İSTANBUL TECHNICAL UNIVERSITY ★ INSTITUTE OF SCIENCE AND TECHNOLOGY

H_∞ CONTROL OF MECHANICAL SYSTEMS WITH INPUT DELAY

M.Sc.Thesis by
Ahmet Furkan EMREHAN

Department : Mathematical Engineering

Programme : Mathematical Engineering

JUNE 2011

İSTANBUL TECHNICAL UNIVERSITY ★ INSTITUTE OF SCIENCE AND TECHNOLOGY

H_∞ CONTROL OF MECHANICAL SYSTEMS WITH INPUT DELAY

M.Sc.Thesis by
Ahmet Furkan EMREHAN
(509081008)

Date of submission	:	**06 May 2011**
Date of the defence examination	:	**06 June 2011**

Thesis Supervisor (Chairman)	:	**Prof. Dr. Ulviye BAŞER (ITU)**
Members of Examining Committee	:	**Prof. Dr. Külmiz ÇEVİK (ITU)**
		Assoc. Prof. Dr. Ali ERCENGİZ (ITU)

JUNE 2011

İSTANBUL TEKNİK ÜNİVERSİTESİ ★ FEN BİLİMLERİ ENSTİTÜSÜ

GİRİŞ GECİKMELİ MEKANİK SİSTEMLERDE H_∞ KONTROLÜ

YÜKSEK LİSANS TEZİ

Ahmet Furkan EMREHAN
(509081008)

Tezin Enstitüye Verildiği Tarih : 06 Mayıs 2011
Tezin Savunulduğu Tarih : 06 Haziran 2011

Tez Danışmanı : Prof. Dr. Ulviye BAŞER (İTÜ)
Diğer Jüri Üyeleri : Prof. Dr. Külmiz ÇEVİK (İTÜ)
Doç. Dr. Ali ERCENGİZ (İTÜ)

HAZİRAN 2011

FOREWORD

I would like to express my deep appreciation and thanks for my advisor, Prof. Dr. Ulviye BAŞER, providing her continuous support and patience throught the studies.

I also would like to thank my family for their unconditional support in every stage of my life.

May 2011 Ahmet Furkan EMREHAN

(Mathematician)

TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

Page

H_∞ CONTROL OF MECHANICAL SYSTEMS WITH INPUT DELAY

SUMMARY

In this thesis, robust H_∞ control problem is examined with state feedback control law in an uncertain mechanical systems with input delay.

State feedback control problem is to investigate the stabilization of the closed system that is obtained by applying state feedback control law which is designed by a regulating matrix and the state of the system to stabilize it. The purpose of the state feedback H_∞ control problem is both to provide the stabilization of the system by using a state feedback and to make the performance index, that is given in terms of both disturbances, exogenous input... etc. and controlled output, less than zero.

Mechanical system considered in this thesis is suggested by Du and Zhang [3], and the problem is solved by Lyapunov-Krasovskii functional defined in Parlakçı [4]. On account of different structure of the uncertainty and time varying delay, sufficient conditions obtained for robust stability is different than the results given in [3] and [4].

It is well known that in the solutions of systems with time-delay obtained by using Linear Matrix Inequalities (LMI's), as number of parameters is increased, it yields better results and greater upper bound on the delay. Thus, the delay interval is divided into equal parts and by using suitable Lyapunov-Krasovskii functionals, which are defined by means of these intervals H_∞ control problem with state feedback is solved for both nominal and uncertain mechanical systems. All results in this study are given in terms of LMI's depending on delay and the derivative of the delay. Robust H_∞ control law is defined by using the solution of these LMI's. Efficiency of the results are shown by a numerical example.

In 1st section of this thesis, uncertain mechanical systems with time varying delay is introduced . H_∞ control problem is described and preliminary informations are given.

In 2nd section, at first H_∞ control problem for closed loop nominal mechanical systems is solved and sufficent conditions is presented in *Lemma* 6. Furthermore, by linearizing the nonlinear terms in *Lemma* 6 with compatible computations, sufficent conditions giving H_∞ control law for nominal mechanical systems with input delay are obtained in *Theorem* 1. In *Theorem* 2, sufficent conditions for problem of robust H_∞ control law are obtained.

In section three, by using Lyapunov function based on uniform partition, sufficent conditions for feedback control law to stabilize linear H_∞ systems with input delay are presented in *Theorem* 3. After that, robust stability of uncertain system is investigated. Results are given in *Theorem* 4.

In section four, two methods for numerical solution of the problem are shown, numerical application of the methods suggested in this study is given and the results are compared with the results in the literature. In final section, the general results are summarized.

GİRİŞ GECİKMELİ MEKANİK SİSTEMLERDE H_∞ KONTROLÜ

ÖZET

Bu tezde, doğrusal, giriş gecikmeli belirsiz katsayılı mekanik bir sistemde geri besleme kuralı ile dayanıklı H_∞ kontrol problemi incelenmiştir.

Durum geri beslemeli kontrol problemi, sistemi kararlı hale getirmek için, sistemin durumu ve düzenleyici bir matris ile oluşturulan kontrol kuralının sisteme uygulanması ile elde edilen kapalı çevrim sisteminin kararlılığının incelenmesi problemidir. H_∞ kontrol probleminde amaç, durum geri beslemesi ile elde edilen kapalı çevrim sisteminin hem karalılığının sağlanması, hem de gürültü, dış etki vb. ve sistemin kontrol edilen çıktısı cinsinden verilen performans indeksinin sıfırdan küçük bir gerçel sayıya eşit kılınmasıdır.

Bu tezde gözönüne alınan mekanik sistem Du and Zhang, [3] tarafından önerilmiş, problem Parlakcı, [4] da tanımlanan Lyapunov-Krazovski fonksiyoneli ile çözülmüştür. Ele alınan mekanik sistemdeki belirsizliklerin farklı yapısı ve gecikmenin zaman bağımlı olması nedenleri ile dayanıklı kararlılık için elde edilen yeter koşullar [3] ve [4] deki sonuçlardan farklıdırlar.

Gecikmeli sistemlerin Lineer Matris Eşitsizlikleri (LMI's) yardımı ile elde edilen çözümlerinde parametre sayısı arttıkça gecikme üzerindeki üst sınırın büyüdüğü ve daha iyi bir sonuç alındığı bilinmektedir. Bu nedenle gecikme aralığı eşit parçalara ayrılmış ve bu aralıklara uygun olarak tanımlanan Lyapunov-Krazovski fonksiyoneli yardımıyla nominal ve belirsiz katsayılı mekanik sistemlerde durum geri beslemeli H_∞ kontrol problemi çözülmüştür. Bu çalışmadaki tüm sonuçlar gecikmeye ve gecikmenin türevine bağımlı LMI's yardımı ile verilmiştir. Dayanıklı H_∞ kontrol kuralı bu LMI's nin çözümü ile tanımlanmaktadır. Sonuçların etkinliği sayısal bir örnek üzerinde gösterilmiştir.

Bu tezin 1. Bölümünde belirsiz katsayılı, zamanla değişen giriş gecikmeli mekanik sistemler tanıtılmış, H_∞ kontrol problemi tanımlanmış ve çalışmada yararlanılan ön bilgiler verilmiştir.

İkinci bölümde, önce giriş bölümünde tanımlanmış olan kapalı çevrim nominal mekanik sistemler için H_∞ kontrol problemi çözülmüş ve elde edilen yeter koşullar *Lemma* 6 te sunulmuştur. Daha sonra, Lemma 6 teki sonuçlarda yer alan lineer olmayan terimler uygun hesaplamalarla giderilerek nominal, giriş gecikmeli mekanik sistemler için H_∞ kontrol kuralının elde edilmesini sağlayan yeter koşullar LMI's formunda *Teorem* 1 de verilmiştir. *Teorem* 2 de ise dayanıklı H_∞ kontrol probleminin çözümü için yeterli koşullar elde edilmiştir.

Üçüncü Bölümde, gecikme aralığının eşit parçalanması göz önüne alınarak seçilmiş Lyapunov-Krasovski fonksiyoneli yardımıyla giriş gecikmeli lineer H_∞ sistemini kararlı hale getirecek geri besleme kuralı için yeter koşullar *Teorem* 3 te sunulmuştur. Ardından belirsiz katsayılı sistemin dayanıklı kararlılığı incelenmiş, sonuçlar *Teorem* 4 te ifade edilmiştir.

Dördüncü bölümde, problemin sayısal çözümü için iki yöntem önerilmiş, bu çalışmada önerilen yöntemlerin sayısal bir uygulmasına yer verilmiş ve sonuçlar literatürdeki sonuçlar ile karşılaştırılmıştır. Son bölümde sonuçlar özet halinde sunulmuştur.

1. INTRODUCTION

1.1 General Background

The solution of differential equations with delay and the stability conditions of them are first examined in [1]. In Mathematical descriptions of the dynamic systems, time delay is a inevitable case that occurs in many times. Especially, time delay is encountered in huge transmission lines, hydraulic systems and chemical processes. For instance, resistance and length of wire cause to a delay in signal transmission. The brake applied to a motor is one of the most concrete example of the system with delay. One can refer to [3] for investigation on time-delay system to develop vehicle suspension with actuator time delay. Networked systems are also example of the system with input delay.

Time delay may happen in state or input variables or both of them at the same time. It can be a constant or time varying. Time delay is one of the reason of instability and bad performance. For example, time delay can cause stable system to be instable. Thus, the investigations on stability and stabilization of the system with delay is important in both theory and practice and many works are related to that subject in the literature, [2], [14], [15], [10] and the references therein. In [4] and [5], which are motivated our work in this thesis stability and stabilization are also investigated for the systems with state and input delays, respectively. In reference [6], stabilization of the system with both state and input delay is also examined.

A control law for a system with time delay works in a certain interval of the delay. In other words, control law can stabilize the system in a certain interval of the delay. So upper the bound of the delay is very important. If the delay exceeds the upper bound in which control law works, the system becomes instable. Thus one of the goals of the studies in this area is to obtain a control law that works in the greatest upper bound as possible under given conditions. In view of the stability, time varying delay is worse than constant ones. Because variation in delay causes to shrink the upper bound of the delay. In reference [7], this condition is investigated.

Stabilization of the system with time delay, employing state feedback control law is a method that encountered in many research. The state feedback H_∞ control problem is, to investigate the stability of the closed closed loop system, obtained with the aim of stabilization of the system, applying a state feedback to the system, by means of a control input which is created with state variables of the system and a regulating matrix. Many articles cited in the thesis are examples of this method.

To minimize γ performance index corresponding upper bound delay is another purpose of the H_∞ control problem. In system with time delay, uncertainties often take place unavoidably. They are also causes of the instability and poor performance degradation. Robust stabilization of the system with delay by using norm bounded LMI's is investigated in [5], [8] and [9]. Hence the robust stabilization of the system is very significant. This is other goal of these research. In the literature, surveys to examine this are vast. References from [7] to [13] examined this branch and sub-branches.

1.2 Problem Statement

In this thesis we consider the following uncertain mechanical systems with an input delay, which is defined in [5].

$$(M+\Delta_M)\ddot{p}(t)+(N+\Delta_N)\dot{p}(t)+(K+\Delta_K)p(t)=B_u u(t-\tau(t)) \quad \textbf{(1.2.1)}$$

where $p(t) \in \Re^n$ is the displacement, M, N and $K \in \Re^{n\times n}$ are the mass, damping and stiffness matrices; Δ_M, Δ_N and Δ_K are the corresponding perturbations. As usual, M is assumed to be non-singular. $B \in \Re^{n\times m}$ is gain matrix for the state feedback control input $u(t)$, which only consists of the displacement and the velocity feedback signals given by

$$u(t)=F_d p(t)+F_v\dot{p}(t) \quad \textbf{(1.2.2)}$$

where $F_d \in \Re^{m\times n}$, $F_v \in \Re^{m\times n}$ are the feedback gain matrices for the displacement and the velocity, respectively. $\tau(t)$ is an unknown time-varying delay in input with

$$0 \le \tau(t) \le \bar{\tau},\ \ 0 \le |\dot{\tau}(t)| \le \mu \quad \textbf{(1.2.3)}$$

where $\bar{\tau}$ represents the magnitude of the delay. Models in the form of the equation (**1.2.1**) arise frequently in a wide variety of applications in vibration and structural analysis. By defining $x(t) := \begin{bmatrix} p^T(t) & \dot{p}^T(t) \end{bmatrix}^T$, the equation (**1.2.1**) can be written as

$$\begin{bmatrix} I & 0 \\ 0 & M+\Delta_M \end{bmatrix}\dot{x}(t)=\begin{bmatrix} 0 & I \\ -K-\Delta_K & -N-\Delta_N \end{bmatrix}x(t)+\begin{bmatrix} 0 \\ B_u \end{bmatrix}u(t-\tau(t)) \quad \textbf{(1.2.4)}$$

It can be further written as

$$(\Sigma+\Delta_\Sigma)\dot{x}(t)=(\hat{A}+\Delta_A)x(t)+\hat{B}u(t-\tau(t)) \quad \textbf{(1.2.5)}$$

where $x(t) \in \Re^n$ is the state, $u(t) \in \Re^m$ is the control input and

$$\begin{aligned} \Sigma &= \begin{bmatrix} I & 0 \\ 0 & M \end{bmatrix}, \Delta_\Sigma = \begin{bmatrix} 0 & 0 \\ 0 & \Delta_M \end{bmatrix}, \hat{A} = \begin{bmatrix} 0 & I \\ -K & -N \end{bmatrix}, \\ \Delta_A &= \begin{bmatrix} 0 & 0 \\ -\Delta_K & -\Delta_N \end{bmatrix}, \hat{B} = \begin{bmatrix} 0 \\ B_u \end{bmatrix} \end{aligned} \quad \textbf{(1.2.6)}$$

The uncertainties Δ_M, Δ_N and Δ_K in (**1.2.6**) are assumed to satisfy the following bounds : $\left\|\Delta_M M^{-1}\right\| \le \delta \le 1$ and $\left\|\begin{bmatrix} \Delta_K & \Delta_N \end{bmatrix}\right\| \le \alpha$. Since Σ is a non-singular matrix these uncertainties can be written as follows;

$$\left\|\Delta_\Sigma \Sigma^{-1}\right\| \le \delta \le 1, \quad \|\Delta_A\| \le \alpha \quad \textbf{(1.2.7)}$$

Now, consider a disturbance signal $w(t) \in L_2[0,\infty)$ (that is, with bounded energy) and controlled output signal $z(t) \in \Re^q$. Then, in general, instead of the equation (**1.2.5**) the following equations describe the uncertain mechanical systems in (**1.2.1**);

$$\begin{aligned} (\Sigma+\Delta_\Sigma)\dot{x}(t) &= (\hat{A}+\Delta_A)x(t)+\hat{B}u(t-\tau(t))+\hat{B}_w w(t) && \textbf{(1.2.8)} \\ z(t) &= Cx(t) && \textbf{(1.2.9)} \\ x(t) &= \Phi(t), \quad \forall t \in [-\bar{\tau},0],\ \ \bar{\tau}>0, && \textbf{(1.2.10)} \end{aligned}$$

where $\hat{B}_w = \begin{bmatrix} 0 & B_{dw}^T \end{bmatrix}^T \in \Re^{2n\times p}$ is the disturbance gain matrix, $C \in \Re^{2n\times q}$ the output matrix and $\Phi(.)$ is a smooth vector-valued initial function. Notice that the first condition in (**1.2.7**) ensures $\Sigma + \Delta_\Sigma$ is non-singular. Uncertainties in mechanical systems (**1.2.1**) satisfying (**1.2.7**) are said to be admissible.The nominal system is given as follows

$$\Sigma\dot{x}(t) = \hat{A}x(t) + \hat{B}u(t-\tau(t)) + \hat{B}_w w(t) \quad (\mathbf{1.2.11})$$

$$z(t) = Cx(t) \quad (\mathbf{1.2.12})$$

$$x(t) = \Phi(t), \quad \forall t \in [-\bar{\tau}, 0], \quad \bar{\tau} > 0, \quad (\mathbf{1.2.13})$$

Let F be the memoryless state-feedback gain matrix given by $F = \begin{bmatrix} F_d & F_v \end{bmatrix}$, and so the state feedback control rule

$$u(t) = Fx(t) \quad (\mathbf{1.2.14})$$

Then, the closed-loop system of (**1.2.8**)- (**1.2.10**) is given by

$$\dot{x}(t) = \tilde{A}x(t) + \tilde{A}_2 x(t-\tau(t)) + \tilde{B}_w w(t) \quad (\mathbf{1.2.15})$$

$$z(t) = Cx(t), \quad (\mathbf{1.2.16})$$

where $\tilde{A} = (\Sigma + \Delta_\Sigma)^{-1}(\hat{A} + \Delta_A)$, $\tilde{A}_2 = (\Sigma + \Delta_\Sigma)^{-1}\hat{B}F$, $\tilde{B}_w = (\Sigma + \Delta_\Sigma)^{-1}\hat{B}_w$, with initial condition in (**1.2.10**). Moreover, Nominal form of the system (**1.2.15**) - (**1.2.16**) is described as,

$$\dot{x}(t) = Ax(t) + A_2 x(t-\tau(t)) + B_w w(t) \quad (\mathbf{1.2.17})$$

$$z(t) = Cx(t) \quad (\mathbf{1.2.18})$$

where $A = \Sigma^{-1}\hat{A}$, $A_2 = \Sigma^{-1}\hat{B}F$, $B_w = \Sigma^{-1}\hat{B}_w$ with initial condition in (**1.2.10**).

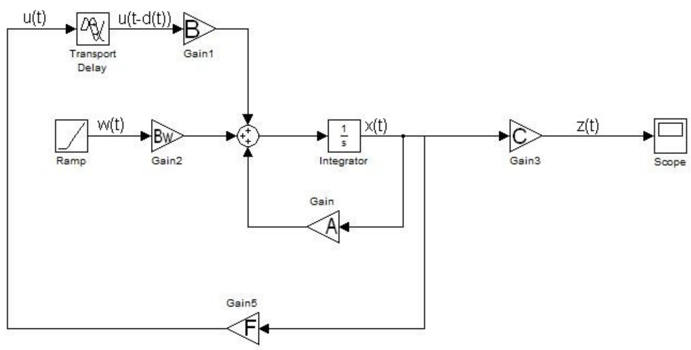

Figure 1.1: Block Diagram of The System in (**1.2.8-1.2.10**).

In **Figure 1.1**, where $A = (\Sigma + \Delta_\Sigma)^{-1}(\hat{A} + \Delta_A)$, $B = (\Sigma + \Delta_\Sigma)^{-1}\hat{B}$, $B_w = (\Sigma + \Delta_\Sigma)^{-1}\hat{B}_w$ and $d(t) = \tau(t)$.

Definition : For a prescribed scalar $\gamma > 0$, we define the performance index

$$J(w(t)) = \int_0^\infty z^T(t)z(t) - \gamma^2 w^T(t)w(t)dt \quad (\mathbf{1.2.19})$$

F is to be designed, such that
(1) The closed-loop system given in (**1.2.15**) is asymptotically stable;
(2) Under zero initial condition, the closed-loop system guarantees $J(w(t)) < 0$, for all non-zero $w(t) \in L_2[0,\infty)$ and some prescribed $\gamma > 0$.

1.3 Outline of The Thesis

In this thesis, H_∞ control problem for uncertain mechanical systems with input delay given in [5] is inevestigated. This problem is solved by the method given in [4]. On account of special structure of the uncertain part of the system, different methods are used for robust stabilization of the uncertain system. Furthermore, to find control law to work in smaller performans index and larger upper bound of the delay, a new Lyapunov functionals are described by uniformly partitioning delay interval. It is observed that the better results are obtained by the application of these functionals.

In the first chapter, a concise introduction to the context of the work is given. Problem statement and the lemmas used in this thesis are given.

