler ekleyerek Einstein alan denklemlerinin sağ tarafını modifiye etmek. İkinci metot “modifiye gravitasyon modeli” olarak adlandırılır ve eğer Lagrajyen sadece Ricci skalerine bağlıysa “ $f(R)$ gravitasyon modeli” olarak adlandırılırlar.

Nötron yıldızlarının yeğin gravitasyon alanı genel göreliliği sınamak ve alternatif gravitasyon modellerine kısıtlamalar getirmek için uygun deney koşulları sunmaktadır. Bu tezde Starobinsky modifiye gravitasyon modelinde,

$$f(R) = R + \alpha R^2,$$

nötron yıldızlarının hidrostatik dengesi incelenmiştir. Starobinsky modelinde küresel simetrik metrik,

$$ds^2 = -\exp(2\phi)dt^2 + \exp(2\lambda)dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2),$$

için alan denklemleri,

$$(1 + \alpha h_R) G_{\mu\nu} - \frac{1}{2}\alpha(h - h_R R) g_{\mu\nu} - \alpha(\nabla_\mu \nabla_\nu - g_{\mu\nu} \square) h_R = 8\pi T_{\mu\nu},$$

burada

$$h = R^2 \quad \text{ve} \quad h_R = dh/dR \quad (0.1)$$

olmak üzere yukarıdaki gibi elde edilmiştir. Enerji-momentum tensörünün korunum denkleminin “r” bileşininden türetilen

$$\frac{dP}{dr} = -(\rho + P)\phi'$$

denklemini ve kütle ile λ parametersi arasındaki ilişkiyi tanımlayan

$$\exp(-2\lambda) = 1 - \frac{m(r)}{r}$$

denklemini kullanılarak hidrostatik denge denklemleri boyutsuz parametreler $\bar{\rho}$ (boyutsuz yoğunluk), \bar{m} (boyutsuz kütle), \bar{P} (boyutsuz basınç), \bar{R} (boyutsuz Ricci skaleri) ve x (boyutsuz radyal koordinat) cinsinden

$$\frac{d\bar{P}}{dx} = -\frac{\bar{\rho} + \bar{P}}{4x(x - \bar{m})(1 + 2\epsilon\bar{R} + \epsilon x\bar{R}')} \times [16\pi x^3 \bar{P} + 2\bar{m} + 4\epsilon\bar{m}\bar{R} - \epsilon x^3 \bar{R}^2 - 8\epsilon x\bar{R}'(x - \bar{m})]$$

ve

$$\begin{aligned}
(1 + 2\epsilon\bar{R} + \epsilon\bar{R}'x) \frac{d\bar{m}}{dx} &= \frac{x^2}{6} (48\pi\bar{P} + (2 + 3\epsilon\bar{R})\bar{R} + 32\pi\bar{\rho}) \\
&+ \frac{1}{(1 + 2\epsilon\bar{R})} \frac{2\epsilon\bar{R}'}{6} [-3\bar{m}(1 + 2\epsilon\bar{R}) + x^3(\bar{R} + 3\epsilon\bar{R}^2 + 16\pi\bar{\rho})] \\
&+ \epsilon^2 x(x - \bar{m}) \frac{4}{(1 + 2\epsilon\bar{R})} \bar{R}^2
\end{aligned}$$

burada α parametresinin boyutsuz hali ϵ olmak üzere yukarıdaki gibi elde edilmiştir. Ayrıca alan denklemlerinin izi

$$\begin{aligned}
\epsilon(1 + 2\epsilon\bar{R})\bar{R}'' &= (-8\pi\bar{\rho} + 24\pi\bar{P} + \bar{R}) \frac{1}{6} \left(\frac{x}{x - \bar{m}} \right) (1 + 2\epsilon\bar{R}) \\
&+ \frac{1}{6} \frac{\epsilon}{x - \bar{m}} [(1 + 2\epsilon\bar{R})6\bar{m}x^{-1} - 12(1 + 2\epsilon\bar{R})] \bar{R}' \\
&+ \frac{1}{6} \frac{\epsilon}{x - \bar{m}} [\epsilon x^2 \bar{R}^2 + x^2 \bar{R} + 16\pi x^2 \bar{\rho}] \bar{R}' \\
&+ 2\epsilon^2 \bar{R}^2
\end{aligned}$$

şeklinde Ricci skalerinin ikinci merteye türevini içeren bir diferansiyel denklem veriyor. Bu denklem en yüksek merteye türevli terimin önünde çok küçük bir parametre olan ϵ çarpanı olduğundan *tekil pertürbasyon problemi* özelliği gösteriyor. Bu nedenle Ricci skaleri, kütle ve basınç için elde ettiğimiz diferansiyel denklem setini çözerken *eşleşen asimptotik açılımlar* metodu kullanılmıştır. Sabit yoğunluklu bir nesne için aşağıdaki sınır koşulları kullanılarak

$$\begin{aligned}
\bar{m}(0) &= 0, & \bar{R}'(0) &= 0 \\
\bar{P}(1) &= 0, & \bar{R}'(1) &= 0
\end{aligned}$$

boyutsuz kütle

$$\bar{m}(x) = \frac{8}{3}\pi x^3 \bar{\rho},$$

boyutsuz basınç

$$\bar{P}(x) = \bar{\rho} \frac{\sqrt{\frac{1}{1 - \frac{8}{3}\pi\bar{\rho}}} - \sqrt{\frac{1}{1 - \frac{8}{3}\pi\bar{\rho}x^2}}}{3\sqrt{\frac{1}{1 - \frac{8}{3}\pi\bar{\rho}x^2}} - \sqrt{\frac{1}{1 - \frac{8}{3}\pi\bar{\rho}}}},$$

ve boyutsuz Ricci skaleri

$$\begin{aligned}
\bar{R}(x) &= 16\pi\bar{\rho} \left(1 - \frac{2\sqrt{\frac{1}{1 - \frac{8}{3}\pi\bar{\rho}}}}{6\sqrt{\frac{1}{1 - \frac{8}{3}\pi\bar{\rho}x^2}} - 2\sqrt{\frac{1}{1 - \frac{8}{3}\pi\bar{\rho}}}} \right) \\
&- \epsilon^{1/2} 2 \frac{(4\pi\bar{\rho})^2}{1 - \frac{8}{3}\pi\bar{\rho}} \sqrt{6 \left(1 - \frac{8}{3}\pi\bar{\rho} \right)} \exp \left(- \frac{1}{\sqrt{6(1 - \frac{8}{3}\pi\bar{\rho})}} \frac{1 - x}{\epsilon^{1/2}} \right)
\end{aligned}$$

olarak bulunmuştur.

Ricci skaleri için bulunan çözümün yüzeyde Schwarzschild çözümüyle eşleşmediği belirlenmiştir. Bundan dolayı Starobinsky gravitasyon modelinde yüzeyde

Schwarzschild çözümüne eşleşen sabit yoğunluklu neutron yıldızı çözümünün olamayacağı sonucuna varılmıştır. Bu sonuç bu gravitasyon modeli çerçevesinde elde edilebilecek bir vakum çözümüne eşleşen bir yıldız çözümü olamayacağı anlamına gelmemektedir.

1. INTRODUCTION

One of the most important discovery in physics in recent times is the acceleration of the universe [1]. Efforts to understand this phenomenon followed two avenues. The first is to add a constant energy-momentum component to the right hand side of Einstein's field equations such as the dark energy. The other one is modifying Einstein's general relativity (GR) by replacing the Lagrangian of the Einstein-Hilbert action R with some terms such as different orders of curvature terms of R , the Ricci scalar [2, 3]. That causes to change the left hand side of Einstein's field equations. In this so called $f(R)$ theories of gravity [4, 5, 6] the action is written as

$$S = \frac{1}{16\pi} \int d^4x \sqrt{-g} f(R) + S_{\text{matter}} \quad (\mathbf{1.1})$$

where fundamental constants $G = c = 1$. There are many models with different orders of curvature terms. Viable models, while providing accelerated expansion of the Universe [7, 8, 9] (see [10] for a review), should be consistent with the solar system and laboratory tests in the weak field limit [11]. It had been shown [12, 13] that the parametrized post-Newtonian (PPN) parameter γ has the value $1/2$ for $R \pm \mu^4/R$ type of gravities (independent of μ !) [2] which is measured to be close to unity [14] and is exactly 1 in general relativity. Mapping such $f(R)$ theories of gravity to scalar-tensor gravity models [15] leads to Brans-Dicke parameters well known to be incompatible with solar systems tests (see [16] for a review of tests of relativistic theories of gravity and PPN parameters).

Any model of gravity in which $f(R)$ is not a linear function of R results in field equations with 4th order derivatives which then would require extra boundary conditions than it is customary in general relativity. If higher curvature terms are perturbatively small, this would lead to a singular perturbation problem where the coefficient of the highest order terms in the system of equations are small so that the zeroth order expansion would artificially reduce the order of the equations.

Starobinsky model of gravity [17] employs

$$f(R) = R + \alpha R^2 \tag{1.2}$$

which provides an accelerated stage in the early universe—inflation— and is motivated by the presence of such a term at the weak energy limit of string theory [18]. Even if such a model of gravity satisfies the laboratory and the solar system tests, it is of interest to test it in the strong gravity regime of neutron stars by examining effect of the higher curvature term on the structure of neutron stars.

