ISTANBUL TECHNICAL UNIVERSITY ★ GRADUATE SCHOOL

BICONSERVATIVE AND BIHARMONIC SURFACES IN EUCLID AND MINKOWSKI SPACES

M.Sc. THESIS

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Department of Mathematics Engineering

Mathematics Engineering Programme

JULY 2024

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To my mom,

FOREWORD

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ABBREVIATIONS

PNMCV : Parallel Normalized Mean Curvature Vector

PMCV : Parallel Mean Curvature Vector

PDE : Partial Differential Equation

SYMBOLS

$\mathbb{E}_r^n$: n- dimensional Semi-Euclidean Space with index r

∇ : Levi- Civita Connection of the Submanifold

$\tilde{\nabla}$: Levi- Civita Connection of the Semi-Euclidean Space

$\nabla^{\perp}$: Normal Connection

Δ : Laplace Operator

A : Shape Operator

h : Second Fundemantel Form

H : Mean Curvature Vector Field

f : Mean Curvature Function

K : Gaussian Curvature Vector

$R(X,Y,Z,W)$: Riemannian Curvature Tensor

w_{ij} : Connection Form

BICONSERVATIVE AND BIHARMONIC SURFACES

IN EUCLID AND MINKOWSKI SPACES

SUMMARY

In 1964, Sampson and Eells formulated the concept of biharmonic maps as an extension of harmonic maps while investigating the energy functional E between Riemannian manifolds, a subject of both geometric and physical significance. Subsequently, numerous mathematicians have shown interest in the study of biharmonic mappings.

By the definition, the bienergy functional between semi-Riemannian manifolds (M^m, g) and $(N^n, \tilde{g})$ is defined by

$$E_2(\varphi) = \frac{1}{2}\int_M \|\tau(\varphi)\|^2 v_g$$

for a smooth map $\varphi : M \to N$, where $\tau(\varphi)$ represents the tension field of φ. $\varphi : M \to N$ is said to be biconservative if it is a critical point of E_2. This condition is equivalent to satisfying the Euler-Lagrange equation associated with the bienergy functional

$$\tau_2(\varphi) = 0,$$

where τ_2 is the bitension field defined by

$$\tau_2(\varphi) := \Delta\tau(\varphi) - \operatorname{tr}\tilde{R}(d\varphi, \tau(\varphi))d\varphi.$$

In the 1980s, B. Y. Chen conducted research on biharmonic submanifolds within Euclidean spaces as a component of B. Y. Chen's initiative to comprehend submanifolds of finite type in semi-Euclidean spaces. B. Y. Chen proposed an other characterization of biharmonic submanifolds in these spaces. Let $x : M \to \mathbb{E}_r^n$ be an isometric immersion. By examining normal and tangential parts of $\tau_2(x)$, the following results can be obtained.

Let $x : M^m \to \mathbb{E}_r^n$ be an isometric immersion of an n-dimensional semi-Riemannian submanifold M^m into the semi-Euclidean space $\mathbb{E}_r^n$. If x satisfies the fourth-order semi-linear PDE system given by the equations

$$\Delta^{\perp}H + \operatorname{trace}h(A_H(\cdot),\cdot) = 0$$

and

$$m\text{grad}\|H\|^2 + 4\text{trace}A_{\nabla^\perp H}(\cdot) = 0,$$

then M^m is biharmonic.

On the other hand, if a mapping $\varphi : M \to N$ satisfies the weaker condition

$$\langle \tau_2(\varphi), d\varphi \rangle = 0,$$

then it is said to be biconservative. Mainly, if $x : M \to N$ is an isometric immersion, then the previous equation is equivalent to

$$\tau_2(x)^\top = 0,$$

where $\tau_2(x)^\top$ represents the tangential part of $\tau_2(x)$. In this case, M is said to be a biconservative submanifold of N.

In this thesis, we mainly focus on biharmonic and biconservative surfaces in four dimensional Euclidean and Minkowski spaces. The first section provides a concise overview of the historical background and underlying principles concerning biharmonic and biconservative submanifolds, as well as an overview of the research conducted far in this field. In the second section, we give some basic notations and basic facts about submanifolds of semi- Euclidean spaces, the definition of biconservative submanifolds and we introduce the rotational surfaces. In the third section, we give biconservative PNMCV surfaces in $\mathbb{E}^4$. We obtain local parameterizations of these surfaces and demonstrate that they are not biharmonic. In the fourth section, we give biconservative rotational surfaces in $\mathbb{E}^4_1$. We study with three different class of rotational surfaces and obtain the condition for each of them to be biconservative. In the concluding section, the derived conclusions are presented, along with recommendations regarding possible future researches.

ÖKLİD VE MİNKOWSKİ UZAYLARINDAKİ

BİKONZÖRVATİF VE BİHARMONİK YÜZEYLER

ÖZET

1960'ların ortalarında Sampson ve Eels, enerji fonksiyonelinin fiziksel ve geometrik özelliklerini araştırırken harmonik tasvirlerin genelleştirilmiş bir versiyonu olan k-harmonik tasvir kavramını diferansiyel geometriye kazandırdılar. Bu tanımın sunulmasının ardından birçok geometrici, biharmonik tasvirler üzerine çalışmalar yapmaya başladı.

(M^m, g) ve $(N^n, \tilde{g})$ yarı Riemann manifoldları arasındaki E_2 bienerji fonksiyoneli her $\varphi : M \to N$ düzgün tasviri için

$$E_2(\varphi) = \frac{1}{2}\int_M \|\tau(\varphi)\|^2 v_g$$

şeklinde tanımlanır ki burada $\tau(\varphi)$ ile φ tasvirinin gerilme alanı gösterilmektedir. φ tasvirine eğer E_2 bienerji fonksiyonelinin bir kritik noktası ise biharmoniktir denir. Bu şart

$$\tau_2(\varphi) = 0$$

şeklindeki Euler- Langrange denkleminin sağlanmasına dentir ki burada τ_2 ile

$$\tau_2(\varphi) := \Delta\tau(\varphi) - \mathrm{tr}\tilde{R}(d\varphi, \tau(\varphi))d\varphi$$

şeklinde tanımlanan bi-gerilme alanı gösterilmiştir.

1980'lerin ortalarında, B. Y. Chen, sonlu tipteki alt manifoldların yapısı üzerine araştırmalar yaparken, ayrıca biharmonik alt manifoldları inceledi ve Öklid uzaylarındaki biharmonik alt manifoldların tanımını sundu. Daha sonra, bu tanım yarı-Öklid uzaylarına genişletildi. $\mathbb{E}_r^n$ yarı Öklid uzayından M^m alt manifolduna giden bir $x : M \to \mathbb{E}_r^n$ izometrik daldırması göz önüne alınsın. Bu tasvirin biharmonik olması için gerek ve yeter koşul

$$\Delta^{\perp}H + \mathrm{trace}h(A_H(\cdot),\cdot) = 0,$$

ve

$$m\mathrm{grad}\|H\|^2 + 4\mathrm{trace}A_{\nabla^{\perp}_{\cdot}H}(\cdot) = 0$$

denklemleriyle verilen dördüncü dereceden yarı-lineer kısmi diferansiyel denklem sisteminin sağlanmasıdır.

Diğer taraftan, eğer bir $\varphi : M \to N$ tasvirine daha zayıf bir koşul olan

$$\langle \tau_2(\varphi), d\varphi \rangle = 0,$$

denklemini sağlıyorsa bikonzörvatiftir denir. Ayrıca, eğer özel olarak $x : M \to N$ bir izometrik daldırma ise bir önceki denklem

$$\tau_2(x)^{\top} = 0$$

halini alır ki burada $\tau_2(x)^{\top}$ ile $\tau_2(x)$ vektör alanının teğet kısmı gösterilmiştir. Bu durumda, M yüzeyi N yüzeyinin bir bikönzervatif alt manifoldudur.

M alt manifoldunun ortalama eğrilik vektörü ve ortalama eğriliği, sırasıyla,

$$H = \frac{1}{m}\text{trace}h$$

ve

$$f = \sqrt{|\langle H, H \rangle|}$$

denklemleriyle tanımlanır ki burada h ile M alt manifoldunun ikinci temel formu gösterillmiştir. $f \neq 0$ olduğu varsayılsın. Eğer $\frac{H}{f}$ vektör alanı paralel ise M alt manifolduna paralel normalleştirilmiş ortalama eğrilik vektörüne (PNMCV) sahiptir denir.

Diğer taraftan, $\mathbb{E}_1^4$ Minkowski uzayındaki hiperbolik, zamansal ve parabolik dönel yüzeylerin parametrizasyonları, sırasıyla,

$$F_1(s,t) = (x(s), y(s), w(s)\sinh t, w(s)\cosh t), s \in I, t \in [0, 2\pi],$$

$$F_2(s,t) = (x(s), y(s), w(s)\cosh t, w(s)\sinh t), s \in I, t \in [0, 2\pi]$$

ve

$$F_3(s,t) = x(s)\eta_1 + \sqrt{2}tw(s)\eta_2 + (z(s) + t^2 w(s))\xi_3 + w(s)\xi_4, s \in I, t \in \mathbb{R}$$

şeklindedir. Burada x, y ve w düzgün birer fonksiyondur ve $\eta_1, \eta_2, \xi_3, \xi_4$ $\mathbb{E}_1^4$ Minkowski uzayındaki sözde- ortonormal çatı alanı oluşturan sabit vektörlerdir.

Bu tezde öncelikli olarak $\mathbb{E}^4$ deki bikonzörvatif PNMCV yüzeylerle çalışılmıştır. Daha sonra ise $\mathbb{E}_1^4$ Minkowski uzayındaki dönel yüzeylerin bikonzörvatif olması için gerek ve yeter koşullar elde edilmiştir.

İlk bölümde, biharmonik ve bikonzörvatif alt manifoldlara ilişkin tarihsel arka plan ve temel ilkelere kısa bir genel bakışın yanı sıra bu alanda şimdiye kadar elde edilmiş sonuçlar sunulmaktadır. İkinci bölümde yarı-Öklid uzaylarının alt manifoldları, bikonzörvatif alt manifoldlar ve dönel yüzeyler hakkında bazı temel gösterimler verilmiştir.

Üçüncü bölümde, $\mathbb{E}^4$ uzayında bikonzörvatif PNMCV yüzeylerle çalışılmıştır. Bu yüzeylerin lokal parametrizasyonları elde edilmiştir. Sonuç olarak, $\mathbb{E}^4$ uzayında biharmonik PNMCV yüzey olmadığı gösterilmiştir.
Dördüncü bölümde, $\mathbb{E}_1^4$ Minkowski uzayındaki dönel yüzeyler çalışılmıştır. Hiperbolik, Zamansal ve Parabolik dönel yüzeylerin her biri için bikonzörvatif olma şartları elde edilmiştir.
Sonuç bölümünde, elde edilen sonuçlar özetlenmiş ve bu sonuçları geliştirmeye yönelik bazı öneriler sunulmuştur.

1. INTRODUCTION

In 1964, while examining the energy functional E between Riemannian manifolds—a topic of both geometric and physical importance—Sampson and Eels extended harmonic maps to biharmonic maps. In the mid-1980s, following the introduction of the concept of k-harmonic maps by Eells and Lemaire to classify harmonic maps between two Riemannian manifolds, [1]. Since then, many mathematicians have become interested in studying biharmonic mappings.

By the definition, the bienergy functional between semi-Riemannian manifold (M,g) and $(N,\tilde{g})$ is given by

$$E_2(\varphi) = \frac{1}{2}\int_M \|\tau(\varphi)\|^2 v_g \tag{1.0.1}$$

for a smooth map $\varphi : M \to N$, where $\tau(\varphi)$ represents the tension field of φ. If a map $\varphi : (M,g) \to (N,\tilde{g})$ is a critical point of E_2 then it is biharmonic. This condition is equivalent to satisfying the Euler-Lagrange equation associated with the bienergy functional given by

$$\tau_2(\varphi) = 0, \tag{1.0.2}$$

where τ_2 is the bitension field defined as

$$\tau_2(\varphi) := \Delta\tau(\varphi) - \operatorname{tr}\tilde{R}(d\varphi, \tau(\varphi))d\varphi. \tag{1.0.3}$$

In [2–4], during G. Y. Jiang's program of studying the biharmonic maps and biharmonic isometric immersions between Riemannian manifolds, G. Y. Jiang obtained first and second variational formulas of E_2. By using these results, conservation laws for biharmonic maps between Riemannian manifolds were obtained.

