İSTANBUL TECHNICAL UNIVERSITY ★ INSTITUTE OF SCIENCE AND TECHNOLOGY

SUZUKI 2-GROUPS

PH.D Thesis by
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SUZUKI 2-GRUPLARI

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FOREWORD

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SUZUKI 2-GROUPS

SUMMARY

Suzuki 2-groups are studied: abelian of arbitrary exponent and nonabelian of exponent 4. For any Suzuki 2-group, one can associate a ground field which makes the theory of Suzuki 2-groups deeper. Let (G,T) be a Suzuki 2-group of exponent 2^n and I be the subgroup of involutions in G. We put $K = T/C_T(I) \cup \{0\}$ where 0 is a new symbol. Then we can define the multiplication on K by extending the group operation of $T/C_T(I)$ and addition on K by pulling back the group operation of I to K. Then K becomes a field where $I \rtimes T/C_T(I)$ is isomorphic to the affine group $K^+ \rtimes K^\star$ with K^+ and $K^\star$ are the additive and the multiplicative groups in K, respectively. Classification of a Suzuki 2-group (G,T), means determining the structure of the group G which admits such an action of T.

We proved uniqueness of an abelian Suzuki 2-group (G,T) of any given exponent 2^n over a perfect ground field K, by showing that G is isomorphic to the algebraic group $K \times ... \times K = K^n$ over K and G is an extension of the field K by K^{n-1}.

Then, we analyzed the role of "perfectness" assumption on the field, in case of exponent 4, we provide a classification of abelian Suzuki 2-groups of exponent 4 over an arbitrary field in terms of a certain cohomological invariant. We proved that there is a one-to-one correspondence between the family of abelian Suzuki 2-groups of exponent 4 over a field K of characteristic 2 and elements of a certain subset of the 2-dimensional cohomology group $H^2(K,K)$.

Nonabelian Suzuki 2-groups G of exponent 4 are classified into several types. One type appears when G is free over a perfect field K such that for any element $g \in G$, the subgroup $< g^T >$ is abelian. We call G a quasi-abelian Suzuki 2-group and give the classification in terms of a map $f : K \times K \to K$ satisfying certain properties. Another type of G, which we call smart Suzuki 2-group, is a nonabelian Suzuki 2-group of exponent 4 where T acts freely and transitively on G/I. In this case, we introduce a pair of fields K and k of characteristic 2 which we call the wide and the narrow fields associated to G, respectively. We describe the group structure in terms of the characteristic map $\alpha : K \to k$. We provide also some examples of nonabelian Suzuki 2-groups and give some criteria for the existence of their linear presentation by 3×3 matrices.

SUZUKİ 2-GRUPLARI

ÖZET

Sonsuz Suzuki 2-gruplarının iki türü incelenmiştir: Herhangi bir mertebede abelyen gruplar ve dördüncü mertebede abelyen olmayan gruplar. Bir Suzuki 2-grubu, Suzuki 2-gruplar teorisini derinleştiren bir temel cisim ile eşleştirilebilir. (G,T) mertebesi 2^n olan bir Suzuki 2-grubu ve I, G'nin involusyon altgrubu olsun. $K = T/C_T(I) \cup \{0\}$ olarak adlandırıp, K üzerinde çarpma işlemini $T/C_T(I)$ grubunun işlemini genişleterek ve toplama işlemini ise, involusyon grubunun işlemini kullanarak tanımlarsak, K bir cisim oluşturur ve $I \rtimes T/C_T(I)$, $K^+ \rtimes K^\star$ ile izomorfik olur. Bir Suzuki 2-grubu (G,T)'nin sınıflandırılması, T'nin üzerinde etki ettiği G grubunun yapısının belirlenmesidir.

Abelyen Suzuki 2-grupları için belli bir kohomolojik değişmez cinsinden sınıflandırma yapılmıştır. Bu sınıflandırmada, özel olarak, yetkin bir cisim K üzerinde, mertebesi 2^n olan Suzuki 2-grupları (G,T)'nin tekliği ispatlanmıştır ve G'nin cebirsel grup $K \times ... \times K = K^n$ ile izomorf oldugu; G'nin, K cisminin K^{n-1} ile bir genişlemesi olduğu gösterilmiştir. Karakteristiği 2 olan herhangi bir cisim K üzerinde, mertebesi 4 olan abelyen Suzuki 2-grupları ile ikinci kohomolojik grup $H^2(K,K)$'nin belli bir altkümesinin elemenları arasında birebir eşleştirme olduğu ispatlanmıştır.

Dördüncü mertebeden abelyen olmayan Suzuki 2-grupları birkaç farklı tipte sınıflandırılmıştır. (G,T), yetkin bir cisim K üzerinde, 4. mertebeden abelyen olmayan bir Suzuki 2-grup ve G'nin her elemanı g için, $< g^T >$ altgrubu abelyen ise, G quasi-abelyen Suzuki 2-grup olarak adlandırılmıştır ve sınıflandırması, belli şartları sağlayan $f : K \times K \to K$ fonksiyonu cinsinden yapılmıştır. Smart Suzuki 2-grup olarak adlandırılan başka bir çeşit grup ise T'nin, G/I üzerinde serbest ve geçişli etki ettiği gruptur. Bir Smart Suzuki 2-grubu için karakteritiği 2 olan bir geniş cisim K ve bir dar cisim k tanımlanmıştır ve bu grupların yapısı grubun karakteristik fonksiyonu $\alpha : K \to k$ kullanılarak açıklanmıştır. Abelyen olmayan Suzuki 2-grup örnekleri verilmiş ve bu grupların 3×3 matrislerle temsil edilebilmeleri için bazı kriterler verilmiştir.

1. INTRODUCTION

1.1 Background

Finite Suzuki 2-groups were introduced in connection with the classification of Zassenhaus groups which was accomplished by M. Suzuki, G. Higman [2], N. Ito [3] and W. Feit [4]. A Zassenhaus group is a permutation group acting doubly transitively on a finite set such that nontrivial elements fix at most two points in the set [5]. The degree of a Zassenhaus group is the number of elements in the set. Suzuki classified Zassenhaus groups of odd degree [6]. During his studies, Suzuki needed the classification of finite nonabelian 2-groups with more than one involution, having a cyclic group of automorphisms which permutes its involutions transitively. Higman classified these groups and called them Suzuki 2-groups [2].

A. Nesin and M. Davis extended Higman's definition to the case of infinite groups. Instead of cyclic group T, they considered an abelian group of automorphisms and defined a Suzuki 2-group as a pair (G,T) of groups where G is a nilpotent 2-group of bounded exponent endowed with an action of an abelian group T that acts on G by group automorphisms and which is transitive on the involutions of G [1]. Classification of abelian Suzuki 2-groups of exponent 4 over a perfect field of characteristic 2 is given by A. Nesin and M. Davis. T. Altınel, A. Borovik and G. Cherlin defined a Suzuki 2-group in a similar way to A. Nesin, sometimes they remove the condition of bounded exponent [7], [8]. They have some results related to model theory. A. Nesin and M. Davis proved that an infinite free Suzuki 2-group of finite Morley rank is abelian. They also obtained some interesting results about abelian Suzuki 2-groups over perfect or quadratically closed fields. T. Altınel, A. Borovik and G. Cherlin proved that an infinite free Suzuki 2-group of finite Morley rank is abelian and homocyclic [8]. In this thesis, we are not involved in model theory and our considerations are purely group theoretic. We make use of A. Nesin's definition of a Suzuki 2-group in this thesis. Classification of a Suzuki 2-group (G,T), means determining the structure of the group G, that is, which group G admits such an action of T.

1.2 An Overview of the Thesis

1.2.1 Principal Results of the Thesis

We developed and generalized the result of A. Nesin and M. Davis [1] from abelian Suzuki 2-groups of exponent 4 over a perfect field to any exponent. We proved that

Theorem 1.2.1 *Let G be an abelian Suzuki 2-group of exponent 2^n, $n \geq 1$, over a perfect field K. Fix an element $g \in G$ of order 2^n and let $g_m = g^{2^m}$ for $m = 1, \ldots, n-1$. Then, for all $y \in K$, we have*

$$gg^y = g^{1+y} \prod_{i=1}^{n-1} g_i^{\sum_{k=1}^{2^{i-1}} y^{(2k-1)/2^i}} \tag{1.1}$$

for $n \geq 2$ (and $gg^y = g^{1+y}$ for $n = 1$).

Then, we analyzed the role of "perfectness" assumption on the field, in case of exponent 4, and we classified abelian Suzuki 2-groups over an arbitrary field obtaining the following results:

Theorem 1.2.2 *Let (G,T) be an abelian free Suzuki 2-group of exponent 4 over the ground field K and f be the 2-cocycle associated to G and $h : K \to K^2$ be a map defined by $h(x) = f(x)^2$ for all $x \in K$. Then h satisfies the following equalities: for all $x \in K$, $y \in K \setminus \{0,1\}$,*

$$h(y) = y^2 h(y^{-1}) \tag{1.2}$$

$$h(x+y) + y^2 h(xy^{-1}) = h(y) + (1+y)^2 h(x(1+y)^{-1}) \tag{1.3}$$

$$h|_{K^2} = Id_{K^2}. \tag{1.4}$$

Conversely, assume that K is a field of characteristic 2 and that $h : K \to K^2$ is a map satisfying the equalities (1.2), (1.3) and (1.4). Let $G = K \times K$, $T = K^{\star}$ and $f : K \to K$ be a map defined by $f(x) = \sqrt{h(x)}$ for all $x \in K$. We define the multiplication operation on G by

$$(x_1, y_1)(x_2, y_2) = (x_1 + x_2, y_1 + y_2 + x_1 f(x_2 x_1^{-1})),$$
$$(x, y_1)(0, y_2) = (0, y_1)(x, y_2)(0, y_1 + y_2) \qquad \textbf{(1.5)}$$

for all $x_i \in K \setminus \{0\}$, $x, y_i \in K$, $i = 1, 2$, and the action of T on G by componentwise multiplication. Then (G, T) is an abelian free Suzuki 2-group of exponent 4 over K.

Corollary 1.2.1 *There is a one-to-one correspondence between the set of maps $h : K \to K^2$ satisfying (1.2), (1.3), (1.4) and the set of equivalence classes of abelian free Suzuki 2-groups G of exponent 4 over a (not necessarily perfect) field K of characteristic 2.*

Furthermore, we made an interpretation of these groups via cohomology theory and made a relation between abelian Suzuki 2-groups of exponent 4 and the second cohomology group by proving the following result:

Theorem 1.2.3 *There is a one-to-one correspondence between the family of abelian Suzuki 2-groups of exponent 4 over a field K of characteristic 2 and elements of a certain subset of the 2-dimensional cohomology group $H^2(K, K)$.*

We study two different types of nonabelian Suzuki 2-groups of exponent 4, that we call *quasi-abelian Suzuki 2-groups* and *smart Suzuki 2-groups*. A *quasi-abelian Suzuki 2-group* (G, T) is a nonabelian free Suzuki 2-group (G, T) of exponent 4 over a perfect field K of characteristic 2, such that for any element $g \in G$, the subgroup $< g^T >$ is abelian. We proved that in a quasi-abelian Suzuki 2-group (G, T), the quotient G/I of G by its involutions I, becomes a vector space over K. If dimension of G/I over K is equal to $n - 1$ for some $n \in \mathbb{N}$, then (G, T) is called an n-dimensional quasi-abelian Suzuki 2-group. We first classify 3-dimensional quasi-abelian Suzuki

2-groups in terms of a map $f : K \times K \to K$ satisfying certain properties and then we extend our results to classify n-dimensional quasi-abelian Suzuki 2-groups by proving the following theorems:

Theorem 1.2.4 *Let (G,T) be 3-dimensional quasi-abelian Suzuki 2-group over K. Assume that $\{\overline{g},\overline{h}\}$ is a basis for G/I over K for $g,h \in G$ with $g^2 = h^2 = a$. Then there exists a K-multiplicative, biadditive, surjective map $f : K \times K \to K$ such that $h^y g^x = g^x h^y a^{f(y,x)}$ and $f(x,y) \neq x+y$ for all $(x,y) \in K^\star \times K^\star$.*

Conversely, let $f : K \times K \to K$ be a K-multiplicative, biadditive, surjective map with $f(x,y) \neq x+y$ for any $x,y \in K^\star \times K^\star$. Let $G = K \times K \times K$ and $T = K^\star$. Define the multiplication operation on G by

$$(x_1,y_1,z_1)(x_2,y_2,z_2) = (x_1+x_2, y_1+y_2, f(y_1,x_2)+\sqrt{x_1x_2}+\sqrt{y_1y_2}+z_1+z_2). \tag{1.6}$$

Then (G,T) is a quasi-abelian Suzuki 2-group over the field K and the action of T on G is componentwise multiplication.

Theorem 1.2.5 *Let (G,T) be an (n+1)-dimensional quasi-abelian Suzuki 2-group over K, $n \in \mathbb{N}$. Then there exist $g_1,...,g_n \in G$ such that*
$g_1^2 = ... = g_n^2 = a$ and $\{\overline{g_1},...,\overline{g_n}\}$ is a basis for G/I over K. Furthermore any element in G can be written uniquely as $g_1^{x_1}...g_n^{x_n}a^y$ for $x_i,y \in K$, $i = 1,...,n$ and there exist K-multiplicative, biadditive, surjective maps $f_{ij} : K \times K \to K$ $i,j = 1,...,n$; $i < j$ such that

$$\begin{aligned}
&f_{12}(x_1,x_2)+...+f_{1n}(x_1,x_n)+f_{23}(x_2,x_3)+...+f_{2n}(x_2,x_n)+...\\
&+f_{n-2,n-1}(x_{n-2},x_{n-1})+f_{n-2,n}(x_{n-2},x_n)+f_{n-1,n}(x_{n-1},x_n)\\
&\neq x_1+...+x_n
\end{aligned} \tag{1.7}$$

for all $(x_i,x_j) \in K \times K$ with at least two nonzero elements $x_m,x_s \in K^\star$ for some $m,s \in \{1,...,n\}$ and

$$g_i^{x_i} g_j^{x_j} = g_j^{x_j} g_i^{x_i} a^{f_{ij}(x_i,x_j)} \tag{1.8}$$

for all $(x_i, x_j) \in K \times K$.

Conversely, assume that $f_{ij} : K \times K \to K$ $i, j = 1, ..., n;$ $i < j$ *are K-multiplicative, biadditive, surjective maps satisfying the inequality (1.7). Let* $G = K \times ... \times K = K^{n+1}$ *and* $T = K^{\star}$. *Define the multiplication operation on G by*

$$(x_1, ..., x_n, z_1)(y_1, ..., y_n, z_2) = (x_1 + y_1, ..., x_n + y_n,$$
$$f_{1,2}(y_1, x_2) + ... + f_{1,n}(y_1, x_n) + f_{2,3}(y_2, x_3)$$
$$+ ... + f_{2,n}(y_2, x_n) + ... + f_{n-1,n}(y_{n-1}, x_n))$$
$$+ \sqrt{x_1 y_1} + ... + \sqrt{x_n y_n} + z_1 + z_2) \quad \textbf{(1.9)}$$

Then T acts on G by componentwise multiplication and (G, T) *is an (n+1)-dimensional quasi-abelian Suzuki 2-group over K.*

A *smart Suzuki 2-group* (G, T) is a nonabelian Suzuki 2-group of exponent 4 where T acts transitively and freely on G/I. We described the group structure of a smart Suzuki 2-group in terms of a certain function $\alpha : K \to k$ relating a pair of fields of characteristic 2 in the following result:

Theorem 1.2.6 *The characteristic map between the wide and narrow fields associated to a smart Suzuki 2-group determine the structure of the Suzuki 2-group. That is, if* (G_i, T_i) *are smart Suzuki 2-groups with the field isomorphisms* $\Psi : K_1 \to K_2$, $\Phi : k_1 \to k_2$ *such that the diagrams*

$$\begin{array}{ccc} K_1 \times K_1 & \longrightarrow & K_2 \times K_2 \\ \beta_1 \downarrow & & \beta_2 \downarrow \\ k_1 & \xrightarrow{\phi} & k_2 \end{array} \quad \textbf{(1.10)}$$

$$\begin{array}{ccc} K_1 & \xrightarrow{\Psi} & K_2 \\ \alpha_1 \downarrow & & \alpha_2 \downarrow \\ k_1 & \xrightarrow{\phi} & k_2 \end{array} \quad \textbf{(1.11)}$$

commute, then G_1 *and* G_2 *are isomorphic Suzuki 2-groups.*

1.2.2 The Content of the Thesis

In the second chapter of the thesis we give basic definitions and facts about Suzuki 2-groups.

In the third chapter, we start with the classification of abelian Suzuki 2-groups of exponent 8 over a perfect field K of characteristic 2. Then we generalize our result to the case of exponent 2^n. We proved that if (G,T) is an abelian Suzuki 2-group of exponent 2^n over a perfect field K, then G is an extension of K by K^{n-1}, in other words, G is an extension of K by the subgroup of G of exponent 2^{n-1}. This classification implies uniqueness of an abelian Suzuki 2-group of any given exponent 2^n over a perfect field K.

In chapter four, we drop the condition on the field that it is perfect and obtain results about abelian Suzuki 2-groups of exponent 4 over a field of characteristic 2. We introduce an invariant $h : K \to K^2$ satisfying certain properties, and give a classification of those groups by this invariant. Alternatively, we provide another classification in terms of a certain cohomological invariant. Namely, we proved that there is a one-to-one correspondence between the family of abelian Suzuki 2-groups of exponent 4 over an arbitrary field K of characteristic 2 and elements of a certain subset of the 2-dimensional cohomology group $H^2(K,K)$.

The fifth chapter of this thesis is devoted to nonabelian Suzuki 2-groups (G,T) of exponent 4 over a field of characteristic 2. We classify these groups into several types. When G is free over a perfect field such that for any element $g \in G$, the subgroup $< g^T > = < \{g^t : t \in T\} >$ is abelian, we call G a quasi-abelian Suzuki 2-group and give a classification in terms of a map $f : K \times K \to K$ satisfying certain properties. When the ground field is not perfect, under the assumption that T acts freely and transitively on the quotient G/I of G by the central subgroup of involutions I, we introduce *smart Suzuki 2-groups* and describe the group structure in terms of a so called *characteristic map* $\alpha : K \to k$ of G relating a pair of fields K and k (the *wide* and *narrow* fields) of characteristic 2 where the additive groups K^+ and $k^\oplus$ of fields are isomorphic to T and

$T/C_T(I)$, respectively. We provide also some examples of nonabelian Suzuki 2-groups and give some criteria for the existence of their linear presentation by 3×3 matrices.

2. PRELIMINARIES

2.1 Basic Definitions and Notations

Recall that a 2-group is a group whose elements have orders a power of 2. An element of order 2 of a group is called an *involution*. If A is a group, $A^\sharp$ denotes the set of nontrivial elements in A. For $a \in A$, $\circ(a)$ denotes the order of the element a. $O_2(A)$ denotes the largest normal 2-subgroup of A, that is, $O_2(A)$ is the product of all normal 2-subgroups of A. This makes sense since the product of two normal 2-subgroups is again a normal 2-subgroup of A.Furthermore, $O_2(A)$ is a characteristic subgroup of A, since automorphisms map normal 2-subgroups to normal 2-subgroups. If A is finite and has a normal Sylow 2-subgroup P, then $P = O_2(A)$.

If A acts on a set X, then *the centralizer of* X *in* A is the subgroup

$$C_A(X) = \{a \in A : x^a = x, \forall x \in X\}. \qquad \textbf{(2.1)}$$

We say that A acts transitively on X if for any $x, y \in X$, there is $a \in A$ such that $x^a = y$.

If B, C are groups, then an *extension* of B by C is a group A having a normal subgroup $B_1 \cong B$ with $A/B_1 \cong C$.

Let G be a (not necessarily normal) subgroup of a group Γ. Then a subgroup Q in Γ is a *complement* of G in Γ if $G \bigcap Q = 1$ and $GQ = \Gamma$. A group Γ is a *semidirect product* of a group G by a group T, denoted $\Gamma = G \rtimes T$, if $G \lhd \Gamma$ and G has a complement $Q \simeq T$. We say that Γ *splits over* G.

Let G be a group.

(i)The series

$$1 = Z_0(G) \subset Z_1(G) \subset ... \subset Z_n(G) \subset ... \qquad \textbf{(2.2)}$$

where $Z_0(G) = 1$, and $Z(G/Z_i(G)) = Z_{i+1}(G)/Z_i(G)$ for $i = 0, 1, ..., n-1$ is called *the upper central series (or ascending central series)* of G.

(ii) The series

$$... \subset G^{(n)} \subset G^{(n-1)} \subset ... \subset G^1 \subset G^{(0)} = G \tag{2.3}$$

where $G^{(0)} = G$ and $G^{(i+1)} = [G^{(i)}, G]$ for $i = 1, ..., n-2$ is called *the lower central series (or descending central series)* of G.

A group G is called *nilpotent*, if $Z_n(G) = G$ (or equivalently f $G^n = 1$) for some $n \in \mathbb{N}$. The smallest such n is called the *nilpotency class* of G.

Let G be a group. If for some $n \in \mathbb{N}$, every element of G has order $\leq n$, then G is said to have *finite exponent* or *bounded exponent*. The smallest such n is called the *exponent* G, denoted by $exp(G) = n$.

A subgroup H of a group G is said to be a *pure subgroup* of G, if for all $h \in H$ and $n \in \mathbb{N}$, if there is $g \in G$ such that $g^n = h$, then there is $k \in H$ such that $k^n = h$.

If K is a field, K^+ and $K^\star$ denote the additive and multiplicative groups in K, respectively. K is said to be *perfect* if either it has characteristic 0 or it has prime characteristic p and every $\lambda \in K$ has a p-th root in K.

Let K be a finite field of characteristic p. Then the map $\sigma_p : L \to K$ defined by $\sigma_p(x) = x^p$, $x \in K$ is map σ_p is called the Frobenius automorphism of K.

2.2 Properties of Suzuki 2-Groups

2.2.1 Notation and Terminology

A *Suzuki 2-group* is a pair (G, T) of groups where G is a nilpotent 2-group of bounded exponent and T is an abelian group that acts on G by group automorphisms and which is transitive on the involutions of G. Sometimes we say that $G \rtimes T$ is a Suzuki 2-group or simply G is a Suzuki 2-group. From now on (G, T) denotes a Suzuki 2-group. For each $n \in \mathbb{N}$, we define

$$I_n = \{g \in G : g^{2^n} = 1\}. \tag{2.4}$$

So, $I_1 = I$ is the set of involutions in G.

(G,T) is a *free Suzuki 2-group* if T acts freely on G, that is, if $g^t = g$ for $g \in G$ and $t \in T$ implies either $g = 1$ or $t = 1$. (G,T) is an *abelian Suzuki 2-group* if G is abelian.

