ISTANBUL TECHNICAL UNIVERSITY ★ GRADUATE SCHOOL OF SCIENCE ENGINEERING AND TECHNOLOGY

NON-ASSOCIATIVE GAUGE THEORY

M.Sc. THESIS

Aytül FİLİZ

Department of Physics Engineering

Physics Engineering Programme

JUNE 2012

ISTANBUL TECHNICAL UNIVERSITY ★ GRADUATE SCHOOL OF SCIENCE ENGINEERING AND TECHNOLOGY

NON-ASSOCIATIVE GAUGE THEORY

M.Sc. THESIS

Aytül FİLİZ
(509071102)

Department of Physics Engineering

Physics Engineering Programme

Thesis Advisor: Prof. Dr. Ömer Faruk DAYI

JUNE 2012

İSTANBUL TEKNİK ÜNİVERSİTESİ ★ FEN BİLİMLERİ ENSTİTÜSÜ

BİRLEŞİMSİZ AYAR TEORİSİ

YÜKSEK LİSANS TEZİ

Aytül FİLİZ
(509071102)

Fizik Mühendisliği Anabilim Dalı

Fizik Mühendisliği Programı

Tez Danışmanı: Prof. Dr. Ömer Faruk DAYI

HAZİRAN 2012

Aytül FİLİZ, a **M.Sc.** student of ITU **Graduate School of Science Engineering and Technology** student ID **509071102**, successfully defended the **thesis** entitled "**NON-ASSOCIATIVE GAUGE THEORY**", which she prepared after fulfilling the requirements specified in the associated legislations, before the jury whose signatures are below.

Thesis Advisor :	**Prof.Dr. Ömer Faruk DAYI** Istanbul Technical University	....
Co-advisor :	**Prof.Dr. Cemsinan DELİDUMAN** Mimar Sinan Fine Arts University	....
Jury Members :	**Assoc. Prof. Dr. Aybike ÖZER** Istanbul Technical University	....
	Assist. Prof. Dr. Hakkı Tuncay ÖZER Istanbul Technical University	....
	Assist. Prof. Dr. Meltem GÜNGÖRMEZ Istanbul Technical University	....

Date of Submission : 04 May 2012
Date of Defense : 08 June 2012

To my mother,

FOREWORD

I am very grateful to Prof. Dr. Cemsinan DELİDUMAN for always having time for me, for his guidance, patience and encouragement. I feel priviliged to have been his student.

I would like to thank Prof. Dr. Ömer Faruk DAYI for his willingness and invaluable supports.

I would like to thank Prof. Dr. Yaşar YILMAZ for his sympathy and sincere helps.

Finally, I would like to thank my family, especially to my dear mother whom I always need to be with me.

June 2012 Aytül FİLİZ

TABLE OF CONTENTS

Page

ABBREVIATIONS

App	: Appendix
$\mathbf{A^{A}_{\mu B}}$	: Gauge field
BLG	: Bagger Lambert Gusstavson
D	: Dimension of spacetime
$\mathbf{D_{\mu}}$	: Covariant derivative
$\mathbf{f^{abc}{}_{d}}$	: Lie algebra structure constants
G	: Finite Lie group
$\mathbf{g_{IJ}}$	: Metric tensor
$\mathbf{g_{YM}}$	: Constant of Yang-Mills
$\mathbf{H_{\mu\nu\lambda A}}$	: Superspace 4-form
SO(8)	: Special Orthogonal Rotation Group
SU	: Special Unitary
SUSY	: Supersymmetry
SYM	: Symmetry
YM	: Yang-Mills
$\mathbf{X^{I}_{A}}$	: Scalar field
$\mathbf{X^{+}, X^{-}}$	: Supermembrane coordinates
$\boldsymbol{\varepsilon^{ijk}}$	: Totally antisymmetric tensor
$\boldsymbol{\varepsilon}$	: Constant anti-commuting spacetime spinor
$\boldsymbol{\Gamma^{\mu}}$	: Spacetime Dirac Matrices
$\boldsymbol{\Psi_{A}}$	: Spinor field

NON-ASSOCIATIVE GAUGE THEORY

SUMMARY

There had been no known Lagrangian description of multiple M2-branes before Bagger and Lambert presented an alternative approach to this system using a new kind of symmetry structure, so called Lie 3-algebra. The multiple M2-branes revolution started in 2007 with a series three papers presented by Jonathan Bagger & Neil Lambert [1–3] and also independtly by Andreas Gustavsson [4]. Motivated by Basu & Harvey [5] proposal, they present a three dimensional N=2 supersymmetric action for the scalar-spinor sector of multiple M2-brane worlvolume theory. In other words, they construct a unique 3 dimensional supersymmetric field theory that is consistent with all the symmetries expected of a multiple M2-brane theory: maximal supersymmetry (16 supersymmetries), conformal invariance and an SO(8) R-symmetry acts on the eight transverse scalars. They first described the supersymmetry transformations of multiple M2-branes and then argued that the M2-brane coordinates are naturally elements of a non-associative algebra. Their field theory model is based on such an algebra with a totally antisymmetric triple product and they have succeeded in formulating Lagrangian description for multiple M2-branes in terms of this product.

Since there has been significant recent progress for multiple M2-branes worldvolume theory, it is natural to wonder whether the results can be extended to multiple M5-branes system [6–9]. For this purpose, Lambert & Papageorgakis proposed [10] an ansatz to construct a Lie-3 algebra theory in six dimensions with (2,0) supersymmetry. They consider the set of supersymmetry transformations for the abelian case first and then propose a non-Abelian generalisation, resulting a system of equations of motion that represent the (2,0) supersymmetric tensor multiplet.

In this thesis we study technical details of the Bagger-Lambert-Gustavsson theory and its extension to the multiple M5-branes, a system of interacting M5-branes manifesting (2,0) supersymmetry.

This thesis consists of five chapters:

In chapter 1, we give a brief introduction to M-theory, to some of the objects that are relevant to its definition and historical progress of multiple M2-brane worldvolume theory.

In chapter 2, we give some algebraic details of the Bagger-Lambert-Gustavsson theory that is based on a nonassociative algebra called Lie-3 algebra, for which the anti-symmetrized associator leads to a natural triple product structure.

In chapter 3, we present the basics of the Bagger-Lambert-Gustavsson theory. We study how they gauge a symmetry that arises from the algebra's triple product and construct a supersymmetric multiple M2-branes worldvolume theory. We first consider

a set of supersymmetry transformations postulated by Bagger & Lambert and check the closures of them. We see that these supersymmetries close into translations and gauge transformations with a set of equations of motion so that the superalgebra closes " on shell " and all the equations of motions are invariant under supersymmetry transformations. We end this chapter presenting a supersymmetric and gauge-invariant Bagger & Lambert action that is consistent all expected continious symmetries.

In chapter 4, we discuss the extension of this model to multiple M5-branes theory whose low energy dynamics are based on a theory in 6 dimensions with (2,0) supersymmetry, SO(5) R-symmetry and conformal symmetry. We study Lambert & Papageorgakis propasal in detail. We proceed by checking the closures of these supersymmetry algebras for both Abelian and non-Abelian cases. The superalgebras close ” on shell ”. The closures yield a set of equations of motion and a number of constraints. In addition, the superalgebra for non-Abelian (2,0) tensor multiplets in six dimensions close ” on shell ” up to translations and gauge transformations in case the structure constants are the elements of a real 3-algebra. We end this chapter reviewing the relation between 6 dimensional this theory and 5 dimensional super Yang-Mills, in other words expanding it around a particular vacuum point how it reduces to five dimensional super-Yang-Mills along with six-dimensional, Abelian (2,0) tensor multiplets.

In chapter 5, we have collected some conclusions.

Finally, we end this study presenting three apendices; Notation & Convention, Fierz reordering formulas and Gamma matrix identities used repeatedly in the thesis.

BİRLEŞİMSİZ AYAR TEORİSİ

ÖZET

Evreni tek bir çerçeve içinde açıklayabilen tutarlı bir teorinin varlığı, modern fiziğin yanıtını aradığı temel sorulardandır. M teori, böyle bir teoriye en muhtemel adaydır. Beş farklı süpersicim teorisini ve onbir boyutlu süperçekim teorisini birleştiren başarılı bir teoridir. Süpersicim teorileri ve onbir boyutlu süperçekim teorisi M teorinin farklı limitlerdeki özel durumları olarak ele alınır.

M teorinin anlamlı olduğu onbir boyuttaki temel cisim zardır. Teoride sicimlerden başka M2-zar, M5-zar, KK6-zar ve 9-zar gibi yüksek boyutlu nesnelerin de olduğu ortaya konmuştur. Ayrıca üzerlerinde çok çalışılmış süpersicim teorilerinin üçünün de (tip I, tip II-A, tip II-B) sicimlerin yanında D-zar olarak adlandırılan yüksek boyutlu nesneler içerdikleri gösterilmiştir.

1995'den bu yana D-zarlar hakkında çok şey öğrenilmiştir. Tek bir D-zar için D-zar üzerindeki teori, süpersimetrik Yang-Mills teorisidir. N tane kesişen D-zarı tanımlayan teorisi ise Abelyan olmayan Born-Infeld teorisidir.

M-zarlar üzerindeki alan teorileri hakkında ise henüz yeterli bilgi bulunmamaktadır. Tek bir M2-zarı ve tek bir M5-zarı tanımlayan eylem uzun zamandır bilinmektedir. Fakat bu zarların üst üste dizildiği durumlar için eylem yazma işi, M teorinin ortaya çıkışından bu yana problem olmuştur. Jonathan Bagger & Neil Lambert ve bağımsız olarak Andreas Gustavsson, Lie-3 cebri formalizmi kullanarak çoklu M2-zarları için eylem yazmayı başarmışlardır. Diğer bir ifade ile çoklu M2 zarları teorisi için üç boyutlu, süpersimetrik bir alan teorisi yazmışlardır. Çoklu M5-zarları eylemi ise hala gizemlidir.

Çoklu M-zarları dinamiği, çoklu D-zarları dinamiğine göre daha zordur. Fakat sevindirici yanı, M-zarlar ile D-zarlar arasındaki benzerliklerdir. M2$\perp$M5 sistemi D1$\perp$D3 sistemine benzerdir. M teorinin tip II-A süpersicim teorisinin şiddetli çiftlenim limiti olarak düşünülmesi ile bu benzerlikler görülmüştür. İşte bu benzerlik "D-zarlar hakkında bilinenler M-zarlara genelleştirilebilir mi?" sorusunu ortaya çıkarmıştır.

Bu amaçla Basu & Harvey, D1$\perp$D3 ve M2$\perp$M5 sistemleri arasındaki dualite ilişkisinden yola çıkarak, D1$\perp$D3 sistemini tanımlayan Nahm denklemini M2$\perp$M5 sistemi için genelleştirmeyi başarmışlardır. Elde ettikleri genelleştirilmiş bu denklem, dörtlü Nambu parantezini içermektedir. Ardından, bu denklemin üçlü Lie parantezi içeren bir eylemden türediği ortaya konmuştur. Bunun üzerine Bagger & Lambert ve bağımsız olarak Gustavsson, Basu & Harvey'in çalışmalarından esinlenerek ve Lie-3 cebri olarak adlandırılan bu yeni simetri yapısını kullanarak çoklu M2-zarları için beklenen;

- N=8 süpersimetri,
- SO(8) R-simetri,
- Konformal simetri

simetrilerine sahip, süpersimetrik ve ayar değişmez bir eylem yazmayı başarmışlardır. Bu başarı, söz konusu teorinin çoklu M5-zarlarına genelleştirilmesi için yapılan yeni girişimleri de beraberinde getirmiştir.

Bu tezde çoklu M2-zarları için sunulan Bagger-Lambert-Gustavsson teorisinin ve bu teorinin çoklu M5-zarlarına uygulamasının teknik detayları üzerinde durulmuştur.

Tez, beş ana bölümden oluşur.

Birinci bölümde; M teori, teorinin ilişkili olduğu nesneler ve teorinin tarihsel gelişiminden kısaca söz edilmiştir. Bagger-Lambert-Gustavsson teorisinin ve bu teorinin çoklu M5-zarlarına uygulamasının motivasyonu sunulmuştur.

İkinci bölümde; teorinin dayandığı bazı cebirsel detaylar hakkında bilgi verilmiştir. Teori, Lie-3 cebir yapısı üzerine kurulmuştur. Bu bölümde cebir, Lie cebir ve Lie-3 cebir yapıları sunulmuştur.

Üçüncü bölümde, Bagger-Lambert-Gustavsson modelinin temelleri sunulmuştur. Çoklu M2-zarları için Bagger ve Lambert tarafından ortaya konan süpersimetri dönüşümleri dikkate alınarak, ayar kuramının nasıl oluşturulduğu incelenmiştir. Ardından ayar kuramının süpersimetrikleştirilmesi üzerinde durulmuştur. Süpersimetri dönüşümleri alınarak cebrin sırası ile skaler alan, spinör alan ve ayar alanı için kapalılığı kontrol edilmiştir. Bu süpercebrin ”on shell” olarak öteleme ve ayar dönüşümlerine kapandığı görülmüştür, buradan hareket denklemleri elde edilmiştir. Fermiyonik hareket denkleminin süper varyasyonu alınarak, bosonik hareket denklemine ulaşılmıştır. Son olarak, beklenen simetrilere uygun, söz konusu dönüşümler altında süpersimetrik ve ayar değişmez eylem sunularak üçüncü bölüm sonlandırılmıştır.

Dördüncü bölümde, Bagger-Lambert-Gustavsson teorisinin sonuçlarının Lie-3 cebri formalizmi ile çoklu M5-zarlarına genişletilmesi sunulmuştur. Altı boyutta çoklu M5-zarlarından beklenen simetriler:

- (2,0) süpersimetri,
- SO(5) R-simetri,
- Konformal simetri

şeklindedir.

Bagger-Lambert-Gustavsson teorisinin çoklu M5-zarlarına uygulaması için Neil Lambert ve Constantinos Papageorgakis, Lie-3 cebri yapısı ile altı boyutta (2,0) süpersimetrik bir teori yazmışlardır. Bu modelin teknik detayları incelenmiş ve sonuçları sunulmuştur. Altı boyutlu Abelyan (2,0) tensör multipletin kovaryant süpersimetri dönüşümlerinden hareket edilerek, non-Abelyan durum için süpersimetri dönüşümleri yazılmıştır ve cebrin kapalılığı her iki durum için kontrol edilmiştir.

Süpercebir ”on shell” olarak kapanmıştır, buradan hareket denklemleri ve kısıtlar elde edilmiştir. Non-Abelyan (2,0) tensör multipleti için süpersimetri dönüşümlerinin, cebrin yapı sabitlerinin gerçel bir 3-cebrin elemanları olması durumunda, ”on shell” olarak öteleme ve ayar dönüşümlerine kapandığı görülmüştür. Fermiyonik hareket denkleminin süper varyasyonu alınarak bosonik hareket denklemi bulunmuştur.

Teorinin vakum noktası civarında açılması ile beş boyutlu süper-Yang-Mills teorisi ile ilişkisi incelenmiş ve yorumlanmıştır.

Beşinci bölümde, sonuçlar sunulunmuştur.

Son olarak, tezde kullanılan notasyon, Fierz özdeşliği ve gamma matris özdeşlikleri ile ilgili ekler verilerek çalışma sonlandırılmıştır.

1. INTRODUCTION

The aim of this chapter is to give an introduction to M-theory, to some of the objects that are relevant to its definition and historical progress of multiple M2-brane worldvolume theory. We begin by reviewing the history of string theory.

There are two theories describing the universe. One is Einstein's theory of General Relativity which explains very large-scale behaviour of the cosmos and the other is Quantum Mechanics, physics of the microcosm. Because both of these theories describe the same universe, it should be possible to understand universe better when the laws of universe can be combined in one theory. In other words, the most outstanding problem of theoretical physics is to unify our picture of two fundamental theories. This was Albert Einstein's dream. He studied relentlessly for a so-called unified field theory, a theory capable of describing nature's forces within a single, all-encompassing coherent framework, during the last years of his life but he never realised this dream since a number of essential features of matter and the forces of nature were either unknown or, at best, not well understood [11]. Neverthless, the physicists sustained their studies for this purpose. However, they have inavoidably been riddled with infinities or violoted some of the cherished princples of physics such as causality [12].

Superstring Theory (and its latest formulation M-theory) is the most promising hope for a truly unified and finite theory [12]. It has been the leading candidate over the past decades for a theory that consistently unifies all fundamental forces of nature, including gravity and forms of matter [13, 14]. It is a contender for a theory of everything (TOE). It arose in the late 1960s with the name of string theory. But this attempt failed first and the theory was replaced by QCD. It was posited later that the string theory is not just a theory of strong interactions, but a theory incorparating all the forces of nature, including gravity [15, 16].

This theory essentially based on fundamental one dimensional oscillating objects called strings rather then point-like particles. The basic idea of the theory is that specific particles correspond to specific oscillation modes of the string [17]. This view then developed to the superstring theory, including supersymmetry which is a symmetry that relates bosons to fermions [18, 19]. Although supersymmetry is initially discovered in the early 1970's, we still don't know how it plays a major role in nature [20]. The discovery of this symmetry leads to the extension of the theories such as string theories, quantum field theories. Theories admitting more than one supersymmetry are known as extended SUSY theories [21]. The generators of this symmetry forms an algebra called superalgebra, that is a super extension of the Poincare Lie algebra [22,23]. Supersymmetry is required and is a generic feature of all potentially realistic superstring theories [24]. The consistency of string theories with fermions depends significantly on local supersymmetry. In the view of Superstring theory all particles, which differ in spin and other quantum numbers are related by a symmetry which reflects the properties of the string. So, it can be considered a particular kind of particle theory in that it unites the particles in the same way that a violin string provides a unified description of the musical tones [12].

It was discovered that quantum mechanical consistency of such a ten dimensional theory with N=1 supersymmetry requires a local Yang-Mills gauge symmetry based on one of two possible Lie algebras : SO (32) or E8 x E8. Only for these two cases do certain quantum mechanical anomalies cancel. "When one uses the superstring formalism for both left moving modes and right moving modes, the supersymmetries associated with the left movers and the right movers can have either opposite handedness or the same handedness. These two possibilities give different theories called the type IIA and IIB superstring theories, respectively. A third possibility, called the type I superstring theory can be derived from the type IIB theory by modding out by its left-right symmetry, a procedure called orientifold projection. The type I and type II superstring theories are described using formalism with world-sheet and space-time supersymmetry, respectively. It is possible to use the formalism of the 26-dimensional bosonic string for the left movers and the formalism of the 10-dimensional superstring for the right movers. The string theories constructed in this way are called "heterotic". The mismatch in space-time dimensions may sound strange but it is actually, exactly what is needed. The extra 16 left moving dimensions must describe a torus with very special properties to give a consistent theory .

Altogether, there appear to be five distinct types of string theory, each in ten dimensions. Three of them, the type I theory and the two heterotic theories, have N=1 supersymmetry in the ten dimensional sense. The minimal spinor in ten dimensions has 16 real components, so these theories have 16 conserved supercharges. The type I superstring theory has the gauge group SO (32), whereas the heterotic theories realize both SO(32) and E8 x E8. Other two theories, type IIA and type IIB, have N=2 supersymmetry or equivalently 32 supercharges.

The realization that there are five different superstring theories was confusing. The search for a single unified theory of all interactions and the attempts to unify gravity and quantum mechanics had led to five different string theories. Definitely, there is only one Universe , so it would be most satisfying if there were only one possible unified theory. In the late 1980s it was realized that there is a property known as T-duality that relates the two type II theories and the two heterotic theories , so that they shouldn't really be regarded as distinct theories. Progress in understanding nonperturbative phenomena peaked in 1995 (This is known as second superstring revolution). Nonperturbative S-dualities and the opening up of an eleventh dimension at strong coupling in certain cases led to new identifications " [17]. In other words, these theories are related by transformations that are called dualities. In the study of superstring theory, one can discover that superstring theories contain various p-branes, objects with p-spatial dimensions, in addition to the fundamental string. In fact, one can say that fundamental strings are only a special type of p-brane with p=1. The branes first appear as solutions of the supergravites which are the low energy effective theories of the superstring theories, and of eleven dimensional super gravity which is assumed to be the low energy effective theory of M-theory. Most of these solutions were known before the dualities in string theory become the object at intensive research. The type I and type II superstrıng theories contain a class of p-branes called D-branes. Their defining property is that they are objects on which fundamental strings can end. The fact that the fundamental strings can end on D-branes implies that quantum field theories are of the Yang-Mills type, like the Standard model reside on the world volumes of D-branes. The Yang-Mills fields arise as the massless modes of open strings attached to the D-branes [17].

Going back to the main problem for relating and unifying five distinct superstring theories, one can say that there has been speculations that there is an underlying more fundamental theory, since S-and T-dualities exist. T-duality implies that in many cases two different geometries for the extra dimensions are physically equivalent and S-duality explains behaviour of the three of the five supersting theories at strong coupling. And the question arises what happens other two superstring theories-type IIA and E8 x E8 heterotic.The answer is that they grow on eleventh dimension. This new dimension in the type IIA case is a circle and in the heterotic case is a line. Therefore, a new type of quantum theory in 11 dimensions called M-theory, emerges [17]. M stands for mother (as in mother of all theories), or membrane or perhaps Matrix theory. M-theory imply that the five superstring theories are connected by a web of dualities [15].

After the advent of the duality existence it was seen that all these five string theories can be regarded as different limits of a single unified theory called M-theory. M-theory is a non-perturbative description of string theory. It suggests a method of relating the five superstring theories and eleven dimensional supergravity which is a field theory that combines supersymmetry and general relativity, involving massless bosons and fermions. It represents the main goal of unification: the ability to have a unique theory capable of describing all physical phenomena.The five different versions of string theory were just M-theory expanded around different vacua. M-theory contains various extended objects such as M2-brane, M5- brane, KK6-brane and 9-brane, propagating in 11 spacetime dimensions. The basic building blocks of M-theory, that preserve half the supersymmetry of the vacuum, are M2-branes and M5-branes. This is due to the fact that 11 dimensional supergravity, which is the low limit of M-theory, contains a single 3-form gauge field, which couples electrically to the M2-brane and magnetically to the M5-brane [25, 26]. These objects can also be seen at the level of the M-theory superalgebra [25, 27].

As we declared before, modern String theory involves the dynamics of the multiple D-branes which are described by the end points of open strings. Open strings streching between D-branes become massless when the D-branes coincide. For a single D-brane, the worldvolume theory is supersymmetric Yang-Mills theory. However, in the case of N coincident D-branes, this leads to a non Abelian gauge theory called Born infeld theory with N^2 degrees of freedom on the worldvolume of the branes.

It is natural to say that the dynamics of the multiple membranes and multiple M5-branes is more complex and not well understood than multiple D-branes. However the action describing a single M2-brane and the action for a single M5-brane are known for a long time, the action for multiple M2 -branes had been an important open problem since the discovery of M-theory, also the case of multiple M5-branes is still mysterious [28].

As it is mentioned before, a D-brane is an extended object on which open fundamental strings can end on it. This fact leads to the derivation of the multiple D-branes theory. It is natural to wonder whether an analogous property holds for branes in M-theory. The analogues are constructed by considering M-theory as the strongly coupled limit of type IIA String theory. Considering a fundamental string ending on a D2-brane in type IIA, the M-theory limit transforms the D2-brane into on M2-brane and the F-string into another M2-brane. In the case of multiple M2-branes they are considered to be connected each other not by strings but by M2-branes. Similarly, starting with a fundamental string ending on a D4-brane and taking the M-theory limit, one has on M2-brane ending on an M5-brane with common part of their worldvolumes being a string [29–32].

The M2⊥M5 relationship is very similar to a relationship among D-branes. It is possible to use S-duality to transform the supersymmetric configuration of an open fundamental string ending on a D3-brane,a configuration known as a "BIon". In that case, the fundamental string turns into a D-string, while the D3-brane remains the same, so one ends up with a supersymmetric configuration of a D string case on a D3-brane. This view can be extended to coincident paralel D strings ending on paralel D3-branes [29].

The relationship between the intersecting D1⊥D3 and M2⊥M5 systems can be explained with the help of the dualities. By compactifying on M2⊥M5 system on a circle within the M5 but not within the M2, a D2⊥D4 system is obtained. Then using T-duality on a common direction, D2 and D4-branes become D-strings and D3-branes respectively, D1⊥D3 system.

There has been two complementary pictures of D1⊥D3 system; one is D3-brane worldvolume Picture: The BIon spike and the other is D1-brane (D-string) worldvolume Picture: The Fuzzy Funnel. In the first Picture,the D1-branes arise as a soliton 'spike' in the D3 worldvolume theory. In the second Picture, the D3-brane arises as a 'fuzzy' sphere in D-string worldvolume theory [33–35].

As much more is known about D-branes, in particular worldvolume theory of multiple D-branes and because they are related by dimensional reduction, D-brane intersections in string theory are useful for inferring properties of the M-theory intersection. For this purpose, it is natural to be focused on the D1⊥D3 system in type IIB theory which is related to M2⊥ M5 system by dimensional reduction and T-duality.

Infact, relation between D1⊥D3 system motivated Basu and Harvey [5] to put forward a proposal for multiple membrane worldvolume theory. Their plan was to conjecture a generalization of the Nahm equation:

$$\frac{\partial X^i}{\partial x^9} = \pm\frac{i}{2}\varepsilon^{ijk}[X^j,X^k], \qquad i,j,k \in 1,2,3 \tag{1.1}$$

that describes the D1⊥D3 system. By using representation of SU(2)

$$\left[\alpha^i,\alpha^j\right] = 2i\varepsilon^{ijk}\alpha^k \tag{1.2}$$

a solution of this equation can be inferred as

$$X^i = \pm\frac{1}{2x^9}\alpha^i, \qquad i = 1,2,3. \tag{1.3}$$

Basu & Harvey proposed a multiple M2-branes worldvolume theory, conjecturing a Nahm equation, along with its solution in terms of a "fuzzy funnel" describing the M2-branes worldvolume opening up onto on M5-brane. They found that this generalized equation describing on M5-brane in the worldvolume theory of multiple M2-branes requires Nambu 4-bracket:

$$\frac{\partial X^i}{\partial x^{10}} = \frac{1}{4!}\frac{b}{8\pi\rho_p^3}[G_5,X^j,X^k,X^l] \tag{1.4}$$

where

$$[A_1,A_2,A_3,A_4] \equiv \sum_{permutations\,\sigma} sign(\sigma)A_{\sigma(1)}A_{\sigma(2)}A_{\sigma(3)}A_{\sigma(4)} \tag{1.5}$$

is the Nambu 4-bracket. Some of the terms in such an action could be

$$S=-T_{M2}\int d^3\sigma tr\left[1+(\partial_a X^M)^2-\frac{b^2}{12}(H^{KLM})^2+\frac{b^2}{48}[\partial_a X^{[K},H^{LMN]}]^2\right]^{\frac{1}{2}}. \quad \mathbf{(1.6)}$$

As a result, it is proposed that this generalized equation of motion arises from an action which involves 3-bracket. Inspired by Basu & Harvey proposal, Bagger & Lambert [1–3] and also independently Gustavsson [4] presented a Lagrangian possessing all the symmetries expected for the theory of multiple M2-branes [36–39]:

- 16 supersymmetries (N=8, d=3) (8 scalars X^I are supplemented by a set of fermion coordinates Ψ^A, $A=1,...8$ with each being a complex 2-component spinor on the worldvolume.
- SO(8) R-Symmetry (which acts on the eight transverse scalars)
- Nontirivial gauge symmetry
- Conformal symmetry.

The expected multiple M2-branes worldvolume theory consists of these continious symmetries.

They proposed a Lagrangian with four supersymmetries that might model come general features of the complete coincident M2-brane worlvolume theory.

In this thesis we study technical details of the Bagger-Lambert-Gustavsson theory that is presented to describe multiple M2-branes and the attempt for the extension of this theory to multiple M5-branes, a system of interacting M5-branes manifesting (2,0) supersymmetry. As we mentioned above, the progress in the formulation of Lagrangian descriptions for multiple M2-branes relied on the introduction of a novel algebraic structure, called Lie 3-algebra [40–43]. So, we first give some algebraic details of this theory. We then study how they gauge a symmetry that arises from the algebra's triple product and construct a supersymmetric multiple M2-branes worldvolume theory. Afterwards we consider a set of supersymmetry transformations postulated by Bagger & Lambert and check the closure of them. We see that these supersymmetries close into translations and gauge transformations with a set of equations of motion so that the superalgebra closes " on shell " and all the equations of motions are invariant under supersymmetry transformations. We end multiple M2-brane worldvolume theory presenting a supersymmetric and gauge-invariant Bagger & Lambert action that is consistent all expected continious symmetries.

We then continue with the extension of this model to multiple M5-branes theory whose low energy dynamics are based on a theory in 6 dimensions with the following symmetries:

- (2,0) Supersymmetry
- SO(5) R-Symmetry
- Conformal symmetry.

Lambert & Papageorgakis [10] proposed an ansatz to construct a Lie-3 algebra theory in six dimensions with (2,0) supersymmetry. They consider the set of supersymmetry transformations for the abelian case first and then propose a non-Abelian generalisation, resulting a system of equations that represents the (2,0) supersymmetric tensor multiplet. We study this propasal in detail. We proceed by checking the closure of the supersymmetry algebra. The superalgebra closes " on shell " up to translations and gauge transformations in case the structure constants are the elements of a real 3-algebra. The closures yield a set of equations of motion and a number of constraints. Finally, we end the thesis reviewing the relation between 6 dimensional this theory and 5 dimensional super Yang-Mills, in other words expanding it around a particular vacuum point how it reduces to five dimensional super-Yang-Mills along with six-dimensional, Abelian (2,0) tensor multiplets.

2. SOME ALGEBRAIC DETAILS

In this chapter we give some algebraic details of BLG model which is discussed in detail in chapter 3. The BLG model is based on a nonassociative algebra, so called Lie-3 algebra [44,45] for which the anti-symmetrized associator leads to a natural triple product structure. In the field theory derived by Bagger and Lambert, one assumes that the scalars and fermions take values in the Lie-3 algebra. However, before presenting Lie-3 algebra, we briefly remind the reader what an algebra is [46].

2.1 Algebra

An algebra is a vector space consisting of vectors $v_1, v_2, v_3 ...... \in V$ and scalar $f_1, f_2, f_3 .... \in F$ on which three types of operations

- (α) vector addition, +
- (β) scalar multiplication, $\circ$
- (γ) vector multiplication, $\times$

are defined satisfying the following postulates A, B, C.

2.1.1 Postulate A

(V , +) is an abelian group.

- $v_i, v_j \in V \Rightarrow v_i + v_j \in V$ (Closure)
- $v_i + (v_j + v_k) = (v_i + v_j) + v_k$ (Associativity)
- $v_0 + v_i = v_i = v_i + v_0$ (Existence of Identity)
- $v_i + (-v_i) = v_0 = (-v_i) + v_i$ (Unique inverse)
- $v_i + v_j = v_j + v_i$ (Commutativity)

2.1.2 Postulate B

- $f_i \in F$, $v_j \in V$, $f_i v_j \in V$ (Closure)
- $f_i \circ (f_j \circ v_k) = (f_i \circ f_j) \circ v_k$ (Associativity)
- $1 \circ v_i = v_i = v_i \circ 1$ (Existence of Identity)
- $f_i \circ (v_k + v_l) = f_i \circ v_k + f_i \circ v_l$ (Distributive law)
- $(f_i + f_j) \circ v_k = f_i \circ v_k + f_j \circ v_j$

2.1.3 Postulate C

- $v_1, v_2 \in V$, $v_1 \times v_2 \in V$ (Closure)
- $(v_1 + v_2) \times v_3 = (v_1 \times v_3) + (v_2 \times v_3)$
  $v_1 \times (v_2 + v_3) = (v_1 \times v_2) + (v_1 \times v_3)$ (Bilinearity)
- $(v_1 \times v_2) \times v_3 = v_1 \times (v_2 \times v_3)$ (Associativity)
- $(v_1 \times 1) = v_1$ (Existence of Identity)
- $v_1 \times v_2 = \pm v_2 \times v_1$(Commutativity and Anti-commutativity)
- $v_1 \times (v_2 \times v_3) = (v_1 \times v_2) \times v_3 + v_2 \times (v_1 \times v_3)$ (Derivation)

The first and second axioms of C are sufficient to define an algebra over V-vector space. The other postulates imply the type of algebra such as commutative algebra, assosiative or non-associative algebra, etc.
The vector multiplication operator "$\times$" can be defined as an antisymmetric, bilinear [-,-] bracket as follows,

$$\begin{aligned} A \times B &= [A, B] \\ &= AB - BA \end{aligned} \qquad (\mathbf{2.1})$$

which has the linearity property

$$[A, \beta B + \gamma C] = \beta [A, B] + \gamma [A, C] \qquad (\mathbf{2.2})$$

and it can be seen that it defines an algebra on V which has no identity and non-Abelian.

2.2 Lie Algebra

An algebra with the antisymmetric multiplication defined by the commutation relations above is called a "Lie Algebra", provided this operation also obeys the last axiom of postulate C.

$$A \times (B \times C) = (A \times B) \times C + B \times (A \times C) \qquad \textbf{(2.3)}$$

This property, called a derivation, can be written more familiarly as

$$[A, [B, C]] = [[A, B], C] + [B, [A, C]] \qquad \textbf{(2.4)}$$

or

$$[A, [B, C]] + [B, [C, A]] + [C, [A, B]] = 0. \qquad \textbf{(2.5)}$$

The latter form is known as the Jacobi's identity.

In other words, a Lie algebra is essentially a vector space with an extra bilinear operation $[-,-] : V \otimes V \to V$ that assigns to every pair of vectors a third one. This extra operation is usually called multiplication and, in the case of Lie algebras, the Lie bracket on the commutator.

It is also possible to define a trilinear operation, an associator on an algebra. This allows a straight forward generalization to a Lie-3 algebra, as a vector space with a 3-bracket $[-,-,-] : V \otimes V \otimes V \to V$. Triple product that is trilinear in each of the entries and satisfies a fundamental identity that generalises the concept of the Jacobi identity.

An associator on an algebra is commonly defined as

$$< A, B, C >= (A \cdot B) \cdot C - A \cdot (B \cdot C) \qquad \textbf{(2.6)}$$

which vanishes in an associative algebra and is written as

$$(A \cdot B) \cdot C = A \cdot (B \cdot C). \qquad \textbf{(2.7)}$$

To define a Lie-3 algebra, one needs a non-vanishing anti-symmetrized associator which leads to a natural triple product structure for this non-associative algebra as in the case of known non-Abelian Lie algebras defined by commutators. The antisymmetrized associator for the construction of a Lie-3 algebra is given as

$$\begin{aligned}[A,B,C] &= <A,B,C> + <B,C,A> + <C,A,B> \\ &\quad - <A,C,B> - <C,B,A> - <B,A,C> \,. \end{aligned} \quad \textbf{(2.8)}$$

Similar to the closure of the Lie algebras defined by the [-,-] commutator operation, which is given as

$$[T^a,T^b] = f^{ab}{}_c T^c \quad \textbf{(2.9)}$$

or

$$[T_a,T_b] = f_{ab}{}^c T_c, \quad \textbf{(2.10)}$$

the closure of triple Lie algebras defined with the [-,-,-] associator operation is given as

$$[T^a,T^b,T^c] = f^{abc}{}_d T^d \quad \textbf{(2.11)}$$

or

$$[T_a,T_b,T_c] = f_{abc}{}^d T_d \quad \textbf{(2.12)}$$

where $\{T^a\}$ is a basis and $\{f^{abc}{}_d\}$ is the structure constants of the Lie algebra.
This is a finite-dimensional complex vector space with a basis T^a which is endowed with a trilinear antisymmetric product above and from which it is clear that $f^{abc}{}_d = f^{[abc]}{}_d$.
Furthermore, if X,Y,Z's are any elements of this algebra and they can be expanded in the basis of algebra and their triple product is calculated as follows,

$$\begin{aligned}[X,Y,Z] &= [X_a T^a, Y_b T^b, Z_c T^c] \\ &= X_a Y_b Z_c [T^a,T^b,T^c] \\ &= X_a Y_b Z_c f^{abc}{}_d T^d. \end{aligned} \quad \textbf{(2.13)}$$

A trace form is required on the 3-algebra to define an action and to construct an invariant Lagrangian. Trace form on an algebra $Tr : V \otimes V \to \mathbb{C}$ is a bilinear map that is symmetric and invariant:

$$Tr(T^a, T^b) = Tr(T^b, T^a) \qquad (symmetry) \tag{2.14}$$

$$Tr(T^a \cdot T^b, T^c) = Tr(T^a, T^b \cdot T^c). \qquad (invariance) \tag{2.15}$$

The invariance property implies

$$\begin{aligned} Tr(<T^a, T^b, T^c>, T^d) &= Tr((T^a \cdot T^b) \cdot T^c, T^d) Tr(T^a \cdot (T^b \cdot T^c), T^d) \\ &= Tr(T^a \cdot T^b, T^c \cdot T^d) Tr(T^a, (T^b \cdot T^c) \cdot T^d) \\ &= Tr(T^a, <T^b, T^c, T^d>) \end{aligned} \tag{2.16}$$

which follows that

$$Tr([T^a, T^b, T^c], T^d) = -Tr(T^a, [T^b, T^c, T^d]). \tag{2.17}$$

This relation on the trace form with antisymmetry of the triple-product implies that

$$f^{abcd} = f^{[abcd]}, \tag{2.18}$$

i.e. f^{abcd} 's are totally antisymmetric, in analogy with the familiar result in Lie algebras. The trace form provides a notion for metric

$$h^{ab} = Tr(T^a, T^b) \tag{2.19}$$

which can be used to raise and lower indices:

$$f^{abcd} = f^{abc}{}_e h^{ed}. \tag{2.20}$$

A notion of 'Hermitian' conjugation † and positivity is also assumed for the trace form which implies that $Tr(A, A^\dagger) \geq 0$ for any element of the algebra (with equality if and only if A=0).