On the other hand, Chen defined the biharmonic submanifold M of $\mathbb{E}_r^n$ as a submanifold whose position vector x satisfying

$$\Delta^2 x = 0, \tag{1.0.4}$$

where Δ is the Laplace operator. In [5], it is shown that this definition is identical to the definition of Sampson and Eells when $x : M^m \to \mathbb{E}_r^n$ is isometric immersion. In [6], Chen studied on submanifolds of finite type.

It is straightforward to demonstrate that a minimal isometric immersion is biharmonic. However, in [5], B. Y. Chen asserted that the converse is also true if the ambient space is Euclidean, and he proposed that a biharmonic submanifold of Euclidean space is minimal.

Let $x : M^m \to \mathbb{E}_r^n$ be an isometric immersion. Considering normal and tangential parts of $\tau_2(x) = \Delta^2 x$, one can obtain that M^m into the semi-Euclidean space $\mathbb{E}_r^n$. A direct computation yields that M^m is biharmonic if and only if the fourth-order semi-linear PDE system given by the equations

$$\Delta^{\perp} H + \text{trace} h(A_H(\cdot), \cdot) = 0 \tag{1.0.5}$$

and

$$m\text{grad}\|H\|^2 + 4\text{trace} A_{\nabla^{\perp} H}(\cdot) = 0 \tag{1.0.6}$$

are satisfied. On the other hand, if a map $\varphi : M \to N$ satisfies

$$\langle \tau_2(\varphi), d\varphi \rangle = 0, \tag{1.0.7}$$

then it is said to be biconservative. In particular, if $x : M \to N$ is an isometric immersion, then the former equation is equivalente to

$$\tau_2(x)^{\top} = 0, \tag{1.0.8}$$

where $\tau_2(x)^{\top}$ denotes the tangential part of $\tau_2(x)$. In this case, M is said to be a biconservative submanifold of N.

B. Y. Chen described a submanifold as having parallel normalized mean curvature vector field (PNMCV) if the unit vector field proportional to the mean curvature vector is parallel on the normal bundle [7]. Note that, B. Y. Chen mentioned a weaker version of B. Y. Chen's biharmonic conjecture about PNMCV submanifolds in [8].

In [9–15], Several geometers have investigated biconservative immersions to understand into the geometry of biharmonic submanifolds. For example, Y. Fu and N. C. Turgay in [15] studied complete classification of biconservative hypersurfaces with diagonalizable shape operator in Minkowski 4-space. After that the second named author in [15] kept up pseudo-Euclidean spaces in [16]. S. Montaldo *et al.* provided a complete classification of biconservative surfaces with constant mean curvature in 4-dimensional Riemannian space forms in [17]. The comprehensive classification of surfaces with parallel mean curvature vectors in the product spaces $\mathbb{S}^n \times \mathbb{R}$ and $\mathbb{H}^n \times \mathbb{R}$ was detailed in [11]. More recently, the general properties of biconservative surfaces in arbitrary manifolds, especially those with constant mean curvature, were explored in [18].

In [19], R. Yeğin Şen and N. C. Turgay demonstrated that a PNMCV biconservative surface is congruent to a specific rotational surface in $\mathbb{E}^4$. Futhermore, in [19] it was shown that there is no biharmonic PNMCV surface in $\mathbb{E}^4$.

In this thesis, biharmonic and biconservative surfaces in 4-dimensional Euclidean and Minkowski spaces were focused. This thesis is contained in five sections and these are planned as follows.

The first section presents a brief summary of the historical background and fundamental principles related to biharmonic and biconservative submanifolds, along with an overview of the research conducted in this field thus far.

The second section basic notations are given before fundamental concepts concerning submanifolds of semi-Euclidean spaces, biconservative submanifolds, and rotational surfaces are summarized.

In the third section, we interest biconservative PNMCV surfaces in $\mathbb{E}^4$. We obtain local parametrizations for these surfaces and show that they are not biharmonic.

The fourth section focuses on biconservative rotational surfaces in $\mathbb{E}_1^4$. We examine three different classes of rotational surfaces and determine the conditions under which each class is biconservative.

In the final section, we present our conclusions and offer recommendations for future research directions.

2. PRELIMINARIES

In this section a brief summary of basic facts and definitions is given.

2.1 Submanifolds of Semi-Euclidean Spaces

Let $\mathbb{E}_r^n$ be the semi-Euclidean n-space equipped by the canonical degenerated metric tensor with index r given by:

$$\tilde{g} = \langle , \rangle = \sum_{i=1}^{n-r-1} dx_i^2 - \sum_{j=n-r}^{n} dx_j^2,$$

where $(x_1, x_2, ..., x_n)$ represents a rectangular coordinate system in $\mathbb{E}_r^n$, the Levi-Civita connection of $\mathbb{E}_r^n$ will be denoted by $\tilde{\nabla}$.

Let M be a submanifold of $\mathbb{E}_r^n$ with Levi Civita connection ∇, second fundamental form h, shape operator A and normal connection $\nabla^\perp$. Then, the Gauss and Weingarten formulas

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X,Y),$$

$$\tilde{\nabla}_X \xi = -A_\xi X + \nabla_X^\perp \xi$$

are satisfied whenever X and Y are tangent to M and ξ is normal to M. Note that a normal vector field ξ is said to be parallel if $\nabla_X^\perp \xi = 0$ whenever X is tangent to M.

On the other hand, the mean curvature vector field H of M is defined by

$$H = \frac{1}{m}\text{trace}h \tag{2.1.1}$$

and the mean curvature (function) is given by

$$f = \sqrt{|\langle H, H \rangle|},$$

M is said to be minimal if H vanishes identically.

Assume $f > 0$, If the normal vector field $\frac{H}{f}$ is parallel, then M is said to have a

parallel normalized mean curvature vector (PNMCV). Note that, being PNMCV is a weaker condition than being non-minimal and PMCV. Therefore, the notion of PNMCV generalizes PMCV submanifolds.

Furthermore, the Gaussian curvature of M is defined by

$$K = \frac{R(X,Y,Y,X)}{\langle X,X\rangle\langle Y,Y\rangle - \langle X,Y\rangle^2}.$$

Now, let M be a surface in $\mathbb{E}_r^n$. The curvature tensor of M is defined by

$$R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z.$$

For any tangent vector fields X, Y and Z on M, the Codazzi and Gauss equations are expressed as follows:

$$(\overline{\nabla}_X h)(Y,Z) = (\overline{\nabla}_Y h)(X,Z) \tag{2.1.2}$$

and

$$R(X,Y)Z = A_{h(Y,Z)}X - A_{h(X,Z)}Y \tag{2.1.3}$$

respectively, where the covariant derivative $(\overline{\nabla}_X h)(Y,Z)$ of the second fundamental form h is defined by

$$(\overline{\nabla}_X h)(Y,Z) = \nabla^{\perp}_X h(Y,Z) - h(\nabla_X Y,Z) - h(Y,\nabla_X Z).$$

On the other hand, the Ricci equation is

$$R^{\perp}(X,Y)\xi = h(X,A_\xi Y) - h(A_\xi X,Y) \tag{2.1.4}$$

whenever X, Y are tangent and ξ is normal to M.

2.2 Biconservative Submanifolds

Recall that the bitension field τ_2 is defined by $\tau_2(x) := \Delta\tau(x) - \mathrm{tr}\tilde{R}(dx, \tau(x))dx$. By considering tangential and normal parts of $\tau_2(x)$, the following results can be obtained:

Proposition 2.1. *An isometric immersion $x : M^m \to N^n$ between two semi-Riemannian manifolds is biharmonic if and only if the equations*

$$m\mathrm{grad}\|H\|^2 + 4\mathrm{trace}(A_{\nabla^\perp_{(\cdot)}H}(\cdot)) + 4\mathrm{trace}(\tilde{R}(\cdot, H)\cdot)^\top = 0 \tag{2.2.1}$$

and

$$\mathrm{trace}h(A_H(\cdot),\cdot) - \Delta^\perp H + \mathrm{trace}(\tilde{R}(\cdot, H).)^\perp) = 0, \tag{2.2.2}$$

are satisfied.

By considering Proposition 2.1, one can obtain the following proposition.

Proposition 2.2. *An isometric immersion $x : M^m \to N^n$ between two Riemannian manifolds is biconservative if and only if the equation* (2.2.1) *is satisfied.*

Before proceeding, we would like to obtain $\mathbb{E}^n$ the necessary and sufficient condition for a PNMCV submanifold M in the Euclidean spaces $\mathbb{E}^n$ to be biconservative. Since the following result was obtained the curvature tensor $\tilde{R}$ of $\mathbb{E}^n$ vanishes, immediately from equation (2.2.1).

Proposition 2.3. *Let M be an m-dimensional PNMCV submanifold of Euclidean space $\mathbb{E}^n$. Then M is biconservative if and only if*

$$A_{e_{m+1}}(\mathrm{grad}f) = \frac{-mf}{2}\mathrm{grad}f, \tag{2.2.3}$$

where $e_{m+1} = \frac{H}{f}$ and A_{m+1} is the shape operator corresponding to e_{m+1}.

Proof. Since $N = \mathbb{E}^n$, we have $\tilde{R} = 0$. Therefore, (2.2.1) turns into

$$m\mathrm{grad}\|H\|^2 + 4\mathrm{trace}(A_{\nabla^\perp_{(\cdot)}H}(\cdot)) = 0. \tag{2.2.4}$$

Moreover, from the definition of e_{m+1}, we have

$$H = fe_{m+1}, \nabla^\perp_{e_i}H = e_i(f)e_{m+1} \tag{2.2.5}$$

and since M is PNMCV, we have

$$\nabla^{\perp}_{e_i} e_{m+1} = 0.$$

By considering (2.2.5), the second term in the left-hand side of the equation (2.2.4) becomes

$$\begin{aligned}
\text{trace}(A_{\nabla^{\perp}_{\cdot} H}(\cdot)) &= A_{\nabla^{\perp}_{e_1} H}(e_1) + A_{\nabla^{\perp}_{e_2} H}(e_2) \\
&= A_{e_1(f)e_{m+1}}(e_1) + A_{e_2(f)e_{m+1}}(e_2) \\
&= e_1(f) A_{e_{m+1}} e_1 + e_2(f) A_{e_{m+1}} e_2 \\
&= A_{e_{m+1}}(e_1(f)e_1 + e_2(f)e_2) \\
&= A_{e_{m+1}}(\text{grad} f). \qquad (2.2.6)
\end{aligned}$$

On the other hand, the first term in the left-hand side of equation (2.2.4) implies

$$\text{grad}\|H\|^2 = \text{grad} f^2 = 2f\,\text{grad} f. \qquad (2.2.7)$$

By combining from equations (2.2.4), (2.2.6) and (2.2.7), one can obtain

$$\begin{aligned}
0 &= m\,\text{grad}\|H\|^2 + 4\,\text{trace}(A_{\nabla^{\perp}_{\cdot} H}(\cdot)) \\
&= 2mf\,\text{grad} f + 4A_{e_{m+1}}(\text{grad} f)
\end{aligned}$$

which yields (2.2.3). □

Remark 2.1. *If e_{m+1} and the mean curvature f is constant, then H is parallel. In this case, the equation* (2.2.3) *is satisfied trivially. Hence, throughout this thesis when referring PNMCV biconservative surface, we consider surface with non-constant mean curvature.*

2.3 Rotational Surfaces in $\mathbb{E}^4$

In this subsection, the class of rotational surfaces with a 2- dimensional rotational axis in Minkowski space $\mathbb{E}^4_1$ will be studied. There exist three types of rotational surfaces namely hyperbolic, timelike or parabolic (See for example [20]). These surfaces remain invariant under hyperbolic rotation, timelike rotation and screw rotation, respectively.