If $I \rtimes T/C_T(I)$ is isomorphic to the affine group $K^+ \rtimes K^\star$ for some field K, then we will say that (G,T) *is a Suzuki 2-group over the field* K and K is denoted by $K(G)$.

2.2.2 Introducing a Field in a Suzuki 2-Group [1]

If (G,T) is a Suzuki 2-group, then we can interpret a field in $G \rtimes T$. Put $K = T/C_T(I) \cup \{0\}$ where 0 is a new symbol. We define the "multiplication" on K as the operation extending the group operation of $T/C_T(I)$ by the rule

$$T/C_T(I).0 = 0 = 0.T/C_T(I). \qquad \textbf{(2.5)}$$

To define the "addition" on K, we start by fixing an involution $i \in I^\sharp$. For $\bar{t} \in T/C_T(I)$, $i^{\bar{t}}$ is already well-defined, extend this to K by $i^0 = 1$. Then we pull back the group operation of I to K defining

$$i^{\bar{t}+\bar{s}} = i^{\bar{t}} i^{\bar{s}} \qquad \textbf{(2.6)}$$

for $\bar{t}, \bar{s} \in K$.

Lemma 2.2.1 *K becomes a field with the operations defined above.*

Proof. Since T acts transitively on I, $i^K = I$. Since $T/C_T(I)$ acts freely on I, the map $\bar{t} \mapsto i^{\bar{t}}$ is a bijection between K and I. We use this map to pull back the group operation of I to K defining $i^{t+s} = i^t i^s$ for $t, s \in K$. Then K becomes a field with $I \rtimes T/C_T(I) \simeq K^+ \times K^\star$. K is of characteristic 2 since $T/C_T(I)$ acts freely on I and so $1 = i^0 = i^{\bar{t}+\bar{t}}$ implies $\bar{t} + \bar{t} = 0$ for all $\bar{t} \in K$. $\square$

We call K the *ground field* associated to G.

2.2.3 Some Fundamentals about Suzuki 2-Groups

The following useful facts are extracted from A. Nesin and M. Davis [1]. We provide proofs whenever they are omitted in the original.

Lemma 2.2.2 *I is a central subgroup of G and for each $n \in \mathbb{N}$, I_n is a T-normal subset of G.*

Proof. Since G is nilpotent, $Z(G)$ is nontrivial. Let $g \in Z(G)$. Since G is a 2-group of bounded exponent, the order of g is 2^k for some $k \in \mathbb{N}$. Then $g^{2^{k-1}}$ is an involution in $Z(G)$. Put $a = g^{2^{k-1}}$. Since T acts transitively on I and $Z(G)$ is a characteristic subgroup of G, for any $b \in I$, there is $t \in T$ with $b = a^t \in Z(G)$. Thus, I is a central subgroup of G.

Since T acts by automorphisms on G, T maps elements of order 2^n, to the elements of the same order, therefore I_n is a T-normal subset of G. □

Lemma 2.2.3 *$(G, T/C_T(G))$ is a Suzuki 2-group. Therefore, replacing T by $T/C_T(G)$ if necessary, without loss of generality we may assume that T acts faithfully on G.*

Proof. It is enough to show that $T/C_T(G)$ acts transitively on I. Put $\bar{t} = tC_T(G)$ for any $t \in T$. If $a, b \in I$, then since T acts transitively on I, $a = b^t = b^{\bar{t}}$ for some $t \in T$. □

Lemma 2.2.4 *Assume $Z(G)$ has an element of order 2^n. Then $I_n \leq Z(G)$ and T acts transitively on I_n/I_{n-1}. Thus, G/I_{n-1} is also a Suzuki 2-group.*

Proof. Proof is by induction on n. Assume that the statement is true for n. Suppose $Z(G)$ has an element z of order 2^{n+1}. We need to show that $I_{n+1} \leq Z(G)$. Now $z \in Z(G)$ implies $z^2 \in Z(G)$, i.e., $Z(G)$ has an element z^2 of order 2^n. Then by induction $I_n \leq Z(G)$, T acts transitively on I_n/I_{n-1} and G/I_{n-1} is a Suzuki 2-group. Take any $g \in I_{n+1}$. Then $g^2, z^2 \in I_n \leq Z(G)$. Since T acts by group automorphisms on G and transitively on I_n/I_{n-1}, there exists $t \in T$ such that $g^2 I_{n-1} = (z^2 I_{n-1})^t = (z^t)^2 I_{n-1}$. Since $z \in Z(G)$, $g^{-2}(z^t)^2 = (g^{-1}z^t)^2 \in I_{n-1}$ and $g^{-1}z^t \in I_n \leq Z(G)$. Now, $z \in Z(G)$ implies $g^{-1} \in Z(G)$, i.e., $g \in Z(G)$. Therefore, $I_{n+1} \leq Z(G)$.

In order to show that T acts transitively on I_{n+1}/I_n, take any $aI_n, bI_n \in I_{n+1}/I_n$. Then $a^2 I_{n-1}, b^2 I_{n-1} \in I_n/I_{n-1}$ and since T acts transitively on I_n/I_{n-1} by induction, there exists $t \in T$ with $(a^2)^t I_{n-1} = b^2 I_{n-1}$ i.e. $(a^t)^2 b^{-2} \in I_{n-1}$ and $(a^t b^{-1})^2 = (a^t)^2 b^{-2} \in$

I_{n-1} since $a,b \in I_{n+1} \leq Z(G)$. Then $a^t b^{-1} \in I_n$ and $a^t I_n = bI_n$. Therefore, T acts transitively on I_{n+1}/I_n and G/I_{n-1} becomes a Suzuki 2-group. □

Lemma 2.2.5 *If A and B are T-normal subgroups of an abelian Suzuki 2-group G, then $exp(A) = exp(B)$ if and only if $A = B$.*

Proof. If $A = B$ then $exp(A) = exp(B)$. Conversely, assume that $exp(A) = exp(B) = 2^n$. Take any $a \in A$, $b \in B$ with $\circ(a) = \circ(b) = 2^n$. Now since G is abelian, by Lemma 2.2.3, T acts transitively on I_n/I_{n-1} and there exists $t \in T$ with $bI_{n-1} = a^t I_{n-1}$, i.e., $b^{-1}a^t \in I_{n-1}$ and for some $i \in I_{n-1}$, $b = a^t i \in A$. Thus, $A = B$. □

Lemma 2.2.6 *Let (G,T) be abelian. Let H be a T-normal subgroup of G. Then*

$$H = I_m = \{[h,t] : h \in H\} =< h^T > \qquad \textbf{(2.7)}$$

for some $m \in \mathbb{N}$, for fixed $t \in T/C_T(I)$ and any $h \in H$ of maximal order.

Proof. Since G is abelian, for $h,x \in H$, $[h,t][x,t] = h^{-1}h^t x^{-1}x^t = h^{-1}x^{-1}h^t x^t = (hx)^{-1}(hx)^t = [hx,t]$. Thus, $\{[h,t] : h \in H\} =< [h,t] : h \in H >$. If $exp(H) = 2^m$, then since $exp(\{[h,t] : h \in H\} = exp(< h^T >) = exp(H) = 2^m = exp(I_m)$, by Lemma 2.2.4, we have $H = I_m = \{[h,t] : h \in H\} =< h^T >$. □

Lemma 2.2.7 *If $t \in T$ centralizes an element $g \in G$ then t centralizes all the involutions of G.*

Proof. Assume $t \in T$ centralizes an element $g \in G$. Then since G is of bounded exponent, $\circ(g) = 2^m$ for some $m \in \mathbb{N}$. Then t centralizes $g^{2^{m-1}}$ which is an involution. Put $a = g^{2^{m-1}}$. If $b \in I$, then $b = a^s$ for some $s \in T$. Then $b^t = (a^s)^t = a^{st} = a^{ts} = (a^t)^s = a^s = b$. Then t centralizes all the involutions of G. Thus, for any $i \in I^\sharp$,

$$C_T(i) = C_T(I) = \bigcup_{g \in G^\sharp} C_T(g). \qquad \textbf{(2.8)}$$

□

Lemma 2.2.8 *If $t \in O_2(T)$ with $t^2 \in C_T(I)$ then $t \in C_T(I)$.*

Proof. If $i^t = i$ then $t \in C_T(i) = C_T(I)$. Otherwise, $(ii^t)^t = i^t i^{t^2} = i^t i = ii^t$, so again $t \in C_T(ii^t) = C_T(I)$. □

Theorem 2.2.1 *Assume (G,T) is faithful and abelian. Then $O_2(T) = C_T(I)$ and $O_2(T)$ has exponent at most exp(G). Furthermore, $T = O_2(T) \oplus S$ for some subgroup S of T and the group (G,S) is a free Suzuki 2-group.*

Proof. For the proof we need the following lemma:

Lemma 2.2.9 *If $B \leq A$ is a pure subgroup of finite exponent of the abelian group A then B splits in A, i.e., there is a $C \leq A$ such that $A = B \oplus C$.*

Proof of Theorem 2.2.1.. Take any nontrivial $t \in C_T(I)$. Then by Lemma 2.2.6, t centralizes a nontrivial element $g \in G^\sharp$. But then t centralizes the nontrivial T-normal subgroup $< g^T >$ as for $s \in T$, $(g^s)^t = g^{st} = g^{ts} = (g^t)^s = g^s$. If g has no square root, then by Lemma 2.2.4, $< g^T >= G$, and since T acts faithfully on G, $t = 1$ which is a contradiction. Thus, g has a square root, say $g_1 \in G$. Then $g = g^t = (g_1^2)^t = g_1^2$ and since G is abelian, $(g_1^2)^{-t} g_1^2 = (g_1^{-t} g_1)^2 = 1$ and the element $g_1^{-t} g_1$ is in I, and so is fixed by t. Hence, $g_1^{-1} g_1^t = (g_1^{-t} g_1)^{-1} = g_1^{-t} g_1 = (g_1^{-t} g_1)^t = g_1^{-t^2} g_1^t$ and t^2 fixes g_1. Continuing this process, we obtain that t^{2^2} fixes g_2 where $g_2^2 = g_1$, and so on. Since G is of finite exponent, after k steps, for some $k \in \mathbb{N}$, we get a square free element $g_k \in G$ which is fixed by t^{2^k}. Now since $< (g_k)^T >= G$, $t^{2^k} = 1$ and $t \in O_2(T)$. Hence, $C_T(I) \leq O_2(T)$.

Now we shall show that $O_2(T) \leq C_T(I)$. Take any $t \in O_2(T)$. Then $t^{2^k} = 1$. Then $t^{2^{k-1}} \in O_2(T)$ with $(t^{2^{k-1}})^2 = 1 \in C_T(I)$. Applying Lemma 2.2.7 $k-1$ times, we obtain that $t \in C_T(I)$. Therefore, $C_T(I) = O_2(T)$.

By Lemma 2.2.8, now since $O_2(T)$ is a pure subgroup of finite exponent in T we have $T = S \oplus O_2(T)$ for some $S \leq T$. Since the action of T on I is induced by S, (G,S) is a Suzuki 2-group. Since $T = S \oplus O_2(T)$ and $O_2(T) = C_T(I) = \bigcup_{g \in G^\sharp} C_T(g)$, (G,S) is free. □

Lemma 2.2.10 *Any nontrivial T-normal subgroup of a (free) Suzuki 2-group contains involutions and therefore is a (free) Suzuki 2-group.*

Proof. Let H be a nontrivial T-normal subgroup of a Suzuki 2-group G. Take any $h \in H$. Since G is of bounded exponent, $\circ(h) = 2^k$, $k \in \mathbb{N}$. Then $h^{2^{k-1}}$ is an involution in H. Let $a \in I^\sharp$. Since T acts transitively on involutions, there is $t \in T$ with $a = (h^{2^{k-1}})^t \in H^t = H$ as H is T-normal. □

Lemma 2.2.11 *Let G be abelian of exponent 4 over K. Let $g \in G$ be a fixed element of order 4 with $a = g^2$. Then every element of G can be written as $g^x a^y$ for unique $x, y \in K$.*

Proof. By Theorem 2.2.1, we may assume that G is free. Take any $h \in G$. Then $h^2 = (g^2)^x$, $x \in K$. Since G is abelian, $(hg^{-x})^2 = h^2(g^{-x})^2 = 1$, so $hg^{-x} \in I$, and $hg^{-x} = a^y$, $y \in K$, i.e., $h = g^x a^y$.

Uniqueness: If $h = g^x a^y = g^t a^s$ for some $x, y, t, s \in K$, then $a^x = (g^x a^y)^2 = (g^t a^s)^2 = a^t$. Since G is free we have $x = t$ which implies $y = s$. □

Lemma 2.2.12 *Let (G,T) be abelian, free of exponent 4 over K. Let $g \in G$ be an element of order 4 with $g^2 = a$. Then $G = < g^T >$ and there is a map $f : K \to K$ such that $gg^x = g^{1+x} a^{f(x)}$ for all $x \in K$.*

Proof. Take any $h \in G$, then $h^2 \in I$ so $h^2 = (g^2)^t$ for some $t \in T$. But then $hg^{-t} \in I$ and $h \in g^t I \subseteq < g^T >$. Thus, $G = < g^T >$. Now, for any $x \in K$, $gg^x = g^y \ mod I$, for some $y \in K$ and squaring gives $aa^x = a^y$, i.e., $a^{1+x} = a^y$. Since G is free, $1 + x = y$ and $gg^x = g^{1+x} a^{f(x)}$. □

Lemma 2.2.13 *For $x, y \in G$, $xI = yI$ if and only if $xy = yx$ and $x^2 = y^2$.*

Proof. Fix $i \in I^\sharp$. Assume $xI = yI$, then $xi = yj$ for some $j \in I$, i.e., $x = yji$. Now since $I \subseteq Z(G)$, $xy = yjiy = yyji = yx$ and since $xi = yj$, $x^2 = x^2 i^2 = (xi)^2 = (yj)^2 = y^2 j^2 = y^2$.

Conversely, $xy = yx$ and $x^2 = y^2$ implies that $(y^{-1}x)^2 = y^{-1}xy^{-1}x = y^{-1}y^{-1}xx = y^{-2}x^2 = y^{-2}y^2 = 1$ that is, $y^{-1}x \in I$ and $xI = yI$. $\square$

Proposition 2.2.1 *Let G be (free) of exponent at least 4 such that I_2 is an abelian group. Then G/I is also a (free) Suzuki 2-group. Furthermore $K(G/I) = K(G)$.*

Proof of Proposition 2.2.1.

Lemma 2.2.14 *G/I is Suzuki 2-group.*

Proof of Lemma 2.2.14. It is enough to show that T acts transitively on the set of involutions of G/I. Let $\bar{x}$, $\bar{y}$ be two involutions of G/I. Then $I = (xI)^2 = x^2I$, $x^2 \in I$, similarly $y^2 \in I = I_1$ so $x, y \in I_2$. Since T acts transitively on I, there exists $t \in T$ with $(x^t)^2 = (x^2)^t = y^2$. Now since $x, y \in I_2$ and I_2 is abelian we have $(x^t y^{-1})^2 = (x^t)^2 y^{-2} = 1$ so $x^t y^{-1} \in I$ and $x^t I = yI$. Hence, T acts transitively on involutions of G/I and G/I is a Suzuki 2-group.

Lemma 2.2.15 *G/I is free when G is free.*

Proof of Lemma 2.2.15. Let $x \in G$, $t \in T$ be such that $\bar{x}^t = \bar{x}$ in G/I. $x^t I = xI$ implies that $x^t = xi$ for some $i \in I$ and so $(x^t)^2 = x^2$. Since T acts on G by automorphisms, $(x^2)^t = (x^t)^2 = x^2$. But G is a free Suzuki 2-group, thus either $t = 1$ or $x^2 = 1$, i.e., $\bar{x} = I$.

Lemma 2.2.16 *$K(G) = K(G/I)$.*

Proof of Lemma 2.2.16. Since $K(G) = (T/C_T)\bigcup\{0\}$ and $K(G/I) = (T/C_T(I_2/I))\bigcup\{0\}$ where I and I_2/I are the sets involutions of G and G/I it is enough to show that $C_T(I) = C_T(I_2/I)$ so that the elements of the fields are the same and in this case the field multiplication induced from the operation in T is the same and that the field addition is the same, i.e., $t + s = u$ in $K(G)$ if and only if $t + s = u$ in $K(G/I)$.

There is a one-to-one correspondence between I_2/I and I mapping $jI \in I_2/I$ onto $j^2 \in I$. Since I_2 is abelian, for $x, y \in I_2$, by Lemma 2.2.13, $x^2 = y^2$ implies that $xI = yI$. Now

$t \in T$ centralizes $xI \in I_2/I$ if and only if $x^{-1}x^t \in I$ if and only if $1 = (x^{-1}x^t)^2 = x^{-2}(x^2)^t$, $(x^2)^t = x^2$ if and only if t centralizes $x^2 \in I$. So, $C_T(I) = C_T(I_2/I)$.

Next assume $t+s=u$ in $K(G)$ and let $x \in I_2 \setminus I$. Then since $x^2 \in I$, $(x^t x^s)^2 = (x^2)^t(x^2)^s = (x^2)^{t+s} = (x^2)^u = (x^u)^2$. Also, $(x^t x^s)x^u = x^u(x^t x^s)$. But then by lemma 2.2.13, $x^t x^s I = x^u I$, that is, $(xI)^t(xI)^s = (xI)^u$ in G/I, hence $t+s=u$ in $K(G/I)$.

Conversely, assume that $s+t=u$ in $K(G/I)$. Let $y \in I$ and $x \in I_2$ be such that $y = x^2$. Since $x^t x^s = x^u$ (mod I), squaring gives $y^t y^s = y^u$, i.e., $t+s=u$ in $K(G)$. □

Theorem 2.2.2 *Let (G,T) be an abelian Suzuki 2-group of exponent 4 over a perfect field K of characteristic 2. Then G is isomorphic to the following algebraic group over K: as a set $G = K \times K$ and the product is given by the rule*

$$(x,x')(y,y') = (x+y, x'+y'+(xy)^{1/2}) \qquad \textbf{(2.9)}$$

for all $(x,x'),(y,y') \in K \times K$. If further G is a free Suzuki 2-group, then the action of $T \simeq K^\star$ on $G = K \times K$ is componentwise multiplication.

Proof of Theorem 2.2.2. By Theorem 2.2.1, we may assume that G is free. Identify K with $T \cup \{0\}$. Let $g \in G$ be a fixed element of order 4 with $g^2 = a$. By Lemma 2.2.11, every element of G can be written as $g^x a^y$ for some unique $x,y \in K$. Then the map $\psi : G \to K \times K$ defined by $\psi(g^x a^y) = (x,y)$ is a well-defined bijection. Identify the set G with $K \times K$ via the map ψ. Then since $\psi((g^x a^y)^t) = \psi(g^{tx} a^{ty}) = (tx,ty)$, the action of T on G corresponds to componentwise multiplication.

It remains to show that the multiplication is as in equality (2.9), i.e., $g^x g^y = g^{x+y} a^{(xy)^{1/2}}$ for all $x,y \in K$. Since T acts on G by automorphisms $g^x g^y = (g g^{x^{-1}y})^x$ and it is enough to prove the equality for $x=1$. Let f be the map defined by $gg^y = g^{1+y}a^{f(y)}$ for all $y \in K$. We need to show that $f(y^2) = y$. First we show that $f(0)=0$ and $f(1)=1$. For $y=0$, $g = gg^0 = ga^{f(0)}$, and for $y=1$, $a = g^2 = gg = a^{f(1)}$, since G is free $f(0)=0$ and $f(1)=1$. We have

$$g^{1+y}a^{f(y)} = gg^y = g^y g = (gg^{y^{-1}})^y = (g^{1+y^{-1}}a^{f(y^{-1})})^y = g^{y+1}a^{yf(y^{-1})}. \qquad \textbf{(2.10)}$$

Equality (2.10) gives

$$f(y) = yf(y^{-1}). \tag{2.11}$$

We also need to express the associativity in G in terms of f. We have the equalities:

$$g^y g^z = (gg^{zy^{-1}})^y \tag{2.12}$$

and

$$(gg^y)g^z = g(g^y g^z). \tag{2.13}$$

$$\begin{aligned}
(gg^y)g^z &= g^{1+y}a^{f(y)}g^z = g^{1+y}g^z a^{f(y)} \\
&= (gg^{z(1+y)^{-1}})^{1+y}a^{f(y)} = [g^{1+z(1+y)^{-1}}a^{f(z(1+y)^{-1})}]^{1+y}a^{f(y)} \\
&= g^{1+y+z}a^{(1+y)f(z(1+y)^{-1})}a^{f(y)} = g^{1+y+z}a^{f(y)+f(z(1+y)^{-1})(1+y)}
\end{aligned} \tag{2.14}$$

and

$$\begin{aligned}
g(g^y g^z) &= g(gg^{zy^{-1}})^y = g[g^{1+zy^{-1}}a^{f(zy^{-1})}]^y = gg^{y+z}a^{yf(zy^{-1})} \\
&= g^{1+y+z}a^{f(y+z)}a^{yf(zy^{-1})} = g^{1+y+z}a^{f(y+z)+yf(zy^{-1})}.
\end{aligned} \tag{2.15}$$

Equalities (2.13), (2.14) and (2.15) imply that

$$g^{1+y+z}a^{f(y)+f(z(1+y)^{-1})(1+y)} = g^{1+y+z}a^{f(y+z)+yf(zy^{-1})}, \tag{2.16}$$

that is,

$$f(y+z) + yf(zy^{-1}) = f(y) + f(z(1+y)^{-1})(1+y). \tag{2.17}$$

Taking $z = y$ in (2.17) we obtain

$$\begin{aligned}
f(y+y) + yf(1) &= f(y) + f(y(1+y)^{-1})(1+y) \\
f(0) + y &= f(y) + f(y(1+y)^{-1})(1+y) \\
y &= f(y) + f(y(1+y)^{-1})(1+y).
\end{aligned} \tag{2.18}$$

Now using equality (2.11) twice in (2.18) we get

$$\begin{aligned}
yf(y^{-1}) + y(1+y)^{-1}f(y^{-1}(1+y))(1+y) &= y \\
yf(y^{-1}) + yf(y^{-1}(1+y)) &= y \\
f(y^{-1}) + f(y^{-1}+1) &= 1
\end{aligned} \tag{2.19}$$

for $y \in K - \{0\}$, i.e., $f(y) + f(1+y) = 1$ for all $y \in K - \{0\}$. In particular, for $y = 0$, we have $f(0) + f(1) = 1$, hence

$$f(1+y) = 1 + f(y) \tag{2.20}$$

for all $y \in K$. In (2.17) replace z by $y + y^2$ to get

$$f(y+y^2+y) + f((y+y^2)y^{-1})y = f(y) + f((1+y)y(1+y)^{-1})(1+y)$$
$$f(y^2) + f(1+y)y = f(y) + f(y)(1+y) = f(y) + f(y) + yf(y) \tag{2.21}$$

which implies

$$f(y^2) + f(1+y)y = yf(y). \tag{2.22}$$

Finally, using (2.20) in (2.21) we obtain

$$f(y^2) + (1 + f(y)) = yf(y)$$
$$f(y^2) + y + yf(y) = yf(y)$$
$$f(y^2) = y \tag{2.23}$$

for all $y \in K$. □

3. ABELIAN SUZUKI 2-GROUPS OVER PERFECT FIELDS

3.1 Notation and Terminology

Throughout this chapter K denotes a perfect field of characteristic 2, (G,T) denotes an abelian Suzuki 2-group, and I denotes the set of involutions in G.