In chapter two, stability condition for closed-loop system is given in *Lemma* 6 by means of Lyapunov-Krosovski functionals which are intoduced in [4]. Stabilization conditions for nominal and uncertain mechanical system with input delay are presented in $Theorem$ 1 and $Theorem$ 2, respectively.

In chapter three, in order to improve the results of previous chapter the new stabilization conditions for nominal and uncertain mechanical systems with input delay are presented in $Theorem$ 3 and $Theorem$ 4, respectively, by constructing new Lyapunov-Krosovski functionals based on the uniform partition of the delay interval.

In chapter four, an algorithm and a remark to solve the problem numerically are presented. A numerical example showing the effectness of the proposed theorems of the theorems is given. It is compared the results obtained here with the similar study in the literature. Finally, the works achieved in this thesis are summarized in the conclusion.

Notations

Throughout this paper, the superscripts $' - 1'$ and T represent the inverse and transpose of a matrix, respectively;$\Re$,$\Re^n$ and $\Re^{n\times m}$ denote the set of real numbers, an n-dimensional Euclidean space and the set of all $n \times m$ real matrices, respectively; $A > 0$ and $A < 0$ denote positive and negative definite matrice respectively. I is identity matrix with an appropriate dimension. 0 is a zero matrix with an appropriate dimension. $diag\{....\}$ stands for a block-diagonal matrix. The symmetric terms in a symmetric matrix are denoted by $*$, to illustrate

$$\begin{bmatrix} A & B \\ B^T & C \end{bmatrix} = \begin{bmatrix} A & B \\ * & C \end{bmatrix}$$

1.4 Preliminaries

In this section, Lemmas and informations used in the thesis are defined.

Lemma 1 : [16] A and $B \in \Re^{n\times n}$, There is a positive and symmetric $T \in \Re^{n\times n}$ such that

$$AB^T + BA^T \leq ATA^T + BT^{-1}B^T$$

Lemma 2 : [4] A and $B \in \Re^{n\times n}$, There is a $\varepsilon > 0$ such that

$$AB^T + BA^T \leq \varepsilon AA^T + \varepsilon^{-1}BB^T$$

Lemma 3 :[5] Let A,L,E and F be real matrices of appropriate dimensions with $\|F\| < 1$. For any real matrix $P > 0$ and scalar $\mu > 0$ such that $\mu I - EPE > 0$,

$$(A+LFE)^T P(A+LFE) \leq APA^T + APE^T(\mu I - EPE)^{-1}EPA^T + \mu LL^T$$

Lemma 4 :[17] For any constant-real matrix $P > 0$,scalar $\tau > 0$ and vector valued function $w : [0,\mu] \longrightarrow \Re^n$. Then this inequality holds,

$$\tau \int_0^\tau w^T(s)Pw(s)ds \geq \left(\int_0^\tau w(s)ds\right)^T P \left(\int_0^\tau w(s)ds\right)$$

Lemma 5 : [18] (Schur Complement) Let A,B and C be real matrices of appropriate dimension with $C^T = C > 0$ and $A = A^T$ then $A + BC^{-1}B^T$ is equivalent to

$$\begin{bmatrix} A & B \\ * & -C \end{bmatrix} < 0 \;\; or \begin{bmatrix} -C & B \\ * & A \end{bmatrix} < 0$$

2. H_∞ CONTROL OF MECHANICAL SYSTEM WITH INPUT DELAY

In this section, the solution of the H_∞-control problem of the nominal system given in (**1.2.11**)-(**1.2.13**) is taken into consideration.

For this purpose, we first examine the problem for the closed-loop system in (**1.2.17**) and (**1.2.18**) by applying the procedure given in [4] and then we stated the sufficient conditions for the solution of the problem for system (**1.2.11**)-(**1.2.13**). For simplicity, throughout this section we let A, B and B_w instead of $\hat{A}$, $\hat{B}$ and $\hat{B}_w$, respectively.

2.1 Lemma 6

Given scalars γ, $\bar{\tau} > 0$ and $\mu > 0$, the closed-loop system given by the equations (**1.2.17**) and (**1.2.18**) is asymptotically stable with H_∞ performance index γ for any time delay $\tau(t)$ satisfying (**1.2.3**) if there exist symmetric and positive definite matrices, P, Q, R, T and the matrices S_i, for $(i = 1,..,4)$ and F with appropriate dimensions, satisfying the following LMI's

$$\Theta = \begin{bmatrix} \theta_{11} & \theta_{12} & \theta_{13} & \theta_{14} & P_{11}B_w & \theta_{16} & \theta_{17} & C^T & \mu P_{12} \\ * & \theta_{22} & \theta_{23} & \theta_{24} & 0 & \theta_{26} & \theta_{27} & 0 & 0 \\ * & * & \theta_{33} & \theta_{34} & P_{12}^T B_w & 0 & 0 & 0 & \mu P_{22} \\ * & * & * & \theta_{44} & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & -\gamma^2 I & \theta_{56} & \theta_{57} & 0 & 0 \\ * & * & * & * & * & -Q_{11} & -Q_{12} & 0 & 0 \\ * & * & * & * & * & * & -Q_{22} & 0 & 0 \\ * & * & * & * & * & * & * & -I & 0 \\ * & * & * & * & * & * & * & * & -\mu T \end{bmatrix} < 0 \qquad \textbf{(2.1.1)}$$

$$P = \begin{bmatrix} P_{11} & P_{12} \\ * & P_{22} \end{bmatrix} > 0 \text{ with } P_{11} > 0 \text{ and } Q = \begin{bmatrix} Q_{11} & Q_{12} \\ * & Q_{22} \end{bmatrix} > 0 \qquad \textbf{(2.1.2)}$$

where

$$\begin{aligned} \theta_{11} &= P_{11}A + A^T P_{11} + R + S_1 + S_1^T \\ \theta_{12} &= P_{11}A_2 - S_1^T + S_2 \\ \theta_{13} &= A^T P_{12} + S_3 \\ \theta_{14} &= P_{12} - S_1^T + S_4 \\ \theta_{16} &= \bar{\tau}Q_{11} + \bar{\tau}A^T Q_{12}^T \\ \theta_{17} &= \bar{\tau}Q_{12} + \bar{\tau}A^T Q_{22} \\ \theta_{22} &= \mu T - (1-\mu)R - S_2 - S_2^T \\ \theta_{23} &= A_2^T P_{12} - S_3 \\ \theta_{24} &= -S_4 - S_2^T \\ \theta_{26} &= \bar{\tau}A_2^T Q_{12}^T \end{aligned}$$

$$
\begin{aligned}
\theta_{27} &= \bar{\tau}A_2^T Q_{22} \\
\theta_{33} &= -Q_{11} \\
\theta_{34} &= P_{22} - Q_{12} - S_3^T \\
\theta_{44} &= -Q_{22} - S_4 - S_4^T \\
\theta_{56} &= \bar{\tau}B_w^T Q_{12}^T \\
\theta_{57} &= \bar{\tau}B_w^T Q_{22}
\end{aligned}
$$

and the other terms θ_{ij}, for $i > j$ of Θ are all zero. Moreover, a desired H_∞ state feedback control law is given by $u(t) = Fx(t)$.

Proof:

Choose a Lyapunov functional $V(x(t),t)$ as

$$V(x(t),t) = V_1(x(t),t) + V_2(x(t),t) + V_3(x(t),t),$$

where

$$
\begin{aligned}
&V_1(x(t),t) = \eta^T(t)P\eta(t), \\
&V_2(x(t),t) = \bar{\tau}\int_{-\bar{\tau}}^{0}\int_{t}^{t+\theta} \xi^T(s)Q\xi(s)dsd\theta, \\
&V_3(x(t),t) = \int_{t-\tau(t)}^{t} x^T(s)Rx(s)ds,
\end{aligned}
$$

and

$$
\begin{aligned}
\eta(t) &= \begin{bmatrix} x^T(t) & (\int_{t-\tau(t)}^{t} x(s)ds)^T \end{bmatrix}^T, \xi(t) = \begin{bmatrix} x^T(t) & \dot{x}^T(t) \end{bmatrix}^T, \\
P &= \begin{bmatrix} P_{11} & P_{12} \\ * & P_{22} \end{bmatrix} > 0,\ Q = \begin{bmatrix} Q_{11} & Q_{12} \\ * & Q_{22} \end{bmatrix} > 0 \text{ and } P > 0
\end{aligned}
$$

We take the derivative of $V_1(x(t),t)$ along the state trajectory system of (**1.2.17**) as $\dot{V}_1(x(t),t) = \dot{\eta}^T(t)P\eta(t) + \eta^T(t)P\dot{\eta}(t)$. One can write $\dot{\eta}(t)$ as follows,

$$\dot{\eta}(t) = \eta_1(t) + \dot{\tau}(t)\eta_2(t)$$

where

$$\eta_1(t) = \begin{bmatrix} Ax(t) + A_2x(t-\tau(t)) + B_w w(t) \\ \int_{t-\tau(t)}^{t} \dot{x}(s)ds \end{bmatrix},\ \eta_2(t) = \begin{bmatrix} 0 \\ I \end{bmatrix} x(t-\tau(t))$$

Define

$$\chi(t) = \begin{bmatrix} x^T(t) & x^T(t-\tau(t)) & \left(\int_{t-\tau(t)}^{t} x(s)ds\right)^T & \left(\int_{t-\tau(t)}^{t} \dot{x}(s)ds\right)^T & w^T(t) \end{bmatrix}^T$$

we obtain

$$\eta(t) = \Gamma_1\chi(t),\ \eta_1(t) = \Gamma_2\chi(t),\ \begin{bmatrix} 0 & I \end{bmatrix} P\eta(t) = \Gamma_3\chi(t),\ x(t-\tau(t)) = \Gamma_4\chi(t)$$

where

$$
\begin{aligned}
\Gamma_1 &= \begin{bmatrix} I & 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 \end{bmatrix}, & \Gamma_2 &= \begin{bmatrix} A & A_2 & 0 & 0 & B_w \\ 0 & 0 & 0 & I & 0 \end{bmatrix}, \\
\Gamma_3 &= \begin{bmatrix} P_{12}^T & 0 & P_{22} & 0 & 0 \end{bmatrix}, & \Gamma_4 &= \begin{bmatrix} 0 & I & 0 & 0 & 0 \end{bmatrix}
\end{aligned}
$$

Thus

$$\eta^T(t)P\eta_1(t)+\eta_1^T(t)P\eta(t)=\Gamma_1^TP\Gamma_2+\Gamma_2^TP\Gamma_1 \qquad \textbf{(2.1.3)}$$

By *Lemma* 1, we obtain

$$\dot{\tau}(t)\eta^T(t)P\eta_2(t)+\dot{\tau}(t)\eta_2^T(t)P\eta(t)\le\chi^T(t)\left(\mu\Gamma_4^TT\Gamma_4+\mu\Gamma_3^TT^{-1}\Gamma_3\right)\chi(t) \qquad \textbf{(2.1.4)}$$

where T is symmetric positive definite matrix. Therefore, we have

$$\dot{V}_1(x(t),t)\le\chi^T(t)(\Gamma_1^TP\Gamma_2+\Gamma_2^TP\Gamma_1+\mu\Gamma_4^TT\Gamma_4+\mu\Gamma_3^TT^{-1}\Gamma_3)\chi(t) \qquad \textbf{(2.1.5)}$$

The derivative of $\dot{V}_2(x(t),t)$ can be computed as

$$\dot{V}_2(x(t),t)=\bar{\tau}^2\xi^T(t)Q\xi(t)-\bar{\tau}\int_{-\bar{\tau}}^{0}\xi^T(t+\theta)Q\xi(t+\theta)d\theta$$

We make the substitution $s=t+\theta$, we obtain $d\theta=ds$ and

$$-\bar{\tau}\int_{\theta=-\bar{\tau}}^{\theta=0}\xi^T(t+\theta)Q\xi(t+\theta)d\theta=-\bar{\tau}\int_{s=t-\bar{\tau}}^{s=t}\xi^T(s)Q\xi(s)ds \qquad \textbf{(2.1.6)}$$

By *Lemma* 3 , we know that

$$-\bar{\tau}\int_{t-\bar{\tau}}^{t}\xi^T(s)Q\xi(s)ds\le-\left(\int_{t-\bar{\tau}(t)}^{t}\xi(s)ds\right)^TQ\left(\int_{t-\bar{\tau}(t)}^{t}\xi(s)ds\right) \qquad \textbf{(2.1.7)}$$

By means of $\chi(t)$, we obtain

$$\xi(t)=\Gamma_5\chi(t),\ \int_{t-\bar{\tau}(t)}^{t}\xi(s)ds=\Gamma_6\chi(t)$$

where

$$\Gamma_5=\begin{bmatrix} I & 0 & 0 & 0 & 0\\ A & A_2 & 0 & 0 & B_w\end{bmatrix},\ \Gamma_6=\begin{bmatrix} 0 & 0 & I & 0 & 0\\ 0 & 0 & 0 & I & 0\end{bmatrix}$$

Thus, we have that

$$\dot{V}_2(x(t),t)\le\chi^T(t)\left(\bar{\tau}^2\Gamma_5^TQ\Gamma_5-\Gamma_6^TQ\Gamma_6\right)\chi(t) \qquad \textbf{(2.1.8)}$$

Finally, the derivative of $V_3(x(t),t)$ is

$$\begin{aligned}\dot{V}_3(x(t),t) &= x^T(t)Rx(t)-[1-\dot{\tau}(t)]x^T(t-\tau(t))Rx(t-\tau(t))\\ &\le x^T(t)Rx(t)-[1-\mu]x^T(t-\tau(t))Rx(t-\tau(t))\\ &= \chi^T(t)diag\{R,[1-\mu]R,0,0,0\}\chi(t)\end{aligned} \qquad \textbf{(2.1.9)}$$

By the Leibnitz-Newton formula we know that

$$\int_{t-\tau(t)}^{t}\dot{x}(s)ds=x(t)-x(t-\tau(t)) \qquad \textbf{(2.1.10)}$$

By means of (**2.1.10**), we obtain a relaxation as

$$\chi^T(t)(S^T\Gamma_7+\Gamma_7^TS)\chi(t)=0 \qquad \textbf{(2.1.11)}$$

where

$$\Gamma_7=\begin{bmatrix} I & -I & 0 & -I & 0\end{bmatrix},\ S=\begin{bmatrix} S_1 & S_2 & S_3 & S_4 & 0\end{bmatrix}$$

with S_1,S_2,S_3 and S_4 matrices of appropriate dimensions. By the inequalities (**2.1.5**),(**2.1.8**),(**2.1.9**) and adding (**2.1.11**), it yields

$$\dot{V}(x(t),t)=\dot{V}_1(x(t),t)+\dot{V}_2(x(t),t)+\dot{V}_3(x(t),t) \qquad \textbf{(2.1.12)}$$

$$\begin{aligned}\dot{V}(x(t),t) &\leq \chi^T(t)(\Gamma_1^T P\Gamma_2 + \Gamma_2^T P\Gamma_1 + \mu\Gamma_4^T T\Gamma_4 \\ &+\mu\Gamma_3^T T^{-1}\Gamma_3 + \bar{\tau}^2\Gamma_5^T Q\Gamma_5 - \Gamma_6^T Q\Gamma_6 \\ &+diag\{R,[1-\mu]R,0,0,0\} + S^T\Gamma_7 + \Gamma_7^T S)\chi(t) \qquad \textbf{(2.1.13)}\end{aligned}$$

The controlled output $z(t)$ can be written as

$$z(t) = \Gamma_8\chi(t), \text{ where } \Gamma_8 = \begin{bmatrix} C & 0 & 0 & 0 & 0 \end{bmatrix}$$

Thus

$$z^T(t)z(t) - \gamma^2 w^T(t)w(t) = \chi^T(t)(\Gamma_8^T\Gamma_8 - diag\{0,0,0,0,-\gamma^2 I\})\chi(t) \qquad \textbf{(2.1.14)}$$

In order to prove that the system with input delay (**1.2.15**)-(**1.2.16**) is asymptotically stable with a H_∞ performance index $\gamma > 0$,it is required that the Hamiltonian

$$H(x(t),w(t),t) = \dot{V}(x(t),t) + z^T(t)z(t) - \gamma^2 w^T(t)w(t) < 0 \qquad \textbf{(2.1.15)}$$

In this context,

$$\begin{aligned}H(x(t),w(t),t) &= \dot{V}(x(t),t) + z^T(t)z(t) - \gamma^2 w^T(t)w(t) \\ &\leq \chi^T(t)(\Gamma_1^T P\Gamma_2 + \Gamma_2^T P\Gamma_1 + \mu\Gamma_4^T T\Gamma_4 \\ &+\mu\Gamma_3^T T^{-1}\Gamma_3 + \bar{\tau}^2\Gamma_5^T Q\Gamma_5 - \Gamma_6^T Q\Gamma_6 \\ &+diag\{R,[1-\mu]R,0,0,0-\gamma^2 I\} \\ &+S^T\Gamma_7 + \Gamma_7^T S + \Gamma_8^T\Gamma_8)\chi(t) \\ &= \chi^T(t)(\theta + \bar{\tau}^2\Gamma_5^T Q\Gamma_5 \\ &+\Gamma_8^T\Gamma_8 + \mu\Gamma_3^T T^{-1}\Gamma_3)\chi(t) < 0, \qquad \textbf{(2.1.16)}\end{aligned}$$

where

$$\begin{aligned}\theta &= \Gamma_1^T P\Gamma_2 + \Gamma_2^T P\Gamma_1 + \mu\Gamma_4^T T\Gamma_4 - \Gamma_6^T Q\Gamma_6 + diag\{R,-[1-\mu]R,0,0,0-\gamma^2 I\} \\ &+S^T\Gamma_7 + \Gamma_7^T S. \qquad \textbf{(2.1.17)}\end{aligned}$$

Applying Schur's complement to (**2.1.16**) the following inequality is obtained

$$\begin{bmatrix} \theta & \bar{\tau}\Gamma_5^T Q & \Gamma_8^T & \Gamma_3^T \\ * & -Q & 0 & 0 \\ * & * & -I & 0 \\ * & * & * & -\mu T \end{bmatrix} < 0. \qquad \textbf{(2.1.18)}$$

For the simplicity in notation let us denote the matrix on the left hand side of the inequality (**2.1.18**) by Θ and denote it as in (**2.1.1**).