2.3.1 Rotational Surfaces of Hyperbolic Type

Let $\beta : I \to \mathbb{E}_1^4$,

$$\beta(s) = (x(s), y(s), 0, w(s))$$

be a smooth regular curve defined on an open interval I. The curve $\beta(s)$ lies in the 3-dimensional subspace

$$\mathbb{E}_1^3 = \text{span}\{\eta_1, \eta_2, \eta_4\}$$

of $\mathbb{E}_1^4$. Now, consider the surface defined by

$$F_1(s,t) = (x(s), y(s), w(s)\sinh t, w(s)\cosh t), s \in I, t \in \mathbb{R}. \tag{2.3.1}$$

This represents a surface whose points are fixed by $\{\eta_1, \eta_2\}$, leaving behind a space-like plane under the influence of orthogonal rotation in $\mathbb{E}_1^4$, centered at $\beta(s)$. The surface F_1 is called a rotational surface of hyperbolic type, and it is regular and timelike if $w(s) > 0$. Note that the profile curve $\beta(s)$ is timelike. Without loss of generality, one can assume that β is parametrized by its arc-length, i.e,

$$x'^2 + y'^2 - w'^2 = 1.$$

2.3.2 Timelike Rotational Surface

Consider the curve β in $\mathbb{E}_1^4$ given by,

$$\beta(s) = (x(s), y(s), w(s), 0),$$

where I is an open interval. The curve $\beta(s)$ lies in the three-dimensional subspace $\mathbb{E}^3 = \text{span}\{\eta_1, \eta_2, \eta_3\}$ of $\mathbb{E}_1^4$. Consider the surface given by the parametrization

$$F_2(s,t) = (x(s), y(s), w(s)\cosh t, w(s)\sinh t), s \in I, t \in \mathbb{R} \tag{2.3.2}$$

in $\mathbb{E}_1^4$. This surface is as a timelike rotational surface when $w(s) > 0$ and the profile curve $\beta(s)$ is spacelike on the interval I. This surface represents the orbit of $\beta(s)$ under the action of the orthogonal transformation of $\mathbb{E}^4$ that pointwise fixes the spacelike 1-plane spanned by $\{\eta_1, \eta_2\}$. Moreover, F_2 is regular and timelike. Without loss of generality, one can assume that β is parametrized by its arc-length, i.e,

$$x'^2 + y'^2 + w'^2 = 1.$$

2.3.3 Rotational Surfaces of Parabolic Type

Now, consider the pseudo-orthonormal basis $\{\eta_1, \eta_2, \xi_3, \xi_4\}$ of $\mathbb{E}_1^4$ such that $\xi_3 = \frac{\eta_4 - \eta_3}{\sqrt{2}}$ and $\xi_4 = \frac{\eta_3 + \eta_4}{\sqrt{2}}$. Note that $\langle \xi_3, \xi_3 \rangle = 0$, $\langle \xi_4, \xi_4 \rangle = 0$ and $\langle \xi_3, \xi_4 \rangle = -1$. Consider a smooth regular curve

$$\gamma(s) = x(s)\eta_1 + \overline{z}(s)\xi_3 + \overline{w}(s)\xi_4,$$

$s \in I$, lying in the subspace $\mathbb{E}_1^3 = \text{span}\{\eta_1, \eta_3, \eta_4\}$. Since $\eta_3 = \frac{\xi_4 - \xi_3}{\sqrt{2}}$ and $\eta_4 = \frac{\xi_3 + \xi_4}{\sqrt{2}}$,

$$\gamma(s) = x(s)\eta_1 + (\frac{\overline{w}(s) - \overline{z}(s)}{\sqrt{2}})\xi_3 + (\frac{\overline{w}(s) + \overline{z}(s)}{\sqrt{2}})\xi_4, \quad s \in I \tag{2.3.3}$$

Now, put $z(s) = \frac{\overline{w}(s) - \overline{z}(s)}{\sqrt{2}}$ and $w(s) = \frac{\overline{w}(s) + \overline{z}(s)}{\sqrt{2}}$. Then $\gamma(s) = x(s)\eta_1 + z(s)\xi_3 + w(s)\xi_4$. Consider the surface defined by

$$F_3(s,t) = x(s)\eta_1 + \sqrt{2}tw(s)\eta_2 + (z(s) + t^2 w(s))\xi_3 + w(s)\xi_4, s \in I, t \in \mathbb{R}. \tag{2.3.4}$$

The orbit of $\gamma(s)$ under the action of the orthogonal transformation T of $\mathbb{E}^4$ defined by $T(\eta_1) = \eta_1$, $T(\eta_2) = \eta_2 + \sqrt{2}t\xi_3$, $T(\xi_3) = \xi_3$ and $T(\xi_4) = \sqrt{2}t\eta_2 + t^2\xi_3 + \xi_4$ that continues the degenerate plane spanned by $\{\eta_1, \xi_3\}$ pointwise fixed. The surface F_3 retains screw invariante and is also referred to as a rotational surface of parabolic type. It remains regular and timelike if $w(s) > 0$ and the profile curve $\gamma(s)$ is timelike on I.

3. BICONSERVATIVE PNMCV SURFACES IN $\mathbb{E}^4$

In this section, we are going to consider biconservative PNMCV surfaces in $\mathbb{E}^4$. Note that the results obtained in this section taken from [19]. Firstly, we obtain the following lemma which provides a necessary and sufficient condition for being biconservative of such surfaces.

Lemma 3.1. *Let (M,g) be a PNMCV surface in $\mathbb{E}^4$. Then, M is biconservative surface if and only if there exists an orthonormal frame field $\{e_1,e_2;e_3,e_4\}$ satisfying the following conditions:*

(1) *The Levi-Civita connection ∇ and the normal connection $\nabla^\perp$ of M satisfy*

$$\nabla_{e_1}e_1 = \nabla_{e_1}e_2 = 0, \tag{3.0.1}$$

$$\nabla_{e_2}e_1 = \frac{-3e_1(f)}{4f}e_2, \quad \nabla_{e_2}e_2 = \frac{3e_1(f)}{4f}e_1, \tag{3.0.2}$$

$$\nabla^\perp e_3 = \nabla^\perp e_4 = 0 \tag{3.0.3}$$

for a function f satisfying $e_1(f)\neq 0$, $e_2(f)=0$.

(2) *Shape operators along e_3 and e_4 have matrix representation given by*

$$A_{e_3} = \begin{pmatrix} -f & 0 \\ 0 & 3f \end{pmatrix} \text{ and } A_{e_4} = \begin{pmatrix} cf^{\frac{3}{2}} & 0 \\ 0 & -cf^{\frac{3}{2}} \end{pmatrix}. \tag{3.0.4}$$

for a constant $c>0$.

Proof. Let the mean curvature vector of M be denoted by H with $\langle H,H\rangle = f^2$. We suppose that M is biconservative. Let $\{e_3,e_4\}$ be the local orthonormal frame field for the normal bundle of M such that $e_3 = \frac{H}{f}$ is parallel. Since M is biconservative , the equation (2.2.3) satisfied for $m=2$.

Now, consider the orthonormal frame field $\{e_1,e_2\}$ of the tangent bundle of M such that $e_1 = \frac{\operatorname{grad} f}{\|\operatorname{grad} f\|}$. From (2.2.3) one can get $A_{e_3}(e_1) = -fe_1$.

Since the co-dimension is 2, e_4 is also parallel. Therefore, the equation (3.0.3) is satisfied.

Since H is proportional to e_3 , the equation $2fe_3 = \text{trace}A_{e_3}e_3 + \text{trace}A_{e_4}e_4$ implies

$$\text{trace}A_{e_3} = 2f \tag{3.0.5}$$

and

$$\text{trace}A_{e_4} = 0. \tag{3.0.6}$$

Now, the first equation in (3.0.5) gives

$$\langle A_{e_3}(e_1), e_1\rangle + \langle A_{e_3}(e_2), e_2\rangle = 2f$$

Therefore, A_{e_3} satisfies

$$A_{e_3}(e_2) = \langle A_{e_3}(e_2), e_1\rangle e_1 + \langle A_{e_3}(e_2), e_2\rangle e_2 = 3fe_2$$

On the other hand, the second equation in (3.0.6) implies

$$\langle A_{e_4}(e_1), e_1\rangle = -\langle A_{e_4}(e_2), e_2\rangle = \lambda$$

for a smooth function λ. Thus, the matrix representations of A_{e_3} and A_{e_4} take the form as the following:

$$A_{e_3} = \begin{pmatrix} -f & 0 \\ 0 & 3f \end{pmatrix} \text{ and } A_{e_4} = \begin{pmatrix} \lambda & \gamma \\ \gamma & -\lambda \end{pmatrix} \tag{3.0.7}$$

for λ and γ. Moreover, by using the Ricci equation (2.1.4) for $X = e_1$, $Y = e_2$ and $\xi = e_3$ we obtain

$$\begin{aligned} &R^{\perp}(e_1, e_2)e_3 \\ &= h(e_1, A_{e_3}e_2) - h(A_{e_3}e_1, e_2) = h(e_1, 3fe_2) - h(-fe_1, e_2) \\ &= 3fh(e_1, e_2) + fh(e_1, e_2) = 4fh(e_1, e_2). \end{aligned}$$

Since e_3 is parallel, $R^{\perp}(e_1, e_2)e_3 = 0$ implies

$$4fh(e_1, e_2) = 0.$$

Because of $f \neq 0$, we obtain $h(e_1, e_2) = 0$.

Morever, $\gamma = \langle A_{e_4}(e_1)e_2 \rangle = \langle h(e_1, e_2), e_4 \rangle = 0$. So, (3.0.7) turns into

$$A_{e_3} = \begin{pmatrix} -f & 0 \\ 0 & 3f \end{pmatrix} \text{ and } A_{e_4} = \begin{pmatrix} \lambda & 0 \\ 0 & -\lambda \end{pmatrix}$$

from which one can get

$$\langle h(e_2, e_2), e_3 \rangle = \langle A_{e_3}(e_2), e_2 \rangle = 3f$$

$$\langle h(e_2, e_2), e_4 \rangle = \langle A_{e_4}(e_2), e_2 \rangle = -\lambda$$

which yields

$$h(e_2, e_2) = 3fe_3 - \lambda e_4. \tag{3.0.8}$$

Now, the Codazzi equation (2.1.2) for $X = e_1$, $Y = Z = e_2$ implies;

$$(\overline{\nabla}_{e_1} h)(e_2, e_2) = (\overline{\nabla}_{e_2} h)(e_1, e_2)$$

which is equivalent to

$$\nabla^{\perp}_{e_1} h(e_2, e_2) - h(\nabla_{e_1} e_2, e_2) - h(e_2, \nabla_{e_1} e_2) = \nabla^{\perp}_{e_2} h(e_1, e_2) - h(\nabla_{e_2} e_1, e_2) - h(e_1, \nabla_{e_2} e_2).$$

By using (2.1.4), the above equation implies

$$\nabla^{\perp}_{e_1}(3fe_3 - \lambda e_4) - 2h(\omega_{12}(e_1)e_1, e_2) = -h(\omega_{12}(e_1)e_2, e_2) - h(e_1, \omega_{21}(e_2)e_1)$$

which gives

$$e_1(3f)e_3 + 3f\nabla^{\perp}_{e_1} e_3 - e_1(\lambda)e_4 - \lambda\nabla^{\perp}_{e_1} e_4 = \omega_{12}(e_2)(-h(e_2, e_2) + h(e_1, e_1)). \tag{3.0.9}$$

Since e_3 and e_4 are parallel, (3.0.9) implies

$$3e_1(f)e_3 - e_1(\lambda)e_4 = \omega_{12}(e_2)(-4fe_3 + 2\lambda e_4).$$

By using the last equation, we have

$$\frac{-3e_1(f)}{4f} = \omega_{12}(e_2) \text{ and } \frac{-e_1(\lambda)}{2\lambda} = \omega_{12}(e_2). \tag{3.0.10}$$

From the above we obtain

$$\frac{3e_1(f)}{2f} = \frac{e_1(\lambda)}{\lambda}.$$

If the integration both sides of the above equation is taken, we have

$$e_1(\ln(f^{\frac{3}{2}})) = e_1 \ln(\lambda)$$

and we find

$$\lambda = cf^{\frac{3}{2}},$$

where c is a smooth function satisfying $e_1(c) = 0$.