A normal subgroup B of a group A determines the factor group A/B. We write $C = A/B$ and call A an *extension* of B by C.

3.2 Classification of Abelian Suzuki 2-Groups of Exponent 8

Theorem 3.2.1 *Let G be of exponent 8 over K. Then G is isomorphic to the following algebraic group over K: as a set $G = K\times K\times K$ and the product is given by the rule*

$$(x,x',x'')(y,y',y'') = (x+y, x'+y'+(xy)^{1/2},$$
$$x''+y''+(x'y')^{1/2}+(x'+y')^{1/2}(xy)^{1/4}+(x+y)^{1/2}(xy)^{1/4}). \tag{3.1}$$

If further G is a free Suzuki 2-group, then the action of $T\simeq K^\star$ on $G = K\times K\times K$ is componentwise multiplication. Thus, G is an extension of the field K by $K\times K$.

Proof of Theorem 3.2.1 By Theorem 2.2.1, we may assume that G is free. Identify K with $T\cup\{0\}$. Let $g\in G$ be a fixed element of order 8 with $g^2 = a$ and $a^2 = b$.

Lemma 3.2.1 *Every element of G can be written as $g^x a^y b^z$ for unique $x,y,z\in K$.*

Proof of Lemma 3.2.1 By Lemma 2.2.9, G^2 is an abelian free Suzuki 2-group of exponent 4. Take any $h\in G$, then $h^2\in G^2$ and by Lemma 2.2.11, $h^2 = a^x b^y = (g^2)^x(a^2)^y$ for unique $x,y\in K$. Now $hg^{-x}a^{-y}$ is an involution in G, so $h = g^x a^y b^z$ for unique $x,y,z\in K$.

Lemma 3.2.2 *G is isomorphic to $K \times K \times K$.*

Proof of Lemma 3.2.2 Identify the set G with $K \times K \times K$ via the map $\varphi : G \to K \times K \times K$, $\varphi(g^x a^y b^z) = (x,y,z)$. Since x,y,z are unique in the presentation $g^x a^y b^z$, φ and φ^{-1} are well-defined so φ is a bijection. T acts by group automorphisms on G by componentwise multiplication since

$$\varphi((g^x a^y b^z)^t) = \varphi(g^{tx} a^{ty} b^{tz}) = (tx, ty, tz). \tag{3.2}$$

Lemma 3.2.3 *The analogous group multiplication is as defined in equality (3.1).*

Proof of Lemma 3.2.3 By Theorem 2.2.2, we have

$$(0,x',x'')(o,y',y'') = (0,x'+y',x''+y''+(x'y')^{1/2}). \tag{3.3}$$

So, it is enough to prove that

$$g^x g^y = g^{x+y} a^{(xy)^{1/2}} b^{(x+y)^{1/2}(xy)^{1/4}}. \tag{3.4}$$

Now since T acts on G by automorphisms we have

$$(g_1 g_2)^x = g_1^x g_2^x \tag{3.5}$$

and

$$(g^x)^y = g^{xy}. \tag{3.6}$$

for all $g, g_1, g_2 \in G$, $x, y \in K$. Let

$$g^x g^y = g^{f(x,y)} a^{l(x,y)} b^{m(x,y)} \tag{3.7}$$

for some maps $f, l, m : K \times K \to K$. Then squaring gives $f(x,y) = x+y$ and $l(x,y) = (xy)^{1/2}$. Therefore, $g^x g^y = g^{x+y} a^{(xy)^{1/2}} b^{m(x,y)}$ and we need to determine $m(x,y)$. Assuming that m is a $K-bilinear$ map it is enough to prove the last equality for $x = 1$. Let $f(y) = m(1,y)$, then

$$g g^y = g^{1+y} a^{y^{1/2}} b^{f(y)}. \tag{3.8}$$

Put $y=1$ in (3.8) to get $g^2 = ab^{f(1)}$, i.e., $a = ab^{f(1)}$ and So $f(1)=0$ Now put $y=0$, to get $g = gb^{f(0)}$ and $f(0)=0$.

$$g^{1+y}a^{y^{1/2}}b^{f(y)} = gg^y = (gg^{y^{-1}})^y = (g^{1+y^{-1}}a^{(y^{-1})^{1/2}}b^{f(y^{-1})})^y$$
$$= g^{1+y}a^{y(y^{-1})^{1/2}}b^{yf(y^{-1})} = g^{1+y}a^{y^{1/2}}b^{yf(y^{-1})}. \qquad \textbf{(3.9)}$$

Equality (3.9) implies

$$f(y) = yf(y^{-1}) \qquad \textbf{(3.10)}$$

for all $y \in K \setminus \{0\}$.

We also need to express the associativity

$$(gg^y)g^z = g(g^yg^z) \qquad \textbf{(3.11)}$$

in G in terms of f.

$$(gg^y)g^z = g^{1+y}a^{y^{1/2}}b^{f(y)}g^z = g^{1+y}g^z a^{y^{1/2}}b^{f(y)}$$
$$= (gg^{z(1+y)^{-1}})^{1+y}a^{y^{1/2}}b^{f(y)}$$
$$= [g^{1+z(1+y)^{-1}}a^{(z(1+y)^{-1})^{1/2}}b^{f(z(1+y)^{-1})}]^{1+y}a^{y^{1/2}}b^{f(y)}$$
$$= g^{1+z+y}a^{(z(1+y))^{1/2}}b^{(1+y)f(z(1+y)^{-1})}a^{y^{1/2}}b^{f(y)}$$
$$= g^{1+z+y}a^{(z(1+y))^{1/2}+y^{1/2}}b^{(zy(1+y))^{1/4}+(1+y)f(z(1+y)^{-1})+f(y)}, \qquad \textbf{(3.12)}$$

$$g(g^yg^z) = g(gg^{zy^{-1}})^y = g[g^{1+zy^{-1}}a^{(zy^{-1})^{1/2}}b^{f(zy^{-1})}]^y$$
$$= gg^{y+z}a^{(zy)^{1/2}}b^{yf(zy^{-1})} = g^{1+y+z}a^{(y+z)^{1/2}}b^{f(y+z)}a^{(zy)^{1/2}}b^{yf(zy^{-1})}$$
$$= g^{1+y+z}a^{(y+z)^{1/2}+(zy)^{1/2}}b^{((y+z)(yz))^{1/4}+f(y+z)+yf(zy^{-1})}. \qquad \textbf{(3.13)}$$

Equalities (3.11), (3.12) and (3.13) imply

$$(zy(1+y))^{1/4} + (1+y)f(z(1+y)^{-1}) + f(y) = ((y+z)(yz))^{1/4} + f(y+z) + yf(zy^{-1}). \qquad \textbf{(3.14)}$$

Taking fourth power of both sides in (3.14) we have

$$zy(1+y) + (1+y)^4 f(z(1+y)^{-1})^4 + f(y)^4 = (y+z)(yz) + f(y+z)^4 + y^4 f(zy^{-1})^4. \qquad \textbf{(3.15)}$$

Substituting $h(x) = f(x)^4$ in (3.15) we get

$$zy(1+y) + (1+y)^4 h(z(1+y)^{-1}) + h(y) = (y+z)(yz) + h(y+z) + y^4 h(zy^{-1}). \tag{3.16}$$

Taking $z = y$ in (3.16) we obtain

$$y^2(1+y) + (1+y)^4 h(y(1+y)^{-1}) + h(y) = h(0) + y^4 h(1) = f(0)^4 + y^4 f(1)^4 = 0. \tag{3.17}$$

That is,

$$(1+y)^4 h(y(1+y)^{-1}) + h(y) = y^2(1+y). \tag{3.18}$$

Using (3.10) in equality (3.18) we get

$$(1+y)^4 y^4 (1+y)^{-4} h(y^{-1}(1+y)) + y^4 h(y^{-1}) = y^2(1+y)$$

$$y^4[h(y^{-1}+1) + h(y^{-1})] = y^2(1+y)$$

$$h(y^{-1}+1) + h(y^{-1}) = y^{-2}(1+y)$$

$$h(y+1) + h(y) = y^2(1+y^{-1})$$

$$(f(1+y) + f(y))^4 = y^2 + y \tag{3.19}$$

and so we get

$$f(1+y) + f(y) = y^{1/2} + y^{1/4}. \tag{3.20}$$

Now substitute $z = y + y^2$ in (3.16) to get

$$y^2(1+y)^2 + (1+y)^4 h(y) + h(y) = y^2 y^2 (1+y) + h(y^2) + y^4 h(1+y)$$

$$y^2(1+y)^2 + y^4 h(y) = y^4(1+y) + h(y^2) + y^4 h(1+y)$$

$$y^4(h(y) + h(1+y)) = y^2(1+y)^2 + y^4(1+y) + h(y^2). \tag{3.21}$$

Equation (3.20) implies that

$$h(y) + h(1+y) = y^2 + y. \tag{3.22}$$

Equalities (3.20) and (3.21) give

$$y^4(y^2+y) = y^2(1+y)^2 + y^4(1+y) + h(y^2)$$

$$y^4(y^2+y) = y^2 + y^4 + y^4 + y^5 + h(y^2)$$

$$y^6 + y^5 = y^2 + y^4 + y^4 + y^5 + h(y^2)$$

$$h(y^2) = y^2 + y^6. \tag{3.23}$$

Since K is perfect of characteristic 2, (3.22) implies

$$h(y) = y + y^3$$
$$f(y) = (y + y^3)^{1/4} = y^{1/4}(1 + y^2)^{1/4} = y^{1/4} + y^{3/4}. \tag{3.24}$$

Thus,

$$m(1,z) = f(z) = z^{1/4} + z^{3/4}$$
$$m(x,xz) = xm(1,z) = xz^{1/4}(1 + z^2)^{1/4} = x^{1/2}(1 + z^2)^{1/4}x^{1/2}z^{1/4}$$
$$m(x,xz) = (x^2 + x^2z^2)^{1/4}x^{1/4}(xz)^{1/4}. \tag{3.25}$$

Substituting $y = xz$ in the equality (3.25) we get

$$m(x,y) = (x + y)^{1/4}(xy)^{1/4}. \tag{3.26}$$

$$g^x g^y = g^{x+y}a^{(xy)^{1/2}}b^{(x+y)^{1/4}(xy)^{1/4}}. \tag{3.27}$$

Since G is an abelian Suzuki 2-group of exponent 8, by Proposition 2.2.1, G/I is an abelian Suzuki 2-group of exponent 4 over the same field K. Then by Theorem 2.2.2, G/I is isomorphic to $K \times K$. Since I is isomorphic to K, G becomes an extension of K by $K \times K$. □

3.3 Classification of Abelian Suzuki 2-Groups of Exponent 2^n

Theorem 3.3.1 *Let G be an abelian Suzuki 2-group of exponent 2^n, $n \geq 1$, over a perfect field K. Fix an element $g \in G$ of order 2^n and let $g_m = g^{2^m}$ for $m = 1, ..., n-1$. Then, for all $y \in K$, we have*

$$gg^y = g^{1+y} \prod_{i=1}^{n-1} g_i^{\sum_{k=1}^{2^{i-1}} y^{(2k-1)/2^i}} \tag{3.28}$$

for $n \geq 2$ (and $gg^y = g^{1+y}$ for $n = 1$).

Proof. For $n = 2$, $gg^y = g^{1+y} g_1^{y^{1/2}}$. Assume G is of exponent 2^{n+1} over K. Let $g \in G$ be an element of order 2^{n+1}. Then by induction,

$$\begin{aligned}
(gg^y)^2 &= g^2(g^2)^y \\
&= (g^2)^{y+1} \prod_{i=1}^{n-1} (g_i^2)^{\sum_{k=1}^{2^{i-1}} y^{(2k-1)/2^i}} \\
&= (g^2)^{y+1} \prod_{i=1}^{n-1} (g^{2^{i+1}})^{\sum_{k=1}^{2^{i-1}} y^{(2k-1)/2^i}}.
\end{aligned} \tag{3.29}$$

Put

$$A_i = \sum_{k=1}^{2^{i-1}} y^{(2k-1)/2^i}. \tag{3.30}$$

Then

$$\begin{aligned}
(gg^y)^2 = (g^2)^{y+1} \prod_{i=1}^{n-1} (g^{2^{i+1}})^{A_i} & \\
&= (g^2)^{y+1} (g^{2^2})^{A_1} (g^{2^3})^{A_2} ... (g^{2^{n-1}})^{A_{n-2}} (g^{2^n})^{A_{n-1}} \\
&= g^{2(1+y)} g^{2(2A_1)} g^{2(2^2 A_2)} ... g^{2(2^{n-1} A_{n-1})} \\
&= (g^{1+y} g^{2A_1} g^{2^2 A_2} ... g^{2^{n-1} A_{n-1}})^2 \\
&= (g^{1+y} g_1^{A_1} g_2^{A_2} ... g_{n-1}^{A_{n-1}})^2 = (g^{1+y} \prod_{i=1}^{n-1} g_i^{A_i})^2.
\end{aligned} \tag{3.31}$$

Hence, $gg^y = g^{1+y} \prod_{i=1}^{n-1} g_i^{\sum_{k=1}^{2^{i-1}} y^{(2k-1)/2^i}}$. □

Theorem 3.3.2 *Let (G,T) be an abelian Suzuki 2-group of exponent 2^n over a perfect field K of characteristic 2. Then G is isomorphic to the algebraic group $K \times ... \times K = K^n$ over K and G is an extension of the field K by K^{n-1}.*

Proof. By Proposition 2.2.1, G/I is an abelian Suzuki 2-group of exponent 2^{n-1} over the same field K. By induction assume that G/I is isomorphic to $K \times ... \times K = K^{n-1}$. Since I is isomorphic to K, G is isomorphic to $K \times ... \times K = K^n$. By Theorem 3.3.1, for all $x, y \in K$ and an element $g \in G$ of order 2^n with $g^{2^m} = g_m$ for $m = 1, ..., n-1$,

$$g^x g^y = g^{x+y} g_1^{(xy)^{1/2}} g_2^{x^{1/4}y^{3/4}+x^{3/4}x^{1/4}} ... g_{n-1}^{x^{1/2^{n-1}}y^{2^{n-1}-1/2n-1}+...+x^{2^{n-1}-1/2n-1}y^{1/2^{n-1}}}. \qquad \textbf{(3.32)}$$

Then the analogous operation in $K \times ... \times K = K^n$ is

$$(x, 0..., 0)(y, 0..., 0) = (x+y, (xy)^{1/2}, ..., x^{1/2^{n-1}}y^{2^{n-1}-1/2n-1} + ... + x^{2^{n-1}-1/2n-1}y^{1/2^{n-1}}). \qquad \textbf{(3.33)}$$

□

4. ABELIAN SUZUKI 2-GROUPS OF EXPONENT 4 OVER A NON-PERFECT FIELD OF CHARACTERISTIC 2

4.1 Notation and Terminology

Throughout this chapter, K denotes a field of characteristic 2 which is not necessarily perfect, K^+ and $K^\star$ denote the additive and the multiplicative groups of K, respectively. (G,T) denotes an abelian free Suzuki 2-group of exponent 4 over K. We will sometimes simply write that G is a Suzuki 2-group. When we say that "f is the 2-cocycle associated to G", we mean the map $f : K \to K$, defined in Lemma 2.2.12, by $gg^x = g^{1+x}a^{f(x)}$ for any $g \in G$ of order 4, for all $x \in K$. I denotes the set of involutions in G. K^2 denotes the subfield $\{x^2 : x \in K\}$ of K. If L is a subfield of K and $h : K \to K$ is a map, then the restriction of h to L is denoted by $h|_L$. The identity map from K to K is denoted by Id_K.

4.2 Classification of Abelian Suzuki 2-Groups of Exponent 4 over an Arbitrary Field

Theorem 4.2.1 *Let (G,T) be an abelian free Suzuki 2-group of exponent 4 over the ground field K and f be the 2-cocycle associated to G and $h : K \to K^2$ be a map defined by $h(x) = f(x)^2$ for all $x \in K$. Then h satisfies the following equalities*

$$h(y) = y^2 h(y^{-1}) \quad \textbf{(4.1)}$$

$$h(x+y) + y^2 h(xy^{-1}) = h(y) + (1+y)^2 h(x(1+y)^{-1}) \quad \textbf{(4.2)}$$

for all $x \in K$, $y \in K \setminus \{0,1\}$,

$$h|_{K^2} = Id_{K^2}. \quad \textbf{(4.3)}$$

Conversely, assume that K is a field of characteristic 2 and that $h : K \to K^2$ is a map satisfying the equalities (4.1), (4.2) and (4.3). Let $G = K \times K$, $T = K^\star$. Define $f : K \to K$ by $f(x) = \sqrt{h(x)}$ for all $x \in K$. Define the multiplication operation on G by

$$(x_1,y_1)(x_2,y_2) = (x_1+x_2, y_1+y_2+x_1 f(x_2 x_1^{-1})),$$
$$(x,y_1)(0,y_2) = (0,y_1)(x,y_2) = (x,y_1+y_2) \qquad \textbf{(4.4)}$$

for all $x_i \in K \setminus \{0\}$, $x, y_i \in K$, $i = 1,2$, and the action of T on G by componentwise multiplication. Then (G,T) is an abelian free Suzuki 2-group of exponent 4 over K.

Proof of Theorem 4.2.1.

Lemma 4.2.1 *The map h satisfies equalities (4.1) and (4.2).*

Proof of Lemma 4.2.2. Equality (3.10), $f(x) = xf(x^{-1})$ implies $h(x) = x^2 h(x^{-1})$. By using the associativity, $(gg^x)g^y = g(g^x g^y)$, in G, we had already obtained the equality (2.17). Since K is of characteristic 2, taking square of both sides in (2.17) we have equality (4.2).

Lemma 4.2.2 $h|_{K^2} = Id_{K^2}$.

Proof of Lemma 4.2.3. $h(0) = f(0)^2 = 0$ and $h(1) = f(1)^2 = 1$ Take $x = y$ in (4.2) to get

$$y^2 = h(y) + h(y(1+y)^{-1})(1+y)^2. \qquad \textbf{(4.5)}$$

Applying twice (4.1) in (4.5) we have

$$y^2 = y^2 h(y^{-1}) + y^2(1+y)^{-2} h(y^{-1}(1+y))(1+y)^2$$
$$1 = h(y^{-1}) + h(1+y^{-1}) \qquad \textbf{(4.6)}$$

$$h(1+y) = 1 + h(y) \qquad \textbf{(4.7)}$$

for all $y \in K \setminus \{0\}$. By using (4.7) and replacing x by $y+y^2$ in (4.2) we get

$$h(y+y^2+y)+y^2h((y+y^2)y^{-1}) = h(y)+h((y+y^2)(1+y)^{-1})(1+y)^2$$

$$h(y^2)+y^2h(1+y) = h(y)+h(y)(1+y)^2$$

$$h(y^2)+y^2+y^2h(y) = h(y)+h(y)+y^2h(y)$$

$$h(y^2) = y^2 \tag{4.8}$$

for all $y \in K$.

Conversely, assume that K is a field of characteristic 2 and that $h : K \to K^2$ is a map satisfying the equalities (4.1), (4.2) and (4.3). Define $f : K \to K$ by $f(x) = \sqrt{h(x)}$ for all $x \in K$.

Lemma 4.2.3 *G is a group together with the operation defined in (4.4).*

Proof of Lemma 4.2.4. For any $x, y \in K$

$$[(1,0)(x,0)](y,0) = (1+x, f(x))(y,0) = (1+x+y, (1+x)f(y(1+x)^{-1})+f(x)) \tag{4.9}$$

$$(1,0)[(x,0)(y,0)] = (1,0)(x+y, xf(yx^{-1})) = (1+x+y, f(x+y)+xf(yx^{-1}) \tag{4.10}$$

The equalities (2.17), (4.9) and (4.10) imply that the operation is associative. $(x,y)^{-1} = (x, x+y)$ for any $x \in K \setminus \{0\}$, $y \in K$.

Lemma 4.2.4 *G is an abelian free Suzuki 2-group of exponent 4 over K.*

Proof of Lemma 4.2.5. By equality (4.1), we get

$$f(x) = \sqrt{h(x)} = \sqrt{x^2h(x^{-1})} = xf(x^{-1}) \tag{4.11}$$

for all $x \in K \setminus \{0\}$. (4.11) implies

$$(x,0)(y,0) = (x+y, xf(yx^{-1}) = (x+y, xyx^{-1}f(xy^{-1}) = (x+y, yf(xy^{-1}) = (y,0)(x,0) \tag{4.12}$$

$$(x,y)^4 = (0,x)^2 = (0,0) \tag{4.13}$$

and

$$(x,0)^t(y,0)^t = (tx,0)(ty,0) = (tx+ty, txf(ty(tx)^{-1})$$
$$= (tx+ty, txf(yx^{-1}) = [(x,0)(y,0)]^t. \tag{4.14}$$

Thus, T acts on G by group automorphisms by componentwise multiplication. Since $I \simeq K^+$, T acts transitively on I.

When K is a perfect field, $K = K^2$ and in this case the equality (4.3) implies that $f(x) = \sqrt{h(x)} = \sqrt{x}$ so that $gg^x = g^{1+x}a^{\sqrt{x}}$.