DeDeo and Psaltis [19] showed the existence of stars in the Starobinsky model by a perturbative method [20, 21]. Arapoğlu et al. [22] employed the same method to constrain the value of α by using neutron star mass-radius measurements [23] and for a representative sample of realistic equations of state (EoSs). Yazadjiev [24] followed a non-perturbative approach where the theory is conformally mapped to a scalar-tensor theory (STT) and argued the inconsistency of the perturbative approach.

The method of mapping $f(R)$ model of gravity to the STT is questioned at ref. [25] who argued that the scalar-tensor theory poses ill defined quantities and that causes singularity and wrong analyses.

In ref. [26] the Starobinsky model given in Equation 1.2 is used for studying the structure of neutron stars. Different from [22], they also used the trace equation which gives a second order differential equation for the Ricci curvature R . Because of the α coefficients on the second derivative of R , the equation poses a singular perturbation problem as also mentioned by the authors. They solve only the trace equation with appropriate analytical method e.g. the matched asymptotic expansion (MAE) method (see Appendix A) and solve the hydrostatic equilibrium equations by numerical methods for various EoSs. Accordingly, the authors of Ref. [26] encountered a problem in matching with the Schwarzschild solution at the surface of neutron stars in their direct numerical solution of the hydrostatic equilibrium equations. In the cosmological settings and for a different $f(R)$ model, authors of ref. [27] show that the MAE method is very useful and it gives very appropriate results.

In this work we employ MAE method for analysing the structure of neutron stars in $f(R) = R + \alpha R^2$ gravity. In § 2 we describe shortly relation between general relativity and hydrostatic equations. In § 3 we derive the field equations for spherical symmetric metric. In § 4 we derive the hydrostatic equilibrium equations. In § 5 we apply MAE method into the hydrostatic equilibrium equations and in § 6 we find solutions to these set of differential equations for uniform density with MAE method.

2. HYDROSTATIC EQUILIBRIUM OF RELATIVISTIC STARS IN GENERAL RELATIVITY

Einstein's field equations

$$G_{\mu\nu} = 8\pi T_{\mu\nu} \quad (2.1)$$

[28, 29] describe the relation between geometry of space-time and density ρ and pressure P where T_{ν}^{μ} is energy-momentum tensor and it equals $diag(\rho, -P, -P, -P)$ for perfect fluid and Einstein tensor is

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}. \quad (2.2)$$

The spherically symmetric and static metric is

$$ds^2 = -\exp(2\phi)dt^2 + \exp(2\lambda)dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \quad (2.3)$$

where λ and ϕ are functions of r . In GR with this metric the “time-time” (tt) and “radial-radial” (rr) components of field equations are

$$8\pi P = \exp(-\lambda) \left(\frac{\phi'}{r} + \frac{1}{r^2} \right) - \frac{1}{r^2} \quad (2.4a)$$

$$8\pi\rho = \exp(-\lambda) \left(\frac{\lambda'}{r} - \frac{1}{r^2} \right) + \frac{1}{r^2}, \quad (2.4b)$$

respectively [30, 31]. From the conservation equation of energy-momentum tensor, $\nabla_{\mu}T_{\nu}^{\mu} = 0$, it can be obtained that

$$\frac{dP}{dr} = -\frac{\rho + P}{2}\phi'. \quad (2.5)$$

These differential equations which describe the hydrostatic quantities are called Tolman-Oppenheimer-Volkoff (TOV) equations [30, 31]. In order to solve the structure of the star these equations must be completed by an equation of state (EoS) which is independent of the gravity model.

3. FIELD EQUATIONS

We start with the action

$$S = \frac{1}{16\pi} \int d^4x \sqrt{-g} f(R) + S_{matter} \quad (3.1)$$

where

$$f(R) = R + \alpha h(R). \quad (3.2)$$

The variation of this action gives the field equations

$$(1 + \alpha h_R) G_{\mu\nu} - \frac{1}{2} \alpha (h - h_R R) g_{\mu\nu} - \alpha (\nabla_\mu \nabla_\nu - g_{\mu\nu} \square) h_R = 8\pi T_{\mu\nu} \quad (3.3)$$

[4, 5] where $h_R = dh/dR$. The trace equation is then

$$3\alpha \square h_R - (1 - \alpha h_R) R - 2\alpha h = 8\pi T \quad (3.4)$$

which, with spherically symmetric and static metric given in Equation 2.3, implies that

$$3\alpha \exp(-2\lambda) h_R'' = 8\pi T + (1 - \alpha h_R) R + 2\alpha h + 3\alpha h_R' \exp(-2\lambda) \left(\lambda' - \frac{2}{r} - \phi' \right). \quad (3.5)$$

where the primes (') denote derivatives with respect to r . Also the “tt” component of the field equation is

$$-8\pi\rho = -r^{-2} + \exp(-2\lambda) (1 - 2r\lambda') r^{-2} + \alpha h_R (-r^{-2} + \exp(-2\lambda) (1 - 2r\lambda') r^{-2}) - \frac{1}{2} \alpha (h - h_R R) + \exp(-2\lambda) \alpha (h_R' r^{-1} (2 - r\lambda') + h_R''), \quad (3.6)$$

and the “rr” component of the field equation is

$$8\pi P = -r^{-2} + \exp(-2\lambda) (1 + 2r\phi') r^{-2} + \alpha h_R (-r^{-2} + \exp(-2\lambda) (1 + 2r\phi') r^{-2}) - \frac{1}{2} \alpha (h - h_R R) + \exp(-2\lambda) \alpha h_R' r^{-1} (2 + r\phi') \quad (3.7)$$

[19].

4. TOV EQUATIONS IN f(R) GRAVITY

We define $m(r)$, mass within radial coordinate r , as

$$\exp(-2\lambda) = 1 - \frac{m(r)}{r}. \quad (4.1)$$

By taking the derivative of this we find

$$\frac{dm}{dr} = 1 - (1 - 2\lambda'r) \exp(-2\lambda) \quad (4.2)$$

which we rearrange as

$$\lambda' = \frac{1}{2} \left(\frac{dm}{dr} - 1 \right) \left(\frac{1}{r - m} \right) + \frac{1}{2r}. \quad (4.3)$$

By plugging this to the “tt” component of the field equation and rearranging it, the first TOV equation is obtained as:

$$\begin{aligned} \left(1 + \alpha h_R + \frac{1}{2} \alpha h'_R r \right) \frac{dm}{dr} &= 8\pi \rho r^2 - \frac{1}{2} \alpha r^2 (h - h_R R) \\ &+ \frac{\alpha h'_R r}{2} \left(-3 \frac{m}{r} + 4 \right) + \left(1 - \frac{m}{r} \right) \alpha r^2 h''_R. \end{aligned} \quad (4.4)$$

Left hand side of the “rr” component of the conservation equation of the energy-momentum tensor, $\nabla_\mu T_1^\mu = 0$, is

$$\begin{aligned} \nabla_\mu T_1^\mu &= \partial_\mu T_1^\mu + \Gamma_{\mu\lambda}^\mu T_1^\lambda - \Gamma_{\mu 1}^\lambda T_\lambda^\mu \\ &= P' + P \left(\lambda' + \frac{1}{r} + \frac{1}{r} + \phi' \right) - \left(\lambda' P + \frac{1}{r} P + \frac{1}{r} P - \phi' \rho \right). \end{aligned} \quad (4.5)$$

By equating above equation to zero, the second TOV equation is derived as

$$\frac{dP}{dr} = -(\rho + P) \phi'. \quad (4.6)$$

Here the ϕ' is found from the “rr” component of the field equation

$$\begin{aligned} \phi' (2 + 2\alpha h_R + \alpha r h'_R) &= 8\pi P \left(\frac{r^2}{r - m} \right) + (1 + \alpha h_R) \frac{m}{r - m} r^{-1} \\ &+ \frac{1}{2} \alpha (h - h_R R) \frac{r^2}{r - m} - 2\alpha h'_R. \end{aligned} \quad (4.7)$$

5. SINGULAR PERTURBATION PROBLEM

We choose $h = R^2$ and define dimensionless parameters are defined as

$$x = \frac{r}{r_s}, \quad \epsilon = \frac{\alpha}{r_s^2}, \quad \bar{R} = r_s^2 R, \quad \bar{P} = r_s^2 P, \quad \bar{\rho} = r_s^2 \rho, \quad \bar{m} = \frac{m}{r_s} \quad (5.1)$$

where r_s is the radial distance from center to the surface of the star and so $0 < x < 1$. The first TOV equation given in Equation 4.4, in dimensionless variables, become

$$\begin{aligned} (1 + 2\epsilon\bar{R} + \epsilon\bar{R}'x) \frac{d\bar{m}}{dx} = & \frac{x^2}{6} (48\pi\bar{P} + (2 + 3\epsilon\bar{R})\bar{R} + 32\pi\bar{\rho}) \\ & + \frac{1}{(1 + 2\epsilon\bar{R})} \frac{2\epsilon\bar{R}'}{6} [-3\bar{m}(1 + 2\epsilon\bar{R}) + x^3(\bar{R} + 3\epsilon\bar{R}^2 + 16\pi\bar{\rho})] \\ & + \epsilon^2 x(x - \bar{m}) \frac{4}{(1 + 2\epsilon\bar{R})} \bar{R}^2. \end{aligned} \quad (5.2)$$

The second TOV equation given in Equation 4.6 then becomes

$$\begin{aligned} \frac{d\bar{P}}{dx} = & - \frac{\bar{\rho} + \bar{P}}{4x(x - \bar{m})(1 + 2\epsilon\bar{R} + \epsilon x\bar{R}')} \times \\ & [16\pi x^3 \bar{P} + 2\bar{m} + 4\epsilon\bar{m}\bar{R} - \epsilon x^3 \bar{R}^2 - 8\epsilon x \bar{R}'(x - \bar{m})]. \end{aligned} \quad (5.3)$$

The trace equation given in Equation 3.4 becomes

$$\begin{aligned} \epsilon(1 + 2\epsilon\bar{R})\bar{R}'' = & (-8\pi\bar{\rho} + 24\pi\bar{P} + \bar{R}) \frac{1}{6} \left(\frac{x}{x - \bar{m}} \right) (1 + 2\epsilon\bar{R}) \\ & + \frac{1}{6} \frac{\epsilon}{x - \bar{m}} [(1 + 2\epsilon\bar{R}) 6\bar{m}x^{-1} - 12(1 + 2\epsilon\bar{R})] \bar{R}' \\ & + \frac{1}{6} \frac{\epsilon}{x - \bar{m}} [\epsilon x^2 \bar{R}^2 + x^2 \bar{R} + 16\pi x^2 \bar{\rho}] \bar{R}' + 2\epsilon^2 \bar{R}^2 \end{aligned} \quad (5.4)$$

[26].