2.2 Design of H_∞ Control Law

Theorem 1

Consider the Mechanical system with input delay (**1.2.11**) to (**1.2.13**) under uncertainity-free condition. Given scalars γ, $\bar{\tau} > 0$ and $\mu > 0$, the closed-loop system is asymptotically stable with H_∞ performance index γ for any time delay satisfying $\tau(t)$, $0 \leq \tau(t) \leq \bar{\tau}$ and $|\dot{\tau}(t)| < \mu$, if there exist symmetric positive definite matrices $X, \bar{P}, \bar{Q}, \bar{R}, \bar{T}, Z$ and matrices $S_i (i = 1,..,4)$, Y with appropriate dimensions, satisfying the following LMI's

$$\bar{P} = \begin{bmatrix} X & \bar{P}_{12} \\ * & \bar{P}_{22} \end{bmatrix} > 0, \ \ with \ \ X > 0, \quad \bar{Q} = \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} \\ * & \bar{Q}_{22} \end{bmatrix} > 0 \qquad \textbf{(2.2.19)}$$

$$\begin{bmatrix} \bar{\Theta}_l & \Pi_1 & \Pi_2 \\ * & -Z & 0 \\ * & * & -XZ^{-1}X \end{bmatrix} < 0, \tag{2.2.20}$$

where

$$\bar{\Theta}_l = \begin{bmatrix} \bar{\theta}_{11} & \bar{\theta}_{12} & \bar{\theta}_{13} & \bar{\theta}_{14} & B_w & \bar{\tau}\bar{Q}_{11} & \bar{\tau}\bar{Q}_{12} & XC^T & \mu\bar{P}_{12} \\ * & \bar{\theta}_{22} & \bar{\theta}_{23} & \bar{\theta}_{24} & 0 & 0 & 0 & 0 & 0 \\ * & * & \bar{\theta}_{33} & \bar{\theta}_{34} & 0 & 0 & 0 & 0 & \mu\bar{P}_{22} \\ * & * & * & \bar{\theta}_{44} & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & -\gamma^2 I & 0 & 0 & 0 & 0 \\ * & * & * & * & * & -\bar{Q}_{11} & -\bar{Q}_{12} & 0 & 0 \\ * & * & * & * & * & * & -\bar{Q}_{22} & 0 & 0 \\ * & * & * & * & * & * & * & -I & 0 \\ * & * & * & * & * & * & * & * & -\mu\bar{T} \end{bmatrix}, \tag{2.2.21}$$

$$\begin{aligned} \Pi_1 &= \begin{bmatrix} AX & BY & 0 & 0 & B_w & 0 & 0 & 0 \end{bmatrix}^T \\ \Pi_2 &= \begin{bmatrix} 0 & 0 & \bar{P}_{12} & 0 & 0 & \bar{\tau}\bar{Q}_{12}^T & \bar{\tau}\bar{Q}_{22}^T & 0 \end{bmatrix}^T \end{aligned} \tag{2.2.22}$$

with

$$\begin{aligned} \bar{\theta}_{11} &= AX + XA^T + \bar{R} + \bar{S}_1 + \bar{S}_1^T \\ \bar{\theta}_{12} &= BY - \bar{S}_1^T + \bar{S}_2 \\ \bar{\theta}_{13} &= \bar{S}_3 \\ \bar{\theta}_{14} &= \bar{P}_{12} - \bar{S}_1^T + \bar{S}_4 \\ \bar{\theta}_{15} &= B_w \\ \bar{\theta}_{22} &= -(1-\mu)\bar{R} + \mu\bar{T} - \bar{S}_2 - \bar{S}_2^T \\ \bar{\theta}_{23} &= -\bar{S}_3 \\ \bar{\theta}_{24} &= -\bar{S}_2^T - \bar{S}_4 \\ \bar{\theta}_{33} &= -\bar{Q}_{11} \\ \bar{\theta}_{34} &= \bar{P}_{22} - \bar{Q}_{12} - \bar{S}_3^T \\ \bar{\theta}_{44} &= -\bar{Q}_{22} - \bar{S}_4 - \bar{S}_4^T \end{aligned} \tag{2.2.23}$$

Moreover, a desired H_∞ state feedback control law is given by $u(t) = Fx(t)$, where $F = YX^{-1}$.

Proof:

Define $X = P_{11}^{-1}$ and

$$\Lambda = diag\{\Lambda_1, X, X, I, X\} \text{ with } \Lambda_1 = diag\{X, X, X, X, I\}. \tag{2.2.24}$$

Pre- and post- multiplying the matrix Θ in (**2.1.1**) by Λ and its transpose, respectively we obtain

$$\bar{\Theta} := \Lambda\Theta\Lambda^T \tag{2.2.25}$$

Then, the entries θ_{ij} of Θ are all multiplied by X from both sides. If we define $\bar{R} := XRX$, $\bar{T} := XTX$, $\bar{S}_i := XS_iX$, for $(i = 1, .., 4)$, $\bar{Q}_{ij} := XQ_{ij}X$, $i, j = 1, 2$, $\bar{P}_{12} := XP_{12}X$

and $\bar{P}_{22} := XP_{22}X$ the matrices $X\theta_{ij}X$, for all i and j and the 5^{th} column of Θ have linear and non-linear parts. Because of that Θ can be divided into linear Θ_l and non-linear parts Θ_{nl} as follows

$$\bar{\Theta} = \bar{\Theta}_l + \bar{\Theta}_{nl}. \tag{2.2.26}$$

where $\bar{\Theta}_l$ and its entries $\bar{\theta}_{ij}$'s are given in (**2.2.21**) and the matrix $\bar{\Theta}_{nl}$ is given below.

$$\bar{\Theta}_{nl} = \begin{bmatrix} 0 & 0 & XA^TX^{-1}\bar{P}_{12} & 0 & 0 & \bar{\tau}XA^TX^{-1}\bar{Q}_{12}^T & \bar{\tau}XA^TX^{-1}\bar{Q}_{22} & 0 & 0 \\ * & 0 & XA_2^TX^{-1}\bar{P}_{12} & 0 & 0 & \bar{\tau}XA_2^TX^{-1}\bar{Q}_{12}^T & \bar{\tau}XA_2^TX^{-1}\bar{Q}_{22} & 0 & 0 \\ * & * & 0 & 0 & \bar{P}_{12}^TX^{-1}B_w & 0 & 0 & 0 & 0 \\ * & * & * & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & 0 & \bar{\tau}B_w^TX^{-1}\bar{Q}_{12}^T & \bar{\tau}B_w^TX^{-1}\bar{Q}_{22} & 0 & 0 \\ * & * & * & * & * & 0 & 0 & 0 & 0 \\ * & * & * & * & * & * & 0 & 0 & 0 \\ * & * & * & * & * & * & * & 0 & 0 \\ * & * & * & * & * & * & * & * & 0 \end{bmatrix} \tag{2.2.27}$$

One can decompose $\bar{\Theta}_{nl}$ as follows

$$\bar{\Theta}_{nl} = \Pi_1 X^{-1}\Pi_2^T + \Pi_2 X^{-1}\Pi_1^T \tag{2.2.28}$$

where Π_1 and Π_2 are the matrices given in (**2.2.22**). By *Lemma* 1, for a symmetric positive definite matrix Z we have

$$\Pi_1 X^{-1}\Pi_2^T + \Pi_2 X^{-1}\Pi_1^T \leq \Pi_1 Z^{-1}\Pi_1^T + \Pi_2\left(XZ^{-1}X\right)^{-1}\Pi_2^T. \tag{2.2.29}$$

Since $\Theta < 0$ in (**2.1.1**) and Λ is symmetric positive definite matrix then, $\bar{\Theta} < 0$ and so, the equations (**2.2.26**)-(**2.2.29**) imply that

$$\bar{\Theta} \leq \bar{\Theta}_l + \Pi_1 Z^{-1}\Pi_2^T + \Pi_2\left(XZ^{-1}X\right)^{-1}\Pi_2^T < 0. \tag{2.2.30}$$

Then, by Schur's complement, the inequality in (**2.2.30**) can be written equivalently as follows.

$$\begin{bmatrix} \bar{\Theta}_l & \Pi_1 & \Pi_2 \\ * & -Z & 0 \\ * & * & -XZ^{-1}X \end{bmatrix} < 0 \tag{2.2.31}$$

This completes the proof.

2.3 Design of Robust H_∞ Control Law

Theorem 2

Consider Uncertain Mechanical system with input delay (**1.2.8**) to (**1.2.10**). Given scalars $\bar{\tau} > 0$ and $\mu > 0$, $\mu_1 > 0$, $\varepsilon_j > 0$ $(j = 1,2,3)$, $\beta_1 > 0$ $\beta_2 > 0$, $\sigma > 0$ the closed-loop system is asymptotically stable with H_∞ performance index γ for any time delay $\tau(t)$ satisfying the conditions given in (**1.2.3**) if there exist symmetric positive definite matrices X, $\bar{P}$, $\bar{Q}$, $\bar{R}$, $\bar{T}$, and the matrices $\bar{S}_i$, $(i = 1,..,4)$ and Y with appropriate dimension satisfying the following LMI's;

$$\bar{P} = \begin{bmatrix} X & \bar{P}_{12} \\ * & \bar{P}_{22} \end{bmatrix} > 0, \; with\; X > 0, \quad \bar{Q} = \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} \\ * & \bar{Q}_{22} \end{bmatrix} > 0, \tag{2.3.32}$$

$$\Sigma\bar{R}\Sigma^T > 0,\ \Sigma(\bar{S}_1+\bar{S}_1^T)\Sigma^T > 0,\ \Sigma Z\Sigma \geq \sigma I,\ \beta_1 I - \Sigma\bar{R}\Sigma^T > 0,$$

$$\beta_2 I - \Sigma(\bar{S}_1+\bar{S}_1^T)\Sigma^T > 0,\ \begin{bmatrix} -\mu_1 I & X\Sigma^T \\ * & -\varepsilon_3 I \end{bmatrix} < 0, \qquad \textbf{(2.3.33)}$$

$$\begin{bmatrix} \Psi_l & \Gamma_2^T\Sigma & \Gamma_3^T \\ * & -\varepsilon_1 I & 0 \\ * & * & -\varepsilon_2 I \end{bmatrix} < 0 \qquad \textbf{(2.3.34)}$$

where

$$\Psi_l = \begin{bmatrix} \Psi & \hat{\Pi}_1 & \Pi_2 & \Pi_3 & \Pi_4 \\ * & -\sigma(1-\delta)^2 I & 0 & 0 & 0 \\ * & * & -XZ^{-1}X & 0 & 0 \\ * & * & * & -\beta_1 I + \Sigma\bar{R}\Sigma^T & 0 \\ * & * & * & * & -\beta_2 I + \Sigma(\bar{S}_1+\bar{S}_1^T)\Sigma^T \end{bmatrix} \qquad \textbf{(2.3.35)}$$

$$\Psi = \begin{bmatrix} \Psi_{11} & \Psi_{12} & \Psi_{13} & \Psi_{14} & B_w & \bar{\tau}\Sigma Q_{11} & \bar{\tau}\Sigma Q_{12} & \Sigma XC^T & \mu\Sigma\bar{P}_{12} \\ * & \bar{\theta}_{22} & \bar{\theta}_{23} & \bar{\theta}_{24} & 0 & 0 & 0 & 0 & 0 \\ * & * & \bar{\theta}_{33} & \bar{\theta}_{34} & 0 & 0 & 0 & 0 & \mu\bar{P}_{22} \\ * & * & * & \bar{\theta}_{44} & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & -\gamma^2 I & 0 & 0 & 0 & 0 \\ * & * & * & * & * & -\bar{Q}_{11} & -\bar{Q}_{12} & 0 & 0 \\ * & * & * & * & * & * & -\bar{Q}_{22} & 0 & 0 \\ * & * & * & * & * & * & * & -I & 0 \\ * & * & * & * & * & * & * & * & -\mu\bar{T} \end{bmatrix} \qquad \textbf{(2.3.36)}$$

$$\begin{aligned} \hat{\Pi}_1 &= \begin{bmatrix} AX\Sigma^T & BY & 0 & 0 & 0 & B_w & 0 & 0 & 0 \end{bmatrix}^T \\ \Pi_2 &= \begin{bmatrix} 0 & 0 & \bar{P}_{12} & 0 & 0 & \bar{\tau}\bar{Q}_{12}^T & \bar{\tau}\bar{Q}_{22}^T & 0 & 0 \end{bmatrix}^T \\ \Pi_3 &= \begin{bmatrix} \Sigma\bar{R}\Sigma^T & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T \\ \Pi_4 &= \begin{bmatrix} \Sigma(\bar{S}_1+\bar{S}_1^T)\Sigma^T & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T \end{aligned} \qquad \textbf{(2.3.37)}$$

$$\begin{aligned} \Psi_{11} &= AX\Sigma + \Sigma XA^T + \Sigma(\bar{R}+\bar{S}_1+\bar{S}_1^T)\Sigma^T + \varepsilon_3\delta^2 I + \mu_1\alpha^2 I + \varepsilon_1\delta^2 I \\ &\quad + \varepsilon_2\alpha^2 I + (\beta_1+\beta_2)\delta^2 I \\ \Psi_{12} &= BY + \Sigma(-\bar{S}_1^T + \bar{S}_2) \\ \Psi_{13} &= \Sigma S_3 \\ \Psi_{14} &= \Sigma(\bar{P}_{12} - \bar{S}_1^T + \bar{S}_4) \end{aligned}$$

$$\begin{aligned} \Gamma_2 &= \begin{bmatrix} XA^T & -\bar{S}_1^T + \bar{S}_2 & \bar{S}_3 & \bar{P}_{12} - \bar{S}_1^T + \bar{S}_4 & 0 & \Gamma_2' \end{bmatrix}, \\ \Gamma_3 &= \begin{bmatrix} X\Sigma^T & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \\ \Gamma_2' &= \begin{bmatrix} \bar{\tau}\bar{Q}_{11} & \bar{\tau}\bar{Q}_{12} & XC^T & \mu\bar{P}_{12} & XA^T & 0 \end{bmatrix}, \end{aligned} \qquad \textbf{(2.3.38)}$$

and the entries $\bar{\theta}_{ij}$, for $i,j = 2,3,4$ of Ψ in (**2.3.36**) are given in (**2.2.23**).Furthermore , a desired H_∞ state feedback control law is given by $u(t) = Fx(t)$, where $F = YX^{-1}$.

Proof:

Consider the uncertain mechanical systems with input delay given in the equations (**1.2.8**)-(**1.2.9**) and the closed-loop system in (**1.2.15**) and (**1.2.16**). By substituting the matrices $\tilde{A}$, $\tilde{A}_2$, $\tilde{B}_w$ and $\tilde{C}$ into the inequality (**2.2.31**), instead of A, A_2 and B_w respectively we obtain the following inequality

$$\tilde{\Theta} = \begin{bmatrix} \tilde{\Theta}_l & \tilde{\Pi}_1 & \Pi_2 \\ * & -Z & 0 \\ * & * & -XZ^{-1}X \end{bmatrix} < 0, \tag{2.3.39}$$

where

$$\tilde{\Theta}_l = \begin{bmatrix} \tilde{\theta}_{11} & \tilde{\theta}_{12} & \bar{\theta}_{13} & \bar{\theta}_{14} & \tilde{B}_w & \bar{\tau}\bar{Q}_{11} & \bar{\tau}\bar{Q}_{12} & X\tilde{C}^T & \mu\bar{P}_{12} \\ * & \bar{\theta}_{22} & \bar{\theta}_{23} & \bar{\theta}_{24} & 0 & 0 & 0 & 0 & 0 \\ * & * & \bar{\theta}_{33} & \bar{\theta}_{34} & 0 & 0 & 0 & 0 & \mu\bar{P}_{22} \\ * & * & * & \bar{\theta}_{44} & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & -\gamma^2 I & 0 & 0 & 0 & 0 \\ * & * & * & * & * & -\bar{Q}_{11} & -\bar{Q}_{12} & 0 & 0 \\ * & * & * & * & * & * & -\bar{Q}_{22} & 0 & 0 \\ * & * & * & * & * & * & * & -I & 0 \\ * & * & * & * & * & * & * & * & -\mu\bar{T} \end{bmatrix}, \tag{2.3.40}$$

$$\tilde{\Pi}_1 = \begin{bmatrix} \tilde{A}X & \tilde{B}Y & 0 & 0 & 0 & \tilde{B}_w & 0 & 0 & 0 \end{bmatrix}^T \tag{2.3.41}$$

with

$$\begin{aligned} \tilde{\theta}_{11} &= \tilde{A}X + X\tilde{A}^T + \bar{R} + \bar{S}_1 + \bar{S}_1^T \\ \tilde{\theta}_{12} &= \tilde{B}Y - \bar{S}_1^T + \bar{S}_2 \end{aligned}$$

By pre- and post-multiplying (**2.3.39**) by

$$\Xi = diag\{(\Sigma+\Delta_\Sigma), I, I, I, I, I, I, I, I, (\Sigma+\Delta_\Sigma), I\}$$

and its transpose respectively, we obtain

$$\Xi\tilde{\Theta}\Xi < 0. \tag{2.3.42}$$

Let $\theta'_{11} := (\Sigma+\Delta_\Sigma)\tilde{\theta}_{11}(\Sigma+\Delta_\Sigma)^T$, which is the $(1,1)^{th}$ block of $\Xi\tilde{\Theta}\Xi$.

$$\begin{aligned} \theta'_{11} &= (\Sigma+\Delta_\Sigma)[(\Sigma+\Delta_\Sigma)^{-1}(A+\Delta_A)X + X(A+\Delta_A)^T(\Sigma+\Delta_\Sigma)^{-T} \\ &\quad +\bar{R} + \bar{S}_1 + \bar{S}_1^T](\Sigma+\Delta_\Sigma)^T \\ &= (A+\Delta_A)X(\Sigma+\Delta_\Sigma)^T + (\Sigma+\Delta_\Sigma)X(A+\Delta_A)^T \\ &\quad +(\Sigma+\Delta_\Sigma)\left[\bar{R}+\bar{S}_1+\bar{S}_1^T\right](\Sigma+\Delta_\Sigma)^T \\ &= AX\Sigma^T + \Sigma XA^T + AX\Delta_\Sigma^T + \Delta_\Sigma XA^T \\ &\quad +\Delta_A X\Sigma^T + \Sigma X\Delta_A^T + \Delta_A X\Delta_\Sigma^T + \Delta_\Sigma X\Delta_A^T \\ &\quad +(\Sigma+\Delta_\Sigma)\left[\bar{R}+\bar{S}_1+\bar{S}_1^T\right](\Sigma+\Delta_\Sigma)^T \end{aligned} \tag{2.3.43}$$

If $\Sigma\bar{R}\Sigma > 0$ and $\beta_1 I - \Sigma\bar{R}\Sigma > 0$, for any $\beta_1 > 0$, then, by *Lemma* 2 we have

$$\begin{aligned}(\Sigma+\Delta_\Sigma)\bar{R}(\Sigma+\Delta_\Sigma)^T &= (I+\Delta_\Sigma\Sigma^{-1})\Sigma\bar{R}\Sigma^T(I+\Delta_\Sigma\Sigma^{-1})^T \\ &\leq \Sigma\bar{R}\Sigma^T + \Sigma\bar{R}\Sigma^T(\beta_1 I - \Sigma\bar{R}\Sigma^T)^{-1}\Sigma\bar{R}\Sigma^T \\ &\quad +\beta_1\Delta_\Sigma\Sigma^{-1}(\Delta_\Sigma\Sigma^{-1})^T \\ &\leq \Sigma\bar{R}\Sigma^T + \Sigma\bar{R}\Sigma^T(\beta_1 I - \Sigma\bar{R}\Sigma^T)^{-1}\Sigma\bar{R}\Sigma^T \\ &\quad +\beta_1\delta^2 I \end{aligned} \quad \mathbf{(2.3.44)}$$

Similarly, under the assumptions $\Sigma(\bar{S}_1+\bar{S}_1^T)\Sigma > 0$ and $\beta_2 I - \Sigma(\bar{S}_1+\bar{S}_1^T)\Sigma > 0$, for any $\beta_2 > 0$ we can write the following inequality

$$\begin{aligned}(\Sigma+\Delta_\Sigma)(\bar{S}_1+\bar{S}_1^T)(\Sigma+\Delta_\Sigma)^T &\leq \Sigma(\bar{S}_1+\bar{S}_1^T)\Sigma^T(\beta_2 I \\ &\quad -\Sigma(\bar{S}_1+\bar{S}_1^T)\Sigma^T)^{-1}\Sigma(\bar{S}_1+\bar{S}_1^T)\Sigma^T \\ &\quad +\Sigma(\bar{S}_1+\bar{S}_1^T)\Sigma^T + \beta_2\delta^2 I \end{aligned} \quad \mathbf{(2.3.45)}$$

Since $\left\|\Delta_\Sigma\Sigma^{-1}\right\| \leq \delta \leq 1$ and $\|I\| = 1$, for any scalars $\varepsilon_3 > 0$ we can write

$$\begin{aligned}\Delta_A X\Delta_\Sigma^T + \Delta_\Sigma X\Delta_A^T &= \Delta_A X\Sigma^T\Sigma^{-T}\Delta_\Sigma^T + \Delta_\Sigma\Sigma^{-1}\Sigma X\Delta_A^T \\ &\leq \varepsilon_3\Delta_\Sigma\Sigma^{-1}\Sigma^{-T}\Delta_\Sigma^T + \frac{1}{\varepsilon_3}\Delta_A X\Sigma^T\Sigma X\Delta_A^T \\ &\leq \varepsilon_3\delta^2 I + \frac{1}{\varepsilon_3}\Delta_A X\Sigma^T\Sigma X\Delta_A^T \end{aligned} \quad \mathbf{(2.3.46)}$$