Therefore, (G,T) is an abelian free Suzuki 2-group of exponent 4 over K. □

4.3 Corollaries of Theorem 4.2.1

Corollary 4.3.1 *Let $h : K \to K^2$ be an additive, $K^2 - linear$ map. Define $f : K \to K$ by $f(x) = \sqrt{h(x)}$ for all $x \in K$. Let $G = K \times K$ and $T = K^\star$. Define the multiplication operation on G as in (4.4). Then T acts on G by componentwise multiplication and (G,T) is an abelian free Suzuki 2-group of exponent 4 over K.*

Proof. Since h is $K^2 - linear$, $h(x^2y) = x^2h(y)$ for all $x,y \in K$. In particular, for $y = 1$ we have $h(x^2) = x^2$ and thus h satisfies the equality (4.3). Also, $x^2h(x^{-1}) = h(x^2x^{-1}) = h(x)$ and h satisfies (4.1). Additivity of h and (4.3) imply

$$h(x+y)+y^2h(xy^{-1}) = h(x)+h(y)+h(xy^{-1}y^2) = h(x)+h(y)+h(xy) \tag{4.15}$$

and

$$h(y)+h(x(1+y)^{-1})(1+y)^2 = h(y)+h(x(1+y)^{-1}(1+y)^2) = h(y)+h(x(1+y))$$
$$= h(y)+h(x)+h(xy). \tag{4.16}$$

Equalities (4.15) and (4.16) imply that h satisfies (4.2). Now since h satisfies all the equalities (4.1), (4.2) and (4.3), the result follows by Theorem 4.2.1. □

Corollary 4.3.2 *There is a one-to-one correspondence between the set of maps $h : K \to K^2$ satisfying (4.1), (4.2), (4.3) and the set of isomorphism classes of abelian free Suzuki 2-groups G of exponent 4 over a (not necessarily perfect) field K of characteristic 2.*

Proof. For $j = 1,2$, assume $h_j : K \to K^2$ be maps satisfying equalities (4.1), (4.2) and (4.3), define $f_j(x) = \sqrt{h_j(x)}$ for all $x \in K$ and let $G_j = K \times K$. Put $T = K^\star$. Then the multiplication operation on G_j by

$$(x_1,y_1)(x_2,y_2) = (x_1+x_2, y_1+y_2+x_1 f_j(x_2x_1^{-1})) \qquad \textbf{(4.17)}$$

for all $x_i, y_i \in K$, $i = 1,2$. Let $g_j \in G_j$ be an element of order 4 with $g_j^2 = a_j$ for $j = 1,2$. Define a group isomorphism $\Phi : G_1 \to G_2$ by $\Phi(g_1) = g_2a_2$. Then there are maps $\varphi : K \to K$ and $\Psi : K \to K$ such that

$$\Phi(g_1^x) = g_2^{\varphi(x)} a_2^{\Psi(x)} \qquad \textbf{(4.18)}$$

for all $x \in K$.

$$\begin{aligned}
&\Phi(a_1) = \Phi(g_1^2) = (\Phi(g_1))^2 = g_2^2 = a_2\\
&\Phi(a_1^x) = \Phi((g_1^2)^x) = \Phi((g_1^x)^2) = \Phi(g_1^x)^2\\
&= (g_2^{\varphi(x)} a_2^{\Psi(x)})^2 = a_2^{\varphi(x)}.
\end{aligned} \qquad \textbf{(4.19)}$$

We have

$$\Phi(g_1g_1^x) = \Phi(g_1)\Phi(g_1^x), \qquad \textbf{(4.20)}$$

$$\Phi(g_1g_1^x) = \Phi(g_1^{1+x}a_1^{f_1(x)}) = g_2^{\varphi(1+x)} a_2^{\Psi(1+x)+\varphi(f_1(x))}, \qquad \textbf{(4.21)}$$

$$\Phi(g_1)\Phi(g_1^x) = g_2a_2g_2^{\varphi(x)}a_2^{\Psi(x)} = g_2^{1+\varphi(x)} a_2^{f_2(\varphi(x))+1+\Psi(x)}. \qquad \textbf{(4.22)}$$

Now (4.20), (4.21) and (4.22) imply that

$$\begin{aligned}
&\varphi(1+x) = 1+\varphi(x),\\
&\Psi(1+x) = 1+\Psi(x),\\
&\varphi(f_1(x)) = f_2(\varphi(x)).
\end{aligned} \qquad \textbf{(4.23)}$$

Also, we have

$$\Phi(g_1^x g_1^y) = \Phi(g_1^x)\Phi(g_1^y), \tag{4.24}$$

$$\Phi(g_1^x g_1^y) = \Phi((g_1 g_1^{yx^{-1}})^x) = \Phi(g_1^{x+y} a_1^{xf_1(x^{-1}y)})) = g_2^{\varphi(x+y)} a_2^{\Psi(x+y)+\varphi(xf_1(x^{-1}y)}, \tag{4.25}$$

$$\Phi(g_1^x)\Phi(g_1^y) = g_2^{\varphi(x)} a_2^{\Psi(x)} g_2^{\varphi(y)} a_2^{\Psi(y)} = g_2^{\varphi(x)+\varphi(y)} a_2^{\varphi(x)f_2(\varphi(x)^{-1}\varphi(y))+\Psi(x)+\Psi(y)}. \tag{4.26}$$

Equalities (4.24), (4.25) and (4.26) imply that

$$\begin{aligned}
&\varphi(x+y) = \varphi(x)+\varphi(y),\\
&\varphi(xy) = \varphi(x)\varphi(y),\\
&\varphi(x^{-1}) = \varphi(x)^{-1},\\
&\Psi(x+y) = \Psi(x)+\Psi(y),\\
&\varphi^{-1} f_2 \varphi = f_1.
\end{aligned} \tag{4.27}$$

Φ is a Suzuki 2-group isomorphism if and only if

$$\Phi(g_1^x a_1^y) = \Phi(g_1)^x \Phi(a_1)^y \tag{4.28}$$

for all $x, y \in K$. Equality (4.28) implies that

$$g_2^{\varphi(x)} a_2^{\varphi(y)+\Psi(x)} = g_2^x a_2^y, \tag{4.29}$$

so $\varphi(x) = x$ and $\varphi(y)+\Psi(x) = y$ for all $x, y \in K$. Thus, $\varphi = Id_K$ and $\Psi = 0$. In this case, equality (4.20) gives that $f_1 = f_2$, i.e., $h_1 = h_2$. □

Example 4.3.1 *Let K be a field of characteristic 2 such that $dim_{K^2} K = 2$. Let $\{1,x\}$ be a basis for K over K^2. Fix $u \in K^2$. Define $h(a+bx) = a+bu$ for all $a,b \in K^2$.*

Then $h : K \to K^2$ is an additive, K^2-linear map. Define $f(x) = \sqrt{h(x)}$ for all $x \in K$. Let $G = K \times K$ and $T = K^\star$. Define the multiplication operation on G by $(x_1,y_1)(x_2,y_2) = (x_1+x_2, y_1+y_2+x_1 f(x_2 x_1^{-1}))$ for all $x_i, y_i \in K$, $i = 1,2$.

Then T acts on G by componentwise multiplication and (G,T) is an abelian free Suzuki 2-group of exponent 4 over K.

4.4 Cohomological Interpretation of Abelian Suzuki 2-Groups of Exponent 4

4.4.1 Some Fundamentals from Cohomology Theory

Recall that If A,B are groups and $\pi : A \to B$ is a group homomorphism then the *kernel* of π is the subgroup $\{a \in A : \pi(a) = 0\}$ of A and it is denoted by $Ker\pi$, the *image* of π is the subgroup $\{\pi(a) : a \in A\}$ of B and it is denoted by $Im\pi$. An abelian group A, written additively, is called a (left) B-module if to each $\sigma \in B$ and $x \in A$, there corresponds a unique element $\sigma(x) \in A$ such that (i) $\sigma(x+y) = \sigma(x) + \sigma(y)$ and (ii) $\sigma\tau(x) = \sigma(\tau(x))$ for all $\sigma, \tau \in B$ and $x, y \in A$. Let A be an additive group and $B \leq A$. Then a (right) *transversal* of B in A (or a complete set of coset representatives) is a subset R of A consisting of one element from each coset of B in A. Then A is the disjoint union $A = \bigcup_{r \in R} B + r$. Thus, every element $a \in A$ has a unique factorization $a = b + r$, $b \in B$, $r \in R$. If $\pi : A \to C$ is surjective, then a *lifting* of $x \in C$ is an element $l(x) \in A$ with $\pi(l(x)) = x$. If we chose a lifting $l(x)$ for each $x \in C$, then the set of all such elements is a transversal of $Ker\pi$. In this case the function $l : C \to A$ is also called a *transversal*, thus both l and its image $l(C)$ are transversals.

Definition 4.4.1 *Let A be a B-module and n a nonnegative integer. By an* n-cochain *of B over A, we mean a function of n-variables from B into A, if $n > 0$, and an element of A if $n = 0$.*

We denote by $C^n(B,A)$ the set of all such n-cochains. We make $C^n(B,A)$ into a group by defining

$$(f+g)(\sigma_1,...,\sigma_n) = f(\sigma_1,...,\sigma_n) + g(\sigma_1,...,\sigma_n) \tag{4.30}$$

for all $f, g \in C^n(B,A)$ and $\sigma_i \in B$.

Definition 4.4.2 *If $f \in C^n(B,A)$, we define an (n+1)-cochain $\delta_{n+1}f$ by*

$$(\delta_{n+1}f)(\sigma_1,...,\sigma_{n+1}) = \sigma_1(f(\sigma_2,...,\sigma_{n+1})) + \sum_{i=1}^{n}(-1)^i f(\sigma_1,...,\sigma_i\sigma_{i+1},...,\sigma_{n+1})$$
$$+(-1)^{n+1} f(\sigma_1,...,\sigma_n). \tag{4.31}$$

Sometimes, we simply write δ instead of δ_{n+1}.

Definition 4.4.3 *Let*

$$Z^n(B,A) = \{f \in C^n(B,A) : \delta f = 0\},$$
$$B^n(B,A) = \{\delta f : f \in C^{n-1}(B,A)\} \quad \textbf{(4.32)}$$

for $n > 0$ *and* $B^0(B,A) = 0$*. The elements of* $Z^n(B,A)$*,* $B^n(B,A)$ *are called* n-cocycles *and* n-coboundaries*, respectively.* $B^n(B,A)$ *is a subgroup of* $Z^n(B,A)$ *since* $\delta : C^{n-1}(B,A) \to C^n(B,A)$ *is a homomorphism. We write* $Ker\delta_{n+1} = Z^n(B,A)$ *and* $Im\delta_n = B^n(B,A)$*. The quotient group*

$$H^n(B,A) = Z^n(B,A)/B^n(B,A) \quad \textbf{(4.33)}$$

is called the n-th cohomology group *of B over A.*

The following facts are from Suzuki [9] and Rotman [10].

Theorem 4.4.1 *Let A be an extension of B by C, and let* $l : C \to A$ *be a transversal. If B is abelian, then there is a homomorphism* $\theta : C \to Aut(B)$ *with*

$$\theta_x(b) = l(x) + b - l(x) \quad \textbf{(4.34)}$$

the conjugate of b by $l(x)$*, for every* $b \in B$*. Moreover, if* $l' : C \to A$ *is another transversal, then*

$$l(x) + b - l(x) = l'(x) + b - l'(x) \quad \textbf{(4.35)}$$

for all $b \in B$ *and* $x \in C$*.*

Proof. Since $B \lhd A$, the restriction $\gamma_a|_B$ of 'conjugation by a' γ_a to B is an automorphism of B for all $a \in A$. The function $\mu : A \to Aut(B)$, given by $a \mapsto \gamma_a|_B$ is a homomorphism: $\mu(a_1 + a_2) = \gamma_{a_1+a_2}|_B = \gamma_{a_1}|_B \circ \gamma_{a_2}|_B = \mu(a_1) + \mu(a_2)$ since $\gamma_{a_1+a_2}(b) = (a_1 + a_2) + b - (a_1 + a_2) = a_1 + a_2 + b - a_2 - a_1 = (\gamma_{a_1} \circ \gamma_{a_2})(b)$ for all $b \in B$.

Moreover $B \leq Ker\mu$, for B being abelian implies that each conjugation by $a \in B$ is the identity. Therefore μ induces a homomorphism $\mu' : A/B \to Aut(B)$, namely, $B + a \mapsto \mu(a)$.

The first isomorphism theorem gives an explicit isomorphism $\lambda : C \to A/B$: if $l : C \to A$ is a transversal, then $\lambda(x) = B + l(x)$. If $l' : C \to A$ is another transversal, then $l(x) - l'(x) \in B$, so that $B + l(x) = B + l'(x)$ for all $x \in C$. It follows that λ does not depend on the choice of transversal. Let $\theta : C \to Aut(B)$ be the composite $\theta = \mu'\lambda$. If $x \in C$ then $\theta_x = \mu'\lambda(x) = \mu'(B + l(x)) = \mu(l(x)) \in Aut(B)$; therefore if $b \in B$, then $\theta_x(b) = \mu(l(x))(b) = l(x) + b - l(x)$ does not depend on the choice of lifting $l(x)$. □

Remark 4.4.1 *(1) A homomorphism $\theta : C \to Aut(B)$ makes B into a C-set, where the action is given by $xb = \theta_x(b)$, written additively. The following formulas are valid for all $x, y, 1 \in C$ and $b_1, b_2 \in B$:*

$$x(b_1 + b_2) = xb_1 + xb_2$$

$$(xy)b_1 = x(yb_1)$$

$$1b_1 = b_1 \tag{4.36}$$

(2) If B is an abelian group and A is an extension of B by C, for every transversal $l : C \to A$

$$xb = \theta_x(b) = l(x) + b - l(x) \tag{4.37}$$

for all $x \in C$ and $b \in B$.

(3) When A is abelian, θ is the trivial homomorphism with $\theta_x = 1$ for all $x \in C$ then $b = xb = l(x) + b - l(x)$ and b commutes with all $l(x)$, hence with all $b' = b + l(x)$ for $b' \in B$.

(4) Let $\pi : A \to C$ be a surjective homomorphism with kernel B and choose a transversal $l : C \to A$ with $l(1) = 0$. Once this transversal has been chosen every element $a \in A$ has a unique expression of the form

$$a = b + l(x) \tag{4.38}$$

$b \in B$, $x \in C$.

There is a formula: for all $x, y \in C$,

$$l(x) + l(y) = f(x, y) + l(xy) \tag{4.39}$$

for some $f(x,y) \in B$*, because* $\pi(l(x)+l(y)) = \pi(l(x))\pi(l(y)) = xy = \pi(l(xy)) = c$ *for some* $c \in C$*, so* $\pi(l(x)+l(y)-l(xy)) = \pi(l(x)+l(y))\pi(l(xy))^{-1} = cc^{-1} = 1$*, hence* $l(x)+l(y)-l(xy) \in B$ *and both* $l(x)+l(y)$ *and* $l(xy)$ *represent the same coset of B.*

Definition 4.4.4 *If* $\pi : A \to C$ *is a surjective homomorphism with kernel B and if* $l : C \to A$ *is a tranversal with* $l(1) = 0$*, then the function* $f : C \times C \to B$*, determined by (4.39) is called a* cocycle *(or* factor set*) associated to A.*

Theorem 4.4.2 *Let* $\pi : A \to C$ *be a surjective homomorphism with kernel B, let* $l : C \to A$ *be a tranversal with* $l(1) = 0$*, and let* $f : C \times C \to B$ *be the corresponding cocycle. Then:*
(i) for all $x, y \in C$*,*

$$f(1,y) = 0 = f(x,1), \tag{4.40}$$

(ii) the cocycle identity *holds for every* $x,y,z \in C$*:*

$$f(x,y) + f(xy,z) = xf(y,z) + f(x,yz). \tag{4.41}$$

Proof. Put $x = 1$ in (4.39) to get $l(1)+l(y) = f(1,y)+l(y)$. Since $l(1) = 0$, we have $f(1,y) = 0$. A similar calculation shows that $f(x,1) = 0$. The cocycle identity follows from associativity:

$$[l(x)+l(y)]+l(z) = f(x,y)+l(xy)+l(z) = f(x,y)+f(xy,z)+l(xyz). \tag{4.42}$$

On the other hand, by (4.37)

$$l(x)+[l(y)+l(z)] = l(x)+f(y,z)+l(yz) = xf(y,z)+l(x)+l(yz)$$
$$= xf(y,z)+f(x,yz)+l(xyz). \tag{4.43}$$

The cocycle identity follows. □

Theorem 4.4.3 *Let B be an abelian group and A be an extension of B by C. A function* $f : C \times C \to B$ *is a cocycle associated to A if and only if it satisfies the cocycle identity*

$$xf(y,z) - f(xy,z) + f(x,yz) - f(x,y) = 0 \tag{4.44}$$

as well as $f(1,y) = 0 = f(x,1)$ *for all* $x,y,z \in C$*.*

Proof. Necessity is Theorem 4.4.2. To prove sufficiency, let A be the set of all ordered pairs $(b_1,x) \in B \times C$ equipped with the operation

$$(b_1,x)+(b_2,y) = (b_1 + xb_2 + f(x,y), xy). \quad \textbf{(4.45)}$$

We need to show that A is a group. The cocycle identity is needed to prove associativity:

$$[(b_1,x)+(b_2,y)]+(b_3,z) = (b_1+xb_2+f(x,y),xy)+(b_3,z)$$
$$= (b_1+xb_2+f(x,y)+xyb_3+f(xy,z),xyz), \quad \textbf{(4.46)}$$

$$(b_1,x)+[(b_2,y)+(b_3,z)] = (b_1,x)+(b_2+yb_3+f(y,z),yz)$$
$$= (b_1+xb_2+xyb_3+xf(y,z)+f(x,yz),xyz). \quad \textbf{(4.47)}$$

The identity is $(0,1)$. Inverses are given by

$$-(b,x) = (-x^{-1}b - x^{-1}f(x,x^{-1}),x^{-1}). \quad \textbf{(4.48)}$$

Define $\pi : A \to C$ by $(a,x) \mapsto x$. π is a surjective homomorphism with kernel $\{(b,1) : b \in K\}$. If we identify B with the kernel of π via $b \mapsto (b,1)$ then $B \triangleleft A$ and A is an extension of B by C.

Next, we must show, for every transversal $l : C \to A$, that $xb = l(x)+b-l(x)$ for all $x \in C$ and $b \in B$. Now we must have $l(x) = (b_1,x)$ for some $b_1 \in B$. So,

$$l(x)+b-l(x) = (b_1,x)+(b,1)-(b_1,x)$$
$$= b_1+xb,x)+(-x^{-1}b_1 - x^{-1}f(x,x^{-1}),x^{-1})$$
$$= (b_1+xb+x[-x^{-1}b_1 - x^{-1}f(x,x^{-1})]+f(x,x^{-1}),1)$$
$$= (b_1+xb-b_1-f(x,x^{-1})+f(x,x^{-1}),1) = (xb,1). \quad \textbf{(4.49)}$$

Finally, define a transversal $l : C \to A$ by $l(x) = (0,x)$ for all $x \in C$. The cocycle F corresponding to this transversal satisfies $F(x,y) = l(x)+l(y)-l(xy)$. Then

$$F(x,y) = l(x)+l(y)-l(xy) = (0,x)+(0,y)-(0,xy)$$
$$= (f(x,y),xy)+(-(xy)^{-1}f(xy,(xy)^{-1}),(xy)^{-1})$$
$$= (f(x,y)-f(xy,(xy)^{-1})+f(xy,(xy)^{-1}),1) = (f(x,y),1) \quad \textbf{(4.50)}$$

and so f is a cocycle as desired. □

Remark 4.4.2 *If B is an abelian group and a group C acts on B then*

$$Z^2(C,B) = \{f \in C^2(C,B) : \delta f = 0\}$$
$$= \{f : C\times C \to B : \delta f(x,y,z) = xf(y,z) - f(xy,z) + f(x,yz) - f(x,y) = 0\}. \quad \textbf{(4.51)}$$

So, $f \in Z^2(C,B)$ if and only if f satisfies the cocycle identity (4.44). Thus, $Z^2(C,B)$ is the set of all cocycles $f : C\times C \to B$. Since B is abelian, $Z^2(C,B)$ is an abelian group under pointwise addition: $(f+g)(x,y) = f(x,y)+g(x,y)$ for $f,g \in Z^2(C,B)$, for all $(x,y) \in C\times C$. If f, g are cocycles, then so is $f+g$ (for $f+g$ also satisfies the cocycle identity and vanishes on $(1,y)$ and $(x,1)$).

Lemma 4.4.1 *Let B be an abelian group and A be an extension of B by C. Let l and l' be transversals with $l(1) = 0 = l'(1)$ giving rise to cocycles f and f', respectively. Then there is a function $h : C \to B$ with $h(1) = 0$ such that*

$$f'(x,y) - f(x,y) = xh(y) - h(xy) + h(x) \quad \textbf{(4.52)}$$

for all $x,y \in C$.

Proof. For each $x \in C$, both $l(x)$ and $l'(x)$ are representatives of the same coset of B in A; thus there is an element $h(x) \in B$ with $l'(x) = h(x)+l(x)$. Since $l(1) = 0 = l'(1)$, we have $h(1) = 0$. The main formula is derived as follows: By using (4.37) and (4.39) we have

$$l'(x)+l'(y) = [h(x)+l(x)]+[h(y)+l(y)] = h(x)+xh(y)+l(x)+l(y)$$
$$= h(x)+xh(y)+f(x,y)+l(xy) = h(x)+xh(y)+f(x,y)-h(xy)+l'(xy). \quad \textbf{(4.53)}$$

Therefore, $f'(x,y) = h(x)+xh(y)+f(x,y)-h(xy)$. The desired formula follows because each term lies in the abelian group B. □

Remark 4.4.3 *If B is an abelian group and a group C acts on B then the set of all coboundaries is*

$$B^2(C,B) = \{\delta h : h \in C^1(C,B)\}. \quad \textbf{(4.54)}$$

Thus, $g \in B^2(C,B)$ if and only if $g(x,y) = \delta h(x,y) = xh(y) - h(xy) + h(x)$ for some $h \in C^1(C,B)$, for all $x,y \in C$. $B^2(C,B)$ is a subgroup of $Z^2(C,B)$.

Recall that if $\{A_n\}$ is a sequence groups and $f_n : A_n \to A_{n+1}$ are homomorphisms with $Imf_{n-1} = Kerf_n$ for each n, then the sequence

$$\ldots \longrightarrow A_{n-1} \xrightarrow{f_{n-1}} A_n \xrightarrow{f_n} A_{n+1} \longrightarrow \ldots \qquad \textbf{(4.55)}$$

is said to be *exact*. A diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & A & \xrightarrow{\kappa} & B & \xrightarrow{\lambda} & C & \longrightarrow & 1 \\ & & \alpha\big\downarrow & & \beta\big\downarrow & & \gamma\big\downarrow & & \\ 1 & \longrightarrow & A' & \xrightarrow{\sigma} & B' & \xrightarrow{\rho} & C' & \longrightarrow & 1 \end{array} \qquad \textbf{(4.56)}$$

with $\sigma\alpha = \beta\kappa$ and $\gamma\lambda = \rho\beta$ is called a commutative diagram.