For satisfying continuity at centre of the star we choose two boundary conditions as $\bar{m}(0) = 0$ and $\bar{R}'(0) = 0$. Since the pressure vanishes at the surface of the star, $\bar{P}(1) = 0$. For matching with the Schwarzschild solution at the surface we choose $\bar{R}'(1) = 0$ [32, 33]. The condition $\bar{R}(1) = 0$ will be used for testing solutions instead of boundary condition.

Because of the ϵ coefficient multiplying the R'' term in the trace equation given in Equation 5.4, the system of equations poses a singular perturbation problem. An appropriate method for handling such singular problems, well known in the fluid dynamics research, is the MAE which we employ in the following. In Appendix A we introduce MAE for solving singular perturbation problems. According to the method, there should be a *boundary layer* which according to the authors of ref. [26] forms near to the surface of the star. The solution which is valid in the boundary layer is called the *inner solution* and for stretching this region a new parameter will be defined in an appropriate way. Outside the boundary layer—the rest of the star—the *outer solution* is valid. In a transition region the two solutions match with each other. The outer solutions of Equations (5.2), (5.3) and (5.4) satisfy the boundary conditions at $x = 0$, and the inner solutions satisfy the boundary conditions at the $x = 1$.

5.1 Equations of Outer Region

The outer solutions are valid in the domain $0 < x < 1$ and they are introduced as perturbative expansions

$$\bar{R}^{\text{out}}(x) = \bar{R}_0^{\text{out}}(x) + \epsilon^{1/2} \bar{R}_1^{\text{out}}(x) + O(\epsilon) \quad (5.5a)$$

$$\bar{m}^{\text{out}}(x) = \bar{m}_0^{\text{out}}(x) + \epsilon^{1/2} \bar{m}_1^{\text{out}}(x) + O(\epsilon) \quad (5.5b)$$

$$\bar{P}^{\text{out}}(x) = \bar{P}_0^{\text{out}}(x) + \epsilon^{1/2} \bar{P}_1^{\text{out}}(x) + O(\epsilon) \quad (5.5c)$$

$$\bar{\rho}^{\text{out}}(x) = \bar{\rho}_0^{\text{out}}(x) + \epsilon^{1/2} \bar{\rho}_1^{\text{out}}(x) + O(\epsilon). \quad (5.5d)$$

We plug these expressions into Equations (5.2), (5.4) and (5.3). The $O(1)$ terms in the equations are

$$\frac{d\bar{m}_0^{\text{out}}}{dx} = \frac{x^2}{6} (48\pi \bar{P}_0^{\text{out}} + 2\bar{R}_0^{\text{out}} + 32\pi \bar{\rho}_0^{\text{out}}) \quad (5.6a)$$

$$4x(x - \bar{m}_0^{\text{out}}) \frac{d\bar{P}_0^{\text{out}}}{dx} = -(\bar{\rho}_0^{\text{out}} + \bar{P}_0^{\text{out}}) (16\pi x^3 \bar{P}_0^{\text{out}} + 2\bar{m}_0^{\text{out}}) \quad (5.6b)$$

$$0 = x(-8\pi \bar{\rho}_0^{\text{out}} + 24\pi \bar{P}_0^{\text{out}} + \bar{R}_0^{\text{out}}). \quad (5.6c)$$

Similarly, the $O(\epsilon^{1/2})$ terms are

$$\frac{d\bar{m}_1^{\text{out}}}{dx} = \frac{x^2}{6} (48\pi\bar{P}_1^{\text{out}} + 2\bar{R}_1^{\text{out}} + 32\pi\bar{\rho}_1^{\text{out}}) \quad (5.7a)$$

$$4x(x - \bar{m}_0^{\text{out}}) \frac{d\bar{P}_1^{\text{out}}}{dx} = -4x\bar{m}_1^{\text{out}} \frac{d\bar{P}_0^{\text{out}}}{dx} - (\bar{\rho}_1^{\text{out}} + \bar{P}_1^{\text{out}}) (16\pi x^3 \bar{P}_0^{\text{out}} + 2\bar{m}_0^{\text{out}}) - (\bar{\rho}_0^{\text{out}} + \bar{P}_0^{\text{out}}) (16\pi x^3 \bar{P}_1^{\text{out}} + 2\bar{m}_1^{\text{out}}) \quad (5.7b)$$

$$0 = \frac{1}{6}x (-8\pi\bar{\rho}_1^{\text{out}} + 24\pi\bar{P}_1^{\text{out}} + \bar{R}_1^{\text{out}}). \quad (5.7c)$$

By solving these equations the outer solutions can be obtained. Also the boundary conditions at $x = 0$ become

$$\bar{m}_0^{\text{out}}(0) = \bar{m}_1^{\text{out}}(0) = 0, \quad (\bar{R}_0^{\text{out}})'(0) = (\bar{R}_1^{\text{out}})'(0) = 0. \quad (5.8)$$

5.2 Equations of Inner Region

We assume that the boundary layer exists near to surface so we define a new variable (coordinate stretching parameter) as $\xi \equiv (1 - x)/\epsilon^\nu$ which implies that

$$\frac{d}{dx} = -\frac{1}{\epsilon^\nu} \frac{d}{d\xi}, \quad \frac{d^2}{dx^2} = \frac{1}{\epsilon^{2\nu}} \frac{d^2}{d\xi^2}. \quad (5.9)$$

Using the stretching parameter, the dimensionless trace equation given in Equation 5.4 is rewritten as

$$\begin{aligned} \epsilon^{1-2\nu} (1 - \epsilon^\nu \xi) (1 - \epsilon^\nu \xi - \bar{m}^{\text{in}}) (1 + 2\epsilon\bar{R}^{\text{in}}) (\bar{R}^{\text{in}})'' = \\ \frac{1}{6} (1 - \epsilon^\nu \xi)^2 (1 + 2\epsilon\bar{R}^{\text{in}}) (-8\pi\bar{\rho}^{\text{in}} + 24\pi\bar{P}^{\text{in}} + \bar{R}^{\text{in}}) \\ - \frac{1}{6} \epsilon^{1-\nu} \left[(1 + 2\epsilon\bar{R}^{\text{in}}) 6\bar{m}^{\text{in}} - 12(1 + 2\epsilon\bar{R}^{\text{in}}) (1 - \epsilon^\nu \xi) + \epsilon (1 - \epsilon^\nu \xi)^3 (\bar{R}^{\text{in}})^2 \right] (\bar{R}^{\text{in}})' \\ - \frac{1}{6} \epsilon^{1-\nu} \left[(1 - \epsilon^\nu \xi)^3 \bar{R}^{\text{in}} + 16\pi (1 - \epsilon^\nu \xi)^3 \bar{\rho}^{\text{in}} \right] (\bar{R}^{\text{in}})' \\ + 2\epsilon^{2-2\nu} (1 - \epsilon^\nu \xi) (1 - \epsilon^\nu \xi - \bar{m}^{\text{in}}) (\bar{R}^{\text{in}})^{\prime 2}. \end{aligned} \quad (5.10)$$

We should, now, choose the value of ν such that R'' term and one of the terms on left hand side of the equation are $O(1)$ while rest of the terms are of higher orders. This can be achieved by choosing

$$\nu = 1/2. \quad (5.11)$$

Thus, the inner solutions which valid for $0 \ll x < 1$ are introduced as

$$\bar{R}^{\text{in}}(\xi) = \bar{R}_0^{\text{in}}(\xi) + \epsilon^{1/2} \bar{R}_1^{\text{in}}(\xi) + O(\epsilon) \quad (5.12a)$$

$$\bar{m}^{\text{in}}(\xi) = \bar{m}_0^{\text{in}}(\xi) + \epsilon^{1/2} \bar{m}_1^{\text{in}}(\xi) + O(\epsilon) \quad (5.12b)$$

$$\bar{P}^{\text{in}}(\xi) = \bar{P}_0^{\text{in}}(\xi) + \epsilon^{1/2} \bar{P}_1^{\text{in}}(\xi) + O(\epsilon) \quad (5.12c)$$

$$\bar{\rho}^{\text{in}}(\xi) = \bar{\rho}_0^{\text{in}}(\xi) + \epsilon^{1/2} \bar{\rho}_1^{\text{in}}(\xi) + O(\epsilon). \quad (5.12d)$$

and the boundary conditions at $x = 1$ then become

$$\bar{P}_0^{\text{in}}(0) = \bar{P}_1^{\text{in}}(0) = (\bar{R}_0^{\text{in}})'(0) = (\bar{R}_1^{\text{in}})'(0) = 0. \quad (5.13)$$