By adding $\mu_1\Delta_A\Delta_A^T - \mu_1\Delta_A\Delta_A^T = 0$ into the last term of (**2.3.46**) we obtain

$$\begin{aligned}\Delta_A X\Delta_\Sigma^T + \Delta_\Sigma X\Delta_A^T &\leq \varepsilon_3\delta^2 I + \Delta_A(\frac{1}{\varepsilon_3}X\Sigma^T\Sigma X - \mu_1 I)\Delta_A^T + \mu_1\Delta_A\Delta_A^T \\ &\leq \varepsilon_3\delta^2 I + \mu_1\alpha^2 I, \end{aligned} \quad \mathbf{(2.3.47)}$$

where $\mu_1 > 0$ is to be chosen such that $\frac{1}{\varepsilon_3}X\Sigma^T\Sigma X - \mu_1 I < 0$. Then, by the equations (**2.3.44**), (**2.3.45**) and (**2.3.47**) the inequality θ'_{11} in (**2.3.43**) becomes

$$\begin{aligned}\theta'_{11} \leq\ & AX\Sigma^T + \Sigma XA^T + AX\Delta_\Sigma^T + \Delta_\Sigma XA^T + \Delta_A X\Sigma^T + \Sigma X\Delta_A^T + \varepsilon_3\delta^2 I + \mu_1\alpha^2 I \\ & +\Sigma(\bar{R}+\bar{S}_1+\bar{S}_1^T)\Sigma^T + \Sigma\bar{R}\Sigma^T(\beta_1 I - \Sigma\bar{R}\Sigma^T)^{-1}\Sigma\bar{R}\Sigma^T + \beta_1\delta^2 I \\ & +\Sigma(\bar{S}_1+\bar{S}_1^T)\Sigma^T(\beta_2 I - \Sigma(\bar{S}_1+\bar{S}_1^T)\Sigma^T)^{-1}\Sigma(\bar{S}_1+\bar{S}_1^T)\Sigma^T + \beta_2\delta^2 I \end{aligned} \quad \mathbf{(2.3.48)}$$

Now, consider the $(10,10)^{th}$ block of Ψ, which is $(\Sigma+\Delta_\Sigma)(-Z)(\Sigma+\Delta_\Sigma)^T$. If $\Sigma Z\Sigma \geq \sigma I$, for any $\sigma > 0$, then

$$\begin{aligned}(\Sigma+\Delta_\Sigma)Z(\Sigma+\Delta_\Sigma)^T &= (I+\Delta_\Sigma\Sigma^{-1})\Sigma Z\Sigma^T(I+\Delta_\Sigma\Sigma^{-1})^T \\ &\geq \sigma(I+\Delta_\Sigma\Sigma^{-1})(I+\Delta_\Sigma\Sigma^{-1})^T \\ &\geq \sigma(1-\delta)^2 I \end{aligned} \quad \mathbf{(2.3.49)}$$

Thus

$$(\Sigma+\Delta_\Sigma)(-Z)(\Sigma+\Delta_\Sigma)^T \leq -\sigma(1-\delta)^2 I \quad \mathbf{(2.3.50)}$$

(**2.3.42**) can be divided into nominal and uncertainty parts as

$$\Xi\tilde{\Theta}\Xi < \Psi_n + \Psi_u < 0, \qquad \textbf{(2.3.51)}$$

where

$$\Psi_n = \begin{bmatrix} \Psi' & \hat{\Pi}_1 & \Pi_2 \\ * & -\sigma(1-\delta)^2 I & 0 \\ * & * & -XZ^{-1}X \end{bmatrix} < 0, \qquad \textbf{(2.3.52)}$$

with

$$\Psi' = \begin{bmatrix} \Psi'_{11} & \Psi_{12} & \Psi_{13} & \Psi_{14} & \Sigma B_w & \bar{\tau}\Sigma Q_{11} & \bar{\tau}\Sigma Q_{12} & \Sigma X C^T & \mu\Sigma\bar{P}_{12} \\ * & \bar{\theta}_{22} & \bar{\theta}_{23} & \bar{\theta}_{24} & 0 & 0 & 0 & 0 & 0 \\ * & * & \bar{\theta}_{33} & \bar{\theta}_{34} & 0 & 0 & 0 & 0 & \mu\bar{P}_{22} \\ * & * & * & \bar{\theta}_{44} & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & -\gamma^2 I & 0 & 0 & 0 & 0 \\ * & * & * & * & * & -\bar{Q}_{11} & -\bar{Q}_{12} & 0 & 0 \\ * & * & * & * & * & * & -\bar{Q}_{22} & 0 & 0 \\ * & * & * & * & * & * & * & -I & 0 \\ * & * & * & * & * & * & * & * & -\mu\bar{T} \end{bmatrix}, \qquad \textbf{(2.3.53)}$$

$$\hat{\Pi}_1 = \begin{bmatrix} AX\Sigma^T & BY & 0 & 0 & 0 & B_w & 0 & 0 & 0 \end{bmatrix}^T, \qquad \textbf{(2.3.54)}$$

$$\begin{aligned} \Psi'_{11} &= AX\Sigma + \Sigma XA^T + \Sigma(\bar{R} + \bar{S}_1 + \bar{S}_1)^T)\Sigma^T + \varepsilon_3\delta^2 I + \mu_1\alpha^2 I + (\beta_1+\beta_2)\delta^2 I \\ &\quad + \Sigma\bar{R}\Sigma^T(\beta_1 I - \Sigma\bar{R}\Sigma^T)^{-1}\Sigma\bar{R}\Sigma^T \\ &\quad + \Sigma(\bar{S}_1 + \bar{S}_1^T)\Sigma^T(\beta_2 I - \Sigma(\bar{S}_1 + \bar{S}_1^T)\Sigma^T)^{-1}\Sigma(\bar{S}_1 + \bar{S}_1^T)\Sigma^T \\ \Psi_{12} &= BY + \Sigma(-\bar{S}_1^T + \bar{S}_2) \\ \Psi_{13} &= \Sigma S_3 \\ \Psi_{14} &= \Sigma(\bar{P}_{12} - \bar{S}_1^T + \bar{S}_4) \end{aligned}$$

and

$$\Psi_u = \Gamma_1^T \Delta_\Sigma \Gamma_2 + \Gamma_2^T \Delta_\Sigma^T \Gamma_1 + \Gamma_1^T \Delta_A \Gamma_3 + \Gamma_3^T \Delta_A^T \Gamma_1, \qquad \textbf{(2.3.55)}$$

where

$$\begin{aligned} \Gamma_1 &= \begin{bmatrix} I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \\ \Gamma_2 &= \begin{bmatrix} XA^T & -\bar{S}_1^T + \bar{S}_2 & \bar{S}_3 & \bar{P}_{12} - \bar{S}_1^T + \bar{S}_4 & 0 & \Gamma'_2 \end{bmatrix}, \\ \Gamma_3 &= \begin{bmatrix} X\Sigma^T & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \qquad \textbf{(2.3.56)} \\ \Gamma'_2 &:= \begin{bmatrix} \bar{\tau}\bar{Q}_{11} & \bar{\tau}\bar{Q}_{12} & XC^T & \mu\bar{P}_{12} & XA^T & 0 \end{bmatrix}, \end{aligned}$$

$$\textbf{(2.3.57)}$$

Since $0 \le \left\|\Delta_\Sigma \Sigma^{-1}\right\| \le \delta \le 1,\ \ 0 \le \|\Delta_A\| \le \alpha$,for some $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ we have

$$\begin{aligned} \Psi_u &= (\delta\Gamma_1)^T(\frac{1}{\delta}\Delta_\Sigma\Sigma^{-1})(\Sigma\Gamma_2) + (\Sigma\Gamma_2)^T(\frac{1}{\delta}\Delta_\Sigma\Sigma^{-1})^T(\delta\Gamma_1) \\ &\quad + (\alpha\Gamma_1)^T((\frac{1}{\alpha}\Delta_A)\Gamma_3 + \Gamma_3^T(\frac{1}{\alpha}\Delta_A)^T(\alpha\Gamma_1) \\ &\le \frac{1}{\varepsilon_1}(\Sigma\Gamma_2)^T(\Sigma\Gamma_2) + \delta^2\varepsilon_1\Gamma_1^T\Gamma_1 + \frac{1}{\varepsilon_2}\Gamma_3^T\Gamma_3 + \alpha^2\varepsilon_2\Gamma_1^T\Gamma_1 \qquad \textbf{(2.3.58)} \end{aligned}$$

Now define

$$\begin{aligned}\Psi_{11} \quad := \quad & AX\Sigma + \Sigma XA^T + \Sigma(\bar{R} + \bar{S}_1 + \bar{S}_1)^T\Sigma^T + \varepsilon_3\delta^2 I + \mu_1\alpha^2 I \\ & +(\beta_1 + \beta_2)\delta^2 I + \mu_1\alpha^2 I + \delta^2\varepsilon_1 I + \alpha^2\varepsilon_2 I, \end{aligned} \qquad (\mathbf{2.3.59})$$

change the $(1,1)^{th}$-entry Ψ'_{11} of Ψ' by Ψ_{11} and denote the matrix obtained by Ψ, which is given in (**2.3.36**). Because of nonlinear part of Ψ'_{11} by Schur's complement the matrix Ψ_n is equivalent to the following matrix

$$\Psi_l = \begin{bmatrix} \Psi & \hat{\Pi}_1 & \Pi_2 & \Pi_3 & \Pi_4 \\ * & -\sigma(1-\delta)^2 I & 0 & 0 & 0 \\ * & * & -XZ^{-1}X & 0 & 0 \\ * & * & * & -\beta_1 I + \Sigma\bar{R}\Sigma^T & 0 \\ * & * & * & * & -\beta_2 I + \Sigma(\bar{S}_1 + \bar{S}_1^T)\Sigma^T \end{bmatrix} < 0, \qquad (\mathbf{2.3.60})$$

where Π_3 and Π_4 are the matrices given in (**2.3.37**)

Then, by Schur's complement, the matrix (**2.3.60**), the inequality (**2.3.58**) and the matrix Ψ_{11} in (**2.3.59**) imply the inequality in (**2.3.34**). Thus, the proof is completed.

3. A UNIFORM DELAY PARTITIONING APPROACH

In this section,a new Lyapunov-Krasovskii functionals are constructed to obtain control law to work in larger delay interval and in smaller performance index, by partitioning the interval as

$$0 < \frac{\bar{\tau}}{m} < \frac{2\bar{\tau}}{m} < \cdots < \frac{k\bar{\tau}}{m} < \cdots < \bar{\tau}.$$

3.1 H_∞ Control in case of Uniform Delay Partitioning

Theorem 3

Consider mechanical system with input delay (**1.2.11**) to (**1.2.13**). Given scalars $\bar{\tau} > 0$ and $\mu > 0$, the closed-loop system is asmptotically stable with H_∞ performance index γ for any time delay satisfying $\tau(t)$, $0 \leq \tau(t) \leq \bar{\tau}$ and $|\dot{\tau}(t)| < \mu$, if there exist symmetric positive definite matrices $X, \bar{P}, \bar{Q}, \bar{R}, \bar{T}, \bar{W}, \bar{T}_k$ $(k = 1,..,m)$, Z and matrices S_i, $(i = 1,..,m+4)$, Y with appropriate dimensions satisfying below LMI's. Then a desired H_∞ state feedback control law is given by $u(t) = Fx(t)$, where $F = YX^{-1}$:

$$\bar{P} = \begin{bmatrix} X & \bar{P}_{12} \\ * & \bar{P}_{22} \end{bmatrix} > 0 \; with \; X > 0, \; \bar{Q} = \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} \\ * & \bar{Q}_{22} \end{bmatrix} > 0,$$

$$\bar{W} = \begin{bmatrix} \bar{W}_{11} & \bar{W}_{12} & \cdots & \bar{W}_{1m} \\ * & \bar{W}_{22} & \cdots & \bar{W}_{2m} \\ \vdots & \vdots & \vdots & \vdots \\ * & * & \cdots & \bar{W}_{mm} \end{bmatrix} > 0, \quad \begin{bmatrix} \bar{\Omega}_l & \Pi_1 & \Pi_2 \\ * & -Z & 0 \\ * & * & -XZ^{-1}X \end{bmatrix} < 0, \qquad \textbf{(3.1.1)}$$

where

$$\bar{\Omega}_l = \begin{bmatrix} \bar{\theta}_l & M_l & \bar{\Gamma}_{10}^T & 0 & \cdots & 0 & \mu\bar{\Gamma}_3^T \\ * & -\bar{Q} & 0 & 0 & \cdots & 0 & 0 \\ * & * & -I & 0 & \cdots & 0 & 0 \\ * & * & * & -\bar{T}_1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ * & * & * & * & \cdots & -\bar{T}_m & 0 \\ * & * & * & * & \cdots & * & -\mu\bar{T} \end{bmatrix} \qquad \textbf{(3.1.2)}$$

$$\bar{\theta}_l = \begin{bmatrix} \bar{\theta}_l^1 & \bar{\theta}^2 & \bar{\theta}_l^3 \\ * & \bar{\theta}^4 & 0 \\ * & * & -\gamma^2 I \end{bmatrix}, \bar{\theta}_l^1 = \begin{bmatrix} \bar{\theta}_{11} & \bar{\theta}_{12} & \bar{\theta}_{13} & \bar{\theta}_{14} \\ * & \bar{\theta}_{22} & \bar{\theta}_{23} & \bar{\theta}_{24} \\ * & * & \bar{\theta}_{33} & \bar{\theta}_{34} \\ * & * & * & \bar{\theta}_{44} \end{bmatrix} \qquad \textbf{(3.1.3)}$$

$$\begin{aligned} \bar{\theta}_{11} &= AX + XA^T + \bar{R} + \bar{W}_{11} - \sum_{k=1}^{m} \bar{T}_k + \bar{S}_1 + \bar{S}_1^T \\ \bar{\theta}_{12} &= BY - \bar{S}_1^T + \bar{S}_2 \\ \bar{\theta}_{13} &= \bar{S}_3 \\ \bar{\theta}_{14} &= \bar{P}_{12} - \bar{S}_1^T + \bar{S}_4 \\ \bar{\theta}_{22} &= -(1-\mu)\bar{R} + \mu\bar{T} - \bar{S}_2 - \bar{S}_2^T \end{aligned}$$

$$
\begin{aligned}
\bar{\theta}_{23} &= -\bar{S}_3\\
\bar{\theta}_{24} &= -\bar{S}_2^T - \bar{S}_4\\
\bar{\theta}_{33} &= -\bar{Q}_{11}\\
\bar{\theta}_{34} &= \bar{P}_{22} - \bar{Q}_{12} - \bar{S}_3^T\\
\bar{\theta}_{44} &= -\bar{Q}_{22} - \bar{S}_4 - \bar{S}_4^T
\end{aligned}
$$

$$
\bar{\theta}^2 = \begin{bmatrix} \bar{H}_1 & \bar{H}_2 & \bar{H}_3 & \cdots & \bar{H}_{m-1} & \bar{T}_m + \bar{S}_{m+4} \\ -\bar{S}_5 & -\bar{S}_6 & -\bar{S}_7 & \cdots & -\bar{S}_{m+3} & -\bar{S}_{m+4} \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ -\bar{S}_5 & -\bar{S}_6 & -\bar{S}_7 & \cdots & -\bar{S}_{m+3} & -\bar{S}_{m+4} \end{bmatrix}, \; \theta_l^3 = \begin{bmatrix} B_w \\ 0 \\ 0 \\ 0 \end{bmatrix}
$$

$$
\bar{\theta}^4 = \begin{bmatrix} \bar{G}_1 & \bar{W}_{23} - \bar{W}_{12} & \bar{W}_{24} - \bar{W}_{13} & \cdots & \bar{W}_{2m} - \bar{W}_{1(m-1)} & -\bar{W}_{1m} \\ * & \bar{G}_2 & \bar{W}_{34} - \bar{W}_{23} & \cdots & \bar{W}_{3m} - \bar{W}_{2(m-1)} & -\bar{W}_{2m} \\ * & * & \bar{G}_3 & \cdots & \bar{W}_{4m} - \bar{W}_{3(m-1)} & -\bar{W}_{3m} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ * & * & * & \cdots & \bar{G}_{m-1} & -\bar{W}_{m-1,m} \\ * & * & * & \cdots & * & -\bar{W}_{mm} - \bar{T}_m \end{bmatrix}
$$

with $\bar{H}_k = \bar{T}_k + \bar{W}_{1(k+1)} + \bar{S}_{k+4}$ and $\bar{G}_k = \bar{W}_{(k+1)(k+1)} - \bar{W}_{kk} - \bar{T}_k$ for $k = 1 \cdots m-1$

$$
\begin{aligned}
\bar{\Gamma}_{10} &= \begin{bmatrix} CX & 0 & 0 & 0 & \mathbb{P}_0^T & 0 \end{bmatrix}^T\\
M_l &= \begin{bmatrix} \tau\bar{Q}_{11} & 0 & 0 & 0 & \mathbb{P}_0^T & 0 \\ \tau\bar{Q}_{12}^T & 0 & 0 & 0 & \mathbb{P}_0^T & 0 \end{bmatrix}^T\\
\bar{\Gamma}_3 &= \begin{bmatrix} P_{12}^T & 0 & P_{22} & 0 & \mathbb{P}_0^T & 0 \end{bmatrix}^T\\
\Pi_1 &= \begin{bmatrix} AX & BY & 0 & 0 & \mathbb{P}_0^T & B_w & 0 & 0 & 0 & \mathbb{P}_0^T & 0 \end{bmatrix}^T\\
\Pi_2 &= \begin{bmatrix} 0 & 0 & P_{12} & 0 & \mathbb{P}_0^T & 0 & \tau\bar{Q}_{12}^T & \tau\bar{Q}_{22} & 0 & \mathbb{P}^T & 0 \end{bmatrix}^T\\
\mathbb{P} &= \begin{bmatrix} \frac{\bar{\tau}}{m}\bar{T}_1 & \cdots & \frac{\bar{\tau}k}{m}\bar{T}_k & \cdots & \bar{\tau}\bar{T}_m \end{bmatrix}^T\\
\mathbb{P}_0 &= \begin{bmatrix} 0 & \cdots & 0 & \cdots & 0 \end{bmatrix}^T
\end{aligned}
$$

$\mathbb{P}_0$ and $\mathbb{P}$ are block matrices with compatible dimension.