Lemma 4.4.2 *Assume that the diagram (4.56) is commutative and that both the upper and lower rows are exact. If α and γ are monomorphisms (epimorphisms or isomorphisms), then so is β.*

Proof. Assume that both α and γ are monomorphisms. We will show that β is a monomorphism. Suppose that $\beta(b) = 1$ for some $b \in B$. Then we have $1 = \rho\beta(b) = \gamma\lambda(b)$ from which we get $\lambda(b) = 1$ since γ is monomorphism. Since the upper row is exact we have $Ker\lambda = Im\kappa$. Therefore, there is an element $a \in A$ such that $b = \kappa(a)$. Then we have $1 = \beta(b) = \beta\kappa(a)$.

The commutativity of diagram gives us $\sigma\alpha = \beta\kappa$ and $\sigma\alpha(a) = 1$. Since the lower row is exact, σ is injective. Now both σ, α are injective, so we get $a = 1$ and $b = \kappa(a) = 1$. This proves that β is a monomorphism.

Assume that both α and γ are surjective. Let $b' \in B'$. We will show that $b' \in \beta(B)$. Since γ is surjective, there is an element $c \in C$ such that $\rho(b') = \gamma(c)$. By assumption, the upper row is exact. We conclude that $c = \lambda(b)$ for some element $b \in B$. So, $\rho(b') = \gamma(\lambda(b)) = \rho(\beta(b))$ by the commutativity of the diagram. Set $b' = \beta(b)b''$. Then we have $\rho(b'') = 1$. Since the lower row is exact, we have $Ker\rho = Im\sigma$. Hence, there is an element $a' \in A'$ such that $\sigma(a') = b''$. As α is also surjective, we have an element $a \in A$ satisfying $\alpha(a) = a'$. Thus, $b'' = \sigma(\alpha(a)) = \beta(\kappa(a))$ and we have $b' = \beta(b\kappa(a)) \in \beta(B)$. This shows that β is an epimorphism. □

Definition 4.4.5 *Let B, C be two fixed groups. Two extensions A, A' of B by C are called* equivalent *if there is a homomorphism $\varphi : A \to A'$ such that the following diagram is commutative.*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & B & \xrightarrow{i} & A & \xrightarrow{\pi} & C & \longrightarrow & 1 \\ & & {\scriptstyle Id_B}\downarrow & & {\scriptstyle \varphi}\downarrow & & {\scriptstyle Id_C}\downarrow & & \\ 0 & \longrightarrow & B & \xrightarrow{i} & A' & \xrightarrow{\pi'} & C & \longrightarrow & 1 \end{array} \qquad \mathbf{(4.57)}$$

In the above diagram, the two rows are the short exact sequences representing the extensions A and A', and the homomorphisms $B \to B$ and $C \to C$ are the identity mappings.

Remark 4.4.4 *By Lemma 4.4.2, if A and A' are equivalent by a homomorphism φ, then φ is an isomorphism. So, A' is equivalent to A by the inverse isomorphism φ^{-1}. Thus, Definition 4.4.11 defines an equivalence relation among the extensions of B by C. Also equivalent extensions are isomorphic.*

Theorem 4.4.4 *Assume that B is an abelian group and a group C acts on B. Then the equivalence classes of the extensions of B by C are in a one-to-one correspondence with the elements of the second cohomology group $H^2(C,B) = Z^2(C,B)/B^2(C,B)$.*

Proof. Let A, A' be two extensions of B by C. Then there are cocycles $f, f' : C \times C \to B$ arising from liftings l, l', respectively.

Assume that A, A' be equivalent extensions of B by C. Consider the diagram (4.57) where $\pi : A \to C$, $\pi' : A' \to C$ are the surjective homomorphisms with kernel B. Since A, A' are equivalent, (4.57) is a commutative diagram, so $\varphi i =$ and $Id_C\pi = \pi'\varphi$. Then $\varphi(b) = \varphi i(b) = iId_B(b) = b$ for all $b \in B$ and $x = \pi(l(x)) = Id_C\pi(l(x)) = \pi'\varphi(l(x))$ for all $x \in C$; that is $\varphi \circ l : C \to A'$ is a lifting. Applying φ to the equation $l(x) + l(y) = f(x,y) + l(xy)$, we obtain that φf is the cocycle determined by the lifting $\varphi \circ l$. But $\varphi f(x,y) = f(x,y)$ for all $x, y \in C$, because $f(x,y) \in B$. Therefore, $\varphi f = f$, that is, f is a cocycle of A'. But f' is also a cocycle of A' (arising from another lifting), and so Lemma 4.4.1 gives $f' - f \in B^2(C,B)$.

Conversely, assume that f, f' lie in the same coset of $B^2(C,B)$ in $Z^2(C,B)$, that is, $f' - f \in B^2(C,B)$. Then there is a function $h : C \to B$ with $h(1) = 0$ such that

$$f'(x,y) - f(x,y) = \delta h(x,y) = xh(y) - h(xy) + h(x) \tag{4.58}$$

for all $x, y \in C$. By (4.38) every element of A has a unique expression of the form $b + l(x)$ for $b \in B$ and $x \in C$. By (4.37) and (4.39) we have

$$[b_1 + l(x)] + [b_2 + l(y)] = b_1 + xb_2 + l(x) + l(y) = b_1 + xb_2 + f(x,y) + l(xy), \tag{4.59}$$

so addition in A is given by

$$[b_1 + l(x)] + [b_2 + l(y)] = b_1 + xb_2 + f(x,y) + l(xy). \tag{4.60}$$

There is a similar description of addition in A'. Define $\varphi : A \to A'$ by $\varphi(b + l(x)) = b - h(x) + l'(x)$. Consider the diagram (4.56). Since $l(1) = l'(1) = 0 = h(1)$,

$$\varphi(b) = \varphi(b + l(1)) = b - h(1) + l'(1) = b, \tag{4.61}$$

so φ fixes B pointwise, thus $\varphi i = iId_B$. Also $Id_C\pi(b + l(x)) = \pi(b + l(x)) = \pi(b) + \pi(l(x)) = x$ and $\pi'\varphi(b + l(x)) = \pi'(b - h(x) + l'(x)) = pi'(l'(x)) = x$ give $Id_C\pi = \pi'\varphi$. We have shown that the diagram commutes.

φ is a homomorphism since

$$\begin{aligned} &\varphi((b_1 + l(x)) + (b_2 + l(y)) = \varphi(b_1 + xb_2 + f(x,y) + l(xy)) \\ &= b_1 + xb_2 + f(x,y) - h(xy) + l'(xy) \end{aligned} \tag{4.62}$$

and

$$\begin{aligned} &\varphi(b_1 + l(x)) + \varphi(b_2 + l(y)) = b_1 - h(x) + l'(x)) + (b_2 - h(y) + l'(y)) \\ &= b_1 - h(x) + xb_2 - xh(y) + f'(x,y) + l'(xy) \\ &= b_1 + xb_2 + f(x,y) - h(xy) + l'(xy). \end{aligned} \tag{4.63}$$

where the last equality follows by (4.58). Note that by Lemma 4.4.2, φ becomes an isomorphism. Therefore, A, and A' are equivalent extensions of B by C. □

4.4.2 Relation between Abelian Suzuki 2-Groups of Exponent 4 and the Second Cohomology Group

Theorem 4.4.5 *There is a one-to-one correspondence between the isomorphism classes of abelian Suzuki 2-groups of exponent 4 over a field K of characteristic 2 and the elements of the 2-dimensional cohomology group $H^2(K,K)$ whose preimages $\tilde{f} \in Z^2(K,K)$ are the cocycles satisfying the following properties:*

(i) $\tilde{f}$ is symmetric and K-multiplicative,

(ii) $\tilde{f}(1,1) = 1$,

(iii) $\tilde{f}(x,0) = 0$ for all $x \in K$.

Proof. Let G be an abelian Suzuki 2-groups of exponent 4 over a field K of characteristic 2. Then by Lemma 2.2.2 and Theorem 2.2.1, we may assume that G is free. But then by Lemma 2.2.12, $G =< g^T >$ for any element $g \in G$ of order 4 and there is a function $f : K \to K$ such that $gg^x = g^{1+x}a^{f(x)}$ for all $x \in K$ where $a = g^2$. Now for all $x, y \in K$,

$$g^x g^y = (gg^{x^{-1}y})^x = (g^{1+x^{-1}y}a^{f(x^{-1}y)})^x = g^{x+y}a^{xf(x^{-1}y)}. \quad \textbf{(4.64)}$$

We define a map $\tilde{f} : K \times K \to K$ by

$$\tilde{f}(x,y) = xf(x^{-1}y) \quad \textbf{(4.65)}$$

for all $x, y \in K \backslash \{0\}$ and

$$\tilde{f}(x,0) = 0 = \tilde{f}(0,x) \quad \textbf{(4.66)}$$

for all $x \in K$. The group of involutions $I \cong K^+$ where K^+ denotes the additive group of K (as described in the proof of Lemma 2.2.12) and by Theorem 2.2.2, $G \cong K \times K$. So, G is an extension of K by K. Since K is abelian, the action of K on K, defined by the equality (4.37) becomes the trivial action. Hence, the cocycle identity (4.41) becomes

$$\tilde{f}(x,y) + \tilde{f}(x+y,z) = \tilde{f}(y,z) + \tilde{f}(x,y+z). \quad \textbf{(4.67)}$$

We will prove that $\tilde{f}$ is a cocycle associated to G. Since we have the equalities (4.66), by Theorem 4.4.4, it is enough to show that $\tilde{f}$ satisfies (4.67). We will use the associativity of G.

$$(g^x g^y)g^z = (gg^{x^{-1}y})^x g^z = (g^{1+x^{-1}y}a^{f(x^{-1}y)})^x g^z = g^{x+y}a^{xf(x^{-1}y)}g^z$$
$$= g^{x+y}g^z a^{xf(x^{-1}y)} = (gg^{(x+y)^{-1}z})^{x+y}a^{xf(x^{-1}y)}$$
$$= (g^{1+(x+y)^{-1}z}a^{f((x+y)^{-1}z)})^{x+y}a^{xf(x^{-1}y)}$$
$$= g^{x+y+z}a^{(x+y)f((x+y)^{-1}z)+xf(x^{-1}y)} = g^{x+y+z}a^{\tilde{f}(x+y,z)+\tilde{f}(x,y)}. \quad \textbf{(4.68)}$$

On the other hand,

$$g^x(g^y g^z) = g^x(gg^{y^{-1}z})^y = g^x(g^{1+y^{-1}z}a^{f(y^{-1}z)})^y = g^x g^{y+z}a^{yf(y^{-1}z)}$$
$$= (gg^{x^{-1}(y+z)})^x a^{yf(y^{-1}z)} = (g^{1+x^{-1}(y+z)}a^{f(x^{-1}(y+z))})^x a^{yf(y^{-1}z)}$$
$$= g^{x+y+z}a^{xf(x^{-1}(y+z))+yf(y^{-1}z)} = g^{x+y+z}a^{\tilde{f}(x,y+z)+\tilde{f}(y,z)} \quad \textbf{(4.69)}$$

for all $x,y,z \in K^\star$. Associativity implies that $\tilde{f} \in Z^2(K,K)$ is a cocycle associated to G.

Since G is abelian and free, $g^{x+y}a^{\tilde{f}(x,y)} = g^x g^y = g^y g^x = g^{x+y}a^{\tilde{f}(y,x)}$ implies that $\tilde{f}(x,y) = \tilde{f}(y,x)$ for all $x,y \in K$, so $\tilde{f}$ is symmetric.

Since T acts on G by group automorphisms, for all $x,y,t \in T$, we have $g^{tx+ty}a^{\tilde{f}(tx,ty)} = g^{tx}g^{ty} = (g^x g^y)^t = g^{tx+ty}a^{t\tilde{f}(x,y)}$ which gives that $\tilde{f}$ is K-multiplicative. The equalities $g^x = g^x g^0 = g^x a^{\tilde{f}(x,0)}$ and $a = gg = a^{\tilde{f}(1,1)}$ imply (ii) and (iii), respectively.

Conversely, assume that $\tilde{f} \in Z^2(K,K)$ is a cocycle satisfying properties (i), (ii) and (iii). Put $G = K \times K$ and $T = K^\star$. We define the multiplication operation on G by

$$(x,t)(y,z) = (x+y, t+z+\tilde{f}(x,y)) \quad \textbf{(4.70)}$$

for all $x,y,t,z \in K$. Then, by (iii), $(0,0)$ becomes the identity element. By (ii), $(x,y)^{-1} = (x, x+y)$ since

$$(x,y)(x,x+y) = (0, x+\tilde{f}(x,x)) = (0, x+x\tilde{f}(1,1)) = (0,x+x) = (0,0). \quad \textbf{(4.71)}$$

In order to prove that the operation defined by equality (4.70) is associative in G, we will use that $\tilde{f}$ is a cocycle.

$$[(x,0)(y,0)](z,0) = (x+y, \tilde{f}(x,y))(z,0) = (x+y+z, \tilde{f}(x,y)+\tilde{f}(x+y,z)). \quad \textbf{(4.72)}$$

On the other hand,

$$(x,0)[(y,0)(z,0)] = (x,0)(y+z,\tilde{f}(y,z)) = ((x+y+z,\tilde{f}(y,z)+\tilde{f}(x,y+z)). \qquad \textbf{(4.73)}$$

Equalities (4.72) and (4.73) together with the cocycle identity (4.67) imply the associativity in G.

Since $\tilde{f}$ is symmetric, G is abelian and nilpotent. The exponent of G is 4. Since $\tilde{f}$ is K-multiplicative, for all $x,y \in K$ and $t \in T$, we have

$$(x,0)^t(y,0)^t = (tx,0)(ty,0) = (tx+ty,\tilde{f}(tx,ty) = (tx+ty,t\tilde{f}(x,y)) = [(x,0)(y,0)]^t, \qquad \textbf{(4.74)}$$

thus T acts on G by group automorphisms by componentwise multiplication. If $(0,x)$, $(0,y)$ are any two nontrivial involutions, then $(0,x) = (0,y)^{xy^{-1}}$. Thus, T acts transitively on I where $I \cong K^+$ and (G,T) is an abelian Suzuki 2-group over K.

Therefore, by Theorem 4.4.5, isomorphism classes of abelian Suzuki 2-groups of exponent 4 over a field K of characteristic 2 are in a one-to-one correspondence with the elements of the 2-dimensional cohomology group $H^2(K,K)$ whose preimages $\tilde{f} \in Z^2(K,K)$ are the cocycles satisfying properties (i), (ii) and (iii). □

5. NONABELIAN SUZUKI 2-GROUPS OF EXPONENT 4

5.1 Quasi-abelian Suzuki 2-Groups

5.1.1 Notation and Terminology

Throughout this section K denotes a perfect field of characteristic 2, K^{+} and $K^{\star}$ denote the additive and the multiplicative groups of K, respectively. I denotes the set of involutions in a Suzuki 2-group G. When we say that a map "$f : K \times K \to K$ is K-multiplicative" we mean that $f(tx,ty) = tf(x,y)$ for all $t,x,y \in K$. If (G,T) is a nonabelian free Suzuki 2-group of exponent 4 such that for any element $g \in G$, the subgroup $< g^{T} >$ is abelian, then we say that (G,T) is a *quasi-abelian Suzuki 2-group*.

5.1.2 Properties of Quasi-abelian Suzuki 2-Groups

Lemma 5.1.1 *Assume that (G,T) is a quasi-abelian Suzuki 2-group over K. Then G/I is a vector space over K by the induced action of T as multiplication of a vector by a scalar.*

Proof. Define the operation of multiplication by a scalar as $x\overline{g} = \overline{g^{x}}$ for $\overline{g} \in G/I, x \in K$. Define the vector addition on G/I by $\overline{g} + \overline{h} = \overline{gh}$ for all $\overline{g}, \overline{h} \in G/I$. Then

$$x(y\overline{g}) = x\overline{g^{y}} = \overline{g^{xy}} = (xy)\overline{g}, \qquad \textbf{(5.1)}$$

$$(x+y)\overline{g} = \overline{g^{x+y}} = \overline{g^{x}g^{y}} = \overline{g^{x}} + \overline{g^{y}}, \qquad \textbf{(5.2)}$$

$$x(\overline{g} + \overline{h}) = x\overline{gh} = \overline{(gh)^{x}} = \overline{g^{x}h^{x}} = \overline{g^{x}} + \overline{h^{x}} = x\overline{g} + x\overline{h}, \qquad \textbf{(5.3)}$$

$$1\overline{g} = \overline{g} \qquad \textbf{(5.4)}$$

for all $x, y \in K$, $\overline{g}, \overline{h} \in G/I$. Equalities (5.2) and (5.3) exhibit that the distributivity laws hold in G/I. Since G is of exponent 4, G/I is abelian and it becomes a vector space over K together with these operations. □

Definition 5.1.1 *If dimension of G/I over K is equal to $n-1$ for some $n \in \mathbb{N}$, then (G,T) is called an n-dimensional quasi-abelian Suzuki 2-group.*

Lemma 5.1.2 *Let (G,T) be a 3-dimensional quasi-abelian Suzuki 2-group over K. Then there are $g, h \in G$ such that $g^2 = h^2 = a$ and $\{\overline{g}, \overline{h}\}$ is a basis for G/I over K.*

Proof. Take any $g \in G \setminus I$ and put $a = g^2$. Since G is nonabelian, the subgroup $< g^T >$ of G is proper. Take any $h \in G \setminus < g^T >$ of order 4. If $h^2 = b$, then since T acts transitively on I, there is $s \in T$ with $a = b^s$. Without loss of generality, we may replace h by h^s to get $g^2 = h^2 = a$. If $x\overline{g} + y\overline{h} = 0$ for some $x, y \in K$ then $\overline{g^x h^y} = I$ which implies $x = y = 0$. Thus, $\{\overline{g}, \overline{h}\}$ is a basis of G/I over K. □

Lemma 5.1.3 *Let (G,T) be a 3-dimensional quasi-abelian Suzuki 2-group over K. Assume that $\{\overline{g}, \overline{h}\}$ is a basis for G/I over K for $g, h \in G$ with $g^2 = h^2 = a$. Then every element in G can be written uniquely as $g^x h^y a^z$ for $x, y, z \in T$.*

Proof. Take any $\eta \in G$. Then $\overline{\eta} \in G/I = \{x\overline{g} + y\overline{h} : x, y \in K\} = \{\overline{g^x h^y} : x, y \in K\}$. So, for some $x, y \in K$, η and $g^x h^y$ belong to the same I-coset in G, i.e., $\eta = g^x h^y a^z$ for some $z \in K$.

We prove uniqueness: If $g^x h^y a^z = g^{x'} h^{y'} a^{z'}$ for some $x, y, z, x', y', z' \in K$. So, $g^x h^y = g^{x'} h^{y'} a^{z+z'}$, then $x\overline{g} + y\overline{h} = x'\overline{g} + y'\overline{h}$ in G/I and since $\{\overline{g}, \overline{h}\}$ is basis we have $x = x'$, $y = y'$. But then since G is free, the equality $g^x h^y a^z = g^x h^y a^{z'}$ implies that $a^z = a^{z'}$, i.e., $z = z'$. □

5.1.3 Classification of 3-dimensional Quasi-abelian Suzuki 2-Groups

Given a K-multiplicative, biadditive, surjective map $f: K \times K \to K$ with $f(x,y) \neq x+y$ for any $x, y \in K^{\star} \times K^{\star}$, let $G_f = K \times K \times K$ and $T = K^{\star}$. Define the multiplication operation on G_f by

$$(x_1, y_1, z_1)(x_2, y_2, z_2) = (x_1 + x_2, y_1 + y_2, f(y_1, x_2) + \sqrt{x_1 x_2} + \sqrt{y_1 y_2} + z_1 + z_2). \tag{5.5}$$

Theorem 5.1.1 *Let (G,T) be 3-dimensional quasi-abelian Suzuki 2-group over K. Assume that $\{\overline{g}, \overline{h}\}$ is a basis for G/I over K for $g, h \in G$ with $g^2 = h^2 = a$. Then there exists a K-multiplicative, biadditive map $f: K \times K \to K$ such that $h^y g^x = g^x h^y a^{f(y,x)}$ and $f(x,y) \neq x+y$ for all $(x,y) \in K^{\star} \times K^{\star}$.*

Conversely, if $f: K \times K \to K$ is a K-multiplicative, biadditive map with $f(x,y) \neq x+y$ for any $x, y \in K^{\star} \times K^{\star}$, then (G_f, T) is a quasi-abelian Suzuki 2-group over the field K and the action of T on G_f is componentwise multiplication.

Proof of Theorem 5.1.1. Since G/I is abelian, $\overline{h^y g^x} = \overline{g^x h^y}$, so $g^x h^y$ and $h^y g^x$ belong to the same I-coset in G thus there exists a map $f: K \times K \to K$ with $g^x h^y = h^y g^x a^{f(y,x)}$. For $x, y \in K^{\star}$, the element $g^x h^y a^z$ is not an involution, so,

$$1 \neq (g^x h^y a^z)^2 = g^x h^y g^x h^y a^{z+z} = g^x g^x h^y h^y a^{f(x,y)} = a^{f(x,y)+x+y}, \tag{5.6}$$

hence $f(x,y) \neq x+y$ for all $(x,y) \in K^{\star} \times K^{\star}$.

Lemma 5.1.4 *f is biadditive.*

Proof of Lemma 5.1.4. By associativity in G

$$\begin{aligned} g^x h^{y_1+y_2} a^{f(y_1+y_2,x)} &= h^{y_1+y_2} g^x = h^{y_1} h^{y_2} g^x a^{\sqrt{y_1 y_2}} = h^{y_1} g^x h^{y_2} a^{f(y_2,x)+\sqrt{y_1 y_2}} \\ &= g^x h^{y_1} h^{y_2} a^{f(y_1,x)+f(y_2,x)+\sqrt{y_1 y_2}} = g^x h^{y_1+y_2} a^{f(y_1,x)+f(y_2,x)}. \end{aligned} \tag{5.7}$$

(5.7) implies that $f(y_1 + y_2, x) = f(y_1, x) + f(y_2, x)$. Similarly,

$$\begin{aligned} g^{x_1+x_2} h^y a^{f(y,x_1+x_2)} &= h^y g^{x_1+x_2} = h^y g^{x_1} g^{x_2} a^{\sqrt{x_1 x_2}} = g^{x_1} h^y g^{x_2} a^{f(y,x_1)+\sqrt{x_1 x_2}} \\ &= g^{x_1} g^{x_2} h^y a^{f(y,x_1)+f(y,x_2)+\sqrt{x_1 x_2}} = g^{x_1+x_2} h^y a^{f(y,x_1)+f(y,x_2)} \end{aligned} \tag{5.8}$$

Equality (5.8) gives $f(y,x_1+x_2)=f(y,x_1)+f(y,x_2)$.