In terms of stretching parameter Equation 5.2 can be written as

$$\begin{aligned} & \frac{1}{\epsilon^{1/2}} \left[1 + 2\epsilon \bar{R}^{\text{in}} - \epsilon^{1/2} (1 - \epsilon^{1/2} \xi) (\bar{R}^{\text{in}})' \right] \frac{d\bar{m}^{\text{in}}}{d\xi} = \\ & - \frac{(1 - \epsilon^{1/2} \xi)^2}{6} (48\pi \bar{P}^{\text{in}} + (2 + 3\epsilon \bar{R}) \bar{R}^{\text{in}} + 32\pi \bar{\rho}^{\text{in}}) \\ & + \epsilon^{1/2} \frac{(\bar{R}^{\text{in}})'}{3(1 + 2\epsilon \bar{R}^{\text{in}})} \left[(1 - \epsilon^{1/2} \xi)^3 (\bar{R}^{\text{in}} + 3\epsilon (\bar{R}^{\text{in}})^2 + 16\pi \bar{\rho}^{\text{in}}) - 3\bar{m}^{\text{in}} (1 + 2\epsilon \bar{R}^{\text{in}}) \right] \\ & - \epsilon (1 - \epsilon^{1/2} \xi) (1 - \epsilon^{1/2} \xi - \bar{m}^{\text{in}}) \frac{4}{(1 + 2\epsilon \bar{R}^{\text{in}})} (\bar{R}^{\text{in}})^2. \end{aligned} \quad (5.14)$$

Similarly, Equation 5.3 can be written as

$$\begin{aligned} & \left[4(1 - \epsilon^{1/2} \xi) (1 - \epsilon^{1/2} \xi - \bar{m}^{\text{in}}) (1 + 2\epsilon \bar{R}^{\text{in}} - (\epsilon^{1/2} - \epsilon \xi) (\bar{R}^{\text{in}})') \right] \frac{d\bar{P}^{\text{in}}}{d\xi} = \\ & \epsilon^{1/2} (\bar{\rho}^{\text{in}} + \bar{P}^{\text{in}}) \left[16\pi (1 - \epsilon^{1/2} \xi)^3 \bar{P}^{\text{in}} + 2\bar{m}^{\text{in}} + 4\epsilon \bar{m}^{\text{in}} \bar{R}^{\text{in}} - \epsilon (1 - \epsilon^{1/2} \xi)^3 (\bar{R}^{\text{in}})^2 \right. \\ & \left. + 8\epsilon^{1/2} (1 - \epsilon^{1/2} \xi) (\bar{R}^{\text{in}})' (1 - \epsilon^{1/2} \xi - \bar{m}^{\text{in}}) \right] \end{aligned} \quad (5.15)$$

and Equation 5.10 becomes

$$\begin{aligned} & (1 - \epsilon^{1/2} \xi) (1 - \epsilon^{1/2} \xi - \bar{m}^{\text{in}}) (1 + 2\epsilon \bar{R}^{\text{in}}) (\bar{R}^{\text{in}})'' = \\ & + \frac{1}{6} (1 - \epsilon^{1/2} \xi)^2 (1 + 2\epsilon \bar{R}^{\text{in}}) (-8\pi \bar{\rho}^{\text{in}} + 24\pi \bar{P}^{\text{in}} + \bar{R}^{\text{in}}) \\ & - \frac{\epsilon^{1/2}}{6} \left[(1 + 2\epsilon \bar{R}^{\text{in}}) 6\bar{m}^{\text{in}} - 12(1 + 2\epsilon \bar{R}^{\text{in}}) (1 - \epsilon^{1/2} \xi) + \epsilon (1 - \epsilon^{1/2} \xi)^3 (\bar{R}^{\text{in}})^2 \right] (\bar{R}^{\text{in}})' \\ & - \frac{\epsilon^{1/2}}{6} \left[(1 - \epsilon^{1/2} \xi)^3 \bar{R}^{\text{in}} + 16\pi (1 - \epsilon^{1/2} \xi)^3 \bar{\rho}^{\text{in}} \right] (\bar{R}^{\text{in}})' \\ & + 2\epsilon (1 - \epsilon^{1/2} \xi) (1 - \epsilon^{1/2} \xi - \bar{m}^{\text{in}}) (\bar{R}^{\text{in}})^2. \end{aligned} \quad (5.16)$$

$O(1)$ terms in these equations are

$$\frac{d\bar{m}_0^{\text{in}}}{d\xi} = 0 \quad (5.17\text{a})$$

$$(1 - \bar{m}_0^{\text{in}}) \frac{d\bar{P}_0^{\text{in}}}{d\xi} = 0 \quad (5.17\text{b})$$

$$6(1 - \bar{m}_0^{\text{in}}) (\bar{R}_0^{\text{in}})'' = -8\pi\bar{\rho}_0^{\text{in}} + 24\pi\bar{P}_0^{\text{in}} + \bar{R}_0^{\text{in}}. \quad (5.17\text{c})$$

Non-trivial solutions of the first and the second equations are

$$\bar{m}_0^{\text{in}} = A_0 \quad \bar{P}_0^{\text{in}} = B_0 \quad (5.18)$$

where A_0 and B_0 are constants. According to boundary condition at Equation 5.13, B_0 should be zero.

Then $O(\epsilon^{1/2})$ terms in the equations of inner solution given in Equations (5.14), (5.15) and (5.16) are

$$\frac{d\bar{m}_1^{\text{in}}}{d\xi} = -\frac{1}{6} (2\bar{R}_0^{\text{in}} + 32\pi\bar{\rho}_0^{\text{in}}) \quad (5.19\text{a})$$

$$4(1 - A_0) \frac{d\bar{P}_1^{\text{in}}}{d\xi} = 2\bar{\rho}_0^{\text{in}} A_0 \quad (5.19\text{b})$$

$$\begin{aligned} (1 - A_0) (\bar{R}_1^{\text{in}})'' &= [\bar{m}_1^{\text{in}} + \xi(2 - A_0)] (\bar{R}_0^{\text{in}})'' \\ &\quad + \frac{1}{6} (-8\pi\bar{\rho}_1^{\text{in}} + 24\pi\bar{P}_1^{\text{in}} + \bar{R}_1^{\text{in}}) \\ &\quad - \frac{1}{3}\xi (-8\pi\bar{\rho}_0^{\text{in}} + \bar{R}_0^{\text{in}}) \\ &\quad - \frac{1}{6} (6A_0 - 12 + \bar{R}_0^{\text{in}} + 16\pi\bar{\rho}_0^{\text{in}}) (\bar{R}_0^{\text{in}})'. \end{aligned} \quad (5.19\text{c})$$

6. UNIFORM DENSITY STARS

For simplicity, density is taken constant in neutron star and have no perturbative part. Accordingly $\rho_1^{\text{out}} = \rho_1^{\text{in}} = 0$.

6.1 Outer Solutions for Uniform Density

Equation 5.6c can be written as

$$\bar{R}_0^{\text{out}}(x) = 8\pi\bar{\rho}_0^{\text{out}} - 24\pi\bar{P}_0^{\text{out}}(x). \quad (6.1)$$

If we plug it into Equation 5.6a, we obtain that

$$\frac{d\bar{m}_0^{\text{out}}}{dx} = 8\pi x^2 \bar{\rho}_0^{\text{out}}. \quad (6.2)$$

The solution of this equation with Equation 5.8 is

$$\bar{m}_0^{\text{out}}(x) = \frac{8}{3}\pi x^3 \bar{\rho}_0^{\text{out}}. \quad (6.3)$$

And the solution of the Equation 5.6b (see Appendix B1) is

$$\bar{P}_0^{\text{out}}(x) = \bar{\rho}_0^{\text{out}} \left(\frac{2(P_c + \bar{\rho}_0^{\text{out}}) \sqrt{\frac{3}{3-8\pi\bar{\rho}_0^{\text{out}}x^2}}}{3(P_c + \bar{\rho}_0^{\text{out}}) \sqrt{\frac{3}{3-8\pi\bar{\rho}_0^{\text{out}}x^2}} - (3P_c + \bar{\rho}_0^{\text{out}})} - 1 \right). \quad (6.4)$$

According to Equation 6.1,

$$\bar{R}_0^{\text{out}}(x) = 16\pi\bar{\rho}_0^{\text{out}} \left(1 - \frac{3P_c + \bar{\rho}_0^{\text{out}}}{3(P_c + \bar{\rho}_0^{\text{out}}) \sqrt{\frac{3}{3-8\pi\bar{\rho}_0^{\text{out}}x^2}} - 3P_c - \bar{\rho}_0^{\text{out}}} \right). \quad (6.5)$$

As shown in Appendix B2 $O(\epsilon^{1/2})$ equations, i.e. Equations (5.7a), (5.7b) and (5.7c), give

$$\bar{m}_1^{\text{out}}(x) = 0 \quad (6.6a)$$

$$\bar{P}_1^{\text{out}}(x) = P_{c1} e^{F(x)} \quad (6.6b)$$

$$\bar{R}_1^{\text{out}}(x) = -24\pi P_{c1} e^{F(x)}. \quad (6.6c)$$

Here $P_{c1} = \bar{P}_1^{\text{out}}(0)$ is a constant and $F(x)$ is given in Equation B2.6. Note that both of \bar{R}_1^{out} and \bar{R}_0^{out} are already consistent with boundary condition given in Equation 5.8.