For this algebra the Jacobi identity takes the form of

$$\begin{aligned} [T^a, T^b, [T^c, T^d, T^e]] &= [[T^a, T^b, T^c], T^d, T^e] \\ &\quad + [T^c, [T^a, T^b, T^d], T^e] + [T^c, T^d, [T^a, T^b, T^e]]. \end{aligned} \tag{2.21}$$

In a basis form this is equivalent to

$$f^{efg}{}_{d}f^{abc}{}_{g} = f^{efa}{}_{g}f^{bcg}{}_{d} + f^{efb}{}_{g}f^{cag}{}_{d} + f^{efc}{}_{g}f^{abg}{}_{d}. \qquad \textbf{(2.22)}$$

3. THE BLG THEORY

3.1 A Brief Introduction to The Basics of The BLG Theory

The aim of this chapter is to redone in details the calculations presented in N = 8 Bagger-Lambert-Gustavsson theory. Bagger-Lambert-Gustavsson have been succesful in formulating Lagrangian for multiple M2-branes in terms of the Lie-3 algebra. We are going to demonstrate how the construction of a M2-brane theory with the correct symmetries naturally leads to a 3-algebra structure.

The multiple M2-branes revolution started in 2007 with a series three papers presented by Jonathan Bagger & Neil Lambert [1–3] and also independtly work by Andreas Gustavsson [4]. Motivated by Basu & Harvey [5] they proposed a field-theory model of multiple M2-branes based on an algebra with a totally antisymmetric triple product and gauge a symmetry that arises from the algebra's triple product [47]. In other words, they constructed a supersymmetric gauge-invariant theory that is consistent with all the symmetries expected of a multiple M2-brane theory. This theory contains 8 scalars, X^I, which take values in the transverse space, and a 16-component real fermion Ψ, which is a two-component real d = 3 spinor in one of the 8-dimensional spinor representations of the SO(8) R-symmetry group. The supersymmetry parameter ε is in the other spinor representation [48, 49].

They showed that the supersymmetry algebra closes (up to a translation and a gauge transformation) on shell and the equations of motion arise from a supersymmetric action consistent with all the known continuous symmetries of the multiple M2-branes.

Bagger & Lambert propose lowest-order supersymmetry transformations in the following form [1]:

$$\begin{aligned}\delta X^I &= i\bar{\varepsilon}\Gamma^I\Psi \\ \delta\Psi &= \partial_\mu X^I\Gamma^\mu\Gamma^I\varepsilon + i\kappa[X^I,X^J,X^K]\Gamma^{IJK}\varepsilon \end{aligned} \qquad \textbf{(3.1)}$$

where $\mu, \nu, \lambda = 0,1,2$ and $I,J,K = 3,4,5,...,10$. In these transformations κ is a dimensionless constant and $[X^I, X^J, X^K]$ is antisymmetric and linear in each of the fields [1]. They found that closure of the 16 component supersymmetry algebra leads to the variation

$$\delta X^I \propto i\bar{\varepsilon}_2 \Gamma_{JK} \varepsilon_1 [X^J, X^K, X^I] \tag{3.2}$$

which can be viewed as a local version of the global symmetry transformation

$$\delta X = [\alpha, \beta, X] \tag{3.3}$$

where $\alpha, \beta \in A$.
For $\delta X = [\alpha, \beta, X]$ to be a symmetry it is required to act as a derivation [3] .

$$\begin{aligned} \delta[X,Y,Z] &= [\delta X, Y, Z] + [X, \delta Y, Z] + [X, Y, \delta Z] \\ &= [[\alpha,\beta,X],Y,Z] + [X,[\alpha,\beta,Y],Z] + [X,Y,[\alpha,\beta,Z]] \end{aligned} \tag{3.4}$$

and this leads to the 'fundamental' identity.

$$[\alpha,\beta,[X,Y,Z]] = [[\alpha,\beta,X],Y,Z] + [X,[\alpha,\beta,Y],Z] + [X,Y,[\alpha,\beta,Z]] \tag{3.5}$$

In BLG theory, the fundamental identity plays a major role analogous to the Jacobi identity in ordinary Lie algebra, where it arises from demanding that the transformation $\delta X = [\alpha, X]$ acts as a derivation and the progress is based on assuming that this identity holds. As discussed before in chapter 2, it can be written in terms of structure constants

$$f^{efg}{}_d f^{abc}{}_g = f^{efa}{}_g f^{bcg}{}_d + f^{efb}{}_g f^{cag}{}_d + f^{efc}{}_g f^{abg}{}_d. \tag{3.6}$$

The symmetry transformation $\delta X = [\alpha, \beta, X]$ can be written as [3]:

$$\delta X_d = f^{abc}{}_d \alpha_a \beta_b X_c. \tag{3.7}$$

Also the notation allows for the more general transformation

$$\delta X_d = f^{abc}{}_d \Lambda_{ab} X_c \tag{3.8}$$

which is assumed from now on.

The transformation $\delta X^I \propto i\bar{\varepsilon}_2\Gamma_{JK}\varepsilon_1[X^J,X^K,X^I]$ corresponds to the choice

$$\Lambda_{ab} \propto \bar{\varepsilon}_1\Gamma_{jk}\varepsilon_2 X_a^J X_b^K. \tag{3.9}$$

Little calculation shows that the action is invariant under global symmetries of these transformations. To show this, we obtain that for any Y,

$$\begin{aligned}
\delta Tr(Y,Y) &= Tr(\delta Y,Y)+Tr(Y,\delta Y) \\
&= h^{de}\delta Y_d Y_e + h^{de} Y_d \delta Y_e \\
&= h^{de}(f^{abc}{}_d\Lambda_{ab}Y_c)Y_e + h^{de}Y_d(f^{abc}{}_e\Lambda_{ab}Y_c) \\
&= h^{de}f^{abc}{}_d\Lambda_{ab}Y_cY_e + h^{de}Y_d f^{abc}{}_e\Lambda_{ab}Y_c \\
&= f^{abce}\Lambda_{ab}Y_cY_e + f^{abcd}\Lambda_{ab}Y_dY_c \\
&= f^{abce}\Lambda_{ab}Y_cY_e + f^{abec}\Lambda_{ab}Y_cY_e \\
&= (f^{abce}+f^{abce})\Lambda_{ab}Y_cY_e \\
&= 0
\end{aligned} \tag{3.10}$$

by the replacement of the indices $c \to e$ and $d \to c$ for the second term in the fifth line and the antisymmetry of f^{abce}.

The trace form is invariant under global symmetries.

Furthermore, the fundamental identity ensures that

$$(\delta[X^I,X^J,X^K])_a = f^{cdb}{}_a\Lambda_{cd}[X^I,X^J,X^K]_b. \tag{3.11}$$

The Lagrangian which is invariant under $\delta X_a^I = f^{abc}{}_d\Lambda_{ab}X_b^I$ transformations can be written as

$$\mathscr{L} = -\frac{1}{2}Tr(\partial_\mu X^I,\partial^\mu X^I) - 3\kappa^2 Tr([X^I,X^J,X^K],[X^I,X^J,X^K]). \tag{3.12}$$

3.2 Gauge Symmetry and Supersymmetry of Multiple M2-Branes

3.2.1 Gauging the symmetry

Since this is a local symmetry, a covariant derivative $D_\mu X$ can be introduced such that

$$\begin{aligned} \delta(D_\mu X) &= D_\mu(\delta X) + (\delta D_\mu)X \\ \delta(D_\mu X)_a &= D_\mu(\delta X_a) + (\delta D_\mu)X_a. \end{aligned} \qquad \textbf{(3.13)}$$

If it is let that

$$\delta X_a = \Lambda_{cd} f^{cdb}{}_a X_b \equiv \widetilde{\Lambda}^b_a X_b \qquad \textbf{(3.14)}$$

the covariant derivative as a natural choice can be written in the form of

$$(D_\mu X)_a = \partial_\mu X_a - \widetilde{A}^b_{\mu a} X_b \qquad \textbf{(3.15)}$$

where $\widetilde{A}^b_{\mu a} = f^{cdb}{}_a A_{\mu cd}$ is a gauge field. The gauge field can be considered as living in the space of linear maps $V \otimes V \to V$, in analogy with the adjoint representation of a Lie algebra. The gauge field acts as an element of gl(N,V), where N is the dimension of the algebra V. Besides, the symmetry algebra is contained in so(N,V) as a consequence of the antisymmetry of f^{abcd} structure constants.
Let us look at how the gauge field transforms under this mentioned symmetry.
According to (**3.15**) we can write

$$\begin{aligned} \delta(D_\mu X)_a &= \delta(\partial_\mu X_a - \widetilde{A}^b_{\mu a} X_b) \\ &= \partial_\mu(\delta X_a) - \widetilde{A}^b_{\mu a}(\delta X_b) - (\delta \widetilde{A}^b_{\mu a}) X_b \\ &= D_\mu(\delta X_a) - (\delta \widetilde{A}^b_{\mu a}) X_b \end{aligned} \qquad \textbf{(3.16)}$$

and we obtain $(\delta \widetilde{A}^b_{\mu a}) X_b$ as:

$$\begin{aligned} (\delta \widetilde{A}^b_{\mu a}) X_b &= D_\mu(\delta X_a) - \delta(D_\mu X)_a \\ &= -(\delta D_\mu) X_a. \end{aligned} \qquad \textbf{(3.17)}$$

We can write the transformation of covariant derivative as:

$$\delta(D_\mu X)_a = \widetilde{\Lambda}^b_a (D_\mu X)_b. \qquad \textbf{(3.18)}$$

Considering $(D_\mu X)_a = \partial_\mu X_a - \widetilde{A}^b_{\mu a} X_b$, we get

$$\delta(D_\mu X)_a = \widetilde{\Lambda}^b_a \partial_\mu X_b - \widetilde{\Lambda}^b_a \widetilde{A}^c_{\mu b} X_c. \tag{3.19}$$

It is possible to write

$$D_\mu(\delta X_a) = \partial_\mu(\delta X_a) - \widetilde{A}^b_{\mu a}(\delta X_b) \tag{3.20}$$

according to (**3.15**).
Placing $\delta X_a = \Lambda_{cd} f^{cdb}{}_a X_b \equiv \widetilde{\Lambda}^b_a X_b$ above, we obtain

$$D_\mu(\delta X_a) = \partial_\mu(\widetilde{\Lambda}^b_a X_b) - \widetilde{A}^c_{\mu a}(\widetilde{\Lambda}^b_c X_b). \tag{3.21}$$

Putting (**3.19**) and (**3.21**) expressions in (**3.13**) we find $(\delta D_\mu)X_a$ as:

$$(\delta D_\mu)X_a = \widetilde{A}^c_{\mu a} \widetilde{\Lambda}^b_c X_b - \widetilde{\Lambda}^b_a \widetilde{A}^c_{\mu b} X_c - (\partial_\mu \widetilde{\Lambda}^b_a) X_b. \tag{3.22}$$

So, the transformation of a gauge field is obtained as follows,

$$\delta \widetilde{A}^b_{\mu a} X_b = -\widetilde{A}^c_{\mu a} \widetilde{\Lambda}^b_c X_b + \widetilde{\Lambda}^b_a \widetilde{A}^c_{\mu b} X_c + (\partial_\mu \widetilde{\Lambda}^b_a) X_b. \tag{3.23}$$

On the other hand, taking into account following expressions,

$$\begin{aligned} D_\mu(\delta X_a) &= D_\mu(\widetilde{\Lambda}^b_a X_b) \\ \partial_\mu(\delta X_a) - \widetilde{A}^b_{\mu a}(\delta X_b) &= (D_\mu \widetilde{\Lambda}^b_a) X_b + \widetilde{\Lambda}^b_a (D_\mu X_b) \\ \partial_\mu(\widetilde{\Lambda}^b_a X_b) - \widetilde{A}^b_{\mu a}(\widetilde{\Lambda}^c_b X_c) &= (D_\mu \widetilde{\Lambda}^b_a) X_b + \widetilde{\Lambda}^b_a \partial_\mu X_b - \widetilde{\Lambda}^b_a \widetilde{A}^c_{\mu b} X_c \end{aligned} \tag{3.24}$$

we get

$$(D_\mu \widetilde{\Lambda}^b_a) X_b = -\widetilde{A}^c_{\mu a} \widetilde{\Lambda}^b_c X_b + \widetilde{\Lambda}^b_a \widetilde{A}^c_{\mu b} X_c + (\partial_\mu \widetilde{\Lambda}^b_a) X_b. \tag{3.25}$$

Finally, we obtain

$$\begin{aligned} \delta \widetilde{A}^b_{\mu a} X_b &= -\widetilde{A}^c_{\mu a} \widetilde{\Lambda}^b_c X_b + \widetilde{\Lambda}^b_a \widetilde{A}^c_{\mu b} X_c + (\partial_\mu \widetilde{\Lambda}^b_a) X_b \\ &= (D_\mu \widetilde{\Lambda}^b_a) X_b \end{aligned} \tag{3.26}$$

and

$$\begin{aligned}\delta\widetilde{A}^{b}_{\mu a} &= -\widetilde{\Lambda}^{b}_{c}\widetilde{A}^{c}_{\mu a}+\widetilde{A}^{b}_{\mu c}\widetilde{\Lambda}^{c}_{a}+(\partial_{\mu}\widetilde{\Lambda}^{b}_{a}) \\ &= (D_{\mu}\widetilde{\Lambda}^{b}_{a}).\end{aligned} \tag{3.27}$$

In fact, this expression is the general form of a gauge transformation.

Before the question how the field strength transforms, let us look at the commutation relation of $\big([D_{\mu},D_{\nu}]\,X\big)_{a}$ in the following,

$$\big([D_{\mu},D_{\nu}]\,X\big)_{a} \;=\; \big(D_{\mu}D_{\nu}X)_{a}-(D_{\nu}D_{\mu}X\big)_{a}. \tag{3.28}$$

Let us calculate first term explicitly.

$$\begin{aligned}D_{\mu}(D_{\nu}X)_{a} &= \partial_{\mu}(D_{\nu}X)_{a}-\widetilde{A}^{b}_{\mu a}(D_{\nu}X)_{b} \\ &= \partial_{\mu}(\partial\nu X_{a}-\widetilde{A}^{b}_{\nu a}X_{b})-\widetilde{A}^{b}_{\mu a}(\partial_{\nu}X_{b}-\widetilde{A}^{c}_{\nu b}X_{c}) \\ &= \partial_{\mu}\partial_{\nu}X_{a}-(\partial_{\mu}\widetilde{A}^{b}_{\nu a})X_{b}-\widetilde{A}^{b}_{\nu a}(\partial_{\mu}X_{b}) \\ &\quad -\widetilde{A}^{b}_{\mu a}(\partial_{\nu}X_{b})+\widetilde{A}^{b}_{\mu a}\widetilde{A}^{c}_{\nu b}X_{c}\end{aligned} \tag{3.29}$$

We can get second term replacing by $\mu \leftrightarrow \nu$ in the first term:

$$\begin{aligned}D_{\nu}(D_{\mu}X)_{a} &= \partial_{\nu}\partial_{\mu}X_{a}-(\partial_{\nu}\widetilde{A}^{b}_{\mu a})X_{b}-\widetilde{A}^{b}_{\mu a}(\partial_{\nu}X_{b}) \\ &\quad -\widetilde{A}^{b}_{\nu a}(\partial_{\mu}X_{b})+\widetilde{A}^{b}_{\nu a}\widetilde{A}^{c}_{\mu b}X_{c}.\end{aligned} \tag{3.30}$$

Then commutator is:

$$\big([D_{\mu},D_{\nu}]\,X\big)_{a}=(\partial_{\nu}\widetilde{A}^{b}_{\mu a}-\partial_{\mu}\widetilde{A}^{b}_{\nu a}-\widetilde{A}^{b}_{\mu c}\widetilde{A}^{c}_{\nu a}+\widetilde{A}^{b}_{\nu c}\widetilde{A}^{c}_{\mu a})X_{b}. \tag{3.31}$$

We define the field strength as

$$\big([D_{\mu},D_{\nu}]\,X\big)_{a}=\widetilde{F}^{b}_{\mu\nu a}X_{b} \tag{3.32}$$

which leads to

$$\widetilde{F}^{b}_{\mu\nu a}=\partial_{\nu}\widetilde{A}^{b}_{\mu a}-\partial_{\mu}\widetilde{A}^{b}_{\nu a}-\widetilde{A}^{b}_{\mu c}\widetilde{A}^{c}_{\nu a}+\widetilde{A}^{b}_{\nu c}\widetilde{A}^{c}_{\mu a}. \tag{3.33}$$

The transformation of the field strength is calculated as follows,

$$\begin{aligned}
\delta\widetilde{F}^b_{\mu\nu a} &= \delta(\partial_\nu\widetilde{A}^b_{\mu a}) - \delta(\partial_\mu\widetilde{A}^b_{\nu a}) - \delta\left(\widetilde{A}^b_{\mu c}\widetilde{A}^c_{\nu a}\right) + \delta\left(\widetilde{A}^b_{\nu c}\widetilde{A}^c_{\mu a}\right) \\
&= \partial_\nu(\delta\widetilde{A}^b_{\mu a}) - \partial_\mu(\delta\widetilde{A}^b_{\nu a}) - (\delta\widetilde{A}^b_{\mu c})\widetilde{A}^c_{\nu a} \\
&\quad -\widetilde{A}^b_{\mu c}(\delta\widetilde{A}^c_{\nu a}) + (\delta\widetilde{A}^b_{\nu c})\widetilde{A}^c_{\mu a} + \widetilde{A}^b_{\nu c}(\delta\widetilde{A}^c_{\mu a}). \qquad (\mathbf{3.34})
\end{aligned}$$

If the transformation of the gauge field (**3.27**) is plugged in (**3.34**) and the expression is rearrenged, the transformation of the field strength is obtained as:

$$\begin{aligned}
\delta\widetilde{F}^b_{\mu\nu a} &= \partial_\nu\partial_\mu\widetilde{\Lambda}^b_a - (\partial_\nu\widetilde{\Lambda}^b_c)\widetilde{A}^c_{\mu a} - \widetilde{\Lambda}^b_c(\partial_\nu\widetilde{A}^c_{\mu a}) + (\partial_\nu\widetilde{A}^b_{\mu c})\widetilde{\Lambda}^c_a + \widetilde{A}^b_{\mu c}(\partial_\nu\widetilde{\Lambda}^c_a) \\
&\quad -\partial_\mu\partial_\nu\widetilde{\Lambda}^b_a + (\partial_\mu\widetilde{\Lambda}^b_c)\widetilde{A}^c_{\nu a} + \widetilde{\Lambda}^b_c(\partial_\mu\widetilde{A}^c_{\nu a}) - (\partial_\mu\widetilde{A}^b_{\nu c})\widetilde{\Lambda}^c_a - \widetilde{A}^b_{\nu c}(\partial_\mu\widetilde{\Lambda}^c_a) \\
&\quad -(\partial_\mu\widetilde{\Lambda}^b_c)\widetilde{A}^c_{\nu a} + (\widetilde{\Lambda}^b_d\widetilde{A}^d_{\mu c})\widetilde{A}^c_{\nu a} - (\widetilde{A}^b_{\mu d}\widetilde{\Lambda}^d_c)\widetilde{A}^c_{\nu a} + (\partial_\nu\widetilde{\Lambda}^b_c)\widetilde{A}^c_{\mu a} \\
&\quad -(\widetilde{\Lambda}^b_d\widetilde{A}^d_{\nu c})\widetilde{A}^c_{\mu a} + (\widetilde{A}^b_{\nu d}\widetilde{\Lambda}^d_c)\widetilde{A}^c_{\mu a} - \widetilde{A}^b_{\mu c}(\partial_\nu\widetilde{\Lambda}^c_a) + \widetilde{A}^b_{\mu c}(\widetilde{\Lambda}^c_d\widetilde{A}^d_{\nu a}) \\
&\quad -\widetilde{A}^b_{\mu c}\widetilde{A}^c_{\nu d}\widetilde{\Lambda}^d_a + \widetilde{A}^b_{\nu c}(\partial_\mu\widetilde{\Lambda}^c_a) - \widetilde{A}^b_{\nu c}(\widetilde{\Lambda}^c_d\widetilde{A}^d_{\mu a}) + \widetilde{A}^b_{\nu c}\widetilde{A}^c_{\mu d}\widetilde{\Lambda}^d_a \\
&= -\widetilde{\Lambda}^b_c\partial_\nu\widetilde{A}^c_{\mu a} + \widetilde{\Lambda}^b_c\partial_\mu\widetilde{A}^c_{\nu a} + \widetilde{\Lambda}^b_c\widetilde{A}^c_{\mu d}\widetilde{A}^d_{\nu a} - \widetilde{\Lambda}^b_c\widetilde{A}^c_{\nu d}\widetilde{A}^d_{\mu a} \\
&\quad +(\partial_\nu\widetilde{A}^b_{\mu c})\widetilde{\Lambda}^c_a - (\partial_\mu\widetilde{A}^b_{\nu c})\widetilde{\Lambda}^c_a - \widetilde{A}^b_{\mu d}\widetilde{A}^d_{\nu c}\widetilde{\Lambda}^c_a + (\widetilde{A}^b_{\nu d}\widetilde{A}^d_{\mu c})\widetilde{\Lambda}^c_a \\
&= -\widetilde{\Lambda}^b_c\widetilde{F}^c_{\mu\nu a} + \widetilde{F}^b_{\mu\nu c}\widetilde{\Lambda}^c_a. \qquad (\mathbf{3.35})
\end{aligned}$$

The resulting Bianchi identity is $D_{[\mu}\widetilde{F}_{\nu\lambda]}{}^b_a = 0$ shown as follows,

$$\begin{aligned}
D_{[\mu}\widetilde{F}^b_{\nu\lambda]a} &= \frac{1}{3!}(D_\mu\widetilde{F}^b_{\nu\lambda a} + D_\nu\widetilde{F}^b_{\lambda\mu a} + D_\lambda\widetilde{F}^b_{\mu\nu a} \\
&\quad -D_\nu\widetilde{F}^b_{\mu\lambda a} - D_\mu\widetilde{F}^b_{\lambda\nu a} - D_\lambda\widetilde{F}^b_{\nu\mu a}). \qquad (\mathbf{3.36})
\end{aligned}$$

where

$$\begin{aligned}
D_\mu\widetilde{F}^b_{\nu\lambda a} &= \partial_\mu\widetilde{F}^b_{\nu\lambda a} - \widetilde{A}^c_{\mu a}\widetilde{F}^b_{\nu\lambda c} \\
&= \partial_\mu\partial_\nu\widetilde{A}^b_{\lambda a} - \partial_\mu\partial_\lambda\widetilde{A}^b_{\nu a} - (\partial_\mu\widetilde{A}^b_{\nu c})\widetilde{A}^c_{\lambda a} - \widetilde{A}^b_{\nu c}(\partial_\mu\widetilde{A}^c_{\lambda a}) \\
&\quad +(\partial_\mu\widetilde{A}^b_{\lambda c})\widetilde{A}^c_{\nu a} + \widetilde{A}^b_{\lambda c}(\partial_\mu\widetilde{A}^c_{\nu a}) - \widetilde{A}_{\mu a}(\partial_\nu\widetilde{A}^b_{\lambda c}) + \widetilde{A}^c_{\mu a}\partial_\lambda\widetilde{A}^b_{\nu c} \\
&\quad +\widetilde{A}^c_{\mu a}\widetilde{A}^b_{\nu d}\widetilde{A}^d_{\lambda c} - \widetilde{A}^c_{\mu a}\widetilde{A}^b_{\lambda d}\widetilde{A}^d_{\nu c}. \qquad (\mathbf{3.37})
\end{aligned}$$

Similarly, all other terms are expanded and put in (**3.36**), it is obtained

$$D_{[\mu}\widetilde{F}_{\nu\lambda]}{}^b_a = 0. \qquad (\mathbf{3.38})$$

Also, using fundamental identity (**3.6**) and considering (**3.37**) where $\widetilde{F}^{b}_{\mu\nu a} = F_{\mu\nu cd}f^{cdb}{}_{a}$, $\widetilde{A}^{b}_{\mu a} = A_{\mu cd}f^{cdb}{}_{a}$, it can be checked the Bianchi identity (**3.38**) is also satisfied.

We repeat the findings of this subsection: There is a natural gauge symmetry on the fields X^I for the $\delta X^I_c = \Lambda_{ab}f^{abc}_{d}X^I_c \equiv \widetilde{\Lambda}^c_d X^I_c$ transformation. There is a covariant derivative $(D_\mu X)_a = \partial_\mu X_a - \widetilde{A}^b_{\mu a}X_b$ for the $\delta\widetilde{A}^b_{\mu a} = D_\mu \widetilde{\Lambda}^b_{\mu a}$ transformation, as well as a gauge-covariant field strength $\widetilde{F}^b_{\lambda\nu a}$. The space of all $\widetilde{\Lambda}_{ab}$ satisfy closure under the ordinary matrix commutator, so it generates a matrix Lie algebra G. From this viewpoint, $\widetilde{A}^b_{\mu a}$ is the usual gauge connection in the adjoint representation of G, whereas the elements of A are in the fundamental representation [2].

3.2.2 Supersymmetrizing gauged theory

In this section, we show how to supersymmetrize the gauged multiple M2-brane model which is consistent with all the symmetries expected from multiple M2-branes: in other words a conformal and gauge invariant action with 16 supersymmetries and SO(8) R-symmetry. As we mentioned before, the expected theory describing multiple M2-branes should have 8 scalar fields X^I, parameterising directions transverse to the worldvolume, as well as their fermionic superpartners, so called Goldstinos, which correspond to broken supersymmetries and a nonpropagating gauge field.

Bagger and Lambert proposed the general form of the supersymmetry trasformations including gauge field as

$$\begin{aligned}
\delta X^I_a &= i\bar{\varepsilon}\Gamma^I\Psi_a \\
\delta\Psi_a &= D_\mu X^I_a \Gamma^\mu\Gamma^I\varepsilon + \kappa X^I_b X^J_c X^K_d f^{bcd}{}_{a}\Gamma^{IJK}\varepsilon \\
\delta\widetilde{A}^b_{\mu a} &= i\bar{\varepsilon}\Gamma_\mu\Gamma_I X^I_c \Psi_d f^{cdb}{}_{a}.
\end{aligned} \qquad (\mathbf{3.39})$$

We check this supersymmetry algebra can be made to close ”on shell” (satisfying the equations of motion) only after including the gauge field and its transformation rule.

We first consider the scalars. We then continue with the closure of the fermions and the gauge field respectively. We see that the closures require equations of motion so that the supersymmetry algebra closes up to a translation and a gauge transformation.

3.2.2.1 The closure of scalars

Let us consider scalar fields first to check the closure and see how the algebra closes.

$$\begin{aligned}
[\delta_1,\delta_2]X^I_a &= \delta_1\delta_2X^I_a-\delta_2\delta_1X^I_a \\
&= \delta_1(i\bar{\varepsilon}_2\Gamma^I\Psi_a)-\delta_2(i\bar{\varepsilon}_1\Gamma^I\Psi_a) \\
&= i\bar{\varepsilon}_2\Gamma^I(\delta_1\Psi_a)-i\bar{\varepsilon}_1\Gamma^I(\delta_2\Psi_a) \\
&= i\bar{\varepsilon}_2\Gamma^I(D_\mu X^J_a\Gamma^\mu\Gamma^J\varepsilon_1+\kappa X^J_bX^K_cX^L_d f^{bcd}{}_a\Gamma^{JKL}\varepsilon_1) \\
&\quad -i\bar{\varepsilon}_1\Gamma^I(D_\mu X^J_a\Gamma^\mu\Gamma^J\varepsilon_2+\kappa X^J_bX^K_cX^L_d f^{bcd}{}_a\Gamma^{JKL}\varepsilon_2) \\
&= iD_\mu X^J_a\bar{\varepsilon}_2\Gamma^I\Gamma^\mu\Gamma^J\varepsilon_1+i\kappa X^J_bX^K_cX^L_d f^{bcd}{}_a\bar{\varepsilon}_2\Gamma^I\Gamma^{JKL}\varepsilon_1 \\
&\quad -iD_\mu X^J_a\bar{\varepsilon}_1\Gamma^I\Gamma^\mu\Gamma^J\varepsilon_2+i\kappa X^J_bX^K_cX^L_d f^{bcd}{}_a\bar{\varepsilon}_1\Gamma^I\Gamma^{JKL}\varepsilon_2 \\
&= iD_\mu X^J_a\bar{\varepsilon}_2(\Gamma^I\Gamma^\mu\Gamma^J+\Gamma^J\Gamma^\mu\Gamma^I)\varepsilon_1 \\
&\quad +i\kappa X^J_bX^K_cX^L_d f^{bcd}{}_a\bar{\varepsilon}_2(\Gamma^I\Gamma^{JKL}+\Gamma^{JKL}\Gamma^I)\varepsilon_1
\end{aligned} \tag{3.40}$$

Using the gamma matrix identities (**A.48**, **A.49**):

$$\Gamma^I\Gamma^\mu\Gamma^J+\Gamma^J\Gamma^\mu\Gamma^I = -2g^{IJ}\Gamma^\mu$$

and

$$\begin{aligned}
\Gamma^I\Gamma^{JKL}+\Gamma^{JKL}\Gamma^I &= 2g^{IJ}\Gamma^{KL}+2g^{IL}\Gamma^{JK}+2g^{IK}\Gamma^{LJ} \\
&= 2g^{IJ}\Gamma^{KL}+2g^{IL}\Gamma^{JK}-2g^{IK}\Gamma^{JL},
\end{aligned} \tag{3.41}$$

the commutation can be written as:

$$\begin{aligned}
[\delta_1,\delta_2]X^I_a &= iD_\mu X^J_a\bar{\varepsilon}_2(-2g^{IJ}\Gamma^\mu)\varepsilon_1+i\kappa X^J_bX^K_cX^L_d f^{bcd}{}_a\bar{\varepsilon}_2(2g^{IJ}\Gamma^{KL} \\
&\quad +2g^{IL}\Gamma^{JK}-2g^{IK}\Gamma^{JL})\varepsilon_1.
\end{aligned} \tag{3.42}$$

Considering the properties that metric tensor raises-lowers the indices and the antisymmetry of $\{f^{bcd}{}_a\}$' s respectively, we can write

$$\begin{aligned}
[\delta_1,\delta_2]X^I_a &= -2iD_\mu X^I_a\bar{\varepsilon}_2\Gamma^\mu\varepsilon_1+2i\kappa(X^I_bX^K_cX^L_d f^{bcd}{}_a\bar{\varepsilon}_2\Gamma^{KL}\varepsilon_1 \\
&\quad +X^J_bX^K_cX^I_d f^{bcd}{}_a\bar{\varepsilon}_2\Gamma^{JK}\varepsilon_1-X^J_bX^I_cX^L_d f^{bcd}{}_a\bar{\varepsilon}_2\Gamma^{JL}\varepsilon_1)
\end{aligned} \tag{3.43}$$

which can be simplified to

$$[\delta_1, \delta_2] X^I_a \;\; = \;\; -2iD_\mu X^I_a \bar{\varepsilon}_2 \Gamma^\mu \varepsilon_1 + 6i\kappa X^J_c X^K_d X^I_b f^{cdb}{}_a \bar{\varepsilon}_2 \Gamma^{JK} \varepsilon_1. \tag{3.44}$$

We find that the transformations proposed by Bagger & Lambert, close into a translation and a gauge transformation;

$$[\delta_1, \delta_2] X^I_a \;\; = \;\; v^\mu D_\mu X^I_a + \widetilde{\Lambda}^b_a X^I_b \tag{3.45}$$

where we define $-2i\bar{\varepsilon}_2 \Gamma^\mu \varepsilon_1 \equiv v^\mu$ and $6i\kappa \bar{\varepsilon}_2 \Gamma^{JK} \varepsilon_1 X^J_c X^K_d f^{cdb}{}_a \equiv \widetilde{\Lambda}^b_a$.

3.2.2.2 The closure of spinors

Continuing with the closure of the spinor fields, we get

$$\begin{aligned}
[\delta_1, \delta_2] \Psi_a &= \delta_1 \delta_2 \Psi_a - \delta_2 \delta_1 \Psi_a \\
&= \delta_1 (D_\mu X^I_a \Gamma^\mu \Gamma^I \varepsilon_2 + \kappa X^I_b X^J_c X^K_d \Gamma^{IJK} f^{bcd}{}_a \varepsilon_2) \\
&\quad -\delta_2 (D_\mu X^I_a \Gamma^\mu \Gamma^I \varepsilon_1 + \kappa X^I_b X^J_c X^K_d \Gamma^{IJK} f^{bcd}{}_a \varepsilon_1) \\
&= \delta_1 (D_\mu X^I_a \Gamma^\mu \Gamma^I \varepsilon_2) + \delta_1 (\kappa X^I_b X^J_c X^K_d \Gamma^{IJK} f^{bcd}{}_a \varepsilon_2) \\
&\quad -\delta_2 (D_\mu X^I_a \Gamma^\mu \Gamma^I \varepsilon_1) - \delta_2 (\kappa X^I_b X^J_c X^K_d \Gamma^{IJK} f^{bcd}{}_a \varepsilon_1).
\end{aligned} \tag{3.46}$$

Each term in paranthesis can be calculated seperately as follows,

$$\begin{aligned}
\delta_1 (\Gamma^\mu \Gamma^I D_\mu X^I_a \varepsilon_2) &= \Gamma^\mu \Gamma^I \delta_1 (D_\mu X^I_a) \varepsilon_2 \\
&= \Gamma^\mu \Gamma^I \delta_1 (\partial_\mu X^I_a - \widetilde{A}^b_{\mu a} X^I_b) \varepsilon_2 \\
&= \Gamma^\mu \Gamma^I \partial_\mu (\delta_1 X^I_a) \varepsilon_2 - \Gamma^\mu \Gamma^I \delta_1 (\widetilde{A}^b_{\mu a} X^I_b) \varepsilon_2 \\
&= \Gamma^\mu \Gamma^I \partial_\mu (\delta_1 X^I_a) \varepsilon_2 - \Gamma^\mu \Gamma^I \widetilde{A}^b_{\mu a} (\delta_1 X^I_b) \varepsilon_2 \\
&\quad -\Gamma^\mu \Gamma^I (\delta_1 \widetilde{A}^b_{\mu a}) X^I_b \varepsilon_2 \\
&= \Gamma^\mu \Gamma^I (\partial_\mu (\delta_1 X^I_a) - \widetilde{A}^b_{\mu a} (\delta_1 X^I_b)) \varepsilon_2 - \Gamma^\mu \Gamma^I (\delta_1 \widetilde{A}^b_{\mu a}) X^I_b \varepsilon_2 \\
&= \Gamma^\mu \Gamma^I D_\mu (\delta_1 X^I_a) \varepsilon_2 - \Gamma^\mu \Gamma^I (\delta_1 \widetilde{A}^b_{\mu a}) X^I_b \varepsilon_2 \\
&= \Gamma^\mu \Gamma^I D_\mu (i\bar{\varepsilon}_1 \Gamma^I \Psi_a) \varepsilon_2 - \Gamma^\mu \Gamma^I (i\bar{\varepsilon}_1 \Gamma_\mu \Gamma_J X^J_c \Psi_d f^{cdb}{}_a) X^I_b \varepsilon_2 \\
&= i\Gamma^\mu \Gamma^I \bar{\varepsilon}_1 \Gamma^I D_\mu \Psi_a \varepsilon_2 - i\Gamma^\mu \Gamma^I \bar{\varepsilon}_1 \Gamma_\mu \Gamma_J \Psi_d \varepsilon_2 X^I_b X^J_c f^{cdb}{}_a
\end{aligned} \tag{3.47}$$

$$\delta_1(\kappa\Gamma^{IJK}X_b^I X_c^J X_d^K f^{bcd}{}_a\varepsilon_2) =$$

$$\begin{aligned}
&= \kappa(\Gamma^{IJK}(\delta_1 X_b^I)X_c^J X_d^K f^{bcd}{}_a\varepsilon_2 + \Gamma^{IJK}X_b^I(\delta_1 X_c^J)X_d^K f^{bcd}{}_a\varepsilon_2 \\
&\quad +\Gamma^{IJK}X_b^I X_c^J(\delta_1 X_d^K)f^{bcd}{}_a\varepsilon_2) \\
&= \kappa(\Gamma^{IJK}(i\bar{\varepsilon}_1\Gamma^I\Psi_b)X_c^J X_d^K f^{bcd}{}_a\varepsilon_2 + \Gamma^{IJK}X_b^I(i\bar{\varepsilon}_1\Gamma^J\Psi_c)X_d^K f^{bcd}{}_a\varepsilon_2 \\
&\quad +\Gamma^{IJK}X_b^I X_c^J(i\bar{\varepsilon}_1\Gamma^K\Psi_d)f^{bcd}{}_a\varepsilon_2) \\
&= \kappa(i\Gamma^{IJK}\bar{\varepsilon}_1\Gamma^I\Psi_b X_c^J X_d^K f^{bcd}{}_a\varepsilon_2 + i\Gamma^{IJK}\bar{\varepsilon}_1\Gamma^J\Psi_c X_b^I X_d^K f^{bcd}{}_a\varepsilon_2 \\
&\quad +i\Gamma^{IJK}\bar{\varepsilon}_1\Gamma^K\Psi_d X_b^I X_c^J f^{bcd}{}_a\varepsilon_2).
\end{aligned} \tag{3.48}$$

Considering $\{f^{bcd}{}_a\}$'s are antisymmetric, this expression can be simplified as:

$$\delta_1(\kappa\Gamma^{IJK}X_b^I X_c^J X_d^K f^{bcd}{}_a\varepsilon_2) = 3i\kappa\Gamma^{IJK}\bar{\varepsilon}_1\Gamma^K\Psi_d\varepsilon_2 X_b^I X_c^J f^{bcd}{}_a. \tag{3.49}$$

The last two term in (**3.46**) can be obtained from the first two term by replacing the indices. Thus, the closure of fermion field is given as:

$$\begin{aligned}
[\delta_1,\delta_2]\Psi_a &= i\Gamma^\mu\Gamma^I\bar{\varepsilon}_1\Gamma^I D_\mu\Psi_a\varepsilon_2 - i\Gamma^\mu\Gamma^I\bar{\varepsilon}_1\Gamma_\mu\Gamma_J\Psi_d\varepsilon_2 X_b^I X_c^J f^{cdb}{}_a \\
&\quad +3i\kappa\Gamma^{IJK}\bar{\varepsilon}_1\Gamma^K\Psi_d\varepsilon_2 X_b^I X_c^J f^{cdb}{}_a - i\Gamma^\mu\Gamma^I\bar{\varepsilon}_2\Gamma^I D_\mu\Psi_a\varepsilon_1 \\
&\quad +i\Gamma^\mu\Gamma^I\bar{\varepsilon}_2\Gamma_\mu\Gamma_J\Psi_d\varepsilon_1 X_b^I X_c^J f^{cdb}{}_a - 3i\kappa\Gamma^{IJK}\bar{\varepsilon}_2\Gamma^K\Psi_d\varepsilon_1 X_b^I X_c^J f^{cdb}{}_a \\
&= -i\Gamma^\mu\Gamma^I(\bar{\varepsilon}_2\Gamma^I D_\mu\Psi_a\varepsilon_1 - \bar{\varepsilon}_1\Gamma^I D_\mu\Psi_a\varepsilon_2) + i\Gamma^\mu\Gamma^I(\bar{\varepsilon}_2\Gamma_\mu\Gamma_J\Psi_d\varepsilon_1 \\
&\quad -\bar{\varepsilon}_1\Gamma_\mu\Gamma_J\Psi_d\varepsilon_2)X_b^I X_c^J f^{cdb}{}_a - 3i\kappa\Gamma^{IJK}(\bar{\varepsilon}_2\Gamma^K\Psi_d\varepsilon_1 \\
&\quad -\bar{\varepsilon}_1\Gamma^K\Psi_d\varepsilon_2)X_b^I X_c^J f^{cdb}{}_a.
\end{aligned} \tag{3.50}$$