On the other hand, from the Codazzi equation (2.1.2) for $X = e_2$, $Y = Z = e_1$ implies;

$$(\overline{\nabla}_{e_2} h)(e_1, e_1) = (\overline{\nabla}_{e_1} h)(e_2, e_1)$$

which is equivalent to

$$\nabla^{\perp}_{e_2} h(e_1, e_1) - 2h(\nabla_{e_2} e_1, e_1) = \nabla^{\perp}_{e_1} h(e_2, e_1) - h(\nabla_{e_1} e_2, e_1) - h(e_2, \nabla_{e_1} e_2).$$

Note that (3.0.8) implies $h(e_1, e_1) = -fe_3 + \lambda e_4$. So the equation becomes

$$\nabla^{\perp}_{e_2}(-fe_3 + \lambda e_4) - 2h(\omega_{12}(e_2)e_2, e_1) = -h(-\omega_{12}(e_1)e_1 - h(e_2, \omega_{12}(e_1)e_2)$$

$$\nabla^{\perp}_{e_2}(-fe_3 + \lambda e_4) - 2\omega_{12}(e_2)h(e_1, e_2) = \omega_{12}(e_1)h(e_1, e_1) - \omega_{12}(e_1)h(e_2, e_2)$$

$$\nabla^{\perp}_{e_2}(-fe_3 + \lambda e_4) = \omega_{12}(e_1)((-fe_3 + \lambda e_4) - (3fe_3 - \lambda e_4))$$

which gives

$$-e_2(f)e_3 + e_2(\lambda)e_4 = w_{12}(e_1)(-4fe_3 + 2\lambda e_4). \tag{3.0.11}$$

Since f is not constant and e_1 is proportional to ∇f. By using the definition of ∇f, one can get

$$\text{grad} f = e_1(f)e_1 + e_2(f)e_2. \tag{3.0.12}$$

Since $f \neq 0$, we find $e_1(f) \neq 0$ and $e_2(f) = 0$. Therefore, the coefficient from e_3 of (3.0.11) implies

$$\omega_{12}(e_1) = 0.$$

Also, by considering the coefficient of e_4 gives

$$e_2(\lambda) = 0. \tag{3.0.13}$$

By replacing e_4 with $-e_4$ if necessary can be assumed to be positive.

Therefore, M satisfies the condition (1) of the Lemma. On the other hand, the second equation in (3.0.7) and the first equation in (3.0.13) give the condition (2) of the Lemma. Thus, the proof of the necessary condition was completed. Conversely, assume the existence of local orthonormal frame field $\{e_1, e_2; e_3, e_4\}$ on M satisfiying condition (1) and (2). Because of condition (2) implies

$$H = \frac{1}{2}(\text{trace}A_{e_3}e_3 + \text{trace}A_{e_4}e_4) = fe_3.$$

From (3.0.3) then M is PNMCV. If $e_1(f) \neq 0$ and $e_2(f) = 0$ then

$$\text{grad}f = e_1(f)e_1.$$

By condition (2)

$$\begin{aligned} A_{e_3}(\text{grad}f) =& e_1(f)A_{e_3}e_1 \\ =& -fe_1(f)e_1 \\ =& -f\text{grad}f. \end{aligned}$$

Consequently, M is a biconservative PNMCV surface. □

Remark 3.1. *For the satisfied conditions for $c = 0$ in Lemma 3.1, one would have $A_{e_4} = 0$ which leads to $\tilde{\nabla}_X e_4 = 0$ whenever X is tangent to M. Hence, e_4 is a constant vector which yields that M is a surface lying in $\mathbb{E}^3$.*

Corollary 3.1. *In $\mathbb{E}^4$, no biharmonic PNMCV surface exists.*

Proof. Consider that there exists a biharmonic PNMCV surface in $\mathbb{E}^4$. Then M satisfy the equation (2.2.2). By considering condition (2) of Lemma 3.1, it is computed the left-handside (2.2.2) to obtain

$$\begin{aligned} \text{trace}h(A_H(\cdot,\cdot) &= h(A_{fe_3}(e_1), e_1) + h(A_{fe_3}(e_2), e_2) \\ &= fh(A_{e_3}(e_1), e_1) + fh(A_{e_3}(e_2), e_2) \\ &= fh(-fe_1, e_1) + fh(3fe_2, e_2) \\ \text{trace}h(A_H(\cdot,\cdot) &= -f^2h(e_1, e_1) + 3f^2h(e_2, e_2). \end{aligned} \tag{3.0.14}$$

By substiting $h(e_1,e_1) = -fe_3 + cf^{\frac{3}{2}}e_4$ and $h(e_2,e_2) = 3fe_3 - cf^{\frac{3}{2}}e_4$, the equation (3.0.14) turns into

$$\text{trace}h(A_H(\cdot),\cdot) = 10f^3e_3 - 4f^2cf^{\frac{3}{2}}e_4.$$

If the above result is substituted into equation (2.2.2), the following equation is found:

$$\text{trace}h(A_H(\cdot),\cdot) - \Delta^{\perp}H + \text{trace}(\tilde{R}(\cdot,H).)^{\perp}) = (10f^3 - \Delta f)e_3 - 4cf^{\frac{7}{2}}e_4. \quad (3.0.15)$$

By combining (2.2.2) and (2.2.3) we are obtained $c = 0$. By Remark 3.1 , M is minimal which yields a contradiction. □

As an immediate consequence of Lemma 3.1, it would like to state the following corollary:

Corollary 3.2. *If the surface M admits a biconservative PNMCV immersion into $\mathbb{E}^4$, then the Gaussian curvature K of M and the mean curvature f of Ψ satisfy the following condition:*

$$K = -3f^2 - c^2f^3$$

for a constant $c \geq 0$.

Proof. The definition for the Gaussian curvature K of the surface is found as follows;

$$K = (R(X,Y),Z,W) = \langle h(Y,Z),h(X,W)\rangle - \langle h(X,Z),h(Y,W)\rangle.$$

For $X = e_1$, $Y = e_2$, $Z = e_2$ and $W = e_1$, obtain;

$$K = (R(e_1,e_2),e_2,e_1) = \langle h(e_2,e_2),h(e_1,e_1)\rangle - \langle h(e_1,e_2),h(e_2,e_1)\rangle$$

$$= \langle(-fe_3 + \lambda e_4),(3fe_3 - \lambda e_4)\rangle - 0$$

$$= -3f^2\langle e_3,e_3\rangle + f\lambda\langle e_3,e_4\rangle + 3f\lambda\langle e_4,e_3\rangle - \lambda^2\langle e_4,e_4\rangle$$

$$= -3f^2 - \lambda^2 = -3f^2 - c^2f^3$$

for a constant $c \geq 0$. □

By utilizing Lemma 3.1 and Corollary 3.2, we arrive at the following rigidity result.

Proposition 3.1. *Let (M,g) be an abstract surface. If $x_1, x_2 : (M,g) \to \mathbb{E}^4$ are two biconservative PNMCV isometric immersions, then there exists an isometry $\Phi : \mathbb{E}^4 \to \mathbb{E}^4$ such that $x_2 = \Phi \circ x_1$.*

Proof. Let ${}^iT^\perp M$, ${}^iH$ and ${}^i\nabla^\perp$ are denoted the normal bundle, the mean curvature and the normal connection of ψ_i, respectively and $f_i = (\langle {}^iH, {}^iH\rangle)^{\frac{1}{2}}$, for $i = 1,2$.

By Corollary 3.2, the Gaussian curvature K of M is satisfied;

$$-K = 3f_1^2 + c^2 f_1^3 = 3f_2^2 + \tilde{c}^2 f_2^3 \tag{3.0.16}$$

for some nonnegative constants c, $\tilde{c}$. Thus, we have

$$XK = (-6f_1 - 3c^2 f_1^2)X(f_1) = (-6f_2 - 3\tilde{c}^2 f_2^2)X(f_2)$$

whenever X is tangent to M which is implied

$$\frac{\text{grad} f_i}{\|\text{grad} f_i\|} = -\frac{\text{grad} K}{\|\text{grad} K\|}, i = 1,2.$$

For this reason, the tangent vector fields e_1, e_2 described in Lemma 3.1 do not depend on the biconservative immersion but only the surface M.

Let $\{e_1, e_2; {}^ie_3, {}^ie_4\}$ denote the frame field corresponding to ψ_i described in Lemma 3.1. Then, it is satisfied $e_2(f_i) = 0$. Furthermore, (3.0.2) is implied

$$\nabla_{e_2} e_1 = \frac{-3e_1(f_1)}{4f_1} e_2 = \frac{-3e_1(f_2)}{4f_2} e_2$$

which is given $f_1 = \zeta f_2$ for a function ζ satisfying $e_1(\zeta) = 0$, $\zeta > 0$. As $e_2(f_i) = 0$, ζ is a constant. Moreover, considering (3.0.8) one can get $\zeta = 1$ and $c = \tilde{c}$. So that, $f_1 = f_2$ is satisfied. Consequently, the shape operators A_i for e_3 and e_4 are made to satisfy the following conditions:

$$A_{1_{e_3}} = A_{2_{e_3}} \quad \text{and} \quad A_{1_{e_4}} = A_{2_{e_4}}. \tag{3.0.17}$$

Now, it is considered the vector bundle isomorphism $\Psi : T_1^\perp M \to T_2^\perp M$ given by $\Psi({}^1H) = {}^2H$ and $\Psi({}^1e_4) = {}^2e_4$. Then, because of $f_1 = f_2$, it is obtained;

$$\langle \Psi({}^1e_\alpha), \Psi({}^1e_\beta)\rangle = \langle {}^2e_\alpha, {}^2e_\beta\rangle; \alpha, \beta = 3,4$$

and (3.0.17) implies

$$\Psi \circ \alpha_{\Psi_1} = \alpha_{\Psi_2}.$$

Moreover, in order to Ψ_1 and Ψ_2 are PNMCV, it is satisfied;

$$\Psi \circ ({}^1\nabla^\perp e_\alpha) = 0 = {}^2\nabla^\perp(\Psi({}^1e_a)), \alpha = 3,4.$$

Accordingly, by the uniqueness part of fundamental theorem of submanifolds, there exists an isometry $\tilde{\Psi}$ of $\mathbb{R}^4$ such that $\tilde{\Psi} \circ \Psi_1 = \Psi_2$. □

Let f be a mean curvature and $(\text{grad} f)(m) \neq 0$ where $m \in M$ with $f(m) \neq 0$. By using Lemma 3.1, I would like to construct a local coordinate system on a neighborhood of $m \in M$ for a biconservative PNMCV surface M in $\mathbb{E}^4$.