6.2 Inner Solutions for Uniform Density

The non-trivial solutions for \bar{m}_0^{in} and \bar{P}_0^{in} are given in Equation 5.18. Then for constant density and $\rho_1^{\text{in}} = 0$, \bar{R}_0^{in} can be found from Equation 5.17c as

$$\bar{R}_0^{\text{in}}(\xi) = C_0 \exp\left(\frac{\xi}{\sqrt{6(1-A_0)}}\right) + D_0 \exp\left(-\frac{\xi}{\sqrt{6(1-A_0)}}\right) + 8\pi\bar{\rho}_0^{\text{in}} \quad (6.7)$$

and the boundary conditions at Equation 5.13 requires $C_0 = D_0$.

With these results, solutions of Equation 5.19a and Equation 5.19b are

$$\bar{m}_1^{\text{in}}(\xi) = C_0 \frac{\sqrt{6(1-A_0)}}{3} \left[-\exp\left(\frac{\xi}{\sqrt{6(1-A_0)}}\right) + \exp\left(-\frac{\xi}{\sqrt{6(1-A_0)}}\right) \right] - 8\pi\bar{\rho}_0^{\text{in}}\xi + A_1 \quad (6.8a)$$

$$\bar{P}_1^{\text{in}}(\xi) = \frac{A_0\bar{\rho}_0^{\text{in}}}{2(1-A_0)}\xi \quad (6.8b)$$

for uniform density. The solution of Equation 5.19c has a singularity unless $C_0 = 0$ and it is found as

$$\begin{aligned} \bar{R}_1^{\text{in}}(\xi) = & (2C_0\xi + C_1) \exp\left(\frac{\xi}{\sqrt{6(1-A_0)}}\right) + (2C_0\xi + D_1) \exp\left(-\frac{\xi}{\sqrt{6(1-A_0)}}\right) \\ & - \frac{C_0^2}{\sqrt{6(1-A_0)}} \exp\left(\frac{2\xi}{\sqrt{6(1-A_0)}}\right) + \frac{D_0^2}{\sqrt{6(1-A_0)}} \exp\left(-\frac{2\xi}{\sqrt{6(1-A_0)}}\right) \\ & - \frac{12\pi\bar{\rho}_0^{\text{in}}A_0}{1-A_0}\xi. \end{aligned} \quad (6.9)$$

The boundary conditions at Equation 5.13 imply that

$$C_1 - D_2 = \frac{4C_0^2}{\sqrt{6(1-A_0)}} - 4C_0\sqrt{6(1-A_0)} + \frac{12\pi\bar{\rho}_0^{\text{in}}}{1-A_0}A_0\sqrt{6(1-A_0)}. \quad (6.10)$$

6.3 Matching Solutions of Uniform Density

For simplifying the equations we define some constants as

$$\frac{8}{3}\pi\bar{\rho}_0^{\text{out}} = \bar{M}, \quad K = P_c + \bar{\rho}_0^{\text{out}}, \quad L = 3P_c + \bar{\rho}_0^{\text{out}}, \quad (6.11a)$$

$$\beta = \sqrt{6(1-A_0)}, \quad Z = (1 - \bar{M}^{\text{out}})^{1/2}, \quad N = 3KZ - L. \quad (6.11b)$$

For matching the solutions we will employ Van Dyke's method [34] :

1. Two term outer solutions are written in terms of the inner variable

$$\bar{m}^{\text{out}}(\xi) = \bar{M} (1 - \epsilon^{1/2}\xi)^3 \quad (6.12a)$$

$$\bar{P}^{\text{out}}(\xi) = \bar{\rho}_0^{\text{out}} \left(\frac{2K [1 - \bar{M} (1 - \epsilon^{1/2}\xi)^2]^{-1/2}}{3K [1 - \bar{M} (1 - \epsilon^{1/2}\xi)^2]^{-1/2} - L} - 1 \right) + \epsilon^{1/2} P_{c1} e^{F(1 - \epsilon^{1/2}\xi)} \quad (6.12b)$$

$$\bar{R}^{\text{out}}(\xi) = 6\bar{M} \left(1 - \frac{L}{3K [1 - \bar{M} (1 - \epsilon^{1/2}\xi)^2]^{-1/2} - L} \right) - 24\epsilon^{1/2} \pi P_{c1} e^{F(1 - \epsilon^{1/2}\xi)}. \quad (6.12c)$$

2. They are expanded for small ξ up to $\epsilon^{1/2}$

$$\bar{m}^{\text{out}}(\xi) = \bar{M} (1 - 3\epsilon^{1/2}\xi) \quad (6.13a)$$

$$\begin{aligned} \bar{P}^{\text{out}}(\xi) = \bar{\rho}_0^{\text{out}} \left(\frac{2K}{NZ^{1/2}} - 1 \right) + \epsilon^{1/2} \frac{2K \bar{\rho}_0^{\text{out}} \bar{M}}{Z^2} \left[\frac{3KN^{-1} - Z^2}{N} \right] \xi \\ + \epsilon^{1/2} P_{c1} \left(e^{F(1)} - \epsilon^{1/2} \xi \frac{\partial F}{\partial \xi} e^{F(1)} \right) \end{aligned} \quad (6.13b)$$

$$\bar{R}^{\text{out}}(\xi) = 6\bar{M} \left(1 - \frac{L}{N} \right) - 18\bar{M}^2 \epsilon^{1/2} \frac{LK}{N^2 Z^3} \xi - 24\epsilon^{1/2} \pi P_{c1} \left(e^{F(1)} - \epsilon^{1/2} \xi e^{F(1)} \frac{\partial F}{\partial \xi} \Big|_1 \right). \quad (6.13c)$$

3. Two term inner solutions are written in terms of the outer variable

$$\begin{aligned} \bar{m}^{\text{in}}(x) = A_0 - 8\pi \bar{\rho}_0^{\text{in}} (1 - x) \\ + \epsilon^{1/2} \left[-C_0 \frac{\beta}{3} \exp\left(\frac{1-x}{\epsilon^{1/2}\beta}\right) + C_0 \frac{\beta}{3} \exp\left(-\frac{1-x}{\epsilon^{1/2}\beta}\right) + A_1 \right] \end{aligned} \quad (6.14a)$$

$$\bar{P}^{\text{in}}(x) = \bar{\rho}_0^{\text{in}} \frac{A_0}{2} \frac{1-x}{1-A_0} \quad (6.14b)$$

$$\begin{aligned} \bar{R}^{\text{in}}(x) = C_0 \exp\left(\frac{1-x}{\sqrt{\epsilon}\beta}\right) + C_0 \exp\left(-\frac{1-x}{\sqrt{\epsilon}\beta}\right) + 8\pi \bar{\rho}_0^{\text{in}} \\ + \epsilon^{1/2} \left[\left(2C_0 \frac{1-x}{\epsilon^{1/2}} + C_1 \right) \exp\left(\frac{1-x}{\sqrt{\epsilon}\beta}\right) + \left(2C_0 \frac{1-x}{\epsilon^{1/2}} + D_1 \right) \exp\left(-\frac{1-x}{\sqrt{\epsilon}\beta}\right) \right] \\ + \epsilon^{1/2} \left[-\frac{C_0^2}{\beta} \exp\left(2\frac{1-x}{\sqrt{\epsilon}\beta}\right) + \frac{C_0^2}{\beta} \exp\left(-2\frac{1-x}{\sqrt{\epsilon}\beta}\right) \right] \\ - \epsilon^{1/2} 12\pi \bar{\rho}_0^{\text{in}} \frac{A_0}{1-A_0} \frac{1-x}{\epsilon^{1/2}}. \end{aligned} \quad (6.14c)$$

4. They will be expanded for small ϵ up to $\epsilon^{1/2}$. Before that, to avoid infinities C_0 and C_1 should be zero which implies $D_1 = -12\pi \bar{\rho}_0^{\text{in}} A_0 \sqrt{6(1-A_0)}/(1-A_0)$ and note that, since C_0 is zero we can use solution of \bar{R}_1^{in} given in Equation 6.9.