Expanding these three terms above with Fierz reordering formula in (**A.20**), we get

$$-i\Gamma^\mu\Gamma^I(\bar{\varepsilon}_2\Gamma^I D_\mu\Psi_a\varepsilon_1 - \bar{\varepsilon}_1\Gamma^I D_\mu\Psi_a\varepsilon_2) =$$

$$\begin{aligned}
&= \frac{i}{16}\Gamma^\mu\Gamma^I[(2\bar{\varepsilon}_2\Gamma_\nu\varepsilon_1)\Gamma^\nu\Gamma^I D_\mu\Psi_a - (\bar{\varepsilon}_2\Gamma_{LM}\varepsilon_1)\Gamma^{LM}\Gamma^I D_\mu\Psi_a \\
&\quad +\frac{1}{4!}(\bar{\varepsilon}_2\Gamma_\nu\Gamma_{LMNO}\varepsilon_1)\Gamma^\nu\Gamma^{LMNO}\Gamma^I D_\mu\Psi_a]
\end{aligned} \tag{3.51}$$

$$i\Gamma^\mu\Gamma^I(\bar{\varepsilon}_2\Gamma_\mu\Gamma_J\Psi_d\varepsilon_1 - \bar{\varepsilon}_1\Gamma_\mu\Gamma_J\Psi_d\varepsilon_2)X_b^I X_c^J f^{cdb}{}_a =$$

$$\begin{aligned}
&= -\frac{i}{16}\Gamma^\mu\Gamma^I[(2\bar{\varepsilon}_2\Gamma_\nu\varepsilon_1)\Gamma^\nu\Gamma_\mu\Gamma_J\Psi_d - (\bar{\varepsilon}_2\Gamma_{LM}\varepsilon_1)\Gamma^{LM}\Gamma_\mu\Gamma_J\Psi_d \\
&\quad +\frac{1}{4!}(\bar{\varepsilon}_2\Gamma_\nu\Gamma_{LMNO}\varepsilon_1)\Gamma^\nu\Gamma^{LMNO}\Gamma_\mu\Gamma_J\Psi_d]X_b^I X_c^J f^{cdb}{}_a
\end{aligned} \tag{3.52}$$

$$-3i\kappa\Gamma^{IJK}(\bar{\varepsilon}_2\Gamma^K\Psi_d\varepsilon_1-\bar{\varepsilon}_1\Gamma^K\Psi_d\varepsilon_2)X^I_bX^J_cf^{cdb}{}_a=$$

$$\begin{aligned}= & \frac{3i\kappa}{16}\Gamma^{IJK}[(2\bar{\varepsilon}_2\Gamma_\nu\varepsilon_1)\Gamma^\nu\Gamma^K\Psi_d-(\bar{\varepsilon}_2\Gamma_{LM}\varepsilon_1)\Gamma^{LM}\Gamma^K\Psi_d \\ & +\frac{1}{4!}(\bar{\varepsilon}_2\Gamma_\nu\Gamma_{LMNO}\varepsilon_1)\Gamma^\nu\Gamma^{LMNO}\Gamma^K\Psi_d]X^I_bX^J_cf^{cdb}{}_a.\end{aligned} \quad \textbf{(3.53)}$$

Putting these expressions in $[\delta_1,\delta_2]\Psi_a$ and rearranging, we find

$$\begin{aligned}[\delta_1,\delta_2]\Psi_a = & \frac{i}{16}\{2(\bar{\varepsilon}_2\Gamma_\nu\varepsilon_1)[\Gamma^\mu\Gamma^I\Gamma^\nu\Gamma^I D_\mu\Psi_a-\Gamma^\mu\Gamma^I\Gamma^\nu\Gamma_\mu\Gamma_J\Psi_dX^I_bX^J_cf^{cdb}{}_a \\ & +3\kappa\Gamma^{IJK}\Gamma^\nu\Gamma^K\Psi_dX^I_bX^J_cf^{cdb}{}_a]-(\bar{\varepsilon}_2\Gamma_{LM}\varepsilon_1)[\Gamma^\mu\Gamma^I\Gamma^{LM}\Gamma^I D_\mu\Psi_a \\ & -\Gamma^\mu\Gamma^I\Gamma^{LM}\Gamma_\mu\Gamma_J\Psi_dX^I_bX^J_cf^{cdb}{}_a+3\kappa\Gamma^{IJK}\Gamma^{LM}\Gamma^K\Psi_dX^I_bX^J_cf^{cdb}{}_a] \\ & +\frac{1}{4!}(\bar{\varepsilon}_2\Gamma_\nu\Gamma_{LMNO}\varepsilon_1)[\Gamma^\mu\Gamma^I\Gamma^\nu\Gamma^{LMNO}\Gamma^I D_\mu\Psi_a \\ & -\Gamma^\mu\Gamma^I\Gamma^\nu\Gamma^{LMNO}\Gamma_\mu\Gamma_J\Psi_dX^I_bX^J_cf^{cdb}{}_a \\ & +3\kappa\Gamma^{IJK}\Gamma^\nu\Gamma^{LMNO}\Gamma^K\Psi_dX^I_bX^J_cf^{cdb}{}_a]\} \\ = & \frac{i}{16}[2(\bar{\varepsilon}_2\Gamma_\nu\varepsilon_1)A-(\bar{\varepsilon}_2\Gamma_{LM}\varepsilon_1)B+\frac{1}{4!}(\bar{\varepsilon}_2\Gamma_\nu\Gamma_{LMNO}\varepsilon_1)C].\end{aligned} \quad \textbf{(3.54)}$$

We simplify A, B, C respectively using gamma matrix identities in (A).

Each term in A can be calculated as follows.

First term is calculated with the help of gamma matrix identities (**A.4**), (**A.1**) and (A) as:

$$\begin{aligned}\Gamma^\mu\Gamma^I\Gamma^\nu\Gamma^I D_\mu\Psi_a &= -\Gamma^I\Gamma^I\Gamma^\mu\Gamma^\nu D_\mu\Psi_a \\ &= -8(-\Gamma^\nu\Gamma^\mu+2g^{\mu\nu})D_\mu\Psi_a \\ &= 8\Gamma^\nu\Gamma^\mu D_\mu\Psi_a-16g^{\mu\nu}D_\mu\Psi_a\end{aligned} \quad \textbf{(3.55)}$$

and considering $\kappa=-\frac{1}{6}$, second term is calculated using (**A.4**), (**A.1**), (A) and (**A.36**) as follows,

$$\begin{aligned}(-\Gamma^\mu\Gamma^I\Gamma^\nu\Gamma_\mu\Gamma_J + 3\kappa\Gamma^{IJK}\Gamma^\nu\Gamma^K)\Psi_dX^I_bX^J_cf^{bcd}{}_a= \\ =(\Gamma^\mu\Gamma^\nu\Gamma^I\Gamma_\mu\Gamma_J+\frac{1}{2}\Gamma^\nu\Gamma^{IJK}\Gamma^K)\Psi_dX^I_bX^J_cf^{bcd}{}_a \\ =[(-\Gamma^\nu\Gamma^\mu+2g^{\mu\nu})\Gamma^I\Gamma_\mu\Gamma_J+\frac{1}{2}\Gamma^\nu\Gamma^{IJK}\Gamma^K]\Psi_dX^I_bX^J_cf^{bcd}{}_a \\ =[(-\Gamma^\nu\Gamma^\mu+2g^{\mu\nu})\Gamma^I\Gamma_\mu\Gamma_J+\frac{1}{2}\Gamma^\nu(6\Gamma^{IJ})]\Psi_dX^I_bX^J_cf^{bcd}{}_a.\end{aligned} \quad \textbf{(3.56)}$$

We consider the $\Gamma^I\Gamma^J$ as a term of symmetric and antisymmetric parts as:

$$\Gamma^{IJ} = \frac{1}{2}(\Gamma^I\Gamma^J + \Gamma^J\Gamma^I) + \frac{1}{2}(\Gamma^I\Gamma^J - \Gamma^J\Gamma^I). \quad \mathbf{(3.57)}$$

Because the multiplication of symmetric part with $X_b^I X_c^J f^{bcd}{}_a$ is zero, we can write $\Gamma^I\Gamma^J$ as Γ^{IJ} and we get

$$\begin{aligned}(-\Gamma^\mu\Gamma^I\Gamma^\nu\Gamma_\mu\Gamma_J &+ 3\kappa\Gamma^{IJK}\Gamma^\nu\Gamma^K)\Psi_d X_b^I X_c^J f^{bcd}{}_a = \\ &= [3\Gamma^\nu\Gamma^{IJ} - 2\Gamma^\nu\Gamma^{IJ} + 3\Gamma^\nu\Gamma^{IJ}]\Psi_d X_b^I X_c^J f^{bcd}{}_a \\ &= 4\Gamma^\nu\Gamma^{IJ}\Psi_d X_b^I X_c^J f^{bcd}{}_a. \quad \mathbf{(3.58)}\end{aligned}$$

Combining all these expressions, we obtain A as follows,

$$A = 8\Gamma^\nu\Gamma^\mu D_\mu\Psi_a - 16g^{\mu\nu}D_\mu\Psi_a + 4\Gamma^\nu\Gamma^{IJ}\Psi_d X_b^I X_c^J f^{bcd}{}_a. \quad \mathbf{(3.59)}$$

Each term in B can be calculated similarly.

First term is (see **A.51**)

$$\Gamma^\mu\Gamma^I\Gamma^{LM}\Gamma_I D_\mu\Psi_a = 4\Gamma^\mu\Gamma^{LM}D_\mu\Psi_a. \quad \mathbf{(3.60)}$$

Second term is (see A)

$$-\Gamma^\mu\Gamma^I\Gamma^{LM}\Gamma_\mu\Gamma_J\Psi_d X_b^I X_c^J f^{bcd}{}_a = 3\Gamma^I\Gamma^{LM}\Gamma_J\Psi_d X_b^I X_c^J f^{bcd}{}_a \quad \mathbf{(3.61)}$$

and the last term is (see **A.55**)

$$\begin{aligned}-\frac{1}{2}\Gamma^{IJK}\Gamma^{LM}\Gamma^K\Psi_d X_b^I X_c^J f^{bcd}{}_a &= -\Gamma^{LM}\Gamma^{IJ}\Psi_d X_b^I X_c^J f^{bcd}{}_a + 3\Gamma^{LJ}\Psi_d X_b^M X_c^J f^{bcd}{}_a \\ &\quad -3\Gamma^{LI}\Psi_d X_b^I X_c^M f^{bcd}{}_a + 3\Gamma^{MI}\Psi_d X_b^I X_c^L f^{bcd}{}_a \\ &\quad -3\Gamma^{MJ}\Psi_d X_b^L X_c^J f^{bcd}{}_a - 2\Psi_d X_b^M X_c^L f^{bcd}{}_a \\ &\quad +2\Psi_d X_b^L X_c^M f^{bcd}{}_a \\ &= -\Gamma^{LM}\Gamma^{IJ}\Psi_d X_b^I X_c^J f^{bcd}{}_a - 6\Gamma^{LI}\Psi_d X_b^I X_c^M f^{bcd}{}_a \\ &\quad +6\Gamma^{MI}\Psi_d X_b^I X_c^L f^{bcd}{}_a - 4\Psi_d X_b^M X_c^L f^{bcd}{}_a.\end{aligned}$$

(3.62)

Hereby, we can write

$$\begin{aligned} B &= 4\Gamma^{\mu}\Gamma^{LM}D_{\mu}\Psi_a + 3\Gamma^{I}\Gamma^{LM}\Gamma_J\Psi_d X_b^I X_c^J f^{bcd}{}_a - \Gamma^{LM}\Gamma^{IJ}\Psi_d X_b^I X_c^J f^{bcd}{}_a \\ &\quad -6\Gamma^{LI}\Psi_d X_b^I X_c^M f^{bcd}{}_a + 6\Gamma^{MI}\Psi_d X_b^I X_c^L f^{bcd}{}_a - 4\Psi_d X_b^M X_c^L f^{bcd}{}_a. \end{aligned} \quad (\mathbf{3.63})$$

Considering (see **A.30**)

$$3\Gamma^{I}\Gamma^{LM}\Gamma_J = 3\Gamma^{LM}\Gamma^{I}\Gamma_J + 6g^{IL}\Gamma^{M}\Gamma_J - 6g^{IM}\Gamma^{L}\Gamma_J, \quad (\mathbf{3.64})$$

we obtain B as (see **A.29**):

$$\begin{aligned} B &= 4\Gamma^{\mu}\Gamma^{LM}D_{\mu}\Psi_a + 3\Gamma^{LM}\Gamma^{I}\Gamma_J\Psi_d X_b^I X_c^J f^{bcd}{}_a + 6g^{IL}\Gamma^{M}\Gamma_J\Psi_d X_b^I X_c^J f^{bcd}{}_a \\ &\quad -6g^{IM}\Gamma^{L}\Gamma_J\Psi_d X_b^I X_c^J f^{bcd}{}_a - \Gamma^{LM}\Gamma^{IJ}\Psi_d X_b^I X_c^J f^{bcd}{}_a - 6\Gamma^{LI}\Psi_d X_b^I X_c^M f^{bcd}{}_a \\ &\quad +6\Gamma^{MI}\Psi_d X_b^I X_c^L f^{bcd}{}_a - 4\Psi_d X_b^M X_c^L f^{bcd}{}_a \\ &= 4\Gamma^{\mu}\Gamma^{LM}D_{\mu}\Psi_a + 2\Gamma^{LM}\Gamma^{I}\Gamma_J\Psi_d X_b^I X_c^J f^{bcd}{}_a + 6(\Gamma^{M}\Gamma^{I} - \Gamma^{MI})\Psi_d X_b^L X_c^I f^{bcd}{}_a \\ &\quad -6(\Gamma^{L}\Gamma^{I} - \Gamma^{LI})\Psi_d X_b^I X_c^M f^{bcd}{}_a - 4\Psi_d X_b^M X_c^L f^{bcd}{}_a \\ &= 4\Gamma^{\mu}\Gamma^{LM}D_{\mu}\Psi_a + 2\Gamma^{LM}\Gamma^{I}\Gamma_J\Psi_d X_b^I X_c^J f^{bcd}{}_a + 6g^{MI}\Psi_d X_b^L X_c^I f^{bcd}{}_a \\ &\quad -6g^{LI}\Psi_d X_b^I X_c^M f^{bcd}{}_a - 4\Psi_d X_b^M X_c^L f^{bcd}{}_a \\ &= 4\Gamma^{\mu}\Gamma^{LM}D_{\mu}\Psi_a + 2\Gamma^{LM}\Gamma^{I}\Gamma_J\Psi_d X_b^I X_c^J f^{bcd}{}_a + 16\Psi_d X_b^L X_c^M f^{bcd}{}_a. \end{aligned} \quad (\mathbf{3.65})$$

Finally, each term of C can be calculated in a similar way.

First term of C is given as follows (see **A.52**),

$$\begin{aligned} \Gamma^{\mu}\Gamma^{I}\Gamma^{\nu}\Gamma^{LMNO}\Gamma^{I}D_{\mu}\Psi_a &= -\Gamma^{\mu}\Gamma^{\nu}\Gamma^{I}\Gamma^{LMNO}\Gamma^{I}D_{\mu}\Psi_a \\ &= 0. \end{aligned} \quad (\mathbf{3.66})$$

Second term is (see **A.4**, **A.42**, **A.1**)

$$\begin{aligned} -\Gamma^{\mu}\Gamma^{I}\Gamma^{\nu}\Gamma^{LMNO}\Gamma_{\mu}\Gamma_J\Psi_d X_b^I X_c^J f^{bcd} &= -\Gamma^{\mu}\Gamma^{\nu}\Gamma_{\mu}\Gamma^{I}\Gamma^{LMNO}\Gamma_J\Psi_d X_b^I X_c^J f^{bcd} \\ &= -\Gamma^{\mu}(-\Gamma_{\mu}\Gamma^{\nu} + 2g^{\nu}{}_{\mu})\Gamma^{I}\Gamma^{LMNO}\Gamma_J \\ &\quad .\Psi_d X_b^I X_c^J f^{bcd} \\ &= \Gamma^{\nu}\Gamma^{I}\Gamma^{LMNO}\Gamma_J\Psi_d X_b^I X_c^J f^{bcd}{}_a \end{aligned} \quad (\mathbf{3.67})$$

and the last term is (see **A.53**)

$$
\begin{aligned}
3\kappa\Gamma^{IJK}\Gamma^{\nu}\Gamma^{LMNO}\Gamma^{K}\Psi_d X_b^I X_c^J f^{cdb}{}_a &= \frac{1}{2}\Gamma^{\nu}\Gamma^{IJK}\Gamma^{LMNO}\Gamma^{K}\Psi_d X_b^I X_c^J f^{cdb}{}_a \\
&= -\frac{1}{2}\Gamma^{\nu}\Gamma^{I}\Gamma^{LMNO}\Gamma^{J}\Psi_d X_b^I X_c^J f^{cdb}{}_a \\
&\quad +\frac{1}{2}\Gamma^{\nu}\Gamma^{J}\Gamma^{LMNO}\Gamma^{I}\Psi_d X_b^I X_c^J f^{cdb}{}_a.
\end{aligned} \tag{3.68}
$$

So, we get

$$
\begin{aligned}
C &= \frac{1}{2}\Gamma^{\nu}(\Gamma^{I}\Gamma^{LMNO}\Gamma^{J} + \Gamma^{J}\Gamma^{LMNO}\Gamma^{I})\Psi_d X_b^I X_c^J f^{cdb}{}_a \\
&= 0.
\end{aligned} \tag{3.69}
$$

We used (**A.32**) considering the fact that $f^{abc}{}_d$'s are antisymmetric above.

If we put all these expression in (**3.54**), we find

$$
\begin{aligned}
[\delta_1,\delta_2]\Psi_a &= -2i(\bar{\varepsilon}_2\Gamma^{\mu}\varepsilon_1)D_{\mu}\Psi_a - i(\bar{\varepsilon}_2\Gamma_{LM}\varepsilon_1)\Psi_d X_b^L X_c^M f^{cdb}{}_a \\
&\quad + i(\bar{\varepsilon}_2\Gamma_{\nu}\varepsilon_1)\Gamma^{\nu}[\Gamma^{\mu}D_{\mu}\Psi_a + \frac{1}{2}\Gamma_{IJ}\Psi_d X_b^I X_c^J f^{cdb}{}_a] \\
&\quad - \frac{i}{4}(\bar{\varepsilon}_2\Gamma_{LM}\varepsilon_1)\Gamma^{LM}[\Gamma^{\mu}D_{\mu}\Psi_a + \frac{1}{2}\Gamma_{IJ}\Psi_d X_b^I X_c^J f^{cdb}{}_a].
\end{aligned} \tag{3.70}
$$

The closure of the spinor fields requires that the second and third lines vanish.

This leads to

$$
\Gamma^{\mu}D_{\mu}\Psi_a + \frac{1}{2}\Gamma_{IJ}X_c^I X_d^J \Psi_b f^{cdb}{}_a = 0. \tag{3.71}
$$

Finally, on shell, we find that

$$
[\delta_1,\delta_2]\Psi_a = v^{\mu}D_{\mu}\Psi_a + \widetilde{\Lambda}_a^b\Psi_b, \tag{3.72}
$$

after the fermionic equation of motion (**3.71**) is satisfied.

As a result, considering fermions we see that the transformation of spinor field closes into a translation and a gauge transformation after satisfying fermionic equation of motion.

3.2.2.3 The closure of the gauge field

Lastly, we will check the closure of the gauge field which give us the field equation of field strength as follows,

$$
\begin{aligned}
[\delta_1,\delta_2]\widetilde{A}^b_{\mu a} &= \delta_1\delta_2\widetilde{A}^b_{\mu a}-\delta_2\delta_1\widetilde{A}^b_{\mu a} \\
&= \delta_1(i\bar{\varepsilon}_2\Gamma_\mu\Gamma_I X^I_c\Psi_d f^{cdb}{}_a)-\delta_2(i\bar{\varepsilon}_1\Gamma_\mu\Gamma_I X^I_c\Psi_d f^{cdb}{}_a) \\
&= i\bar{\varepsilon}_2\Gamma_\mu\Gamma_I(\delta_1 X^I_c)\Psi_d f^{cdb}{}_a+i\bar{\varepsilon}_2\Gamma_\mu\Gamma_I X^I_c(\delta_1\Psi_d)f^{cdb}{}_a \\
&\quad -i\bar{\varepsilon}_1\Gamma_\mu\Gamma_I(\delta_2 X^I_c)\Psi_d f^{cdb}{}_a-i\bar{\varepsilon}_1\Gamma_\mu\Gamma_I X^I_c(\delta_2\Psi_d)f^{cdb}{}_a \\
&= i\bar{\varepsilon}_2\Gamma_\mu\Gamma_I(i\bar{\varepsilon}_1\Gamma^I\Psi_c)\Psi_d f^{cdb}{}_a+i\bar{\varepsilon}_2\Gamma_\mu\Gamma_I X^I_c(\Gamma^\nu\Gamma^J D_\nu X^J_d\varepsilon_1 \\
&\quad +\kappa X^J_e X^K_f X^L_g\Gamma^{JKL}\varepsilon_1 f^{efg}{}_d)f^{cdb}{}_a-i\bar{\varepsilon}_1\Gamma_\mu\Gamma_I(i\bar{\varepsilon}_2\Gamma^I\Psi_c)\Psi_d f^{cdb}{}_a \\
&\quad -i\bar{\varepsilon}_1\Gamma_\mu\Gamma_I X^I_c(\Gamma^\nu\Gamma^J D_\nu X^J_d\varepsilon_2+\kappa X^J_e X^K_f X^L_g\Gamma^{JKL}\varepsilon_2 f^{efg}{}_d)f^{cdb}{}_a \\
&= -\bar{\varepsilon}_2\Gamma_\mu\Gamma_I\bar{\varepsilon}_1\Gamma^I\Psi_c\Psi_d f^{cdb}{}_a+i\bar{\varepsilon}_2\Gamma_\mu\Gamma_I\Gamma^\nu\Gamma^J\varepsilon_1 X^I_c D_\nu X^J_d f^{cdb}{}_a \\
&\quad +i\kappa\bar{\varepsilon}_2\Gamma_\mu\Gamma_I\Gamma^{JKL}\varepsilon_1 X^I_c X^J_e X^K_f X^L_g f^{efg}{}_d f^{cdb}{}_a \\
&\quad +\bar{\varepsilon}_1\Gamma_\mu\Gamma_I\bar{\varepsilon}_2\Gamma^I\Psi_c\Psi_d f^{cdb}{}_a-i\bar{\varepsilon}_1\Gamma_\mu\Gamma_I\Gamma^\nu\Gamma^J\varepsilon_2 X^I_c D_\nu X^J_d f^{cdb}{}_a \\
&\quad -i\kappa\bar{\varepsilon}_1\Gamma_\mu\Gamma_I\Gamma^{JKL}\varepsilon_2 X^I_c X^J_e X^K_f X^L_g f^{efg}{}_d f^{cdb}{}_a \\
&= -\bar{\varepsilon}_2\Gamma_\mu\Gamma_I(\bar{\Psi}_c\Gamma^I\varepsilon_1\Psi_d-\bar{\Psi}_d\Gamma^I\varepsilon_1\Psi_c)f^{cdb}{}_a \\
&\quad +i\bar{\varepsilon}_2\Gamma_\mu\Gamma_I\Gamma^\nu\Gamma^J\varepsilon_1 X^I_c D_\nu X^J_d f^{cdb}{}_a-i\bar{\varepsilon}_1\Gamma_\mu\Gamma_I\Gamma^\nu\Gamma^J\varepsilon_2 X^I_c D_\nu X^J_d f^{cdb}{}_a \\
&\quad -\frac{i}{6}(\bar{\varepsilon}_2\Gamma_\mu\Gamma_I\Gamma^{JKL}\varepsilon_1-\bar{\varepsilon}_1\Gamma_\mu\Gamma_I\Gamma^{JKL}\varepsilon_2)X^I_c X^J_e X^K_f X^L_g f^{efg}{}_d f^{cdb}{}_a.
\end{aligned}
\tag{3.73}
$$

Expanding the first term in (**3.73**) with Fierz (**A.20**) and rearrenging, we obtain

$$
\begin{aligned}
[\delta_1,\delta_2]\widetilde{A}^b_{\mu a} &= \frac{1}{16}\bar{\varepsilon}_2\Gamma_\mu\Gamma_I\{2(\bar{\Psi}_c\Gamma_\nu\Psi_d)\Gamma^\nu\Gamma^I\varepsilon_1-(\bar{\Psi}_d\Gamma_{LM}\Psi_c)\Gamma^{LM}\Gamma^I\varepsilon_1 \\
&\quad +\frac{1}{4!}(\bar{\Psi}_c\Gamma_\nu\Gamma_{LMNO}\Psi_d)\Gamma^\nu\Gamma^{LMNO}\Gamma^I\varepsilon_1\}f^{cdb}{}_a \\
&\quad +i\bar{\varepsilon}_2(\Gamma_\mu\Gamma_I\Gamma^\nu\Gamma^J-\Gamma^J\Gamma^\nu\Gamma_I\Gamma_\mu)\varepsilon_1 X^I_c D_\nu X^J_d f^{cdb}{}_a \\
&\quad -\frac{i}{3}\bar{\varepsilon}_2\Gamma_\mu\Gamma_I\Gamma^{JKL}\varepsilon_1 X^I_c D_\nu X^J_d f^{cdb}{}_a \\
&= -(\bar{\varepsilon}_2\Gamma^\nu\varepsilon_1)\varepsilon_{\mu\nu\lambda}(\bar{\Psi}_c\Gamma^\lambda\Psi_d)f^{cdb}{}_a+2i\bar{\varepsilon}_2\Gamma^\nu\varepsilon_1\varepsilon_{\mu\nu\lambda}X^I_c D^\lambda X^I_d f^{cdb}{}_a \\
&\quad -2i\bar{\varepsilon}_2\Gamma_{IJ}\varepsilon_1 X^I_c D_\mu X^J_d f^{cdb}{}_a \\
&= 2i(\bar{\varepsilon}_2\Gamma^\nu\varepsilon_1)\varepsilon_{\mu\nu\lambda}(X^I_c D^\lambda X^I_d+\frac{i}{2}\bar{\Psi}_c\Gamma^\lambda\Psi_d)f^{cdb}{}_a \\
&\quad -2i(\bar{\varepsilon}_2\Gamma_{IJ}\varepsilon_1)X^I_c D_\mu X^J_d f^{cdb}{}_a
\end{aligned}
\tag{3.74}
$$

where we used respectively (**A.51**), (**A.52**) and the property that $f^{cdb}{}_a$'s are antisymmetric. Fortunately, $f^{efg}{}_b f^{cdb}{}_a$ term contribute null to commutator since it vanishes as a result of the fundamental identity (**3.6**).

We used also

$$\Gamma^\mu\Gamma^I\Gamma^\nu\Gamma^J + \Gamma^J\Gamma^\nu\Gamma^I\Gamma^\mu = -2\Gamma^\mu\Gamma^\nu g^{IJ} + 2g^{\mu\nu}\Gamma^J\Gamma^I \tag{3.75}$$

and consider again $\Gamma^I\Gamma^J$ as Γ^{IJ} since the multiplication of symmetric part with $X^I_b X^J_c f^{bcd}{}_a$ is zero.

To close the algebra we get the $\widetilde{A}^b_{\mu a}$ equation of motion as:

$$\widetilde{F}^b_{\mu\nu a} + \varepsilon_{\mu\nu\lambda}(X^J_c D^\lambda X^J_d f^{cdb}{}_a + \frac{i}{2}\bar{\Psi}_c\Gamma^\lambda\Psi_d) f^{cdb}{}_a = 0 \tag{3.76}$$

So that, "on shell" the algebra closes as required:

$$[\delta_1,\delta_2]\widetilde{A}^b_{\mu a} = v^\nu \widetilde{F}^b_{\mu\nu a} + D_\mu\widetilde{\Lambda}^b_a. \tag{3.77}$$

Notice that $\widetilde{A}^b_{\mu a}$ contains no local degrees of freedom, as required. It is shown that the supersymmetry algebra closes "on shell" under the global transformations [3].

3.2.2.4 Bosonic equation of motion

The supervariation of the fermionic equation can be used to find the bosonic equation with the help of gauge field equation.

So, to obtain the bosonic equation of motion, we first take the supervariation of the fermionic equation of motion. We found fermion equation of motion (**3.71**) as:

$$\Gamma^\mu D_\mu \Psi_a + \frac{1}{2}\Gamma_{IJ} X^I_c X^J_d \Psi_b f^{cdb}{}_a = 0.$$

The supervariation of this equation gives

$$\begin{aligned} 0 &= \delta(\Gamma^\mu D_\mu \Psi_a + \frac{1}{2}\Gamma_{IJ} X^I_c X^J_d \Psi_b f^{cdb}{}_a) \\ &= \Gamma^\nu \delta(D_\nu \Psi_a) + \frac{1}{2}\Gamma_{IJ}(\delta X^I_c) X^J_d \Psi_b f^{cdb}{}_a + \frac{1}{2}\Gamma_{IJ} X^I_c (\delta X^J_d)\Psi_b f^{cdb}{}_a \\ &\quad + \frac{1}{2}\Gamma_{IJ} X^I_c X^J_d (\delta\Psi_b) f^{cdb}{}_a. \end{aligned} \tag{3.78}$$

It is known that the covariant derivative transforms as:

$$\delta(D_\nu \Psi_a) = D_\nu(\delta\Psi_a) - (\delta\widetilde{A}^b_{\mu a})\Psi_b.$$

Substituting $\delta\Psi_a$ and $\delta\widetilde{A}^b_{\mu a}$ transformations (**3.39**) in this expression, we get

$$\begin{aligned}
\delta(D_\nu \Psi_a) &= D_\nu(D_\mu X^I_a \Gamma^\mu \Gamma^I \varepsilon + \kappa X^I_b X^J_c X^K_d f^{bcd}{}_a \Gamma^{IJK}\varepsilon) \\
&\quad -i\bar{\varepsilon}\Gamma_\mu \Gamma_I X^I_c \Psi_d \Psi_b f^{cdb}{}_a \\
&= D_\nu D_\mu X^I_a \Gamma^\mu \Gamma^I \varepsilon + \kappa(D_\nu X^I_b) X^J_c X^K_d f^{bcd}{}_a \Gamma^{IJK}\varepsilon \\
&\quad +\kappa X^I_b (D_\nu X^J_c) X^K_d f^{bcd}{}_a \Gamma^{IJK}\varepsilon + \kappa X^I_b X^J_c (D_\nu X^K_d) f^{bcd}{}_a \Gamma^{IJK}\varepsilon \\
&\quad -i\bar{\varepsilon}\Gamma_\mu \Gamma_I X^I_c \Psi_d \Psi_b f^{bcd}{}_a \\
&= D_\nu D_\mu X^I_a \Gamma^\mu \Gamma^I \varepsilon + 3\kappa(D_\nu X^I_b) X^J_c X^K_d f^{bcd}{}_a \Gamma^{IJK}\varepsilon \\
&\quad -i\bar{\varepsilon}\Gamma_\mu \Gamma_I X^I_c \Psi_d \Psi_b f^{cdb}{}_a. \qquad \textbf{(3.79)}
\end{aligned}$$

So, the first term in (**3.78**) can be written as:

$$\begin{aligned}
\Gamma^\nu \delta(D_\nu \Psi_a) &= \Gamma^\nu \Gamma^\mu \Gamma^I D_\nu D_\mu X^I_a \varepsilon - \frac{1}{2}\Gamma^\nu \Gamma^{IJK}(D_\nu X^I_b) X^J_c X^K_d \varepsilon f^{bcd}{}_a \\
&\quad -i\bar{\varepsilon}\Gamma^\nu \Gamma_\mu \Gamma_I X^I_c \Psi_d \Psi_b f^{cdb}{}_a. \qquad \textbf{(3.80)}
\end{aligned}$$

Second and third terms in (**3.78**) are

$$\begin{aligned}
\frac{1}{2}\Gamma_{IJ}(\delta X^I_c) X^J_d \Psi_b f^{cdb}{}_a + \frac{1}{2}\Gamma_{IJ} X^I_c (\delta X^J_d)\Psi_b f^{cdb}{}_a &= \Gamma_{IJ}(\delta X^I_c) X^J_d \Psi_b f^{cdb}{}_a \\
&= \Gamma_{IJ}(i\bar{\varepsilon}\Gamma^I \Psi_c) X^J_d \Psi_b f^{cdb}{}_a
\end{aligned} \qquad \textbf{(3.81)}$$

and the last term in (**3.78**) is

$$\begin{aligned}
\frac{1}{2}\Gamma_{IJ} X^I_c X^J_d (\delta\Psi_b) f^{cdb}{}_a &= \frac{1}{2}\Gamma_{IJ} X^I_c X^J_d D_\mu X^K_b \Gamma^\mu \Gamma^K \varepsilon f^{cdb}{}_a \\
&\quad + \frac{\kappa}{2}\Gamma_{IJ}\Gamma^{KLM} X^I_c X^J_d X^K_e X^L_f X^M_g f^{efg}{}_b f^{cdb}{}_a. \qquad \textbf{(3.82)}
\end{aligned}$$

Combining all these expressions, we get

$$\begin{aligned}
0 &= \Gamma^\nu \Gamma^\mu \Gamma^I D_\nu D_\mu X^I_a \varepsilon - \frac{1}{2}\Gamma^\nu \Gamma^{IJK}(D_\nu X^I_b) X^J_c X^K_d \varepsilon f^{bcd}{}_a \\
&\quad -i\bar{\varepsilon}\Gamma^\nu \Gamma_\mu \Gamma_I X^I_c \Psi_d \Psi_b f^{cdb}{}_a + i\Gamma_{IJ}\bar{\varepsilon}\Gamma^I \Psi_c X^d_J \Psi_b f^{cdb}{}_a \\
&\quad + \frac{1}{2}\Gamma_{IJ}\Gamma^\mu \Gamma^K X^I_c X^J_d D_\mu X^K_b \varepsilon f^{cdb}{}_a - \frac{1}{12}\Gamma_{IJ}\Gamma^{KLM} X^I_c X^J_d X^K_e X^L_f X^M_g f^{efg}{}_b f^{cdb}{}_a \varepsilon.
\end{aligned} \qquad \textbf{(3.83)}$$

Now we want to rearrange each term in (**3.83**).

We have shown (**3.32**) that

$$[D_\mu, D_\nu]X^I{}_a = \widetilde{F}^b_{\mu\nu a}X^I_b$$

which can be written as:

$$D_\nu D_\mu X^I{}_a = D_\mu D_\nu X^I{}_a - \widetilde{F}^b_{\mu\nu a}X^I_b. \tag{3.84}$$

Using (**3.84**) and (**A.1**), the first term can be written as:

$$\begin{aligned}
\Gamma^\nu\Gamma^\mu\Gamma^I D_\nu D_\mu X^I{}_a\varepsilon &= -\Gamma^\mu\Gamma^\nu\Gamma^I D_\mu D_\nu X^I_a\varepsilon + 2\Gamma^I D^\nu D_\nu X^I_a\varepsilon \\
&\quad -\Gamma^\nu\Gamma^\mu\Gamma^I \widetilde{F}^b_{\mu\nu a}X^I_b\varepsilon \\
&= -\Gamma^\nu\Gamma^\mu\Gamma^I D_\nu D_\mu X^I_a\varepsilon + 2\Gamma^I D^2 X^I_a\varepsilon \\
&\quad -\Gamma^\nu\Gamma^\mu\Gamma^I \widetilde{F}^b_{\mu\nu a}X^I_b\varepsilon
\end{aligned} \tag{3.85}$$

and we get

$$\Gamma^\nu\Gamma^\mu\Gamma^I D_\nu D_\mu X^I_a\varepsilon = \Gamma^I D^2 X^I_a\varepsilon - \frac{1}{2}\Gamma^\nu\Gamma^\mu\Gamma^I \widetilde{F}^b_{\mu\nu a}X^I_b\varepsilon. \tag{3.86}$$

Second and fifth terms which are similar in (**3.78**) can be written as:

$-\frac{1}{2}\Gamma^\nu\Gamma^{IJK}(D_\nu X^I_b)X^J_c X^K_d f^{bcd}{}_a\varepsilon + \frac{1}{2}\Gamma_{JK}\Gamma^\nu\Gamma^I(D_\nu X^I_b)X^J_c X^K_d f^{bcd}{}_a\varepsilon$

$$\begin{aligned}
&= \frac{1}{2}\Gamma^\nu(-\Gamma^{IJK} + \Gamma_{JK}\Gamma^I)(D_\nu X^I_b)X^J_c X^K_d f^{bcd}{}_a\varepsilon \\
&= \frac{1}{2}\Gamma^\nu(-g^{IJ}\Gamma^K + g^{IK}\Gamma^J)(D_\nu X^I_b)X^J_c X^K_d f^{bcd}{}_a\varepsilon \\
&= -\frac{1}{2}\Gamma^\nu\Gamma^K(D_\nu X^J_b)X^J_c X^K_d f^{bcd}{}_a\varepsilon + \frac{1}{2}\Gamma^\nu\Gamma^K(D_\nu X^J_b)X^K_d X^J_c f^{bdc}{}_a\varepsilon \\
&= -\Gamma^\nu\Gamma^K(D_\nu X^J_b)X^J_c X^K_d f^{bcd}{}_a\varepsilon \\
&= \Gamma^K\Gamma^\nu(D_\nu X^J_b)X^J_c X^K_d f^{bcd}{}_a\varepsilon \\
&= -\Gamma^K\Gamma^\nu X^K_d X^J_c D_\nu X^J_b f^{dcb}{}_a\varepsilon \\
&= -\Gamma^I\Gamma_\lambda X^I_d X^J_c D_\lambda X^J_b f^{bcd}{}_a\varepsilon.
\end{aligned} \tag{3.87}$$

Plugging these into (**3.78**), we obtain

$$
\begin{aligned}
0 \;=\;& \Gamma^I D^2 X^I_a \varepsilon - \frac{1}{2}\Gamma^\nu\Gamma^\mu\Gamma^I \widetilde{F}^b_{\mu\nu a} X^I_b \varepsilon - \Gamma^I\Gamma_\lambda X^I_d X^J_c D_\lambda X^J_b f^{bcd}{}_a \varepsilon \\
& - i\bar{\varepsilon}\Gamma^\nu\Gamma_\mu\Gamma_I X^I_c \Psi_d \Psi_b f^{cdb}{}_a + i\Gamma_{IJ}\bar{\varepsilon}\Gamma^I\Psi_c X^J_d \Psi_b f^{bcd}{}_a \\
& - \frac{1}{12}\Gamma_{IJ}\Gamma^{KLM} X^I_c X^J_d X^K_e X^L_f X^M_g f^{efg}{}_b f^{bcd}{}_a \varepsilon. \qquad (\mathbf{3.88})
\end{aligned}
$$

We now want to deal with the last term in this expression. We use the gamma matrix identity (see **A.29**, **A.47**)

$$
\begin{aligned}
\Gamma_{IJ}\Gamma^{KLM} &= \Gamma_{IJ}(\Gamma^K\Gamma^{LM} - g^{KL}\Gamma^M + g^{KM}\Gamma^L) \\
&= (\Gamma_I\Gamma_J - g_{IJ})[\Gamma^K(\Gamma^L\Gamma^M - g^{LM}) - g^{KL}\Gamma^M + g^{KM}\Gamma^L)] \\
&= \Gamma_I\Gamma_J\Gamma^K\Gamma^L\Gamma^M - \Gamma_I\Gamma_J\Gamma^K g^{LM} - \Gamma_I\Gamma_J g^{KL}\Gamma^M \\
&\quad + \Gamma_I\Gamma_J g^{KM}\Gamma^L g_{IJ}\Gamma^K\Gamma^L\Gamma^M + g_{IJ}\Gamma^K g^{KL} \\
&\quad + g_{IJ} g^{KL}\Gamma^M - g_{IJ} g^{KM}\Gamma^L \qquad (\mathbf{3.89})
\end{aligned}
$$

and make the definition

$$
X^I_c\Gamma^I = X_c. \qquad (\mathbf{3.90})
$$

Then, we see that the terms such as $\Gamma_I\Gamma_J X^I_c X^J_d f^{bcd}{}_a$ vanishes because $f^{bcd}{}_a$ is antisymmetric.