Lemma 3.2. *Let M is a proper biconservative PNMCV surface. And so, around any point of M, there exists a local coordinate system (s,t) such that*

$$f = f(s),\ e_1 = \frac{\partial}{\partial s} \text{ and } e_2 = f(s)^{\frac{3}{4}} \frac{\partial}{\partial t}.$$

Proof. Let $\{e_1, e_2; e_3, e_3\}$ be a local orthonormal frame field given in Lemma 3.1. Because of $\nabla_{e_1} e_2 = 0$ and $\nabla_{e_2} e_1 = \frac{-3e_1(f)}{4f} e_2$, then $[e_1, e_2] = \frac{3e_1(f)}{4f} e_2$ which implies $[e_1, Ee_2] = 0$ for any function E satisfying

$$e_1(E) = \frac{-3e_1(f)}{4f} E. \tag{3.0.18}$$

Therefore, there exists a local coordinate system (s,t) such that $e_1 = \frac{\partial}{\partial s}$ and $Ee_2 = \frac{\partial}{\partial t}$. This is a proper PNMCV biconservative then by Lemma 3.1 there exists $\{e_1, e_2; e_3, e_4\}$;

$$[e_1, e_2] = \nabla_{e_1} e_2 - \nabla_{e_2} e_1 = \frac{3e_1(f)}{4f} e_2$$

$$\begin{aligned} [e_1, Ee_2] &= e_1(E)e_2 + E[e_1, e_2] \\ &= e_1(E)e_2 + E\frac{3e_1(f)}{4f} e_2 \\ &= (e_1(E) + \frac{3e_1(f)}{4f} E)e_2 \end{aligned}$$

$\exists E : [e_1, Ee_2] = o$ by the way $\exists(s,t)$: $e_1 = \frac{\partial}{\partial s}$ and $Ee_2 = \frac{\partial}{\partial t}$. Owing to $e_2(f) = 0$ $f = f(s)$.

$$\frac{\partial E(s,t)}{\partial s} + \frac{3f'}{4f}E(s,t) = 0$$

The equation is found as;

$$\frac{E_s(s,t)}{E} = \frac{-3f'}{4f} \text{ then } E(s,t) = f^{\frac{-3}{4}}$$

Furthermore, $e_2(f) = 0$ which yields $f = f(s)$ and due to (3.0.18) it can be chosen E as $E = f(s)^{\frac{-3}{4}}$. □

Corollary 3.3. *Let M be a proper biconservative PNMCV surface in $\mathbb{E}^4$. Thus, the mean curvature of M implies the following ordinary differential equation.*

$$\frac{9f'(s)^2}{16f(s)^{\frac{7}{2}}} + c^2 f(s)^{\frac{3}{2}} + 9f(s)^{\frac{1}{2}} = c_2^2, \tag{3.0.19}$$

for a positive constant c_2, which s is the local coordinate given in Lemma 3.2.

Proof. Considering a local orthonormal frame field $\{e_1, e_2; e_3, e_4\}$ given in Lemma 3.1 and local coordinate system (s,t) given in Lemma 3.2, if the equation (2.1.3) is considered for: $Y = \partial t$ and $X = Z = \partial s$ gives

$$R(e_1,e_2)e_1 = A_{h(e_1,e_1)}e_2,$$

$$R(e_1,e_2)e_1 = \nabla_{e_1}\nabla_{e_2}e_1 - \nabla_{[e_1,e_2]}e_1 = -A_{-fe_3+cf^{\frac{3}{2}}e_4}e_2.$$

If the previously found results are substituted in the equation above, the result below is obtained

$$\nabla_{e_1}(\frac{-3e_1(f)}{4f}e_2) - \nabla_{[e_1,e_2]}e_1 = fA_{e_3}e_2 - cf^{\frac{3}{2}}A_{e_4}e_2.$$

At that point one can obtain

$$\begin{aligned} 3f^2e_2 + c^2f^3e_2 &= (\frac{-3f'}{4f})'e_2 - \nabla_{[e_1,e_2]}e_1 \\ &= \frac{-12f''f + 12f'^2}{16f^2}e_2 - \nabla_{[e_1,e_2]}e_1 \\ &= \frac{-12f''f + 12f'^2}{16f^2}e_2 - \nabla_{\nabla_{e_1}e_2 - \nabla_{e_2}e_1}e_1. \end{aligned}$$

If the previously found results are substituted in the equation above, the result below is obtained

$$\begin{aligned}(3f^2+c^2f^3)e_2 &= \frac{-12f''f+12f'^2}{16f^2}e_2 - \nabla_{\frac{3e_1(f)}{4f}e_2}e_1 \\ &= \frac{-12f''f+12f'^2}{16f^2}e_2 - \frac{(3f')^2}{(4f)^2}e_2\end{aligned}$$

which is equivalent to;

$$3f^2+c^2f^3 = \frac{-3f''f+3f'^2}{4f^2} - \frac{9f'^2}{16f^2}.$$

By multiplying both sides by $-\frac{4f^2}{3}$, one can get the following equation

$$f''f - \frac{7}{4}f'^2 + 4f^4 + \frac{4c^2f^5}{3} = 0$$

By multiplying this equation with $\frac{9f'}{8f^{\frac{9}{2}}}$

$$\frac{9f'f''}{8f^{\frac{7}{2}}} - \frac{63f'^3}{32f^{\frac{9}{2}}} + \frac{9f'}{2f^{\frac{1}{2}}} + \frac{3c^2f'f^{\frac{1}{2}}}{2} = 0.$$

Then, by integrating this equation, one can get (3.0.19) for some constant c_2 which can be assumed to be positive. □

4. BICONSERVATIVE ROTATIONAL SURFACES IN $\mathbb{E}_1^4$

In this section, some classifications of biconservative rotational surfaces are obtained. Note that these results are presented in [21].

4.1 Rotational Surfaces of Hyperbolic Type

Let M be a surface in $\mathbb{E}_1^4$ given by (2.3.1) for given functions $x(s)$, $y(s)$ and $w(s)$ satisfying $x'^2 + y'^2 - w'^2 = 1$. Let $N = (n_1, n_2, n_3)$ and $B = (b_1, b_2, b_3)$ denote the binormal vector and normal vector fields of curve $\beta = (x, y, w)$ in $\mathbb{E}_1^3$. Then, since

$$\frac{\partial F_1}{\partial s} = (x'(s), y'(s), w'(s)\sinh t, w'(s)\cosh t)$$
$$\frac{\partial F_1}{\partial t} = (0, 0, w(s)\cosh t, w(s)\sinh t)$$

$F = F_1$ satisfies, $\langle F_t, F_t\rangle = w^2$, $\langle F_t, F_s\rangle = \langle F_s, F_t\rangle = 0$ and $\langle F_s, F_s\rangle = x'^2 + y'^2 - w'^2 = 1$ are obtained. So, M is a Riemannian surface. Next, consider the vector fields

$$e_1 = \frac{\partial}{\partial s} = (x', y', w'\sinh t, w'\cosh t)$$
$$e_2 = \frac{1}{w}\frac{\partial}{\partial t} = (0, 0, \cosh t, \sinh t)$$

which from an orthonormal base for the tangent bundle of M. Note that the

$$e_3 = (n_1, n_2, n_3\sinh t, n_3\cosh t)$$

$$e_4 = (b_1, b_2, b_3\sinh t, b_3\cosh t)$$

satisfied

$$\langle e_1, e_3\rangle = x'n_1 + y'n_2 + w'n_3(\sinh t^2 - \cosh t^2) = \langle T, N\rangle = 0.$$

$$\langle e_2, e_3\rangle = 0+0+n_3(\sinh t\cosh t - \sinh t\cosh t) = 0,$$

$$\langle e_3, e_3\rangle = \langle N, N\rangle = 1,$$

$$\langle e_3, e_4\rangle = \langle N, B\rangle = 0,$$

$$\langle e_4, e_4\rangle = \langle B, B\rangle = -1.$$

Therefore, $\{e_3, e_4\}$ is an orthonormal base for the normal bundle of M.

The coefficients of the second fundamental form are determined as follows.

$$\tilde{\nabla}_{e_1} e_1 = \frac{\partial F_1}{\partial s}(x', y', w'\sinh t, w'\cosh t)$$

$$= (x'', y'', w''\sinh t, w''\cosh t) = (t_1', t_2', t_3'\sinh t, t_3'\cosh t).$$

By considering the Frenet equation $T' = \kappa N$ one can obtain.

$$\tilde{\nabla}_{e_1} e_1 = (\kappa n_1, \kappa n_2, \kappa n_3\sinh t, \kappa n_3\cosh t) = \kappa e_3.$$

Therefore h and ∇ satisfy $h(e_1, e_1) = \kappa e_3$ and $\nabla_{e_1} e_1 = 0$.

$$\tilde{\nabla}_{e_1} e_2 = \frac{\partial}{\partial s}(0, 0, \cosh t, \sinh t) = (0, 0, 0, 0) = h(e_1, e_2) + \nabla e_1 e_2$$

from which one can obtain

$$h(e_1, e_2) = 0 \text{ and } \nabla_{e_1} e_2 = 0$$

$$\tilde{\nabla}_{e_1} e_3 = \frac{\partial}{\partial s}(n_1, n_2, n_3\sinh t, n_3\cosh t) = (n_1', n_2', n_3'\sinh t, n_3'\cosh t)$$

By considering the Frenet equation $N' = -\kappa T + \tau B$ one can get.

$$\tilde{\nabla}_{e_1} e_3 = (-\kappa t_1 + \tau b_1, -\kappa t_2 + \tau b_2, (-\kappa t_3 + \tau b_3)\sinh t, (-\kappa t_3 + \tau b_3)\cosh t)$$

$$= -\kappa(t_1, t_2, t_3\sinh t, t_3\cosh t) + \tau(b_1, b_2, b_3\sinh t, b_3\cosh t)$$

$$= -\kappa e_1 + \tau e_4.$$

By combining this equation and

$$\tilde{\nabla}_{e_1} e_3 = -A_{e_3} e_1 + \nabla^{\perp}_{e_1} e_3,$$

one can obtain

$$A_{e_3}e_1 = \kappa e_1 \text{ and } \nabla^{\perp}_{e_1}e_3 = \tau e_4. \tag{4.1.1}$$

By a similar way, the equation

$$\tilde{\nabla}_{e_1}e_4 = \frac{\partial}{\partial s}(b_1, b_2, b_3\sinh t, b_3\cosh t)$$

implies

$$A_{e_4}e_1 = 0 \text{ and } \nabla^{\perp}_{e_1}e_4 = \tau e_3 \tag{4.1.2}$$

$$\tilde{\nabla}_{e_2}e_1 = \frac{1}{w}\frac{\partial}{\partial t}(x', y', w'\sinh t, w'\cosh t) = \frac{w'}{w}e_2$$

gives

$$h(e_2, e_1) = 0 \text{ and } \nabla_{e_2}e_1 = \frac{w'}{w}e_2.$$

Furthermore, the equation

$$\tilde{\nabla}_{e_2}e_2 = \frac{1}{w}\frac{\partial}{\partial t}(0, 0, \cosh t, \sinh t)$$

implies

$$h(e_2, e_2) = \frac{1}{w}(0, 0, \sinh t, \cosh t) \text{ and } \nabla_{e_2}e_2 = \frac{-w'}{w}e_1. \tag{4.1.3}$$

On the other hand, by considering Weingarten equation and

$$\tilde{\nabla}_{e_2}e_3 = \frac{1}{w}\frac{\partial}{\partial t}(n_1, n_2, n_3\sinh t, n_3\cosh t),$$

one can obtain

$$A_{e_3}e_2 = -\frac{n_3}{w}e_2 \text{ and } \nabla^{\perp}_{e_2}e_3 = 0. \tag{4.1.4}$$

Similarly,

$$\tilde{\nabla}_{e_2}e_4 = \frac{1}{w}\frac{\partial}{\partial t}(b_1, b_2, b_3\sinh t, b_3\cosh t)$$

implies

$$A_{e_4}e_2 = -\frac{b_3}{w}e_2 \text{ and } \nabla^{\perp}_{e_2}e_4 = 0. \tag{4.1.5}$$

By summing up (4.1.1), (4.1.2), (4.1.4) and (4.1.5), one can get

$$A_{e_3} = \begin{pmatrix} \kappa & 0 \\ 0 & -\frac{n_3}{w} \end{pmatrix} \text{ and } A_{e_4} = \begin{pmatrix} 0 & 0 \\ 0 & -\frac{b_3}{w} \end{pmatrix}. \tag{4.1.6}$$