Then the expansions of the inner solutions for small ϵ are

$$\bar{m}^{\text{in}}(x) = A_0 - 8\pi\bar{\rho}_0^{\text{in}}(1-x) + \epsilon^{1/2}A_1 \quad (6.15a)$$

$$\bar{P}^{\text{in}}(x) = \frac{A_0\bar{\rho}_0^{\text{in}}}{2(1-A_0)}(1-x) \quad (6.15b)$$

$$\bar{R}_0^{\text{in}}(x) = 8\pi\bar{\rho}_0^{\text{in}} - \frac{12\pi\bar{\rho}_0^{\text{in}}A_0}{1-A_0}(1-x). \quad (6.15c)$$

5. Matching the inner and outer expressions gives

$$\frac{8}{3}\pi(1-3\epsilon^{1/2}\xi)\bar{\rho}_0^{\text{out}} = A_0 - 8\pi\bar{\rho}_0^{\text{in}}(1-x) + \epsilon^{1/2}A_1 \quad (6.16a)$$

$$\bar{\rho}_0^{\text{out}}\left(\frac{2K}{NZ} - 1\right) + 2\epsilon^{1/2}\xi\bar{\rho}_0^{\text{out}}\bar{M}\frac{K}{Z^4}\left(\frac{3KN^{-1} - Z^2}{N}\right) + \epsilon^{1/2}P_{c1}e^{F(1)} = \frac{A_0\bar{\rho}_0^{\text{in}}}{2(1-A_0)}(1-x) \quad (6.16b)$$

$$6\bar{M}\left(1 - \frac{L}{N}\right) - 18\epsilon^{1/2}\xi\bar{M}^2\frac{KL}{N^2Z^{3/2}} - 24\epsilon^{1/2}\pi P_{c1}e^{F(1)} = 8\pi\bar{\rho}_0^{\text{in}} - \frac{12\pi\bar{\rho}_0^{\text{in}}A_0}{1-A_0}(1-x). \quad (6.16c)$$

The second and the third equations give $P_{c1} = 0$. Note that $\epsilon^{1/2}\xi = 1-x$. Then we should equate the coefficients of x on the right and left hand side of the equations with each other and the constant terms on both sides with each other. By combining the constant terms in the second and the third equations

$$\bar{\rho}_0^{\text{out}}\left(\frac{2K(1-\bar{M})^{-1/2}}{3K(1-\bar{M})^{-1/2}-L} - 1\right) = 0 \quad (6.17)$$

can be obtained. Then

$$P_c = \bar{\rho}_0^{\text{out}}\frac{\sqrt{\frac{1}{1-\frac{8}{3}\pi\bar{\rho}_0^{\text{out}}}-1}}{3-\sqrt{\frac{1}{1-\frac{8}{3}\pi\bar{\rho}_0^{\text{out}}}}}. \quad (6.18)$$

With that result the coefficients of x in the second and the third equations are

$$3\bar{M} - \frac{9\bar{M}^2}{4}(1-\bar{M})^{-1/2} = 8\pi\bar{\rho}_0^{\text{in}} - \frac{12\pi\bar{\rho}_0^{\text{in}}A_0}{1-A_0} \quad (6.19a)$$

$$\frac{9\bar{M}^2}{2}(1-\bar{M})^{-1/2} = \frac{12\pi\bar{\rho}_0^{\text{in}}A_0}{1-A_0}. \quad (6.19b)$$

By combining these equations we obtain

$$8\pi\bar{\rho}_0^{\text{out}} = 8\pi\bar{\rho}_0^{\text{in}} \quad (6.20)$$

which implies $\bar{\rho}_0^{\text{out}} = \bar{\rho}_0^{\text{in}}$. Also matching condition of \bar{m}^{in} and \bar{m}^{out} gives

$$\frac{8}{3}\pi(-2+3x)\bar{\rho}_0^{\text{out}} = A_0 - 8\pi\bar{\rho}_0^{\text{in}}(1-x) + \epsilon^{1/2}A_1. \quad (6.21)$$

The above equation can be satisfied if

$$\bar{\rho}_0^{\text{in}} = \bar{\rho}_0^{\text{out}}, \quad A_1 = 0, \quad \frac{8}{3}\pi\bar{\rho}_0^{\text{in}} = A_0. \quad (6.22)$$

In this way we have determined all constants in terms of $\bar{\rho}_0^{\text{out}}$. Since $\bar{\rho}_0^{\text{out}}$ equal to $\bar{\rho}_0^{\text{in}}$, we can denote both of them with $\bar{\rho}$.

6.4 Composite Solutions for Uniform Density

After matching the solutions we can construct the composite solution by subtracting from sum of the solutions the overlapping parts as described in the Appendix by Equation A2.3.28. Accordingly, the dimensionless mass is found as

$$\bar{m}(x) = \frac{8}{3}\pi x^3 \bar{\rho} + \frac{8}{3}\pi \bar{\rho} - 8\pi \bar{\rho}(1-x) - \left[\frac{8}{3}\pi \bar{\rho} - 8\pi \bar{\rho}(1-x) \right] = \frac{8}{3}\pi x^3 \bar{\rho}. \quad (6.23)$$

The dimensionless pressure is found as

$$\begin{aligned} \bar{P}(x) &= \bar{\rho}_0^{\text{out}} \left(\frac{2K(1-\bar{M}x^2)^{-1/2}}{3K(1-\bar{M}x^2)^{-1/2} - L} - 1 \right) + \frac{A_0 \bar{\rho}_0^{\text{in}}}{2(1-A_0)}(1-x) - \frac{A_0 \bar{\rho}_0^{\text{in}}}{2(1-A_0)}(1-x) \\ &= \bar{\rho} \frac{\sqrt{\frac{1}{1-\frac{8}{3}\pi\bar{\rho}} - \sqrt{\frac{1}{1-\frac{8}{3}\pi\bar{\rho}x^2}}} - \sqrt{\frac{1}{1-\frac{8}{3}\pi\bar{\rho}x^2}}}{3\sqrt{\frac{1}{1-\frac{8}{3}\pi\bar{\rho}x^2} - \sqrt{\frac{1}{1-\frac{8}{3}\pi\bar{\rho}}}}}. \end{aligned} \quad (6.24)$$

Finally, the dimensionless Ricci scalar is found as

$$\begin{aligned} \bar{R}(x) &= 16\pi\bar{\rho}_0^{\text{out}} \left(1 - \frac{L}{3K(1-\bar{M}x^2)^{-1/2} - L} \right) + 8\pi\bar{\rho}_0^{\text{in}} \\ &\quad - \epsilon^{1/2}\beta \frac{9\bar{M}A_0}{2-2A_0} \exp\left(-\frac{1-x}{\sqrt{\epsilon}\beta}\right) - \frac{9\bar{M}A_0}{2-2A_0}(1-x) - \left[3\bar{M} - \frac{9\bar{M}A_0}{2-2A_0}(1-x) \right] \\ &= 16\pi\bar{\rho} \left(1 - \frac{2\sqrt{\frac{1}{1-\frac{8}{3}\pi\bar{\rho}}}}{6\sqrt{\frac{1}{1-\frac{8}{3}\pi\bar{\rho}x^2} - 2\sqrt{\frac{1}{1-\frac{8}{3}\pi\bar{\rho}}}}} \right) \\ &\quad - \epsilon^{1/2} 2 \frac{(4\pi\bar{\rho})^2}{1-\frac{8}{3}\pi\bar{\rho}} \sqrt{6\left(1-\frac{8}{3}\pi\bar{\rho}\right)} \exp\left(-\frac{1}{\sqrt{6\left(1-\frac{8}{3}\pi\bar{\rho}\right)}} \frac{1-x}{\epsilon^{1/2}}\right). \end{aligned} \quad (6.25)$$

Note that, the negative pressure is unphysical and for avoiding this situation we impose

$$3 > \sqrt{\frac{1}{1-\frac{8}{3}\pi\bar{\rho}}} > 1 \quad (6.26)$$

which implies that $0 < \bar{\rho} < 0.1$.