Proof:

Choose a Lyapunov functional $V(x(t),t)$ as

$$V(x(t),t) = V_1(x(t),t) + V_2(x(t),t) + V_3(x(t),t) + V_4(x(t),t) + V_5(x(t),t), \quad \textbf{(3.1.4)}$$

where

$$V_1(x(t),t) = \eta^T(t)P\eta(t) \quad \textbf{(3.1.5)}$$

$$V_2(x(t),t) = \bar{\tau}\int_{-\bar{\tau}}^{0}\int_{t}^{t+\theta} \xi^T(s)Q\xi(s)dsd\theta \quad \textbf{(3.1.6)}$$

$$V_3(x(t),t) = \sum_{k=1}^{m} \frac{k\bar{\tau}}{m}\int_{-\frac{k\bar{\tau}}{m}}^{0}\int_{t+\theta}^{t} \dot{x}^T(s)T_k\dot{x}(s)dsd\theta \quad \textbf{(3.1.7)}$$

$$V_4(x(t),t) = \int_{t-\tau(t)}^{t} x^T(s)Rx(s)ds \quad \textbf{(3.1.8)}$$

$$V_5(x(t),t) = \int_{t-\frac{\bar{\tau}}{m}}^{t} \gamma^T(s) W \gamma(s) ds \qquad \textbf{(3.1.9)}$$

$$\begin{aligned} \eta(t) &= \begin{bmatrix} x^T(t) & (\int_{t-\tau(t)}^{t} x(s)ds)^T \end{bmatrix}^T \\ \xi(t) &= \begin{bmatrix} x^T(t) & \dot{x}^T(t) \end{bmatrix}^T \\ \gamma(t) &= \begin{bmatrix} x^T(t) & x(t-\frac{\bar{\tau}}{m})^T & x(t-\frac{2\bar{\tau}}{m})^T & \cdots & x(t-\frac{(m-1)\bar{\tau}}{m})^T \end{bmatrix}^T \end{aligned}$$

where

$$P = \begin{bmatrix} P_{11} & P_{12} \\ * & P_{22} \end{bmatrix} > 0, \text{ with } P_{11} > 0, \; Q = \begin{bmatrix} Q_{11} & Q_{12} \\ * & Q_{22} \end{bmatrix} > 0,$$

$$R > 0, \; T_k > 0, (k = 1,..,m), \; W = \begin{bmatrix} W_{11} & W_{12} & \cdots & W_{1m} \\ * & W_{22} & \cdots & W_{2m} \\ \vdots & \vdots & \vdots & \vdots \\ * & * & \cdots & W_{mm} \end{bmatrix} > 0.$$

The derivative of $V_1(x(t),t)$ is

$$\dot{V}_1(x(t),t) = \dot{\eta}^T(t) P \eta(t) + \eta^T(t) P \dot{\eta}(t) \qquad \textbf{(3.1.10)}$$

One can write $\dot{\eta}(t)$ as follows,

$$\dot{\eta}(t) = \eta_1(t) + \dot{\tau}(t)\eta_2(t), \qquad \textbf{(3.1.11)}$$

where

$$\eta_1(t) = \begin{bmatrix} Ax(t) + A_2 x(t-\tau(t)) + B_w w(t) \\ \int_{t-\tau(t)}^{t} \dot{x}(s)ds \end{bmatrix}, \eta_2(t) = \begin{bmatrix} 0 \\ I \end{bmatrix} x(t-\tau(t)) \qquad \textbf{(3.1.12)}$$

Define

$$\chi(t) = \begin{bmatrix} x^T(t) & x^T(t-\tau(t)) & \left(\int_{t-\tau(t)}^{t} x(s)ds\right)^T & \left(\int_{t-\tau(t)}^{t} \dot{x}(s)ds\right)^T & \mathbb{P}^T(t) & w^T(t) \end{bmatrix}^T$$

with $\mathbb{P}(t) = \begin{bmatrix} x(t-\frac{\bar{\tau}}{m})^T & \cdots & x(t-\frac{k\bar{\tau}}{m})^T & \cdots & x(t-\bar{\tau})^T \end{bmatrix}^T$, we obtain

$$\eta(t) = \Gamma_1 \chi(t), \eta_1(t) = \Gamma_2 \chi(t), \begin{bmatrix} 0 & I \end{bmatrix} P \eta(t) = \Gamma_3 \chi(t), x(t-\tau(t)) = \Gamma_4 \chi(t),$$

where

$$\begin{aligned} \Gamma_1 &= \begin{bmatrix} I & 0 & 0 & 0 & \mathbb{P}_0^T & 0 \\ 0 & 0 & I & 0 & \mathbb{P}_0^T & 0 \end{bmatrix}, \Gamma_2 = \begin{bmatrix} A & A_2 & 0 & 0 & \mathbb{P}_0^T & B_w \\ 0 & 0 & 0 & I & \mathbb{P}_0^T & 0 \end{bmatrix}, \\ \Gamma_3 &= \begin{bmatrix} P_{12}^T & 0 & P_{22} & 0 & \mathbb{P}_0^T & 0 \end{bmatrix}, \Gamma_4 = \begin{bmatrix} 0 & I & 0 & 0 & \mathbb{P}_0^T & 0 \end{bmatrix} \end{aligned}$$

with $\mathbb{P}_0 = \begin{bmatrix} 0 & \cdots & 0 & \cdots & 0 \end{bmatrix}^T$. Thus

$$\eta^T(t) P \eta_1(t) + \eta_1^T(t) P \eta(t) = \chi^T(t)(\Gamma_1^T P \Gamma_2 + \Gamma_2^T P \Gamma_1)\chi(t) \qquad \textbf{(3.1.13)}$$

By *Lemma* 1, for any symmetric positive definite matrix we obtain,

$$\dot{\tau}(t)\eta^T(t)P\eta_2(t)+\dot{\tau}(t)\eta_2^T(t)P\eta(t)\leq\chi^T(t)\left(\mu\Gamma_4^TT\Gamma_4+\mu\Gamma_3^TT^{-1}\Gamma_3\right)\chi(t), \quad \textbf{(3.1.14)}$$

where T is symmetric positive definite matrice. Therefore, we have

$$\dot{V}_1(x(t),t)\leq\chi^T(t)(\Gamma_1^TP\Gamma_2+\Gamma_2^TP\Gamma_1+\mu\chi^T\Gamma_4^TT\Gamma_4\chi+\mu\chi^T\Gamma_3^TT^{-1}\Gamma_3)\chi(t) \quad \textbf{(3.1.15)}$$

The derivative of $V_2(x(t),t)$ can be computed as

$$\dot{V}_2(x(t),t)=\bar{\tau}^2\xi^T(t)Q\xi(t)-\bar{\tau}\int_{-\bar{\tau}}^{0}\xi^T(t+\theta)Q\xi(t+\theta)d\theta \quad \textbf{(3.1.16)}$$

By changing the variable as $s=t+\theta$, we obtain $d\theta=ds$ and

$$-\bar{\tau}\int_{-\bar{\tau}}^{0}\xi^T(t+\theta)Q\xi(t+\theta)d\theta=-\bar{\tau}\int_{t-\bar{\tau}}^{t}\xi^T(s)Q\xi(s)ds \quad \textbf{(3.1.17)}$$

By *Lemma* 3 , we know that

$$-\bar{\tau}\int_{t-\bar{\tau}}^{t}\xi^T(s)Q\xi(s)ds\leq-\left(\int_{t-\bar{\tau}(t)}^{t}\xi(s)ds\right)^TQ\left(\int_{t-\bar{\tau}(t)}^{t}\xi(s)ds\right) \quad \textbf{(3.1.18)}$$

By means of $\chi(t)$, we obtain

$$\xi(t)=\Gamma_5\chi(t),\int_{t-\bar{\tau}(t)}^{t}\xi(s)ds=\Gamma_6\chi(t), \quad \textbf{(3.1.19)}$$

where

$$\Gamma_5=\begin{bmatrix} I & 0 & 0 & 0 & \mathbb{P}_0^T & 0\\ A & A_2 & 0 & 0 & \mathbb{P}_0^T & B_w\end{bmatrix},\Gamma_6=\begin{bmatrix} 0 & 0 & I & 0 & \mathbb{P}_0^T & 0\\ 0 & 0 & 0 & I & \mathbb{P}_0^T & 0\end{bmatrix} \quad \textbf{(3.1.20)}$$

Thus, we have

$$\dot{V}_2(x(t),t)\leq\chi^T(t)\left(\bar{\tau}^2\Gamma_5^TQ\Gamma_5-\Gamma_6^TQ\Gamma_6\right)\chi(t) \quad \textbf{(3.1.21)}$$

The derivative of $V_3(x(t),t)$ can be computed as

$$\dot{V}_3(x(t),t)=\sum_{k=1}^{m}\left(\frac{k\bar{\tau}^2}{m}\dot{x}^T(t)T_k\dot{x}(t)-\frac{k\bar{\tau}}{m}\int_{-\frac{k\bar{\tau}}{m}}^{0}\dot{x}^T(t+\theta)T_k\dot{x}(t+\theta)d\theta\right) \quad \textbf{(3.1.22)}$$

For $1\leq k\leq m$, we make the substitution $s=t+\theta$, we obtain $d\theta=ds$ and

$$-\frac{k\bar{\tau}}{m}\int_{-\frac{k\bar{\tau}}{m}}^{0}\dot{x}^T(t+\theta)T_k\dot{x}(t+\theta)d\theta=-\frac{k\bar{\tau}}{m}\int_{t-\frac{k\bar{\tau}}{m}}^{t}\dot{x}^T(s)T_k\dot{x}(s)ds \quad \textbf{(3.1.23)}$$

By *Lemma* 3 , we know that

$$-\frac{k\bar{\tau}}{m}\int_{t-\frac{k\bar{\tau}}{m}}^{t}\dot{x}^T(s)T_k\dot{x}(s)ds\leq-\left(\int_{t-\frac{k\bar{\tau}}{m}}^{t}\dot{x}(s)ds\right)^TT_k\left(\int_{t-\frac{k\bar{\tau}}{m}}^{t}\dot{x}(s)ds\right) \quad \textbf{(3.1.24)}$$

By the Leibnitz-Newton formula we have,

$$-\left(\int_{t-\frac{k\bar{\tau}}{m}}^{t}\dot{x}(s)ds\right)^{T}T_k\left(\int_{t-\frac{k\bar{\tau}}{m}}^{t}\dot{x}(s)ds\right)=-\left(x(t)-x(t-\frac{k\bar{\tau}}{m})\right)^{T}T_k\left(x(t)-x(t-\frac{k\bar{\tau}}{m})\right) \quad \textbf{(3.1.25)}$$

Thus, one can write,

$$\begin{aligned}\dot{V}_3(x(t),t) &\leq \sum_{k=1}^{m}\left[\frac{k\bar{\tau}^2}{m}\dot{x}^T(t)T_k\dot{x}(t)-\left(x(t)-x(t-\frac{k\bar{\tau}}{m})\right)^{T}T_k\left(x(t)-x(t-\frac{k\bar{\tau}}{m})\right)\right]\\ &= \sum_{k=1}^{m}\frac{k\bar{\tau}^2}{m}\dot{x}^T(t)T_k\dot{x}(t)=\sum_{k=1}^{m}\chi^T(t)\Theta_k^T T_k^{-1}\Theta_k\chi(t), \qquad \textbf{(3.1.26)}\end{aligned}$$

where

$$\Theta_k := \frac{k\bar{\tau}}{m}T_k\left[\begin{array}{cccccc} A & A_2 & 0 & 0 & \mathbb{P}_0^T & B_w \end{array}\right]$$

Let $x(t) =: x$ and $x(t-\frac{k\bar{\tau}}{m}) =: x_{k\bar{\tau}}$. One can compute as

$$\begin{aligned}-\sum_{k=1}^{m}(x-x_{k\bar{\tau}})^T T_k(x-x_{k\bar{\tau}}) &= \chi^T(t)\Big[diag\Big\{-\sum_{k=1}^{m}T_k,0,0,0,-T_1,\cdots,-T_m,0\Big\}\\ &\qquad +\Phi\Big]\chi(t), \qquad \textbf{(3.1.27)}\end{aligned}$$

where

$$\Phi := \begin{bmatrix} 0 & 0 & 0 & 0 & T_1 & T_2 & \cdots & T_m & 0\\ * & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0\\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\ * & * & * & * & * & * & \cdots & * & 0 \end{bmatrix}$$

Then we get,

$$\begin{aligned}\dot{V}_3(x(t),t) &\leq \chi^T(t)\Big(\sum_{k=1}^{m}\Theta_k^T T_k^{-1}\Theta_k+diag\Big\{-\sum_{k=1}^{m}T_k,0,0,0,-T_1,\cdots,-T_m,0\Big\}\\ &\qquad +\Phi\Big)\chi(t) \qquad \textbf{(3.1.28)}\end{aligned}$$

The derivative of $V_4(x(t),t)$ is

$$\begin{aligned}\dot{V}_4(x(t),t) &= x^T(t)Rx(t)-[1-\dot{\tau}(t)]x^T(t-\tau(t))Rx(t-\tau(t))\\ &\leq x^T(t)Rx(t)-[1-\mu]x^T(t-\tau(t))Rx(t-\tau(t))\\ &= \chi^T(t)diag\{R,[1-\mu]R,0,0,0,\cdots,0,0\}\chi(t)\end{aligned} \qquad \textbf{(3.1.29)}$$

Finally, the derivative of $V_5(x(t),t)$ is

$$\dot{V}_5(x(t),t)=\gamma^T(t)W\gamma(t)-\gamma^T(t-\frac{\bar{\tau}}{m})W\gamma(t-\frac{\bar{\tau}}{m}) \qquad \textbf{(3.1.30)}$$

$$\gamma(t) = \Gamma_7\chi(t) \text{ and } \gamma(t-\frac{\bar{\tau}}{m}) = \Gamma_8\chi(t),$$

where

$$\Gamma_7 = \begin{bmatrix} I & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & I & 0 & 0 \end{bmatrix}, \Gamma_8 = \begin{bmatrix} 0 & 0 & 0 & 0 & I & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & I & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & I & 0 \end{bmatrix} \quad \textbf{(3.1.31)}$$

Then we obtain,

$$\dot{V}_5(x(t),t) = \chi^T(t)(\Gamma_7^T W\Gamma_7 - \Gamma_8^T W\Gamma_8)\chi(t) \quad \textbf{(3.1.32)}$$

By the Leibnitz-Newton formula we know that

$$\int_{t-\tau(t)}^{t} \dot{x}(s)ds = x(t) - x(t-\tau(t)) \quad \textbf{(3.1.33)}$$

By means of (**3.1.33**), we obtain a relaxation as

$$\chi^T(t)(S^T\Gamma_9 + \Gamma_9^T S)\chi(t) = 0 \quad \textbf{(3.1.34)}$$

where

$$\Gamma_9 = \begin{bmatrix} I & -I & 0 & -I & \mathbb{P}_0^T & 0 \end{bmatrix}$$

$$S = \begin{bmatrix} S_1 & S_2 & S_3 & S_4 & S_5 & S_6 & \cdots & S_{m+4} & 0 \end{bmatrix}$$

with $S_i, (i = 1....(m+4))$ matrices of appropriate dimensions. The controlled output $z(t)$ can be written as $z(t) = \Gamma_{10}\chi(t)$, where

$$\Gamma_{10} = \begin{bmatrix} C & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix} \quad \textbf{(3.1.35)}$$

Thus

$$\begin{aligned} z^T(t)z(t) - \gamma^2 w^T(t)w(t) &= \chi^T(t)\Gamma_{10}^T\Gamma_{10}\chi(t) - \gamma^2 w^T(t)w(t) \\ &= \chi^T(t)(\Gamma_{10}^T\Gamma_{10} - diag\{0,0,0,0,0,\cdots,0,-\gamma^2 I\})\chi(t) \end{aligned} \quad \textbf{(3.1.36)}$$

In order to prove that system (**1.2.15**)-(**1.2.16**) is asymptotically stable with a H_∞ performance index $\gamma > 0$,it is required that the Hamiltonian

$$H(x(t),w(t),t) = \dot{V}(x(t),t) + z^T(t)z(t) - \gamma^2 w^T(t)w(t) < 0 \quad \textbf{(3.1.37)}$$

$$\begin{aligned} H(x(t),w(t),t) &= \dot{V}(x(t),t) + z^T(t)z(t) - \gamma^2 w^T(t)w(t) \\ &\leq \chi^T(t)(\Gamma_1^T P\Gamma_2 + \Gamma_2^T P\Gamma_1 + \mu\Gamma_4^T T\Gamma_4 \\ &\quad +\mu\Gamma_3^T T^{-1}\Gamma_3 + \bar{\tau}^2\Gamma_5^T Q\Gamma_5 - \Gamma_6^T Q\Gamma_6 \\ &\quad +\sum_{k=1}^{m} \Theta_k^T T_k^{-1}\Theta_k \\ &\quad +diag\left\{-\sum_{k=1}^{m} T_k, 0,0,0,-T_1,-T_2,\cdots,-T_m,0\right\} + \Phi \\ &\quad +diag\{R,-[1-\mu]R,0,0,0,-\gamma^2 I\} + \Gamma_7^T W\Gamma_7 - \Gamma_8^T W\Gamma_8 \\ &\quad +S^T\Gamma_9 + \Gamma_9^T S + \Gamma_{10}^T\Gamma_{10})\chi(t) < 0 \end{aligned}$$

$\Leftrightarrow$

$$
\begin{aligned}
&\Gamma_1^T P\Gamma_2 + \Gamma_2^T P\Gamma_1 + \mu\Gamma_4^T T\Gamma_4 \\
&+\mu\Gamma_3^T T^{-1}\Gamma_3 + \bar{\tau}^2\Gamma_5^T Q\Gamma_5 - \Gamma_6^T Q\Gamma_6 \\
&+\sum_{k=1}^{m}\Theta_k^T T_k^{-1}\Theta_k + diag\left\{-\sum_{k=1}^{m} T_k, 0,0,0,-T_1,-T_2,\cdots,-T_m,0\right\} + \Phi \\
&+diag\left\{R,-[1-\mu]R,0,0,0-\gamma^2 I\right\} + \Gamma_7^T W\Gamma_7 - \Gamma_8^T W\Gamma_8 \\
&+S^T\Gamma_9 + \Gamma_9^T S + \Gamma_{10}^T\Gamma_{10} < 0
\end{aligned}
\tag{3.1.38}
$$

Define

$$
\begin{aligned}
\theta \;:=\;& \Gamma_1^T P\Gamma_2 + \Gamma_2^T P\Gamma_1 + \mu\Gamma_4^T T\Gamma_4 - \Gamma_6^T Q\Gamma_6 \\
&+\Phi + \Gamma_7^T W\Gamma_7 - \Gamma_8^T W\Gamma_8 + S^T\Gamma_9 + \Gamma_9^T S \\
&+diag\left\{R-\sum_{k=1}^{m} T_k, [1-\mu]R, 0,0,-T_1,-T_2,\cdots,-T_m,-\gamma^2 I\right\}
\end{aligned}
$$

Then, (**3.1.38**) turns into

$$
\theta + \bar{\tau}^2\Gamma_5^T Q\Gamma_5 + \Gamma_{10}^T\Gamma_{10} + \sum_{k=1}^{m}\Theta_k^T T_k^{-1}\Theta_k + \mu\Gamma_3^T T^{-1}\Gamma_3 < 0 \tag{3.1.39}
$$

and one can write (**3.1.39**) as

$$
\theta + [\bar{\tau}\Gamma_5^T Q]Q^{-1}[\bar{\tau}\Gamma_5^T Q]^T + \Gamma_{10}^T I\Gamma_{10} + \sum_{k=1}^{m}\Theta_k^T T_k^{-1}\Theta_k + \mu\Gamma_3^T T^{-1}\Gamma_3 < 0 \tag{3.1.40}
$$