Consequently, the mean curvature vector of H takes the form

$$H = \frac{1}{2}[(\kappa - \frac{n_3}{w})e_3 + \frac{b_3}{w}e_4]. \tag{4.1.7}$$

In another case, the Gaussian curvature of M is

$$K = \det A_{e_3} - \det A_{e_4} = -\frac{\kappa n_3}{w}. \tag{4.1.8}$$

Now, let M be a biconservative surface, i.e, (1.0.6) is satisfied. Then (4.1.7) implies

$$\begin{aligned}\nabla^{\perp}_{e_1}H &= \nabla^{\perp}_{e_1}(\frac{1}{2}[(\kappa - \frac{n_3}{w})e_3 + (\frac{b_3}{w})e_4]) \\ &= \frac{1}{2}(\kappa' - (\frac{n_3}{w})')e_3 + (\frac{b_3}{2w})'e_4 + \frac{1}{2}(\kappa - \frac{n_3}{w})\nabla^{\perp}_{e_1}e_3 + \frac{b_3}{2w}\nabla^{\perp}_{e_1}e_4.\end{aligned}$$

By combining this equation (4.1.1), (4.1.2) and (4.1.6) one can obtain $A_{\nabla^{\perp}_{e_1}H}e_1$ as

$$\begin{aligned}A_{\nabla^{\perp}_{e_1}H}e_1 &= \frac{1}{2}(\kappa' - (\frac{n_3}{w})')A_{e_3}e_1 + (\frac{b_3}{2w})'A_{e_4}e_1 + \frac{1}{2}(\kappa - \frac{n_3}{w})A_{\nabla^{\perp}_{e_1}e_3}e_1 + \frac{b_3}{2w}A_{\nabla^{\perp}_{e_1}e_4}e_1 \\ &= \frac{1}{2}(\kappa' - (\frac{n_3}{w})')\kappa e_1 + \frac{1}{2}(\kappa - \frac{n_3}{w})A_{\tau e_4}e_1 + \frac{b_3}{2w}A_{\tau e_3}e_1 \\ &= \frac{1}{2}(\kappa' - (\frac{n_3}{w})')\kappa e_1 + \frac{1}{2}(\kappa - \frac{n_3}{w})\tau A_4 e_1 + \frac{b_3}{2w}\tau A_3 e_1 \\ &= \frac{1}{2}(\kappa' - (\frac{n_3}{w})')\kappa e_1 + \frac{b_3}{2w}\tau A_3 e_1 \\ &= \frac{1}{2}(\kappa\kappa' - [\frac{(\kappa n_3' w - \kappa n_3 w'}{w^2}])e_1 + \frac{b_3}{2w}\tau\kappa e_1.\end{aligned}$$

By using the Frenet formulas, one can observe that the last equation implies

$$A_{\nabla^{\perp}_{e_1}H}e_1 = \frac{1}{2}\kappa\kappa' e_1 + \frac{\kappa^2 t_3}{2w}e_1 + \frac{\kappa n_3 w'}{2w^2}e_1.$$

Furthermore, since

$$\begin{aligned}\nabla^{\perp}_{e_2}H &= \nabla^{\perp}_{e_2}(\frac{1}{2}[(\kappa - \frac{n_3}{w}e_3) + (\frac{b_3}{w})e_4]) \\ &= e_2(\frac{1}{2}(\kappa - \frac{n_3}{w}))e_3 + e_2(\frac{b_3}{2w})e_4 + \frac{1}{2}(\kappa - \frac{n_3}{w})\nabla^{\perp}_{e_2}e_3 + \frac{b_3}{2w}\nabla^{\perp}_{e_2}e_4 = 0,\end{aligned}$$

one can conclude that

$$A_{\nabla^{\perp}_{e_2}H}e_2 = 0.$$

On the other hand, since

$$f = \langle H, H\rangle = \frac{1}{4}[(\kappa - \frac{n_3}{w})^2 - (\frac{b_3}{w})^2],$$

H satisfies $\text{grad} f = A(s)e_1$ for the function A computed as

$$\begin{aligned}
&A'(s)e_1([\frac{1}{4}(\kappa - \frac{n_3}{w})^2 - \frac{1}{4}(\frac{b_3}{w})^2]')e_1 \\
&= (\frac{1}{2}(\kappa - \frac{n_3}{w})(\kappa' - \frac{n_3' w + n_3 w'}{w^2}) - \frac{b_3}{2w}(\frac{b_3' w - b_3 w'}{w^2}))e_1 \\
&= (\frac{1}{2}\kappa\kappa' + \frac{\kappa^2 t_3}{2w} - \frac{\kappa\tau b_3}{2w} + \frac{\kappa n_3 w'}{2w^2} - \frac{\kappa' n_3}{2w} - \frac{\kappa n_3 t_3}{2w^2} + \frac{(b_3^2 - n_3^2)w'}{2w^3} + \frac{\tau n_3 b_3}{w^2})e_1.
\end{aligned}$$

Now, by combining the equations above one can conclude that (1.0.6) is equivalent to;

$$\kappa\kappa' + \frac{\kappa^2 t_3}{w} + \frac{\kappa n_3 w'}{w^2} - \frac{\kappa' n_3}{2w} + \frac{(b_3^2 - n_3^2)w'}{2w^3} - \frac{\kappa\tau b_3}{2w} - \frac{\kappa n_3 t_3}{2w^2} + \frac{\tau n_3 b_3}{w^2} = 0. \quad (4.1.9)$$

Thus, we have the theorem below:

Theorem 4.1. *Let M be the rotational surface in $\mathbb{E}_1^4$ parametrized by* (2.3.1)*. Then M is biconservative if and only if the equation* (4.1.9) *is satisfied.*

4.2 Timelike Rotational Surfaces

Let M be a surface in $\mathbb{E}_1^4$ given by (2.3.2) for given functions $x(s)$, $y(s)$ and $w(s)$ satisfying $x'^2 + y'^2 + w'^2 = 1$. Let $N = (n_1, n_2, n_3)$ and $B = (b_1, b_2, b_3)$ denote the binormal and normal vector fields of the curve $\beta = (x, y, z)$ in $\mathbb{E}^3$. Then, since

$$\begin{aligned}
\frac{\partial F_2}{\partial s} &= (x'(s), y'(s), w'(s)\cosh t, w'(s)\sinh t) \\
\frac{\partial F_2}{\partial t} &= (0, 0, w(s)\sinh t, w(s)\cosh t).
\end{aligned}$$

$F = F_2$ satisfies $\langle F_s, F_s\rangle = x'^2 + y'^2 + w'^2 = 1$, $\langle F_t, F_t\rangle = -w^2$ and $\langle F_t, F_s\rangle = \langle F_s, F_t\rangle = 0$ are obtained. So, M is a Lorentzian surface. Next, consider the vector fields

$$\begin{aligned}
e_1 = \frac{\partial}{\partial s} &= (x', y', w'\cosh t, w'\sinh t) \\
e_2 = \frac{1}{w}\frac{\partial}{\partial t} &= (0, 0, \sinh t, \cosh t)
\end{aligned}$$

which from an orthonormal base for tangent bundle of M.

$$\langle e_1, e_1\rangle = 1,$$

$$\langle e_2, e_2\rangle = -1,$$

$$\langle e_1, e_2\rangle = 0.$$

In another case, the vector fields

$$e_3 = (n_1, n_2, n_3 \cosh t, n_3 \sinh t)$$

and

$$e_4 = (b_1, b_2, b_3 \cosh t, b_3 \sinh t).$$

satisfy

$$\langle e_1, e_3 \rangle = x' n_1 + y' n_2 + w' n_3 \sinh t^2 + w' n_3 \cosh t^2 = \langle T, N \rangle = 0,$$

$$\langle e_2, e_3 \rangle = n_3(\sinh t \cosh t - \sinh t \cosh t) = 0,$$

$$\langle e_1, e_4 \rangle = 0,$$

$$\langle e_2, e_4 \rangle = 0,$$

$$\langle e_3, e_3 \rangle = 1,$$

$$\langle e_4, e_3 \rangle = 0,$$

$$\langle e_4, e_4 \rangle = 1.$$

Therefore, $\{e_3, e_4\}$ is an orthonormal base for the normal bundle of M. The coefficients of the second fundamental form are found as follows.

$$\begin{aligned} \tilde{\nabla}_{e_1} e_1 &= \frac{\partial}{\partial s}(x', y', w' \cosh t, w' \sinh t) \\ &= (x'', y'', w'' \cosh t, w'' \sinh t) = (t_1', t_2', t_3' \cosh t, t_3' \sinh t). \end{aligned}$$

By considering the Frenet equation $T' = \kappa N$, one can obtain.

$$\tilde{\nabla}_{e_1} e_1 = (\kappa n_1, \kappa n_2, \kappa n_3 \cosh t, \kappa n_3 \sin t) = \kappa e_3.$$

Therefore, h and ∇ satisfy $h(e_1, e_1) = \kappa e_3$ and $\nabla_{e_1} e_1 = 0$.

$$\tilde{\nabla}_{e_1} e_2 = \frac{\partial}{\partial s}(0, 0, \sinh t, \cosh t) = (0, 0, 0, 0) = h(e_1, e_2) + \nabla_{e_1} e_2$$

which implies

$$h(e_1, e_2) = 0 \text{ and } \nabla_{e_1} e_2 = 0.$$

Furthermore, one can find

$$\tilde{\nabla}_{e_1} e_3 = \frac{\partial}{\partial s}(n_1, n_2, n_3 \cosh t, n_3 \sinh t) = (n_1', n_2', n_3' \cosh t, n_3' \sinh t).$$

By considering the Frenet equation $N' = -\kappa T + \tau B$, one can obtain

$$\begin{aligned}\tilde{\nabla}_{e_1} e_3 &= (-\kappa t_1 + \tau b_1, -\kappa t_2 + \tau b_2, (-\kappa t_3 + \tau b_3)\cosh t, (-\kappa t_3 + \tau b_3)\sinh t) \\ &= -\kappa(t_1, t_2, t_3 \cosh t, t_3 \sinh t) + \tau(b_1, b_2, b_3 \cosh t, b_3 \sinh t) = -\kappa e_1 + \tau e_4\end{aligned}$$

By combining this equation and

$$\tilde{\nabla}_{e_1} e_3 = -A_3 e_1 + \nabla^{\perp}_{e_1} e_3,$$

one can obtain

$$A_{e_3} e_1 = \kappa e_1 \text{ and } \nabla^{\perp}_{e_1} e_3 = \tau e_4. \tag{4.2.1}$$

By a similar way, the equation

$$\tilde{\nabla}_{e_1} e_4 = \frac{\partial}{\partial s}(b_1, b_2, b_3 \cosh t, b_3 \sinh t)$$

implies

$$A_{e_4} e_1 = 0 \text{ and } \nabla^{\perp}_{e_1} e_4 = -\tau e_3. \tag{4.2.2}$$

and

$$\tilde{\nabla}_{e_2} e_1 = \frac{1}{w}\frac{\partial}{\partial t}(x', y', w' \cosh t, w' \sinh t) = \frac{w'}{w} e_2$$

gives

$$h(e_2, e_1) = 0 \text{ and } \nabla e_2 e_1 = \frac{w'}{w} e_2.$$

Furthermore, the equation

$$\tilde{\nabla}_{e_2} e_2 = \frac{1}{w}\frac{\partial}{\partial t}(0, 0, \sinh t, \cosh t)$$

implies

$$h(e_2, e_2) = \frac{1}{w}(0, 0, \cosh t, \sinh t) \text{ and } \nabla e_2 e_2 = \frac{-w'}{w} e_1. \tag{4.2.3}$$

In another case, by considering Weingarten equation and

$$\tilde{\nabla}_{e_2} e_3 = \frac{1}{w}\frac{\partial}{\partial t}(n_1, n_2, n_3 \cosh t, n_3 \sinh t),$$

one can obtain

$$A_3 e_2 = -\frac{n_3}{w} e_2 \text{ and } \nabla^{\perp}_{e_2} e_3 = 0. \tag{4.2.4}$$

Similarly,

$$\tilde{\nabla}_{e_2} e_4 = \frac{1}{w}\frac{\partial}{\partial t}(b_1, b_2, b_3 \cosh t, b_3 \sinh t)$$

implies

$$A_{e_4} e_2 = -\frac{b_3}{w} e_2 \text{ and } \nabla^{\perp}_{e_2} e_4 = 0 \tag{4.2.5}$$