Finally, we get supervariation of fermionic equation of motion as:

$$
\begin{aligned}
0 \;=\;& \Gamma^I(D^2 X^I_a - \frac{i}{2}\bar{\Psi}_c\Gamma^{IJ}X^J_d\Psi_b f^{cdb}{}_a + \frac{1}{2} f^{bcd}{}_a f^{efg}{}_d X^J_b X^K_c X^I_e X^J_f X^K_g)\varepsilon \\
& + \Gamma^I\Gamma_\lambda X^I_b(\frac{1}{2}\varepsilon^{\mu\nu\lambda}\widetilde{F}^b_{\mu\nu a} - X^J_c D^\lambda X^J_d f^{cdb}{}_a - \frac{i}{2}\bar{\Psi}_c\Gamma^\lambda\Psi_d f^{bcd}{}_a)\varepsilon. \qquad (\mathbf{3.91})
\end{aligned}
$$

As a consequence of the vector equation of motion, the last term vanishes and the first term gives the scalar equations of motion,

$$
D^2 X^I_a - \frac{i}{2}\bar{\Psi}_c\Gamma^{IJ}X^J_d\Psi_b f^{cdb}{}_a - \frac{\partial V}{\partial X^{I\alpha}} = 0 \qquad (\mathbf{3.92})
$$

where the potential is

$$
\begin{aligned}
V \;&=\; \frac{1}{12} f^{abcd} f^{efg}{}_d X^I_a X^J_b X^K_c X^I_e X^J_f X^K_g \\
&=\; \frac{1}{2.3!} Tr([X^I, X^J, X^K], [X^I, X^J, X^K]). \qquad (\mathbf{3.93})
\end{aligned}
$$

Let us summarize our calculations and their results. The supersymmetry transformations proposed by Bagger & Lambert are:

$$\begin{aligned}
\delta X^I_a &= i\bar{\varepsilon}\Gamma^I\Psi_a \\
\delta\Psi_a &= D_\mu X^I_a\Gamma^\mu\Gamma^I\varepsilon + \kappa X^I_b X^J_c X^K_d f^{bcd}{}_a\Gamma^{IJK}\varepsilon \\
\delta\widetilde{A}^b_{\mu a} &= i\bar{\varepsilon}\Gamma_\mu\Gamma_I X^I_c\Psi_d f^{cdb}{}_a.
\end{aligned} \tag{3.94}$$

We see that these supersymmetries close into translations and gauge transformations,

$$\begin{aligned}
[\delta_1,\delta_2]X^I_a &= v^\mu\partial_\mu X^I_a + (\widetilde{\Lambda}^b_a - v^\nu\widetilde{A}^b_{\nu a}X^I_b) \\
[\delta_1,\delta_2]\Psi_a &= v^\mu\partial_\mu\Psi_a + (\widetilde{\Lambda}^b_a - v^\nu\widetilde{A}^b_{\nu a}\Psi_b) \\
[\delta_1,\delta_2]\widetilde{A}^b_{\mu a} &= v^\nu\partial_\nu\widetilde{A}^b_{\mu a} + \widetilde{D}_\mu(\widetilde{\Lambda}^b_a - v^\nu\widetilde{A}^b_{\nu a})
\end{aligned} \tag{3.95}$$

after using the equations of motion

$$\begin{aligned}
\Gamma^\mu D_\mu\Psi_a + \frac{1}{2}\Gamma_{IJ}X^I_c X^J_d\Psi_b f^{cdb}{}_a &= 0 \\
D^2X^I_a - \frac{i}{2}\bar{\Psi}_c\Gamma^{IJ}X^J_d\Psi_b f^{cdb}{}_a - \frac{\partial V}{\partial X^{I\alpha}} &= 0 \\
\widetilde{F}^b_{\mu\nu a} + \varepsilon_{\mu\nu\lambda}(X^J_c D^\lambda X^J_d f^{cdb}{}_a + \frac{i}{2}\bar{\Psi}_c\Gamma^\lambda\Psi_d)f^{cdb}{}_a &= 0.
\end{aligned} \tag{3.96}$$

It is openly seen that the supersymmetry variation of the fermion equation of motion vanishes, and that the algebra closes on shell. From here, we see that all the equations of motion are invariant under supersymmetry.

We end this chapter by presenting an action for this system. One can have the equations of motion directly from the Lagrangian:

$$\begin{aligned}
\mathscr{L} = &-\frac{1}{2}(D_\mu X^I_a)(D_\mu X^I_a) + \frac{i}{2}\bar{\Psi}_a\Gamma^\mu D_\mu\Psi_a + \frac{i}{4}\bar{\Psi}_b\Gamma_{IJ}X^I_c X^J_d\Psi_a f^{abcd} - V \\
&+\frac{1}{2}\varepsilon^{\mu\nu\lambda}(f^{abcd}A_{\mu ab}\partial_\nu A_{\lambda cd} + \frac{2}{3}f^{cda}{}_g f^{efgb}A_{\mu ab}A_{\nu cd}A_{\lambda ef}).
\end{aligned} \tag{3.97}$$

The action is gauge invariant and supersymmetric under the transformations mentioned above.

The Bagger-Lambert-Gustavsson theory presents a model that was expected from multiple M2-brane theory. According to this model, the action is invariant under 16 supersymmetries and SO(8) R-symmetry. Besides, it is conformally invariant. These symmetries are the continuous symmetries that are expected of coincident M2-brane worldvolume theory.

4. M5-BRANE ATTEMPT FOR THE EXTENSION OF THE BLG THEORY FORMULATION

In this section, the attempt for the extension of the revolutionary Bagger-Lambert-Gustavsson theory to the M5-brane theory is discussed. As it mentioned before, 3-algebra formalism plays a crucial role in the analysis of the dynamics of the M theory. Since formulation for a system of N coincident M2-branes which can be condensed to a single M5-brane by means of Lie 3-algebra was achieved, it is natural to ask whether it would also be possible to use 3-algebra formalism to formulate a system of interacting M5-branes manifesting (2,0) supersymmetry and the results from M2-branes extend to the case of multiple M5-branes. For this purpose, Lambert and Papageorgakis [10] proposed a set of supersymmetry transformations for non-Abelian (2,0) tensor multiplets in six dimensions, using a 3-algebraic structure. In other words, they obtain a non Abelian system of equations that furnish a representation of the (2,0) supersymmetric tensor multiplet and the result was an interacting system of equations where the gauge structure arises from a 3-algebra. They attained the on-shell conditions are quite restrictive so that the system can be reduced to five dimensional gauge theory along with six-dimensional Abelian (2,0) tensor multiplets. Compared to M2-brane system the formulation of M5-brane system is more complex. Even for the case of a single fivebrane it seems hard to write down a six dimensional action with conformal symmetry because of the selfduality of the three form field strength [29]. Moreover, multiple M5-brane theory is given by a conformal field theory in six dimensions with mutually local electric and magnetic states and no coupling constant. All of these features make the Lagrangian description for M5-brane theory difficult to pacify with a Lagrangian definition [29, 50, 51].

Here, we studied the method in detail which is used to obtain Abelian and non-Abelian equations for the six-dimensional tensor multiplet as the method discussed in chapter 3 to derive Bagger-Lambert-Gustavsson model for multiple M2-branes. We first consider the set of supersymmetry transformations for the Abelian M5-brane and then study the construction of a non-Abelian generalisation proposed by Lambert & Papageorgakis.

Apart from the expected non-Abelian scalars, fermions and the field strength, this model includes a gauge field as well as a non-propagating vector field which transforms nontrivially under the non-Abelian gauge symmetry.

We proceed by checking the closures of the supersymmetry algebra for both Abelian and non-Abelian cases. We see that the superalgebra closes "on shell" up to translations and gauge transformations in case the structure constants are the elements of real 3-algebras, they are totally antisymmetric and obey the fundamental identity. The "on shell" conditions lead to a set of equations of motion and constraints. Finally, expanding this theory around a vacuum point, we see that it reduces to five dimensional super-Yang-Mills along with six-dimensional, Abelian (2,0) tensor multiplets [10].

The supersymmetry transformations for the Abelian case are [52]:

$$\begin{aligned} \delta X^I &= i\bar{\varepsilon}\Gamma^I\Psi \\ \delta\Psi &= \Gamma^\mu\Gamma^I\partial_\mu X^I\varepsilon + \frac{1}{3!}\frac{1}{2}\Gamma^{\mu\nu\lambda}H_{\mu\nu\lambda}\varepsilon \\ \delta B_{\mu\nu} &= i\bar{\varepsilon}\Gamma_{\mu\nu}\Psi \end{aligned} \tag{4.1}$$

where $\mu = 0,.....5$, $I = 6,.....10$ and $H_{\mu\nu\lambda} = 3\partial_{[\mu}B_{\nu\lambda]}$ is selfdual:

$$H_{\mu\nu\lambda A} = \frac{1}{3!}\varepsilon_{\mu\nu\lambda\tau\sigma\rho}H^{\tau\sigma\rho}{}_A.$$

The supersymmetry generator ε is chiral: $\Gamma_{012345}\varepsilon = \varepsilon$ and the fermions Ψ are antichiral: $\Gamma_{012345}\Psi = -\Psi$ [10].

In general, it is preferred to write the algebra purely in terms of $H_{\mu\nu\lambda}$ to $B_{\nu\lambda}$ as:

$$\begin{aligned} \delta X^I &= i\bar{\varepsilon}\Gamma^I\Psi \\ \delta\Psi &= \Gamma^\mu\Gamma^I\partial_\mu X^I\varepsilon + \frac{1}{3!}\frac{1}{2}\Gamma^{\mu\nu\lambda}H_{\mu\nu\lambda}\varepsilon \\ \delta H_{\mu\nu\lambda} &= 3i\bar{\varepsilon}\Gamma_{[\mu\nu}\partial_{\lambda]}\Psi. \end{aligned} \tag{4.2}$$

The scalar fields X^I, the fermions Ψ, and the self-dual field $H_{\mu\nu\lambda}$ form a (2, 0) tensor multiplet in six dimensions.

On the other hand, the six-dimensional (2,0) transformation rules reduce to those of the five-dimensional super-Yang-Mills which are given by

$$\begin{aligned}
\delta X^I &= i\bar{\varepsilon}\Gamma^I\Psi \\
\delta\Psi &= \Gamma^\alpha\Gamma^I D_\alpha X^I\varepsilon + \frac{1}{2}\Gamma^{\alpha\beta}\Gamma^5 F_{\alpha\beta}\varepsilon - \frac{i}{2}[X^I, X^J]\Gamma^{IJ}\Gamma^5\varepsilon \\
\delta A_\alpha &= i\bar{\varepsilon}\Gamma_\alpha\Gamma_5\Psi.
\end{aligned} \qquad \textbf{(4.3)}$$

To find a term analogous to the $[X^I, X^J]$ for the transformation of the fermions in (**4.3**), non-Abelian generalisation requires a new field C^μ. So, the non-Abelian generalisation is given as [10]:

$$\begin{aligned}
\delta X^I{}_A &= i\bar{\varepsilon}\Gamma^I\Psi_A \\
\delta\Psi_A &= \Gamma^\mu\Gamma^I D_\mu X^I{}_A\varepsilon + \frac{1}{3!}\frac{1}{2}\Gamma_{\mu\nu\lambda}H^{\mu\nu\lambda}{}_A\varepsilon - \frac{1}{2}\Gamma_\lambda\Gamma^{IJ}C^\lambda_B X^I_C X^J_D f^{CDB}{}_A\varepsilon \\
\delta H_{\mu\nu\lambda A} &= 3i\bar{\varepsilon}\Gamma_{[\mu\nu}D_{\lambda]}\Psi_A + i\bar{\varepsilon}\Gamma^I\Gamma_{\mu\nu\lambda\kappa}C^\kappa_B X^I_C\Psi_D g^{CDB}{}_A \\
\delta\tilde{A}^B_{\mu A} &= i\bar{\varepsilon}\Gamma_{\mu\lambda}C^\lambda_C\Psi_D h^{CDB}{}_A \\
\delta C^\mu_A &= 0.
\end{aligned} \qquad \textbf{(4.4)}$$

4.1 Abelian (2,0) Tensor Multiplets with 3-Algebras

We first consider the covariant supersymmetry transformations (**4.1**) of a free six-dimensional (2,0) tensor multiplet for the Abelian case. Then, we would like to investigate whether the same algebra can be expanded for the case of non-Abelian fields.

4.1.1 Closure on X^I

We start by checking the closure of supersymmetry algebra on the scalars:

$$\begin{aligned}
[\delta_1, \delta_2]X^I &= \delta_1\delta_2 X^I - \delta_2\delta_1 X^I \\
&= \delta_1(i\bar{\varepsilon}_2\Gamma^I\Psi) - \delta_2(i\bar{\varepsilon}_1\Gamma^I\Psi) \\
&= i\bar{\varepsilon}_2\Gamma^I\Gamma^\mu\Gamma^J\partial_\mu X^J\varepsilon_1 - i\bar{\varepsilon}_1\Gamma^I\Gamma^\mu\Gamma^J\partial_\mu X^J\varepsilon_2 \\
&\quad + \frac{i}{12}(\bar{\varepsilon}_2\Gamma^I\Gamma^{\mu\nu\lambda}H_{\mu\nu\lambda}\varepsilon_1 - \bar{\varepsilon}_1\Gamma^I\Gamma^{\mu\nu\lambda}H_{\mu\nu\lambda}\varepsilon_2)
\end{aligned} \qquad \textbf{(4.5)}$$

Because $\bar{\varepsilon}_2\Gamma^I\Gamma^{\mu\nu\lambda}\varepsilon_1 = \bar{\varepsilon}_1\Gamma^I\Gamma^{\mu\nu\lambda}\varepsilon_2$, the last line of (**4.5**) vanishes (see **A.13**). Then we have (see **A.48**)

$$
\begin{aligned}
[\delta_1, \delta_2]X^I &= i\bar{\varepsilon}_2(\Gamma^I\Gamma^\mu\Gamma^J + \Gamma^J\Gamma^\mu\Gamma^I)\partial_\mu X^J\varepsilon_1 \\
&= 2i\bar{\varepsilon}_2\Gamma^\mu\partial_\mu X^J\varepsilon_1 \\
&= \vartheta^\mu\partial_\mu X^J \qquad (4.6)
\end{aligned}
$$

where $\vartheta^\mu = 2i\bar{\varepsilon}_2\Gamma^\mu\varepsilon_1$.

4.1.2 Closure on Ψ

We then continue with the spinor fields,

$$
[\delta_1, \delta_2]\Psi = \delta_1\delta_2\Psi - \delta_2\delta_1\Psi \qquad (4.7)
$$

We consider each term above seperately.

The first term can be written as follows,

$$
\begin{aligned}
\delta_1\delta_2\Psi &= \delta_1(\Gamma^\mu\Gamma^I\partial_\mu X^I\varepsilon_2 + \frac{1}{12}\Gamma^{\mu\nu\lambda}H_{\mu\nu\lambda}\varepsilon_2) \\
&= \Gamma^\mu\Gamma^I\partial_\mu(\delta_1 X^I)\varepsilon_2 + \frac{1}{12}\Gamma^{\mu\nu\lambda}(\delta_1 H_{\mu\nu\lambda})\varepsilon_2 \\
&= i\Gamma^\mu\Gamma^I(\bar{\varepsilon}_1\Gamma^I\partial_\mu\Psi)\varepsilon_2 + \frac{i}{4}\Gamma^{\mu\nu\lambda}(\bar{\varepsilon}_1\Gamma_{[\mu\nu}\partial_{\lambda]}\Psi)\varepsilon_2 \\
&= i(\bar{\varepsilon}_1\Gamma^I\partial_\mu\Psi)(\Gamma^\mu\Gamma^I\varepsilon_2) + \frac{i}{4}((\bar{\varepsilon}_1\Gamma_{[\mu\nu}\partial_{\lambda]}\Psi)(\Gamma^{\mu\nu\lambda}\varepsilon_2) \qquad (4.8)
\end{aligned}
$$

where $H_{\mu\nu\lambda} = 3\partial_{[\mu}B_{\nu\lambda]}$ and $\delta B_{\mu\nu} = i\bar{\varepsilon}\Gamma_{\mu\nu}\Psi$.
Now we write the second term by replacing $1 \leftrightarrow 2$ in the first term.

$$
\begin{aligned}
\delta_2\delta_1\Psi &= i\Gamma^\mu\Gamma^I(\bar{\varepsilon}_2\Gamma^I\partial_\mu\Psi)\varepsilon_1 + \frac{i}{4}\Gamma^{\mu\nu\lambda}(\bar{\varepsilon}_2\Gamma_{[\mu\nu}\partial_{\lambda]}\Psi)\varepsilon_1 \\
&= i(\bar{\varepsilon}_2\Gamma^I\partial_\mu\Psi)(\Gamma^\mu\Gamma^I\varepsilon_1) + \frac{i}{4}((\bar{\varepsilon}_2\Gamma_{[\mu\nu}\partial_{\lambda]}\Psi)(\Gamma^{\mu\nu\lambda}\varepsilon_1) \qquad (4.9)
\end{aligned}
$$

Combining (**4.8**) and (**4.9**), we obtain $[\delta_1, \delta_2]\Psi$ as follows:

$$
\begin{aligned}
[\delta_1, \delta_2]\Psi = &-i\Gamma^\mu\Gamma^I(\bar{\varepsilon}_2\Gamma^I\partial_\mu\Psi\varepsilon_1 - \bar{\varepsilon}_1\Gamma^I\partial_\mu\Psi\varepsilon_2) \\
&-\frac{i}{4}\Gamma^{\mu\nu\lambda}[(\bar{\varepsilon}_2\Gamma_{[\mu\nu}\partial_{\lambda]}\Psi\varepsilon_1 - \bar{\varepsilon}_1\Gamma_{[\mu\nu}\partial_{\lambda]}\Psi\varepsilon_2)]. \qquad (4.10)
\end{aligned}
$$

We are going to expand this expression with Fierz reordering formula and rearrange with Gamma matrix identities.

Using the Fierz identity in (**A.22**), we obtain

$$\begin{aligned}[\delta_1,\delta_2]\Psi &= \frac{i}{16}\{2(\bar{\varepsilon}_2\Gamma_\rho\varepsilon_1)[\Gamma^\mu\Gamma^I\Gamma^\rho\Gamma^I\partial_\mu\Psi+\frac{1}{4}\Gamma^{\mu\nu\lambda}\Gamma^\rho\Gamma_{\nu\lambda}\partial_\mu\Psi] \\ &-2(\bar{\varepsilon}_2\Gamma_\rho\Gamma^K\varepsilon_1)[\Gamma^\mu\Gamma^I\Gamma^\rho\Gamma^K\Gamma^I\partial_\mu\Psi+\frac{1}{4}\Gamma^{\mu\nu\lambda}\Gamma^\rho\Gamma^K\Gamma_{\nu\lambda}\partial_\mu\Psi] \\ &+\frac{1}{12}(\bar{\varepsilon}_2\Gamma_{\rho\sigma\tau}\Gamma^{KL}\varepsilon_1)[\Gamma^\mu\Gamma^I\Gamma^{\rho\sigma\tau}\Gamma^{KL}\Gamma^I\partial_\mu\Psi \\ &+\frac{1}{4}\Gamma^{\mu\nu\lambda}\Gamma^{\rho\sigma\tau}\Gamma^{KL}\Gamma_{\nu\lambda}\partial_\mu\Psi]\}. \qquad (\mathbf{4.11})\end{aligned}$$

We now make some definitions:

$$\begin{aligned}[\delta_1,\delta_2]\Psi &= \frac{i}{16}\{2(\bar{\varepsilon}_2\Gamma_\rho\varepsilon_1)K-2(\bar{\varepsilon}_2\Gamma_\rho\Gamma^K\varepsilon_1)L \\ &+\frac{1}{12}(\bar{\varepsilon}_2\Gamma_{\rho\sigma\tau}\Gamma^{KL}\varepsilon_1)M \qquad (\mathbf{4.12})\end{aligned}$$

where

$$K = \Gamma^\mu\Gamma^I\Gamma^\rho\Gamma^I\partial_\mu\Psi+\frac{1}{4}\Gamma^{\mu\nu\lambda}\Gamma^\rho\Gamma_{\nu\lambda}\partial_\mu\Psi \qquad (\mathbf{4.13})$$

$$L = \Gamma^\mu\Gamma^I\Gamma^\rho\Gamma^K\Gamma^I\partial_\mu\Psi+\frac{1}{4}\Gamma^{\mu\nu\lambda}\Gamma^\rho\Gamma^K\Gamma_{\nu\lambda}\partial_\mu\Psi \qquad (\mathbf{4.14})$$

$$M = \Gamma^\mu\Gamma^I\Gamma^{\rho\sigma\tau}\Gamma^{KL}\Gamma^I\partial_\mu\Psi+\frac{1}{4}\Gamma^{\mu\nu\lambda}\Gamma^{\rho\sigma\tau}\Gamma^{KL}\Gamma_{\nu\lambda}\partial_\mu\Psi. \qquad (\mathbf{4.15})$$

We are going to simplify those expressions using gamma matrix identities.

To simplifiy K, we need

$$\begin{aligned}\Gamma^\mu\Gamma^I\Gamma^\rho\Gamma^I\partial_\mu\Psi &= 5\Gamma^\rho\Gamma^\mu\partial_\mu\Psi-10g^{\mu\rho}\partial_\mu\Psi \qquad (\mathbf{4.16}) \\ \frac{1}{4}\Gamma^{\mu\nu\lambda}\Gamma^\rho\Gamma_{\nu\lambda}\partial_\mu\Psi &= \Gamma^\rho\Gamma^\mu\partial_\mu\Psi-6g^{\rho\mu}\partial_\mu\Psi. \qquad (\mathbf{4.17})\end{aligned}$$

(see **A.1**, A, **A.30**, **A.41**)

Using them we get

$$K = 6\Gamma^\rho\Gamma^\mu\partial_\mu\Psi-16g^{\rho\mu}\partial_\mu\Psi. \qquad (\mathbf{4.18})$$

We used following expressions to simplify L (see **A.1**, **A.42**, A, **A.30**, **A.41**).

$$\begin{aligned}\Gamma^\mu\Gamma^I\Gamma^\rho\Gamma^K\Gamma^I\partial_\mu\Psi &= -3\Gamma^K\Gamma^\rho\Gamma^\mu\partial_\mu\Psi+6g^{\mu\rho}\Gamma^K\partial_\mu\Psi \\ \frac{1}{4}\Gamma^{\mu\nu\lambda}\Gamma^\rho\Gamma^K\Gamma_{\nu\lambda}\partial_\mu\Psi &= \Gamma^K\Gamma^\rho\Gamma^\mu\partial_\mu\Psi-6g^{\rho\mu}\Gamma^K\partial_\mu\Psi \qquad (\mathbf{4.19})\end{aligned}$$

and we get

$$L = -2\Gamma^K\Gamma^\rho\Gamma^\mu\partial_\mu\Psi. \quad \textbf{(4.20)}$$

Lastly, to simplify M, we need (see **A.42**,**A.31**, **A.34**, **A.57**, **A.41**)

$$\begin{aligned}
\Gamma^\mu\Gamma^I\Gamma^{\rho\sigma\tau}\Gamma^{KL}\Gamma^I\partial_\mu\Psi &= -\Gamma^{KL}\Gamma^\mu\Gamma^{\rho\sigma\tau}\partial_\mu\Psi \\
\frac{1}{4}\Gamma^{\mu\nu\lambda}\Gamma^{\rho\sigma\tau}\Gamma^{KL}\Gamma_{\nu\lambda}\partial_\mu\Psi &= 5\Gamma^{KL}\Gamma^\mu\Gamma^{\rho\sigma\tau}\partial_\mu\Psi + \Gamma^{KL}\Gamma^{\mu\rho\sigma}\Gamma^\tau\partial_\mu\Psi \\
&\quad +\Gamma^{KL}\Gamma^{\mu\rho\tau}\Gamma^\sigma\partial_\mu\Psi + \Gamma^{KL}\Gamma^{\mu\tau\sigma}\Gamma^\rho\partial_\mu\Psi. \quad \textbf{(4.21)}
\end{aligned}$$

Then we get M as follows:

$$\begin{aligned}
M &= 4\Gamma^{KL}\Gamma^\mu\Gamma^{\rho\sigma\tau}\partial_\mu\Psi + \Gamma^{KL}\Gamma^{\mu\rho\sigma}\Gamma^\tau\partial_\mu\Psi \\
&\quad +\Gamma^{KL}\Gamma^{\mu\rho\tau}\Gamma^\sigma\partial_\mu\Psi + \Gamma^{KL}\Gamma^{\mu\tau\sigma}\Gamma^\rho\partial_\mu\Psi \\
&= -4\Gamma^{KL}\Gamma^{\rho\sigma\tau}\Gamma^\mu\partial_\mu\Psi + 24\Gamma^{KL}g^{\mu[\rho}\Gamma^{\sigma\tau]}\partial_\mu\Psi + \Gamma^{KL}\Gamma^{\mu\rho\sigma}\Gamma^\tau\partial_\mu\Psi \\
&\quad +\Gamma^{KL}\Gamma^{\mu\rho\tau}\Gamma^\sigma\partial_\mu\Psi + \Gamma^{KL}\Gamma^{\mu\tau\sigma}\Gamma^\rho\partial_\mu\Psi \\
&= 0. \quad \textbf{(4.22)}
\end{aligned}$$

Combining all these expressions in $[\delta_1,\delta_2]\Psi$ and simplifying, we obtain

$$\begin{aligned}
[\delta_1,\delta_2]\Psi &= -2i(\bar{\varepsilon}_2\Gamma^\mu\varepsilon_1)\partial_\mu\Psi + \frac{3i}{4}\left(\bar{\varepsilon}_2\Gamma_\rho\varepsilon_1\right)\Gamma^\rho(\Gamma^\mu\partial_\mu\Psi) \\
&\quad +\frac{i}{4}\left(\bar{\varepsilon}_2\Gamma_\rho\Gamma^K\varepsilon_1\right)\Gamma^K\Gamma^\rho(\Gamma^\mu\partial_\mu\Psi). \quad \textbf{(4.23)}
\end{aligned}$$

Therefore the algebra closes on-shell with the equation of motion

$$\Gamma^\mu\partial_\mu\Psi = 0 \quad \textbf{(4.24)}$$

where we define $-2i(\bar{\varepsilon}_2\Gamma^\mu\varepsilon_1) = \vartheta^\mu$.

Finally, we get the closure of fermion algebra as follows,

$$[\delta_1,\delta_2]\Psi = \vartheta^\mu\partial_\mu\Psi. \quad \textbf{(4.25)}$$

This algebra closes "on shell" up to translation, with (**4.25**).

4.1.3 Closure on $H_{\mu\nu\lambda}$

We then continue with $H_{\mu\nu\lambda}$.

$$[\delta_1,\delta_2]H_{\mu\nu\lambda} \quad = \quad \delta_1\delta_2 H_{\mu\nu\lambda} - \delta_2\delta_1 H_{\mu\nu\lambda} \tag{4.26}$$

We first consider

$$\begin{aligned}
\delta_1(\delta_2 H_{\mu\nu\lambda}) &= 3i\delta_1(\bar{\varepsilon}_2\Gamma_{[\mu\nu}\partial_{\lambda]}\Psi) \\
&= 3i\bar{\varepsilon}_2\Gamma_{[\mu\nu}\partial_{\lambda]}\delta_1\Psi \\
&= 3i\bar{\varepsilon}_2\Gamma_{[\mu\nu}\partial_{\lambda]}(\Gamma^\rho\Gamma^I\partial_\rho X^I\varepsilon_1 + \frac{1}{3!}\frac{1}{2}\Gamma^{\rho\sigma\tau}H_{\rho\sigma\tau}\varepsilon_1) \\
&= i\bar{\varepsilon}_2\Gamma_{\mu\nu}\Gamma^\rho\Gamma^I\varepsilon_1\partial_\lambda\partial_\rho X^I\varepsilon_1 - \text{a cyclic (in }\mu\nu\lambda) \\
&\quad + \frac{i}{12}\bar{\varepsilon}_2\Gamma_{\mu\nu}\Gamma^{\rho\sigma\tau}\varepsilon_1\partial_\lambda H_{\rho\sigma\tau} - \text{a cyclic (in }\mu\nu\lambda).
\end{aligned} \tag{4.27}$$

The first term of (**4.27**) will contribute following result to $[\delta_1,\delta_2]H_{\mu\nu\lambda}$ (see **A.13**):

$$\begin{aligned}
\bar{\varepsilon}_2\Gamma_{\mu\nu}\Gamma^\rho\Gamma^I\varepsilon_1 - \bar{\varepsilon}_1\Gamma_{\mu\nu}\Gamma^\rho\Gamma^I\varepsilon_2 &= \bar{\varepsilon}_2(\Gamma^\rho_{\mu\nu} + g^\rho_\nu\Gamma_\mu - g^\rho_\mu\Gamma_\nu)\Gamma^I\varepsilon_1 \\
&\quad - \bar{\varepsilon}_1(\Gamma^\rho_{\mu\nu} + g^\rho_\nu\Gamma_\mu - g^\rho_\mu\Gamma_\nu)\Gamma^I\varepsilon_2 \\
&= 2g^\rho_\nu\bar{\varepsilon}_2\Gamma_\mu\Gamma^I\varepsilon_1 - 2g^\rho_\mu\bar{\varepsilon}_2\Gamma_\nu\Gamma^I\varepsilon_1.
\end{aligned} \tag{4.28}$$

So, terms that contain $\partial_\rho X^I$ in (**4.26**) :

$$\begin{aligned}
& i\bar{\varepsilon}_2\Gamma_{\mu\nu}\Gamma^\rho\Gamma^I\varepsilon_1\partial_\lambda\partial_\rho X^I - i\bar{\varepsilon}_2\Gamma_{\mu\nu}\Gamma^\rho\Gamma^I\varepsilon_1\partial_\nu\partial_\rho X^I - i\bar{\varepsilon}_2\Gamma_{\lambda\nu}\Gamma^\rho\Gamma^I\varepsilon_1\partial_\mu\partial_\rho X^I \\
& - i\bar{\varepsilon}_1\Gamma_{\mu\nu}\Gamma^\rho\Gamma^I\varepsilon_2\partial_\lambda\partial_\rho X^I + i\bar{\varepsilon}_1\Gamma_{\mu\nu}\Gamma^\rho\Gamma^I\varepsilon_2\partial_\nu\partial_\rho X^I + i\bar{\varepsilon}_1\Gamma_{\lambda\nu}\Gamma^\rho\Gamma^I\varepsilon_2\partial_\mu\partial_\rho X^I \\
& = 0.
\end{aligned} \tag{4.29}$$

As a result, closure of $H_{\mu\nu\lambda}$ is given as:

$$\begin{aligned}
[\delta_1,\delta_2]H_{\mu\nu\lambda} &= \frac{i}{4}(\bar{\varepsilon}_2\Gamma_{[\mu\nu}\Gamma^{\rho\sigma\tau}\varepsilon_1\partial_{\lambda]}H_{\rho\sigma\tau} - \bar{\varepsilon}_1\Gamma_{[\mu\nu}\Gamma^{\rho\sigma\tau}\varepsilon_2\partial_{\lambda]}H_{\rho\sigma\tau}) \\
&= i(\bar{\varepsilon}_2\Gamma^\rho\varepsilon_1)\partial_\lambda H_{\mu\nu\rho}.
\end{aligned} \tag{4.30}$$

(see **A.56**)

To obtain the equation of motion for X^I and $H_{\mu\nu\lambda}$, we take the supervariation of fermionic equation of motion,

$$\begin{aligned}
0 &= \delta(\Gamma^\mu\partial_\mu\Psi) = \Gamma^\mu\partial_\mu(\delta\Psi_a) \\
&= \Gamma^\mu\partial_\mu(\Gamma^\nu\Gamma^I\partial_\nu X^I\varepsilon) + \Gamma^\mu\partial_\mu\left(\frac{1}{3!}\frac{1}{2}\Gamma^{\rho\sigma\tau}H_{\rho\sigma\tau}\varepsilon\right).
\end{aligned} \tag{4.31}$$

Therefore,

$$0 = \Gamma^{\mu}\Gamma^{\nu}\Gamma^{I}\varepsilon\partial_{\mu}\partial_{\nu}X^{I} + \frac{1}{3!}\frac{1}{2}\Gamma^{\mu}\Gamma^{\rho\sigma\tau}\varepsilon\partial_{\mu}H_{\rho\sigma\tau}. \quad (\mathbf{4.32})$$

Exchanging the name of the indices ($\mu \leftrightarrow \nu$) in the first term:

$$0 = \Gamma^{\nu}\Gamma^{\mu}\Gamma^{I}\varepsilon\partial_{\nu}\partial_{\mu}X^{I} + \frac{1}{3!}\frac{1}{2}\Gamma^{\mu}\Gamma^{\rho\sigma\tau}\varepsilon\partial_{\mu}H_{\rho\sigma\tau}. \quad (\mathbf{4.33})$$

Adding (**4.32**) to (**4.33**), we get

$$0 = 2\Gamma^{I}\varepsilon\partial_{\mu}\partial^{\mu}X^{I} + \frac{1}{3!}\Gamma^{\mu}\Gamma^{\rho\sigma\tau}\varepsilon\partial_{\mu}H_{\rho\sigma\tau}. \quad (\mathbf{4.34})$$

Multiplying (**4.34**) with $\bar{\varepsilon}$ from left, we find

$$0 = 2\bar{\varepsilon}\Gamma^{I}\varepsilon\partial_{\mu}\partial^{\mu}X^{I} + \frac{1}{3!}\bar{\varepsilon}\Gamma^{\mu}\Gamma^{\rho\sigma\tau}\varepsilon\partial_{\mu}H_{\rho\sigma\tau}. \quad (\mathbf{4.35})$$

Recall that: $\bar{\varepsilon}\Gamma^{\mu}\varepsilon = 0$, but $\bar{\varepsilon}\Gamma^{I}\varepsilon \neq 0$ or $\bar{\varepsilon}\Gamma^{\mu\rho\sigma\tau}\varepsilon \neq 0$

Therefore, we obtain

$$\begin{aligned} \partial_{\mu}\partial^{\mu}X^{I} &= 0 \\ &\text{and} \\ \partial_{[\mu}H_{\rho\sigma\tau]} &= 0. \end{aligned} \quad (\mathbf{4.36})$$

Continuing with the second term in (**4.27**), we can write

$$\Gamma_{[\mu\nu}\Gamma^{\rho\sigma\tau}\partial_{\lambda]}H_{\rho\sigma\tau} = 2\Gamma_{[\mu}\Gamma_{\nu}\Gamma^{\rho\sigma\tau}\partial_{\lambda]}H_{\rho\sigma\tau} - 2g_{[\mu\nu}\Gamma^{\rho\sigma\tau}\partial_{\lambda]}H_{\rho\sigma\tau}. \quad (\mathbf{4.37})$$

The second term of (**4.37**) gives zero contribution and we get

$$\begin{aligned} \Gamma_{[\mu\nu}\Gamma^{\rho\sigma\tau}\partial_{\lambda]}H_{\rho\sigma\tau} &= 2\Gamma_{[\mu}\Gamma_{\nu}\Gamma^{\rho\sigma\tau}\partial_{\lambda]}H_{\rho\sigma\tau} + 2\Gamma_{[\mu}g^{\rho}_{\nu}\Gamma^{\sigma\tau}\partial_{\lambda]}H_{\rho\sigma\tau} \\ &\quad -2\Gamma_{[\mu}g^{\sigma}_{\nu}\Gamma^{\rho\tau}\partial_{\lambda]}H_{\rho\sigma\tau} + 2\Gamma_{[\mu}g^{\tau}_{\nu}\Gamma^{\rho\sigma}\partial_{\lambda]}H_{\rho\sigma\tau}. \end{aligned} \quad (\mathbf{4.38})$$

(Excluding zero contributions)

Thus, we obtain

$$\begin{aligned} \Gamma_{[\mu\nu}\Gamma^{\rho\sigma\tau}\partial_{\lambda]}H_{\rho\sigma\tau} &= 2\Gamma_{[\mu\nu}{}^{\rho\sigma\tau}\partial_{\lambda]}H_{\rho\sigma\tau} - 12\Gamma^{\tau}\partial_{[\lambda}H_{\mu\nu]\tau} \\ &= 2\Gamma_{\mu\nu}{}^{\rho\sigma\tau}\partial_{[\lambda}H_{\rho\sigma\tau]} - 12\Gamma^{\tau}\partial_{[\lambda}H_{\mu\nu]\tau}. \end{aligned} \quad (\mathbf{4.39})$$

Second term gives translation contribution and first term leads to $\partial_{[\lambda}H_{\rho\sigma]} = 0$ should be an equation of motion.