Lemma 5.1.5 *f is K-multiplicative.*

Proof of Lemma 5.1.5.

$$g^{tx}h^{ty}a^{f(tx,ty)}=h^{ty}g^{tx}=(h^y g^x)^t=(g^x h^y)^t(a^{f(xy)})^t=g^{tx}h^{ty}a^{tf(x,y)}. \tag{5.9}$$

(5.9) implies that f is K-multiplicative.

Lemma 5.1.6 *The operation defined in (5.5) is associative.*

Proof of Lemma 5.1.6. Since f is biadditive, we obtain the following equalities:

$$\begin{aligned}
&[(x_1,y_1,z_1)(x_2,y_2,z_2)](x_3,y_3,z_3)\\
&=(x_1+x_2,y_1+y_2,f(y_1,x_2)+\sqrt{x_1x_2}+\sqrt{y_1y_2}+z_1+z_2)(x_3,y_3,z_3)\\
&=(x_1+x_2+x_3,y_1+y_2+y_3,f(y_1+y_2,x_3)+\sqrt{(x_1+x_2)x_3}+\sqrt{(y_1+y_2)y_3}\\
&+f(y_1,x_2)+\sqrt{x_1x_2}+\sqrt{y_1y_2}+z_1+z_2+z_3)\\
&=(x_1+x_2+x_3,y_1+y_2+y_3,f(y_1,x_3)+f(y_2,x_3)\\
&+\sqrt{x_1x_3}+\sqrt{x_2x_3}+\sqrt{y_1y_3}+\sqrt{y_2y_3}+f(y_1,x_2)+\sqrt{x_1x_2}+\sqrt{y_1y_2}+z_1+z_2+z_3)
\end{aligned} \tag{5.10}$$

On the other hand,

$$\begin{aligned}
&(x_1,y_1,z_1)[(x_2,y_2,z_2)(x_3,y_3,z_3]\\
&=(x_1,y_1,z_1)(x_2+x_3,y_2+y_3,f(y_2,x_3)+\sqrt{x_2x_3}+\sqrt{y_2y_3}+z_2+z_3)\\
&=(x_1+x_2+x_3,y_1+y_2+y_3,(y_1,x_2+x_3)+\sqrt{(x_2+x_3)x_1}+\sqrt{(y_2+y_3)y_1}\\
&+z_1+z_2+z_3+\sqrt{x_2x_3}+\sqrt{y_2y_3}+f(y_2,x_3))\\
&=(x_1+x_2+x_3,y_1+y_2+y_3,f(y_1,x_2)+f(y_1,x_3)+f(y_2,x_3)\\
&+\sqrt{x_1x_2}+\sqrt{x_1x_3}+\sqrt{y_1y_2}+\sqrt{y_1y_3}+\sqrt{x_2x_3}+\sqrt{y_2y_3}+z_1+z_2+z_3)
\end{aligned} \tag{5.11}$$

Equalities (5.10) and (5.11) give the associativity in G.

Lemma 5.1.7 *(G,T) is a quasi-abelian Suzuki 2-group over K.*

Proof of Lemma 5.1.7. Every element (x,y,z) in G has an inverse $(x,y,z)^{-1} = (x,y,f(y,x)+x+y+z)$, since for any $x,y,z \in K$, the equality

$$(x,y,z,)(a,b,c) = (x+a,y+b,f(y,a)+\sqrt{xa}+\sqrt{yb}+z+c) = (0,0,0) \quad \textbf{(5.12)}$$

implies $a = x$, $b = y$, and $c = f(y,x)+x+y+z$.

Note that G is nonabelian since $f(y_1,x_2) \neq f(y_2,x_1)$ in general.

Since $f(y,x) \neq x+y$ for any $x,y \in K^\star \times K^\star$, for $(x,y,z) \in G$,
$(x,y,z)(x,y,z) = (0,0,f(y,x)+x+y) = (0,0,0)$ if and only if $f(y,x) = x+y$ if and only if $x = y = 0$ and hence, the set of involutions I is equal to $\{(0,0,z) : z \in K\} \simeq K^+$. Since G/I is is of exponent 2, it is abelian, so G is nilpotent of class 2.

Note that since f is biadditive, $f(0,x) = f(0+0,x) = f(0,x)+f(0,x) = 0 = f(y,0)$ for all $x,y \in K$ and therefore, the center of G consists of involutions since $(x,y,z,)(a,b,c) = (x+a,y+b,f(y,a)+\sqrt{xa}+\sqrt{yb}+z+c) = (a+x,b+y,f(b,x)+\sqrt{xa}+\sqrt{yb}+z+c) = (a,b,c)(x,y,z,)$ for all $x,y,z \in K$, if and only if $f(y,a) = f(b,x)$ for all $x,y \in K$ if and only if $a = b = 0$

Since f is K-multiplicative, for all $x,y,z \in K$, $t \in T$, we have

$$\begin{aligned}
&(x_1,y_1,z_1)^t(x_2,y_2,z_2)^t = (tx_1,ty_1,tz_1)(tx_2,ty_2,tz_2) \\
&= (tx_1+tx_2,ty_1+ty_2,f(ty_1,tx_2)+t\sqrt{x_1x_2}+t\sqrt{y_1y_2}+tz_1+tz_2) \\
&= (x_1+x_2,y_1+y_2,f(y_1,x_2)+\sqrt{x_1x_2}+\sqrt{y_1y_2}+z_1+z_2)^t \\
&= ((x_1,y_1,z_1)(x_2,y_2,z_2))^t. \qquad \textbf{(5.13)}
\end{aligned}$$

Hence, T acts on G by automorphisms.

T acts transitively on the set of involutions for if $(0,0,s),(0,0,z)$ are nontrivial involutions then $(0,0,s) = (0,0,sz^{-1}z) = (0,0,z)^{sz^{-1}}$. Thus, (G,T) is a nonabelian free Suzuki 2-group of exponent 4 over K. Finally, for all $x \in K$, the subgroup $< (x,0,0)^T >$ of G is abelian and therefore (G,T) is a 3-dimensional quasi-abelian Suzuki 2-group.
□

5.1.4 An Example of a 3-dimensional Quasi-abelian Suzuki 2-Group

Example 5.1.1 *Let K be the polynomial field $\mathbb{F}_2(t,t^{1/2},...,t^{1/2^n},...)$, $G=K\times K\times K$ and $T=K^\star$. Let $f(x,y)=\sqrt{xy}$ for all $x,y\in K$. Define the multiplication operation in G by*

$$(x_1,y_1,z_1)(x_2,y_2,z_2)=(x_1+x_2,y_1+y_2,\sqrt{y_1x_2}+\sqrt{x_1x_2}+\sqrt{y_1y_2}+z_1+z_2). \qquad \textbf{(5.14)}$$

If $f(x,y)=\sqrt{xy}=x+y$ for some $x,y\in K$, then $y^{-1}\sqrt{xy}=y^{-1}x+1$ and $\sqrt{xy^{-1}}=y^{-1}x+1$. Substituting $z=xy^{-1}$, we have

$$z=z^2+1. \qquad \textbf{(5.15)}$$

Since $\mathbb{F}_4\not\subseteq K$, there is no element in K satisfying equality (5.15). Therefore, $f(x,y)=\sqrt{xy}=x+y$ if and only if $x=y=0$. Clearly, f is K-multiplicative and since characteristic of K is 2, it is biadditive. The action of T on G is componentwise multiplication. So, (G,T) is a 3-dimensional quasi-abelian Suzuki 2-group over the field K.

5.1.5 Classification of n-dimensional Quasi-abelian Suzuki 2-Groups

Theorem 5.1.2 *Let (G,T) be an (n+1)-dimensional quasi-abelian Suzuki 2-group over K, $n\in\mathbb{N}$. Then there are $g_1,...,g_n\in G$ such that $g_1^2=...=g_n^2=a$ and $\{\overline{g_1},...,\overline{g_n}\}$ is a basis for G/I over K. Furthermore any element in G can be written uniquely as $g_1^{x_1}...g_n^{x_n}a^y$ for $x_i,y\in K$, $i=1,...,n$ and there exist K-multiplicative, biadditive maps $f_{ij}:K\times K\to K$ $i,j=1,...,n$; $i<j$ such that*

$$\begin{aligned}&f_{12}(x_1,x_2)+...+f_{1n}(x_1,x_n)+f_{23}(x_2,x_3)+...+f_{2n}(x_2,x_n)+...\\&+f_{n-2,n-1}(x_{n-2},x_{n-1})+f_{n-2,n}(x_{n-2},x_n)+f_{n-1,n}(x_{n-1},x_n)\\&\neq x_1+...+x_n\end{aligned} \qquad \textbf{(5.16)}$$

for all $(x_i,x_j)\in K\times K$ with at least two nonzero $x_m,x_s\in K^\star$ for some $m,s\in\{1,...,n\}$ and

$$g_i^{x_i}g_j^{x_j}=g_j^{x_j}g_i^{x_i}a^{f_{ij}(x_i,x_j)} \qquad \textbf{(5.17)}$$

for all $(x_i, x_j) \in K \times K$.

Conversely, assume that $f_{ij} : K \times K \to K$ $i, j = 1, ..., n;$ $i < j$ *are K-multiplicative, biadditive maps satisfying inequality (5.16). Let* $G = K \times ... \times K = K^{n+1}$ *and* $T = K^{\star}$. *Define the multiplication operation on G by*

$$(x_1, ..., x_n, z_1)(y_1, ..., y_n, z_2) = (x_1 + y_1, ..., x_n + y_n,$$
$$f_{1,2}(y_1, x_2) + ... + f_{1,n}(y_1, x_n) + f_{2,3}(y_2, x_3)$$
$$+ ... + f_{2,n}(y_2, x_n) + ... + f_{n-1,n}(y_{n-1}, x_n))$$
$$+ \sqrt{x_1 y_1} + ... + \sqrt{x_n y_n} + z_1 + z_2). \quad \textbf{(5.18)}$$

Then T acts on G by componentwise multiplication and (G, T) *is an (n+1)-dimensional quasi-abelian Suzuki 2-group over K.*

Proof of Theorem 5.1.2.

Lemma 5.1.8 *There exist maps* $f_{ij} : K \times K \to K$, $i, j = 1, ..., n;$ $i < j$, *satisfying equality (5.17).*

Proof of Lemma 5.1.8. It follows from Theorem 5.1.1.

Lemma 5.1.9 *There exist* $g_1, ..., g_n \in G$ *such that* $g_1^2 = ... = g_n^2 = a$ *and* $\{\overline{g_1}, ..., \overline{g_n}\}$ *is a basis for* G/I *over K. Furthermore, any element in G can be written uniquely as* $g_1^{x_1} ... g_n^{x_n} a^y$ *for* $x_i, y \in K$, $i = 1, ..., n$.

Proof of Lemma 5.1.9. By Lemma 5.1.1, G/I is a vector space over K. The rest of the proof is analogous to the proofs of Lemma 5.1.2 and Lemma 5.1.3. Take any $g_1 \in G \setminus I$. Let $g_1^2 = a$. Take any $g_2 \in G \setminus < g_1^T >$. If $g_2^2 \neq a$ then we may replace g_2 by g_2^s for some $s \in T$ to obtain $g_1, g_2 \in G$ with $g_1^2 = g_2^2 = a$. Continuing this process, after n steps we obtain $g_1, ..., g_n \in G$ such that $\{\overline{g_1}, ..., \overline{g_n}\}$ is a basis for G/I over K. Then every element of G can be written uniquely as $g_1^{x_1} ... g_n^{x_n} a^z$ for $x_i, z \in K$, $i = 1, ..., n$.

Lemma 5.1.10 *The maps* f_{ij}*'s* $i, j = 1, ..., n;$ $i < j$ *are biadditive and K-multiplicative..*

Proof of Lemma 5.1.10. Proof is analogous to the proofs of Lemma 5.1.4, Lemma 5.1.5 and Lemma 5.1.6. By associativity of G, the maps f_{ij}'s $i,j=1,...,n$; $i<j$ are biadditive and since T acts by automorphisms on G, they are K-multiplicative.

Lemma 5.1.11 *Inequality (5.16) holds.*

Proof of Lemma 5.1.11. For $x_i, y \in K$, $i=1,...,n$, such that $x_m.x_s \neq 0$ for some $m,s \in \{1,...,n\}$, we have

$$\begin{aligned}
&1 \neq (g_1^{x_1} g_2^{x_2} ... g_n^{x_n} a^y)^2 \\
&= g_1^{x_1} g_2^{x_2} ... g_n^{x_n} g_1^{x_1} g_2^{x_2} ... (g_{n-1}^{x_{n-1}} g_n^{x_n}) a^{y+y} \\
&= g_1^{x_1} g_2^{x_2} ... g_n^{x_n} g_1^{x_1} g_2^{x_2} ... g_n^{x_n} g_{n-1}^{x_{n-1}} a^{f_{n-1,n}(x_{n-1},x_n)} \\
&= g_1^{x_1} ... g_{n-1}^{x_{n-1}} g_1^{x_1} ... g_{n-1}^{x_{n-1}} a^{f_{n-1,n}(x_{n-1},x_n)+...+f_{1,n}(x_1,x_n)} \\
&= ... \\
&= a^{f_{n-1,n}(x_{n-1},x_n)+...+f_{1,n}(x_1,x_n)+...+f_{1,2}(x_1,x_2)+x_1+...+x_n}. \qquad \textbf{(5.19)}
\end{aligned}$$

Since (G,T) is a free Suzuki 2-group, inequality (5.19) implies (5.16).

Lemma 5.1.12 *G is a nonabelian group of exponent 4 with the operation defined in (5.18).*

Proof of Lemma 5.1.12. Analogous to Lemma 5.1.7, associativity in G, follows by biadditivity of the maps f_{ij}'s $i,j=1,...,n$; $i<j$.

Every element in G has an inverse

$$\begin{aligned}
&(x_1,...,x_n,z)^{-1} = (x_1,...,x_n, f_{1,2}(x_1,x_2)+...+f_{1,n}(x_1,x_n)+...+f_{n-1,n}(x_{n-1},x_n) \\
&+x_1+...+x_n+z). \qquad \textbf{(5.20)}
\end{aligned}$$

G is nonabelian since,

$$\begin{aligned}
&f_{1,2}(y_1,x_2)+...+f_{1,n}(y_1,x_n)+...+f_{n-1,n}(y_{n-1},x_n) \neq \\
&f_{1,2}(x_1,y_2)+...+f_{1,n}(x_1,y_n)+...+f_{n-1,n}(x_{n-1},y_n), \qquad \textbf{(5.21)}
\end{aligned}$$

in general, for $x_i, y_i \in K$, $i = 1, ..., n$. G is a group of exponent 4, since

$$(x_1, ..., x_n, z)^4 = (0, ..., 0, f_{1,2}(x_1 x_2) + ... + f_{1,n}(x_1, x_n) + ... + f_{n-1,n}(x_{n-1}, x_n)$$
$$+ x_1 + ... + x_n)^2 = (0, ...0, 0). \quad \textbf{(5.22)}$$

Lemma 5.1.13 (G,T) *is an (n+1)-dimensional quasi-abelian Suzuki group.*

Proof of Lemma 5.1.13. Since the center of G consists of involutions, G is nilpotent of class 2. By componentwise multiplication, T acts freely on G by group automorphisms. T acts transitively on I, for if $(0, ..., 0, s), (0, ..., 0, z)$ are nontrivial involutions then $(0, ..., 0, s) = (0, ..., 0, sz^{-1}z) = (0, 0, z)^{sz^{-1}}$, $s, z \in K$. For any $x_i \in K$, the subgroups $< (0, ..., x_i, 0, ..., 0)^T >$ are abelian for $i = 1, ..., n$. Therefore, (G,T) is a quasi-abelian Suzuki 2-group over K. □

5.1.6 An Example of an n-dimensional Quasi-abelian Suzuki 2-Group

Example 5.1.2 *Fix* $n \in \mathbb{N}$. *Let* $K = F_2(t_1, ..., t_{n-1})(t_i^{1/2^m} : i = 1, ..., n-1; m \in \mathbb{N})$. *Define* $f_{ij} : K \times K \to K$ *by* $f_{ij}(x_i, x_j) = \sqrt{x_i x_j}$ *for* $i, j = 1, ..., n-1$; $i < j$. *Then* $f_{ij}'s$ *are biadditive, K-multiplicative maps satisfying (5.16) and (5.17). Let* $G = K \times ... \times K = K^n$ *and* $T = K^\star$. *Define the multiplication operation on G as in (5.18). Then T acts on G by componentwise multiplication and* (G,T) *is an n-dimensional quasi-abelian Suzuki 2-group over K.*

5.2 Smart Suzuki 2-Groups

5.2.1 Notation and Terminology

For any field K, we denote the additive and multiplicative groups in K by K^+ and $K^\star$, respectively. Let (G,T) be a nonabelian Suzuki 2-group of exponent 4. If T acts transitively and freely on G/I, then we say that (G,T) is a *smart Suzuki 2-group*.

If (G,T) is a smart Suzuki 2-group, by Lemma 2.2.1, $T \cup \{0\}$ and $T/C_T(I) \cup \{\tilde{0}\}$ become fields where 0 and $\tilde{0}$ are new symbols. Throughout this section, K denotes $T \cup \{0\}$ and k denotes $T/C_T(I) \cup \{\tilde{0}\}$. We have $K^+ \simeq G/I$ and $k^\oplus \simeq I$, where $+$ and $\oplus$

denote the induced additions on K and k, respectively. We say that K is the *wide field* and k is the *narrow field* associated to G.

5.2.2 Properties of Smart Suzuki 2-Groups

We define maps $\alpha : K \to k$ by $\alpha(t) = \bar{t}$ for all $t \in T$ and $\alpha(0) = \tilde{0}$; and $\beta : K \times K \to k$ by $a^{\beta(x,y)} = [g^x, g^y] = (g^x)^{-1}(g^y)^{-1}g^x g^y$ for all $x, y \in T$ and $\beta(0,0) = \tilde{0}$, for any $g \in G$ of order 4 with $a = g^2$.

Lemma 5.2.1 *Such β exists and unique, it is independent of the choice of the element $g \in G$.*

Proof. Since $G' = I$, such β exists and as $T/C_T(I)$ acts freely on I, β is unique. Moreover, since T acts transitively on G/I, β is independent of choice of the element $g \in G$, that is, if $\overline{h} \in G/I$, then $\overline{h} = \overline{g}^s$ for some $s \in T$, so

$$[h^x, h^y] = [(g^s)^x, (g^s)^y] = [g^x, g^y]^s = (a^s)^{\beta(x,y)} = (h^2)^{\beta(x,y)}. \quad \textbf{(5.23)}$$

□

α is called the *characteristic map* of G, and β is called the *commutator map* of G.

Lemma 5.2.2 *Let (G, T) be a smart Suzuki 2-group. Then G/I becomes a one dimensional vector space over K.*

Proof. We define the operation of multiplication by a scalar by $x.\overline{g} = \overline{g^x}$ for all $g \in G$, $x \in K$ and the vector addition addition on G/I by $\overline{g} + \overline{h} = \overline{gh}$ for all $g, h \in G$. Now G/I is an additive abelian group. We need distributivity: For any $t, x, y \in K$ and $g \in G$, $t(\overline{g}^x + \overline{g}^y) = t(\overline{g^x} + \overline{g^y}) = t(\overline{g^x g^y}) = \overline{(g^x g^y)^t} = \overline{g^{xt} g^{yt}} = \overline{g^{xt}} + \overline{g^{yt}} = t\overline{g}^x + t\overline{g}^y$. On the other hand, $(t+y)\overline{g}^x = (t+y)\overline{g^x} = \overline{g^{x(t+y)}} = \overline{g^{xt+xy}} = \overline{g}^{xt+xy} = \overline{g}^{xt}\overline{g}^{xy} = \overline{g^{xt}}\,\overline{g^{xy}} = t\overline{g}^x + y\overline{g}^x$. □

Let us fix an element $g \in G$ of order 4 and put $a = g^2$. We define a map $f_g : K \times K \to k$ by $g^x g^y = g^{x+y} a^{f_g(x,y)}$, for all $x, y \in K$.

Lemma 5.2.3 *Let (G, T) be a smart Suzuki 2-group and $g \in G$ be an element of order 4. Then* *(i) f_g is well-defined.*

(ii) f_g is a 2-cocycle.

Proof.

(i) Let $x,y,t \in K$ and put $a = g^2$. By definition of addition on K, $\overline{g}^{x+y} = \overline{g}^x\overline{g}^y$ so $g^x g^y = g^{x+y}a^{f_g(x,y)}$ and

$$g^{tx+ty}a^{f(tx,ty)} = g^{tx}g^{ty} = (g^x g^y)^t = (g^{x+y}a^{f_g(x,y)})^t = g^{tx+ty}a^{\bar{t}f_g(x,y)} \quad \textbf{(5.24)}$$

implies

$$a^{f_g(tx,ty)} = a^{\bar{t}f_g(x,y)}. \quad \textbf{(5.25)}$$

Let $h \in G$ be any other element of order 4. Since G is a smart Suzuki 2-group, $\overline{h} = \overline{g}^t$ for some $t \in T$. Then $h = g^t b$ for some $b \in I$, $h^2 = a^t$ and $g^t = hb$ so by (5.25)

$$\begin{aligned}
h^x h^y &= (g^t b)^x (g^t b)^y = g^{tx}g^{ty}b^x b^y = g^{tx}g^{ty}b^{\overline{x}}b^{\overline{y}} = g^{tx+ty}a^{f_g(tx,ty)}b^{\overline{x}\oplus\overline{y}} \\
&= g^{tx+ty}a^{\bar{t}f_g(x,y)}b^{\overline{x}\oplus\overline{y}} = (g^t)^{x+y}(a^t)^{f_g(x,y)}b^{\overline{x}\oplus\overline{y}} = (hb)^{x+y}(h^2)^{f_g(x,y)}b^{\overline{x}\oplus\overline{y}} \\
&= h^{x+y}b^{x+y}(h^2)^{f_g(x,y)}b^{\overline{x}\oplus\overline{y}} = h^{x+y}(h^2)^{f_g(x,y)}b^{x+y}b^{\overline{x}\oplus\overline{y}} \\
&= h^{x+y}(h^2)^{f_g(x,y)}b^{\overline{x+y}}b^{\overline{x}\oplus\overline{y}} = h^{x+y}(h^2)^{f_g(x,y)}b^{\overline{x+y}\oplus\overline{x}\oplus\overline{y}}.
\end{aligned} \quad \textbf{(5.26)}$$

Since T acts transitively on I, $b^{\overline{x+y}\oplus\overline{x}\oplus\overline{y}} = (h^2)^s$ for some $\overline{s} \in T/C_T(I)$. Therefore, (5.26) implies that

$$h^x h^y = h^{x+y}(h^2)^{f_g(x,y)}(h^2)^{\overline{s}} = h^{x+y}(h^2)^{f_g(x,y)\oplus\overline{s}}. \quad \textbf{(5.27)}$$

Since $h^x h^y = h^{x+y}(h^2)^{f_h(x,y)}$, we have $f_h(x,y) = f_g(x,y) \oplus \overline{s}$.

