İSTANBUL TECHNICAL UNIVERSITY ★ INSTITUTE OF SCIENCE AND TECHNOLOGY

DYNAMIC ANALYSIS OF A FLEXIBLE FOUR BAR MECHANISM WITH USING MATLAB SIMULINK

M.Sc. Thesis by
Elif SAKAR KILINÇ

Department : Aeronautical And Astronautical Engineering

Programme : Aeronautical And Astronautical Engineering

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M.Sc. Thesis by
Elif SAKAR KILINÇ
(511061033)

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Supervisor (Chairman) : **Prof. Dr. İbrahim ÖZKOL (ITU)**
Members of the Examining Committee : **Prof. Dr. Metin O. KAYA (ITU)**
Assoc. Prof. Dr. Erol UZAL (IU)

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MATLAB SIMULINK KULLANARAK ELASTİK DÖRT ÇUBUK MEKANIZMASININ DİNAMİK ANALİZİ

YÜKSEK LİSANS TEZİ
Elif SAKAR KILINÇ
(511061033)

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Tez Danışmanı : Prof. Dr. İbrahim ÖZKOL(İTÜ)
Diğer Jüri Üyeleri : Prof. Dr. Metin O. KAYA (İTÜ)
Doç. Dr. Erol UZAL (IU)

AĞUSTOS 2010

FOREWORD

I am dedicating this thesis to my dear parents who provided their all support to me during my lifetime.

I want to express my deep appreciation and thanks for my advisor Prof. Dr. İbrahim ÖZKOL due to the supports and contributions in preparing this thesis.

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May 2010

Elif SAKAR KILINÇ

Mechatronic Engineer

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ABBREVIATIONS

FBD **:** Free Body Diagram

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DYNAMIC ANALYSIS OF AN ELASTIC FOUR BAR MECHANISM WITH USING MATLAB SIMULINK

SUMMARY

In this study, dynamic analysis of an elastic and rijid planar four bar mechanism carried out. In analysis:

-Completely rijid four bar mechanism,

-Elastic coupler link four bar mechanism,

-Elastic output link four bar mechanism are used. Driving force is applied by a motor from the same point – crank link – for all types of four-bar mechanism in this study.

In the first step of the study, kinematics of a rijid planar four mechanism is observed. Analyses results provided by using Matlab Simulink. The results are resemble to the results in literature.

In the second and the third step of this study, dynamic differential equations obtained for all four bar mechanisms which are used in this study. Analyses solutions derived by files which are prepared with using Matlab Simulink. All files are particular for that mechanism. The graphs compared which are obtained by analyses results. This comparison gives idea about working efficency of rijid and elastic four bar mechanisms.

MATLAB SIMULINK KULLANARAK ELASTİK DÖRT ÇUBUK MEKANİZMASININ DİNAMİK ANALİZİ

ÖZET

Bu çalışmada, elastik ve rijid düzlemsel bir dört çubuk mekanizmasının kinematik ve dinamik analizi gerçekleştirilmiştir. Analizlerde:

-Dört çubuğun rijid,

-Yalnızca kuplör çubuğunun elastik,

-Yalnızca çıkış çubuğunun elastik, olduğu durumlar göz önüne alınmıştır. Bu çalışmada ele alınan tüm dört çubuk mekanizmalarında tahrik tek bir noktadan – krank çubuğu- bir motor vasıtasıyla verilmiştir.

Çalışmanın ilk aşamasında, rijid bir düzlemsel dört çubuk mekanizmasının kinematiği incelenmiştir. Matlab Simulink programı kullanarak analiz sonuçlarına ulaşılmıştır. Elde edilen sonuçlar literatürde daha önceden yapılmış çalışmaların sonuçları ile karşılaştırılmış olup uyumun çok yüksek seviyede olduğu görülmüştür.

Çalışmanın ikinci ve üçüncü bölümünde, her mekanizma için dinamik denklemler çıkartılmış olup, her mekanizma için ayrı ayrı hazırlanan Matlab Simulink program dosyaları ile analiz sonuçlarına ulaşılmıştır. Bu sonuçlar yoluyla elde edilen grafikler kıyaslanmıştır. Bu kıyaslama rjid ve elastik dört çubuk mekanizmasının çalışma verimliliği hakkında fikir vermektedir.

1. INTRODUCTION

1.1 Purpose of the Thesis

In this study, a constrained modelling approach is used. Dynamic modelling for a flexible four-bar mechanism is studied. The fully coupled non-linear equations of motions are obtained through a constrained Lagrangian approach. Resulting differential-algebraic equations are solved numerically to obtain the system response.

High operating speeds, superior reliability and accurate performance are major characteristics of modern industrial machinery and commercial equipment.The traditional rigid body analysiswhich presumes low operating speeds becomes insufficient for describing the performance ofsuch highspeedmachinery. A thorough understanding of the dynamic behaviors of machineelements undergoing highspeedoperations is necessary in this situation Erdman and Sandor and Lowen and Chassapis have demonstrated in their surveys of past works that thedynamic behaviors of elastic linkagesare quite different from those of rigid linkages. Thestudies made by Golebiewski and Sadler, Alexander and Lawrance, and Stamps andBagci have revealed the phenomenon of lower order harmonic resonances in dynamic responses of elastic linkages. A four-bar mechanism in high-speed motion may become unstable when its inputlink rotates at a speed falling within an unstable range. Significant deformations, strains andstresses may develop in one or more flexible links. Both forced and parametric excitations may cause significant responses in a flexible mechanism when resonance occurs.Flexible links in amechanism are commonly modeled as elastic beams with and without consideration of the effects of large deformations, shear deformations, rotary inertia and axialdeformations.

The boundary conditions at the joint locations are usually unknown at this stage. Once modeling of an unconstrained link is completed, the Lagrange multiplier method or the augmented Lagrange equations, may be used to formulate the equations of motion for the entire mechanismby enforcing continuity conditions across the interfaces. These differential equations governing the kineto-elastodynamic behaviors of a mechanism may be solved directly using numerical or analytical methods. Because of the large size of the dynamical equations, computing time for an accurate dynamic analysis becomesa concern.The nodal coordinates are transformed intomodal coordinates through a linear modal transformation.

All three-nodal displacements aremeasured with respect to its rigid-body configuration. Within each element, the shape function is aquintic polynomial for the lateral displacement and a quadratic polynomial for the longitudinal displacement. Once equations of unconstrained links are obtained, the global equations of motion for theentire mechanism are formulated using the Lagrange multiplier method.

Traditionally, dynamic analysis and control of mechanisms have been basedon the assumption that the links behave as rigid bodies.In mechanical systemsoperating at high speeds, some oscillatory elastic motion is inevitable.Thismotion becomes a major concern when performance requirements are such thathigh precision is important. Because of the trend toward increasing operatingspeeds and reducing weights in modern machinery, it may be inaccurate to treatcertain links in such systems as rigid bodies. The effect of flexibility on thedynamic behavior of mechanical systems has been the subject of numerous investigations. Mathematical modelling of flexiblemechanisms is complicated. The rigid body motion involves changing geometriesresulting in varying system parameters, and these influence the elasticdeformations. The elastic deformations themselves in turn influence the rigidbody motion. Though the importance of including the effect of elastic motion in dynamic modelling has been recognized for quite some time, most models were based on a prescribed rigid body motion.The assumption here is that the elastic deformations do not have significant effect on rigid body motion.

1.2 Content of the Study

Kinematic and dynamic modelling for a planar four-bar is observed.

In Section 2 : Kinematic analysis of the four-bar mechanism is examined. General analysis of a rigid four-bar mechanism is examined in Section 2.1. Subheads of that section are; General configuration of the mechanism, Velocity and acceleration analysis, and Position analysis using the vector cross product. In Section 2.2 we get kinematic analysis datas by using Matlab Simulink.

Dynamic analysis of a rigid four-bar mechanism is explored in Section 3. Section 3.1 is named as forces on four-bar mechanism. And subheads of this section are; FBD for each bar of the four-bar mechanism, Reaction forces acting on each bar of the mechanism, Simulink model for a four-bar mechanism. In Section 3.2 Dynamic analyses of rigid four-bar mechanism is examined.

Elastic four-bar mechanism's dynamic analyses is examined in Section 4. In that section elastic coupler link and elastic output link dynamic analyses is examined. All analyses solution is discovered by using Matlab Simulink.

Conclusions and recommendations take part at last section.

2. KINEMATIC ANALYSIS OF THE FOUR-BAR MECHANISM

A four-bar mechanism is shown in Figure 2.1. The mechanism actually consists of three moving bars (or links) AB (with $\left|\vec{AB}\right| = L_2$ **)**, BC (with $\left|\vec{BC}\right| = L_3$**)**, and CD (with $\left|\vec{CD}\right| = L_4$**)** joined by pins at B and C, and pinned to the ground (i.e. to the plane of the paper) at A and at D. The fourth bar is AD, is with $\left|\vec{AD}\right| = L_1$, which givs rise to the name of the mechanism. The bars are approximated as rigid bodies, and their lengths are essentially approximated as L_2, L_3, and L_4, respectively. In practice, the mechanism is constructed so that the bars move along different, but parallel, planes to avoid collisions; the pins connecting the bars would be long like a nail but perpendicular to the plane of the paper. Points A and D are the centers of rotation of bars AB and CD, respectively.

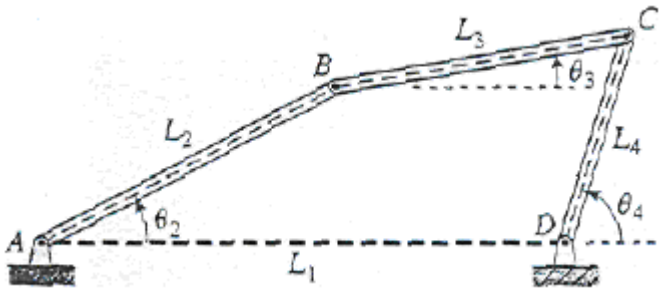

Figure 2.1A four-bar mechanism (the angles θ_2, θ_3, and $\theta_4$are changing with time)

Either link AB or link CD is driven by a motor (which is not shown in the figure) in order to make either $\theta_2$or θ_4 a desired function of time t. In such a case, the motion of the mechanism is completely determined by the knowledge of either θ_2(t)or θ_4(t). Taking the angle θ_2(t) to be the known variable, the objective of the analysis is to determine how the other angles shown in Figure 2.1 and 2.2 and their first and second time derivatives (i.e. the angular velocity and angular acceleration of the other links) depend on θ_2, $\dot{\theta}_2$ and $\ddot{\theta}_2$**.** There are no different equations to be

integrated here since the motion is specified. This type of analysis is called kinematic analysis, as opposed to dynamic analysis where the objective would have been to determine the motion of the system when known forces are applied to it and/or the reaction forces that any of the bars are subjected to during the motion.