By Schur complement, (**3.1.40**) is equivalent to

$$
\Omega = \begin{bmatrix}
\theta & \bar{\tau}\Gamma_5^T Q & \Gamma_{10}^T & \Theta_1^T & \cdots & \Theta_m^T & \mu\Gamma_3^T \\
* & -Q & 0 & 0 & \cdots & 0 & 0 \\
* & * & -I & 0 & \cdots & 0 & 0 \\
* & * & * & -T_1 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
* & * & * & * & \cdots & -T_m & 0 \\
* & * & * & * & \cdots & * & -\mu T
\end{bmatrix} < 0, \tag{3.1.41}
$$

where (for $k = 1,2,....m$)

$$
\theta = \begin{bmatrix}
\theta^1 & \theta^2 & \theta^3 \\
* & \theta^4 & 0 \\
* & * & -\gamma^2 I
\end{bmatrix}, \theta^1 = \begin{bmatrix}
\theta_{11} & \theta_{12} & \theta_{13} & \theta_{14} \\
* & \theta_{22} & \theta_{23} & \theta_{24} \\
* & * & \theta_{33} & \theta_{34} \\
* & * & * & \theta_{34}
\end{bmatrix} \tag{3.1.42}
$$

$$
\begin{aligned}
\theta_{11} &= P_{11}A + A^T P_{11} + R - \sum_{k=1}^{m} T_k + W_{11} + S_1 + S_1^T \\
\theta_{12} &= P_{11}A_2 - S_1^T + S_2 \\
\theta_{13} &= A^T P_{12} + S_3 \\
\bar{\theta}_{14} &= \bar{P}_{12} - S_1^T + S_4 \\
\theta_{22} &= \mu T - [1-\mu]R - S_2 - S^T \\
\theta_{23} &= A_2 P_{12} - S_3 \\
\theta_{24} &= -S_2^T - S_4 \\
\theta_{33} &= -Q_{11} \\
\theta_{34} &= P_{22} - Q_{12} - S_3^T \\
\theta_{44} &= -Q_{22} - S_4 - S_4^T
\end{aligned}
$$

$$
\theta^2 = \begin{bmatrix} H_1 & H_2 & H_3 & \cdots & H_{m-1} & T_m + S_{m+4} \\ -S_5 & -S_6 & -S_7 & \cdots & -S_{m+4} & -S_{m+4} \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ -S_5 & -S_6 & S_7 & \cdots & -S_{m+4} & -S_{m+4} \end{bmatrix}, \theta^3 = \begin{bmatrix} P_{11}B_w \\ 0 \\ P_{12}^T B_w \\ 0 \end{bmatrix}
$$

$$
\theta^4 = \begin{bmatrix} G_1 & W_{23} - W_{12} & W_{24} - W_{13} & \cdots & W_{2m} - W_{1(m-1)} & -W_{1m} \\ * & G_2 & \cdots & W_{3m} - W_{2(m-1)} & -W_{2m} & \\ * & * & G_3 & \cdots & W_{4m} - W_{3(m-1)} & -W_{3m} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ * & * & * & \cdots & G_{m-1} & -W_{m-1,m} \\ * & * & * & \cdots & * & -W_{mm} - T_m \end{bmatrix}
$$

with $H_k := T_k + W_{1(k+1)} + S_{k+4}$ and $G_k := W_{(k+1)(k+1)} - W_{kk} - T_k$ for $k = 1 \cdots m-1$

Define $X := P_{11}^{-1}$ and $\Lambda := diag\{\Lambda_1, \Lambda_2, I, X, X, I, \Lambda_2, X\}$, $\Lambda_1 := diag\{X, X, X, X\}$, $\Lambda_2 := diag\{\underbrace{X, ..., X}_{m-times}\}$ Pre- and post- multiplying Ω by Λ and its tranpose, respectively, we have

$$
\bar{\Omega} = \Lambda \Omega \Lambda^T \qquad \textbf{(3.1.43)}
$$

Then

$$
\bar{\Omega} = \begin{bmatrix} \bar{\theta} & \bar{\tau}\bar{\Gamma}_5^T \bar{Q}^T & \bar{\Gamma}_8^T & \bar{\Theta}_1^T & \cdots & \bar{\Theta}_m^T & \bar{\Gamma}_3^T \\ * & -\bar{Q} & 0 & 0 & \cdots & 0 & 0 \\ * & * & -I & 0 & \cdots & 0 & 0 \\ * & * & * & -\bar{T}_1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ * & * & * & * & \cdots & -\bar{T}_m & 0 \\ * & * & * & * & \cdots & * & -\mu\bar{T} \end{bmatrix} < 0 \qquad \textbf{(3.1.44)}
$$

$$
\bar{\theta} = \begin{bmatrix} \bar{\theta}^1 & \bar{\theta}^2 & \bar{\theta}^3 \\ * & \bar{\theta}^4 & 0 \\ * & * & -\gamma^2 I \end{bmatrix} \qquad \textbf{(3.1.45)}
$$

where

$$\begin{aligned}
\bar{\theta}^1 &:= \Lambda_1\theta^1\Lambda_1\\
\bar{\theta}^2 &:= \Lambda_1\theta^2\Lambda_2\\
\bar{\theta}^3 &:= \Lambda_1\theta^3\\
\bar{\theta}_4 &:= \Lambda_2\theta_4\Lambda_2\\
M^T &:= diag\{\Lambda_1,\Lambda_2,I\}\left[\bar{\tau}\Gamma_5^T Q\right] diag\{X,X\}\\
\bar{\Theta}_k^T &:= diag\{\Lambda_1,\Lambda_2,I\}\Theta_k^T\Lambda_2(k=1,2,...m)\\
\bar{\Gamma}_9^T &:= diag\{\Lambda_1,\Lambda_2,I\}\Gamma_9^T\\
\bar{\Gamma}_3^T &:= diag\{\Lambda_1,\Lambda_2,I\}\Gamma_3^T X\\
\bar{Q} &:= diag\{X,X\}Qdiag\{X,X\}
\end{aligned} \tag{3.1.46}$$

and $\bar{T} := XTX$, $\bar{R} := XRX$, $\bar{T}_k := XT_kX$, $\bar{W}_{ij} := XW_{ij}X$, $\bar{S} = XS_iX$ $\bar{\Omega}$ can be decomposed as follows;

$$\bar{\Omega} = \bar{\Omega}_l + \bar{\Omega}_{nl} < 0 \tag{3.1.47}$$

where $\bar{\Omega}_l$ and $\bar{\Omega}_{nl}$ denotes linear and non-linear parts of $\bar{\Omega}$,respectively.

$$\bar{\Omega}_l = \begin{bmatrix} \bar{\theta}_l & M^T & \bar{\Gamma}_9^T & 0 & \cdots & 0 & \bar{\Gamma}_3^T\\ * & -\bar{Q} & 0 & 0 & \cdots & 0 & 0\\ * & * & -I & 0 & \cdots & 0 & 0\\ * & * & * & -\bar{T}_1 & \cdots & 0 & 0\\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\ * & * & * & * & \cdots & -\bar{T}_m & 0\\ * & * & * & * & \cdots & * & -\mu\bar{T} \end{bmatrix} < 0, \bar{\theta}_l = \begin{bmatrix} \bar{\theta}_l^1 & \bar{\theta}^2 & \bar{\theta}_l^3\\ * & \bar{\theta}^4 & 0\\ * & * & -\gamma^2 I \end{bmatrix} \tag{3.1.48}$$

$$\bar{\theta}_l^1 = \begin{bmatrix} \bar{\theta}_{11} & \bar{\theta}_{12} & \bar{\theta}_{13} & \bar{\theta}_{14}\\ * & \bar{\theta}_{22} & \bar{\theta}_{23} & \bar{\theta}_{24}\\ * & * & \bar{\theta}_{33} & \bar{\theta}_{34}\\ * & * & * & \bar{\theta}_{44} \end{bmatrix} \tag{3.1.49}$$

$$\begin{aligned}
\bar{\theta}_{11} &= AX + XA^T + \bar{R} + \bar{W}_{11} - \sum_{k=1}^{m}\bar{T}_k + \bar{S}_1 + \bar{S}_1^T\\
\bar{\theta}_{12} &= BY - \bar{S}_1^T + \bar{S}_2\\
\bar{\theta}_{13} &= \bar{S}_3\\
\bar{\theta}_{14} &= \bar{P}_{12} - \bar{S}_1^T + \bar{S}_4\\
\bar{\theta}_{22} &= -(1-\mu)\bar{R} + \mu\bar{T} - \bar{S}_2 - \bar{S}_2^T\\
\bar{\theta}_{23} &= -\bar{S}_3\\
\bar{\theta}_{24} &= -\bar{S}_2^T - \bar{S}_4\\
\bar{\theta}_{33} &= -\bar{Q}_{11}\\
\bar{\theta}_{34} &= \bar{P}_{22} - \bar{Q}_{12} - \bar{S}_3^T\\
\bar{\theta}_{44} &= -\bar{Q}_{22} - \bar{S}_4 - \bar{S}_4^T
\end{aligned}$$

$$\bar{\theta}^2 = \begin{bmatrix} \bar{H}_1 & \bar{H}_2 & \bar{H}_3 & \cdots & \bar{H}_{m-1} & \bar{T}_m + \bar{S}_{m+4} \\ -\bar{S}_5 & -\bar{S}_6 & -\bar{S}_7 & \cdots & -\bar{S}_{m+3} & -\bar{S}_{m+4} \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ -\bar{S}_5 & -\bar{S}_6 & -\bar{S}_7 & \cdots & -\bar{S}_{m+3} & -\bar{S}_{m+4} \end{bmatrix}, \ \theta_l^3 = \begin{bmatrix} B_w \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\bar{\theta}^4 = \begin{bmatrix} \bar{G}_1 & \bar{W}_{23} - \bar{W}_{12} & \bar{W}_{24} - \bar{W}_{13} & \cdots & \bar{W}_{2m} - \bar{W}_{1(m-1)} & -\bar{W}_{1m} \\ * & \bar{G}_2 & \bar{W}_{34} - \bar{W}_{23} & \cdots & \bar{W}_{3m} - \bar{W}_{2(m-1)} & -\bar{W}_{2m} \\ * & * & \bar{G}_3 & \cdots & \bar{W}_{4m} - \bar{W}_{3(m-1)} & -\bar{W}_{3m} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ * & * & * & \cdots & \bar{G}_{m-1} & -\bar{W}_{m-1,m} \\ * & * & * & \cdots & * & -\bar{W}_{mm} - \bar{T}_m \end{bmatrix}$$

with $\bar{H}_k = \bar{T}_k + \bar{W}_{1(k+1)} + \bar{S}_{k+4}$ and $\bar{G}_k = \bar{W}_{(k+1)(k+1)} - \bar{W}_{kk} - \bar{T}_k$ for $k = 1 \cdots m-1$

$$\begin{aligned} \bar{\Gamma}_{10} &= \begin{bmatrix} CX & 0 & 0 & 0 & \mathbb{P}_0^T & 0 \end{bmatrix}^T, \\ M_l &= \begin{bmatrix} \tau\bar{Q}_{11} & 0 & 0 & 0 & \mathbb{P}_0^T & 0 \\ \tau\bar{Q}_{12}^T & 0 & 0 & 0 & \mathbb{P}_0^T & 0 \end{bmatrix}^T, \\ \bar{\Gamma}_3 &= \begin{bmatrix} P_{12}^T & 0 & P_{22} & 0 & \mathbb{P}_0^T & 0 \end{bmatrix}^T, \end{aligned}$$

Moreover $k = 1, 2, ...m$

$$\bar{\Omega}_{nl} = \begin{bmatrix} \bar{\theta}_{nl} & M_{nl} & 0 & \bar{\Theta}_1^T & \cdots & \bar{\Theta}_m^T & 0 \\ * & 0 & 0 & 0 & \cdots & 0 & 0 \\ * & * & 0 & 0 & \cdots & 0 & 0 \\ * & * & * & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ * & * & * & * & \cdots & 0 & 0 \\ * & * & * & * & \cdots & * & 0 \end{bmatrix} < 0$$

$$\bar{\theta}_{nl} = \begin{bmatrix} \bar{\theta}_{nl}^1 & 0 & \bar{\theta}_{nl}^3 \\ * & 0 & 0 \\ * & * & 0 \end{bmatrix}, \bar{\theta}_{nl}^1 = \begin{bmatrix} 0 & 0 & XA^TX^{-1}\bar{P}_{12} & 0 \\ * & 0 & XA_2^TX^{-1}\bar{P}_{12} & 0 \\ * & * & 0 & 0 \\ * & * & * & 0 \end{bmatrix}, \theta_{nl}^3 = \begin{bmatrix} 0 \\ 0 \\ \bar{P}_{12}^TX^{-1}B_w \\ 0 \end{bmatrix}$$

$$[\bar{\tau}\bar{\Gamma}_5^T\bar{Q}^T]_{nl} = \begin{bmatrix} \bar{\tau}XA^TX^{-1}\bar{Q}_{12}^T & \bar{\tau}XA^TX^{-1}\bar{Q}_{22} \\ \bar{\tau}XA_2^TX^{-1}\bar{Q}_{12}^T & \bar{\tau}XA_2^TX^{-1}\bar{Q}_{22} \\ 0 & 0 \\ 0 & 0 \\ \mathbb{P}_0 & \mathbb{P}_0 \\ \bar{\tau}B_w^TX^{-1}\bar{Q}_{12}^T & \bar{\tau}B_w^TX^{-1}\bar{Q}_{22} \end{bmatrix}, \bar{\Theta}_k^T = \frac{k\bar{\tau}}{m} \begin{bmatrix} XA^TX^{-1}\bar{T}_k \\ XA_2^TX^{-1}\bar{T}_k \\ 0 \\ 0 \\ \mathbb{P}_0 \\ B_w^TX^{-1}\bar{T}_k \end{bmatrix}$$

One can decompose

$$\bar{\Omega}_{nl} = \Pi_1 X^{-1} \Pi_2^T + \Pi_2 X^{-1} \Pi_1^T, \qquad \textbf{(3.1.50)}$$

where

$$\begin{aligned}
\Pi_1 &= \left[\begin{array}{cccccccccc} AX & BY & 0 & 0 & \mathbb{P}_0^T & B_w & 0 & 0 & 0 & \mathbb{P}_0^T & 0 \end{array}\right]^T, \\
\Pi_2 &= \left[\begin{array}{ccccccccccc} 0 & 0 & P_{12} & 0 & \mathbb{P}_0^T & 0 & \tau\bar{Q}_{12}^T & \tau\bar{Q}_{22} & 0 & \mathbb{P}^T & 0 \end{array}\right]^T, \\
\mathbb{P} &= \left[\begin{array}{ccccc} \frac{\bar{\tau}}{m}\bar{T}_1 & \cdots & \frac{\bar{\tau}k}{m}\bar{T}_k & \cdots & \bar{\tau}\bar{T}_m \end{array}\right]^T, \\
\mathbb{P}_0 &= \left[\begin{array}{ccccc} 0 & \cdots & 0 & \cdots & 0 \end{array}\right]^T
\end{aligned}$$

By *Lemma* 1, we have

$$\Pi_1 X^{-1}\Pi_2^T + \Pi_2 X^{-1}\Pi_1^T \leq \Pi_1 Z^{-1}\Pi_1^T + \Pi_2\left(XZ^{-1}X\right)^{-1}\Pi_2^T \qquad \textbf{(3.1.51)}$$

where Z is a symmetric positive-definite matrice. Consider (**3.1.47**), we have

$$\bar{\Omega} \leq \bar{\Omega}_l + \Pi_1 Z^{-1}\Pi_2^T + \Pi_2 X Z^{-1} X^{-1}\Pi_2^T < 0 \qquad \textbf{(3.1.52)}$$

By Schur complement,(**3.1.52**) is equivalent to

$$\left[\begin{array}{ccc} \bar{\Omega}_l & \Pi_1 & \Pi_2 \\ * & -Z & 0 \\ * & * & -XZ^{-1}X \end{array}\right] < 0 \qquad \textbf{(3.1.53)}$$

it completes our proof.

3.2 Robust H_∞ Control in case of Uniform Delay Partitioning

Theorem 4

Consider Uncertain system with input delay (**1.2.8**) to (**1.2.10**). Given scalars $\bar{\tau} > 0$ and $\mu > 0$, the closed-loop system is asmptotically stable with H_∞ performance index γ for any time delay satisfying $\tau(t)$, $0 \leq \tau(t) \leq \bar{\tau}$ and $|\dot{\tau}(t)| < \mu$, if there exist symmetric positive definite matrices $X, \bar{P}, \bar{Q}, \bar{R}, \bar{T}, \bar{W}, \bar{T}_k(k = 1,..,m), Z$ and matrices $S_i(i = 1,..,m+4)$, Y with appropriate dimensions and scalars $\mu_1 > 0$, $\varepsilon_j > 0(j = 1,2,3)$,$\beta_j > 0(j = 1,2,3)$, $\sigma > 0$ satisfying below Lineer Matrix Inequalities (LMI). Then a desired H_∞ state feedback control law is given by $u(t) = Fx(t)$, where $F = YX^{-1}$:

$$\bar{P} = \left[\begin{array}{cc} X & \bar{P}_{12} \\ * & \bar{P}_{22} \end{array}\right] > 0, \; with\, X > 0, \quad \bar{Q} = \left[\begin{array}{cc} \bar{Q}_{11} & \bar{Q}_{12} \\ * & \bar{Q}_{22} \end{array}\right] > 0,$$

$$\bar{W} = \left[\begin{array}{cccc} \bar{W}_{11} & \bar{W}_{12} & \cdots & \bar{W}_{1m} \\ * & \bar{W}_{22} & \cdots & \bar{W}_{2m} \\ \vdots & \vdots & \vdots & \vdots \\ * & * & \cdots & \bar{W}_{mm} \end{array}\right] > 0$$

$$\Sigma\bar{R}\Sigma^T > 0, \; \Sigma(\bar{S}_1 + \bar{S}_1^T)\Sigma^T > 0, \Sigma O_{55}\Sigma^T > 0, \qquad \textbf{(3.2.54)}$$

$$\beta_1 I - \Sigma\bar{R}\Sigma^T > 0, \; \beta_2 I - \Sigma(\bar{S}_1 + \bar{S}_1^T)\Sigma^T > 0, \; \beta_3 I - \Sigma O_{55}\Sigma^T > 0, \qquad \textbf{(3.2.55)}$$

$$\Sigma Z\Sigma \geq \sigma I, \; \left[\begin{array}{cc} -\mu_1 I & X\Sigma^T \\ * & -\varepsilon_3 I \end{array}\right] < 0, \qquad \textbf{(3.2.56)}$$

$$\begin{bmatrix} \Psi_l & \Upsilon_2^T\Sigma & \Upsilon_3^T \\ * & -\varepsilon_1 I & 0 \\ * & * & -\varepsilon_2 I \end{bmatrix} < 0, \tag{3.2.57}$$

where

$$\Psi_l = \begin{bmatrix} \Psi & \hat{\Pi}_1 & \Pi_2 & \Pi_3 & \Pi_4 & \Pi_5 \\ * & -\sigma(1-\delta)^2 I & 0 & 0 & 0 & 0 \\ * & * & -XZ^{-1}X & 0 & 0 & 0 \\ * & * & * & N_1 & 0 & 0 \\ * & * & * & * & N_2 & 0 \\ * & * & * & * & * & N_3 \end{bmatrix} \tag{3.2.58}$$

$$\Psi = \begin{bmatrix} \tilde{\theta}_n & M_n & \tilde{\Gamma}_{10}^T & 0 & \cdots & 0 & \mu\tilde{\Gamma}_3^T \\ * & -\bar{Q} & 0 & 0 & \cdots & 0 & 0 \\ * & * & -I & 0 & \cdots & 0 & 0 \\ * & * & * & -\bar{T}_1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ * & * & * & * & \cdots & -\bar{T}_m & 0 \\ * & * & * & * & \cdots & * & -\mu\bar{T} \end{bmatrix} \tag{3.2.59}$$

$$\tilde{\theta}_n = \begin{bmatrix} \tilde{\theta}_n^1 & \bar{\theta}_n^2 & \tilde{\theta}_l^3 \\ * & \bar{\theta}^4 & 0 \\ * & * & -\gamma^2 I \end{bmatrix}, \ \tilde{\theta}_n^1 = \begin{bmatrix} \Psi_{11} & \Psi_{12} & \Psi_{13} & \Psi_{14} \\ * & \bar{\theta}_{22} & \bar{\theta}_{23} & \bar{\theta}_{24} \\ * & * & \bar{\theta}_{33} & \bar{\theta}_{34} \\ * & * & * & \bar{\theta}_{44} \end{bmatrix}, \ \theta_l^3 = \begin{bmatrix} B_w \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{3.2.60}$$

$$\bar{\theta}_n^2 = \begin{bmatrix} \Sigma\bar{H}_1 & \Sigma\bar{H}_2 & \Sigma\bar{H}_3 & \cdots & \Sigma\bar{H}_{m-1} & \Sigma(\bar{T}_m + \bar{S}_{m+4}) \\ -\bar{S}_5 & -\bar{S}_6 & -\bar{S}_7 & \cdots & -\bar{S}_{m+3} & -\bar{S}_{m+4} \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ -\bar{S}_5 & -\bar{S}_6 & -\bar{S}_7 & \cdots & -\bar{S}_{m+3} & -\bar{S}_{m+4} \end{bmatrix},$$