This result justifies vanishing of each term of (**4.36**) separately.

4.2 Non-Abelian (2,0) Tensor Multiplets with 3-Algebras

Now, we wish to apply the same method as in the Abelian case to find suitable non-Abelian equations of motion for the six dimensional tensor multiplet. As we mentioned before, to get a term analogous to the $[X^I, X^J]$ for $\delta\Psi$ above, it is needed to introduce a Γ_μ matrix to account for the fact that ε and Ψ have opposite chirality. As a result of this requirement, non-Abelian generalisation includes the existence of a new field C^μ so that [10]:

$$\begin{aligned}
\delta X^I{}_A &= i\bar{\varepsilon}\Gamma^I\Psi_A \\
\delta\Psi_A &= \Gamma^\mu\Gamma^I D_\mu X^I{}_A\varepsilon + \frac{1}{3!}\frac{1}{2}\Gamma^{\mu\nu\lambda}H_{\mu\nu\lambda}\varepsilon - \frac{1}{2}\Gamma_\lambda\Gamma^{IJ}C^\lambda_B X^I_C X^J_D f^{CDB}{}_A\varepsilon \\
\delta H_{\mu\nu\lambda A} &= 3i\bar{\varepsilon}\Gamma_{[\mu\nu}D_{\lambda]}\Psi_A + i\bar{\varepsilon}\Gamma^I\Gamma_{\mu\nu\lambda\kappa}C^\kappa_B X^I_C\Psi_D g^{CDB}{}_A \\
\delta\tilde{A}^B_{\mu A} &= i\bar{\varepsilon}\Gamma_{\mu\lambda}C^\lambda_C\Psi_D h^{CDB}{}_A \\
\delta C^\mu_A &= 0.
\end{aligned} \qquad (\mathbf{4.40})$$

4.2.1 Closure on X^I_A

We first check the closure of supersymmetry using the supersymmetry transformations of X^I_A field:

$$[\delta_1, \delta_2]X^I_A = \delta_1\delta_2 X^I_A - \delta_2\delta_1 X^I_A. \qquad (\mathbf{4.41})$$

The first term is found as:

$$\begin{aligned}
\delta_1\delta_2 X^I_A &= \delta_1\left(i\bar{\varepsilon}_2\Gamma^I\Psi_A\right) \\
&= i\bar{\varepsilon}_2\Gamma^I\left(\delta_1\Psi_A\right) \\
&= i\bar{\varepsilon}_2\Gamma^I\Gamma^\mu\Gamma^J D_\mu X^J_A\varepsilon_1 + \frac{i}{12}\bar{\varepsilon}_2\Gamma^I\Gamma_{\mu\nu\lambda}H^{\mu\nu\lambda}{}_A\varepsilon_1 \\
&\quad -\frac{i}{2}\bar{\varepsilon}_2\Gamma^I\Gamma_\lambda\Gamma^{JK}C^\lambda_B X^J_C X^K_D f^{CDB}{}_A\varepsilon_1.
\end{aligned} \qquad (\mathbf{4.42})$$

The second term is obtained analogously,

$$\delta_2\delta_1 X_A^I = i\bar{\varepsilon}_1\Gamma^I\Gamma^\mu\Gamma^J D_\mu X_A^J\varepsilon_2 + \frac{i}{12}\bar{\varepsilon}_1\Gamma^I\Gamma_{\mu\nu\lambda}H^{\mu\nu\lambda}{}_A\varepsilon_2 - \frac{i}{2}\bar{\varepsilon}_1\Gamma^I\Gamma_\lambda\Gamma^{JK}C_B^\lambda X_C^J X_D^K f^{CDB}{}_A\varepsilon_2. \quad (\mathbf{4.43})$$

Combining them we get

$$\begin{aligned}[\delta_1,\delta_2]X_A^I &= i\bar{\varepsilon}_2(\Gamma^I\Gamma^\mu\Gamma^J + \Gamma^J\Gamma^\mu\Gamma^I)D_\mu X_A^J\varepsilon_1 \\ &\quad + \frac{i}{12}(\bar{\varepsilon}_2\Gamma^I\Gamma_{\mu\nu\lambda}\varepsilon_1 - \bar{\varepsilon}_1\Gamma^I\Gamma_{\mu\nu\lambda}\varepsilon_2)H^{\mu\nu\lambda}{}_A \\ &\quad - \frac{i}{2}\bar{\varepsilon}_2(\Gamma^I\Gamma_\lambda\Gamma^{JK} + \Gamma^{JK}\Gamma_\lambda\Gamma^I)C_B^\lambda X_C^J X_D^K f^{CDB}{}_A\varepsilon_1. \end{aligned} \quad (\mathbf{4.44})$$

Second line of (**4.44**) vanishes, because $\bar{\varepsilon}_2\Gamma^I\Gamma_{\mu\nu\lambda}\varepsilon_1 = \bar{\varepsilon}_1\Gamma^I\Gamma_{\mu\nu\lambda}\varepsilon_2$ (see **A.13**). The gamma matrix structure in the first and third terms can be simplified as follows,

$$\begin{aligned}[\delta_1,\delta_2]X_A^I &= -2i\bar{\varepsilon}_2\Gamma^\mu\varepsilon_1 D_\mu X_A^I + i\bar{\varepsilon}_2 g^{IJ}\Gamma_\lambda\Gamma^K C_B^\lambda X_C^J X_D^K f^{CDB}{}_A\varepsilon_1 \\ &\quad - i\bar{\varepsilon}_2 g^{IK}\Gamma_\lambda\Gamma^J C_B^\lambda X_C^J X_D^K f^{CDB}{}_A\varepsilon_1. \end{aligned} \quad (\mathbf{4.45})$$

We substitue (**A.48**) and (**A.50**) in (**4.45**).

Thus, we get the following expression.

$$\begin{aligned}[\delta_1,\delta_2]X_A^I &= -2i\bar{\varepsilon}_2\Gamma^\mu\varepsilon_1 D_\mu X_A^I + 2i\bar{\varepsilon}_2\Gamma_\lambda\Gamma^J\varepsilon_1 C_D^\lambda X_B^I X_C^J f^{CDB}{}_A \\ &= v^\mu D_\mu X_A^I + \widetilde{\Lambda}_A^B X_B^I \end{aligned} \quad (\mathbf{4.46})$$

where $v^\mu = -2i(\bar{\varepsilon}_2\Gamma^\mu\varepsilon_1)$ and $\widetilde{\Lambda}_A^B = 2i\left(\bar{\varepsilon}_2\Gamma_\lambda\Gamma^J\varepsilon_1\right)C_D^\lambda X_C^J f^{CDB}{}_A$.

We find that the transformations close into a translation and a gauge transformation.

4.2.2 Closure on C_A^μ

Then, we investigate the closure on the new field C_A^μ. It is clear that from the transformation $\delta C_A^\mu = 0$ such that

$$\begin{aligned}[\delta_1,\delta_2]C_A^\mu &= \delta_1\delta_2 C_A^\mu - \delta_2\delta_1 C_A^\mu \\ &= 0. \end{aligned} \quad (\mathbf{4.47})$$

But what we expect is, to close the algebra in the form of (**4.45**)

$$[\delta_1, \delta_2] C^\mu_A = v^\nu D_\nu C^\mu_A + \widetilde{\Lambda}^B_A C^\mu_B. \quad \textbf{(4.48)}$$

This algebra closes on shell providing the constraints $D_\nu C^\mu_A = 0$ and $C^\lambda_B C^\rho_C f^{CDB}{}_A = 0$. We can also get the second constraint from supersymmetrising the first constraint.

4.2.3 Closure on $\widetilde{A}^B_{\mu A}$

We then continue with the closure on the gauge field.

$$[\delta_1, \delta_2] \widetilde{A}^B_{\mu A} = \delta_1 \delta_2 \widetilde{A}^B_{\mu A} - \delta_2 \delta_1 \widetilde{A}^B_{\mu A} \quad \textbf{(4.49)}$$

The first term is calculated as follows,

$$\begin{aligned}
\delta_1 \delta_2 \widetilde{A}^B_{\mu A} &= \delta_1 (i\bar{\varepsilon}_2 \Gamma_{\mu\lambda} C^\lambda_C \Psi_D h^{CDB}{}_A) \qquad\qquad (\delta C^\lambda_C = 0) \\
&= i\bar{\varepsilon}_2 \Gamma_{\mu\lambda} \Gamma^\nu \Gamma^I C^\lambda_C D_\nu X^I_D h^{CDB}{}_A \varepsilon_1 \\
&\quad + \frac{i}{12} \bar{\varepsilon}_2 \Gamma_{\mu\lambda} \Gamma_{\nu\rho\sigma} C^\lambda_C H^{\nu\rho\sigma}{}_D h^{CDB}{}_A \varepsilon_1 \\
&\quad - \frac{i}{2} \bar{\varepsilon}_2 \Gamma_{\mu\lambda} \Gamma_\sigma \Gamma^{IJ} C^\lambda_C C^\sigma_G X^I_E X^J_F f^{EFG}{}_D h^{CDB}{}_A \varepsilon_1.
\end{aligned} \quad \textbf{(4.50)}$$

And the second term is calculated analogously,

$$\begin{aligned}
\delta_2 \delta_1 \widetilde{A}^B_{\mu A} &= i\bar{\varepsilon}_1 \Gamma_{\mu\lambda} \Gamma^\nu \Gamma^I C^\lambda_C D_\nu X^I_D h^{CDB}{}_A \varepsilon_2 \\
&\quad + \frac{i}{12} \bar{\varepsilon}_1 \Gamma_{\mu\lambda} \Gamma_{\nu\rho\sigma} C^\lambda_C H^{\nu\rho\sigma}{}_D h^{CDB}{}_A \varepsilon_2 \\
&\quad - \frac{i}{2} \bar{\varepsilon}_1 \Gamma_{\mu\lambda} \Gamma_\sigma \Gamma^{IJ} C^\lambda_C C^\sigma_G X^I_E X^J_F f^{EFG}{}_D h^{CDB}{}_A \varepsilon_2.
\end{aligned} \quad \textbf{(4.51)}$$

Combining them we obtain,

$$\begin{aligned}
[\delta_1, \delta_2] \widetilde{A}^B_{\mu A} &= i\bar{\varepsilon}_2 \left(\Gamma_{\mu\lambda} \Gamma^\nu \Gamma^I + \Gamma^I \Gamma^\nu \Gamma_{\mu\lambda} \right) C^\lambda_C D_\nu X^I_D h^{CDB}{}_A \varepsilon_1 \\
&\quad + \frac{i}{12} \bar{\varepsilon}_2 \left(\Gamma_{\mu\lambda} \Gamma_{\nu\rho\sigma} + \Gamma_{\nu\rho\sigma} \Gamma_{\mu\lambda} \right) C^\lambda_C H^{\nu\rho\sigma}{}_D h^{CDB}{}_A \varepsilon_1 \\
&\quad - \frac{i}{2} \bar{\varepsilon}_2 \left(\Gamma_{\mu\lambda} \Gamma_\sigma \Gamma^{IJ} + \Gamma^{IJ} \Gamma_\sigma \Gamma_{\mu\lambda} \right) C^\lambda_C C^\sigma_G X^I_E X^J_F f^{EFG}{}_D h^{CDB}{}_A \varepsilon_1.
\end{aligned} \quad \textbf{(4.52)}$$

Each gamma matrix structure in parentheses can be calculated with the help of following Gamma Matrix identities (see **A.30**, **A.56**):

$$\begin{aligned}
\left[\Gamma^{\nu},\Gamma_{\mu\lambda}\right] &= 2g^{\nu}_{\mu}\Gamma_{\lambda}-2g^{\nu}_{\lambda}\Gamma_{\mu} \\
\left\{\Gamma_{\mu\lambda},\Gamma_{\nu\rho\sigma}\right\} &= 2g_{\lambda\nu}g_{\mu\rho}\Gamma_{\sigma}-2g_{\lambda\nu}g_{\mu\sigma}\Gamma_{\rho}+2g_{\lambda\rho}g_{\mu\sigma}\Gamma_{\nu}-2g_{\lambda\rho}g_{\mu\nu}\Gamma_{\sigma} \\
&\quad +2g_{\lambda\sigma}g_{\mu\nu}\Gamma_{\rho}-2g_{\lambda\sigma}g_{\mu\rho}\Gamma_{\nu} \\
\left\{\Gamma_{\mu\lambda}\Gamma_{\sigma},\Gamma_{\sigma}\Gamma_{\mu\lambda}\right\} &= 2\Gamma_{\mu\lambda\sigma}.
\end{aligned}$$

Using these identities, we get

$$\begin{aligned}
[\delta_1,\delta_2]\widetilde{A}^{B}_{\mu A} &= i\bar{\varepsilon}_2\Gamma^{I}(2g^{\nu}_{\mu}\Gamma_{\lambda}-2g^{\nu}_{\lambda}\Gamma_{\mu})\varepsilon_1 C^{\lambda}_{C}D_{\nu}X^{I}_{D}h^{CDB}{}_{A} \\
&\quad +\frac{i}{12}\bar{\varepsilon}_2(24\Gamma^{\nu}H_{\mu\nu\lambda D})\varepsilon_1 C^{\lambda}_{C}h^{CDB}{}_{A} \\
&\quad -i(\bar{\varepsilon}_2\Gamma_{\mu\lambda\sigma}\Gamma^{IJ}\varepsilon_1)C^{\lambda}_{C}C^{\sigma}_{G}X^{I}_{E}X^{J}_{F}f^{EFG}{}_{D}h^{CDB}{}_{A} \\
&= -2i\left(\bar{\varepsilon}_2\Gamma_{\lambda}\Gamma^{I}\varepsilon_1\right)C^{\lambda}_{C}D_{\mu}X^{I}_{D}h^{CDB}{}_{A} && \textbf{(4.53)} \\
&\quad +2i\left(\bar{\varepsilon}_2\Gamma_{\mu}\Gamma^{I}\varepsilon_1\right)C^{\lambda}_{C}D_{\lambda}X^{I}_{D}h^{CDB}{}_{A}-v^{\nu}C^{\lambda}_{C}H_{\mu\nu\lambda D}h^{CDB}{}_{A} && \textbf{(4.54)} \\
&\quad -i\left(\bar{\varepsilon}_2\Gamma_{\mu\nu\lambda}\Gamma^{IJ}\varepsilon_1\right)C^{\nu}_{C}C^{\lambda}_{G}X^{I}_{E}X^{J}_{F}f^{EFG}{}_{D}h^{CDB}{}_{A} && \textbf{(4.55)} \\
&= v^{\nu}\widetilde{F}^{B}_{\mu\nu A}+D_{\mu}\widetilde{\Lambda}^{B}_{A}. && \textbf{(4.56)}
\end{aligned}$$

We have written what $[\delta_1,\delta_2]\widetilde{A}^{B}_{\mu A}$ should amount to in the last expression above.

$\widetilde{F}^{B}_{\mu\nu A}$ is defined as,

$$\widetilde{F}^{B}_{\mu\nu A} = \partial_{\nu}\widetilde{A}^{B}_{\mu A}-\partial_{\mu}\widetilde{A}^{B}_{\nu A}-\widetilde{A}^{B}_{\mu C}\widetilde{A}^{C}_{\nu A}+\widetilde{A}^{B}_{\nu C}\widetilde{A}^{C}_{\mu A}. \qquad \textbf{(4.57)}$$

First term (**4.53**) implies that $h^{CDB}{}_{A} = f^{DBC}{}_{A}$ to provide the correct gauge transformation.

Therefore, we see that $v^{\nu}C^{\lambda}_{C}H_{\mu\nu\lambda D}h^{CDB}{}_{A}$ term is a translation so that

$$\widetilde{F}^{B}_{\mu\nu A}=C^{\lambda}_{C}H_{\mu\nu\lambda D}f^{BDC}{}_{A} \qquad \textbf{(4.58)}$$

and the (**4.55**) gives,

$$C^{\nu}_{C}C^{\lambda}_{G}f^{EFG}{}_{D}f^{BDC}{}_{A}=0. \qquad \textbf{(4.59)}$$

Thus, we see that $f^{ABC}{}_{D}$ must satisfy fundamental identity for 3- algebras and as a result (**4.59**) follows from $C^{\lambda}_{B}C^{\rho}_{C}f^{CDB}{}_{A}=0$.

For the first term of (**4.54**) line, we can write

$$C_C^{\lambda} D_{\lambda} X_D^I f^{BDC}{}_A = 0. \tag{4.60}$$

4.2.4 Closure on $H_{\mu\nu\lambda A}$

Then, we investigate the closure on the antisymmetric tensor field-strength $H_{\mu\nu\lambda A}$;

$$\begin{aligned}
[\delta_1, \delta_2] H_{\mu\nu\lambda A} &= v^{\rho} D_{\rho} H_{\mu\nu\lambda A} - 2i(\bar{\varepsilon}_2 \Gamma_{\rho} \Gamma^I \varepsilon_1) C_C^{\rho} X_D^I g^{DBC}{}_A H_{\mu\nu\lambda B} \\
&\quad -6i(\bar{\varepsilon}_2 \Gamma_{[\mu} \Gamma^I \varepsilon_1)(\widetilde{F}^C_{\nu\lambda]A} - C_B^{\rho} H_{\nu\lambda]\rho D} g^{CDB}{}_A) X_C^I \\
&\quad -6i(\bar{\varepsilon}_2 \Gamma_{\rho[\mu\nu} \Gamma^{IJ} \varepsilon_1) C_B^{\rho} X_C^I D_{\lambda]} X_D^J (f^{CDB}{}_A - g^{CDB}{}_A) \\
&\quad -\frac{3i}{8}(\bar{\varepsilon}_2 \Gamma^{\sigma} \Gamma^J \varepsilon_1)(\bar{\Psi}_C \Gamma_{\mu\nu\lambda\rho\sigma} \Gamma^J \Psi_D) C_B^{\rho} (h^{DBC}{}_A - g^{CDB}{}_A) \\
&\quad +2i(\bar{\varepsilon}_2 \Gamma^{\tau} \Gamma^K \varepsilon_1) \varepsilon_{\mu\nu\lambda\rho\sigma\tau} C_B^{\rho} C_E^{\sigma} X_C^I X_F^J X_G^K g^{D[B]C}{}_A f^{FG[E]}{}_D \\
&\quad +i(\bar{\varepsilon}_2 \Gamma_{\mu\nu\lambda} \Gamma_{LM} \varepsilon_1) \varepsilon^{IJKLM} C_B^{\kappa} C_{\kappa E} X_C^I X_F^J X_G^K g^{DB[C}{}_A f^{FG]E}{}_D \\
&\quad +3i(\bar{\varepsilon}_2 \Gamma_{\rho[\mu\nu} \Gamma_{LM} \varepsilon_1) \varepsilon^{IJKLM} C_B^{\rho} C_{\lambda]E} X_C^I X_F^J X_G^K g^{DB[C}{}_A f^{FG]E}{}_D \\
&\quad +v^{\rho}(4D_{[\mu} H_{\nu\lambda\rho]A} + \varepsilon_{\mu\nu\lambda\rho\sigma\tau} C_B^{\sigma} X_C^I D^{\tau} X_D^I g^{CDB}{}_A \\
&\quad +\frac{i}{2} \varepsilon_{\mu\nu\lambda\rho\sigma\tau} C_B^{\sigma} \bar{\Psi}_C \Gamma^{\tau} \Psi_D g^{CDB}{}_A \\
&= v^{\rho} D_{\rho} H_{\mu\nu\lambda A} + \tilde{\Lambda}_A^B H_{\mu\nu\lambda B}
\end{aligned} \tag{4.61}$$

where in the last line again we have given the required expression for the closure of the algebra as before.

According to above expression, it can be seen that the second term of the first line gives the correct gauge transformation if the relation

$$g^{CDB}{}_A = f^{CDB}{}_A \tag{4.62}$$

is satisfied.

Also it can be seen that the second and third lines vanish for the correct gauge transformation. The fourth line vanishes if the relation

$$h^{DBC}{}_A = g^{CDB}{}_A \tag{4.63}$$

is satisfied.

Obtained the previous conditions this implies that $f^{CDB}{}_{A} = -f^{CBD}{}_{A}$ and thus $f^{CDB}{}_{A}$ is totally antisymmetric in C, D, B. As in the case of multiple M2-branes, consistency of the gauge symmetries $\tilde{\Lambda}^{B}_{A}$ implies that the structure constants satisfy the fundamental identity [10] :

$$f^{[ABC}{}_{E}f^{D]EF}{}_{G} = 0. \qquad \textbf{(4.64)}$$

Considering that the seventh line vanishes gives the H-equation of motion as:

$$\begin{aligned} &D_{[\mu}H_{\nu\lambda\rho]A} + \tfrac{1}{4}\varepsilon_{\mu\nu\lambda\rho\sigma\tau}C^{\sigma}_{B}X^{I}_{C}D^{\tau}X^{I}_{D}f^{CDB}{}_{A} \\ &\quad + \tfrac{i}{8}\varepsilon_{\mu\nu\lambda\rho\sigma\tau}C^{\sigma}_{B}\bar{\Psi}_{C}\Gamma^{\tau}\Psi_{D}f^{CDB}{}_{A} = 0. \end{aligned} \qquad \textbf{(4.65)}$$

Furthermore, it can be seen that the Bianchi identity $D_{[\lambda}\widetilde{F}^{A}_{\mu\nu]B} = 0$, along with the H-equation of motion.

This implies that

$$C^{\rho}_{C}D_{\rho}H_{\mu\nu\lambda D}f^{CDB}{}_{A} = 0. \qquad \textbf{(4.66)}$$

It could be introduced a field $B_{\mu\nu A}$ such that $H_{\mu\nu\lambda} = 3\partial_{[\mu}B_{\nu\lambda]}$. This result would lead to the algebraic constraint [10]:

$$\begin{aligned} &\widetilde{F}^{B}_{[\mu\nu A}B_{\lambda\rho]B} + \tfrac{1}{6}\varepsilon_{\mu\nu\lambda\rho\sigma\tau}C^{\sigma}_{B}X^{I}_{C}D^{\tau}X^{I}_{D}f^{CDB}{}_{A} \\ &\quad + \tfrac{i}{12}\varepsilon_{\mu\nu\lambda\rho\sigma\tau}C^{\sigma}_{B}\bar{\Psi}_{C}\Gamma^{\tau}\Psi_{D}f^{CDB}{}_{A} = 0, \end{aligned} \qquad \textbf{(4.67)}$$

but this over-constrains the fields and so there cannot exist a suitable $B_{\mu\nu A}$ as we will verify this in the next steps.

We check that the closure of the antisymmetric tensor field-strength $H_{\mu\nu\lambda A}$ as required in case the equations of motion and algebraic constraints above are satisfied. $H_{\mu\nu\lambda A}$ is self-dual, are invariant under the six dimensional (2,0) supersymmetry transformations [10].

4.2.5 Closure on Ψ_A

Let us look at the closure on the fermions. We expect to find that the algebra closes into a translation and a gauge transformation. Applying the ideas of the previous calculation, we will get the closure on the fermions to gauge this symmetry:

$$[\delta_1, \delta_2]\Psi_A = \delta_1\delta_2\Psi_A - \delta_2\delta_1\Psi_A. \tag{4.68}$$

We calculate first term as follows.

$$\begin{aligned}
\delta_1\delta_2\Psi_A &= \delta_1(\Gamma^\mu\Gamma^I D_\mu X^I_A\varepsilon_2 + \frac{1}{12}\Gamma_{\mu\nu\lambda}H^{\mu\nu\lambda}{}_A\varepsilon_2 - \frac{1}{2}\Gamma_\lambda\Gamma^{IJ}C^\lambda_B X^I_C X^J_D f^{CDB}{}_A\varepsilon_2) \\
&= i\Gamma^\mu\Gamma^I\bar{\varepsilon}_1\Gamma^I D_\mu\Psi_A\varepsilon_2 - i\Gamma^\mu\Gamma^I\bar{\varepsilon}_1\Gamma_{\mu\lambda}C^\lambda_C\Psi_D h^{CDB}{}_A X^I_B\varepsilon_2 \\
&\quad + \frac{i}{4}\Gamma^{\mu\nu\lambda}\bar{\varepsilon}_1\Gamma_{[\mu\nu}D_{\lambda]}\Psi_A\varepsilon_2 + \frac{i}{12}\Gamma^{\mu\nu\lambda}\bar{\varepsilon}_1\Gamma^I\Gamma_{\mu\nu\lambda\kappa}C^\kappa_B X^I_C\Psi_D g^{CDB}{}_A\varepsilon_2 \\
&\quad - \frac{i}{2}\Gamma_\lambda\Gamma^{IJ}C^\lambda_B\bar{\varepsilon}_1\Gamma^I\Psi_C X^J_D f^{CDB}{}_A\varepsilon_2 \\
&\quad - \frac{i}{2}\Gamma_\lambda\Gamma^{IJ}C^\lambda_B X^I_C\bar{\varepsilon}_1\Gamma^J\Psi_D f^{CDB}_A\varepsilon_2
\end{aligned} \tag{4.69}$$

And the second term is written as

$$\begin{aligned}
\delta_2\delta_1\Psi_A &= i\Gamma^\mu\Gamma^I\bar{\varepsilon}_2\Gamma^I D_\mu\Psi_A\varepsilon_1 - i\Gamma^\mu\Gamma^I\bar{\varepsilon}_2\Gamma_{\mu\lambda}C^\lambda_C\Psi_D h^{CDB}{}_A X^I_B\varepsilon_1 \\
&\quad + \frac{i}{4}\Gamma^{\mu\nu\lambda}\bar{\varepsilon}_2\Gamma_{[\mu\nu}D_{\lambda]}\Psi_A\varepsilon_1 + \frac{i}{12}\Gamma^{\mu\nu\lambda}\bar{\varepsilon}_2\Gamma^I\Gamma_{\mu\nu\lambda\kappa}C^\kappa_B X^I_C\Psi_D g^{CDB}{}_A\varepsilon_1 \\
&\quad - \frac{i}{2}\Gamma_\lambda\Gamma^{IJ}C^\lambda_B\bar{\varepsilon}_2\Gamma^I\Psi_C X^J_D f^{CDB}{}_A\varepsilon_1 \\
&\quad - \frac{i}{2}\Gamma_\lambda\Gamma^{IJ}C^\lambda_B X^I_C\bar{\varepsilon}_2\Gamma^J\Psi_D f^{CDB}_A\varepsilon_1.
\end{aligned} \tag{4.70}$$

Combining them we obtain

$$\begin{aligned}
[\delta_1, \delta_2]\Psi_A &= -i\Gamma^\mu\Gamma^I\left(\bar{\varepsilon}_2\Gamma^I D_\mu\Psi_A\varepsilon_1 - \bar{\varepsilon}_1\Gamma^I D_\mu\Psi_A\varepsilon_2\right) \\
&\quad + i\Gamma^\mu\Gamma^I(\bar{\varepsilon}_2\Gamma_{\mu\lambda}C^\lambda_C\Psi_D h^{CDB}{}_A X^I_B\varepsilon_1 \\
&\quad - \bar{\varepsilon}_1\Gamma_{\mu\lambda}C^\lambda_C\Psi_D h^{CDB}{}_A X^I_B\varepsilon_2) \\
&\quad - \frac{i}{4}\Gamma^{\mu\nu\lambda}\left(\bar{\varepsilon}_2\,\Gamma_{[\mu\nu}D_{\lambda]}\Psi_A\varepsilon_1 - \bar{\varepsilon}_1\Gamma_{[\mu\nu}D_{\lambda]}\Psi_A\varepsilon_2\right) \\
&\quad - \frac{i}{12}\Gamma^{\mu\nu\lambda}(\bar{\varepsilon}_2\Gamma^I\Gamma_{\mu\nu\lambda\kappa}C^\kappa_B X^I_C\Psi_D g^{CDB}{}_A\varepsilon_1 \\
&\quad - \bar{\varepsilon}_1\Gamma^I\Gamma_{\mu\nu\lambda\kappa}C^\kappa_B X^I_C\Psi_D g^{CDB}{}_A\varepsilon_2) \\
&\quad + \frac{i}{2}\Gamma_\lambda\Gamma^{IJ}C^\lambda_B(\bar{\varepsilon}_2\Gamma^I\Psi_C X^J_D f^{CDB}{}_A\varepsilon_1 - \bar{\varepsilon}_1\Gamma^I\Psi_C X^J_D f^{CDB}{}_A\varepsilon_2) \\
&\quad + \frac{i}{2}\Gamma_\lambda\Gamma^{IJ}C^\lambda_B X^I_C\left(\bar{\varepsilon}_2\Gamma^J\Psi_D f^{CDB}{}_A\varepsilon_1 - \bar{\varepsilon}_1\Gamma^J\Psi_D f^{CDB}{}_A\varepsilon_2\right).
\end{aligned} \tag{4.71}$$

We now name each term of $[\delta_1,\delta_2]\,\Psi_A$ as:

$$A \;=\; -i\Gamma^\mu\Gamma^I\left(\bar{\varepsilon}_2\Gamma^I D_\mu\Psi_A\varepsilon_1-\bar{\varepsilon}_1\Gamma^I D_\mu\Psi_A\varepsilon_2\right) \tag{4.72}$$

$$B \;=\; i\Gamma^\mu\Gamma^I\left(\bar{\varepsilon}_2\Gamma_{\mu\lambda}C^\lambda_C\Psi_D h^{CDB}{}_A X^I_B\varepsilon_1-\bar{\varepsilon}_1\Gamma_{\mu\lambda}C^\lambda_C\Psi_D h^{CDB}{}_A X^I_B\varepsilon_2\right) \tag{4.73}$$

$$C \;=\; -\frac{i}{4}\Gamma^{\mu\nu\lambda}\left(\bar{\varepsilon}_2\,\Gamma_{[\mu\nu}D_{\lambda]}\Psi_A\varepsilon_1-\bar{\varepsilon}_1\Gamma_{[\mu\nu}D_{\lambda]}\Psi_A\varepsilon_2\right) \tag{4.74}$$

$$\begin{aligned} D \;=\;& -\frac{i}{12}\Gamma^{\mu\nu\lambda}(\bar{\varepsilon}_2\Gamma^I\Gamma_{\mu\nu\lambda\kappa}C^\kappa_B X^I_C\Psi_D g^{CDB}{}_A\varepsilon_1\\ &-\bar{\varepsilon}_1\Gamma^I\Gamma_{\mu\nu\lambda\kappa}C^\kappa_B X^I_C\Psi_D g^{CDB}{}_A\varepsilon_2) \end{aligned} \tag{4.75}$$

$$E \;=\; \frac{i}{2}\Gamma_\lambda\Gamma^{IJ}C^\lambda_B\left(\bar{\varepsilon}_2\Gamma^I\Psi_C X^J_D f^{CDB}{}_A\varepsilon_1-\bar{\varepsilon}_1\Gamma^I\Psi_C X^J_D f^{CDB}{}_A\varepsilon_2\right) \tag{4.76}$$

$$F \;=\; \frac{i}{2}\Gamma_\lambda\Gamma^{IJ}C^\lambda_B X^I_C\left(\bar{\varepsilon}_2\Gamma^J\Psi_D f^{CDB}{}_A\varepsilon_1-\bar{\varepsilon}_1\Gamma^J\Psi_D f^{CDB}{}_A\varepsilon_2\right). \tag{4.77}$$

We rearrenge each term above with Fierzing (**A.22**). As a result, we obtain

$$\begin{aligned} A \;=\;& \frac{i}{16}[2\left(\bar{\varepsilon}_2\Gamma_\nu\varepsilon_1\right)\Gamma^\mu\Gamma^I\Gamma^\nu\Gamma^I D_\mu\Psi_A\\ &-2\left(\bar{\varepsilon}_2\Gamma_\nu\Gamma^J\varepsilon_1\right)\Gamma^\mu\Gamma^I\Gamma^\nu\Gamma^J\Gamma^I D_\mu\Psi_A\\ &+\frac{1}{12}\left(\bar{\varepsilon}_2\Gamma_{\nu\lambda\sigma}\Gamma^{JK}\varepsilon_1\right)\Gamma^\mu\Gamma^I\Gamma^{\nu\lambda\sigma}\Gamma^{JK}\Gamma^I D_\mu\Psi_A] \end{aligned} \tag{4.78}$$

$$\begin{aligned} B \;=\;& \frac{i}{16}[2\left(\bar{\varepsilon}_2\Gamma_\nu\varepsilon_1\right)\left(-\Gamma^\mu\Gamma^I\Gamma^\nu\Gamma_{\mu\kappa}C^\kappa_C\Psi_D h^{CDB}{}_A X^I_B\right)\\ &-2\left(\bar{\varepsilon}_2\Gamma_\nu\Gamma^J\varepsilon_1\right)\left(-\Gamma^\mu\Gamma^I\Gamma^\nu\Gamma^J\Gamma_{\mu\kappa}C^\kappa_C\Psi_D h^{CDB}{}_A X^I_B\right)\\ &+\frac{1}{12}\left(\bar{\varepsilon}_2\Gamma_{\nu\lambda\sigma}\Gamma^{JK}\varepsilon_1\right)\left(-\Gamma^\mu\Gamma^I\Gamma^{\nu\lambda\sigma}\Gamma^{JK}\Gamma_{\mu\kappa}C^\kappa_C\Psi_D h^{CDB}{}_A X^I_B\right)] \end{aligned} \tag{4.79}$$

$$\begin{aligned} C \;=\;& \frac{i}{16}[2\left(\bar{\varepsilon}_2\Gamma_\nu\varepsilon_1\right)(\frac{1}{4}\Gamma^{\rho\gamma\tau}\Gamma^\nu\Gamma_{[\rho\gamma}D_{\tau]}\Psi_A)\\ &-2\left(\bar{\varepsilon}_2\Gamma_\nu\Gamma^J\varepsilon_1\right)(\frac{1}{4}\Gamma^{\rho\gamma\tau}\Gamma^\nu\Gamma^J\Gamma_{[\rho\gamma}D_{\tau]}\Psi_A)\\ &+\frac{1}{12}(\bar{\varepsilon}_2\Gamma_{\nu\lambda\sigma}\Gamma^{JK}\varepsilon_1)(\frac{1}{4}\Gamma^{\rho\gamma\tau}\Gamma^{\nu\lambda\sigma}\Gamma^{JK}\Gamma_{[\rho\gamma}D_{\tau]}\Psi_A)] \end{aligned} \tag{4.80}$$

$$D = \frac{i}{16}[2\,(\bar{\varepsilon}_2\Gamma_\nu\varepsilon_1)\,(\frac{1}{12}\Gamma^{\rho\sigma\tau}\Gamma^\nu\Gamma^I\Gamma_{\rho\sigma\tau\kappa}C^\kappa_B X^I_C\Psi_D g^{CDB}{}_A)$$
$$-2\left(\bar{\varepsilon}_2\Gamma_\nu\Gamma^J\varepsilon_1\right)(\frac{1}{12}\Gamma^{\rho\sigma\tau}\Gamma^\nu\Gamma^J\Gamma^I\Gamma_{\rho\sigma\tau\kappa}C^\kappa_B X^I_C\Psi_D g^{CDB}{}_A)$$
$$+\frac{1}{12}\left(\bar{\varepsilon}_2\Gamma_{\nu\lambda\sigma}\Gamma^{JK}\varepsilon_1\right)(\frac{1}{12}\Gamma^{\rho\sigma\tau}\Gamma^{\nu\lambda\sigma}\Gamma^{JK}\Gamma^I\Gamma_{\rho\sigma\tau\kappa}C^\kappa_B X^I_C\Psi_D g^{CDB}{}_A)] \quad \textbf{(4.81)}$$

$$E = \frac{i}{16}[2\,(\bar{\varepsilon}_2\Gamma_\nu\varepsilon_1)\,(-\frac{1}{2}\Gamma_\kappa\Gamma^{IM}C^\kappa_B\Gamma^\nu\Gamma^I\Psi_C X^M_D f^{CDB}{}_A)$$
$$-2\left(\bar{\varepsilon}_2\Gamma_\nu\Gamma^J\varepsilon_1\right)(-\frac{1}{2}\Gamma_\kappa\Gamma^{IM}C^\kappa_B\Gamma^\nu\Gamma^J\Gamma^I\Psi_C X^M_D f^{CDB}{}_A)$$
$$+\frac{1}{12}\left(\bar{\varepsilon}_2\Gamma_{\nu\lambda\sigma}\Gamma^{JK}\varepsilon_1\right)(-\frac{1}{2}\Gamma_\kappa\Gamma^{IM}C^\kappa_B\Gamma^{\nu\lambda\sigma}\Gamma^{JK}\Gamma^I\Psi_C X^M_D f^{CDB}{}_A)] \quad \textbf{(4.82)}$$

$$F = \frac{i}{16}[2\,(\bar{\varepsilon}_2\Gamma_\nu\varepsilon_1)\,(-\frac{1}{2}\Gamma_\kappa\Gamma^{IM}C^\kappa_B X^I_C\Gamma^\nu\Gamma^M\Psi_D f^{CDB}{}_A)$$
$$-2\left(\bar{\varepsilon}_2\Gamma_\nu\Gamma^J\varepsilon_1\right)(-\frac{1}{2}\Gamma_\kappa\Gamma^{IM}C^\kappa_B X^I_C\Gamma^\nu\Gamma^J\Gamma^M\Psi_D f^{CDB}{}_A)$$
$$+\frac{1}{12}\left(\bar{\varepsilon}_2\Gamma_{\nu\lambda\sigma}\Gamma^{JK}\varepsilon_1\right)(-\frac{1}{2}\Gamma_\kappa\Gamma^{IM}C^\kappa_B X^I_C\Gamma^{\nu\lambda\sigma}\Gamma^{JK}\Gamma^M\Psi_D f^{CDB}{}_A)]. \quad \textbf{(4.83)}$$

Combining all these Fierz reordered terms, we obtain

$$[\delta_1,\delta_2]\Psi_A = \frac{i}{16}[2\,(\bar{\varepsilon}_2\Gamma_\nu\varepsilon_1)A' - 2\left(\bar{\varepsilon}_2\Gamma_\nu\Gamma^J\varepsilon_1\right)B' + \frac{1}{12}\left(\bar{\varepsilon}_2\Gamma_{\nu\lambda\sigma}\Gamma^{JK}\varepsilon_1\right)C' \quad \textbf{(4.84)}$$

where A',B' and C' are given as

$$A' = \Gamma^\mu\Gamma^I\Gamma^\nu\Gamma^I D_\mu\Psi_A - \Gamma^\mu\Gamma^I\Gamma^\nu\Gamma_{\mu\kappa}C^\kappa_C\Psi_D h^{CDB}{}_A X^I_B$$
$$+\frac{1}{4}\Gamma^{\rho\gamma\tau}\Gamma^\nu\Gamma_{[\rho\gamma}D_{\tau]}\Psi_A + \frac{1}{12}\Gamma^{\rho\gamma\tau}\Gamma^\nu\Gamma^I\Gamma_{\rho\gamma\tau\kappa}C^\kappa_B X^I_C\Psi_D g^{CDB}{}_A$$
$$-\frac{1}{2}\Gamma_\kappa\Gamma^{IM}C^\kappa_B\Gamma^\nu\Gamma^I\Psi_C X^M_D f^{CDB}{}_A - \frac{1}{2}\Gamma_\kappa\Gamma^{IM}C^\kappa_B X^I_C\Gamma^\nu\Gamma^M\Psi_D f^{CDB}{}_A \quad \textbf{(4.85)}$$

$$B' = \Gamma^\mu\Gamma^I\Gamma^\nu\Gamma^J\Gamma^I D_\mu\Psi_A - \Gamma^\mu\Gamma^I\Gamma^\nu\Gamma^J\Gamma_{\mu\kappa}C^\kappa_C\Psi_D X^I_B h^{CDB}{}_A$$
$$+\frac{1}{4}\Gamma^{\rho\gamma\tau}\Gamma^\nu\Gamma^J\Gamma_{[\rho\gamma}D_{\tau]}\Psi_A + \frac{1}{12}\Gamma^{\rho\gamma\tau}\Gamma^\nu\Gamma^J\Gamma^I\Gamma_{\rho\gamma\tau\kappa}C^\kappa_B X^I_C\Psi_D g^{CDB}{}_A$$
$$-\frac{1}{2}\Gamma_\kappa\Gamma^{IM}C^\kappa_B\Gamma^\nu\Gamma^J\Gamma^I\Psi_C X^M_D f^{CDB}{}_A$$
$$-\frac{1}{2}\Gamma_\kappa\Gamma^{IM}C^\kappa_B X^I_C\Gamma^\nu\Gamma^J\Gamma^M\Psi_D f^{CDB}{}_A \quad \textbf{(4.86)}$$

$$
\begin{aligned}
C' &= \Gamma^\mu\Gamma^I\Gamma^{\nu\lambda\sigma}\Gamma^{JK}\Gamma^I D_\mu\Psi_A - \Gamma^\mu\Gamma^I\Gamma^{\nu\lambda\sigma}\Gamma^{JK}\Gamma_{\mu\kappa}C^\kappa_C\Psi_D h^{CDB}{}_A X^I_B \\
&\quad +\frac{1}{4}\Gamma^{\rho\gamma\tau}\Gamma^{\nu\lambda\sigma}\Gamma^{JK}\Gamma_{[\rho\gamma}D_{\tau]}\Psi_A + \frac{1}{12}\Gamma^{\rho\gamma\tau}\Gamma^{\nu\lambda\sigma}\Gamma^{JK}\Gamma^I\Gamma_{\rho\gamma\tau\kappa}C^\kappa_B X^I_C\Psi_D g^{CDB}{}_A \\
&\quad -\frac{1}{2}\Gamma_\kappa\Gamma^{IM}C^\kappa_B\Gamma^{\nu\lambda\sigma}\Gamma^{JK}\Gamma^I\Psi_C X^M_D f^{CDB}{}_A \\
&\quad -\frac{1}{2}\Gamma_\kappa\Gamma^{IM}C^\kappa_B X^I_C\Gamma^{\nu\lambda\sigma}\Gamma^{JK}\Gamma^M\Psi_D f^{CDB}{}_A. \qquad \textbf{(4.87)}
\end{aligned}
$$

Let us calculate each term in A' separately.