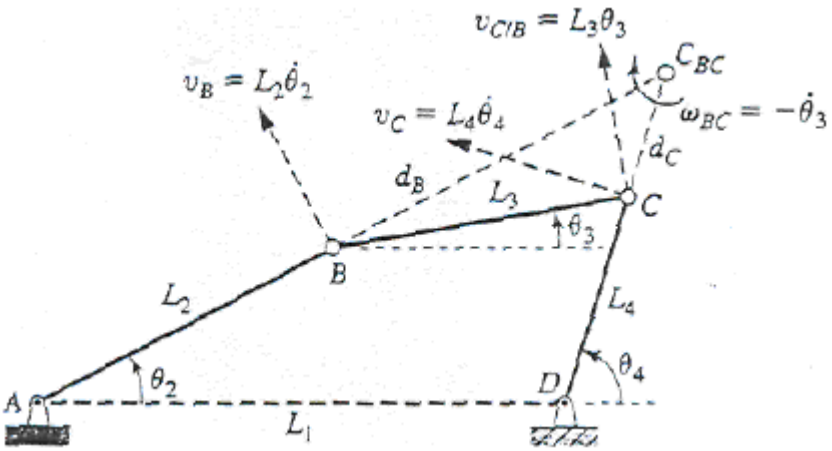

Figure 2.2 A four-bar mechanism and the instantaneous center of rotation of bar BC

2.1 General Analysis of the Four-Bar Mechanism

The general kinematic analysis of four-bar mechanisms, which is based on equations that are valid for all instants of time, is presented in section 2.1.

2.1.1 General configuration of the four-bar mechanism

The kinematic analysis of any mechanism involves first drawing a simple sketch of the mechanism ina general configuration, and then writing a set of geometric relations by inspection of the sketch. Such relations consist of equations for appropriate distances along perpendicular directions. The general sketch for the four-bar mechanism is shown in Figure 2.1. The following two equations (which are called the loop equations for the four-bar mechanism) can be written by inspecting Figure 2.2. They are simply the scalar components along line AD, and along the perpendicular to line AD, of the vector identity $A\vec{D} = A\vec{B} + B\vec{C} + C\vec{D}$.

$$L_2\cos\theta_2 + L_3\cos\theta_3 - L_4\cos\theta_4 = L_1 \qquad \textbf{(2.1)}$$

$$L_4 \sin\theta_4 = L_2 \sin\theta_2 + L_3 \sin\theta_3 \qquad \textbf{(2.2)}$$

The left and right-hand sides of **(2.1)** repsent two expressions for the same distance $\left|\vec{AD}\right|$ and **(2.2)** represents two expressions for the perpendicular distance form point C to line AD. Notice that the mechanism was drawn in a general configuration in Figure 2.1 and 2.2 for generating **(2.1)** and **(2.2)**.

2.1.2 Velocity and acceleration analysis

Once **(2.1)** and **(2.2)** are obtained, the problem of determining the angular velocities and accelerations of any of the bars of the mechanism is essentially solved. That is done simply by taking the first and second time derivatives of those equations to obtain a set of equations that involve the angular velocities $\dot{\theta}_2$, $\dot{\theta}_3$, and $\dot{\theta}_4$ and the angular accelerations $\ddot{\theta}_2$, $\ddot{\theta}_3$, and $\ddot{\theta}_4$. This general procedure yields the equations that allow one to determine the angular velocities and the angular accelerations of all members of a four-bar mechanism in any configuration.

The following equations that involve angular velocities are obtained by taking the first time derivative of **(2.1)** and **(2.2)**.

$$L_2\dot{\theta}_2 \sin\theta_2 + L_3\dot{\theta}_3 \sin\theta_3 - L_4\dot{\theta}_4 \sin\theta_4 = 0 \qquad \textbf{(2.3)}$$

$$L_2\dot{\theta}_2 \cos\theta_2 + L_3\dot{\theta}_3 \cos\theta_3 - L_4\dot{\theta}_4 \cos\theta_4 = 0 \qquad \textbf{(2.4)}$$

(2.3) and **(2.4)** are linear in the angular velocities and, thus, can be easily solved for $\dot{\theta}_3$ and $\dot{\theta}_4$ in terms of $\dot{\theta}_2$ and the angles θ_2, θ_3, and θ_4 when their values are obtained from **(2.1)** and **(2.2)**. Having the numerical values of the angles θ_2, θ_3, and θ_4, the solution for the angular velocities $\dot{\theta}_3$ and $\dot{\theta}_4$ can be easily obtained with a simple matrix inversion and a matrix multiplication as follows. The solution can also be readily obtained by hand or with a calculator.

$$\begin{bmatrix} L_3\sin\theta_3 & -L_4\sin\theta_4 \\ L_3\cos\theta_3 & -L_4\cos\theta_4 \end{bmatrix} \begin{bmatrix} \dot{\theta}_3 \\ \dot{\theta}_4 \end{bmatrix} = \begin{bmatrix} -L_2\dot{\theta}_2\sin\theta_2 \\ -L_2\dot{\theta}_2\cos\theta_2 \end{bmatrix}$$

or

$$\begin{bmatrix}\dot{\theta}_3\\ \dot{\theta}_4\end{bmatrix}=\begin{bmatrix}L_3\sin\theta_3 & -L_4\sin\theta_4\\ L_3\cos\theta_3 & -L_4\cos\theta_4\end{bmatrix}^{-1}\begin{bmatrix}-L_2\dot{\theta}_2\sin\theta_2\\ -L_2\dot{\theta}_2\cos\theta_2\end{bmatrix} \tag{2.5}$$

The angular accelerations $\ddot{\theta}_3$, and $\ddot{\theta}_4$ of links BC and DC, respectively, are obtained by simply taking the time derivative of **(2.3)** and **(2.4)**. This yields

$$L_2(\ddot{\theta}_2\sin\theta_2+\dot{\theta}_2{}^2\cos\theta_2)+L_3(\ddot{\theta}_3\sin\theta_3+\dot{\theta}_3{}^2\cos\theta_3)=L_4(\ddot{\theta}_4\sin\theta_4+\dot{\theta}_4{}^2\cos\theta_4) \tag{2.6}$$

$$L_2(\ddot{\theta}_2\cos\theta_2-\dot{\theta}_2{}^2\sin\theta_2)+L_3(\ddot{\theta}_3\cos\theta_3-\dot{\theta}_3{}^2\sin\theta_3)=L_4(\ddot{\theta}_4\cos\theta_4-\dot{\theta}_4{}^2\sin\theta_4) \tag{2.7}$$

(2.6) and **(2.7)** are linear in the angular accelerations and therefore, can be easily solved for $\ddot{\theta}_3$, and $\ddot{\theta}_4$ in terms of the angular accelerations $\ddot{\theta}_2$; of the angular velocities $\dot{\theta}_2$, $\dot{\theta}_3$, and $\dot{\theta}_4$; and of the angles θ_2, θ_3, and θ_4 once their values are known. In matrix form, the solution for the angular accelerations $\ddot{\theta}_3$, and $\ddot{\theta}_4$ is obtained as follows. Again, the solution can readily be obtained by hand or with a calculator.

$$\begin{bmatrix}\ddot{\theta}_3\\ \ddot{\theta}_4\end{bmatrix}=\begin{bmatrix}L_3\sin\theta_3 & -L_4\sin\theta_4\\ L_3\cos\theta_3 & -L_4\cos\theta_4\end{bmatrix}^{-1}\times\begin{bmatrix}L_4\dot{\theta}_4{}^2\cos\theta_4-L_3\dot{\theta}_3{}^2\cos\theta_3-L_2(\ddot{\theta}_2\sin\theta_2+\dot{\theta}_2{}^2\cos\theta_2)\\ -L_4\dot{\theta}_4{}^2\sin\theta_4+L_3\dot{\theta}_3{}^2\sin\theta_3-L_2(\ddot{\theta}_2\cos\theta_2-\dot{\theta}_2{}^2\sin\theta_2)\end{bmatrix} \tag{2.8}$$

Of all the equations for this mechanism [**(2.1)** and **(2.2)**, **(2.5)** and **(2.8)**]. **(2.1)** and **(2.2)** are the most difficult to solve because they are nonlinear in the unknown angles θ_3 and θ_4 (with the nonlinearities being a trigonometric function). Although it is possible to find an analytical solution to those equations by making a number of algebraic manipulations and using trigonometric identities, a numerical solution is generally much more convenient and practical to obtain.

Numerical solutions for θ_3 and θ_4 can be obtained graphically by simply drawing the figure to scale and measuring the resulting angles. This approach is a simple one,

but it is also one that may not give answers that are very accurate. Its main disadvantage, however, is that a new drawing has to be made when a new solution is desired for a different configuration of the mechanism at some other time. Another way to obtain a solution is to plot the expressions $\sin\theta_3$ and $\cos\theta_3$ given by **(2.1)** and **(2.2)** versus θ_4, with $0 \le \theta_4 \le 2\pi$, and then select, from the plot, only the values of $\sin\theta_3$ and $\cos\theta_3$ that satisfy the relationship $\sin^2\theta_3 + \cos^2\theta_3 = 1$. This strategy has the same disadvantage previously mentioned.

A more efficient and direct way to solve **(2.1)** and **(2.2)** is given below. After the calculations presented so far are illustrated, the sequence of calculations is automated in Simulink. The model can, of course, be used for solving any four-bar mechanism problem.