(ii) We need to show that f_g satisfies the cocycle identity (4.67). We have

$$(g^x g^y)g^z = g^{x+y}a^{f_g(x,y)}g^z = g^{x+y}g^z a^{f_g(x,y)} = g^{x+y+z}a^{f_g(x,y)\oplus f_g(x+y,z)}. \quad \textbf{(5.28)}$$

On the other hand,

$$g^x(g^y g^z) = g^x g^{y+z}a^{f_g(y,z)} = g^{x+y+z}a^{f_g(y,z)\oplus f_g(x,y+z)}. \quad \textbf{(5.29)}$$

The result follows by the associativity in G. □

We say that f_g is a *2-cocycle associated to G*.

Lemma 5.2.4 *We have the following commutator properties for any $g,h \in G$ of order 4 and $x \in K$:*

$$[g,h] = g^{-1}h^{-1}gh = (gh)^2g^2h^2, \tag{5.30}$$

$$[g_1g_2,h] = [g_1,h][g_2,h], \tag{5.31}$$

$$[g^x,h^x] = [g,h]^x. \tag{5.32}$$

Proof. Since $g^2,h^2 \in I = Z(G)$,

$$[g,h] = g^{-1}h^{-1}gh = g^3h^3gh = gg^2h^2hgh = gg^2hghh^2 = ghghg^2h^2 = (gh)^2g^2h^2. \tag{5.33}$$

By using (5.30) and the equality $gh = hg[g,h]$ repeatedly, we have

$$\begin{aligned}
[g_1g_2,h] &= (g_1g_2h)^2(g_1g_2)^2h^2 = g_1g_2hg_1g_2h(g_1g_2)^2h^2\\
&= g_2g_1[g_1,g_2]hg_1g_2h(g_1g_2)^2h^2 = g_2g_1[g_1,g_2]hg_1hg_2[g_2,h](g_1g_2)^2h^2\\
&= g_2g_1hg_1hg_2[g_1,g_2][g_2,h](g_1g_2)^2h^2 = g_2(g_1h)^2g_2[g_1,g_2][g_2,h](g_1g_2)^2h^2\\
&= (g_1h)^2g_2^2[g_1,g_2][g_2,h](g_1g_2)^2h^2\\
&= (g_1h)^2g_2^2[g_1,g_2][g_2,h]g_1^2g_2^2[g_1,g_2]h^2\\
&= (g_1h)^2[g_1,g_2][g_2,h]g_1^2[g_1,g_2]h^2 = (g_1h)^2[g_2,h]g_1^2h^2\\
&= (g_1h)^2g_1^2h^2[g_2,h] = [g_1,h][g_2,h].
\end{aligned} \tag{5.34}$$

Finally, again by using (5.30) repeatedly, we obtain

$$[g^x,h^x] = (g^xh^x)^2(g^x)^2(h^x)^2 = ((gh)^2)^x(g^2)^x(h^2)^x = ((gh)^2g^2h^2)^x = [g,h]^x \tag{5.35}$$

since T acts on G by automorphisms.

Proposition 5.2.1 *Let (G,T) be a smart Suzuki 2-group. Then the following are satisfied for all $x,y,z \in K$:*
(i) $\beta(x,y) = \delta\alpha(x,y) = \alpha(x+y) \oplus \alpha(x) \oplus \alpha(y)$.
(ii) β is biadditive,

(iii) $\beta(zx,zy) = \alpha(z)\beta(x,y)$,

(iv) $\beta(x,y) = f_g(x,y) \oplus f_g(y,x)$, *for any element* $g \in G$ *of order 4.*

(v) $\alpha(x) = f_g(x,x)$ *for any element* $g \in G$ *of order 4.*

Proof.

(i)

$$a^{\beta(x,y)} = [g^x, g^y] = (g^x)^{-1}(g^y)^{-1}g^x g^y = (g^x g^y)^2(g^x)^2(g^y)^2$$
$$= (g^{x+y})^2(g^x)^2(g^y)^2 = a^{x+y}a^x a^y = a^{\overline{x+y}}a^{\bar{x}}a^{\bar{y}}$$
$$= a^{\alpha(x+y)}a^{\alpha(x)}a^{\alpha(y)} = a^{\alpha(x+y)\oplus\alpha(x)\oplus\alpha(y)} = a^{\delta\alpha(x,y)}. \tag{5.36}$$

Since $T/C_T(I)$ acts freely on I, $\beta(x,y) = \delta\alpha(x,y)$.

(ii) By (5.31),

$$a^{\beta(x+y,z)} = [g^{x+y}, g^z] = [g^x g^y, g^z] = [g^x, g^z][g^y, g^z]$$
$$= a^{\beta(x,z)}a^{\beta(y,z)} = a^{\beta(x,z)\oplus\beta(y,z)}. \tag{5.37}$$

Thus $\beta(x+y,z) = \beta(x,z) \oplus \beta(y,z)$.

(iii) By (5.32),

$$a^{\beta(zx,zy)} = [g^{zx}, g^{zy}] = [g^x, g^y]^z = (a^{\beta(x,y)})^z = (a^z)^{\beta(x,y)} = (a^{\alpha(z)})^{\beta(x,y)} = a^{\alpha(z)\beta(x,y)}. \tag{5.38}$$

Thus, $\beta(zx,zy) = \alpha(z)\beta(x,y)$.

(iv)Equality

$$a^{\beta(x,y)} = [g^x, g^y] = (g^x)^{-1}(g^y)^{-1}g^x g^y \tag{5.39}$$

implies

$$g^y g^x = g^x g^y a^{\beta(x,y)}. \tag{5.40}$$

On the other hand, we have

$$g^y g^x = g^{y+x}a^{f_g(y,x)}. \tag{5.41}$$

Now (5.40) and (5.41) together imply that

$$g^{y+x}a^{f_g(y,x)} = g^{x+y}a^{f_g(x,y)}a^{\beta(x,y)}, \tag{5.42}$$

so $a^{f_g(y,x)} = a^{f_g(x,y)\oplus\beta(x,y)}$, i.e., $\beta(x,y) = f_g(x,y) \oplus f_g(y,x)$.

If $h \in G$ is any other element of order 4, then $\overline{h}^t = \overline{g}$ for some $t \in T$ and $g = h^t b$, $b \in I$. Now by (5.32),

$$a^{\beta(x,y)} = [g^x, g^y] = [h^{tx}, h^{ty}] = [h^x, h^y]^t = h^{-tx}h^{-ty}h^{tx}h^{ty}, \tag{5.43}$$

so,

$$h^{ty}h^{tx}a^{\beta(x,y)} = h^{tx}h^{ty} \tag{5.44}$$

while by (5.27)

$$h^{tx}h^{ty} = h^{tx+ty}(h^2)^{\bar{t}f_g(x,y)\oplus\bar{t}\bar{s}} \tag{5.45}$$

for some $s \in T$. Thus, (5.44) and (5.45) together imply that

$$h^{ty+tx}(h^2)^{\bar{t}f_g(y,x)\oplus\bar{t}\bar{s}}a^{\beta(x,y)} = h^{tx+ty}(h^2)^{\bar{t}f_g(x,y)\oplus\bar{t}\bar{s}}. \tag{5.46}$$

Since $h^2 = a^{t^{-1}}$, (5.46) implies that

$$(a^{t^{-1}})^{\bar{t}f_g(y,x)\oplus\bar{t}\bar{s}}a^{\beta(x,y)} = (a^{t^{-1}})^{\bar{t}f_g(x,y)\oplus\bar{t}\bar{s}}, \tag{5.47}$$

that is,

$$(a^{t^{-1}t})^{f_g(y,x)\oplus\bar{s}}a^{\beta(x,y)} = (a^{t^{-1}t})^{f_g(x,y)\oplus\bar{s}} \tag{5.48}$$

which gives

$$a^{f_g(y,x)\oplus\bar{s}}a^{\beta(x,y)} = a^{f_g(x,y)\oplus\bar{s}}. \tag{5.49}$$

We have

$$a^{\beta(x,y)} = a^{f_g(x,y)\oplus\bar{s}}a^{f_g(y,x)\oplus\bar{s}} = a^{f_g(x,y)\oplus\bar{s}\oplus f_g(y,x)\oplus\bar{s}} = a^{f_g(x,y)\oplus f_g(y,x)} \tag{5.50}$$

and again we obtain $\beta(x,y) = f_g(x,y) \oplus f_g(y,x)$.

(v) Since K is of characteristic 2,

$$a^{\alpha(x)} = a^{\bar{x}} = a^{x} = (g^2)^x = (g^x)^2 = g^{x+x}a^{f_g(x,x)} = a^{f_g(x,x)}, \tag{5.51}$$

so $\alpha(x) = f_g(x,x)$. $\square$

Let $g,h \in G$ be elements of order 4 with $g^2 = a$, and $h = g^t b$ for some $t \in T$, $b \in I$ with $b = (h^2)^r$, $r \in T$.

Lemma 5.2.5 *Any two 2-cocycles f_g and f_h associated to G, differ by a multiple of the commutator map β of G, that is*

$$f_h(x,y) = f_g(x,y) \oplus \bar{r}\beta(x,y). \tag{5.52}$$

Proof. Equality (5.26) implies that

$$\begin{aligned} h^x h^y &= h^{x+y}(h^2)^{f_g(x,y)} b^{\overline{x+y} \oplus \bar{x} \oplus \bar{y}} \\ &= h^{x+y}(h^2)^{f_g(x,y)} b^{\alpha(x+y)\oplus\alpha(x)\oplus\alpha(y)} = h^{x+y}(h^2)^{f_g(x,y)} b^{\delta\alpha(x,y)} \\ &= h^{x+y}(h^2)^{f_g(x,y)} b^{\beta(x,y)} = h^{x+y}(h^2)^{f_g(x,y)} (h^2)^{\bar{r}\beta(x,y)} \\ &= h^{x+y}(h^2)^{f_g(x,y)\oplus\bar{r}\beta(x,y)}. \end{aligned} \tag{5.53}$$

Since we also have $h^x h^y = h^{x+y}(h^2)^{f_h(x,y)}$, the result follows by equality (5.54).

5.2.3 Classification of Smart Suzuki 2-Groups

Theorem 5.2.1 *The wide and narrow fields and the characteristic map associated to a smart Suzuki 2-group determine the structure of the Suzuki 2-group. That is, if (G_i, T_i) are smart Suzuki 2-groups for $i = 1,2$, then G_1 and G_2 are isomorphic smart Suzuki 2-groups if and only if there exist field isomorphisms $\Psi : K_1 \to K_2$, $\Phi : k_1 \to k_2$ such that the diagrams*

$$\begin{array}{ccc} K_1 \times K_1 & \longrightarrow & K_2 \times K_2 \\ \beta_1 \downarrow & & \beta_2 \downarrow \\ k_1 & \xrightarrow{\phi} & k_2 \end{array} \tag{5.54}$$

$$\begin{array}{ccc} K_1 & \xrightarrow{\Psi} & K_2 \\ \alpha_1 \big\downarrow & & \alpha_2 \big\downarrow \\ k_1 & \xrightarrow{\phi} & k_2 \end{array} \tag{5.55}$$

commute.

Proof. Assume that $\theta : G_1 \to G_2$ is a smart Suzuki 2-group isomorphism. Let $g_1 \in G_1$ be an element of order 4, put $a_1 = g_1^2$, $g_2 = \theta(g_1)$ and $a_2 = g_2^2$. Then there are maps $\Psi : K_1 \to K_2$ and $\Phi : k_1 \to k_2$ such that

$$\theta(g_1^{x_1}) = \theta(g_1)^{\Psi(x_1)} = g_2^{\Psi(x_1)}, \tag{5.56}$$

and that

$$\theta(a_1^{x_1}) = \theta(a_1^{\alpha_1(x_1)}) = a_2^{\Phi(\alpha_1(x_1))} \tag{5.57}$$

for all $x_1 \in K_1$. There is an isomorphism between G_1/I_1 and G_2/I_2 induced by θ. Thus,

$$\theta(\overline{g_1}^{x_1}\overline{g_1}^{y_1}) = \theta(\overline{g_1}^{x_1+y_1}) = \overline{g_2}^{\Psi(x_1+y_1)}. \tag{5.58}$$

On the other hand,

$$\theta(\overline{g_1}^{x_1})\theta(\overline{g_1}^{y_1}) = \overline{g_2}^{\Psi(x_1)}\overline{g_2}^{\Psi(y_1)} = \overline{g_2}^{\Psi(x_1)+\Psi(y_1)}, \tag{5.59}$$

now since T_2 acts freely on G_2/I_2, equalities (5.58) and (5.59) imply

$$\Psi(x_1+y_1) = \Psi(x_1)+\Psi(y_1) \tag{5.60}$$

for all $x_1, y_1 \in K_1$. Also,

$$\theta((\overline{g_1}^{x_1})^{y_1}) = \theta(\overline{g_1}^{x_1y_1}) = \overline{g_2}^{\Psi(x_1y_1)} \tag{5.61}$$

and that

$$\theta((\overline{g_1}^{x_1})^{y_1}) = \theta(\overline{g_1}^{x_1})^{\Psi(y_1)} = (\overline{g_2}^{\Psi(x_1)})^{\Psi(y_1)} = \overline{g_2}^{\Psi(x_1)\Psi(y_1)} \tag{5.62}$$

since T_1 acts freely on G_1/I_1, equalities (5.61) and (5.62) give

$$\Psi(x_1y_1) = \Psi(x_1)\Psi(y_1) \tag{5.63}$$

for all $x_1, y_1 \in K_1$. Equalities (5.60) and (5.63) give that $\Psi : K \to K$ is a field homomorphism. Since θ is a smart Suzuki 2-group isomorphism, $\theta^{-1} : G_2 \to G_1$ is also a Suzuki 2-group isomorphism such that

$$\overline{g_1}^{x_1} = \theta^{-1}(\overline{g_2}^{\Psi(x_1)}) = \theta^{-1}(\overline{g_2})^{\eta(\Psi(x_1))} = \overline{g_1}^{\eta(\Psi(x_1))}, \tag{5.64}$$

hence $\eta = \Psi^{-1}$, and $\Psi : K_1 \to K_2$ is a field isomorphism. Moreover, we have

$$\theta(a_1^{x_1} a_1^{y_1}) = \theta(a_1^{\alpha_1(x_1)} a_1^{\alpha_1(y_1)}) = \theta(a_1^{\alpha_1(x_1) \oplus \alpha_1(y_1)}) = a_2^{\phi(\alpha_1(x_1) \oplus \alpha_1(y_1))} \tag{5.65}$$

$$\theta(a_1^{x_1} a_1^{y_1}) = \theta(a_1^{\alpha_1(x_1)})\theta(a_1^{\alpha_1(y_1)}) = a_2^{\phi(\alpha_1(x_1))} a_2^{\phi(\alpha_1(y_1))} = a_2^{\phi(\alpha_1(x_1)) \oplus \phi(\alpha_1(y_1))}. \tag{5.66}$$

Since $T_2/C_{T_2}(I_2)$ acts freely on I_2, equalities (5.65) and (5.66) imply that

$$\phi(\overline{x_1} \oplus \overline{y_1}) = \phi(\alpha_1(x_1) \oplus \alpha_1(y_1)) = \phi(\alpha_1(x_1)) \oplus \phi(\alpha_1(y_1)) = \phi(\overline{x_1}) \oplus \phi(\overline{y_1}) \tag{5.67}$$

for all $\overline{x_1}, \overline{y_1} \in k_1$. We have

$$\begin{aligned} a_2^{\phi(\alpha_1(x_1)\alpha_1(y_1))} &= \theta(a_1^{\alpha_1(x_1)\alpha_1(y_1)}) = \theta((a_1^{x_1})^{y_1}) = \theta((a_1^{\alpha_1(x_1)})^{\alpha_1(y_1)}) \\ &= (a_2^{\phi(\alpha_1(x_1))})^{\phi(\alpha_1(y_1))} = a_2^{\phi(\alpha_1(x_1))\phi(\alpha_1(y_1))} \end{aligned} \tag{5.68}$$

so, ϕ is multiplicative and thus, ϕ is a field homomorphism. Moreover, $\phi : k_1 \to k_2$ is a field isomorphism since

$$a_1^{\overline{x_1}} = a_1^{\alpha_1(x_1)} = \theta^{-1}(a_2^{\phi(\alpha_1(x_1))}) = a_1^{\rho(\phi(\alpha_1(x_1)))} = a_1^{\rho(\phi(\overline{x_1}))} \tag{5.69}$$

for some homomorphism $\rho : k_2 \to k_1$. Since $T_1/C_{T_1}(I_1)$ acts freely on I_1, we have $\rho = \phi^{-1}$.

Finally, we will prove that diagrams (5.54) and (5.55) commute:

$$\begin{aligned} a_2^{\phi(\beta_1(x_1,y_1))} &= \theta(a_1^{\beta_1(x_1,y_1)}) = \theta([g_1^{x_1}, g_1^{y_1}]) = \theta(g_1^{-x_1})\theta(g_1^{-y_1})\theta(g_1^{x_1})\theta(g_1^{y_1}) \\ &= g_2^{-\Psi(x_1)} g_2^{-\Psi(y_1)} g_2^{\Psi(x_1)} g_2^{\Psi(y_1)} = [g_2^{\Psi(x_1)}, g_2^{\Psi(y_1)}] = a_2^{\beta_2(\Psi(x_1),\Psi(y_1))}. \end{aligned} \tag{5.70}$$

Since $T_2/C_{T_2}(I_2)$ acts freely on I_2, equality (5.70) implies that

$$\phi(\beta_1(x_1, y_1)) = \beta_2(\Psi(x_1), \Psi(y_1)) \tag{5.71}$$

for all $x_1, y_1 \in K_1$ and diagram (5.54) commutes. Also,

$$a_2^{\phi(\alpha_1(x_1))} = \theta(a_1)^{\phi(\alpha_1(x_1))} = \theta(a_1^{\alpha_1(x_1)}) = \theta(a_1^{x_1}) = \theta(a_1)^{\Psi(x_1)}$$
$$= a_2^{\Psi(x_1)} = a_2^{\alpha_2(\Psi(x_1))} \quad \textbf{(5.72)}$$

Equality (5.72) gives

$$\phi(\alpha_1(x_1)) = \alpha_2(\Psi(x_1)) \quad \textbf{(5.73)}$$

for all $x_1 \in K_1$ and diagram (5.55) commutes.

Conversely, assume that there exist the field isomorphisms $\Psi : K_1 \to K_2$, $\phi : k_1 \to k_2$ such that diagrams (5.54) and (5.55) commute. For $j = 1,2$, let I_j denote the set of involutions in G_j, fix an element $g_1 \in G_1$ of order 4. Since G_j are smart Suzuki 2-groups, $K_j^+ \cong G_j/I_j$. Identify K_j by G_j/I_j. Put

$$\overline{g_2} = \Psi(\overline{g_1}) \quad \textbf{(5.74)}$$

and

$$a_j = g_j^2. \quad \textbf{(5.75)}$$

Since $I_j \cong k_j^{\oplus}$, the field isomorphism $\Phi : k_1 \to k_2$ induces an isomorphism $I_1 \to I_2$ such that $a_1^{\overline{x_1}} \mapsto a_2^{\phi(\overline{x_1})}$. By abuse of notation we will write $\phi(a_1^{x_1})$ instead of $a_2^{\phi(\overline{x_1})}$.

Since characteristic of K_i is 2, K_i is a vector space over Z_2. Let Ω be an index set, and choose a basis $\{\overline{g_{1i}}\}_{i\in\Omega}$ for K_1.

Put

$$\overline{g_{2i}} = \Psi(\overline{g_{1i}}) \quad \textbf{(5.76)}$$

for $i \in \Omega$. Let $g_{1i} \in G_1$ and $g_{2i} \in G_2$ be the liftings of $\overline{g_{1i}}$ and $\overline{g_{2i}}$, respectively. Let $g \in G_1$. Then

$$\overline{g} = \overline{g_{11}} + \ldots + \overline{g_{1k}}. \quad \textbf{(5.77)}$$

So, $g = g_{11} \ldots g_{1k} a$ for some $a \in I_1$. We define $\theta : G_1 \to G_2$ by

$$\theta(g) = g_{21} \ldots g_{2k} \phi(a). \quad \textbf{(5.78)}$$

Lemma 5.2.6 *θ is well-defined.*

Proof of Lemma 5.2.4. Consider the case when $k = 2$. Let $\overline{g} = \overline{g_{11}} + \overline{g_{12}}$. Since G_1 is a smart Suzuki 2-group, T_1 acts transitively on G_1/I_1, so $\overline{g_{11}} = \overline{g_1}^{x_1}$, $\overline{g_{12}} = \overline{g_1}^{y_1}$ and $\overline{g_{21}} = \overline{g_2}^{\Psi(x_1)}$, $\overline{g_{22}} = \overline{g_2}^{\Psi(y_1)}$.

By commutativity of diagram (5.54), we have

$$\phi(\beta_1(x_1,y_1)) = \beta_2(\Psi(x_1),\Psi(y_1)) \tag{5.79}$$

for all $x_1, y_1 \in K_1$. Then

$$\begin{aligned} g_{21}g_{22} &= \theta(g_{11}g_{12}) = \theta(g) = \theta(g_{12}g_{11}[g_{11},g_{12}]) = g_{22}g_{21}\Phi([g_{11},g_{12}]) \\ &= g_{22}g_{21}\Phi([g_1^{x_1},g_1^{y_1}]) = g_{22}g_{21}\Phi(a_1^{\beta_1(x_1,y_1)}) \\ &= g_{22}g_{21}a_2^{\Phi(\beta_1(x_1,y_1))} = g_{22}g_{21}a_2^{\beta_2(\Psi(x_1),\Psi(y_1))} \\ &= g_{22}g_{21}[g_2^{\Psi(x_1)},g_2^{\Psi(y_1)}] = g_{22}g_{21}[g_{21},g_{22}]. \end{aligned} \tag{5.80}$$

The general case follows by induction.

Lemma 5.2.7 *θ is a homomorphism.*

Proof of Lemma 5.2.5. Let

$$g = g_{11}g_{12}\cdots g_{1k}, \tag{5.81}$$

$$g' = g'_{11}g'_{12}\cdots g'_{1m} \tag{5.82}$$

in G_1 for $k,m \in \Omega$. If g_{1i}'s are different than g'_{1j} for all i, j then

$$\begin{aligned} \theta(gg') &= \theta(g_{11}g_{12}\cdots g_{1k}g'_{11}g'_{12}\cdots g'_{1m}) = g_{21}g_{22}\cdots g_{2k}g'_{21}g'_{22}\cdots g'_{2m} \\ &= \theta(g_{11}g_{12}\cdots g_{1k})\theta(g'_{11}g'_{12}\cdots g'_{1m}) = \theta(g)\theta(g'). \end{aligned} \tag{5.83}$$

If there is a common factor in g and g', we consider the case when $k = 1$ and $m = 2$. Put $g = g_{11}$, $g' = g_{11}g_{12}$.