The first and second terms are: (see **A.1**, A, **A.30**, **A.41**)

$$
\Gamma^\mu\Gamma^I\Gamma^\nu\Gamma^I D_\mu\Psi_A = 5\Gamma^\nu\Gamma^\mu D_\mu\Psi_A - 10g^{\mu\nu}D_\mu\Psi_A, \qquad \textbf{(4.88)}
$$

$-\Gamma^\mu\Gamma^I\Gamma^\nu\Gamma_{\mu\kappa}C^\kappa_C\Psi_D X^I_B h^{CDB}{}_A =$

$$
\begin{aligned}
&= \Gamma^I\left(-\Gamma^\nu\Gamma^\mu + 2g^{\mu\nu}\right)\Gamma_{\mu\kappa}C^\kappa_C\Psi_D X^I_B h^{CDB}{}_A \\
&= -5\Gamma^I\Gamma^\nu\Gamma_\kappa C^\kappa_C\Psi_D X^I_B h^{CDB}{}_A + 2\Gamma^I\Gamma^\nu_\kappa C^\kappa_C\Psi_D X^I_B h^{CDB}{}_A \\
&= +3\Gamma^\nu\Gamma^I\Gamma_\kappa C^\kappa_C\Psi_D X^I_B f^{CDB}{}_A - 2g^\nu_\kappa\Gamma^I C^\kappa_C\Psi_D X^I_B f^{CDB}{}_A. \qquad \textbf{(4.89)}
\end{aligned}
$$

Third term is:

$$
\begin{aligned}
\frac{1}{4}\Gamma^{\rho\gamma\tau}\Gamma^\nu\Gamma_{[\rho\gamma}D_{\tau]}\Psi_A &= \frac{1}{24}\left(-\Gamma^\nu\Gamma^{\rho\gamma\tau} + 2g^{\nu\rho}\Gamma^{\gamma\tau} + 2g^{\nu\tau}\Gamma^{\rho\gamma} + 2g^{\nu\gamma}\Gamma^{\tau\rho}\right) \\
&\quad .\left(2\Gamma_{\rho\gamma}D_\tau + 2\Gamma_{\gamma\tau}D_\rho + 2\Gamma_{\tau\rho}D_\gamma\right)\Psi_A \\
&= +\frac{5}{3}\Gamma^\nu\Gamma^\tau D_\tau\Psi_A + \frac{5}{3}\Gamma^\nu\Gamma^\rho D_\rho\Psi_A + \frac{5}{3}\Gamma^\nu\Gamma^\gamma D_\gamma\Psi_A \\
&\quad -\frac{2}{3}\Gamma^\nu\Gamma^\tau D_\tau\Psi_A + \frac{3}{2}g^{\tau\nu}D_\tau\Psi_A - 5g^{\nu\rho}D_\rho\Psi_A \\
&\quad -\frac{2}{3}\Gamma^\nu\Gamma^\gamma D_\gamma\Psi_A + \frac{3}{2}g^{\gamma\nu}D_\gamma\Psi_A - 5g^{\nu\tau}D_\tau\Psi_A \\
&\quad -\frac{2}{3}\Gamma^\nu\Gamma^\rho D_\rho\Psi_A + \frac{3}{2}g^{\rho\nu}D_\rho\Psi_A - \frac{2}{3}\Gamma^\nu\Gamma^\gamma D_\gamma\Psi_A \\
&\quad +\frac{3}{2}g^{\gamma\nu}D_\gamma\Psi_A - \frac{2}{3}\Gamma^\nu\Gamma^\tau D_\tau\Psi_A + \frac{3}{2}g^{\tau\nu}D_\tau\Psi_A \\
&\quad -\frac{2}{3}\Gamma^\nu\Gamma^\rho D_\rho\Psi_A + \frac{3}{2}g^{\rho\nu}D_\rho\Psi_A - 5g^{\nu\gamma}D_\gamma\Psi_A \\
&= \Gamma^\nu\Gamma^\rho D_\rho\Psi_A - 6g^{\rho\nu}D_\rho\Psi_A. \qquad \textbf{(4.90)}
\end{aligned}
$$

We used (**A.31**, **A.41**) above.

Continuing with fourth term, we get (see **A.31**, **A.41**)

$$\begin{aligned}\Gamma^{\rho\gamma\tau}\Gamma^{\nu}\Gamma^{I}\Gamma_{\rho\gamma\tau\kappa}C^{\kappa}_{B}X^{I}_{C}\Psi_{D}g^{CDB}{}_{A} &= -12\Gamma^{I}\Gamma^{\nu}\Gamma_{\kappa}C^{\kappa}_{B}X^{I}_{C}\Psi_{D}g^{CDB}{}_{A} \\ &\quad +72g^{\nu}{}_{\kappa}C^{\kappa}_{B}X^{I}_{C}\Psi_{D}g^{CDB}{}_{A} \qquad (\mathbf{4.91})\end{aligned}$$

and the last two terms can be arrenged as (**A.35**)

$$\begin{aligned}-\frac{1}{2}\Gamma_{\kappa}\Gamma^{IM}C^{\kappa}_{B}\Gamma^{\nu}\Gamma^{I}\Psi_{C}X^{M}_{D}f^{CDB}{}_{A} &- \frac{1}{2}\Gamma_{\kappa}\Gamma^{IM}C^{\kappa}_{B}X^{I}_{C}\Gamma^{\nu}\Gamma^{M}\Psi_{D}f^{CDB}{}_{A} \\ &= -\frac{1}{2}(\Gamma_{\kappa}\Gamma^{IM}\Gamma^{\nu}\Gamma^{I}\Psi_{C}X^{M}_{D}C^{\kappa}_{B}f^{CDB}{}_{A} \\ &+ \Gamma_{\kappa}\Gamma^{IM}\Gamma^{\nu}\Gamma^{M}\Psi_{D}X^{I}_{C}C^{\kappa}_{B}f^{CDB}{}_{A}). \qquad (\mathbf{4.92})\end{aligned}$$

Replacing $I \leftrightarrow M$, $C \leftrightarrow D$ and taking into account the antisymmetry of $f^{CDB}{}_{A}$ for the second term in (**4.92**) expression above, we obtain

$$\begin{aligned}-\frac{1}{2}\Gamma_{\kappa}\Gamma^{IM}C^{\kappa}_{B}\Gamma^{\nu}\Gamma^{I}\Psi_{C}X^{M}_{D}f^{CDB}{}_{A} &- \frac{1}{2}\Gamma_{\kappa}\Gamma^{IM}C^{\kappa}_{B}X^{I}_{C}\Gamma^{\nu}\Gamma^{M}\Psi_{D}f^{CDB}{}_{A} \\ &= -\Gamma_{\kappa}\Gamma^{\nu}\Gamma^{IM}\Gamma^{I}\Psi_{C}X^{M}_{D}C^{\kappa}_{B}f^{CDB}{}_{A}. \qquad (\mathbf{4.93})\end{aligned}$$

Using $\Gamma^{IM}\Gamma^{I} = -4\Gamma^{M}$ (**A.35**) above, we get the following expression,

$$-\frac{1}{2}\Gamma_{\kappa}\Gamma^{IM}C^{\kappa}_{B}\Gamma^{\nu}\Gamma^{I}\Psi_{C}X^{M}_{D}f^{CDB}{}_{A} - \frac{1}{2}\Gamma_{\kappa}\Gamma^{IM}C^{\kappa}_{B}X^{I}_{C}\Gamma^{\nu}\Gamma^{M}\Psi_{D}f^{CDB}{}_{A} =$$

$$\begin{aligned}&= -4\Gamma^{M}\Gamma^{\nu}\Gamma_{\kappa}C^{\kappa}_{B}\Psi_{C}X^{M}_{D}f^{CDB}{}_{A} + 8g^{\nu}{}_{\kappa}\Gamma^{M}C^{\kappa}_{B}\Psi_{C}X^{M}_{D}f^{CDB}{}_{A} \\ &= 4\Gamma^{\nu}\Gamma^{I}\Gamma_{\kappa}C^{\kappa}_{B}\Psi_{C}X^{I}_{D}f^{CDB}{}_{A} + 8g^{\nu}{}_{\kappa}\Gamma^{I}C^{\kappa}_{B}\Psi_{C}X^{I}_{D}f^{CDB}{}_{A}. \qquad (\mathbf{4.94})\end{aligned}$$

Putting all these expressions in A' expression, we obtain

$$A' = 6\Gamma^{\nu}\Gamma^{\mu}D_{\mu}\Psi_{A} - 16g^{\mu\nu}D_{\mu}\Psi_{A} + 6\Gamma^{\nu}\Gamma^{I}\Gamma_{\kappa}C^{\kappa}_{C}\Psi_{D}X^{I}_{B}f^{CDB}{}_{A}. \qquad (\mathbf{4.95})$$

Now we calculate the second term B' (**4.86**):

$$\begin{aligned}B' &= \Gamma^{\mu}\Gamma^{I}\Gamma^{\nu}\Gamma^{J}\Gamma^{I}D_{\mu}\Psi_{A} - \Gamma^{\mu}\Gamma^{I}\Gamma^{\nu}\Gamma^{J}\Gamma_{\mu\kappa}C^{\kappa}_{C}\Psi_{D}X^{I}_{B}h^{CDB}{}_{A} \\ &\quad +\frac{1}{4}\Gamma^{\rho\gamma\tau}\Gamma^{\nu}\Gamma^{J}\Gamma_{[\rho\gamma}D_{\tau]}\Psi_{A} + \frac{1}{12}\Gamma^{\rho\gamma\tau}\Gamma^{\nu}\Gamma^{J}\Gamma^{I}\Gamma_{\rho\gamma\tau\kappa}C^{\kappa}_{B}X^{I}_{C}\Psi_{D}g^{CDB}{}_{A} \\ &\quad -\frac{1}{2}\Gamma_{\kappa}\Gamma^{IM}C^{\kappa}_{B}\Gamma^{\nu}\Gamma^{J}\Gamma^{I}\Psi_{C}X^{M}_{D}f^{CDB}{}_{A} - \frac{1}{2}\Gamma_{\kappa}\Gamma^{IM}C^{\kappa}_{B}X^{I}_{C}\Gamma^{\nu}\Gamma^{J}\Gamma^{M}\Psi_{D}f^{CDB}{}_{A}.\end{aligned}$$

Every term can be arrenged with the help of Gamma matrix identities as is done for A' above.

First term is written as follows (see **A.42**),

$$\Gamma^{\mu}\Gamma^{I}\Gamma^{\nu}\Gamma^{J}\Gamma^{I}D_{\mu}\Psi_{A} \;=\; 3\Gamma^{\nu}\Gamma^{J}\Gamma^{\mu}D_{\mu}\Psi_{A}+6g^{\mu\nu}\Gamma^{J}D_{\mu}\Psi_{A}. \qquad \textbf{(4.96)}$$

Second term is (see **A.35**):

$$\begin{aligned}
-\Gamma^{\mu}\Gamma^{I}\Gamma^{\nu}\Gamma^{J}\Gamma_{\mu\kappa}C^{\kappa}_{C}\Psi_{D}X^{I}_{B}h^{CDB}{}_{A} \;&=\; -3\Gamma^{I}\Gamma^{J}\Gamma^{\nu}\Gamma_{\kappa}C^{\kappa}_{C}\Psi_{D}X^{I}_{B}h^{CDB}{}_{A}\\
&\quad -2g^{\nu}{}_{\kappa}\Gamma^{I}\Gamma^{J}C^{\kappa}_{C}\Psi_{D}X^{I}_{B}h^{CDB}{}_{A}\\
&=\; 3\Gamma^{\nu}\Gamma^{J}\Gamma^{I}\Gamma_{\kappa}C^{\kappa}_{C}\Psi_{D}X^{I}_{B}f^{CDB}{}_{A}\\
&\quad -6g^{IJ}\Gamma^{\nu}\Gamma_{\kappa}C^{\kappa}_{C}\Psi_{D}X^{I}_{B}f^{CDB}{}_{A}\\
&\quad +2g^{\nu}{}_{\kappa}\Gamma^{J}\Gamma^{I}C^{\kappa}_{C}X^{I}_{B}\Psi_{D}f^{CDB}{}_{A}\\
&\quad -4g^{\nu}{}_{\kappa}g^{IJ}C^{\kappa}_{C}X^{I}_{B}\Psi_{D}f^{CDB}{}_{A}. \qquad \textbf{(4.97)}
\end{aligned}$$

Third and fourth terms can be calculated with the help of **A.31** and **A.41** as follows,

$$\frac{1}{4}\Gamma^{\rho\gamma\tau}\Gamma^{\nu}\Gamma^{J}\Gamma_{[\rho\gamma}D_{\tau]}\Psi_{A} \;=\; -\Gamma^{\nu}\Gamma^{J}\Gamma^{\rho}D_{\rho}\Psi_{A}-6g^{\rho\nu}\Gamma^{J}D_{\rho}\Psi_{A}, \qquad \textbf{(4.98)}$$

$$\frac{1}{12}\Gamma^{\rho\gamma\tau}\Gamma^{\nu}\Gamma^{J}\Gamma^{I}\Gamma_{\rho\gamma\tau\kappa}C^{\kappa}_{B}X^{I}_{C}\Psi_{D}g^{CDB}{}_{A}$$

$$\begin{aligned}
&=\; \Gamma^{J}\Gamma^{I}(-\Gamma^{\nu}\Gamma_{\kappa}+6g^{\nu}{}_{\kappa})C^{\kappa}_{B}X^{I}_{C}\Psi_{D}g^{CDB}{}_{A}\\
&=\; \Gamma^{\nu}\Gamma^{J}\Gamma^{I}\Gamma_{\kappa}C^{\kappa}_{B}X^{I}_{C}\Psi_{D}f^{CDB}{}_{A}-6g^{\nu}{}_{\kappa}\Gamma^{J}\Gamma^{I}C^{\kappa}_{B}X^{I}_{C}\Psi_{D}f^{CDB}{}_{A}. \qquad \textbf{(4.99)}
\end{aligned}$$

And the last term is

$$-\frac{1}{2}\Gamma_{\kappa}\Gamma^{IM}C^{\kappa}_{B}\Gamma^{\nu}\Gamma^{J}\Gamma^{I}\Psi_{C}X^{M}_{D}f^{CDB}{}_{A}-\frac{1}{2}\Gamma_{\kappa}\Gamma^{IM}C^{\kappa}_{B}X^{I}_{C}\Gamma^{\nu}\Gamma^{J}\Gamma^{M}\Psi_{D}f^{CDB}{}_{A}$$

$$
\begin{aligned}
&= -\Gamma_\kappa\Gamma^\nu\Gamma^{IM}\Gamma^J\Gamma^I\Psi_C X^M_D C^\kappa_B f^{CDB}{}_A \\
&= -\Gamma_\kappa\Gamma^\nu(-2\Gamma^J\Gamma^M + 6g^{MJ})\Psi_C X^M_D C^\kappa_B f^{CDB}{}_A \\
&= 2\Gamma_\kappa\Gamma^\nu\Gamma^J\Gamma^M\Psi_C X^M_D C^\kappa_B f^{CDB}{}_A - 6g^{MJ}\Gamma_\kappa\Gamma^\nu\Psi_C X^M_D C^\kappa_B f^{CDB}{}_A \\
&= 2\Gamma^\nu\Gamma^J\Gamma_\kappa\Gamma^M\Psi_C X^M_D C^\kappa_B f^{CDB}{}_A + 4g^\nu{}_\kappa\Gamma^J\Gamma^M\Psi_C X^M_D C^\kappa_B f^{CDB}{}_A \\
&\quad +6g^{MJ}\Gamma^\nu\Gamma_\kappa\Psi_C X^M_D C^\kappa_B f^{CDB}{}_A - 12g^{MJ}g^\nu{}_\kappa\Psi_C X^M_D C^\kappa_B f^{CDB}{}_A \\
&= -2\Gamma^\nu\Gamma^J\Gamma^I\Gamma_\kappa\Psi_D X^I_B C^\kappa_C f^{CDB}{}_A + 4g^\nu{}_\kappa\Gamma^J\Gamma^I\Psi_D X^I_B C^\kappa_C f^{CDB}{}_A \\
&\quad +6g^{IJ}\Gamma^\nu\Gamma_\kappa\Psi_D X^I_B C^\kappa_C f^{CDB}{}_A - 12g^{IJ}g^\nu{}_\kappa\Psi_D X^I_B C^\kappa_C f^{CDB}{}_A. \quad \textbf{(4.100)}
\end{aligned}
$$

Replacing all these expressions in (**4.86**), we get B', we get

$$
\begin{aligned}
B' &= 2\Gamma^\nu\Gamma^J\Gamma^\mu D_\mu\Psi_A + 2\Gamma^\nu\Gamma^J\Gamma^I\Gamma_\kappa C^\kappa_C X^I_B\Psi_D f^{CDB}{}_A \\
&\quad -16g^\nu{}_\kappa g^{IJ} C^\kappa_C\Psi_D X^I_B f^{CDB}{}_A. \qquad \textbf{(4.101)}
\end{aligned}
$$

The last term C' (**4.87**) is:

$$
\begin{aligned}
C' &= \frac{1}{12}\left(\bar{\varepsilon}_2\Gamma_{\nu\lambda\sigma}\Gamma^{JK}\varepsilon_1\right)(\Gamma^\mu\Gamma^I\Gamma^{\nu\lambda\sigma}\Gamma^{JK}\Gamma^I D_\mu\Psi_A \\
&\quad -\Gamma^\mu\Gamma^I\Gamma^{\nu\lambda\sigma}\Gamma^{JK}\Gamma_{\mu\kappa}C^\kappa_C\Psi_D h^{CDB}{}_A X^I_B + \frac{1}{4}\Gamma^{\rho\gamma\tau}\Gamma^{\nu\lambda\sigma}\Gamma^{JK}\Gamma_{[\rho\gamma}D_{\tau]}\Psi_A \\
&\quad +\frac{1}{12}\Gamma^{\rho\gamma\tau}\Gamma^{\nu\lambda\sigma}\Gamma^{JK}\Gamma^I\Gamma_{\rho\gamma\tau\kappa}C^\kappa_B X^I_C\Psi_D g^{CDB}{}_A \\
&\quad -\frac{1}{2}\Gamma_\kappa\Gamma^{IM}C^\kappa_B\Gamma^{\nu\lambda\sigma}\Gamma^{JK}\Gamma^I\Psi_C X^M_D f^{CDB}{}_A \\
&\quad -\frac{1}{2}\Gamma_\kappa\Gamma^{IM}C^\kappa_B X^I_C\Gamma^{\nu\lambda\sigma}\Gamma^{JK}\Gamma^M\Psi_D f^{CDB}{}_A)].
\end{aligned}
$$

To simplify this expression, we can rearrange the terms associated with each other and using gamma matrix identities.

Firstly,we can combine the terms first and fourth just as a term which contain covariant derivative.

$$
-\Gamma^I\Gamma^{JK}\Gamma^I\Gamma^\mu\Gamma^{\nu\lambda\sigma}D_\mu\Psi_A + \frac{1}{4}\Gamma^{JK}\Gamma^{\rho\gamma\tau}\Gamma^{\nu\lambda\sigma}\Gamma_{[\rho\gamma}D_{\tau]}\Psi_A \qquad \textbf{(4.102)}
$$

$$
\begin{aligned}
&= -\Gamma^{JK}\Gamma^\mu\Gamma^{\nu\lambda\sigma}D_\mu\Psi_A + \frac{1}{4}\Gamma^{JK}\Gamma^{\rho\gamma\mu}\Gamma^{\nu\lambda\sigma}\Gamma_{[\rho\gamma}D_{\mu]}\Psi_A \\
&= -\Gamma^{JK}\Gamma^\mu\Gamma^{\nu\lambda\sigma}D_\mu\Psi_A + \frac{1}{12}\Gamma^{JK}(\Gamma^{\rho\gamma\mu}\Gamma^{\nu\lambda\sigma}\Gamma_{\rho\gamma}D_\mu \\
&\quad +\Gamma^{\rho\gamma\mu}\Gamma^{\nu\lambda\sigma}\Gamma_{\gamma\mu}D_\rho + \Gamma^{\rho\gamma\mu}\Gamma^{\nu\lambda\sigma}\Gamma_{\mu\rho}D_\gamma)\Psi_A \qquad \textbf{(4.103)}
\end{aligned}
$$

Using $\Gamma^{\mu\rho\gamma}\Gamma^{\nu\lambda\sigma}\Gamma_{\rho\gamma} = 4\Gamma^{\mu}\Gamma^{\nu\lambda\sigma}$ (**A.57**), we get the second term in (**4.103**) as follows,

$$\frac{1}{12}\Gamma^{JK}(\Gamma^{\rho\gamma\mu}\Gamma^{\nu\lambda\sigma}\Gamma_{\rho\gamma}D_{\mu} + \Gamma^{\rho\gamma\mu}\Gamma^{\nu\lambda\sigma}\Gamma_{\gamma\mu}D_{\rho} + \Gamma^{\rho\gamma\mu}\Gamma^{\nu\lambda\sigma}\Gamma_{\mu\rho}D_{\gamma})\Psi_A$$

$$\begin{aligned} &= \frac{1}{12}\Gamma^{JK}(4\Gamma^{\mu}\Gamma^{\nu\lambda\sigma}D_{\mu} + 4\Gamma^{\rho}\Gamma^{\nu\lambda\sigma}D_{\rho} + 4\Gamma^{\gamma}\Gamma^{\nu\lambda\sigma}D_{\gamma})\Psi_A \\ &= \Gamma^{JK}\Gamma^{\mu}\Gamma^{\nu\lambda\sigma}D_{\mu}\Psi_A. \end{aligned} \qquad (\mathbf{4.104})$$

Then (**4.103**) becomes

$$-\Gamma^{I}\Gamma^{JK}\Gamma^{I}\Gamma^{\mu}\Gamma^{\nu\lambda\sigma}D_{\mu}\Psi_A + \frac{1}{4}\Gamma^{JK}\Gamma^{\rho\gamma\tau}\Gamma^{\nu\lambda\sigma}\Gamma_{[\rho\gamma}D_{\tau]}\Psi_A =$$

$$\begin{aligned} &= -\Gamma^{JK}\Gamma^{\mu}\Gamma^{\nu\lambda\sigma}D_{\mu}\Psi_A + \Gamma^{JK}\Gamma^{\mu}\Gamma^{\nu\lambda\sigma}D_{\mu}\Psi_A \\ &= 0. \end{aligned} \qquad (\mathbf{4.105})$$

If we look at the second term in C', we get the following expression using Gamma matrix identity (see **A.56**):

$$\begin{aligned} -\Gamma^{\mu}\Gamma^{I}\Gamma^{\nu\lambda\sigma}\Gamma^{JK}\Gamma_{\mu\kappa}C^{\kappa}_{C}\Psi_D h^{CDB}{}_{A}X^{I}_{B} &= \Gamma^{I}\Gamma^{JK}\Gamma^{\mu}\Gamma^{\nu\lambda\sigma}\Gamma_{\mu\kappa}C^{\kappa}_{C}\Psi_D h^{CDB}{}_{A}X^{I}_{B} \\ &= \Gamma^{I}\Gamma^{JK}(-\Gamma_{\kappa}\Gamma^{\nu\lambda\sigma})C^{\kappa}_{C}\Psi_D h^{CDB}{}_{A}X^{I}_{B} \\ &= -\Gamma^{I}\Gamma^{JK}\Gamma_{\kappa}\Gamma^{\nu\lambda\sigma}C^{\kappa}_{C}\Psi_D f^{DBC}{}_{A}X^{I}_{B}. \end{aligned} \qquad (\mathbf{4.106})$$

And the fourth term can be written

$$\frac{1}{12}\Gamma^{\rho\gamma\tau}\Gamma^{\nu\lambda\sigma}\Gamma^{JK}\Gamma^{I}\Gamma_{\rho\gamma\tau\kappa}C^{\kappa}_{B}X^{I}_{C}\Psi_D g^{CDB}{}_{A} =$$

$$\begin{aligned} &= \frac{1}{12}\Gamma^{JK}\Gamma^{I}\Gamma^{\rho\gamma\tau}\Gamma^{\nu\lambda\sigma}\Gamma_{\rho\gamma\tau\kappa}C^{\kappa}_{B}X^{I}_{C}\Psi_D g^{CDB}{}_{A} \\ &= \frac{1}{12}\Gamma^{JK}\Gamma^{I}\left(-12\Gamma_{\kappa}\Gamma^{\nu\lambda\sigma}\right)C^{\kappa}_{B}X^{I}_{C}\Psi_D g^{CDB}{}_{A} \\ &= -\Gamma^{JK}\Gamma^{I}\Gamma_{\kappa}\Gamma^{\nu\lambda\sigma}C^{\kappa}_{B}X^{I}_{C}\Psi_D f^{CDB}{}_{A}. \end{aligned} \qquad (\mathbf{4.107})$$

We used $\Gamma^{\rho\gamma\tau}\Gamma^{\nu\lambda\sigma}\Gamma_{\rho\gamma\tau\kappa} = -12\Gamma_{\kappa}\Gamma^{\nu\lambda\sigma}$ above (**A.58**).

And the last term can be found replacing $I \leftrightarrow M$, $C \leftrightarrow D$ and taking into account the antisymmetry of $f^{CDB}{}_A$ as:

$$\begin{aligned}
&-\frac{1}{2}\Gamma_\kappa \Gamma^{IM} C^\kappa_B \Gamma^{\nu\lambda\sigma}\Gamma^{JK}(\Gamma^I \Psi_C X^M_D + \Gamma^M \Psi_D X^I_C) f^{CDB}{}_A = \\
&= -\Gamma^{IM}\Gamma^{JK}\Gamma^I \Gamma_\kappa \Gamma^{\nu\lambda\sigma}\Psi_C X^M_D C^\kappa_B f^{CDB}{}_A \\
&= -\Gamma^{JK}\Gamma^{M}\Gamma_\kappa \Gamma^{\nu\lambda\sigma}\Psi_C X^M_D C^\kappa_B f^{CDB}{}_A + \Gamma^{M}\Gamma^{JK}\Gamma_\kappa \Gamma^{\nu\lambda\sigma}\Psi_C X^M_D C^\kappa_B f^{CDB}{}_A .
\end{aligned} \tag{4.108}$$

We used $-\Gamma^{IM}\Gamma^{JK}\Gamma^I = \Gamma^M\Gamma^{JK} - \Gamma^{JK}\Gamma^M$ (see **A.30**) above.

Finally, we can combine all these expressions in C' and replace $M \leftrightarrow I$, $B \to C$, $C \to D$, $D \to B$ in the third term and $M \leftrightarrow I$ $C \leftrightarrow D$ in the fourth term. Then we get

$$\begin{aligned}
C' &= -\Gamma^I\Gamma^{JK}\Gamma_\kappa \Gamma^{\nu\lambda\sigma}\Psi_D X^I_B C^\kappa_C f^{DBC}{}_A - \Gamma^{JK}\Gamma^I\Gamma_\kappa \Gamma^{\nu\lambda\sigma} C^\kappa_B X^I_C \Psi_D f^{CDB}{}_A \\
&\quad + \Gamma^M\Gamma^{JK}\Gamma_\kappa \Gamma^{\nu\lambda\sigma}\Psi_C X^M_D C^\kappa_B f^{CDB}{}_A - \Gamma^{JK}\Gamma^M\Gamma_\kappa \Gamma^{\nu\lambda\sigma}\Psi_C X^M_D C^\kappa_B f^{CDB}{}_A \\
&= 0.
\end{aligned} \tag{4.109}$$

There is no contribution to $[\delta_1, \delta_2]\Psi_A$ from the last term, C'.

Using the relations that we found in A', B', C' above, one gets $[\delta_1, \delta_2]\Psi_A$ as follows,

$$\begin{aligned}
[\delta_1, \delta_2]\Psi_A &= -2i(\bar{\varepsilon}_2\Gamma^\mu\varepsilon_1)D_\mu\Psi_A + 2i\left(\bar{\varepsilon}_2\Gamma_\kappa\Gamma^I\varepsilon_1\right)C^\kappa_C \Psi_D X^I_B f^{CDB}{}_A \\
&\quad + \frac{3i}{4}[(\bar{\varepsilon}_2\Gamma_\nu\varepsilon_1)\Gamma^\nu(\Gamma^\mu D_\mu\Psi_A + \Gamma^I\Gamma_\kappa C^\kappa_C \Psi_D X^I_B f^{CDB}{}_A)] \\
&\quad - \frac{i}{4}[\left(\bar{\varepsilon}_2\Gamma_\nu\Gamma^J\varepsilon_1\right)\Gamma^\nu\Gamma^J(\Gamma^\mu D_\mu\Psi_A + \Gamma^I\Gamma_\kappa C^\kappa_C X^I_B \Psi_D f^{CDB}{}_A)] \\
&= v^\mu D_\mu\Psi_A + \widetilde{\Lambda}^D_A\Psi_D \\
&\quad + \frac{3i}{4}[(\bar{\varepsilon}_2\Gamma_\nu\varepsilon_1)\Gamma^\nu(\Gamma^\mu D_\mu\Psi_A + \Gamma^I\Gamma_\kappa C^\kappa_C \Psi_D X^I_B f^{CDB}{}_A)] \\
&\quad - \frac{i}{4}[\left(\bar{\varepsilon}_2\Gamma_\nu\Gamma^J\varepsilon_1\right)\Gamma^\nu\Gamma^J(\Gamma^\mu D_\mu\Psi_A + \Gamma^I\Gamma_\kappa C^\kappa_C X^I_B \Psi_D f^{CDB}{}_A)]
\end{aligned} \tag{4.110}$$

where $-2i(\bar{\varepsilon}_2\Gamma^\mu\varepsilon_1) = v^\mu$ and $2i\left(\bar{\varepsilon}_2\Gamma_\kappa\Gamma^I\varepsilon_1\right)C^\kappa_C X^I_B f^{CDB}{}_A = \widetilde{\Lambda}^D_A$.

Closure of the algebra requires that the lines in parentheses vanish. Therefore,this determines the fermionic equation of motion;

$$\Gamma^\mu D_\mu\Psi_A + \Gamma^I\Gamma_\kappa C^\kappa_C \Psi_D X^I_B f^{CDB}{}_A = 0. \tag{4.111}$$

So that on shell,

$$[\delta_1,\delta_2]\Psi_A = v^{\mu}D_{\mu}\Psi_A + \widetilde{\Lambda}_A^D\Psi_B. \tag{4.112}$$

We see that transformation of the spinor field close into a translation and a gauge transformation again.

4.2.6 Bosonic equation of motion

To obtain the bosonic equation of motion, we take the supervariation of the fermion equation of motion. We found fermionic equation of motion as (**4.111**). Taking supervariation of this equation, we obtain

$$\begin{aligned} 0 &= \delta(\Gamma^{\mu}D_{\mu}\Psi_A + X_C^I C_B^{\nu}\Gamma^I\Psi_D f^{CDB}{}_A) \\ 0 &= \Gamma^{\nu}\delta(D_{\nu}\Psi_A) + (\delta X_C^I)C_B^{\nu}\Gamma_{\nu}\Gamma^I\Psi_D f^{CDB}{}_A + X_C^I C_B^{\nu}\Gamma_{\nu}\Gamma^I(\delta\Psi_D)f^{CDB}{}_A. \end{aligned} \tag{4.113}$$

Let us calculate each term in (**4.113**) separately. Using the transformation of covariant derivative

$$\delta(D_{\nu}\Psi_A) = D_{\nu}(\delta\Psi_{A)-}(\delta\widetilde{A}_{\nu A}^B)\Psi_B \tag{4.114}$$

and considering (**4.4**) we obtain (**4.113**) as:

$$\begin{aligned} 0 \;=\;& \Gamma^{\nu}\Gamma^{\mu}\Gamma^I D_{\nu}D_{\mu}X_A^I\varepsilon + \frac{1}{12}\Gamma^{\nu}\Gamma_{\mu\rho\lambda}D_{\nu}H^{\mu\rho\lambda}{}_A\varepsilon \\ & -\Gamma^{\nu}\Gamma_{\lambda}\Gamma^{IJ}C_B^{\lambda}X_D^J D_{\nu}X_C^I f^{CDB}{}_A\varepsilon + i\bar{\varepsilon}\Gamma^I\Psi_C C_B^{\nu}\Gamma_{\nu}\Gamma^I\Psi_D f^{CDB}{}_A \\ & +X_C^I C_B^{\nu}\Gamma_{\nu}\Gamma^I\Gamma^{\mu}\Gamma^J D_{\mu}X_D^J f^{CDB}{}_A\varepsilon + \frac{1}{12}X_C^I C_B^{\nu}\Gamma_{\nu}\Gamma^I\Gamma_{\mu\rho\lambda}H^{\mu\rho\lambda}{}_D f^{CDB}{}_A\varepsilon \\ & -\frac{1}{2}X_C^I C_B^{\nu}\Gamma_{\nu}\Gamma^I\Gamma_{\lambda}\Gamma^{JK}C_E^{\lambda}X_F^J X_G^K f^{FGE}{}_D f^{CDB}{}_A\varepsilon. \end{aligned} \tag{4.115}$$

As we calculated before (**3.86**) in chapter 3, we can write

$$\Gamma^{\nu}\Gamma^{\mu}\Gamma^I D_{\nu}D_{\mu}X_A^I\varepsilon \;=\; \Gamma^I D^2 X_A^I\varepsilon - \frac{1}{2}\Gamma^{\nu}\Gamma^{\mu}\Gamma^I\widetilde{F}_{\mu\nu A}^B X_B^I\varepsilon. \tag{4.116}$$

Considering the property that $H^{\mu\rho\lambda}{}_D$ is antisymmetric, we obtain

$$\begin{aligned}
\Gamma_\nu\Gamma_{\mu\rho\lambda}H^{\mu\rho\lambda}{}_D &= \Gamma_\nu(\Gamma_\mu\Gamma_\rho\Gamma_\lambda - g_{\mu\rho}\Gamma_\lambda + g_{\mu\lambda}\Gamma_\rho - g_{g\lambda}\Gamma_\mu)H^{\mu\rho\lambda}{}_D \\
&= \Gamma_\nu\Gamma_\mu\Gamma_\rho\Gamma_\lambda H^{\mu\rho\lambda}{}_D - g_{\mu\rho}\Gamma_\nu\Gamma_\lambda H^{\mu\rho\lambda}{}_D + g_{\mu\lambda}\Gamma_\nu\Gamma_g H^{\mu\rho\lambda}{}_D \\
&\quad - g_{g\lambda}\Gamma_\nu\Gamma_\mu H^{\mu\rho\lambda}{}_D \\
&= \Gamma_\nu\Gamma_\mu\Gamma_\rho\Gamma_\lambda H^{\mu\rho\lambda}{}_D \qquad (\mathbf{4.117})
\end{aligned}$$

$$-\Gamma^\nu\Gamma_\lambda\Gamma^{IJ}C^\lambda_B D_\nu X^I_C f^{CDB}{}_A + \Gamma_\nu\Gamma^I\Gamma^\mu\Gamma^J D_\mu X^J_D X^I_C C^\nu_B f^{CDB}{}_A$$

$$\begin{aligned}
&= -\ \Gamma^{IJ}\Gamma^\nu\Gamma_\lambda C^\lambda_B X^J_D D_\nu X^I_C f^{CDB}{}_A - \Gamma^I\Gamma^J\Gamma_\nu\Gamma^\mu(D_\mu X^J_D)X^I_C C^\nu_B f^{CDB}{}_A \\
&\quad -\ \Gamma^{IJ}\Gamma^\nu\Gamma_\lambda C^\lambda_B X^J_D D_\nu X^I_C f^{CDB}{}_A - (\Gamma^{IJ} + g^{IJ})(-\Gamma^\mu\Gamma_\nu + 2g^\mu_\nu)D_\mu X^J_D \\
&\quad\ . X^I_C C^\nu_B f^{CDB}{}_A \\
&= -\ \Gamma^{IJ}\Gamma^\nu\Gamma_\lambda C^\lambda_B X^J_D D_\nu X^I_C f^{CDB}{}_A + \Gamma^{IJ}\Gamma^\mu\Gamma_\nu C^\nu_B X^I_C D_\mu X^J_D f^{CDB}{}_A \\
&= -\ 2\Gamma^{IJ}C^\nu_B D_\nu X^J_D X^I_C f^{CDB}{}_A + \Gamma^\mu\Gamma_\nu C^\nu_B X^I_C D_\mu X^J_D f^{CDB}{}_A \\
&\quad -\ C^\nu_B D_\nu X^I_D X^I_C f^{CDB}{}_A \\
&= \Gamma^\mu\Gamma_\nu C^\nu_B X^I_C D_\mu X^I_D f^{CDB}{}_A \qquad (\mathbf{4.118})
\end{aligned}$$

where we used the constaint (**4.60**) $C^\nu_B D_\nu X^I_D f^{CDB}{}_A = 0$.