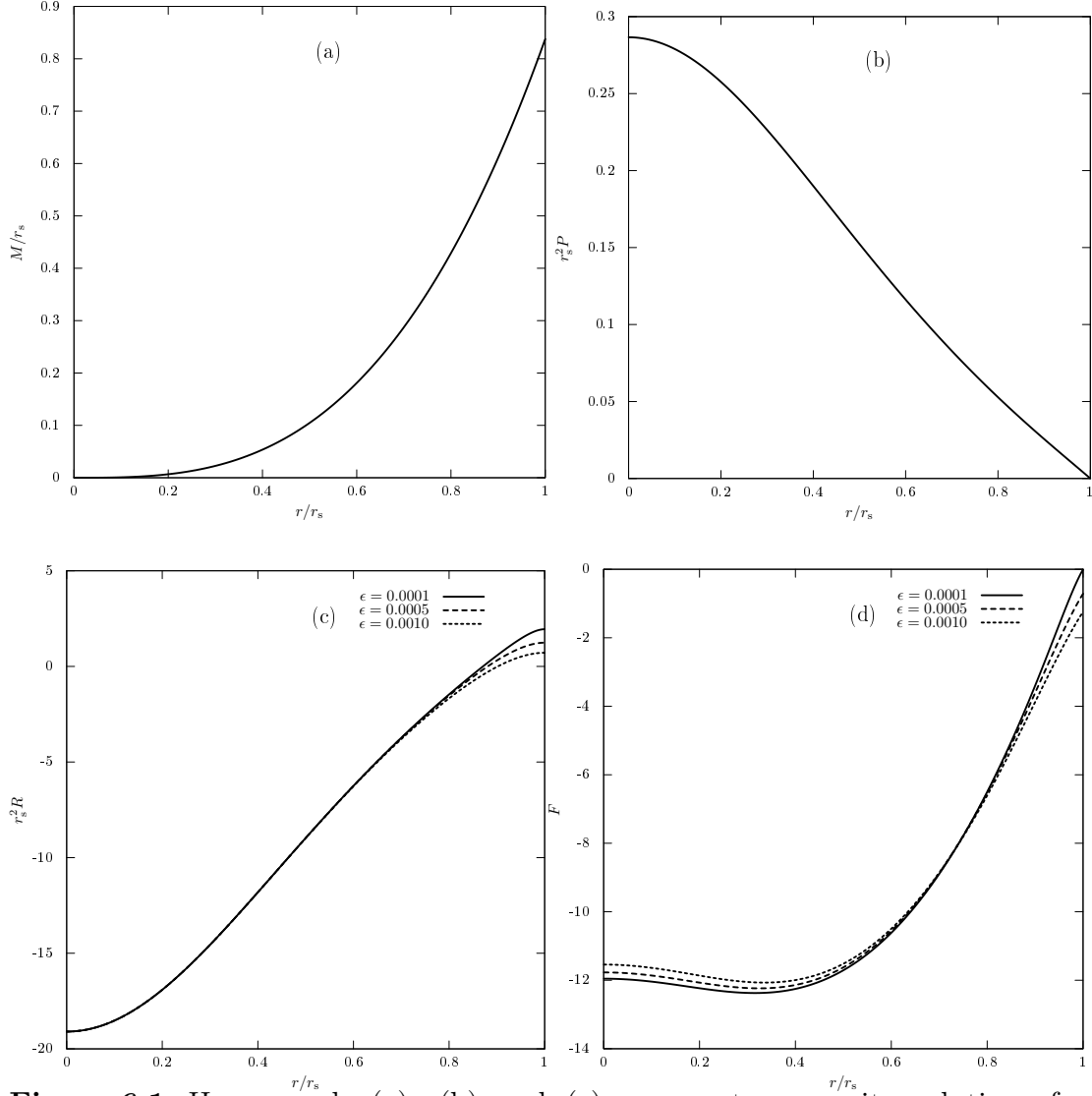


Figure 6.1: Here panels (a), (b) and (c) represent composite solutions for dimensionless mass \bar{m} , pressure \bar{P} and Ricci scalar \bar{R} . For all figures $\bar{\rho} = 0.1$. Also panel (d) represents the coefficient of R' at Equation 5.4

In Figure 6.1 we show the composite solutions. As shown in the panel (c) of the Figure 6.1 \bar{R} does not equal to zero at the surface. That prevents matching Schwarzschild solution at the exterior. It can be possible for $\bar{\rho} \cong 0.11$ which exceeds limit for positive pressure or $\bar{\rho} = 0$ which means there is no neutron star.

Also the panel (d) of Figure 6.1 shows the coefficient of R' at Equation 5.4 is always negative. According to MAE method, if the coefficient of the first order derivative term is negative, boundary layer occurs at $x = 1$. So our assumption, boundary layer occurs at $x = 1$, is valid.

7. CONCLUSION AND DISCUSSION

By starting from a modified action of gravity, we derived field equations and the hydrostatic equilibrium equations for Starobinsky model with static and spherically symmetric metric. We obtained exact differential equations for Ricci scalar, mass and pressure. Since the trace equation poses a singular perturbation problem, we solved these set of differential equations with matched asymptotic expansion. We obtained solution for uniform density stars.

The solutions found at Equation **6.23** and Equation **6.24** for \bar{m} and \bar{P} do not have any perturbation part and are similar to the solutions that would be obtained within general relativity. Yet, the solution found for the Ricci scalar in Equation **6.25** does not match, at the surface, to the Ricci scalar that would be found from the Schwarzschild solution.

Within the framework of general relativity Birkhoff-Jebsen theorem states that for a spherically symmetric body the exterior solution should be Schwarzschild solution [29, 35, 36, 37]. We did not search for a vacuum solution within Starobinsky model and we can not state confidently whether Schwarzschild solution would be the vacuum solution in this model of gravity.

If the Schwarzschild solution is unique to this model of gravity, as well as general relativity, we conclude that our solutions for uniform density object can not match to the vacuum and hence can not describe neutron stars. It does not mean that neutron stars could not exist in Starobinsky model since we found our solutions only for uniform density and uniform density case is not realistic assumption for neutron stars. With more realistic equations of state, solutions which match with Schwarzschild solution at the surface, can be found. Yet, in this thesis we show that the MAE is useful and applicable method for solving TOV equations.

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APPENDICES

APPENDIX A: Method of Matched Asymptotic Expansions

APPENDIX B: Solutions of Uniform Density

A. METHOD OF MATCHED ASYMPTOTIC EXPANSIONS

In this chapter a brief introduction to the method of matched asymptotic expansions is presented. For more detailed descriptions of the method the classical texts given in references [38] and [34] can be consulted.

A1. Asymptotic Expansions

For $\epsilon \rightarrow 0$ an asymptotic expansion of a function $y = y(x; \epsilon)$ where variable x is independent of ϵ in terms of an asymptotic sequence of $\delta_n(\epsilon)$ can be written as

$$y(x; \epsilon) = \sum_{n=0}^{\infty} \delta_n(\epsilon) y_n(x). \quad (\text{A1.1})$$

If

$$y(x; \epsilon) = \sum_{n=0}^{N-1} \delta_n(\epsilon) y_n(x) + R_N(x, \epsilon) \quad (\text{A1.2})$$

where $R_N(x, \epsilon) = O(\delta_N(\epsilon) y_N(x))$ is satisfied for all values of x , the expansion is called uniformly valid. Otherwise the expansion is called non-uniformly valid or singular perturbation expansion.

If $y(x; \epsilon)$ is the solution of a problem and the expansion in Equation A1.1 is not uniformly valid for all x values, the problem cannot be described by the expansion in Equation A1.1. Even in that case, the solution can be used within the interval of x where the expansion is uniformly valid. Also, other expansions can be uniformly valid in other regions and they can be used for describing the solution in those regions.

A2. Singular Perturbation Problem

A simple example for singular perturbation problem is the differential equation

$$\epsilon y'' + a(x)y'(x) + y(x) = 0 \quad (\text{A2.3})$$

where $\epsilon \rightarrow 0$. The boundary conditions are given as $y(0) = 0$ and $y(1) = 1$. and $a(x) = 1$ for simplicity. We are looking for an approximate asymptotic solution in the interval $0 < x < 1$.

Let us first try to solve Equation A2.3 as a regular perturbation problem. The solution can be expanded as a perturbative series

$$y(x) = \sum_{n=0}^{\infty} \epsilon^n y_n(x). \quad (\text{A2.4})$$

The boundary conditions become

$$y_n(0) = 0 \quad n = 0, 1, 2, \dots \quad (\text{A2.5a})$$

$$y_0(1) = 1 \quad (\text{A2.5b})$$

$$y_n(1) = 0 \quad n = 1, 2, \dots \quad (\text{A2.5c})$$

If we plug it into Equation A2.3 we obtain

$$\sum_{n=0} \epsilon^{n+1} y_n''(x) + \sum_{n=0} \epsilon^n y_n'(x) + \sum_{n=0} \epsilon^n y_n(x) = 0. \quad (\text{A2.6})$$

The order $O(\epsilon^0)$ equation

$$y_0'(x) + y_0(x) = 0 \quad (\text{A2.7})$$

is of first order and has the solution

$$y_0(x) = A \exp(-x). \quad (\text{A2.8})$$

This solution cannot satisfy both boundary conditions of $y_0(x)$ as it has only one arbitrary constant. The problem, thus, cannot be solved as a regular perturbation problem. One of the methods for solving these kinds of problems is the matched asymptotic expansion (MAE).

The solution found in Equation A2.8 cannot be valid all over the interval $0 < x < 1$; it can be valid near to only one of the boundaries. Near to the other boundary another solution can be valid and it is assumed that at this boundary a ‘‘boundary layer’’ occurs. Boundary layer is a region where the function rapidly converges to the boundary value. The solution which is valid in the boundary layer is called the inner solution and the solution which is valid outside of the boundary layer is called the outer solution. In an intermediate region two solutions should match with each other.

The region where the boundary layer occurs can be determined by examining the function $a(x)$. In the interval $0 < x < 1$ if $a(x) > 0$, the boundary layer occurs at $x = 0$. If $a(x) < 0$, the boundary layer occurs at $x = 1$. If $a(x)$ changes sign at some points, boundary layer occurs at these points. For the problem in Equation A2.3 boundary layer should occur at $x = 0$.

A2.1. Outer solution

The method for deriving the outer solution is to solve the problem as a regular perturbation problem. The outer solution can be expanded as an asymptotic series in powers of ϵ

$$y^{\text{out}}(x) = \sum_{n=0} \epsilon^n y_n^{\text{out}}(x). \quad (\text{A2.1.9})$$

By plugging this into Equation A2.3 we find the zeroth order and the first order terms as

$$y_0^{\text{out}}(x) = A_0 \exp(-x) \quad (\text{A2.1.10a})$$

$$y_1^{\text{out}}(x) = -A_0 x \exp(-x) + A_1 \exp(-x). \quad (\text{A2.1.10b})$$

The outer solution is valid in the interval $0 < x < 1$ except near $x = 0$ and it yields the boundary condition for $x = 1$. The terms should satisfy the boundary conditions

$$y_0^{\text{out}}(1) = 1 \quad y_1^{\text{out}}(1) = 0. \quad (\text{A2.1.11})$$

According to that, the two term outer solution is found as

$$y^{\text{out}}(x) = \exp(1-x) + \epsilon(1-x)\exp(1-x) + O(\epsilon^2). \quad (\text{A2.1.12})$$

A2.2. Inner solution

For deriving the inner solution we stretch the x coordinate and define a new boundary layer coordinate as

$$\xi = \frac{x}{\epsilon^\nu} \quad (\text{A2.2.13})$$

which implies

$$\frac{d}{dx} = \frac{d\xi}{dx} \frac{d}{d\xi} = \epsilon^{-\nu} \frac{d}{d\xi} \quad (\text{A2.2.14a})$$

$$\frac{d^2}{dx^2} = \epsilon^{-2\nu} \frac{d^2}{d\xi^2}. \quad (\text{A2.2.14b})$$