The characteristic map α_1 of G_1 can be also considered as a map $\alpha : G_1/I_1 \to I_1$ via the following diagram

$$\begin{array}{ccc} K_1 & \xrightarrow{\alpha_1} & k_1 \\ t & & \alpha_1(t) = \bar{t} \\ \big\downarrow & & \big\downarrow \\ \bar{g}^t & & a^{\bar{t}} \\ G_1/I_1 & \longrightarrow & I_1 \end{array} \quad \textbf{(5.84)}$$

where

$$\alpha_1(\bar{g}^t) = a^{\bar{t}} = a^t = (g^2)^t = (g^t)^2. \quad \textbf{(5.85)}$$

Then by commutativity of diagram (5.55) we have $\alpha_2 = \phi \alpha_1 \Psi^{-1}$, so

$$\begin{aligned} &\theta(gg') = \theta(g_{11}g_{11}g_{12}) = \theta(a_{11}g_{12}) = \phi(a_{11})g_{22} \\ &= \phi(g_{11}^2)g_{22} = \phi(\alpha_1(\overline{g_{11}}))g_{22} = \phi(\alpha_1(\Psi^{-1}(\overline{g_{21}})))g_{22} \\ &= \alpha_2(\overline{g_{21}})g_{22} = g_{21}^2 g_{22} = g_{21}g_{21}g_{22} = \theta(g_{11})\theta(g_{11}g_{12}) = \theta(g)\theta(g'). \end{aligned} \quad \textbf{(5.86)}$$

The general case follows by induction.

Lemma 5.2.8 *θ is an isomorphism.*

Proof of Lemma 5.2.6. By replacing indices 1 and 2, we can obtain another homomorphism $\Upsilon : G_2 \to G_1$ and by construction, $\theta \circ \Upsilon = Id$. Hence, $\Upsilon = \theta^{-1}$. □

5.2.4 Examples of Smart Suzuki 2-Groups

Example 5.2.1 *(cf. [1].) Given a field L of characteristic 2 and a nontrivial field automorphism $\Phi : L \to L$ with the property that the maps $x \mapsto x^{\Phi^2}x^{-1}$ and $x \mapsto x^{\Phi}x^{-1}$ are surjective, let*

$$\Gamma = \Gamma(L,\Phi) = \left\{ \begin{pmatrix} x & y & z \\ 0 & x^{\Phi} & y^{\Phi} \\ 0 & 0 & x^{\Phi^2} \end{pmatrix} : x,y,z \in L, x \neq 0 \right\}. \quad \textbf{(5.87)}$$

Then Γ is a nonabelian Suzuki 2-group of exponent 4 whose center is the set of scalar matrices in Γ. The quotient of Γ by its center is a faithful smart Suzuki 2-group where

T corresponds to the image of diagonal elements and G to the image of the strictly upper triangular matrices.

Γ is closed under matrix multiplication:

$$\begin{pmatrix} x & y & z \\ 0 & x^{\Phi} & y^{\Phi} \\ 0 & 0 & x^{\Phi^2} \end{pmatrix} \begin{pmatrix} a & b & c \\ 0 & a^{\Phi} & b^{\Phi} \\ 0 & 0 & a^{\Phi^2} \end{pmatrix} = \begin{pmatrix} xa & xb+ya^{\Phi} & xc+yb^{\Phi}+za^{\Phi^2} \\ 0 & x^{\Phi}a^{\Phi} & x^{\Phi}b^{\Phi}+y^{\Phi}a^{\Phi^2} \\ 0 & 0 & x^{\Phi^2}a^{\Phi^2} \end{pmatrix} \in \Gamma. \quad \textbf{(5.88)}$$

For $\begin{pmatrix} x & y & z \\ 0 & x^{\Phi} & y^{\Phi} \\ 0 & 0 & x^{\Phi^2} \end{pmatrix} \in \Gamma,$

$$\begin{pmatrix} x & y & z \\ 0 & x^{\Phi} & y^{\Phi} \\ 0 & 0 & x^{\Phi^2} \end{pmatrix}^{-1} = \begin{pmatrix} x^{-1} & -x^{-1}x^{-\Phi}y & x^{-1}x^{-\Phi}x^{-\Phi^2}yy^{\Phi} - x^{-1}x^{-\Phi^2}z \\ 0 & x^{-\Phi} & -x^{-\Phi}x^{-\Phi^2}y^{\Phi} \\ 0 & 0 & x^{-\Phi^2} \end{pmatrix} \in \Gamma. \quad \textbf{(5.89)}$$

The center $Z(\Gamma)$ of Γ is the set of scalar matrices in Γ,

$$Z(\Gamma) = \left\{ \begin{pmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{pmatrix} : a \in L - \{0\} \right\}. \quad \textbf{(5.90)}$$

Consider the quotient group $\Gamma/Z(\Gamma)$. Let T be the image of diagonal elements, i.e.,

$$T = \left\{ \begin{pmatrix} x & 0 & 0 \\ 0 & x^{\Phi} & 0 \\ 0 & 0 & x^{\Phi^2} \end{pmatrix} Z(\Gamma) : x \in L - \{0\} \right\}. \quad \textbf{(5.91)}$$

and let G be the image of strictly uppertriangular matrices, i.e.,

$$G = \left\{ \begin{pmatrix} 1 & y & z \\ 0 & 1 & y^{\Phi} \\ 0 & 0 & 1 \end{pmatrix} Z(\Gamma) : y, z \in L \right\}. \quad \textbf{(5.92)}$$

T acts on G by group automorphisms via conjugation:

$$\begin{pmatrix} x^{-1} & 0 & 0 \\ 0 & x^{-\Phi} & 0 \\ 0 & 0 & x^{-\Phi^2} \end{pmatrix} \begin{pmatrix} 1 & y & z \\ 0 & 1 & y^{\Phi} \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x & 0 & 0 \\ 0 & x^{\Phi} & 0 \\ 0 & 0 & x^{\Phi^2} \end{pmatrix} Z(\Gamma)$$

$$= \begin{pmatrix} 1 & x^{\phi}x^{-1}y & x^{\phi^2}x^{-1}z \\ 0 & 1 & x^{\phi^2}x^{-\phi}y^{\phi} \\ 0 & 0 & 1 \end{pmatrix} Z(\Gamma) \in G. \quad \textbf{(5.93)}$$

$\Gamma/Z(\Gamma) = GT$ *since any element* $\begin{pmatrix} a & b & c \\ 0 & a^\Phi & b^\Phi \\ 0 & 0 & a^{\Phi^2} \end{pmatrix} \in \Gamma/Z(\Gamma)$ *can be written as a product of an element in G and an element in T as*

$$\begin{pmatrix} a & b & c \\ 0 & a^\Phi & b^\Phi \\ 0 & 0 & a^{\Phi^2} \end{pmatrix} Z(\Gamma) = \begin{pmatrix} 1 & a^{-\Phi}b & a^{-\Phi^2}c \\ 0 & 1 & a^{-\Phi^2}b^\Phi \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a & 0 & 0 \\ 0 & a^\Phi & 0 \\ 0 & 0 & a^{\Phi^2} \end{pmatrix} Z(\Gamma). \quad \mathbf{(5.94)}$$

Therefore T is a complement of G in $\Gamma/Z(\Gamma)$*. Hence,* $\Gamma/Z(\Gamma) = G \rtimes T$*.*

G is a group of exponent 4:

$$\begin{pmatrix} 1 & y & z \\ 0 & 1 & y^\Phi \\ 0 & 0 & 1 \end{pmatrix}^4 Z(\Gamma) = \begin{pmatrix} 1 & 0 & yy^\Phi \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}^2 Z(\Gamma) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} Z(\Gamma). \quad \mathbf{(5.95)}$$

The center $Z(G)$ *of G is*

$$Z(G) = \left\{ \begin{pmatrix} 1 & 0 & b \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} Z(\Gamma) : b \in L \right\}. \quad \mathbf{(5.96)}$$

$G/Z(G)$ *is (a group of exponent 2) abelian so* $Z_2(G)/Z_1(G) = Z(G/Z_1(G)) = Z(G/Z(G)) = G/Z(G)$*,* $Z_2(G) = G$*, i.e., G is nilpotent of class 2. The set of involutions of G is*

$$I = \left\{ \begin{pmatrix} 1 & 0 & z \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} Z(\Gamma) : z \in L \right\}. \quad \mathbf{(5.97)}$$

T is an abelian group.

We will show that T acts transitively on involutions. For $\begin{pmatrix} x & 0 & 0 \\ 0 & x^\Phi & 0 \\ 0 & 0 & x^{\Phi^2} \end{pmatrix} Z(\Gamma) \in T$ *and* $\begin{pmatrix} 1 & 0 & y \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} Z(\Gamma)$*,* $\begin{pmatrix} 1 & 0 & z \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} Z(\Gamma) \in I$ *we have*

$$\begin{pmatrix} x^{-1} & 0 & 0 \\ 0 & x^{-\Phi} & 0 \\ 0 & 0 & x^{-\Phi^2} \end{pmatrix} \begin{pmatrix} 1 & 0 & y \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x & 0 & 0 \\ 0 & x^\Phi & 0 \\ 0 & 0 & x^{\Phi^2} \end{pmatrix} Z(\Gamma) = \begin{pmatrix} 1 & 0 & z \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} Z(\Gamma) \quad \mathbf{(5.98)}$$

if and only if

$$\begin{pmatrix} 1 & 0 & x^{\phi^2}x^{-1}y \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} Z(\Gamma) = \begin{pmatrix} 1 & 0 & z \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} Z(\Gamma) \quad \mathbf{(5.99)}$$

if and only if for any $z, y \in L \setminus \{0\}$ *there exists* $x \in L \setminus \{0\}$ *such that* $x^{\phi^2} x^{-1} = z^{-1} y^{-1}$. *This is true since the map* $a \mapsto a^{\phi^2} a^{-1}$ *is surjective in L. Therefore,* (G, T) *is a nonabelian Suzuki 2-group of exponent 4 over L.*

For smartness of G we need to show that T acts transitively and freely on G/I. T is transitive on G/I since

$$\begin{pmatrix} x^{-1} & 0 & 0 \\ 0 & x^{-\Phi} & 0 \\ 0 & 0 & x^{-\Phi^2} \end{pmatrix} \begin{pmatrix} 1 & y & 0 \\ 0 & 1 & y^{\phi} \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x & 0 & 0 \\ 0 & x^{\Phi} & 0 \\ 0 & 0 & x^{\Phi^2} \end{pmatrix} Z(\Gamma) = \begin{pmatrix} 1 & z & 0 \\ 0 & 1 & z^{\phi} \\ 0 & 0 & 1 \end{pmatrix} Z(\Gamma) \tag{5.100}$$

if and only if

$$\begin{pmatrix} 1 & x^{\phi} x^{-1} y & 0 \\ 0 & 1 & x^{\phi^2} x^{-\phi} y^{\phi} \\ 0 & 0 & 1 \end{pmatrix} Z(\Gamma) = \begin{pmatrix} 1 & z & 0 \\ 0 & 1 & z^{\phi} \\ 0 & 0 & 1 \end{pmatrix} Z(\Gamma) \tag{5.101}$$

which is true since the map $a \mapsto a^{\phi} a^{-1}$ *is surjective in L.*

T is free on G/I since

$$\begin{pmatrix} x^{-1} & 0 & 0 \\ 0 & x^{-\Phi} & 0 \\ 0 & 0 & x^{-\Phi^2} \end{pmatrix} \begin{pmatrix} 1 & y & 0 \\ 0 & 1 & y^{\phi} \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x & 0 & 0 \\ 0 & x^{\Phi} & 0 \\ 0 & 0 & x^{\Phi^2} \end{pmatrix} Z(\Gamma) = \begin{pmatrix} 1 & y & 0 \\ 0 & 1 & y^{\phi} \\ 0 & 0 & 1 \end{pmatrix} Z(\Gamma) \tag{5.102}$$

if and only if

$$\begin{pmatrix} 1 & x^{\phi} x^{-1} y & 0 \\ 0 & 1 & x^{\phi^2} x^{-\phi} y^{\phi} \\ 0 & 0 & 1 \end{pmatrix} Z(\Gamma) = \begin{pmatrix} 1 & y & 0 \\ 0 & 1 & y^{\phi} \\ 0 & 0 & 1 \end{pmatrix} Z(\Gamma) \tag{5.103}$$

if and only if

$$x^{\phi} x^{-1} y = y. \tag{5.104}$$

Since $y \neq 0$, *this implies* $x^{\phi} x^{-1} = 1$ *and* $x \in Fix\phi$, *that is*

$$\begin{pmatrix} x & 0 & 0 \\ 0 & x^{\Phi} & 0 \\ 0 & 0 & x^{\Phi^2} \end{pmatrix} Z(\Gamma) = \begin{pmatrix} x & 0 & 0 \\ 0 & x & 0 \\ 0 & 0 & x \end{pmatrix} Z(\Gamma) = Z(\Gamma). \tag{5.105}$$

Example 5.2.2 *Let L be an algebraically closed field of characteristic 2 and let* Φ *be a power of the Frobenius automorphism on L, say,* $\Phi(a) = a^{2^n}$, $n \in Z^+$, *all* $a \in L$. *Then* *the maps* $x \mapsto x^{\Phi^2}x^{-1}$ *and* $x \mapsto x^{\Phi}x^{-1}$ *are surjective: For any* $b \in L$ *the polynomials* $x^{2^{2n}-1} - b = 0$ *and* $x^{2^n-1} - b = 0$ *have zeros in* L *since* L *is algebraically closed. Thus, for any* $b \in L$ *there exist* $a, c \in L$ *such that* $a^{\phi^2}a^{-1} = a^{2^{2n}-1} = b$ *and* $c^{\Phi}c^{-1} = c^{2^n-1} = b$. *Then the group* (G,T), *defined by (5.92) and (5.91), is a smart Suzuki 2-group.*

Example 5.2.3 *(cf. [2].) Let* K *be a field of characteristic 2,* $\Phi : K \to K$ *be a field automorphism. Put* $k = \{t^{\Phi}t : t \in K\}$. *Assume that* k *is a subfield of* K. *Let* $h : K \to k$ *be a* $k-linear$*, additive map. Let*

$$G = \left\{ \begin{pmatrix} 1 & x & y \\ 0 & 1 & x^{\Phi} \\ 0 & 0 & 1 \end{pmatrix} : x \in K, y \in k \right\};$$

$$T = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & t & 0 \\ 0 & 0 & t^{\Phi}t \end{pmatrix} : t \in K^{\star} \right\}. \qquad \mathbf{(5.106)}$$

We define the group operation on G *by*

$$\begin{pmatrix} 1 & x & t \\ 0 & 1 & x^{\Phi} \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & y & z \\ 0 & 1 & y^{\Phi} \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & x+y & h(t+z+xy^{\Phi}) \\ 0 & 1 & x^{\Phi}+y^{\Phi} \\ 0 & 0 & 1 \end{pmatrix} \qquad \mathbf{(5.107)}$$

for all $x, y \in K$, $t, z \in k$. *Since* $h = h^2$, *associativity holds in* G:

$$\left[\begin{pmatrix} 1 & x & y \\ 0 & 1 & x^{\Phi} \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & a & b \\ 0 & 1 & a^{\Phi} \\ 0 & 0 & 1 \end{pmatrix} \right] \begin{pmatrix} 1 & z & t \\ 0 & 1 & z^{\Phi} \\ 0 & 0 & 1 \end{pmatrix} =$$

$$\begin{pmatrix} 1 & x+a+z & t+y+b+h((x+a)z^{\Phi})+h^2(xa^{\Phi}) \\ 0 & 1 & x^{\Phi}+a^{\Phi}+z^{\Phi} \\ 0 & 0 & 1 \end{pmatrix} =$$

$$\begin{pmatrix} 1 & x+a+z & t+y+b+h(xz^{\Phi}+az^{\Phi}+xa^{\Phi}) \\ 0 & 1 & x^{\Phi}+a^{\Phi}+z^{\Phi} \\ 0 & 0 & 1 \end{pmatrix} =$$

$$\begin{pmatrix} 1 & x+a+z & t+y+b+h(xa^{\Phi}+xz^{\Phi})+h^2(az^{\Phi}) \\ 0 & 1 & x^{\Phi}+a^{\Phi}+z^{\Phi} \\ 0 & 0 & 1 \end{pmatrix} =$$

$$\begin{pmatrix} 1 & x & y \\ 0 & 1 & x^{\Phi} \\ 0 & 0 & 1 \end{pmatrix} \left[\begin{pmatrix} 1 & a & b \\ 0 & 1 & a^{\Phi} \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & z & t \\ 0 & 1 & z^{\Phi} \\ 0 & 0 & 1 \end{pmatrix} \right] \qquad \mathbf{(5.108)}$$

T *acts on* G *by conjugation. Furthermore,* T *acts transitively on involutions*

$$I = \left\{ \begin{pmatrix} 1 & 0 & xx^{\Phi} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} : x \in K \right\} \qquad \mathbf{(5.109)}$$

since

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & t^{-1} & 0 \\ 0 & 0 & t^{-\Phi}t^{-1} \end{pmatrix} \begin{pmatrix} 1 & 0 & xx^{\Phi} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & t & 0 \\ 0 & 0 & t^{\Phi}t \end{pmatrix} = \begin{pmatrix} 1 & 0 & zz^{\Phi} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{5.110}$$

if and only if $(xt)(xt)^{\Phi} = zz^{\Phi}$, $t = zx^{-1}$.

G is a nonabelian Suzuki 2-group of exponent 4. T acts transitively on

$$G/I \simeq \left\{ \begin{pmatrix} 1 & x & 0 \\ 0 & 1 & x^{\Phi} \\ 0 & 0 & 1 \end{pmatrix} : x \in K \right\} \tag{5.111}$$

since for any $x, y \in K - \{0\}$,

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & (yx^{-1})^{-1} & 0 \\ 0 & 0 & (yx^{-1}y^{\Phi}x^{-\Phi})^{-1} \end{pmatrix} \begin{pmatrix} 1 & x & 0 \\ 0 & 1 & x^{\Phi} \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & yx^{-1} & 0 \\ 0 & 0 & yx^{-1}y^{\Phi}x^{-\Phi} \end{pmatrix} = \begin{pmatrix} 1 & y & 0 \\ 0 & 1 & y^{\Phi} \\ 0 & 0 & 1 \end{pmatrix} \tag{5.112}$$

T acts freely on G/I, *since*

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & t^{-1} & 0 \\ 0 & 0 & t^{-\Phi}t^{-1} \end{pmatrix} \begin{pmatrix} 1 & x & 0 \\ 0 & 1 & x^{\Phi} \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & t & 0 \\ 0 & 0 & t^{\Phi}t \end{pmatrix} = \begin{pmatrix} 1 & xt & 0 \\ 0 & 1 & x^{\Phi}t^{\Phi} \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{5.113}$$

implies $xt = 0$. *Hence,* (G,T) *is a smart Suzuki 2-group with associated 2-cocycle* $f : K \times K \to k$ *satisfying*

$$a^{f(x,y)} = \begin{pmatrix} 1 & 0 & h(xy^{\phi}) \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{5.114}$$

the characteristic map $\alpha(x) = xx^{\Phi}$ *and the commutator map* $\beta : K \times K \to k$ *satisfying*

$$a^{\beta(x,y)} = \begin{pmatrix} 1 & 0 & h(xy^{\phi} + yx^{\phi}) \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{5.115}$$

for all $x, y \in K$.

Example 5.2.4 *Let* $G = Q_8 = \{1,-1,i,-i,j,-j,k,-k\}$ *and* $T = \{1,x,x^2\}$. *The subgroup of involutions in G is* $I = \{1,-1\}$. *We define the action of T on G by* $i^x = j$, $j^x = k$, $k^x = i$, $1^x = 1$ *and* $(-1)^x = -1$. *Then* $C_T(I) = T$. *Put* $K = T \cup \{0\}$ *and* $k = T/C_T(I) \cup \{\tilde{0}\}$. *Then the wide field* $K \cong \mathbb{F}_4$ *and the narrow field* $k \cong \mathbb{F}_2$. *Since* $\bar{i}^x = \overline{i^x} = \bar{j}$, $\bar{j}^x = \bar{k}$, $\bar{i}^{x^2} = \bar{k}$, $\bar{j}^{x^2} = \bar{i}$ *and* $\bar{k}^{x^2} = \bar{j}$, *T acts freely and transitively on* G/I. *Therefore* (G,T) *is a smart Suzuki 2-group.*

Consider the following matrix representation of G: Let $1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $-1 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $i = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$, $j = \begin{pmatrix} 1 & x & 0 \\ 0 & 1 & x^2 \\ 0 & 0 & 1 \end{pmatrix}$, *and* $k = \begin{pmatrix} 1 & x^2 & 0 \\ 0 & 1 & x \\ 0 & 0 & 1 \end{pmatrix}$.
Let $\phi : K \to K$ *be the field automorphism defined by* $\phi(x) = x^2$ *for all* $x \in K$. *We define a map* $h : K \to k$ *by* $h(0) = 0$, $h(1) = 1$, $h(x) = 1$ *and* $h(x^2) = h(x+1) = 0$. *Then h is k-linear and additive. The group operation in G is defined by equality (5.107). The characteristic map* $\alpha : K \to k$ *of G is defined by* $\alpha(0) = 0$, $\alpha(1) = \alpha(x) = \alpha(x^2) = 1$ *where* $Ker\alpha = T = C_T(I)$ *and* $Im\alpha = k = T/C_T(I) \cup \{\tilde{0}\} = T/Ker\alpha \cup \{\tilde{0}\} \cong \mathbb{F}_2$.

Example 5.2.5 *For* $n \in \mathbb{N}$, *let* $K = \mathbb{F}_{2^{2n}}$, $k = \mathbb{F}_{2^n}$ *and let* $\phi : K \to k$ *be the field automorphism defined by* $\phi(x) = x^{2^n}$ *for all* $x \in K$. *Let* $h : K \to k$ *be a k-linear, additive map. Let G, T be the groups defined by equality (5.106) where the group operation on G is defined by equality (5.107). Then* (G,T) *is a smart Suzuki 2-group with the characteristic map* $\alpha : K \to k$ *defined by* $\alpha(x) = x^{\phi}x$ *for all* $x \in K$. *Note that for* $n = 1$, *we obtain the example of quarternions.*

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