The rest terms can be rearranged as follows:

$$-\frac{1}{2}X^I_C C^\nu_B\Gamma_\nu\Gamma^I\Gamma_\lambda\Gamma^{JK}C^\lambda_E X^J_F X^K_G f^{FGE}{}_D f^{CDB}{}_A$$

$$\begin{aligned}
&= \frac{1}{2}\Gamma^I(-\Gamma_\lambda\Gamma_\nu + 2g_{\nu\lambda})(\Gamma^J\Gamma^K - g^{JK})X^I_C C^\nu_B C^\lambda_E X^J_F X^K_G f^{FGE}{}_D f^{CDB}{}_A \\
&= -\frac{1}{2}\Gamma^I\Gamma_\lambda\Gamma_\nu\Gamma^J\Gamma^K X^I_C C^\nu_B C^\lambda_E X^J_F X^K_G f^{FGE}{}_D f^{CDB}{}_A \\
&\quad + \frac{1}{2}\Gamma^I\Gamma_\lambda\Gamma_\nu X^I_C C^\nu_B C^\lambda_E X^J_F X^J_G f^{FGE}{}_D f^{CDB}{}_A \\
&\quad + \Gamma^I\Gamma^J\Gamma^K X^I_C C^\nu_B C^\nu_E X^J_F X^K_G f^{FGE}_D f^{CDB}{}_A \\
&\quad - \Gamma^I X^I_C C^\nu_B C^\nu_E X^J_F X^K_G f^{FGE}{}_D f^{CDB}{}_A \qquad (\mathbf{4.119})
\end{aligned}$$

Considering the property that $f^{CDB}{}_A$ is antisymmetric, the only nonvanishing term is:

$$-\Gamma^I C^\nu_E C^\nu_B X^J_F X^K_G X^I_C C^\nu_B f^{FGE}{}_D f^{CDB}{}_A.$$

Combining all these expressions in (**4.113**) and rearranging, we get

$$\begin{aligned}0 = \quad & \Gamma^I \left\{ D^2 X^I_A - \frac{i}{2}\bar{\Psi}_C C^\nu_B \Gamma_\nu \Gamma^I \Psi_D f^{CDB}{}_A - C^\nu_B C^\nu_E X^I_C X^J_F X^K_G f^{FGE}_D f^{CDB}{}_A \right\} \varepsilon \\ & -\frac{1}{2}\Gamma^I \Gamma^\nu \Gamma^\mu \left\{ \widetilde{F}^B_{\mu\nu A} - C^\lambda_C H_{\mu\nu\lambda D} f^{CDB}{}_A \right\} \varepsilon + D_{[\mu} H_{\nu\lambda\rho]A} \\ & +\frac{1}{4}\varepsilon_{\mu\nu\lambda\rho\sigma\tau} C^\sigma_B X^I_C D^\tau X^I_D f^{CDB}{}_A + \frac{i}{8}\varepsilon_{\mu\nu\lambda\rho\sigma\tau} C^\sigma_B \bar{\Psi}_C \Gamma^\tau \Psi_D f^{CDB}{}_A\end{aligned}$$

where we used (**4.58**) and (**4.65**).

From here we obtain scalar equation of motion as:

$$0 = D^2 X^I_A - \frac{i}{2}\bar{\Psi}_C C^\nu_B \Gamma_\nu \Gamma^I \Psi_D f^{CDB}{}_A - C^\nu_B C^\nu_E X^I_C X^J_F X^K_G f^{FGE}_D f^{CDB}{}_A \tag{4.120}$$

Let us summarize our calculation: We required the transformations to close into a translation and a gauge transformation so that this way we obtained equations of motion and constraints. Taking supervariation of fermionic equation of motion, we get bosonic equation of motion.

All of the equations and constraints are:

$$\begin{aligned} 0 &= D^2 X^I_A - \frac{i}{2}\bar{\Psi}_C C^\nu_B \Gamma^\nu \Gamma^I \Psi_D f^{CDB}{}_A - C_{\nu G} X^J_C X^J_E X^I_F f^{EFG}{}_D f^{CDB}{}_A \\ 0 &= D_{[\mu} H_{\nu\lambda\rho]A} + \frac{1}{4}\varepsilon_{\mu\nu\lambda\rho\sigma\tau} C^\sigma_B X^I_C D^\tau X^I_D f^{CDB}{}_A \\ &\quad + \frac{i}{8}\varepsilon_{\mu\nu\lambda\rho\sigma\tau} C^\sigma_B \bar{\Psi}_C \Gamma^\tau \Psi_D f^{CDB}{}_A \\ 0 &= \Gamma^\mu D_\mu \Psi_A + \Gamma^I \Gamma_\kappa C^\kappa_C \Psi_D X^I_B f^{CDB}{}_A \\ 0 &= \widetilde{F}^B_{\mu\nu A} - C^\lambda_C H_{\mu\nu\lambda D} f^{CDB}{}_A \\ 0 &= D_\mu C^\nu_A = C^\mu_C C^\nu_D f^{BCD}{}_A \\ 0 &= C^\rho_C D_\rho X^I_D f^{CDB}{}_A = C^\rho_C D_\rho \Psi_D f^{CDB}{}_A = C^\rho_C D_\rho H_{\mu\nu\lambda A} f^{CDB}{}_A. \end{aligned} \tag{4.121}$$

4.3 Relation to Five-Dimensional Symmetry

In group space, the signature of the metric determines whether the algebra is Euclidean or Lorentzian. There is exactly one Euclidean four-dimensional 3-algebra as well as an infinite set of Lorentzian 3-algebras [10]. We proceeded to investigate the vacuum solutions of the theory for these two possibilities, but one can also consider three algebras with more than one timelike directions. The goal of this section is to relate gauge theory of these algebras with five dimensional super Yang-Mills theory.

4.3.1 Lorentzian case

The Lorentzian 3-algebras can be set up by starting with an ordinary Lie algebra $\mathscr{G}$ and adding two directions to $\mathscr{G}$ as a vector space, which are called $+$ and $-$, two lightlike generators $T^{\pm}$ such that $A,B = +,-,a,b...$, raising the total dimension of the algebra to dim($\mathscr{G}$) + 2 [10]. Using indices $A,B = +,-,a,b...$, the structure constant are given in terms of the Lie algebra $\mathscr{G}$-structure constants $f^{ab}{}_{c}$ as:

$$f^{+ab}{}_{c} = -f^{a+b}{}_{c} = f^{ab+}{}_{c} = f^{ab}{}_{c}\,, \qquad f^{abc}{}_{-} = f^{abc} \qquad \textbf{(4.122)}$$

with all other remaining components of $f^{ABC}{}_{D}$ vanishing. The invariant metric in terms of standard metric on $\mathscr{G}$ is given as:

$$h_{AB} = \left(\begin{array}{cc|ccc} 0 & -1 & 0 & \dots & 0 \\ -1 & 0 & 0 & \dots & 0 \\ \hline 0 & 0 & & & \\ \vdots & \vdots & & h_{\mathcal{G}} & \\ 0 & 0 & & & \end{array}\right).$$

The dim($\mathscr{G}$) + 2 scalar fields X^I and fermion fields Ψ transform in the SO(8). Two fields ($X^I_{\pm}$ and $\Psi_{\pm}$) are singlets of $\mathscr{G}$, while the other fields X^I_a, Ψ_a transform in the adjoint representation [37]. Since the fields X^I_a transform in the adjoint of $\mathscr{G}$ we can introduce the matrices $T^a = \{T^+, T^-, T^A\}$ such that

$$\begin{aligned} X^I &= X^I_a T^a \\ \left[T^a, T^b\right] &= f^{ab}{}_{c} T^c \\ Tr(T^a T^b) &= g^{ab}. \end{aligned} \qquad \textbf{(4.123)}$$

We focus on vacua of this theory in the particular case of $\mathscr{G} = su(N)$ by expanding around a particular point

$$< C^{\lambda}_{A} >= g\delta^{\lambda}_{5}\delta^{+}_{A} \qquad \textbf{(4.124)}$$

while all other fields are set to zero. And the fourth line of (**4.121**) equations according to this assumption has the form of

$$\begin{aligned}
0 &= \widetilde{F}^{B}_{\mu\nu A} - C^{\lambda}_{C} H_{\mu\nu\lambda D} f^{CDB} \\
&= \widetilde{F}^{B}_{\mu\nu A} - (g\delta^{\lambda}_{5}\delta^{+}_{C}) H_{\mu\nu\lambda D} f^{CDB} \\
&= \widetilde{F}^{B}_{\mu\nu A} - g H_{\mu\nu 5 D} f^{+DB}{}_{A} \\
&= \widetilde{F}^{B}_{\mu\nu A} - g H_{\mu\nu 5 D} f^{DB}{}_{A}. \qquad (\mathbf{4.125})
\end{aligned}$$

From this equation we obtain

$$\begin{aligned}
0 &= \widetilde{F}^{b}_{\alpha\beta a} - g H_{\alpha\beta 5} f^{db}{}_{a} \\
\widetilde{F}^{b}_{\alpha\beta a} &= g H_{\alpha\beta 5} f^{db}{}_{a} \qquad (\mathbf{4.126})
\end{aligned}$$

with $\mu = \alpha, 5$ and all other remaning components of $\widetilde{F}^{B}_{\mu\nu A}$ zero.
The constraint

$$0 = D_{\mu} C^{\nu}_{A} = C^{\mu}_{C} C^{\nu}_{D} f^{BCD}{}_{A}$$

takes the form of

$$0 = D_{\mu}(g\delta^{\nu}_{5}\delta^{+}_{A})$$

and we get

$$\partial_{\mu} g = 0. \qquad (\mathbf{4.127})$$

The rest of (**4.121**) equations respectively have the following forms.
The bosonic equation of motion:

$$\begin{aligned}
0 &= D^{2} X^{I}_{A} - \frac{i}{2}\bar{\Psi}_{C} C^{\nu}_{B} \Gamma^{\nu} \Gamma^{I} \Psi_{D} f^{CDB}{}_{A} - C_{\nu G} X^{J}_{C} X^{J}_{E} X^{I}_{F} f^{EFG}{}_{D} f^{CDB}{}_{A} \\
&= D^{2} X^{I}_{a} - \frac{i}{2}\bar{\Psi}_{c} (g\delta^{\nu}_{5}\delta^{+}_{A}) \Gamma_{\nu} \Gamma^{I} \Psi_{D} f^{CDB}{}_{A} \\
&\quad - (g\delta^{\nu}_{5}\delta^{+}_{A})(g\delta_{5\nu}\delta^{+}_{G}) X^{J}_{c} X^{J}_{e} X^{I}_{f} f^{EFG}{}_{D} f^{CDB}{}_{A} \\
&= \widetilde{D}^{\alpha} \tilde{D}_{\alpha} X^{I}_{a} - i\frac{g}{2}\bar{\Psi}_{c} \Gamma_{5} \Gamma^{I} \Psi_{d} f^{cd}{}_{a} - g^{2} X^{J}_{c} X^{J}_{e} X^{I}_{f} f^{ef}{}_{d} f^{cd}{}_{a}. \qquad (\mathbf{4.128})
\end{aligned}$$

The H-equation of motion:

$$\begin{aligned}
0 &= D_{[\mu}H_{\nu\lambda\rho]A}+\frac{1}{4}\varepsilon_{\mu\nu\lambda\rho\sigma\tau}C^{\sigma}_{B}X^{I}_{C}D^{\tau}X^{I}_{D}f^{CDB}{}_{A}\\
&+\frac{i}{8}\varepsilon_{\mu\nu\lambda\rho\sigma\tau}C^{\sigma}_{B}\bar{\Psi}_{C}\Gamma^{\tau}\Psi_{D}f^{CDB}{}_{A}\\
&= D_{[\mu}H_{\nu\lambda\rho]A}+\frac{1}{4}\varepsilon_{\mu\nu\lambda\delta\sigma\tau}(g\delta^{\sigma}_{5}\delta^{+}_{B})X^{I}_{C}D^{\tau}X^{I}_{D}f^{CDB}{}_{A}\\
&+\frac{i}{8}\varepsilon_{\mu\nu\lambda\delta\sigma\tau}(g\delta^{\sigma}_{5}\delta^{+}_{B})\bar{\Psi}_{c}\Gamma^{\tau}\Psi_{D}f^{CDB}{}_{A}\\
&= D_{[\mu}H_{\nu\lambda\rho]A}+\frac{g}{4}\varepsilon_{\mu\nu\lambda\delta 5\tau}X^{I}_{c}D^{\tau}X^{I}_{d}f^{CD}{}_{A}\\
&+i\frac{g}{8}\varepsilon_{\mu\nu\lambda\delta 5\tau}\bar{\Psi}_{C}\Gamma^{\tau}\Psi_{D}f^{CD}{}_{A}.
\end{aligned} \tag{4.129}$$

There are two possibilities;

i) If any of the indices is equal to 5, it is obtained

$$0 = D_{[\alpha}H_{\beta\gamma]5a}. \tag{4.130}$$

ii) None of the indices is equal to 5.

Recall that,

$$S_{[a_1.....a_n]}=\frac{1}{(n-i)!}\varepsilon_{a_1.....a_ib_1.....b_{n-i}}\frac{1}{i!}\varepsilon^{c_1.....c_ib_1.....b_{n-i}}S_{c_1.....c_i}. \tag{4.131}$$

So, $D_{[\alpha}H_{\beta\gamma]A}$ can be written as:

$$D_{[\alpha}H_{\beta\gamma]A}=\frac{1}{1!}\varepsilon_{\mu\nu\lambda\rho\alpha\beta}\frac{1}{4!}\varepsilon^{\gamma\delta\eta\kappa\alpha\beta}D_{\gamma}H_{\delta\eta\kappa}. \tag{4.132}$$

Recall that $H_{\delta\eta\kappa}$ is self dual,

$$H_{\delta\eta\kappa}=\frac{1}{3!}\varepsilon_{\delta\eta\kappa abc}H^{abc}.$$

Then, it is obtained

$$D_{[\alpha}H_{\beta\gamma]A}=\frac{1}{3!4!}\varepsilon_{\mu\nu\lambda\rho\alpha\beta}\varepsilon^{\gamma\delta\eta\kappa\alpha\beta}\varepsilon_{\delta\eta\kappa abc}D_{\gamma}H^{abc}. \tag{4.133}$$

The Levi-Civita tensor multiplication is given as follows,

$$\begin{aligned}
\varepsilon^{\gamma\delta\eta\kappa\alpha\beta}\varepsilon_{\delta\varepsilon\kappa} &= -\varepsilon^{\delta\eta\kappa\gamma\alpha\beta}\varepsilon_{\delta\eta\kappa abc}\\
&= =-(-3!3!)\{\delta^{\gamma}_{[a}\delta^{\alpha}_{b}\delta^{\beta}_{c]}\}\\
&= 3!3!\frac{1}{3!}\{\delta^{\gamma}_{a}\delta^{\alpha}_{b}\delta^{\beta}_{c}-\delta^{\gamma}_{a}\delta^{\alpha}_{c}\delta^{\beta}_{b}+\delta^{\gamma}_{b}\delta^{\alpha}_{c}\delta^{\beta}_{a}\\
&\quad -\delta^{\gamma}_{b}\delta^{\alpha}_{a}\delta^{\beta}_{c}+\delta^{\gamma}_{c}\delta^{\alpha}_{a}\delta^{\beta}_{b}-\delta^{\gamma}_{c}\delta^{\alpha}_{b}\delta^{\beta}_{a}\}.
\end{aligned} \tag{4.134}$$

So,

$$\varepsilon^{\gamma\delta\eta\kappa\alpha\beta}\varepsilon_{\delta\eta\kappa abc}D_\gamma H^{abc} = 3!3!D_\gamma H^{\alpha\beta\gamma} \quad (\mathbf{4.135})$$

and

$$D_{[\mu}H_{\nu\lambda\rho]A} = \frac{1}{4}\varepsilon_{\mu\nu\lambda\rho\alpha\beta}D_\gamma H^{\alpha\beta\gamma}. \quad (\mathbf{4.136})$$

Multiplying the last line of the (**4.129**) equations with $\varepsilon^{\mu\nu\lambda\rho5\eta}$ from left and writing above expression in it, we get

$$\begin{aligned}
0 &= \varepsilon^{\mu\nu\lambda\rho5\eta}D_{[\mu}H_{\nu\lambda\rho]A} + \frac{g}{4}\varepsilon^{\mu\nu\lambda\rho5\eta}\varepsilon_{\mu\nu\lambda\rho5\tau}X^I_C D^\tau X^I_D f^{CD}{}_A \\
&\quad + \frac{ig}{8}\varepsilon^{\mu\nu\lambda\rho5\eta}\varepsilon_{\mu\nu\lambda\rho5\tau}\bar{\Psi}_C\Gamma^\tau\Psi_D f^{CD}{}_A \\
&= \frac{1}{4}\varepsilon^{\mu\nu\lambda\rho5\eta}\varepsilon_{\mu\nu\lambda\rho\alpha\beta}D_\gamma H^{\alpha\beta\gamma} + \frac{g}{4}\varepsilon^{\mu\nu\lambda\rho5\eta}\varepsilon_{\mu\nu\lambda\rho5\tau}X^I_C D^\tau X^I_D f^{CD}{}_A \\
&\quad + \frac{ig}{8}\varepsilon^{\mu\nu\lambda\rho5\eta}\varepsilon_{\mu\nu\lambda\rho5\tau}\bar{\Psi}_C\Gamma^\tau\Psi_D f^{CD}{}_A. \quad (\mathbf{4.137})
\end{aligned}$$

Recall that,

$$\varepsilon^{\mu\nu\lambda\rho5\eta}\varepsilon_{\mu\nu\lambda\rho\alpha\beta} = -4(\delta^5_\alpha\delta^\eta_\beta - \delta^5_\beta\delta^\eta_\alpha) \quad (\mathbf{4.138})$$

and

$$\varepsilon^{\mu\nu\lambda\rho5\eta}\varepsilon_{\mu\nu\lambda\rho5\tau} = 4!\delta^\eta_\tau. \quad (\mathbf{4.139})$$

Using above expressions we find

$$\begin{aligned}
0 &= \frac{1}{4}4!(D_\gamma H^{5\eta\alpha} - D_\gamma H^{\eta5\alpha}) + \frac{g}{4}.4!X^I_C D^\eta X^I_D f^{CD}{}_A \\
&\quad + i\frac{g}{8}4!\bar{\Psi}_C\Gamma^\eta\Psi_D f^{CD}{}_A \\
&= \frac{1}{4}(-2D_\gamma H^{\gamma\eta5}) + \frac{g}{4}X^I_C D^\eta X^I_D f^{CD}{}_A \\
&\quad + i\frac{g}{8}\bar{\Psi}_C\Gamma^\eta\Psi_D f^{CD}{}_A \\
&= \frac{1}{2}D_\gamma H^{\gamma\eta5} + \frac{g}{4}X^I_C D^\eta X^I_D f^{CD}{}_A \\
&\quad + i\frac{g}{8}\bar{\Psi}_C\Gamma^\eta\Psi_D f^{CD}{}_A \\
&= D_\gamma H^{\gamma\eta5} + \frac{g}{2}X^I_C D^\eta X^I_D f^{CD}{}_A + i\frac{g}{4}\bar{\Psi}_C\Gamma^\eta\Psi_D f^{CD}{}_A \\
&= \tilde{D}^\alpha H_{\alpha\beta5a} + \frac{1}{2}gf^{cd}{}_a(X^I_c\tilde{D}_\beta X^I_d + \frac{i}{2}\bar{\Psi}_c\Gamma_\beta\Psi_d). \quad (\mathbf{4.140})
\end{aligned}$$

The fermionic equation of motion becomes:

$$0 = \Gamma^{\mu}D_{\mu}\Psi_A + gX^I_C\Gamma_5\Gamma^I\Psi_d f^{cd}{}_a. \quad \textbf{(4.141)}$$

And the constraints:

$$0 = C^{\rho}_C D_{\rho}X^I_D f^{CDB}{}_A = C^{\rho}_C D_{\rho}\Psi_D f^{CDB}{}_A = C^{\rho}_C D_{\rho}H_{\mu\nu\lambda A} f^{CDB}{}_A$$

become respectively

$$\begin{aligned} 0 &= C^{\rho}_C D_{\rho}X^I_D f^{CDB}{}_A \\ &= (g\delta^{\rho}_5\delta^+_C)D_{\rho}X^I_D f^{CDB}{}_A \\ &= gD_5X^I_d f^{+db}{}_a \\ &= gD_5X^I_d f^{db}{}_a \qquad \Rightarrow \qquad \partial_5 X^I_d = 0, \end{aligned} \quad \textbf{(4.142)}$$

$$\begin{aligned} 0 &= C^{\rho}_C D_{\rho}\Psi_D f^{CDB}{}_A \\ &= (g\delta^{\rho}_5\delta^+_C)D_{\rho}\Psi_D f^{CDB}{}_A \\ &= gD_5\Psi_d f^{db}{}_a \qquad \Rightarrow \qquad \partial_5\Psi_d = 0, \end{aligned} \quad \textbf{(4.143)}$$

$$\begin{aligned} 0 &= C^{\rho}_C D_{\rho}H_{\mu\nu\lambda A} f^{CDB}_A \\ &= (g\delta^{\rho}_5\delta^+_C)D_{\rho}H_{\mu\nu\lambda A} f^{CDB}{}_A \\ &= gD_5H_{\mu\nu\lambda A} f^{db}{}_a \qquad \Rightarrow \qquad \partial_5 H_{\mu\nu\lambda a} = 0. \end{aligned} \quad \textbf{(4.144)}$$

To summarize, the equations of motion become

$$\begin{aligned} 0 &= \widetilde{F}^b_{\alpha\beta a} - gH_{\alpha\beta 5} f^{db}{}_a \\ 0 &= \tilde{D}^{\alpha}\tilde{D}_{\alpha}X^I_a - i\frac{g}{2}\bar{\Psi}_c\Gamma_5\Gamma^I\Psi_d f^{cd}{}_a - g^2X^J_cX^J_eX^I_f f^{ef}{}_d f^{cd}{}_a \\ 0 &= D_{[\alpha}H_{\beta\gamma]5a} \\ 0 &= \tilde{D}^{\alpha}H_{\alpha\beta 5a} + \frac{1}{2}gf^{cd}{}_a(X^I_c\tilde{D}_{\beta}X^I_d + \frac{i}{2}\bar{\Psi}_c\Gamma_{\beta}\Psi_d) \\ 0 &= \Gamma^{\mu}D_{\mu}\Psi_A + gX^I_C\Gamma_5\Gamma^I\Psi_d f^{cd}{}_a \\ 0 &= \partial_5 X^I_d = \partial_5\Psi_d = \partial_5 H_{\mu\nu\lambda a} \end{aligned} \quad \textbf{(4.145)}$$

where $\tilde{D}_\alpha X^I_a = \partial_\alpha X^I_a - \widetilde{A}^b_{\alpha a} X^I_b$. Whereas one also get from (**4.3**) that

$$\begin{aligned}
\delta X^I_a &= i\bar{\varepsilon}\Gamma^I\Psi_a \\
\delta\Psi_a &= \Gamma^\alpha\Gamma^I\tilde{D}_\alpha X^I_a\varepsilon + \frac{1}{2}\Gamma_{\alpha\beta}\Gamma_5 H^{\alpha\beta 5}{}_a\varepsilon - \frac{i}{2}\Gamma_5\Gamma^{IJ}\delta X^I_c\delta X^J_d f^{cd}{}_a\varepsilon \\
\delta\widetilde{A}^b_{\alpha a} &= i\bar{\varepsilon}\Gamma_\alpha\Gamma_5\Psi_d f^{db}{}_a.
\end{aligned} \quad (\mathbf{4.146})$$

It is seen that with the identifications [10]:

$$g = g^2_{YM} \qquad H^a{}_{\alpha\beta 5} = \frac{1}{g^2_{YM}}F^a_{\alpha\beta} \qquad \widetilde{A}^b_{\alpha a} = A_{\alpha c}f^{cb}{}_a. \quad (\mathbf{4.147})$$

So, it is recovered the equations of motion, Bianchi identity and supersymmetry transformations of five-dimensional SU(N) super Yang-Mills theory. In particular since g has scaling dimension 1, we see that g^2_{YM} also has the correct scaling dimension (**4.3**).

On the other hand, we also have the additional equations

$$\begin{aligned}
0 &= \partial^\mu\partial_\mu X^I_\pm \\
0 &= \partial_{[\mu}H_{\nu\lambda\rho]A} \\
0 &= \Gamma^\mu\partial_\mu\Psi_A
\end{aligned} \quad (\mathbf{4.148})$$

with transformations

$$\begin{aligned}
\delta X^I_\pm &= i\bar{\varepsilon}\Gamma^I\Psi_\pm \\
\delta\Psi_\pm &= \Gamma^\mu\Gamma^I\partial_\mu X^I_\pm\varepsilon + \frac{1}{3!}\frac{1}{2}\Gamma_{\mu\nu\lambda}H^{\mu\nu\lambda}_\pm\varepsilon \\
\delta H_{\mu\nu\lambda\pm} &= 3i\bar{\varepsilon}\Gamma_{[\mu\nu}\partial_{\lambda]}\Psi_\pm.
\end{aligned} \quad (\mathbf{4.149})$$

These contain two free, abelian (2,0) multiplets in six dimensions.

Finally, investigation of the existence of a 2-form $B_{\mu\nu A}$ is given as follows. Before the calculation, we remind that (**4.67**) only acts on the non Abelian fields. For the Abelian case, one has $\partial_{[\mu}H_{\nu\lambda\rho]\pm} = 0$ and it is natural to write locally $H_{\nu\lambda\rho\pm} = 3\partial_{[\mu}B_{\nu\lambda]\pm}$. From (**4.126**) we have, assuming $D_5 B_{\alpha\beta\alpha} = 0$

$$\begin{aligned}
\widetilde{F}^b_{\alpha\beta a} &= gH_{\alpha\beta 5d}f^{db}{}_a \\
&= 2g\tilde{D}_{[\alpha}B_{\beta]5c}f^{cb}{}_a \\
&= \frac{1}{2!}2g(\tilde{D}_\alpha B_{\beta 5c} - \tilde{D}_\beta B_{\alpha 5c}).
\end{aligned} \quad (\mathbf{4.150})$$

Considering,

$$\tilde{D}_\alpha B_{\beta 5a} = \partial_\alpha B_{\beta 5a} - \widetilde{A}^b_{\alpha a} B_{\beta 5b}. \tag{4.151}$$

We find

$$\begin{aligned}
\widetilde{F}^b_{\alpha\beta a} &= g[(\partial_\alpha B_{\beta 5c} - \widetilde{A}^d_{\alpha c} B_{\beta 5d}) - (\partial_\beta B_{\alpha 5c} - \widetilde{A}^d_{\beta c} B_{\alpha 5d}] f^{cb}{}_a \\
&= g\partial_\alpha B_{\beta 5c} f^{cb}{}_a - g\widetilde{A}^d_{\alpha c} B_{\beta 5d} f^{cb}{}_a - g\partial_\alpha B_{\beta 5c} f^{cb}{}_a \\
&\quad + g\widetilde{A}^d_{\alpha c} B_{\beta 5d} f^{cb}{}_a.
\end{aligned} \tag{4.152}$$

However we should compare this to

$$\widetilde{F}^b_{\alpha\beta a} = \partial_\beta \widetilde{A}^b_{\alpha a} - \partial_\alpha \widetilde{A}^b_{\beta a} - \widetilde{A}^b_{\alpha c} \widetilde{A}^c_{\beta a} + \widetilde{A}^b_{\beta c} \widetilde{A}^c_{\alpha a}.$$

Comparing the derivative terms leads to

$$\widetilde{A}^b_{\alpha a} = -gB_{\alpha 5c} f^{cb}{}_a. \tag{4.153}$$

If the nonlinear terms are compared, it is required that

$$-\widetilde{A}^b_{\alpha c} \widetilde{A}^c_{\beta a} + \widetilde{A}^b_{\beta c} \widetilde{A}^c_{\alpha a} = -g\widetilde{A}^d_{\alpha c} B_{\beta 5d} f^{cb}{}_a + g\widetilde{A}^d_{\beta c} B_{\alpha 5d} f^{cb}{}_a. \tag{4.154}$$

Considering $\widetilde{A}^b_{\alpha a} = -gB_{\alpha 5c} f^{cb}_a$, we get

$$\begin{aligned}
-g^2 B_{\alpha 5d} f^{db}_c B_{\beta 5e} f^{ec}{}_a + g^2 B_{\beta 5f} f^{fb}{}_c B_{\alpha 5g} f^{gc}{}_a &= g^2 B_{\alpha 5h} B_{\beta\beta 5d} f^{hd}{}_c f^{cb}{}_a \\
&\quad - g^2 B_{\beta 5k} B_{\alpha 5d} f^{kd}{}_c f^{cb}{}_a,
\end{aligned} \tag{4.155}$$

$$\begin{aligned}
-B_{\alpha 5d} B_{\beta 5e} f^{db}{}_c f^{ec}{}_a + B_{\beta 5f} B_{\alpha 5g} f^{fb}{}_c f^{gc}{}_a &= B_{\alpha 5h} B_{\beta 5d} f^{hd}{}_c f^{cb}{}_a \\
&\quad - B_{\beta 5k} B_{\alpha 5d} f^{kd}{}_c f^{cb}{}_a.
\end{aligned} \tag{4.156}$$

If the indices $g \to d$, $f \to e$ are changed for the second term, $h \to d$, $d \to e$ for the third term and $k \to e$ for the fourth term above, we obtain

$$\begin{aligned}
-B_{\alpha 5d} B_{\beta 5e} f^{db}{}_c f^{ec}{}_a + B_{\beta 5e} B_{\alpha 5d} f^{eb}{}_c f^{dc}{}_a &= B_{\alpha 5d} B_{\beta 5e} f^{de}{}_c f^{cb}{}_a \\
&\quad - B_{\beta 5e} B_{\alpha 5d} f^{ed}{}_c f^{cb}{}_a.
\end{aligned} \tag{4.157}$$

Therefore, it is required that

$$-f^{db}{}_c f^{ec}{}_a + f^{eb}{}_c f^{dc}{}_a \;=\; f^{de}{}_c f^{cb}{}_a - f^{ed}{}_c f^{cb}{}_a \tag{4.158}$$

and

$$\begin{aligned} f^{eb}{}_c f^{dc}{}_a - f^{db}{}_c f^{ec}{}_a &= f^{de}{}_c f^{cb}{}_a + f^{de}{}_c f^{cb}{}_a \\ &= 2 f^{de}{}_c f^{cb}{}_a. \end{aligned} \tag{4.159}$$

whereas using the Jacobi identity one can find instead

$$f^{eb}{}_c f^{dc}{}_a - f^{db}{}_c f^{ec}{}_a = f^{de}{}_c f^{cb}{}_a. \tag{4.160}$$

This result indicates that there is no $B_{\mu\nu}$ in general.
As we have seen the nonabelian sector of the theory is essentially five-dimensional super Yang-Mills.
In summary, in the case of Lorentzian 3-algebra the vacua of the theory correspond to five-dimensional super-Yang-Mills along with two free, abelian (2, 0) multiplets which are six-dimensional.

4.3.2 Euclidean case

The other possibility is an Euclidean signature 3-algebra which is rather similar. The structure constants conform with the invariant tensor of SO(4), $f^{CDBA} = \varepsilon^{CDBA}$ for the A_4 Euclidean 3-algebra. Regardless of one of the SO(4) directions, considering $A = \alpha, 4$ and expanding the theory around $< C^{\mu}_{A} >= v\delta^{\mu}_{5}\delta^{4}_{A}$ leads to

$$\begin{aligned} 0 &= \widetilde{F}^{b}_{\alpha\beta a} - vH_{\alpha\beta 5} f^{4db}{}_a \\ 0 &= \tilde{D}^{\alpha}\tilde{D}_{\alpha}X^{I}_{a} - i\frac{v}{2}\bar{\Psi}_c\Gamma_5\Gamma^{I}\Psi_d f^{cdb}{}_4 - v^2 X^{J}_{c}X^{J}_{e}X^{I}_{f} f^{ef4}{}_d f^{cdb}{}_4 \\ 0 &= D_{[\alpha}H_{\beta\gamma]5a} \\ 0 &= \tilde{D}^{\alpha}H_{\alpha\beta 5a} + \frac{1}{2}v f^{cd4}{}_a(X^{I}_{c}\tilde{D}_{\beta}X^{I}_{d} + \frac{i}{2}\bar{\Psi}_c\Gamma_{\beta}\Psi_d) \\ 0 &= \Gamma^{\mu}D_{\mu}\Psi_A + vX^{I}_{C}\Gamma_5\Gamma^{I}\Psi_d f^{cd4}{}_a \\ 0 &= \partial_5 X^{I}_{d} = \partial_5\Psi_d = \partial_5 H_{\mu\nu\lambda a} \end{aligned} \tag{4.161}$$

where $\tilde{D}_\alpha X^I_a = \partial_\alpha X^I_a - \widetilde{A}^b_{\alpha a} X^I_b$, whereas one also has from (**4.3**) that

$$\begin{aligned} \delta X^I_a &= i\bar{\varepsilon}\Gamma^I\Psi_a \\ \delta\Psi_a &= \Gamma^\alpha\Gamma^I\tilde{D}_\alpha X^I_a\varepsilon + \frac{1}{2}\Gamma_{\alpha\beta}\Gamma_5 H^{\alpha\beta 5}_a\varepsilon - \frac{i}{2}\Gamma_5\Gamma^{IJ}\delta X^I_c\delta X^J_d f^{cd4}{}_a\varepsilon \\ \delta\widetilde{A}^b_{\alpha a} &= i\bar{\varepsilon}\Gamma_\alpha\Gamma_5\Psi_d f^{4db}{}_a \end{aligned} \qquad (\mathbf{4.162})$$

It is seen that with the identifications

$$v = v^2_{YM} \qquad H^a_{\alpha\beta 5} = \frac{1}{v^2_{YM}} F^a_{\alpha\beta} \qquad \widetilde{A}^b_{\alpha a} = A_{\alpha c} f^{c4b}{}_a \qquad (\mathbf{4.163})$$

where f^{abc} are now the structure constants of SU(2). So, the theory is once again identified around the vacuum as five-dimensional SU(2) super Yang-Mills [53], [54]. In the Euclidean 3-algebra case, one has only a single six-dimensional (2, 0) tensor multiplet because of the replacement ($\pm \to 4$) such that [10]:

$$\begin{aligned} 0 &= \partial^\mu\partial_\mu X^I_4 \\ 0 &= \partial_{[\mu} H_{\nu\lambda\rho]A} \\ 0 &= \Gamma^\mu\partial_\mu\Psi_A \end{aligned} \qquad (\mathbf{4.164})$$

with transformations

$$\begin{aligned} \delta X^I_4 &= i\bar{\varepsilon}\Gamma^I\Psi_4 \\ \delta\Psi_4 &= \Gamma^\mu\Gamma^I\partial_\mu X^I_4\varepsilon + \frac{1}{3!}\frac{1}{2}\Gamma_{\mu\nu\lambda}H^{\mu\nu\lambda}_4\varepsilon \\ \delta H_{\mu\nu\lambda 4} &= 3i\bar{\varepsilon}\Gamma_{[\mu\nu}\partial_{\lambda]}\Psi_4. \end{aligned} \qquad (\mathbf{4.165})$$

The Euclidean 3-algebra is qualitatively similar compared to the Lorentzian 3-algebra, unlike 3 dimensional 3-algebra theories with 16 supercharges. On the contrary, it would have been possible to reach this ansatz by working backwards. Considering super Yang- Mills theory in five dimensions, the set of equations of motion and supersymmetry transformations, YM coupling can be renamed as $g^2_{YM} \equiv C^5_+$ and the gauge field $F^a_{\alpha\beta} = \frac{1}{g^2_{YM}} H^a_{\alpha\beta 5}$, $\widetilde{A}^b_{\alpha a} = A_{\alpha c} f^{cb}{}_a$.