$$\bar{\theta}^4 = \begin{bmatrix} \bar{G}_1 & \bar{W}_{23} - \bar{W}_{12} & \bar{W}_{24} - \bar{W}_{13} & \cdots & \bar{W}_{2m} - \bar{W}_{1(m-1)} & -\bar{W}_{1m} \\ * & \bar{G}_2 & \bar{W}_{34} - \bar{W}_{23} & \cdots & \bar{W}_{3m} - \bar{W}_{2(m-1)} & -\bar{W}_{2m} \\ * & * & \bar{G}_3 & \cdots & \bar{W}_{4m} - \bar{W}_{3(m-1)} & -\bar{W}_{3m} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ * & * & * & \cdots & \bar{G}_{m-1} & -\bar{W}_{m-1,m} \\ * & * & * & \cdots & * & -\bar{W}_{mm} - \bar{T}_m \end{bmatrix}$$

with $\bar{H}_k = \bar{T}_k + \bar{W}_{1(k+1)} + \bar{S}_{k+4}$ and $\bar{G}_k = \bar{W}_{(k+1)(k+1)} - \bar{W}_{kk} - \bar{T}_k$ for $k = 1 \cdots m-1$

$$\begin{aligned} \hat{\Pi}_1 &= \begin{bmatrix} AX\Sigma^T & BY & 0 & 0 & \mathbb{P}_0^T & B_w & 0 & 0 & 0 & \mathbb{P}_0^T & 0 \end{bmatrix}^T \\ \Pi_2 &= \begin{bmatrix} 0 & 0 & P_{12} & 0 & \mathbb{P}_0^T & 0 & \tau\bar{Q}_{12}^T & \tau\bar{Q}_{22} & 0 & \mathbb{P}^T & 0 \end{bmatrix}^T \\ \Pi_3 &= \begin{bmatrix} \Sigma\bar{R}\Sigma^T & 0 & 0 & 0 & \mathbb{P}_0^T & 0 & 0 & 0 & 0 & \mathbb{P}_0^T & 0 \end{bmatrix}^T \\ \Pi_4 &= \begin{bmatrix} \Sigma(\bar{S}_1 + \bar{S}_1^T)\Sigma^T & 0 & 0 & 0 & \mathbb{P}_0^T & 0 & 0 & 0 & 0 & \mathbb{P}_0^T & 0 \end{bmatrix}^T \end{aligned}$$

$$
\begin{aligned}
\Pi_5 &= \begin{bmatrix} \Sigma O_{55}\Sigma^T & 0 & 0 & 0 & \mathbb{P}_0^T & 0 & 0 & 0 & 0 & \mathbb{P}_0^T & 0 \end{bmatrix}^T \\
\tilde{\Gamma}_{10} &= \begin{bmatrix} CX\Sigma^T & 0 & 0 & 0 & \mathbb{P}_0^T & 0 \end{bmatrix}^T \\
M_n &= \begin{bmatrix} \tau\bar{Q}_{11}\Sigma^T & 0 & 0 & 0 & \mathbb{P}_0^T & 0 \\ \tau\bar{Q}_{12}^T\Sigma^T & 0 & 0 & 0 & \mathbb{P}_0^T & 0 \end{bmatrix}^T \\
\tilde{\Gamma}_3 &= \begin{bmatrix} P_{12}^T\Sigma^T & 0 & P_{22} & 0 & \mathbb{P}_0^T & 0 \end{bmatrix}^T \\
\Upsilon_2 &= \begin{bmatrix} XA^T & -\bar{S}_1^T+\bar{S}_2 & \bar{S}_3 & \bar{P}_{12}-\bar{S}_1^T+\bar{S}_4 & \tilde{\mathbb{P}} & 0 & \Upsilon_2' \end{bmatrix} \\
\Upsilon_3 &= \begin{bmatrix} X\Sigma^T & 0 & 0 & 0 & \mathbb{P}_0^T & 0 & 0 & 0 & 0 & \mathbb{P}_0^T & 0 & 0 & 0 \end{bmatrix} \\
\Upsilon_2' &= \begin{bmatrix} \bar{\tau}\bar{Q}_{11} & \bar{\tau}\bar{Q}_{12} & XC^T & \mathbb{P}_0^T & \mu\bar{P}_{12} & XA^T & 0 \end{bmatrix}
\end{aligned}
$$

with $\tilde{\mathbb{P}} = \begin{bmatrix} \bar{H}_1 & \cdots & \bar{H}_k & \cdots & \bar{H}_{m-1} & \bar{T}_m+\bar{S}_{m+4} \end{bmatrix}$

$$
\begin{aligned}
\Psi_{11} &= AX\Sigma+\Sigma XA^T+\Sigma(\bar{R}+\bar{S}_1+\bar{S}_1+\bar{W}_{11}-\sum_{k=1}^{m}\bar{T}_k)^T\Sigma^T \\
&\quad +\varepsilon_3\delta^2 I+\mu_1\alpha^2 I+(\beta_1+\beta_2+\beta_3)\delta^2 I+\varepsilon_1\delta^2 I+\varepsilon_2\alpha^2 I \\
\Psi_{12} &= BY+\Sigma(-\bar{S}_1^T+\bar{S}_2) \\
\Psi_{13} &= \Sigma S_3 \\
\Psi_{14} &= \Sigma(\bar{P}_{12}-\bar{S}_1^T+\bar{S}_4) \\
O_{55} &= \bar{W}_{11}-\sum_{k=1}^{m}\bar{T}_k \\
N_1 &= -\beta_1 I+\Sigma\bar{R}\Sigma^T \\
N_2 &= -\beta_2 I+\Sigma(\bar{S}_1+\bar{S}_1^T)\Sigma^T \\
N_3 &= -\beta_3 I+\Sigma O_{55}\Sigma^T
\end{aligned}
$$

and the entries $\bar{\theta}_{ij}$, for $i,j=2,3,4$ of Ψ in (**3.2.60**) are given in (**3.1.4**). $\mathbb{P}_0$ and $\mathbb{P}$ are presented in *Theorem* 3.

Proof:

Consider the uncertain mechanical systems with input delay given in the equations (**1.2.8**) to (**1.2.10**) and the closed-loop system in (**1.2.15**) and (**1.2.16**). By substituting the matrices $\tilde{A}$, $\tilde{A}_2$, $\tilde{B}_w$ and $\tilde{C}$ into the inequality (**3.1.1**), instead of A, A_2 and B_w respectively we obtain the following inequality

$$
\Psi = \begin{bmatrix} \tilde{\Omega}_l & \tilde{\Pi}_1 & \Pi_2 \\ * & -Z & 0 \\ * & * & -XZ^{-1}X \end{bmatrix} < 0 \qquad \textbf{(3.2.61)}
$$

where

$$
\tilde{\Omega}_l = \begin{bmatrix} \tilde{\theta}_l & M_l & \bar{\Gamma}_{10}^T & 0 & \cdots & 0 & \mu\bar{\Gamma}_3^T \\ * & -\bar{Q} & 0 & 0 & \cdots & 0 & 0 \\ * & * & -I & 0 & \cdots & 0 & 0 \\ * & * & * & -\bar{T}_1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ * & * & * & * & \cdots & -\bar{T}_m & 0 \\ * & * & * & * & \cdots & * & -\mu\bar{T} \end{bmatrix} \qquad \textbf{(3.2.62)}
$$

$$\tilde{\theta}_l = \begin{bmatrix} \tilde{\theta}_l^1 & \bar{\theta}^2 & \tilde{\theta}_l^3 \\ * & \bar{\theta}^4 & 0 \\ * & * & -\gamma^2 I \end{bmatrix}, \tilde{\theta}_l^1 = \begin{bmatrix} \tilde{\theta}_{11} & \tilde{\theta}_{12} & \bar{\theta}_{13} & \bar{\theta}_{14} \\ * & \bar{\theta}_{22} & \bar{\theta}_{23} & \bar{\theta}_{24} \\ * & * & \bar{\theta}_{33} & \bar{\theta}_{34} \\ * & * & * & \bar{\theta}_{44} \end{bmatrix}, \tilde{\theta}_l^3 = \begin{bmatrix} \tilde{B}_w \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{aligned} \Pi_1 &= \begin{bmatrix} \tilde{A}X & \tilde{B}Y & 0 & 0 & \mathbb{P}_0^T & \tilde{B}_w & 0 & 0 & 0 & \mathbb{P}_0^T & 0 \end{bmatrix}^T, \\ \tilde{\theta}_{11} &= \tilde{A}X + X\tilde{A}^T + \bar{R} + \bar{W}_{11} - \sum_{k=1}^{m} \bar{T}_k + \bar{S}_1 + \bar{S}_1^T \\ \tilde{\theta}_{12} &= \tilde{B}Y - \bar{S}_1^T + \bar{S}_2 \end{aligned}$$

By pre- and post-multiplaying (**3.2.70**) by

$$\Xi = diag\{(\Sigma + \Delta_\Sigma), I, I, I, \mathbb{P}_I, I, I, I, \mathbb{P}_I, I, (\Sigma + \Delta_\Sigma), I\}, \quad \textbf{(3.2.63)}$$

where

$$\mathbb{P}_I := \begin{bmatrix} I & \cdots & I & \cdots & I \end{bmatrix}^T \quad \textbf{(3.2.64)}$$

with compatible dimension, and its tranpose respectively, we obtain

$$\Xi\Psi\Xi < 0 \quad \textbf{(3.2.65)}$$

Define $\theta_{11}' := (\Sigma + \Delta_\Sigma)\tilde{\theta}_{11}(\Sigma + \Delta_\Sigma)^T$, which is the $(1,1)^{th}$ block of $\Xi\tilde{\Theta}\Xi$. Then

$$\begin{aligned} \theta_{11}' &= (\Sigma + \Delta_\Sigma)[(\Sigma + \Delta_\Sigma)^{-1}(A + \Delta_A)X + X(A + \Delta_A)^T(\Sigma + \Delta_\Sigma)^{-T} + \\ &\quad + \bar{R} + \bar{W}_{11} - \sum_{k=1}^{m} \bar{T}_k + \bar{S}_1 + \bar{S}_1^T](\Sigma + \Delta_\Sigma)^T \end{aligned}$$

Similar to inequalities from (**2.3.43**) to (**2.3.48**), under the assumptions,for any $\beta_j > 0$ (j=1,2,3) and $\varepsilon_3 > 0$

$$\begin{array}{cc} \Sigma\bar{R}\Sigma^T > 0 & \beta_1 I - \Sigma\bar{R}\Sigma^T > 0 \\ \Sigma(\bar{S}_1 + \bar{S}_1^T)\Sigma^T > 0 & \beta_2 I - \Sigma(\bar{S}_1 + \bar{S}_1^T)\Sigma^T > 0 \\ \Sigma(\bar{W}_{11} - \sum_{k=1}^{m} \bar{T}_k)\Sigma^T > 0 & \beta_3 I - \Sigma(\bar{W}_{11} - \sum_{k=1}^{m} \bar{T}_k)\Sigma^T > 0 \end{array} \quad \textbf{(3.2.66)}$$

we have

$$\begin{aligned} \theta_{11}' \leq\ & AX\Sigma^T + \Sigma XA^T + AX\Delta_\Sigma^T + \Delta_\Sigma XA^T \\ & + \Delta_A X\Sigma^T + \Sigma X\Delta_A^T + \varepsilon_3\delta^2 I + \mu_1\alpha^2 I \\ & + \Sigma(\bar{R} + \bar{S}_1 + \bar{S}_1^T + O_{55})\Sigma^T + \Sigma\bar{R}\Sigma^T(\beta_1 I - \Sigma\bar{R}\Sigma^T)^{-1}\Sigma\bar{R}\Sigma^T + \beta_1\delta^2 I \\ & + \Sigma(\bar{S}_1 + \bar{S}_1^T)\Sigma^T(\beta_2 I - \Sigma(\bar{S}_1 + \bar{S}_1^T)\Sigma^T)^{-1}\Sigma(\bar{S}_1 + \bar{S}_1^T)\Sigma^T + \beta_2\delta^2 I \\ & + \Sigma O_{55}\Sigma^T(\beta_3 I - \Sigma O_{55}\Sigma^T)^{-1}\Sigma O_{55}\Sigma^T + \beta_3\delta^2 I \end{aligned} \quad \textbf{(3.2.67)}$$

where $O_{55} := \bar{W}_{11} - \sum_{k=1}^{m} \bar{T}_k$. By inequality at (**2.3.49**), providing $\Sigma Z \Sigma \geq \sigma I$, for any $\sigma > 0$, we obtain,

$$(\Sigma + \Delta_\Sigma)(-Z)(\Sigma + \Delta_\Sigma)^T \leq -\sigma(1-\delta)^2 I \qquad \textbf{(3.2.68)}$$

(**3.2.65**) can be seperated into nominal and uncertain parts as

$$\Xi \Psi \Xi < \tilde{\Psi}_n + \tilde{\Psi}_u < 0, \qquad \textbf{(3.2.69)}$$

where

$$\tilde{\Psi} = \begin{bmatrix} \tilde{\Omega}_n & \hat{\Pi}_1 & \Pi_2 \\ * & -\sigma(1-\delta)^2 I & 0 \\ * & * & -XZ^{-1}X \end{bmatrix} < 0 \qquad \textbf{(3.2.70)}$$

$$\tilde{\Omega}_n = \begin{bmatrix} \tilde{\theta}_n & M_n & \tilde{\Gamma}_{10}^T & 0 & \cdots & 0 & \mu\tilde{\Gamma}_3^T \\ * & -\bar{Q} & 0 & 0 & \cdots & 0 & 0 \\ * & * & -I & 0 & \cdots & 0 & 0 \\ * & * & * & -\bar{T}_1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ * & * & * & * & \cdots & -\bar{T}_m & 0 \\ * & * & * & * & \cdots & * & -\mu\bar{T} \end{bmatrix} \qquad \textbf{(3.2.71)}$$

$$\tilde{\theta}_n = \begin{bmatrix} \tilde{\theta}_n^1 & \bar{\theta}_n^2 & \tilde{\theta}_l^3 \\ * & \bar{\theta}^4 & 0 \\ * & * & -\gamma^2 I \end{bmatrix}, \ \tilde{\theta}_n^1 = \begin{bmatrix} \Psi'_{11} & \Psi_{12} & \Psi_{13} & \Psi_{14} \\ * & \bar{\theta}_{22} & \bar{\theta}_{23} & \bar{\theta}_{24} \\ * & * & \bar{\theta}_{33} & \bar{\theta}_{34} \\ * & * & * & \bar{\theta}_{44} \end{bmatrix}, \ \theta_l^3 = \begin{bmatrix} B_w \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad \textbf{(3.2.72)}$$

$$\begin{aligned} \hat{\Pi}_1 &= \begin{bmatrix} AX\Sigma^T & BY & 0 & 0 & \mathbb{P}_0^T & B_w & 0 & 0 & 0 & \mathbb{P}_0^T & 0 \end{bmatrix}^T, \\ \tilde{\Gamma}_{10} &= \begin{bmatrix} CX\Sigma^T & 0 & 0 & 0 & \mathbb{P}_0^T & 0 \end{bmatrix}^T, \\ M_n &= \begin{bmatrix} \tau\bar{Q}_{11}\Sigma^T & 0 & 0 & 0 & \mathbb{P}_0^T & 0 \\ \tau\bar{Q}_{12}^T\Sigma^T & 0 & 0 & 0 & \mathbb{P}_0^T & 0 \end{bmatrix}^T, \\ \tilde{\Gamma}_3 &= \begin{bmatrix} P_{12}^T\Sigma^T & 0 & P_{22} & 0 & \mathbb{P}_0^T & 0 \end{bmatrix}^T, \end{aligned} \qquad \textbf{(3.2.73)}$$

$$\begin{aligned} \Psi'_{11} &= AX\Sigma + \Sigma X A^T + \Sigma(\bar{R} + \bar{S}_1 + \bar{S}_1^T + O_{55})\Sigma^T \\ &\quad + \varepsilon_3\delta^2 I + \mu_1\alpha^2 I + (\beta_1 + \beta_2 + \beta_2)\delta^2 I \\ &\quad + \Sigma\bar{R}\Sigma^T(\beta_1 I - \Sigma\bar{R}\Sigma^T)^{-1}\Sigma\bar{R}\Sigma^T \\ &\quad + \Sigma(\bar{S}_1 + \bar{S}_1^T)\Sigma^T(\beta_2 I - \Sigma(\bar{S}_1 + \bar{S}_1^T)\Sigma^T)^{-1}\Sigma(\bar{S}_1 + \bar{S}_1^T)\Sigma^T \\ &\quad + \Sigma O_{55}\Sigma^T(\beta_3 I - \Sigma O_{55}\Sigma^T)^{-1}\Sigma O_{55}\Sigma^T \\ \Psi_{12} &= BY + \Sigma(-\bar{S}_1^T + \bar{S}_2) \\ \Psi_{13} &= \Sigma S_3 \\ \Psi_{14} &= \Sigma(\bar{P}_{12} - \bar{S}_1^T + \bar{S}_4) \end{aligned}$$

and

$$\Psi_u = \Upsilon_1^T \Delta_\Sigma \Upsilon_2 + \Upsilon_2^T \Delta_\Sigma^T \Upsilon_1 + \Upsilon_1^T \Delta_A \Upsilon_3 + \Upsilon_3^T \Delta_A^T \Upsilon_1, \qquad \textbf{(3.2.74)}$$

where

$$
\begin{aligned}
\Upsilon_1 &= \begin{bmatrix} I & 0 & 0 & 0 & \mathbb{P}_0^T & 0 & 0 & 0 & 0 & \mathbb{P}_0^T & 0 & 0 & 0 \end{bmatrix}, \\
\Upsilon_2 &= \begin{bmatrix} XA^T & -\bar{S}_1^T + \bar{S}_2 & \bar{S}_3 & \bar{P}_{12} - \bar{S}_1^T + \bar{S}_4 & \tilde{\mathbb{P}} & 0 & \Upsilon_2' \end{bmatrix} \\
\Upsilon_3 &= \begin{bmatrix} X\Sigma^T & 0 & 0 & 0 & \mathbb{P}_0^T & 0 & 0 & 0 & 0 & \mathbb{P}_0^T & 0 & 0 & 0 \end{bmatrix}.
\end{aligned}
$$

with

$$
\begin{aligned}
\tilde{\mathbb{P}} &:= \begin{bmatrix} \bar{H}_1 & \cdots & \bar{H}_k & \cdots & \bar{H}_{m-1} & \bar{T}_m + \bar{S}_{m+4} \end{bmatrix} \\
\Upsilon_2' &:= \begin{bmatrix} \bar{\tau}\bar{Q}_{11} & \bar{\tau}\bar{Q}_{12} & XC^T & \mathbb{P}_0^T & \mu\bar{P}_{12} & XA^T & 0 \end{bmatrix}
\end{aligned}
$$

Since $0 \le \left\|\Delta_\Sigma \Sigma^{-1}\right\| \le \delta \le 1,\ 0 \le \|\Delta_A\| \le \alpha$, we have

$$
\begin{aligned}
\Psi_u &= (\delta\Upsilon_1)^T(\frac{1}{\delta}\Delta_\Sigma\Sigma^{-1})(\Sigma\Upsilon_2) + (\Sigma\Upsilon_2)^T(\frac{1}{\delta}\Delta_\Sigma\Sigma^{-1})^T(\delta\Upsilon_1) \\
&\quad +(\alpha\Upsilon_1)^T((\frac{1}{\alpha}\Delta_A)\Upsilon_3 + \Upsilon_3^T(\frac{1}{\alpha}\Delta_A)^T(\alpha\Upsilon_1) \\
&\le \frac{1}{\varepsilon_1}(\Sigma\Upsilon_2)^T(\Sigma\Upsilon_2) + \delta^2\varepsilon_1\Upsilon_1^T\Upsilon_1 + \frac{1}{\varepsilon_2}\Upsilon_3^T\Upsilon_3 + \alpha^2\varepsilon_2\Upsilon_1^T\Upsilon_1 \qquad \textbf{(3.2.75)}
\end{aligned}
$$

Now, define

$$
\begin{aligned}
\Psi_{11} &:= AX\Sigma + \Sigma XA^T + \Sigma(\bar{R} + \bar{S}_1 + \bar{S}_1^T + \bar{W}_{11} - \sum_{k=1}^{m} \bar{T}_k)^T\Sigma^T \\
&\quad +\varepsilon_3\delta^2 I + \mu_1\alpha^2 I + (\beta_1+\beta_2+\beta_3)\delta^2 I + \varepsilon_1\delta^2 I + \varepsilon_2\alpha^2 I, \qquad \textbf{(3.2.76)}
\end{aligned}
$$

change the $(1,1)^{th}$-entry Ψ_{11}' of Ψ' by Ψ_{11} and denote the matrix obtained by Ψ, which is given in (**2.3.36**). Because of nonlinear part of Ψ_{11}' by Schur's complement the matrix Ψ_n is equivalent to the following matrix

$$
\Psi_l = \begin{bmatrix}
\Psi & \hat{\Pi}_1 & \Pi_2 & \Pi_3 & \Pi_4 & \Pi_5 \\
* & -\sigma(1-\delta)^2 I & 0 & 0 & 0 & 0 \\
* & * & -XZ^{-1}X & 0 & 0 & 0 \\
* & * & * & N_1 & 0 & 0 \\
* & * & * & * & N_2 & 0 \\
* & * & * & * & * & N_3
\end{bmatrix} < 0, \qquad \textbf{(3.2.77)}
$$

with $N_1 := -\beta_1 I + \Sigma\bar{R}\Sigma^T$, $N_2 := -\beta_2 I + \Sigma(\bar{S}_1 + \bar{S}_1^T)\Sigma^T$ and $N_3 := -\beta_3 I + \Sigma O_{55}\Sigma^T$, where $\Pi_i's$ (i=1...5) are the matrices given in (**3.2.61**). Then, by Schur's complement, the matrix (**3.2.77**), the inequality (**3.2.75**) and the matrix Ψ_{11} in (**3.2.76**) imply the inequality in (**3.2.70**). Thus, the proof is completed.