2.1.3 Position analysis using the vector cross product

The solution methodology presented here makes use of the angles α, β and γ shown in Figure 2.3 and of simple trigonometric relations that are obtained by inspection of that figure. First, determine the angle α by using the following expressions that are obtained by inspection of Figure 2.3:

$$\sin\alpha = \frac{L_2 \sin\theta_2}{L} \quad \cos\alpha = \frac{L_1 - L_2 \sin\theta_2}{L} \qquad \textbf{(2.9)}$$

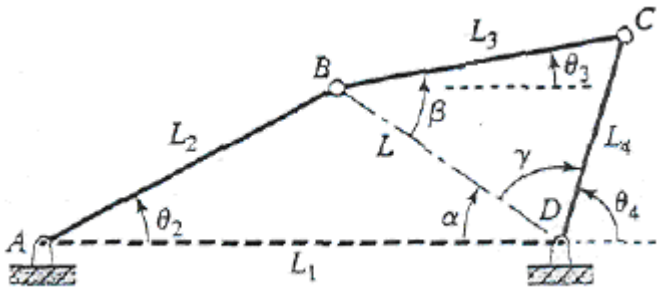

Figure 2.3 The angles α, β and γ needed for calculating θ_3 and θ_4

where

$$L = \sqrt{(L_2 \sin\theta_2)^2 + (L_1 - L_2 \sin\theta_2)^2} = \sqrt{L_1^2 + L_2^2 - 2L_1 L_2 \cos\theta_2} \qquad \textbf{(2.10)}$$

Notice that point B in Figure2.3 will be below line AD, which is a physical possibility, if the value is found for $\sin\alpha$ is negative.

Next, determine the angles β and γ by using the following expressions that are obtained by inspection of the triangle BCD in Fig. 2.3.

$$L_3 \sin\beta = L_4 \sin\gamma \quad \textbf{(2.11)}$$

$$L_3 \cos\beta + L_4 \cos\gamma = L \quad \textbf{(2.12)}$$

To determine β, start by combining **(2.11)** and **(2.12)** as $(L_3\cos\beta - L)^2 + (L_3\sin\beta)^2 = L_4^2$ to obtain

$$\cos\beta = \frac{L_3^2 + L^2 - L_4^2}{2LL_3} \quad \textbf{(2.13)}$$

Notice that a real solution for β is possible only if $\left|L_3^2 + L^2 - L_4^2\right| \le 2LL_3$, of course. If this condition is violated for certain ranges of angle $\theta_2(t)$, then the mechanism cannot operate in that range. The same is true when the calculation of the sine and cosine of any angle yields a value that is outside the range [-1, 1].

If the value obtained for $\cos\beta$ is in the physical range [-1, 1], then there are two possibilities for $\sin\beta$, each one corresponding to a possible configuration of the mechanism (i.e. with point C located either above or below line BD in Figure2.3). The solution to use is determined by the initial configuration that is chosen for the mechanism to start its motion.

Solution number 1: $$\sin\beta = \sqrt{1-\cos^2\beta} \quad \textbf{(2.14)}$$

Solution number 2: $$\sin\beta = -\sqrt{1-\cos^2\beta} \quad \textbf{(2.15)}$$

The angle β corresponding to these two solutions is then determined by using the values that are found for $\sin\beta$ and $\cos\beta$. Having determined β and α, the angle θ_3 is simply determined as

$$\theta_3 = \beta - \alpha \quad \textbf{(2.16)}$$

Finally, the angle γ (which is needed for calculating θ_4) can now be determined by using **(2.11)** and **(2.12)** to calculate $\sin\gamma = (L_3\sin\beta)/L_4$ and $\cos\gamma = (L - L_3\cos\beta)/L_4$ for each of the two possible solutions for β. The angle θ_4 is then determined as

$$\theta_4 = \pi - (\alpha + \gamma) \quad \textbf{(2.17)}$$

2.2 Automated solution for the four-bar mechanism using Matlab Simulink

For practical applications, the most efficent way to do all the calculations presented so far is to automate them in a computer. This can be done either by writing a program using any computer language of one's choice or by using a graphical interface software that replaces the code writing by connecting operational blocks that perform all the required calculations. The latter is what is presented in this book. The user can change parameters in the block diagram and see, almost immediately, how the mechanism would respond to such changes.

Figure 2.4 shows a Simulink block diagram model for automating the calculations of θ_3, and θ_4 presented in Section 2.1.3 and of $\dot{\theta}_3, \dot{\theta}_4, \ddot{\theta}_3$, and $\ddot{\theta}_4$ presented in Section 2.1.2 and for animating motion. The solutions are displayed in a digital manner using digital display blocks. The model can be used for investigating and designing four-bar mechanisms. Its inputs are the lengths L_1, L_2, L_3, and L_4; the angle $\theta_2(t)$ (which may also be a constant), and the angle's first and second derivatives $\dot{\theta}_2$ and $\ddot{\theta}_2$.

Notice that θ_3 and θ_4 could have been determined by using the three nondimensional quantities L_2/L_1, L_3/L_1, and L_4/L_1 instead of the four lengths L_1, L_2, L_3, and L_4. This is seen by simply dividing **(2.1)** and **(2.2)** by L_1 ; such nondimensional lengths will appear. The same is true for the calculation of the angular velcities and angular accelerations.

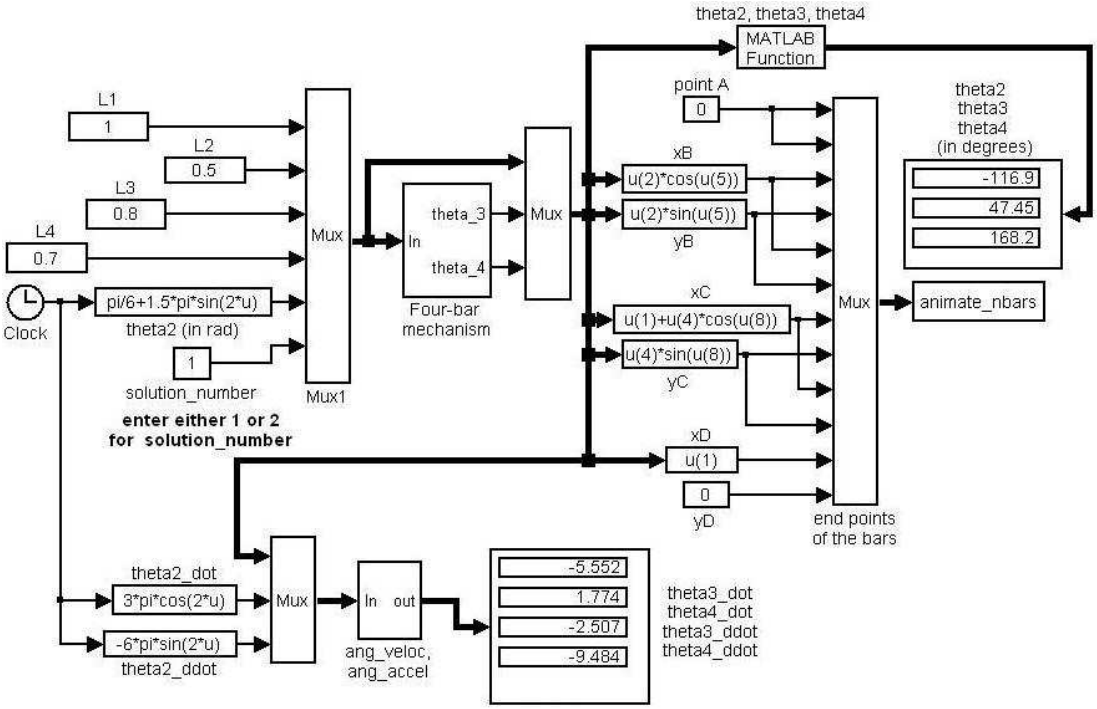

Figure 2.4 A simulink block diagram model for the four-bar mechanism

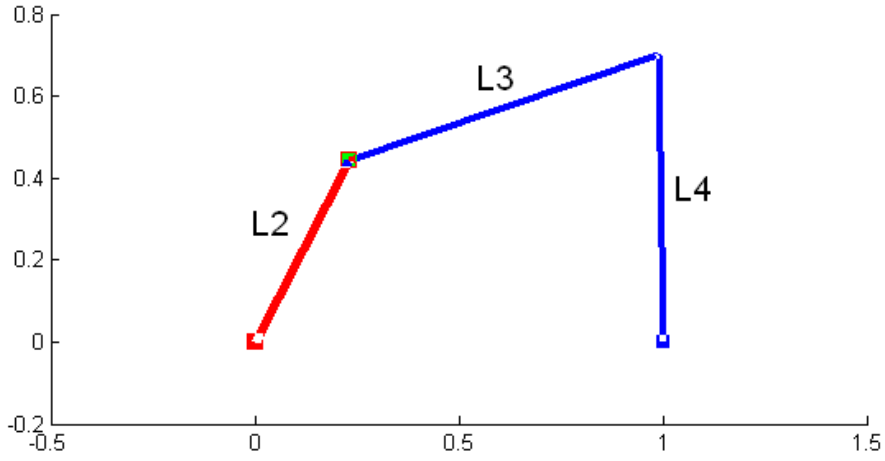

Figure 2.5 The drawing generated by the block diagram by animate_nbars block

From fig. 2.6 to 2.9 the desired outputs θ_2, θ_3, θ_4, $\dot{\theta}_2$, $\dot{\theta}_3$, $\dot{\theta}_4$, $\ddot{\theta}_2$, $\ddot{\theta}_3$, $\ddot{\theta}_4$ can be seen in graphical form. For more convenience some graphs are combined.