5. CONCLUSION

Although a great deal of the physics is known, we don't still know of any simple covariant descriptions of M-theory in 11 dimensions, multiple M2-branes and multiple M5-branes. However the low energy limit of a single M2-brane and a single M5-brane is understood for a long time, the case of multiple M2-branes is not understood or at least, it wasn't understood until the end of 2007.

Motivated by Basu & Harvey, Bagger & Lambert and also independently Gustavsson presented an alternative approach to multiple M2-branes system using three algebra formalism. They have constructed a supersymmetric field theory model that is consistent with all the symmetries expected of a multiple M2-brane theory.

After this revolution, there has been important progress for understanding multiple M2-branes. Furthermore, the BLG model for multiple M2-branes motivates an M5-brane theory with a novel gauge symmetry defined by the Lie 3-algebra. The description of multiple M5-branes is expected to be a non-Abelian generalisation of the single M5-brane theory. For this purpose, there have been significant attempts for the new Lagrangian formulation of a single M5-brane and multiple M5-branes.

All of these significant recent progresses are achieved in our understanding of multiple M2 and M5 branes using a new kind of symmetry structure so called Lie 3 algebra, but there is still a lot to be learned about them.

REFERENCES

[1] **Bagger, J. and Lambert, N.**, (2007). Modeling Multiple M2's, *Phys.Rev.*, **D75**, 045020.

[2] **Bagger, J. and Lambert, N.**, (2008). Comments on multiple M2-branes, *JHEP*, **0802**, 105.

[3] **Bagger, J. and Lambert, N.**, (2008). Gauge symmetry and supersymmetry of multiple M2-branes, *Phys.Rev.*, **D77**, 065008.

[4] **Gustavsson, A.**, (2009). Algebraic structures on parallel M2-branes, *Nucl.Phys.*, **B811**, 66–76.

[5] **Basu, A. and Harvey, J.A.**, (2005). The M2-M5 brane system and a generalized Nahm's equation, *Nucl.Phys.*, **B713**, 136–150.

[6] **Chu, C.S. and Ko, S.L.**, (2012). Non-abelian Action for Multiple Five-Branes with Self-Dual Tensors, *JHEP*, **1205**, 028.

[7] **Huang, K.W. and Huang, W.H.**, (2010). Lie 3-Algebra Non-Abelian (2,0) Theory in Loop Space, `1008.3834`.

[8] **Kawamoto, S.**, **Takimi, T. and Tomino, D.**, (2011). Branes from a non-Abelian (2,0) tensor multiplet with 3-algebra, *J.Phys.A*, **A44**, 325402.

[9] **Ho, P.M. and Matsuo, Y.**, (2008). M5 from M2, *JHEP*, **0806**, 105.

[10] **Lambert, N. and Papageorgakis, C.**, (2010). Nonabelian (2,0) Tensor Multiplets and 3-algebras, *JHEP*, **1008**, 083.

[11] **Greene, B.**, (2011). The Elegant Universe, Random House.

[12] **Kaku, M.**, (1999). Introduction to Superstrings and M-Theory, Graduate Texts in Contemporary Physics, Springer.

[13] **Kiritsis, E.**, 2007. String Theory in a Nutshell, In a Nutshell, Princeton University Press.

[14] **McAllister, L.**, (2010). Lectures on String Theory, arrenged by Tanedo, F.

[15] **McMahon, D.**, (2008). String Theory Demystified, Demystified Series, McGraw Hill.

[16] **Zwiebach, B.**, (2004). A First Course in String Theory, Cambridge University Press.

[17] **Becker, K.**, **Becker, M. and Schwarz, J.**, (2007). String Theory And M-Theory: A Modern Introduction, Cambridge University Press.

[18] **Tanedo, F.**, (2010). Fun and supersymmetry, Lecture Notes, Institute for High Energy Phenomenology, Cornell University, Ithaca,.

[19] **Lambert, N.** Supersymmetry, Lecture Notes, Department of Mathematics, Kings College London.

[20] **Argurio, R.**, (2010). Introduction to Supersymmetry, Notes on supersymmetry, Universite Libre de Bruxelles.

[21] **Lambert, N.**, (2011). Introduction to Supersymmetry, Lectures at the 2011 Balkan Summer School.

[22] **Müller-Kirsten, H. and Wiedemann, A.**, (2010). Introduction to Supersymmetry, World Scientific Lecture Notes in Physics, World Scientific.

[23] **Van Proeyen, A.**, (1999). Tools for supersymmetry.

[24] **Green, M.B. and Schwarz, J.H.**, (1982). Supersymmetrical string theories, *Physics Letters B*, **109**, 444 – 448.

[25] **Low, A.M.**, 2010. Worldvolume Superalgebra Of BLG Theory With Nambu-Poisson Structure, *JHEP*, **1004**, 089, `0909.1941`.

[26] **Miemiec, A. and Schnakenburg, I.**, (2006). Basics of M-theory, *Fortsch.Phys.*, **54**, 5–72.

[27] **Furuuchi, K.**, **Shih, S.Y.D. and Takimi, T.**, (2008). M-Theory Superalgebra From Multiple Membranes, *JHEP*, **0808**, 072.

[28] **Bergshoeff, E.A.**, **de Roo, M.**, **Hohm, O. and Roest, D.**, (2008). Multiple Membranes from Gauged Supergravity, *JHEP*, **0808**, 091.

[29] **Bagger, J.**, **Lambert, N.**, **Mukhi, S. and Papageorgakis, C.**, (2012). Membranes in M-theory.

[30] **Gran, U.**, **Nilsson, B.E. and Petersson, C.**, (2008). On relating multiple M2 and D2-branes, *JHEP*, **0810**, 067.

[31] **Ho, P.M.**, **Imamura, Y. and Matsuo, Y.**, (2008). M2 to D2 revisited, *JHEP*, **0807**, 003.

[32] **Mukhi, S. and Papageorgakis, C.**, (2008). M2 to D2, *JHEP*, **0805**, 085.

[33] **Copland, N.B.**, (2007). Aspects of M-Theory Brane Interactions and String Theory Symmetries.

[34] **Copland, N.B.**, (2010). Introductory Lectures on Multiple Membranes.

[35] **Berman, D.S. and Copland, N.B.**, (2006). A Note on the M2-M5 brane system and fuzzy spheres, *Phys.Lett.*, **B639**, 553–559.

[36] **Bandos, I.A.**, (2009). On multiple M2-brane model(s) and its N=8 superspace formulation(s).

[37] **Benvenuti, S.**, **Rodriguez-Gomez, D.**, **Tonni, E. and Verlinde, H.**, 2009. N=8 superconformal gauge theories and M2 branes, *JHEP*, **0901**, 078.

[38] **Berman, D.S.**, **Tadrowski, L.C. and Thompson, D.C.**, (2008). Aspects of Multiple Membranes, *Nucl.Phys.*, **B802**, 106–120.

[39] **Berman, D.S.**, (2008). M-theory branes and their interactions, *Phys.Rept.*, **456**, 89–126.

[40] **Gomis, J.**, **Milanesi, G. and Russo, J.G.**, (2008). Bagger-Lambert Theory for General Lie Algebras, *JHEP*, **0806**, 075.

[41] **Ho, P.M.**, **Matsuo, Y. and Shiba, S.**, (2009). Lorentzian Lie (3-)algebra and toroidal compactification of M/string theory, *JHEP*, **0903**, 045.

[42] **Ho, P.M.**, **Hou, R.C. and Matsuo, Y.**, (2008). Lie 3-Algebra and Multiple M2-branes, *JHEP*, **0806**, 020.

[43] **de Medeiros, P.**, **Figueroa-O'Farrill, J.M. and Mendez-Escobar, E.**, (2008). Metric Lie 3-algebras in Bagger-Lambert theory, *JHEP*, **0808**, 045.

[44] **Gilmore, R.**, (2006). Lie Groups, Lie Algebras, And Some of Their Applications, Dover Books on Mathematics Series, Dover Publications.

[45] **Gilmore, R.**, (2008). Lie Groups, Physics, and Geometry: An Introduction for Physicists, Engineers and Chemists, Cambridge University Press.

[46] **Hall, B.**, (2003). Lie Groups, Lie Algebras, and Representations: An Elementary Introduction, Graduate Texts in Mathematics, Springer-Verlag.

[47] **Mendez-Escobar, E.**, 2010. Metric 3-Leibniz algebras and M2-branes, ph.D.Thesis.

[48] **Bagger, J. and Bruhn, G.**, 2011. Three-Algebras in N = 5, 6 Superconformal Chern-Simons Theories: Representations and Relations, *Phys.Rev.*, **D83**, 025003.

[49] **Bagger, J. and Lambert, N.**, 2009. Three-Algebras and N=6 Chern-Simons Gauge Theories, *Phys.Rev.*, **D79**, 025002.

[50] **Lambert, N.**, (2012). M-Theory and Maximally Supersymmetric Gauge Theories, invited submission to Annual Reviews, comments welcome.

[51] **Pasti, P.**, **Samsonov, I.**, **Sorokin, D. and Tonin, M.**, (2009). BLG-motivated Lagrangian formulation for the chiral two-form gauge field in D=6 and M5-branes, *Phys.Rev.*, **D80**, 086008.

[52] **Howe, P.S.**, **Sezgin, E. and West, P.C.**, 1997. Covariant field equations of the M theory five-brane, *Phys.Lett.*, **B399**, 49–59.

[53] **Green, M.**, **Schwarz, J. and Witten, E.**, 1988. Superstring Theory: Introduction, Cambridge Monographs on Mathematical Physics, Cambridge University Press.

[54] **Gomis, J.**, **Rodriguez-Gomez, D.**, **Van Raamsdonk, M. and Verlinde, H.**, (2008). Supersymmetric Yang-Mills Theory From Lorentzian Three-Algebras, *JHEP*, **0808**, 094.

[55] **Nishi, C.**, (2005). Simple derivation of general Fierz-like identities, *Am.J.Phys.*, **73**, 1160–1163.

[56] **Nieves, J.F. and Pal, P.B.**, (2004). Generalized Fierz identities, *Am.J.Phys.*, **72**, 1100–1108.

[57] **Laenen E.**, `http://www.nikhef.nl/~t45/ftip/AppendixE.pdf`, date: 12.08.2011.

[58] **Avramis, S.D.**, 2006. Anomaly-free supergravities in six dimensions, `hep-th/arXiv:0611133`.

APPENDICES

APPENDIX A

Notation and Convention:

In this section, we summarize notation used in the thesis. The indices are:

Worldvolume coordinates:	$\mu, \nu = 0,1,2$ for M2-branes
	$\mu, \nu = 0,1,2,4,5$ for M5-branes
Transverse space coordinates:	$I,J = 3, \ldots\ldots, 10$ for M2-branes
	$I,J = 6, \ldots\ldots, 10$ for M5-branes
All 11D coordinates:	$m,n = 0,1,\ldots\ldots,10$
Spin(1, 10) spinor indices:	$\alpha, \beta = 1,\ldots\ldots,32$
Basis of Lie 3-algebra V:	a, b,....., dimV.

Spinorial quantities and fields are Grassmann variables, i.e. anti-commuting. We typically use Greek symbols such as Ψ, ε for Fermionic Grassmann fields and Roman symbols for Bosonic fields.

For 11 dimensions, the dimension of the spinor is $2^5 = 32$. In M-theory, the spinor must be Majorana. So we work 32-component Majorana spinors. But, for an M2-brane breaks the Lorentz symmetry as:

$$SO(10,1) \to SO(2,1) \times SO(8),$$

so that we can get a Weyl spinor of $SO(8)$ [4]. In the M5-brane theory, the fermions are Goldstinos and breaks the symmetry as:

$$SO(10,1) \to SO(5,1) \times SO(5).$$

The Dirac matrices Γ's are a representation of the 11-dimensional Clifford algebra, i.e. given m, n = 0, . . . , 10 it results

$$\{\Gamma^m, \Gamma^n\} = 2\eta^{mn}. \qquad \textbf{(A.1)}$$

We consider the "mostly minus" convention for the Minkowski metric:

$$\eta_{mn} = \begin{pmatrix} 1 & & & & & \\ & -1 & & & & \\ & & -1 & & & \\ & & & \cdots & & \\ & & & & \cdots & \\ & & & & & -1 \end{pmatrix}.$$

By defining the chirality matrix of SO(1,2) as:

$$\Gamma \equiv \Gamma_{012} \tag{A.2}$$

for M2 branes, the convention for the chirality condition we have is

$$\Gamma\varepsilon = \varepsilon \quad ; \quad \overline{\varepsilon}\Gamma = -\overline{\varepsilon} \quad ; \quad \Gamma\Psi = -\Psi \quad ; \quad \overline{\Psi}\Gamma = -\overline{\Psi}. \tag{A.3}$$

The (anti)commutation relations are:

$$[\Gamma, \Gamma^{\mu}] = 0, \quad \{\Gamma, \Gamma^{I}\} = 0. \tag{A.4}$$

Analogously, the chirality matrix of SO(5,1) is defined as:

$$\Gamma \equiv \Gamma_{012345} \tag{A.5}$$

which satisfies the following chirality conditions

$$\Gamma\varepsilon = \varepsilon \quad ; \quad \overline{\varepsilon}\Gamma = -\overline{\varepsilon} \quad ; \quad \Gamma\Psi = -\Psi \quad ; \quad \overline{\Psi}\Gamma = -\overline{\Psi}. \tag{A.6}$$

The (anti)commutation relations are:

$$\{\Gamma^{\mu}, \Gamma^{I}\} = 0 \quad ; \quad \{\Gamma, \Gamma^{I}\} = 0 \quad ; \quad \{\Gamma, \Gamma^{\mu}\} = 0. \tag{A.7}$$

The conjugate spinors are defined with the charge conjugation matrix

$$\overline{\Psi} = \Psi^{T} C \tag{A.8}$$

and for our representation we can choose $C = \Gamma^0$ [10]. This makes it antisymmetric $C^T = -C$ and antihermitian $C^{\dagger} = -C$ with $C^{-1} = -C$.
As $C^T = C^{-1} = -C$, one also has that

$$C\Gamma_{\mu}C^{-1} = -\Gamma_{\mu}^{T} \tag{A.9}$$

and

$$\{C, \Gamma^{I}\} = 0. \tag{A.10}$$

In the general form,

$$\begin{aligned} (C\Gamma_{m_1\ldots m_n})^T &= (\Gamma_{m_1\ldots m_n})^T C^T \\ &= (-1)^{n-1} C\Gamma_{m_n\ldots m_1} \\ &= (-1)^{n-1}(-1)^{[1+2+\ldots+(n-1)]} C\Gamma_{m_1\ldots m_n} \end{aligned} \tag{A.11}$$

and we have the following symmetry properties for Γ matrices [7].

symmetric matrix: $C\Gamma_{m_1}, C\Gamma_{m_1m_2}, C\Gamma_{m_1m_2m_3m_4m_5}, C\Gamma_{m_1m_2m_3m_4m_5m_6}, \cdots$
anti-symmetric matrix: $C\Gamma_{m_1m_2m_3}, C\Gamma_{m_1m_2m_3m_4}, C\Gamma_{m_1m_2m_3m_4m_5m_6m_7}, \cdots$

Using the results that we mentioned above, we can find a simple rule of the sign under exchanging locations of spinors :

$$\begin{aligned}
\bar{\varepsilon}_2\gamma^m\varepsilon_1 &= (-1)\bar{\varepsilon}_1\gamma^m\varepsilon_2 \\
\bar{\varepsilon}_2\gamma^{mn}\varepsilon_1 &= (-1)\bar{\varepsilon}_1\gamma^{mn}\varepsilon_2 \\
\bar{\varepsilon}_2\gamma^{mno}\varepsilon_1 &= (+1)\bar{\varepsilon}_1\gamma^{mno}\varepsilon_2 \\
\bar{\varepsilon}_2\gamma^{mnop}\varepsilon_1 &= (+1)\bar{\varepsilon}_1\gamma^{mnop}\varepsilon_2 \\
\bar{\varepsilon}_2\gamma^{mnopq}\varepsilon_1 &= (+1)\bar{\varepsilon}_1\gamma^{mnopq}\varepsilon_2. \qquad \textbf{(A.12)}
\end{aligned}$$

Using the chirality property $\Gamma\varepsilon = \varepsilon$ and $\bar{\varepsilon}\Gamma = -\bar{\varepsilon}$ we also see that

$$\begin{aligned}
\bar{\varepsilon}\Gamma^{\mu_1\ldots\mu_k}\Gamma^{I_1\ldots I_n}\varepsilon &= \bar{\varepsilon}\Gamma^{\mu_1\ldots\mu_k}\Gamma^{I_1\ldots I_n}\Gamma\varepsilon \\
&= (-1)^k\bar{\varepsilon}\Gamma\Gamma^{\mu_1\ldots\mu_k}\Gamma^{I_1\ldots I_n}\varepsilon \\
&= (-1)^{k+1}\bar{\varepsilon}\Gamma^{\mu_1\ldots\mu_k}\Gamma^{I_1\ldots I_n}\varepsilon. \qquad \textbf{(A.13)}
\end{aligned}$$

So, only if k is odd we could have no-zero value for $\bar{\varepsilon}\Gamma^{\mu_1\ldots\mu_k}\Gamma^{I_1\ldots I_n}\varepsilon$ [7]. We expressed symmetry properties and chiral properties above to derive the following Fierz identities and simplify several calculations used in the thesis.

APPENDIX B

The Fierz identities:

In this section the Fierz identities used repeatedly in the thesis are introduced.
The Fierz-reordering formula exchanges the order of anticommuting spinors. In other words, it is used to express a product of two Dirac bilinears (which are the matrix elements of gamma matrices and products of them) as a linear combination of other products of bilinears with the four constituent Dirac spinors in different order. Since Fierz identities relate Dirac bilinears, that are the objects with well defined transformation properties under the Lorentz group and also gamma matrices form a representation of the Lorentz group generators, it is natural that the Fierz identities imply the existence of a basis of four-vectors formed by Dirac bilinears and guarantee its orthogonality and completeness features. Furthermore, the reverse is possible, that is, recovering the spinor from that basis, which yields the equivalence between spinor and tensorial representation of various quantities [55]. Fierz identities are only a specific set of matrix identities, valid for the Dirac bilinears that span the space of complex matrices. Generalized Fierz identities can be obtained for any real or complex square matrices [55,56].
As is well known, considering the standard 4×4 Dirac matrices γ^μ and taking suitable products of them, a set of 16 matrices can be obtained that, among other features, span the 16 dimensional space of all the 4×4 matrices, the gamma matrices basis [56]. This

process can be generalized to D dimensions. The 2^D matrices Γ^A form a complete set of $2^{\frac{1}{2}D} \times 2^{\frac{1}{2}D}$ matrices for even D. In the odd D case, the 2^{D-1} matrices Γ^A of degree less than or equal to 2(D-1) are also a complete set of $2^{\frac{1}{2}(D-1)} \times 2^{\frac{1}{2}(D-1)}$ matrices. As a result, any matrix of the corresponding dimensionality can be expressed in terms of the Γ^A:

$$\begin{aligned}
M^{\alpha\beta} &= 2^{-\frac{1}{2}D} \sum_{n=0}^{D} \frac{1}{n!} Tr(M\gamma_{a_n\ldots a_1})(\gamma_{a_1\ldots a_n})_{\alpha\beta} \qquad for \quad even \quad D, \\
M^{\alpha\beta} &= 2^{-\frac{1}{2}D} \sum_{n=0}^{D} \frac{1}{n!} Tr(M\gamma_{a_n\ldots a_1})(\gamma_{a_1\ldots a_n})_{\alpha\beta} \qquad for \quad odd \quad D. \qquad \textbf{(A.14)}
\end{aligned}$$

The right hand side is split by factors $2^{\frac{1}{2}(D-1)} \times 2^{\frac{1}{2}(D-1)}$, which represent the trace of the unit matrix for even and odd D, respectively; the factor $\frac{1}{n!}$ is included to avoid summing $n!$ times over the same matrix $\gamma_{a_1\ldots a_n}$. Furthermore, notice that the indices $\{a_1 \ldots a_n\}$ appear twice but in opposite order to avoid extra minus signs [57]. Consequently, the completeness relation can be used to reorder spinors in expressions such as $(\bar{\psi}\Gamma^A\chi)(\bar{\xi}\Gamma^B\zeta)$. In the even D case, it can be derived

$$(\bar{\psi}\Gamma^A\chi)(\bar{\xi}\Gamma^B\zeta) = 2^{-\frac{1}{2}D} \sum_{n=0}^{D} \frac{1}{n!} (\bar{\psi}\Gamma^A\gamma_{a_n\ldots a_1}\Gamma^B\zeta)(\bar{\xi}\gamma_{a_1\ldots a_n}\chi) \qquad \textbf{(A.15)}$$

for commuting spinors (for anticommuting spinors there is an extra factor -1).
After these results, we can obtain the Fierz identities for M2-branes and M5-branes in 11 dimensions.

In 11 D spacetime the basis $\{T^a\}$ of the vector space of $2^{\frac{11}{2}} \times 2^{\frac{11}{2}}$ matrices are:

$$\{T^a\} = \{1, \Gamma_m, i\Gamma_{mn}, i\Gamma_{mnp}, \Gamma_{mnpq}, \Gamma_{mnpqr}\} \qquad \textbf{(A.16)}$$

which satisfies the condition

$$Tr\{T^a, T^b\} = 2^{\frac{11}{2}} \delta_{ab}. \qquad \textbf{(A.17)}$$

Using the basis $\{T^a\}$ in 11D, we have the following generalized expression,

$$\begin{aligned}
(\bar{\varepsilon}_2\chi)\varepsilon_1 &= -2^{-[\frac{11}{2}]}[(\bar{\varepsilon}_2\varepsilon_1)\chi + (\bar{\varepsilon}_2\Gamma_m\varepsilon_1)\Gamma^m\chi - \frac{1}{2!}(\bar{\varepsilon}_2\Gamma_{mn}\varepsilon_1)\Gamma^{mn}\chi \\
&\quad - \frac{1}{3!}(\bar{\varepsilon}_2\Gamma_{mnp}\varepsilon_1)\Gamma^{mnp}\chi + \frac{1}{4!}(\bar{\varepsilon}_2\Gamma_{mnpq}\varepsilon_1)\Gamma^{mnpq}\chi \\
&\quad + \frac{1}{5!}(\bar{\varepsilon}_2\Gamma_{mnpqr}\varepsilon_1)\Gamma^{mnpqr}\chi]. \qquad \textbf{(A.18)}
\end{aligned}$$

From the symmetry property that mentioned above, one can have the following expression in eleven-dimensions:

$$\begin{aligned}
(\bar{\varepsilon}_2\chi)\varepsilon_1 - (\bar{\varepsilon}_1\chi)\varepsilon_2 &= -\frac{1}{16}[(\bar{\varepsilon}_2\Gamma_m\varepsilon_1)\Gamma^m\chi - \frac{1}{2!}(\bar{\varepsilon}_2\Gamma_{mn}\varepsilon_1)\Gamma^{mn}\chi \\
&\quad + \frac{1}{5!}(\bar{\varepsilon}_2\Gamma_{mnpqr}\varepsilon_1)\Gamma^{mnpqr}\chi] \qquad \textbf{(A.19)}
\end{aligned}$$

where m = 0, ..., 10 [7].

In the case of $\mu, \nu = 0,1,2$ and $I,J = 3,.....,10$. ε_1 and ε_2 have the same chirality with respect to Γ_{012}, so the only terms that contribute must have an even number of I indices. Besides, in this case the generalized expression is only nonvanishing when χ has the same Γ_{012} chirality as ε_1 and ε_2. When this is the case for M2 branes, the general expression reduces to [3]:

$$\begin{aligned}(\bar{\varepsilon}_2\chi)\varepsilon_1 - (\bar{\varepsilon}_1\chi)\varepsilon_2 &= -\frac{1}{16}[2\bar{\varepsilon}_2\Gamma_\mu\varepsilon_1)\Gamma^\mu\chi - (\bar{\varepsilon}_2\Gamma_{IJ}\varepsilon_1)\Gamma^{IJ}\chi \\ &\quad +\frac{1}{4!}(\bar{\varepsilon}_2\Gamma_\mu\Gamma_{IJKL}\varepsilon_1)\Gamma^\mu\Gamma^{IJKL}\chi. \qquad \textbf{(A.20)}\end{aligned}$$

In the case of $\mu, \nu = 0,1,2,3,4,5$ and $I,J = 6,.....,10$ for M5 branes, the general expression can be obtained similarly. Splitting SO(10,1)$\rightarrow$SO(5,1)$\otimes$ SO(5) it can be obtained that, since ε_1 and ε_2 have the same chirality with respect to Γ_{012345} (while $\overline{\varepsilon}_1$ and $\overline{\varepsilon}_2$ the opposite) the nonvanishing terms must involve odd powers of Γ_μ's. Also, the expression is only nonvanishing in case ε_1 has the opposite chirality as ε_2. Considering the chirality properties, one then gets

$$\begin{aligned}(\overline{\varepsilon}_2\chi)\varepsilon_1 - (\overline{\varepsilon}_1\chi)\varepsilon_2 &= -\frac{1}{16}[(\overline{\varepsilon}_2\Gamma_\mu\varepsilon_1)\Gamma^\mu\chi - (\overline{\varepsilon}_2\Gamma_\mu\Gamma^I\varepsilon_1)\Gamma^\mu\Gamma^I\chi \\ &\quad +\frac{1}{3!}\frac{1}{2!}(\overline{\varepsilon}_2\Gamma_{\mu\nu\lambda}\Gamma^{IJ}\varepsilon_1)\Gamma^{\mu\nu\lambda}\Gamma^{IJ}\chi \\ &\quad +\frac{1}{4!}(\overline{\varepsilon}_2\Gamma_\mu\Gamma^{IJKL}\varepsilon_1)\Gamma^\mu\Gamma^{IJKL}\chi \\ &\quad +\frac{1}{5!}(\overline{\varepsilon}_2\Gamma_{\mu\nu\lambda\rho\sigma}\varepsilon_1)\Gamma^{\mu\nu\lambda\rho\sigma}\chi]. \qquad \textbf{(A.21)}\end{aligned}$$

Rearranging the last line above in terms of fewer Γ-matrices with the help of ε-tensors. The final form of Fierz identity in 11 dimensions becomes [10],

$$\begin{aligned}(\overline{\varepsilon}_2\chi)\varepsilon_1 - (\overline{\varepsilon}_1\chi)\varepsilon_2 &= -\frac{1}{16}[2(\overline{\varepsilon}_2\Gamma_\mu\varepsilon_1)\Gamma^\mu\chi - 2(\overline{\varepsilon}_2\Gamma_\mu\Gamma^I\varepsilon_1)\Gamma^\mu\Gamma^I\chi \\ &\quad +\frac{1}{3!}\frac{1}{2!}(\overline{\varepsilon}_2\Gamma_{\mu\nu\lambda}\Gamma^{IJ}\varepsilon_1)\Gamma^{\mu\nu\lambda}\Gamma^{IJ}\chi]. \qquad \textbf{(A.22)}\end{aligned}$$

APPENDIX C

Gamma matrix identities:

In this section, we provide some useful Gamma Matrix identities used in the thesis.

Since the antisymmetrized matrices (**A.16**) form a basis, any gamma matrix product can be expressed as a linear combination of these matrices. Using the Clifford algebra, one can easily derive the relations:

$$\Gamma^a\Gamma^{b_1.....b_n} = \Gamma^{ab_1.....b_n} + ng^{a[b_1}\Gamma^{b_2.....b_n]} \qquad \textbf{(A.23)}$$

$$\Gamma^{b_1.....b_n}\Gamma^a = \Gamma^{b_1.....b_na} + n\Gamma^{[b_1.....b_{n-1}}g^{b_n]a}. \qquad \textbf{(A.24)}$$

The most general such formula is [7]:

$$\begin{aligned}\Gamma^{b_1.....b_n}\Gamma_{a_1.....a_m} &= \sum_{p=0}^{min(n,m)} \frac{n!m!}{(n-p)!(m-p)!p!} \\ &.\Gamma^{[b_1.....b_{n-p}}{}_{[a_{p+1}.....a_m}g^{b_{n-p+1}.....b_n]}{}_{a_1.....a_p}\end{aligned} \quad \textbf{(A.25)}$$

$$\begin{aligned}\Gamma_{b_1.....b_n}\Gamma^{a_1.....a_n} &= \sum_{p=0}^{min(n,m)} \frac{n!m!}{(n-p)!(m-p)!p!} \\ &.\Gamma_{[b_1.....b_{n-p}}{}^{[a_{p+1}.....a_m}g_{b_{n-p+1}.....b_n]}{}^{a_1.....a_p}\end{aligned} \quad \textbf{(A.26)}$$

from which it also follows that

$$[\Gamma^a,\Gamma^{b_1.....b_n}] = (1-(-1)^n)\Gamma^{ab_1.....b_n} + n(1+(-1)^n)g^{a[b_1}\Gamma^{b_2.....b_n]} \quad \textbf{(A.27)}$$

$$\{\Gamma^a,\Gamma^{b_1.....b_n}\} = (1+(-1)^n)\Gamma^{ab_1.....b_n} + n(1-(-1)^n)g^{a[b_1}\Gamma^{b_2.....b_n]}. \quad \textbf{(A.28)}$$

Firstly, we can write some simple commutation relations [7], [58]:

$$\Gamma^a\Gamma^b = \Gamma^{ab} + g^{ab} \quad \textbf{(A.29)}$$

$$\Gamma^a\Gamma^{bc} = \Gamma^{bc}\Gamma^a + 4g^{a[b}\Gamma^{c]} \quad \textbf{(A.30)}$$

$$\Gamma^a\Gamma^{bcd} = -\Gamma^{bcd}\Gamma^a + 6g^{a[b}\Gamma^{cd]} \quad \textbf{(A.31)}$$

$$\Gamma^a\Gamma^{bcde} = \Gamma^{abcde} + g^{ab}\Gamma^{cde} - g^{ac}\Gamma^{bde} + g^{ad}\Gamma^{bce} - g^{ae}\Gamma^{bcd} \quad \textbf{(A.32)}$$

$$\Gamma^{ab}\Gamma^{cd} = \Gamma^{abcd} + g^{ad}\Gamma^{bc} - g^{ac}\Gamma^{bd} + g^{bc}\Gamma^{ad} - g^{bd}\Gamma^{ac} + g^{ad}g^{bc} - g^{ac}g^{bd} \quad \textbf{(A.33)}$$

$$\begin{aligned}\Gamma^{ab}\Gamma^{cde} &= \Gamma^{abcde} - g^{ac}\Gamma^{bde} + g^{ad}\Gamma^{bce} - g^{ae}\Gamma^{bcd} + g^{bc}\Gamma^{ade} - g^{bd}\Gamma^{ace} \\ &+ g^{be}\Gamma^{acd} + (g^{ad}g^{bc} - g^{ac}g^{bd}) - (g^{ae}g^{bc} \\ &- g^{ac}g^{be}) + (g^{ae}g^{bd} - g^{ad}g^{be})\Gamma^c.\end{aligned} \quad \textbf{(A.34)}$$

From the relations above, we can obtain the following useful relations that are used during the calculation of the closures in the thesis repeatedly. (Here a,b,c,...is the general index and n is defined by $\sum_a \Gamma^a\Gamma^a = n$) of the supersymmetry algebra.

For example, considering $\sum_a \Gamma^a\Gamma^a = n$, we can get [7]

$$\Gamma_a\Gamma^{ab} = \Gamma^{ba}\Gamma_a = (n-1)\Gamma^b \quad \textbf{(A.35)}$$

$$\Gamma_a\Gamma^{abc} = \Gamma^{bca}\Gamma_a = (n-2)\Gamma^{bc} \quad \textbf{(A.36)}$$

$$\Gamma_a\Gamma^{abcd} = \Gamma^{bcda}\Gamma_a = (n-3)\Gamma^{bcd} \quad \textbf{(A.37)}$$

$$\Gamma_{ba}\Gamma^{ac} = (n-2)\Gamma_b{}^c + (n-1)g_b{}^c \quad \textbf{(A.38)}$$

$$\Gamma_{ab}\Gamma^{ab} = -n(n-1) \quad \textbf{(A.39)}$$

$$\Gamma_{de}\Gamma^{eabc} = (n-4)\Gamma_d\Gamma^{abc} - 3g_d{}^{[a}\Gamma^{bc]} \quad \textbf{(A.40)}$$

$$\Gamma_{ab}\Gamma^{abc} = \Gamma^{cab}\Gamma_{ab} = -(n-1)(n-2)\Gamma^c. \quad \textbf{(A.41)}$$

Some triple summation relations are:

$$\begin{aligned}
\Gamma^a\Gamma^b\Gamma^a &= (2-n)\Gamma^b && \textbf{(A.42)}\\
\Gamma^a\Gamma^{bc}\Gamma^a &= (n-4)\Gamma^{bc} && \textbf{(A.43)}\\
\Gamma^a\Gamma^{bcd}\Gamma^a &= (6-n)\Gamma^{bcd}. && \textbf{(A.44)}
\end{aligned}$$

The following identities used in the thesis are also found to be useful:

$$\begin{aligned}
\Gamma^{abc}\Gamma^d\Gamma^a &= (4-n)\Gamma^d\Gamma^{bc} - (12-4n)g^{d[b}\Gamma^{c]} && \textbf{(A.45)}\\
\Gamma^{abc}\Gamma^{de}\Gamma^a &= (n-6)\Gamma^{de}\Gamma^{bc} + (2n-10)[g^{dc}\Gamma^{be} - g^{ec}\Gamma^{bd} + g^{bd}\Gamma^{ec} \\
&\quad -g^{be}\Gamma^{dc}] + 4[g^{dc}g^{be} - g^{ec}g^{bd}]. && \textbf{(A.46)}
\end{aligned}$$

We used also the relations:

$$\begin{aligned}
\Gamma^{KLM} &= \Gamma^K\Gamma^{LM} - g^{KL}\Gamma^M + g^{KM}\Gamma^L && \textbf{(A.47)}\\
\Gamma^I\Gamma^\mu\Gamma^J + \Gamma^J\Gamma^\mu\Gamma^I &= -2g^{IJ}\Gamma^\mu && \textbf{(A.48)}\\
\Gamma^I\Gamma^{JKL} + \Gamma^{JKL}\Gamma^I &= 2g^{IJ}\Gamma^{KL} + 2g^{IL}\Gamma^{JK} - 2g^{IK}\Gamma^{JL} && \textbf{(A.49)}\\
\Gamma^I\Gamma_\lambda\Gamma^{JK} + \Gamma_\lambda\Gamma^{JK}\Gamma^I &= -\Gamma_\lambda\Gamma^I\Gamma^{JK} + \Gamma_\lambda\Gamma^{JK}\Gamma^I \\
&= \Gamma_\lambda\left(\Gamma^{JK}\Gamma^I - \Gamma^I\Gamma^{JK}\right) \\
&= \Gamma_\lambda\left(-2g^{IJ}\Gamma^K + 2g^{IK}\Gamma^J\right) \\
&= -2g^{IJ}\Gamma_\lambda\Gamma^K + 2g^{IK}\Gamma_\lambda\Gamma^J && \textbf{(A.50)}\\
\Gamma_M\Gamma^{IJ}\Gamma^M &= 4\Gamma^{IJ} && \textbf{(A.51)}\\
\Gamma_M\Gamma^{IJKL}\Gamma^M &= 0 && \textbf{(A.52)}\\
\Gamma^{IJP}\Gamma^{KLMN}\Gamma_P &= -\Gamma^I\Gamma^{KLMN}\Gamma^J + \Gamma^J\Gamma^{KLMN}\Gamma^I && \textbf{(A.53)}\\
\Gamma^I\Gamma^{KL}\Gamma^J - \Gamma^J\Gamma^{KL}\Gamma^I &= 2\Gamma^{KL}\Gamma^{IJ} - 2\Gamma^{KJ}\delta^{IL} + 2\Gamma^{KI}\delta^{JL} \\
&\quad -2\Gamma^{LI}\delta^{JK} + 2\Gamma^{LJ}\delta^{IK} + 4\delta^{KI}\delta^{JL} - 4\delta^{KJ}\delta^{IL} && \textbf{(A.54)}\\
\Gamma^{IJK}\Gamma^{LM}\Gamma_K &= 2\Gamma^{LM}\Gamma^{IJ} - 6\Gamma^{LJ}\delta^{IM} + 6\Gamma^{LI}\delta^{JM} \\
&\quad -6\Gamma^{MI}\delta^{JL} + 6\Gamma^{MJ}\delta^{IL} + 4\delta^{LJ}\delta^{IM} - 4\delta^{LI}\delta^{JM} && \textbf{(A.55)}
\end{aligned}$$

$$\begin{aligned}
\Gamma^{\rho\sigma\tau}\Gamma_{\nu\lambda} &= -\Gamma_{\nu\lambda}\Gamma^{\rho\sigma\tau} + 2g^{\lambda\rho}g^{\nu\sigma}\Gamma^\tau - 2g^{\lambda\rho}g^{\nu\tau}\Gamma^\sigma + 2g^{\lambda\sigma}g^{\nu\tau}\Gamma^\rho \\
&\quad -2g^{\lambda\sigma}g^{\nu\rho}\Gamma^\tau + 2g^{\lambda\tau}g^{\nu\rho}\Gamma^\sigma - 2g^{\lambda\tau}g^{\nu\sigma}\Gamma^\rho && \textbf{(A.56)}\\
\Gamma^{\mu\nu\lambda}\Gamma^{\rho\sigma\tau}\Gamma_{\nu\lambda} &= -30\Gamma^\mu\Gamma^{\rho\sigma\tau} + 12\Gamma^\mu\Gamma^{\sigma\tau}\Gamma^\rho + 12\Gamma^\mu\Gamma^{\rho\sigma}\Gamma^\tau \\
&\quad +12\Gamma^\mu\Gamma^{\tau\rho}\Gamma^\sigma && \textbf{(A.57)}\\
\Gamma^{\rho\gamma\tau}\Gamma^{\nu\lambda\sigma}\Gamma_{\rho\gamma\tau\kappa} &= -12\Gamma_\kappa\Gamma^{\nu\lambda\sigma.} && \textbf{(A.58)}
\end{aligned}$$

CURRICULUM VITAE

Name Surname: Aytül FİLİZ

Place and Date of Birth: Erzincan, 27.03.1983

Adress: Istanbul Technical University, Faculty of Science and Letters, Department of Physics Engineering, Maslak, Istanbul, Turkey

E-Mail: aytulfiliz@gmail.com

B.Sc.: Physics, Istanbul University

B.Sc.: Mathematics, Istanbul University