Accordingly, Equation A2.3 becomes

$$\epsilon^{-2\nu+1} \frac{d^2 y^{\text{in}}(\xi)}{d\xi^2} + \epsilon^{-\nu} \frac{d y^{\text{in}}(\xi)}{d\xi} + y^{\text{in}}(\xi) = 0. \quad (\text{A2.2.15})$$

The inner solution, expanded in asymptotic series of ϵ in terms of ξ , is

$$y^{\text{in}}(\xi) = \sum_{n=0} \epsilon^n y_n^{\text{in}}(\xi). \quad (\text{A2.2.16})$$

By plugging the above into Equation A2.2.15 we obtain

$$\epsilon^{-2\nu+1} \frac{d^2}{d\xi^2} (y_0^{\text{in}} + \dots) + \epsilon^{-\nu} \frac{d}{d\xi} (y_0^{\text{in}} + \dots) + (y_0^{\text{in}} + \dots) = 0. \quad (\text{A2.2.17})$$

The appropriate value of ν is determined by the condition that the term with the highest derivative can be balanced with some other term in the equation. The outer solution corresponds to $\nu = 0$. In that case, the second and the third terms balance each other and the first term is the highest order term, implying that we should consider other possibilities. There are two possibilities:

(i) The second term has the highest order and the first term balances the third term. This implies that $-2\nu + 1 = 0$ and so $\nu = 1/2$. Accordingly, the orders of the terms, respectively, become $O(1)$, $O(\epsilon^{-1/2})$ and $O(1)$. Thus the assumption that the second term has the highest order is violated.

(ii) The third term has the highest order and the first term balances with the second term implying that $-2\nu + 1 = -\nu$. This implies that $\nu = 1$ and the orders of the terms, respectively, become $O(1)$, $O(1)$ and $O(\epsilon)$. So the correct value of ν is 1 and Equation A2.2.15 becomes

$$\frac{d^2}{d\xi^2} (y_0^{\text{in}} + \dots) + \frac{d}{d\xi} (y_0^{\text{in}} + \dots) + \epsilon (y_0^{\text{in}} + \dots) = 0. \quad (\text{A2.2.18})$$

The zeroth order and the first order terms of the inner solution are found as

$$y_0^{\text{in}}(\xi) = B_0 \exp(-\xi) + C_0 \quad (\text{A2.2.19a})$$

$$y_1^{\text{in}}(\xi) = (B_1 + \xi B_0) \exp(-\xi) - C_0 \xi + C_1, \quad (\text{A2.2.19b})$$

and they should satisfy the boundary conditions

$$y_0^{\text{in}}(0) = 0 = y_1^{\text{in}}(0). \quad (\text{A2.2.20})$$

Thus the inner solution is

$$y^{\text{in}}(\xi) = B_0(\exp(-\xi) - 1) + \epsilon ((\xi B_0 + B_1) \exp(-\xi) + B_0 \xi - B_1) + O(\epsilon^2) \quad (\text{A2.2.21})$$

where B_0, B_1 are constants that will be determined by the matching process.

A2.3. Matching

For matching the two term inner and outer solutions we follow Van Dyke's matching process.

1. The two term outer solution is written in terms of the inner variable

$$y^{\text{out}}(\xi) = \exp(1 - \epsilon\xi) + \epsilon(1 - \epsilon\xi)\exp(1 - \epsilon\xi) + O(\epsilon^2) \quad (\text{A2.3.22})$$

2. It is expanded for small ξ

$$y^{\text{out}}(\xi) = e(1 + \epsilon(1 - \xi)) + \dots \quad (\text{A2.3.23})$$

3. The two term inner solution is written in terms of the outer variable

$$y^{\text{in}}(x) = B_0(\exp(-\frac{x}{\epsilon}) - 1) + \epsilon\left(\left(\frac{x}{\epsilon}B_0 + B_1\right)\exp(-\frac{x}{\epsilon}) + B_0\frac{x}{\epsilon} - B_1\right) + O(\epsilon^2) \quad (\text{A2.3.24})$$

4. It is expanded for small ϵ

$$y^{\text{in}}(x) = -B_0 + (xB_0 - \epsilon B_1) + \dots \quad (\text{A2.3.25})$$

5. The two expressions are matched

$$-B_0 + (xB_0 - \epsilon B_1) = e(1 + \epsilon(1 - \xi)) \quad (\text{A2.3.26a})$$

$$B_0(x - 1) - \epsilon B_1 = e(1 - x) + \epsilon e. \quad (\text{A2.3.26b})$$

According to these equations, the coefficients should be

$$B_0 = -e \quad B_1 = -e \quad (\text{A2.3.27})$$

The composite solution $y^c(x)$ can be constructed by adding the inner and outer solutions and subtracting the overlapping part

$$y^c(x) = y^{\text{out}}(x) + y^{\text{in}}(x) - y^{\text{overlap}}(x). \quad (\text{A2.3.28})$$

Here y^{overlap} is the matched parts of y^{out} and y^{in} . Thus

$$\begin{aligned} y^c(x) &= y^{\text{out}}(x) + y^{\text{in}}(x) - [e(1 - x) + \epsilon e] \\ &= \exp(1 - x) + \epsilon(1 - x)\exp(1 - x) - (1 + x)\exp\left(1 - \frac{x}{\epsilon}\right) - \epsilon\exp\left(1 - \frac{x}{\epsilon}\right). \end{aligned} \quad (\text{A2.3.29})$$

B: SOLUTIONS FOR UNIFORM DENSITY

B1. Solution of P_0^{out}

By using Equation 6.3, Equation 5.6b can be written as

$$\frac{d\bar{P}_0^{\text{out}}}{(\bar{\rho}_0^{\text{out}} + \bar{P}_0^{\text{out}})(\bar{P}_0^{\text{out}} + \bar{\rho}_0^{\text{out}}/3)} = -4\pi \frac{xdx}{\left(1 - \frac{8}{3}\pi x^2 \bar{\rho}_0^{\text{out}}\right)}. \quad (\text{B1.1})$$

By integrating both sides from P_c to \bar{P}_0^{out} and from 0 to x

$$\ln \left(\frac{(3\bar{P}_0^{\text{out}} + \bar{\rho}_0^{\text{out}})(P_c + \bar{\rho}_0^{\text{out}})}{(\bar{P}_0^{\text{out}} + \bar{\rho}_0^{\text{out}})(3P_c + \bar{\rho}_0^{\text{out}})} \right)^{3/2\bar{\rho}_0^{\text{out}}} = \ln \left(\frac{8\pi\bar{\rho}_0^{\text{out}}x^2 - 3}{-3} \right)^{3/4\bar{\rho}_0^{\text{out}}} \quad (\text{B1.2})$$

is obtained. This equation can be re-arranged as

$$\bar{P}_0^{\text{out}}(x) = \bar{\rho}_0^{\text{out}} \left(\frac{2(P_c + \bar{\rho}_0^{\text{out}}) \sqrt{\frac{1}{1 - \frac{8\pi}{3}\bar{\rho}_0^{\text{out}}x^2}}}{3(P_c + \bar{\rho}_0^{\text{out}}) \sqrt{\frac{1}{1 - \frac{8\pi}{3}\bar{\rho}_0^{\text{out}}x^2}} - (3P_c + \bar{\rho}_0^{\text{out}})} - 1 \right) \quad (\text{B1.3})$$

where P_c is some constant corresponding to $\bar{P}_0^{\text{out}}(0)$.

B2. Solutions of $O(\epsilon^{1/2})$ Equations

The $O(\epsilon^{1/2})$ equations of $\bar{R}_1^{\text{out}}(x)$ in the case of uniform density star Equation 5.7c implies that

$$\bar{R}_1^{\text{out}}(x) = -24\pi\bar{P}_1^{\text{out}}. \quad (\text{B2.4})$$

By plugging it into Equation 5.7a

$$\frac{d\bar{m}_1^{\text{out}}}{dx} = 0 \quad (\text{B2.5})$$

is obtained. The solution of this equation with boundary conditions given in Equation 5.8 is $\bar{m}_1^{\text{out}}(x) = 0$. By applying these results, the solution of Equation 5.7b is $\bar{P}_1^{\text{out}}(x) = P_{c1}e^{F(x)}$ where P_{c1} is some constant corresponding to $\bar{P}_1^{\text{out}}(0)$ and

$$F(x) = \int \frac{8\pi\bar{\rho}_0^{\text{out}}x}{8\pi\bar{\rho}_0^{\text{out}}x^2 - 3} \frac{3P_c \sqrt{\frac{3}{3-8\pi\bar{\rho}_0^{\text{out}}x^2}} + 3\bar{\rho}_0^{\text{out}} \sqrt{\frac{3}{3-8\pi\bar{\rho}_0^{\text{out}}x^2}} + 3P_c + \bar{\rho}_0^{\text{out}}}{3P_c \sqrt{\frac{3}{3-8\pi\bar{\rho}_0^{\text{out}}x^2}} + 3\bar{\rho}_0^{\text{out}} \sqrt{\frac{3}{3-8\pi\bar{\rho}_0^{\text{out}}x^2}} - 3P_c - \bar{\rho}_0^{\text{out}}} dx. \quad (\text{B2.6})$$

Accordingly we determine

$$\bar{R}_1^{\text{out}}(x) = -24\pi P_{c1}e^{F(x)}. \quad (\text{B2.7})$$

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