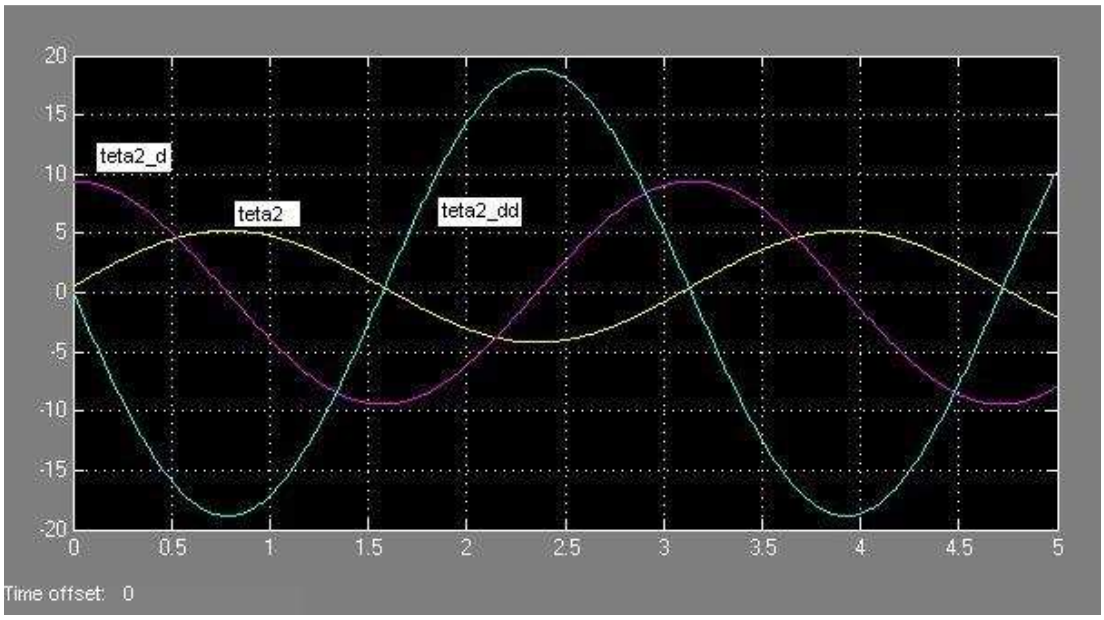

Figure 2.6 θ_2, $\dot{\theta}_2$, $\ddot{\theta}_2$ graph

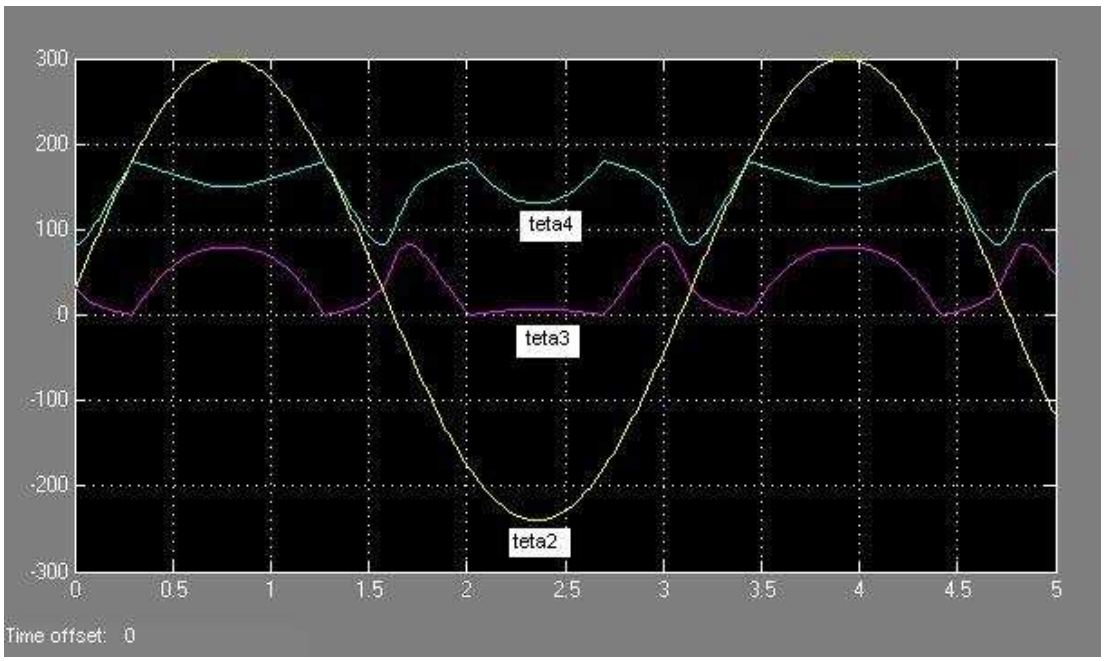

Figure 2.7 θ_2, θ_3, θ_4 graph

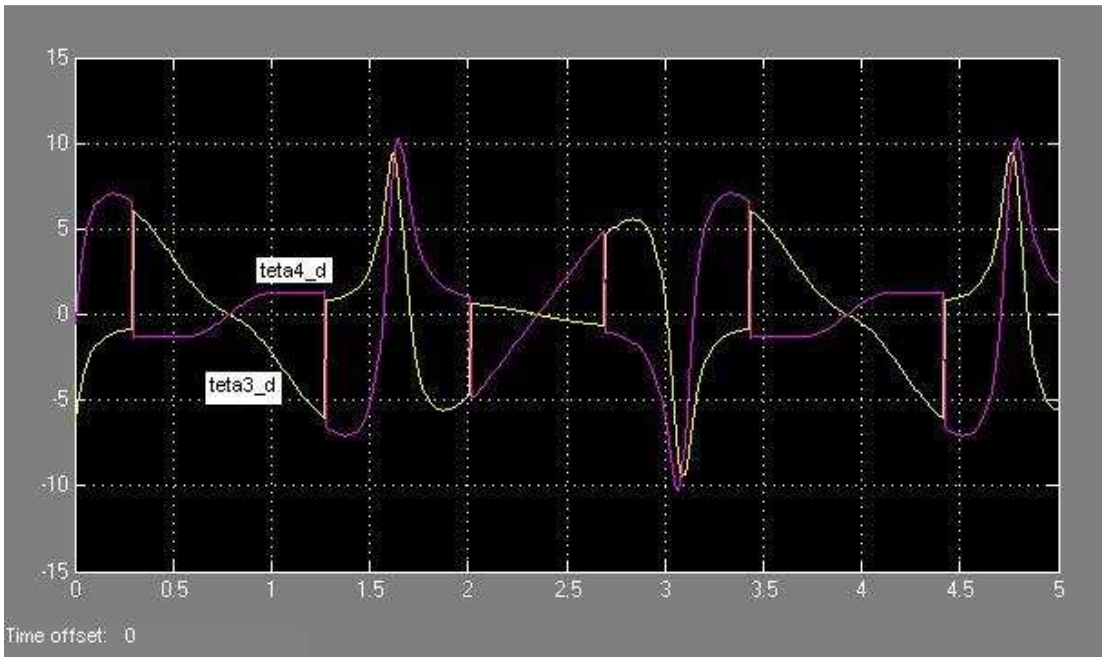

Figure 2.8 $\dot{\theta}_3$, $\dot{\theta}_4$ graph

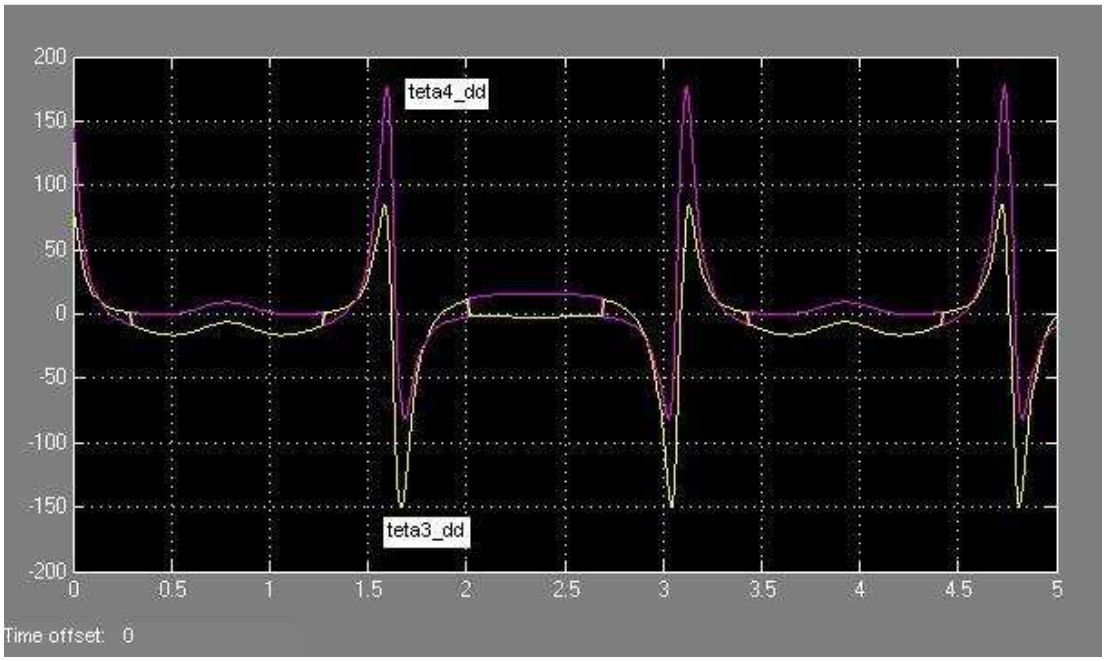

Figure 2.9 $\ddot{\theta}_3$, $\ddot{\theta}_4$ graph

3. DYNAMICS OF RIGID FOUR-BAR MECHANISM

3.1 Forces on Four-Bar Mechanism

This section is devoted to the calculation of the reaction forces acting on each member of a mechanism. These calculations are typically are very lengthy and require the use of the free-body diagram for each member of the mechanism. However, the formulation of the required equations is trivial and it only involves the use of the basic equation $\vec{F} = m\vec{a}_C$, for the motion of the center of mass *C* of a rigid body, and either one of the moment equations $\vec{M}_O = d\vec{H}_O / dt$ (if point *O* is inertial) or $\vec{M}_C = d\vec{H}_C / dt$ for the rotational motion of the body. The kinematical analysis of mechanism was presented before in Section 2, and the results obtained are needed here. A four-bar mechanism is used for illustrating the calculations involved in determining the forces acting on each bar of the mechanism, but the procedure shown here is the same for the other mechanisms.