By summing up (4.2.1), (4.2.2), (4.2.4) and (4.2.5), one can get

$$A_{e_3} = \begin{pmatrix} \kappa & 0 \\ 0 & -\frac{n_3}{w} \end{pmatrix} \text{ and } A_{e_4} = \begin{pmatrix} 0 & 0 \\ 0 & -\frac{b_3}{w} \end{pmatrix}. \tag{4.2.6}$$

Consequently, the mean curvature vector of M takes the form

$$H = \frac{1}{2}[(\kappa - \frac{n_3}{w})e_3 - \frac{b_3}{w} e_4]. \tag{4.2.7}$$

On the other hand, the Gaussian curvature of M is

$$K = -\det A_{e_3} - \det A_{e_4} = \frac{\kappa n_3}{w}. \tag{4.2.8}$$

Now, let M be a biconservative surface, i.e, (1.0.6) is satisfied. Then (4.2.7) implies

$$\begin{aligned} \nabla^{\perp}_{e_1} H &= \nabla^{\perp}_{e_1}(\frac{1}{2}[(\kappa - \frac{n_3}{w})e_3 - (\frac{b_3}{w})e_4]) \\ &= \frac{1}{2}[(\kappa' - (\frac{n_3}{w})')e_3] - (\frac{b_3}{2w})' e_4 + \frac{1}{2}(\kappa - \frac{n_3}{w})\nabla^{\perp}_{e_1} e_3 - \frac{b_3}{2w}\nabla^{\perp}_{e_1} e_4. \end{aligned}$$

By combining this equation (4.2.1), (4.2.5) and (4.2.6) one can obtain $A_{\nabla^{\perp}_{e_1} H} e_1$ as

$$\begin{aligned} A_{\nabla^{\perp}_{e_1} H} e_1 &= \frac{1}{2}(\kappa - \frac{n_3}{w})' A_{e_3} e_1 - e_1(\frac{b_3}{2w})' A_{e_4} e_1 + \frac{1}{2}(\kappa - \frac{n_3}{w}) A_{\nabla^{\perp}_{e_1} e_3} e_1 - \frac{b_3}{2w} A_{\nabla^{\perp}_{e_1} e_4} e_1 \\ &= \frac{1}{2}(\kappa' - (\frac{n_3}{w})')\kappa e_1 + \frac{1}{2}(\kappa - \frac{n_3}{w}) A_{\tau e_4} e_1 - \frac{b_3}{2w} A_{-\tau e_3} e_1 \\ &= \frac{1}{2}(\kappa' - (\frac{n_3}{w})')\kappa e_1 + \frac{1}{2}(\kappa - \frac{n_3}{w})\tau A_{e_4} e_1 + \frac{b_3}{2w}\tau A_{e_3} e_1 \\ &= \frac{1}{2}(\kappa' - (\frac{n_3}{w})')\kappa e_1 + \frac{b_3}{2w}\tau A_{e_3} e_1 \\ &= \frac{1}{2}(\kappa\kappa' - [\frac{(\kappa n_3' w - \kappa n_3 w'}{w^2}])e_1 + \frac{b_3}{2w}\tau\kappa e_1. \end{aligned}$$

By using the Frenet formulas, the last equation implies

$$A_{\nabla^{\perp}_{e_1}H}e_1 = \frac{1}{2}\kappa\kappa' e_1 + \frac{\kappa^2 t_3}{2w}e_1 + \frac{\kappa n_3 w'}{2w^2}e_1.$$

Furthermore, since

$$\begin{aligned}\nabla^{\perp}_{e_2}H &= \nabla^{\perp}_{e_2}(\frac{1}{2}[(\kappa - \frac{n_3}{w}e_3) - (\frac{b_3}{w})e_4]) \\ &= e_2(\frac{1}{2}(\kappa - \frac{n_3}{w}))e_3 - e_2(\frac{b_3}{2w})e_4 + \frac{1}{2}(\kappa - \frac{n_3}{w})\nabla^{\perp}_{e_2}e_3 - \frac{b_3}{2w}\nabla^{\perp}_{e_2}e_4 = 0.\end{aligned}$$

One can conclude that

$$A_{\nabla^{\perp}_{e_2}H}e_2 = 0.$$

In another case, since

$$f = \langle H, H\rangle = \frac{1}{4}[(\kappa - \frac{n_3}{w})^2 - (\frac{b_3}{w})^2],$$

H satisfies $\mathrm{grad} f = A(s)e_1$ for the function A computed as

$$\begin{aligned}A'(s)e_1 &= ([\frac{1}{4}(\kappa - \frac{n_3}{w})^2 + \frac{1}{4}(\frac{b_3}{w})^2]')e_1 \\ &= \frac{1}{2}(\kappa - \frac{n_3}{w})(\kappa' - \frac{n_3' w - n_3 w'}{w^2})e_1 + \frac{b_3}{2w}(\frac{b_3' w - b_3 w'}{w^2})e_1 \\ &= \frac{\kappa\kappa'}{2}e_1 + \frac{\kappa n_3 w'}{2w^2}e_1 + \frac{\kappa^2 t_3}{2w}e_1 - \frac{\kappa\tau b_3}{2w}e_1 - \frac{\kappa' n_3}{2w} - \frac{\kappa n_3 t_3}{2w^2}e_1 - \frac{(b_3^2 + n_3^2)w'}{2w^3}e_1.\end{aligned}$$

By considering biconservative equation by (1.0.6); one can get

$$A'(s)e_1 + \mathrm{trace} A_{\nabla^{\perp}.}H = 0$$

which implies

$$\kappa\kappa' + \frac{\kappa^2 t_3}{w} - \frac{\kappa\tau b_3}{2w} + \frac{\kappa n_3 w'}{w^2} - \frac{\kappa' n_3}{2w} - \frac{\kappa n_3 t_3}{2w^2} - \frac{(b_3^2 + n_3^2)w'}{2w^3} = 0. \tag{4.2.9}$$

Therefore, we have the theorem below:

Theorem 4.2. *Let M be the rotational surface in $\mathbb{E}_1^4$ parametrized by* (2.3.2)*. Then M is biconservative if and only if the equation* (4.2.9) *is satisfied.*

4.3 Rotational Surfaces of Parabolic Type

Let M be a surface in $\mathbb{E}_1^4$ given by (2.3.4) for given functions $x(s)$, $y(s)$ and $w(s)$ satisfying $x'^2+y'^2-w'^2=1$. Let $N=(n_1,n_2,n_3)$ and $B=(b_1,b_2,b_3)$ denote the normal and binormal vector fields of the curve(x,y,w) in $\mathbb{E}_1^3$. Consider $\eta_1=(1,0,0,0)$,$\eta_2=(0,1,0,0)$,$\eta_3=(0,0,1,0)$ and $\eta_4=(0,0,0,1)$ and the rest as accordingly $\xi_3=\frac{(\eta_4-\eta_3)}{\sqrt{2}}=\frac{1}{\sqrt{2}}(0,0,-1,1)$, $\xi_4=\frac{(\eta_4+\eta_3)}{\sqrt{2}}=\frac{1}{\sqrt{2}}(0,0,1,1)$ which satisfy $\langle\eta_1,\eta_1\rangle=\langle\eta_2,\eta_2\rangle=1$, $\langle\eta_1,\eta_2\rangle=0$ and $\langle\xi_3,\xi_3\rangle=\langle\xi_4,\xi_4\rangle=0$ and $\langle\xi_3,\xi_4\rangle=-1$. Then since

$$\frac{\partial F_3}{\partial s}=x'(s)\eta_1+\sqrt{2}tw'\eta_2+(z'+t^2w')\xi_3+w'\xi_4$$
$$\frac{\partial F_3}{\partial t}=\sqrt{2}w\eta_2+2tw\xi_3.$$

$F=F_3$ satisfies, $\langle F_s,F_s\rangle=x'^2-2z'w'=E$, $\langle F_t,F_t\rangle=2w^2$ and $\langle F_s,F_t\rangle=\langle F_t,F_s\rangle=0$. Define $\gamma(s)=x(s)\eta_1+z(s)\xi_3+w(s)\xi_4$ which satisfy $\gamma'(s)=x'(s)\eta_1+z'(s)\xi_3+w'(s)\xi_4$ and $\langle\gamma',\gamma'\rangle=x'^2-2z'w'=E$. There are two different cases depending on whether E is positive or negative.

Case 4.1. *Let E be positive. In this case, by re- defining s properly, one can assume* $E=1$.

Case 4.2. *Let E be negative. In this case, by re- defining s properly, one can assume* $E=-1$.

Then, the surface F_3, vector fields (e_1,e_2,e_3,e_4) given by

$$e_1=\frac{1}{\sqrt{-E}}\frac{\partial}{\partial s}=x'\eta_1+\sqrt{2}tw'\eta_2+(z'+t^2w')\xi_3+w'\xi_4$$
$$e_2=\frac{1}{\sqrt{2}w}\frac{\partial}{\partial t}=\eta_2+\sqrt{2}t\xi_3$$
$$e_3=x'\eta_1+\sqrt{2}tw'\eta_2+(\frac{-1}{w'}+t^2w'+z')\xi_3+w'\xi_4$$
$$e_4=\frac{x'}{w'}\xi_3+\eta_1$$

form an orthonormal frame field on M because they satisfy

$$\langle e_1, e_1\rangle = \frac{1}{\sqrt{-E}}(x'^2 + 2t^2w'^2 - 2w'(z' + t^2w')) = -1$$

$$\langle e_2, e_2\rangle = 1$$

$$\langle e_1, e_2\rangle = \langle (x'\eta_1 + \sqrt{2}tw'\eta_2 + (z' + t^2w')\xi_3 + w'\xi_4), (\eta_2 + \sqrt{2}t\xi_3)\rangle = 0$$

$$\langle e_3, e_3\rangle = x'^2 + 2t^2w'^2 + 2\frac{w'}{w'} - 2t^2w'^2 - 2z'w' = x'^2 - 2z'w' + 2 = 1$$

$$\langle e_3, e_1\rangle = 0$$

$$\langle e_3, e_2\rangle = \langle (x'\eta_1 + \sqrt{2}tw'\eta_2 + (\frac{-1}{w'} + t^2w' + z')\xi_3 + w'\xi_4), (\eta_2 + \sqrt{2}t\xi_3)\rangle = 0$$

$$\langle e_4, e_4\rangle = \langle (\frac{x'}{w'}\xi_3 + \eta_1), (\frac{x'}{w'}\xi_3 + \eta_1)\rangle = 1$$

$$\langle e_4, e_1\rangle = \langle (\frac{x'}{w'}\xi_3 + \eta_1), (x'\eta_1 + \sqrt{2}tw'\eta_2 + (z' + t^2w')\xi_3 + w'\xi_4)\rangle = 0$$

$$\langle e_4, e_2\rangle = \langle (\frac{x'}{w'}\xi_3 + \eta_1), (\eta_2 + \sqrt{2}t\xi_3)\rangle = 0$$

$$\langle e_4, e_3\rangle = \langle (\frac{x'}{w'}\xi_3 + \eta_1), (x'\eta_1 + \sqrt{2}tw'\eta_2 + (\frac{-1}{w'} + t^2w' + z')\xi_3 + w'\xi_4)\rangle = 0.$$

The coefficients of the second fundamental form are determined as follows: the equation

$$\tilde{\nabla}_{e_1}e_1 = \frac{\partial}{\partial s}(x'\eta_1 + \sqrt{2}tw'\eta_2 + (z' + t^2w')\xi_3 + w'\xi_4)$$
$$= x''\eta_1 + \sqrt{2}tw''\eta_2 + (z'' + t^2w'')\xi_3 + w'\xi_4$$

implies

$$h(e_1, e_1) = h^3_{11}e_3 + h^4_{11}e_4,$$

from which one can get

$$\langle h(e_1, e_1), e_3\rangle = h^3_{11} = \frac{w''}{w'},$$
$$\langle h(e_1, e_1), e_4\rangle = h^4_{11} = \frac{x'w'' - x''w'}{w'}.$$