4. ALGORITHM AND EXAMPLE

In this part, it is going to be stated an algorithm for the numerical solution of the example in terms of the theorems in this thesis.

4.1 Linearization of LMI's

In $Theorem$ 1, there is a nonconvex term which is $-XZ^{-1}X$ in the LMI (**2.2.20**). To remove thxis structure, we introduced a new positive definite matrix L satisfying $-XZ^{-1}X < -L$. By taking the inverse of this inequality and by defining $J := X^{-1}$,$N := Z^{-1}$ and $M := L^{-1}$ the inequality $M - JN^{-1}J \geq 0$ is obtained. Because of Schur complement, this inequality is written following form

$$\begin{bmatrix} M & J \\ * & N \end{bmatrix} \geq 0 \qquad \textbf{(4.1.1)}$$

Since the equations

$$LM = XJ = ZN = I$$

are all nonlinear the following LMI's are obtained

$$\begin{bmatrix} L & I \\ * & M \end{bmatrix} \geq 0, \begin{bmatrix} X & I \\ * & J \end{bmatrix} \geq 0, \begin{bmatrix} Z & I \\ * & N \end{bmatrix} \geq 0 \qquad \textbf{(4.1.2)}$$

Now, we refer to the cone complementarity linearization (CCL) algorithm given in [19]. By the CCL algorithm, we suggested the following non linear minimization problem:

$$\begin{aligned} &Minimize\ \ trace(XJ + ZN + LM), \\ &Subject\ to\ (\textbf{2.2.20}),\ (\textbf{4.1.1})\ and\ (\textbf{4.1.2}) \end{aligned}$$

Because of the dimensions of the matrices $L, M, X, J, Z, N \in \Re^{n\times n}$ we expect that

$$trace(LM + XJ + ZN) = 3n$$

Then the solution is feasible and

$$LM = XJ = ZN = I$$

In terms of the CCL method, we can find the suboptimal maximal delay using an iterative algorithm presented below.

4.2 Alghoritm

Step 1. Find a feasible set

$$(X,J,Z,N,L,M)$$

satisfying the inequality in (**2.2.20**), (**4.1.1**) and (**4.1.2**). Set $k=0$.
Step 2. Solve the following LMI problem for the variables (X,J,Z,N,L,M)

$$\begin{array}{l} Minimize\ \ trace(X_kJ+XJ_k+Z_kN+ZN_k+L_kM+LM_k) \\ Subject\ to\ (\mathbf{2.2.20}),\ (\mathbf{4.1.1})\ and\ (\mathbf{4.1.2}) \end{array}$$

Step 3. If a stopping criterion is satisfied, exit. Otherwise set $X_k=X$,$Z_k=Z$,$L_k=L$,$J_k=J$,$N_k=N$,$M_k=M$ and go to $Step\ 2$
Same algorithm is applicable for other theorems in the thesis.

4.3 Remark 1

To remove nonlinearity of $-XZ^{-1}X$, we can use the following inequality

$$-XZ^{-1}X<Z-2X \qquad (\mathbf{4.3.3})$$

In the numerical solutions, it is observed that the Algorithm 4.2 gives better results than this inequality. But, because of the data of some problems Algorithm 4.2 does not give any result when the amount of the upper bound of delay is comparable big. That case the entries of positive definite matrices are also great numbers and then Algorithm 4.2 cannot minimize the trace(XJ+ZN+LM). In the following the inequality given in (**4.3.3**) is proved.
Assume that X and Z are two symmetric positive definite matrices of same dimension. Then, we have

$$[X-Z]^TZ^{-1}[X-Z]>0$$

Thus

$$[X-Z]^TZ^{-1}[X-Z]=XZ^{-1}X-X-X+Z=XZ^{-1}X-2X+Z>0$$

it implies

$$-(XZ^{-1}X-2X+Z)=-XZ^{-1}X+2X-Z<0\Rightarrow -XZ^{-1}X<Z-2X.$$

4.4 Example

This section represent a example to demonstrate the applicability of the theorems in the thesis. Consider the system (**1.2.1**) with

$$M=\begin{bmatrix} 1.1 & 0 & 0 \\ 0 & 1.8 & 0 \\ 0 & 0 & 1.6 \end{bmatrix}, C=\begin{bmatrix} 1.2 & -0.6 & 0 \\ -0.6 & 1.2 & -0.6 \\ 0 & -0.6 & 0.6 \end{bmatrix}, K=\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

The input and output matrices in (**1.2.8**) and (**1.2.9**)

$$\begin{array}{rcl} B &=& \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}^T,\ B_w=\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0.1 \end{bmatrix}^T, \\ C &=& \begin{bmatrix} 0.1 & 0.1 & 0.5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.1 & 0.1 & 0.5 \end{bmatrix} \end{array}$$

We first consider the situations without uncertainties, that is, $\alpha = 0$ and $\delta = 0$. For each γ, one can find the limiting (largest) the time delay $\bar{\tau}$ satisfying the LMI's in Theorems in The Thesis are feasible. Similarly, for each $\bar{\tau}$, one can find the limiting (smallest) performance level γ and the corresponding results are given in Tables.

Table 4.1: Theorem 1

$\alpha = 0, \delta = 0$					
γ	0.1	0.2	0.5	0.8	1.0
Du at all, [3] max $\bar{\tau}$	0.029	0.072	0.140	0.158	0.164
$Theorem1^{*}$ max $\bar{\tau}$	0.272	0.953	1.278	1.327	1.341
$Theorem1^{**}$ max $\bar{\tau}$	0.365	0.989	1.786	2.198	2.406
τ	0.01	0.02	0.05	0.10	0.15
Du at all, [3] min γ	0.052	0.079	0.147	0.278	0.624
$Theorem1^{*}$ min γ	0.053	0.070	0.069	0.075	0.074
$Theorem1^{**}$ min γ	0.071	0.071	0.073	0.076	0.080

In Table 1 , $Theorem1^{*}$ and $Theorem1^{**}$ denote the results of the Theorem 1 with the CCL Alghoritm and Remark in 4.2,respectively . For the CCL Alghoritm, iteration number is taken as 4 and error is taken as lesser than 0.01 (expected value; trace(LM+XJ+ZN)=18). As it is seen in Figure 2 and Figure 3, close-loop system is more stable than open-loop system.

Table 4.2: Theorem 3

$\alpha = 0, \delta = 0$				
Theorem 3	Du [3]	m=1	m=2	m=4
γ=0.1 , max $\bar{\tau}$	0.029	0.405	0.447	0.457
$\bar{\tau}$=0.15 ,min γ	0.624	0.074	0.075	0.077

In Table 2 , some results of the Theorem 3 are presented. Results in Table 2 are computed by using Remark in 4.2 . Moreover, it is seen that as number of the partition m increases, it yields better upper bound, the system turns into more stable condition. In Tables 1 and 2, $\mu = 0$. Some results are graphically illustrated below. (In all graphics, it is taking as $w(t) = 0$.)

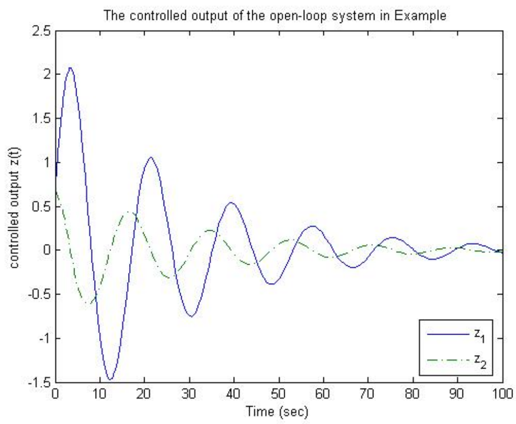

Figure 4.1: Open-loop System.

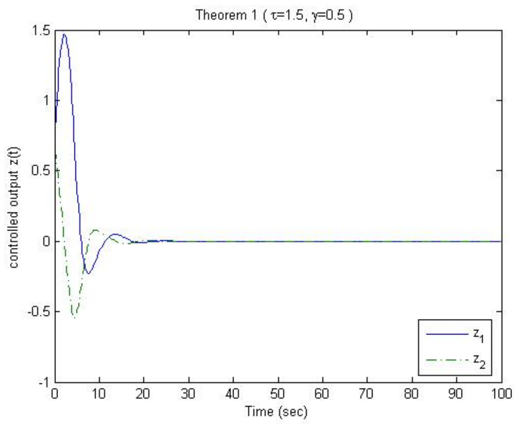

Figure 4.2: Closed-loop System (Theorem 1,$\bar{\tau}$=1.5, γ=0.5).

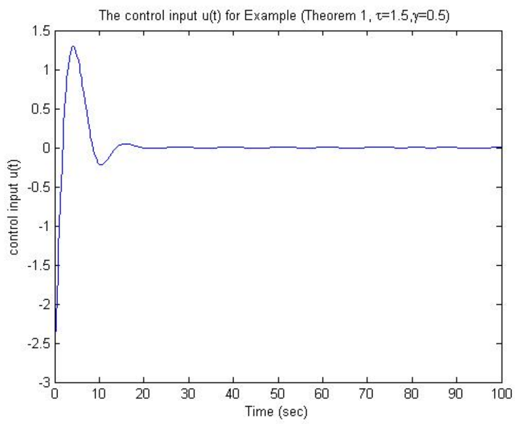

Figure 4.3: The control input (Theorem 1,$\bar{\tau}$=1.5, γ=0.5).

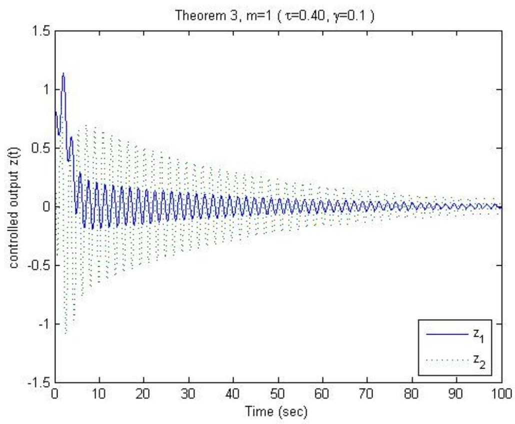

Figure 4.4: Theorem 3,m=1,$\bar{\tau}$=0.40, γ=0.1.

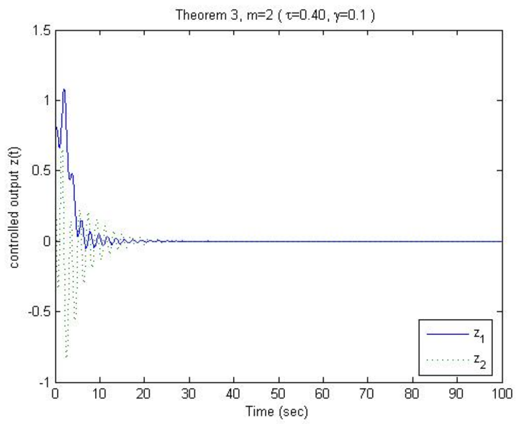

Figure 4.5: Theorem 3,m=2,$\bar{\tau}$=0.40, γ=0.1.

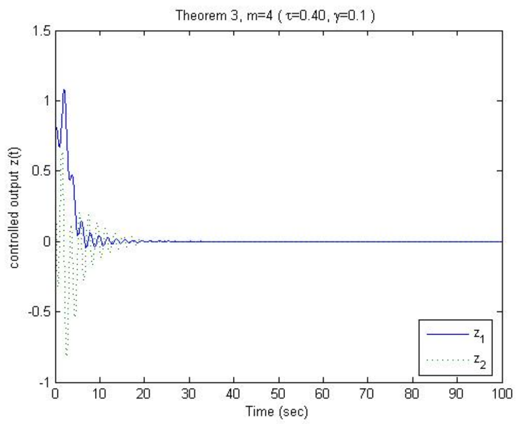

Figure 4.6: Theorem 3, m=4,$\bar{\tau}$=0.40, γ=0.1.

4.5 Remark 2

Finally, during the computations it is observed that the results do not depend on the upper bound of the derivative of the delay, or the difference is of the minimum level. In that case variations on the upper bound of the derivative of the delay effect the stabilization of the system very slightly. To illustrate this situation, consider the example given here for $\gamma = 1$ and $\bar{\tau} = 2.406$ and solve the problem by using $Theorem$ 1 and *Remark* 1 for $\mu = 0$ and $\mu = 2000$. That case feedback matrices are obtained as follows.

$$F_1 = \begin{bmatrix} 0.0046 & 0.0101 & 0.0227 & -0.1818 & -0.5545 & -0.6090 \end{bmatrix} \; (\mu = 0)$$
$$F_2 = \begin{bmatrix} 0.0045 & 0.0101 & 0.0231 & -0.1816 & -0.5544 & -0.6094 \end{bmatrix} \; (\mu = 2000)$$

5. CONCLUSION

In this thesis, robust H_∞-control of the mechanical system with input delay is investigated by means of augmented Lyapunov-Krasovskii functionals in delay dependent case. In order to improve the solution, a new augmented Lyapunov-Krasovskii functionals are introduced according to partitioned delay interval. The results are compared with the results in the previous works. It is observed that the results are better than previous ones in the literature. By finding sufficent conditions as Linear Matrix Inequalities for stabilization appropriate state feedback control law to work in certain delay interval is obtained. All these inequalities are solved by using *MATLAB*.

Finally, relevant algorithm and remark is introduced to linearize some nonlinear terms in LMI's. The results are compared with each other and the results in [3]. At the end, the results are graphically illustrated.

REFERENCES

[1] **Hale J**, Berlin 1977. Theory of Functional Differantial Equations, *Springer*.

[2] **Malek-Zavarei M., Jamshidi M.**,North-Holland: Amsterdam 1987. Time-Delay Systems: Analysis Optimization and Applications.

[3] **Du, H. and Zhang, N.**, 2007. H_∞ control of active vehicle suspensions with actuator time delay, *Journal of Sound and Vibration*, **301**, 236–252.

[4] **Parlakçı,A.M.N.**, 2006. Improved robust stability criteria and design of robust stabilizing controller for uncertain linear time-delay systems, *Int. J. Robust Nonlinear Control*, **16**(13), 599–636.

[5] **Du, H., Lam, J. and Sze, K.Y.**, 2005. H_∞ disturbance attentuation for uncertain mechanical sytems with time delay , *Trans.of the Ins. of Measurement and Control*, **27**, 37–52.

[6] **Zhang, X.M., Wu, M., She, J.H. and He Y.**, 2005. Delay-dependant stabilization of linear systems with time-varying state and input delays, *Automatica* , **41**, 1405-1412.

[7] **Wu M, He Y, She J-H Liu G-P**, 2004. Delay-dependent criteria for robust stability of time-varying delay systems.*Automatica*, **40**,1435-1439.

[8] **Kim J-H.**, 2001. Delay and its derivative dependent robust stability of time-delayed linear systems with uncertainity *IEEE Transactions on Automatic Control*, **46**, 789-792.

[9] **Han Q-L.**, 2002.Robust stability of uncertian delay-differential systems of neutral type, *IEE Transactions on Automatic Control*, **34**, 1199-1203.

[10] **Park P.**, 1999. A delay-dependent stability criterion for systems with uncertain time-invariant delays*IEEE Transactions on Automatic Control*, **44**, 876-877.

[11] **Cheres E, Gutman S, Palmor ZJ.**, 1989.Robust stabilization of uncertian dynamic systems including state delay, *IEEE Transactions on Automatic Control*, **34**, 1199-1203.

[12] **Gu K**, 1999. A generalized discretization scheme of Lyapunov functional in the stability of linear uncertian time-delay systems, *International Journal of Robust and Nonlinear Control*, **9**(1), 1-14.

[13] **Gu K**, 1997. Discretized LMI set in the stability problem of linear uncertian time-delay systems, *International Journal of Control*, **68**(4), 923-934.

[14] **Skelton RE, Iwasaki T, Grigoriadis K.**, Bristol, U.S.A. 1998. A Unified Algebraic Approach to Linear Control Desing, *Taylor and Fransic*.

[15] **Boukas E.** , June 2008. Free-Weighting Matrices Delay-Dependant Stabilization For System With Time-Varying Delays *ICIC Express Letters*, **2**,2. 167-173.

[16] **Xu S., Lam J., Yang C., Verriest E.I.**,2003.An LMI Approach To Guaranteed Cost Control For Uncertain Linear Neutral Delay Systems. *Int.J. of Robust and Nonlinear Control* , **13**, 35-53.

[17] **Park J.H.** ,2004. Design of a Dynamic Output Feedback Controller For A Class Of Neutral Systems With Discrete And Distributed Delays. *IEE Proc. Control And App.*, **151**, 610-614.

[18] **Boyd S.,Ghaoui L.E., Feron E., Balakrishnan V.** ,1994. Linear Matrix Inequalities System And Control Theory, *Society for Industrial and Applied Mathematics*,Philadelphia.

[19] **El Ghaoui L, Oustry F, Ait Rami M**, 1997.A cone complementarity linearization algrithm for static output-feedback and related problems, *IEEE Transactions on Automatic Control*, **42**(8), 1171-1176.

CURRICULUM VITAE

Candidate's full name : Ahmet Furkan EMREHAN

Place and date of birth : İSTANBUL-01.02.1985

Permanent Address : Şeyhraşit Cad. No:51/3 Düğmeciler Mah.
EYÜP/İSTANBUL

Universities and
Colleges attended : İSTANBUL UNIVERSITY 2003-2007

İSTANBUL TECHNİCAL UNIVERSITY 2009-2011