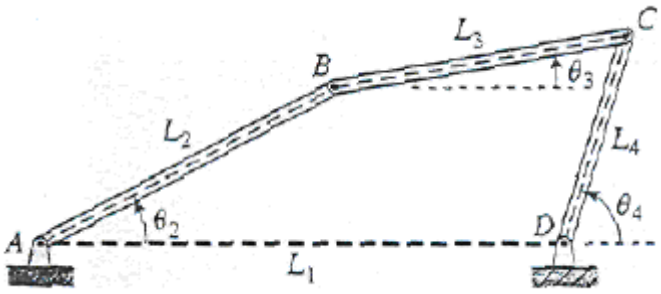

Figure 3.1 A four-bar mechanism (the angles θ_2, θ_3, and θ_4 are changing with time)

A four-bar mechanism is shown in Figure3.1. It is assumed that bar *AB* is driven by an external motor so that the angle $\theta_2(t)$ is known at all times. The reaction forces acting on each bar of the mechanism will be determined in terms of the mass, length, angular position, angular velocity, and angular acceleration of each of its members.

3.1.1 FBD for each bar of the four-bar mechanism

Figure3.2 shows free-body diagrams for each bar of the four-bar mechanism for the simpler case where the mechanism is either in a horizontal plane or the effect of the gravitational forces acting on each member is disregarded.

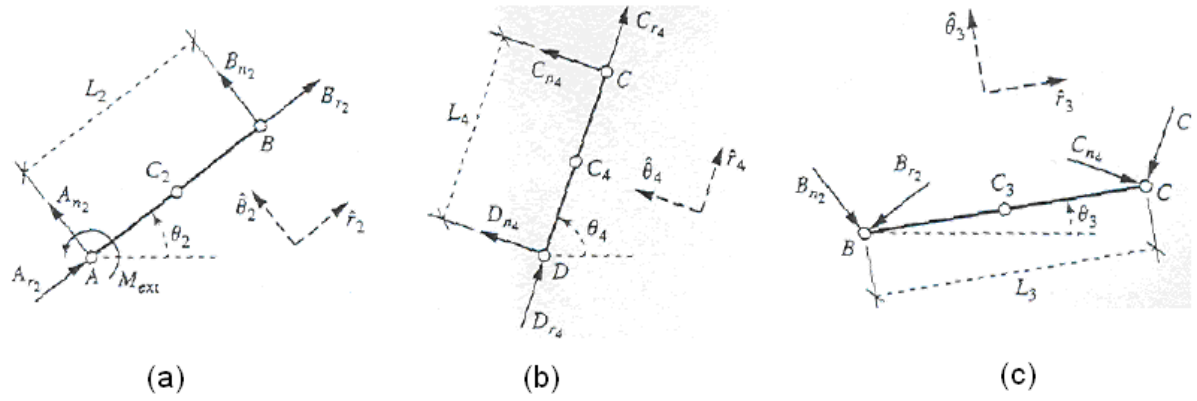

Figure 3.2 Free-body diagram for each bar of the four-bar mechanism of Figure 3.1

3.1.2 Reaction forces acting on each bar of the mechanism

The reaction forces in the plane perpendicular to the plane of the motion of the mechanism are not shown since they do not affect the solution to the problem. The moment M_{ext} is applied to bar AB by a motor in order to make $\theta_2(t)$ a desired function of time. Points C_2, C_3 and C_4 shown in Figure 3.2 are the centers of mass of bars AB, BC, and CD, respectively. Three sets of orthogonal unit vectors, which will be used in the formulation of the equations for obtaining the reaction forces, are also shown in the figure.

In terms of the vector triad $\{\hat{r}_2, \hat{\theta}_2, \hat{z} \overset{\Delta}{=} \hat{r}_2 \times \hat{\theta}_2\}$ that rotates with bar AB shown in Figure 3.2, we then have

Absolute position vector for the center of mass C_2 of bar AB:

$$\vec{r}_{C_2} = A\vec{C}_2 = \frac{L_2}{2}\hat{r}_2$$

Absolute velocity of C_2:

$$\vec{v}_{C_2} = \frac{d}{dt}\vec{r}_{C_2} = \dot{\theta}_2\hat{z} \times \left(\frac{L_2}{2}\hat{r}\right) = \frac{L_2}{2}\dot{\theta}_2\hat{\theta}_2$$

Absolute acceleration of C_2:

$$\vec{a}_{C_2} = \frac{d}{dt}\vec{v}_{C_2} = \frac{L_2}{2}\ddot{\theta}_2\hat{\theta}_2 + \dot{\theta}_2\hat{z}\times\vec{v}_{C_2} = \frac{L_2}{2}\left(\ddot{\theta}_2\hat{\theta}_2 - \dot{\theta}_2^2\hat{r}_2\right)$$

Resultant force acting on bar AB:

$$\vec{F}_{AB} = \left(A_{r_2} + B_{r_2}\right)\hat{r}_2 + \left(A_{n_2} + B_{n_2}\right)\vec{\theta}_2$$

The motion of the center of mass of the bar is governed by Newton's second law $\vec{F} = m_2\vec{a}_{C_2}$, and this yields the following scalar equations:

$$A_{r_2} + B_{r_2} = -m_2\frac{L_2}{2}\dot{\theta}_2^2 \qquad \textbf{(3.1)}$$

$$A_{n_2} + B_{n_2} = m_2\frac{L_2}{2}\ddot{\theta}_2 \qquad \textbf{(3.2)}$$

Since point A is inertial, let us take moments about A and use the simple moment equation $\vec{M}_A = d\vec{H}_A/dt$ where $\vec{H}_A = I_A\dot{\theta}_2\hat{z}$ and $I_A = m_2L_2^2/3$. This moment equation yields the following scalar equation:

$$M_{ext} + L_2B_{n_2} = \frac{1}{3}m_2L_2^2\ddot{\theta}_2 \qquad \textbf{(3.3)}$$

So far we have three equations, but they involve four unknown reaction forces and one unknown moment M_{ext}. To be able to solve the problem, more equations have to be generated until the number of equations and unknowns are the same.

Before proceeding, notice that the form of the **(3.1)** to **(3.3)** are relatively simple because force components A_{r_2}, B_{r_2}, A_{n_2} and B_{n_2} in the directions $\hat{r}_2$ along bar *AB*, and $\hat{\theta}_2$ normal to *AB*, are being used in the analysis. Because of such a choice, the coefficients of these force components in **(3.1)** and **(3.3)** are equal to one, and the expressions for the acceleration components of $\vec{a}_{C_2}$ are simply $\frac{L_2}{2}\ddot{\theta}_2\hat{\theta}_2$ and $-\frac{L_2}{2}\dot{\theta}_2^2\hat{r}_2$

Therefore, in order to obtain more equations that are as simple as **(3.1)** to **(3.3)**, consider bar *CD* next, for which point *D* is inertial. Treat that bar the same way bar *AB* was treated, but using the unit vectors $\hat{r}_4$ and $\hat{\theta}_4$ that rotate with bar *CD*. With the absolute position vector of the center of mass *C4* of bar *CD* being $D\vec{C}_4 = \left(L_4/2\right)\hat{r}_4$, the following equations are then obtained for the bar:

$$C_{r_4} + D_{r_4} = -m_4\frac{L_4}{2}\dot{\theta}_4^2 \qquad \textbf{(3.4)}$$

$$C_{n_4} + D_{n_4} = m_4 \frac{L_4}{2} \ddot{\theta}_4 \quad \textbf{(3.5)}$$

$$L_4 C_{n_4} = \frac{1}{3} m_4 L_4^2 \ddot{\theta}_4 \quad \textbf{(3.6)}$$

Finally, consider the last bar, which is the bar *BC*. The reaction forces acting on that bar are B_{r_2}, B_{n_2}, C_{r_4} and C_{n_4}, and they appear as shown in the free-body diagram in Fig. 3.2c. Since these forces are not in the directions along the bar and perpendicular to the bar, and since neither point *B* nor point *C* is inertial, the equations in which the unknown reaction forces B_{r_2}, B_{n_2}, C_{r_4} and C_{n_4} will appear will be a little more complicated than the equations that were obtained for the other two bars. The information of the equations involve obtaining expressions for the absolute acceleration of the center of mass of bar *BC* and the resultant force acting on the bar.

Absolute position vector for the center of mass C_3 of bar *BC*:

$$\vec{r}_{C_3} = A\vec{C}_3 = A\vec{B} + \frac{L_3}{2}\hat{r}_3$$

Absolute velocity of C_3:

$$\vec{v}_{C_3} = \frac{d}{dt}\vec{r}_{C_3} = \underbrace{\frac{d}{dt} A\vec{B}}_{=L_2\dot{\theta}_2\hat{\theta}_2} + \dot{\theta}_3\hat{z}\times\left(\frac{L_3}{2}\hat{r}\right) = L_2\dot{\theta}_2\hat{\theta}_2 + \frac{L_3}{2}\dot{\theta}_3\hat{\theta}_3$$

Absolute acceleration of C_3:

$$\vec{a}_{C_3} = \frac{d}{dt}\vec{v}_{C_3} = L_2\ddot{\theta}_2\hat{\theta}_2 + \dot{\theta}_2\hat{z}\times\left(L_2\dot{\theta}_2\hat{\theta}_2\right) + \frac{L_3}{2}\ddot{\theta}_3\hat{\theta}_3 + \dot{\theta}_3\hat{z}\times\left(\frac{L_3}{2}\dot{\theta}_3\hat{\theta}_3\right) = L_2\left(\ddot{\theta}_2\hat{\theta}_2 - \dot{\theta}_2^2\hat{r}_2\right) + \frac{L_3}{2}\left(\ddot{\theta}_3\hat{\theta}_3 - \dot{\theta}_3^2\hat{r}_3\right)$$

Resultant force acting on bar *BC*:

$$\vec{F}_{BC} = -B_{r_2}\hat{r}_2 - B_{n_2}\vec{\theta}_2 - C_{r_4}\hat{r}_4 - C_{n_4}\vec{\theta}_4$$

Newton's second law is now applied to bar *BC*, i.e. $\vec{F}_{BC} = m_3\vec{a}_{C_3}$. To do this, first express $\vec{F}_{BC}$ and $\vec{a}_C$ in the same orthogonal set of unit vectors. By arbitrarily choosing to work with vector components along $\hat{r}_2$ and $\hat{\theta}_2$, the transformations that relate $\{\hat{r}_3, \hat{\theta}_3\}$ (which appear in the expression for $\vec{a}_{C_3}$) and $\{\hat{r}_4, \hat{\theta}_4\}$ (which appear in the expression for $\vec{F}_{BC}$) to $\{\hat{r}_3, \hat{\theta}_3\}$ are obtained by inspection of Figure 3.3.