In another case, the equation

$$\tilde{\nabla}_{e_1}e_2 = \frac{\partial}{\partial s}(\eta_2 + \sqrt{2}t\xi_3) = 0 = h(e_1, e_2) + \nabla e_1 e_2$$

implies

$$h(e_1,e_2)=0 \text{ and } \nabla_{e_1}e_2=0.$$

Furthermore, by a direct computation one can get

$$\tilde{\nabla}_{e_1}e_3=\frac{\partial}{\partial s}(x'\eta_1+\sqrt{2}tw'\eta_2+(\frac{-1}{w'}+t^2w'+z')\xi_3+w'\xi_4)$$
$$=(x''\eta_1+\sqrt{2}tw''\eta_2+(\frac{w''}{w'2}+t^2w''+z'')\xi_3+w''\xi_4)$$
$$=-\langle\tilde{\nabla}_{e_1}e_3,e_1\rangle e_1+\langle\tilde{\nabla}_{e_1}e_3,e_2\rangle e_2+\langle\tilde{\nabla}_{e_1}e_3,e_4\rangle e_4=-(\frac{-w''}{w'})e_1+0e_2+(\frac{x''w'-x'w''}{w'})e_4.$$

By combining this equation and

$$\tilde{\nabla}_{e_1}e_3=-A_3e_1+\nabla^{\perp}_{e_1}e_3,$$

one can obtain

$$A_{e_3}e_1=-\frac{w''}{w'}e_1 \text{ and } \nabla^{\perp}_{e_1}e_3=\frac{x''w'-x'w''}{w'}e_4. \tag{4.3.1}$$

By a similar way, the equation

$$\tilde{\nabla}_{e_1}e_4=\frac{\partial}{\partial s}(\frac{x'}{w'}\xi_3+\eta_1)$$
$$=-\langle\tilde{\nabla}_{e_1}e,e_1\rangle e_1+\langle\tilde{\nabla}_{e_1}e_4,e_2\rangle e_2+\langle\tilde{\nabla}_{e_1}e_4,e_3\rangle e_3$$
$$=-(\frac{x''w'-x'w''}{w'})e_4+0e_2+\frac{x''w'-x'w''}{w'}e_3$$

implies

$$A_{e_4}e_1=-\frac{x''w'-x'w''}{w'}e_1 \text{ and } \nabla^{\perp}_{e_1}e_4=\frac{x''w'-x'w''}{w'}e_3. \tag{4.3.2}$$

and

$$\tilde{\nabla}_{e_2}e_1=\frac{1}{w\sqrt{2}}\frac{\partial}{\partial t}(x'\eta_1+\sqrt{2}tw'\eta_2+(z'+t^2w')\xi_3+w'\xi_4)$$
$$=\frac{w'}{w}\eta_2+\frac{tw'\sqrt{2}}{w}\xi_3=\frac{w'}{w}(\eta_2+t\sqrt{2}\xi_3)=\frac{w'}{w}e_2$$
$$=h(e_2,e_1)+\nabla_{e_2}e_1$$

gives

$$h(e_2,e_1)=0 \text{ and } \nabla_{e_2}e_1=\frac{w'}{w}e_2.$$

Furthermore, the equation

$$\tilde{\nabla}_{e_2} e_2 = \frac{1}{w\sqrt{2}} \frac{\partial}{\partial t}(\eta_2 + t\sqrt{2}\xi_3) = \frac{1}{w}\xi_3 \tag{4.3.3}$$

implies

$$h(e_2, e_2) = h^3_{22} e_3 + h^4_{22} e_4 = -\frac{w'}{w} e_3 + 0 e_4$$

which gives

$$\langle h(e_2, e_2) e_3 \rangle = h^3_{22} = -\frac{w'}{w},$$

$$\langle h(e_2, e_2) e_4 \rangle = h^4_{22} = 0.$$

On the other hand, by considering Weingarten equation and

$$\tilde{\nabla}_{e_2} e_3 = \frac{1}{w\sqrt{2}} \frac{\partial}{\partial t}(x'\eta_1 + \sqrt{2}tw'\eta_2 + (\frac{-1}{w'} + t^2 w' + z')\xi_3 + w'\xi_4)$$
$$= \frac{w'}{w}\eta_2 + \sqrt{2}t\frac{w'}{w}\xi_3 = \frac{w'}{w}(\eta_2 + \sqrt{2}t\xi_3) = \frac{w'}{w}\eta_2$$

one can obtain

$$A_{e_3} e_2 = -\frac{-w'}{w} e_2 \text{ and } \nabla^{\perp}_{e_2} e_3 = 0. \tag{4.3.4}$$

Similarly,

$$\tilde{\nabla}_{e_2} e_4 = -A_{e_4} e_2 + \nabla^{\perp}_{e_2} e_4$$

implies

$$A_4 e_2 = 0 \text{ and } \nabla^{\perp}_{e_2} e_4 = 0. \tag{4.3.5}$$

By summing up (4.3.1)- (4.3.5), one can get

$$A_{e_3} = \begin{pmatrix} -\frac{w''}{w'} & 0 \\ 0 & -\frac{w'}{w} \end{pmatrix} \text{ and } A_{e_4} = \begin{pmatrix} \frac{x''w' - w''x'}{w'} & 0 \\ 0 & 0 \end{pmatrix}. \tag{4.3.6}$$

Consequently, the mean curvature vector of M takes the form

$$H = -\frac{1}{2}[(\frac{w'}{w} + \frac{w''}{w'})e_3 + (\frac{x''w' - x'w''}{w'})e_4]. \tag{4.3.7}$$

In another case, the Gaussian curvature of M is

$$K = -\det A_3 - \det A_4 = -\frac{w''}{w}. \tag{4.3.8}$$

Now, let M be a biconservative surface, i.e, (1.0.6) is satisfied. Then (4.3.7) implies

$$\begin{aligned}\nabla^{\perp}_{e_1}H &= \nabla^{\perp}_{e_1}(\frac{1}{2}[(\frac{w''}{w'}+\frac{w'}{w})e_3+(\frac{x''w'-w''x'}{w'})e_4])\\ &= e_1(\frac{1}{2}(\frac{w''}{w'}+\frac{w'}{w}))e_3+e_1(\frac{1}{2}(\frac{x''w'-w''x'}{w'}))e_4\\ &+\frac{1}{2}(\frac{w''}{w'}+\frac{w'}{w})\nabla^{\perp}_{e_1}+\frac{1}{2}(\frac{x''w'-w''x'}{w'})\nabla^{\perp}_{e_1}e_4\end{aligned}$$

$$\begin{aligned}A_{\nabla^{\perp}_{e_1}H}e_1 &= e_1(\frac{1}{2}(\frac{w''}{w'}+\frac{w'}{w}))A_{e_3}e_1\\ &= e_1(\frac{1}{2}(\frac{w''}{w'}+\frac{w'}{w}))A_{e_3}e_1+e_1(\frac{1}{2}(\frac{x''w'-w''x'}{w'}))A_{e_4}e_1\\ &+\frac{1}{2}(\frac{w''}{w'}+\frac{w'}{w})A_{\nabla^{\perp}_{e_1}e_3}e_1+\frac{1}{2}(\frac{x''w'-w''x'}{w'})A_{\nabla^{\perp}_{e_1}e_4}e_1\\ &= -w''(\frac{w'''-w''}{w'^4})e_1-(\frac{w'''w-w''w'}{2w^2})e_1\\ &+((\frac{x''w'-w''x'}{w'})(\frac{x'''w'+x''w''-w'''w'+w''}{w'}))e_1\\ &+(\frac{(x''w'-w''x')^2}{w'^3})w''e_1.\end{aligned}$$

Furthermore, since

$$\begin{aligned}\nabla^{\perp}_{e_2}H &= \nabla^{\perp}_{e_2}(\frac{1}{2}(\frac{w''}{w'}+\frac{w'}{w})e_3+(\frac{x''w'-w''x'}{w'})e_4)\\ &= e_2(\frac{1}{2}(\frac{w''}{w'}+\frac{w'}{w})e_3+(\frac{x''w'-w''x'}{w'})e_4)+\frac{1}{2}(\frac{w''}{w'}+\frac{w'}{w})\nabla^{\perp}_{e_2}e_3\\ &+\frac{1}{2}(\frac{x''w'-w''x'}{w'})\nabla^{\perp}_{e_2}e_4=0.\end{aligned}$$

One can conclude that

$$A_{\nabla^{\perp}_{e_1}H}e_2=0.$$

In another case, since

$$f=\langle H,H\rangle=\frac{1}{4}[(\frac{w''}{w'}+\frac{w'}{w})^2+(\frac{x''w'-x'w''}{w'})^2]$$

H satisfies $\mathrm{grad}f=A(s)e_1$ for the function A computed as

$$\begin{aligned}A'(s)e_1 &= (\frac{1}{4}[(\frac{w''}{w'}+\frac{w'}{w})^2+(\frac{x''w'-x'w''}{w'})^2]\\ &\frac{1}{2}(\frac{w'''w''w'-w''^3}{w'^3}+\frac{w''^2w-w''w'^2}{w^2})\\ &+\frac{1}{2}(\frac{w'''w'^2-w''^2w'}{ww'^2}+\frac{w''w'w-w'^3}{w^3})\\ &+\frac{1}{2}(\frac{x''w'-w''x'}{w'})(\frac{(x'''+x'')w'^2-(w'''x'+2x''w'')w'+w''^2x'}{2w'^2}))e_1.\end{aligned}$$

Now, by combining the equations above one can conclude that (1.0.6) is equivalent to

$$-w''\left(\frac{w'''-w''}{w'^4}\right)-\left(\frac{w'''w-w''w'}{2w^2}\right)+\left(\left(\frac{x''w'-w''x'}{w'}\right)\left(\frac{x'''w'+x''w''-w'''w'+w''}{w'}\right)\right)$$
$$+\left(\frac{(x''w'-w''x')^2}{w'^3}\right)w''+\frac{1}{2}\left[\left(\frac{w'''w''w'-w''^3}{w'^3}+\frac{w''^2w-w''w'^2}{w^2}\right)+\frac{w'''w'^2-w''^2w'}{ww'^2}\right.$$
$$\left.+\left(\frac{x''w'-w''x'}{w'}\right)\left(\frac{x'''w'^2+x''w'^2-w'''x'w'-2x''w''w'+w''^2x'}{2w'^2}\right)\right]+\frac{w''w'w-w'^3}{w^3}=0. \tag{4.3.9}$$

Hence, we have the following theorem:

Theorem 4.3. *Let M be the rotational surface in $\mathbb{E}_1^4$ parametrized by* (2.3.4)*. Then the equation* (4.3.9) *is satisfied if and only if M is biconservative.*

5. CONCLUSIONS AND RECOMMENDATIONS

In this thesis, biconservative and biharmonic surfaces in Euclid and Minkowski spaces were studied. In section three, we aimed to achieve a complete local classification of biconservative surfaces with a parallel normalized mean curvature vector in the Euclidean 4-space. First, in Lemma 3.1, we determined the form of shape operators and the Levi-Civita connections of a biconservative PNMCV surface in $\mathbb{E}^4$. Then, we demonstrated that there are two necessary and sufficient conditions for M to be biconservative PNMCV. In Corollary 3.2, we showed that there are not biharmonic PNMCV surface in $\mathbb{E}^4$.

In section four, we obtained the classifications of biconservative rotational surfaces of hyperbolic, timelike and parabolic types. In Theorem 4.1, Theorem 4.2 and Theorem 4.3, we examined the necessary and sufficient conditions for such surfaces to be biconservative.

Studies about the generalization of the results appearing in Section 4 have been completed in several articles. However, it would be crucial to study higher dimensional submanifolds.

On the other hand, classifying biconservative rotational surfaces in the arbitrary dimensional space forms is still an open problem.

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