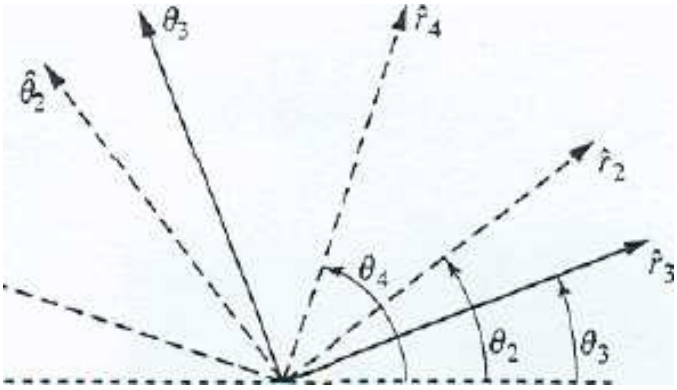

Figure 3.3 Three unit vector systems used in Figure 3.2

These relations are as follows:

$$\hat{r}_3 = \hat{r}_2\cos(\theta_2-\theta_3) - \hat{\theta}_2\sin(\theta_2-\theta_3)\ \hat{\theta}_3 = \hat{r}_2\sin(\theta_2-\theta_3) + \hat{\theta}_2\cos(\theta_2-\theta_3)$$

$$\hat{r}_4 = \hat{r}_2\cos(\theta_4-\theta_2) + \hat{\theta}_2\sin(\theta_4-\theta_2)\ \hat{\theta}_4 = \hat{\theta}_2\cos(\theta_4-\theta_2) - \hat{r}_2\sin(\theta_4-\theta_2)$$

With $\hat{r}_3, \hat{\theta}_3, \hat{r}_4$ and $\hat{\theta}_4$ as expressed in these equations, the following scalar equation are obtained from the vector equation $\vec{F}_{BC} = m_3\vec{a}_{C_3}$:

$$\begin{aligned} &B_{r_2} + C_{r_4}\cos(\theta_4-\theta_2) - C_{n_4}\sin(\theta_4-\theta_2) \\ &\quad = m_3\frac{L_3}{2}\left[\dot{\theta}_3^2\cos(\theta_2-\theta_3) - \ddot{\theta}_3\sin(\theta_2-\theta_3)\right] + m_3L_2\dot{\theta}_2^2 \end{aligned} \qquad \textbf{(3.7)}$$

$$\begin{aligned} &B_{n_2} + C_{r_4}\cos(\theta_4-\theta_2) + C_{n_4}\sin(\theta_4-\theta_2) \\ &\quad = -m_3\frac{L_3}{2}\left[\dot{\theta}_3^2\sin(\theta_2-\theta_3) - \ddot{\theta}_3\cos(\theta_2-\theta_3)\right] - m_3L_2\ddot{\theta}_2^2 \end{aligned} \qquad \textbf{(3.8)}$$

Since either point B nor point C is inertial, let us take moments about the center of mass C_3 of bar BC and use the moment equation $\vec{M}_{C_3} = d\vec{H}_{C_3}/dt$, where $\vec{H}_{C_3} = I_{C_3}\dot{\theta}_3\hat{z}$ and $I_{C_3} = m_3L_3^2/12$. This moment equation yields the following scalar equation:

$$\frac{L_3}{2}\left[B_{r_2}\sin(\theta_2-\theta_3) + B_{n_2}\cos(\theta_2-\theta_3) - C_{r_4}\sin(\theta_4-\theta_3) - C_{n_4}\cos(\theta_4-\theta_3)\right] = I_{C_3}\ddot{\theta}_3 \qquad \textbf{(3.9)}$$

where the moment $\vec{M}_{C_3}$ is obtained by inspecting Fig. 3.2c.

(3.1) to **(3.9)** constitute a system of nine equations involving the nine unknowns M_{ext}, A_{r_2}, A_{n_2}, B_{r_2}, B_{n_2}, C_{r_4}, C_{n_4}, D_{r_4} and D_{n_4}. They are linear in these variables and can be easily solved for. They could, for example, be put in matrix form and then

solved by inverting a 9×9 matrix, as done in previous section for a 2×2 matrix. However, they can be solved in a simpler manner.

(3.5) and **(3.6)** can immediately be solved for the reaction forces C_{n_4} and D_{n_4} to obtain following results:

$$C_{n_4} = \frac{1}{3} m_4 L_4 \ddot{\theta}_4 \qquad \textbf{(3.10)}$$

$$D_{n_4} = \frac{1}{2} C_{n_4} \qquad \textbf{(3.11)}$$

with $\ddot{\theta}_4$ determined as shown in kinematical analysis.

Having solved for C_{n_4} and D_{n_4}, **(3.7)** to **(3.9)** now only involve the unknowns B_{r_2}, B_{n_2} and C_{r_4}. They may be solved with simple matrix inversion and a matrix multiplication as follows, and numerical values may be readily obtained either by hand or with an inexpensive calculator.

$$\begin{bmatrix} B_{r_2} \\ B_{n_2} \\ C_{r_4} \end{bmatrix} = \begin{bmatrix} 1 & 0 & \cos(\theta_4-\theta_2) \\ 0 & 1 & \sin(\theta_4-\theta_2) \\ \sin(\theta_2-\theta_3) & \cos(\theta_2-\theta_3) & -\sin(\theta_4-\theta_3) \end{bmatrix}^{-1} \times \begin{bmatrix} \frac{1}{2} m_3 L_3 \left(\dot{\theta}_3^2 \cos(\theta_2-\theta_3) - \ddot{\theta}_3 \sin(\theta_2-\theta_3)\right) + m_3 L_2 \dot{\theta}_2^2 + C_{n_4} \sin(\theta_4-\theta_2) \\ -\frac{1}{2} m_3 L_3 \left(\dot{\theta}_3^2 \sin(\theta_2-\theta_3) + \ddot{\theta}_3 \cos(\theta_2-\theta_3)\right) - m_3 L_2 \ddot{\theta}_2 - C_{n_4} \cos(\theta_4-\theta_2) \\ \frac{1}{6} m_3 L_3 \ddot{\theta}_3 + C_{n_4} \cos(\theta_4-\theta_3) \end{bmatrix} \qquad \textbf{(3.12)}$$

The remaining unknowns A_{r_2}, A_{n_2}, M_{ext} and D_{r_4} are readily obtained from **(3.1)** to **(3.4)**

3.1.3 Simulink model for a four-bar mechanism

Figure3.4 shows a Simulink block diagram model for automating the calculations just presented and for animating the motion of the mechanism. The default inputs are the same as Section 2.2 it can be seen easily from Figure 2.4 and the density of each bar is set to $\rho = 0.075\, kg/m$. The solutions displayed in Figure 3.4 are for a single time instant, which is essentially $t = 5$.

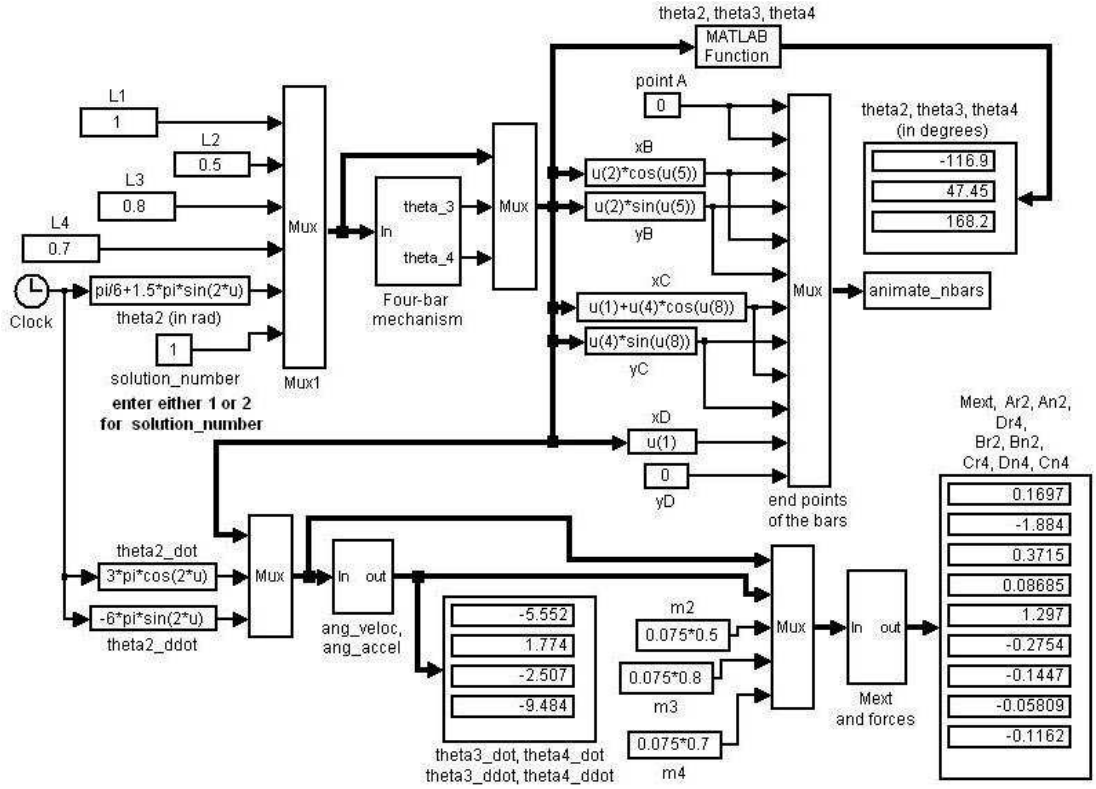

Figure 3.4 Simulink model fourbarforces.mdl for a four-bar mechanism

In Figure 3.5 it is given the moment M_{ext} graph, that is applied to bar *AB* by a motor in order to make $\theta_2(t)$ a desired function of time.

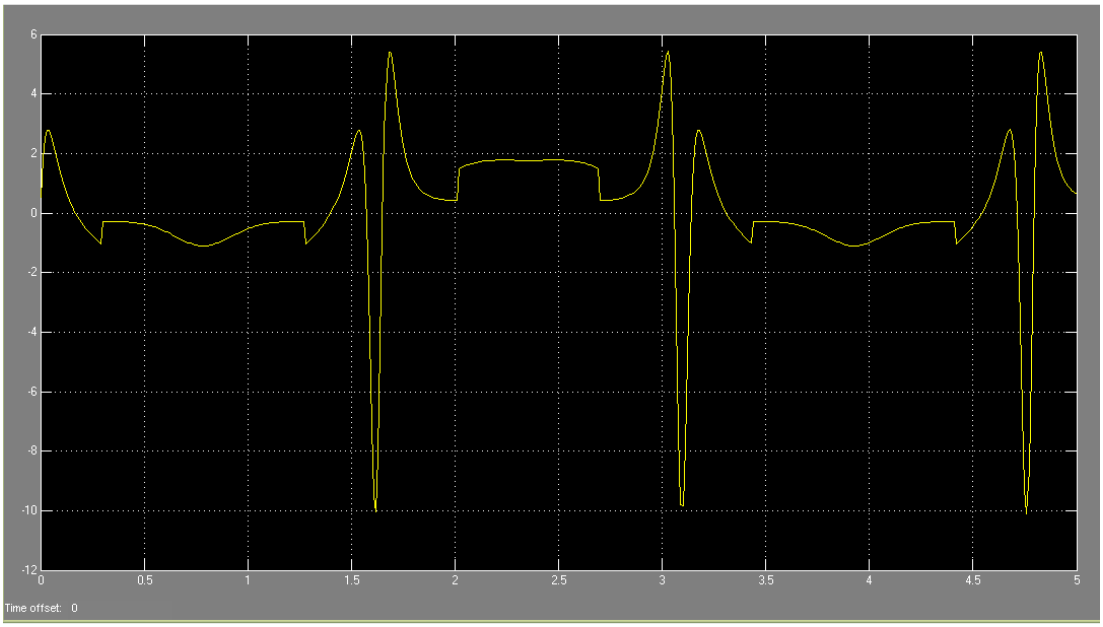

Figure 3.5 Driving moment graph

3.2 Dynamic Analysis of Rigid Four-Bar Mechanism

3.2.1 Lagrange's equation

In this section, we utilize the concepts introduced in the preceding section to develop Lagrange's equation of motion for a system of particles. The obtained Lagrange's equation, however, can be applied to the dynamics of rigid bodies as well, since a rigid body can be considered as a collection of a large number of particles.

In the following discussion, we assume that the system consists of n particles. The displacement r^i of the ith particle is assumed to depend on a set of generalized coordinates q_j, where j=1,2,…,n. Hence

$$r^i = r^i(q_1, q_2, \dots, q_n, t) \quad \textbf{(3.13)}$$

Differentiating **(3.13)** with respect to time using the chain rule of differentiation yields

$$\dot{r}^i = \frac{\partial r^i}{\partial q_1}\dot{q}_1 + \frac{\partial r^i}{\partial q_2}\dot{q}_2 + \cdots + \frac{\partial r^i}{\partial q_n}\dot{q}_n + \frac{\partial r^i}{\partial t} = \sum_{j=1}^{\infty} \frac{\partial r^i}{\partial q_j}\dot{q}_J + \frac{\partial r^i}{\partial t} \quad \textbf{(3.14)}$$

The virtual displacement δr^i can be expressed in terms of the virtual change of the generalized coordinates as

$$\delta r^i = \sum_{j=1}^{n} \frac{\partial r^i}{\partial q_j}\delta q_j \quad \textbf{(3.15)}$$

The dynamic equilibrium of the particle can be defined using Newton's second law as

$$\dot{p}^i = F^i \quad \textbf{(3.16)}$$

Where p^i is the vector of linear momentum and F^i is the vector of the total forces acting on the particles i. the linear momentum vector p^i is defined as

$$p^i = m^i \dot{r}^i \quad \textbf{(3.17)}$$

Where m^i is the mass of the particle I which is assumed to be constant.

Consequently,

$$p^i = m^i \ddot{r}^i \quad \textbf{(3.18)}$$

Substituting **(3.18)** into **(3.17)** yields

$$m^i\ddot{r}^i - F^i = 0$$

Which yields

$$(m^i\ddot{r}^i - F^i)^T \delta r^i = 0 \quad \textbf{(3.19)}$$

By summing up these expressions for the particles and using the definition of δr^i given by **(3.15)**, one obtains

$$\sum_{i=1}^{n_p}\left(m^i\ddot{r}^i - F^i\right)^T \left(\sum_{j=1}^{n}\frac{\partial r^i}{\partial q_j}\delta q_j\right) = 0$$

Which can also be written as

$$\sum_{j=1}^{n}\sum_{i=1}^{n_p}\left(m^i\ddot{r}^i - F^i\right)^T \left(\frac{\partial r^i}{\partial q_j}\delta q_j\right) = 0 \quad \textbf{(3.20)}$$

Define the generalized force Q_j associated with the generalized coordinate q_j as

$$Q_j = \sum_{i=1}^{n_p} F^{iT}\frac{\partial r^i}{\partial q_j} \quad \textbf{(3.21)}$$

One can also show that

$$\sum_{i=1}^{n_p}\left(m^i\ddot{r}^{iT}\frac{\partial r^i}{\partial q_j}\right) = \sum_{i=1}^{n_p}\left[\frac{d}{dt}\left(m^i\dot{r}^{iT}\frac{\partial r^i}{\partial q_j}\right) - m^i\dot{r}^{iT}\frac{d}{dt}\left(\frac{\partial r^i}{\partial q_j}\right)\right] \quad \textbf{(3.22)}$$

It is, however, clear from **(3.14)** that

$$\frac{\partial \dot{r}^i}{\partial \dot{q}_j} = \frac{\partial r^i}{\partial q_j} \quad \textbf{(3.23)}$$

Furthermore,

$$\frac{d}{dt}\left(\frac{\partial r^i}{\partial q_j}\right) = \sum_{k=1}^{n}\frac{\partial^2 r^i}{\partial q_j \partial q_k}\dot{q}_k + \frac{\partial^2 r^i}{\partial q_j \partial t} = \frac{\partial \dot{r}^i}{\partial q_j} \quad \textbf{(3.24)}$$

Substituting the results of **(3.23)** and **(3.24)** into **(3.22)** yields

$$\begin{aligned}\sum_{i=1}^{n_p} m^i\ddot{r}^{iT}\frac{\partial r^i}{\partial q_j} &= \sum_{i=1}^{n_p}\frac{d}{dt}\left[\frac{\partial}{\partial \dot{q}_j}\left(\frac{1}{2}m^i\dot{r}^{iT}\dot{r}^i\right)\right] - \frac{\partial}{\partial q_j}\left(\frac{1}{2}m^i\dot{r}^{iT}\dot{r}^i\right) \\ &= \sum_{i=1}^{n_p}\frac{d}{dt}\left(\frac{\partial T^i}{\partial \dot{q}_j}\right) - \frac{\partial T^i}{\partial q_j}\end{aligned} \quad \textbf{(3.25)}$$

Where T^i is the kinetic energy of the particle i defined as

$$T^i = \frac{1}{2} m^i \dot{r}^{iT} \dot{r}^i \quad \textbf{(3.26)}$$

The kinetic energy of the system of particles is given by

$$T = \sum_{i=1}^{n_p} T^i \quad \textbf{(3.27)}$$

Therefore, **(3.25)** can be written in terms of the kinetic energy of the system of particles as

$$\sum_{i=1}^{n_p} m^i \ddot{r}^{iT} \frac{\partial r^i}{\partial q_j} = \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} \quad \textbf{(3.28)}$$

Substituting **(3.21)** and **(3.27)** into **(3.20)** yields

$$\sum_{j=1}^{n} \left[\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial \dot{q}_j} - Q_j\right] \delta q_j = 0 \quad \textbf{(3.29)}$$

Since the generalized coordinates $q_1, q_2, \ldots, q_n$ are assumed to be linearly independent, Equation (3.29) yields a set of n equations defined as

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} - Q_j = 0, \quad j = 1,2,\ldots,n$$

Or

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} = Q_j, \qquad j = 1,2,\ldots,n \quad \textbf{(3.30)}$$

This equation is called Lagrange's equation of motion. Clearly, there are as many equations as the number of generalized coordinates. These equations can be derived using scalar quantities, mainly the kinetic energy of the system and virtual work of the applied and elastic forces. The use of Lagrange's equation for developing the differential equations of motion for single and two degree of freedom systems is demonstrated by the following examples.

3.2.2 Dynamic equation of motion with Lagrange's equation

The four-bar mechanism shown in Figure 3.6 is driven by a motor that applies to bar AB a known moment $M_{motor}(t)$. The motion takes place in a horizontal plane and is started from rest [i.e. with $\dot{\theta_2} = 0$]with a given initial value of θ_2. The mechanism consists of three moving thin homogenous bars AB, BC, and CD of lengths L_2, L_3 and L_4 and masses m_2, m_3, and m_4, respectively. We will obtain the differential equation of motion for this constrained system, and integrate it numerically.

In terms of the unit vectors $\hat{x}\ and\ \hat{y}$ shown in Figure 3.6, the absolute velocity $\vec{v}$ of the center of mass C_3 of bar BC may be written as:

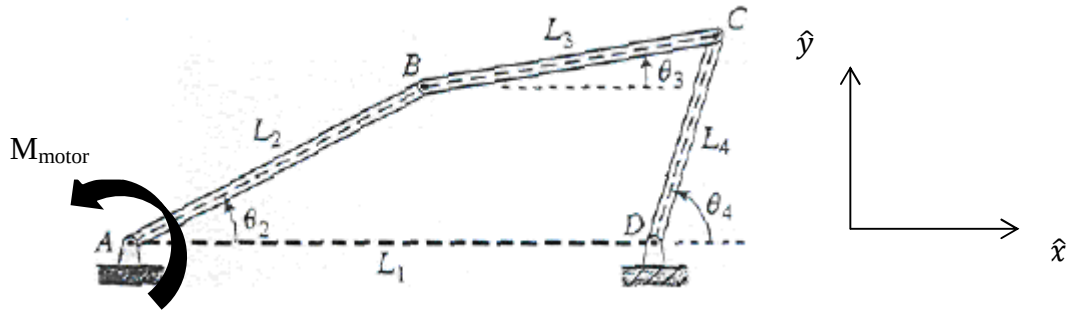

Figure 3.6 A four-bar mechanism subjected to a known moment

M_{motor} applied to bar AB

$$\vec{V}_{C_3}=\frac{d}{dT}\overrightarrow{DC}+\frac{d}{dt}\overrightarrow{CC_3}=\frac{d}{dt}\left[L_4(\hat{x}cos\theta_4+\hat{y}sin\theta_4)\right]+\frac{d}{dt}\left[\frac{L_3}{2}(-\hat{x}cos\theta_3-\hat{y}sin\theta_3)\right]$$

$$=\left(\frac{L_3}{2}\dot{\theta}_3 sin\theta_3-L_4\dot{\theta}_4 sin\theta_4\right)\hat{x}+\left(L_4\dot{\theta}_4 cos\theta_4-\frac{L_3}{2}\dot{\theta}_3 cos\theta_3\right)\hat{y}$$

(3.31)

The following expression is then obtained for the kinetic energy of the motion:

$$T=\frac{1}{2}I_A\dot{\theta}_2^2+\frac{1}{2}I_D\dot{\theta}_4^2+\frac{1}{2}I_3\dot{\theta}_3^2+\frac{1}{2}m_3\vec{v}_{C_3}$$

$$=\frac{1}{2}I_A\dot{\theta}_2^2+\frac{1}{2}I_D\dot{\theta}_4^2+\frac{1}{2}I_3\dot{\theta}_3^2$$

$$+\frac{1}{2}m_3\left[\left(L_4\dot{\theta}_4 cos\theta_4-\frac{L_3}{2}\dot{\theta}_3 cos\theta_3\right)^2+\left(L_4\dot{\theta}_4 sin\theta_4-\frac{L_3}{2}\dot{\theta}_3 sin\theta_3\right)^2\right]$$

$$=\frac{1}{2}I_A\dot{\theta}_2^2+\frac{1}{2}I_D\dot{\theta}_4^2+\frac{1}{2}I_3\dot{\theta}_3^2+\frac{1}{2}m_3\left[L_4^2\dot{\theta}_4^2+\frac{1}{4}L_3^2\dot{\theta}_3^2-L_3L_4\dot{\theta}_3\dot{\theta}_4 cos(\theta_4-\theta_3)\right]$$ **(3.32)**

In **(3.32)**, $I_A=m_2L_2^2/3$ is the moment of inertia of bar AB about the vertical axis that passes through A, $I_D=m_4L_4^2/3$ is the moment of inertia of bar CD about the vertical axis that passes through D, and $I_3=m_3L_3^2/12$ is the moment of inertia of bar BC about the vertical axis that passes through its center of mass. The use of these expressions for the moments of inertia imply that the thin bars are being approximated as lines AB, CD, and BC, respectively.

The angles θ_2, θ_3,and θ_4, are dependent on each other. As first presented in Section2,the following constraint relations between them are obtained by inspection of Figure 3.6:

$$L_2cos\theta_2 + L_3cos\theta_3 - L_4cos\theta_4 - L_1 = 0 \quad \triangleq \alpha_1(\theta_2, \theta_3, \theta_4) \qquad \textbf{(3.33)}$$

$$L_2sin\theta_2 + L_3sin\theta_3 - L_4sin\theta_4 = 0 \quad \triangleq \alpha_2(\theta_2, \theta_3, \theta_4) \qquad \textbf{(3.34)}$$

The virtual work δW done by all forces acting on the system is simply

$$\delta W = M_{motor}\delta\theta_2 \triangleq Q_{\theta_2}\delta\theta_2 \qquad \textbf{(3.35)}$$

Generally, it is more convenient to use Lagrange multipliers to handle constraint equations. Since there are two of such equations, two multipliers $\lambda_1(t)$ and $\lambda_2(t)$ are needed. For the three variables θ_2, θ_3,and θ_4, then implies

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{\theta}_2} - \frac{\partial T}{\partial \theta_2} = Q_{\theta_2} + \lambda_1\frac{\partial a_1}{\partial \theta_2} + \lambda_2\frac{\partial a_2}{\partial \theta_2} \ (\mathbf{a})$$

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{\theta}_3} - \frac{\partial T}{\partial \theta_3} = Q_{\theta_3} + \lambda_1\frac{\partial a_1}{\partial \theta_3} + \lambda_2\frac{\partial a_2}{\partial \theta_3} \ (\mathbf{b})$$

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{\theta}_4} - \frac{\partial T}{\partial \theta_4} = Q_{\theta_4} + \lambda_1\frac{\partial a_1}{\partial \theta_4} + \lambda_2\frac{\partial a_2}{\partial \theta_4} \ (\mathbf{c}) \qquad \textbf{(3.36)}$$

Where $Q_{\theta_2} = M_{motor}$, $Q_{\theta_3} = Q_{\theta_4} = 0$, and the functions $\alpha_1(\theta_2, \theta_3, \theta_4)$ and $\alpha_2(\theta_2, \theta_3, \theta_4)$ are defined in **(3.33)** and **(3.34)**.

(3.36a) to **(3.36c)** yield the following second-order differential equations:

$$\frac{1}{3}m_2L_2^2\ddot{\theta}_2 = M_{motor} - \lambda_1L_2sin\theta_2 + \lambda_2L_2cos\theta_2 (\mathbf{a})$$

$$\frac{1}{3}m_3L_3^2\ddot{\theta}_3 - \frac{1}{2}m_3L_3L_4\left[\ddot{\theta}_4cos(\theta_4 - \theta_3) - \dot{\theta}_4^2\sin(\theta_4 - \theta_3)\right]$$

$$= -\lambda_1 L_3 sin\theta_3 + \lambda_2 L_3 cos\theta_3 (\mathbf{b})$$

$$\left(\frac{1}{3}m_4 + m_3\right) L_4^2 \ddot{\theta}_4 - \frac{1}{2} m_3 L_3 L_4 \left[\ddot{\theta}_3 cos(\theta_4 - \theta_3) + \dot{\theta}_3^2 sin(\theta_4 - \theta_3)\right]$$

$$= \lambda_1 L_4 sin\theta_4 - \lambda_2 L_4 cos\theta_4 (\mathbf{c})$$

(3.37)

(3.37a) to **(3.37c)** involve five unknowns, which are $\theta_2, \theta_3, \theta_4$ and the two Lagrange multipliers λ_1 and λ_2. The strategy involves augmenting **(3.37a)** to **(3.37c)** with **(3.38a)** and **(3.38b)** that follow, which are obtained by taking the second time derivative of the algebraic constraint equations. The five equations are combined into a single matrix equation, **(3.39)** which is easily solved numerically by Simulink using a Matlab Function Block.

$$L_2 \ddot{\theta}_2 sin\theta_2 + L_3 \ddot{\theta}_3 sin\theta_3 - L_4 \ddot{\theta}_4 sin\theta_4 + L_2 \dot{\theta}_2^2 cos\theta_2 + L_3 \dot{\theta}_3^2 cos\theta_3 - L_4 \dot{\theta}_4^2 cos\theta_4 = 0$$

(3.38a)

$$L_2 \ddot{\theta}_2 cos\theta_2 + L_3 \ddot{\theta}_3 cos\theta_3 - L_4 \ddot{\theta}_4 cos\theta_4 - L_2 \dot{\theta}_2^2 sin\theta_2 - L_3 \dot{\theta}_3^2 sin\theta_3 + L_4 \dot{\theta}_4^2 sin\theta_4 = 0$$

(3.38b)

$$\begin{bmatrix} \frac{1}{3}m_2 L_2^2 & 0 & 0 & L_2 sin\theta_2 & -L_2 cos\theta_2 \\ 0 & \frac{1}{3}m_3 L_3^2 & -\frac{1}{2}m_3 L_3 L_4 cos(\theta_4 - \theta_3) & L_3 sin\theta_3 & -L_3 cos\theta_3 \\ 0 & -\frac{1}{2}m_3 L_3 L_4 cos(\theta_4 - \theta_3) & \left(\frac{1}{3}m_4 + m_3\right) L_4^2 & -L_4 sin\theta_4 & L_4 cos\theta_4 \\ L_2 sin\theta_2 & L_3 sin\theta_3 & -L_4 sin\theta_4 & 0 & 0 \\ L_2 cos\theta_2 & L_3 cos\theta_3 & -L_4 cos\theta_4 & 0 & 0 \end{bmatrix}$$

$$\text{x} \begin{bmatrix} \ddot{\theta}_2 \\ \ddot{\theta}_3 \\ \ddot{\theta}_4 \\ \lambda_1 \\ \lambda_2 \end{bmatrix} = \begin{bmatrix} M_{motor} \\ -\frac{1}{2}m_3 L_3 L_4 \dot{\theta}_4^2 sin(\theta_4 - \theta_3) \\ \frac{1}{2}m_3 L_3 L_4 \dot{\theta}_3^2 sin(\theta_4 - \theta_3) \\ L_4 \dot{\theta}_4^2 cos\theta_4 - L_3 \dot{\theta}_3^2 cos\theta_3 - L_2 \dot{\theta}_2^2 cos\theta_2 \\ -L_4 \dot{\theta}_4^2 sin\theta_4 + L_3 \dot{\theta}_3^2 sin\theta_3 + L_2 \dot{\theta}_2^2 sin\theta_2 \end{bmatrix}$$

(3.39)

3.2.3 Dynamic analysis solutions for four-bar mechanism using Matlab Simulink

The simulink implementation of this strategy is summarized in the flowchart shown in Figure 3.7.

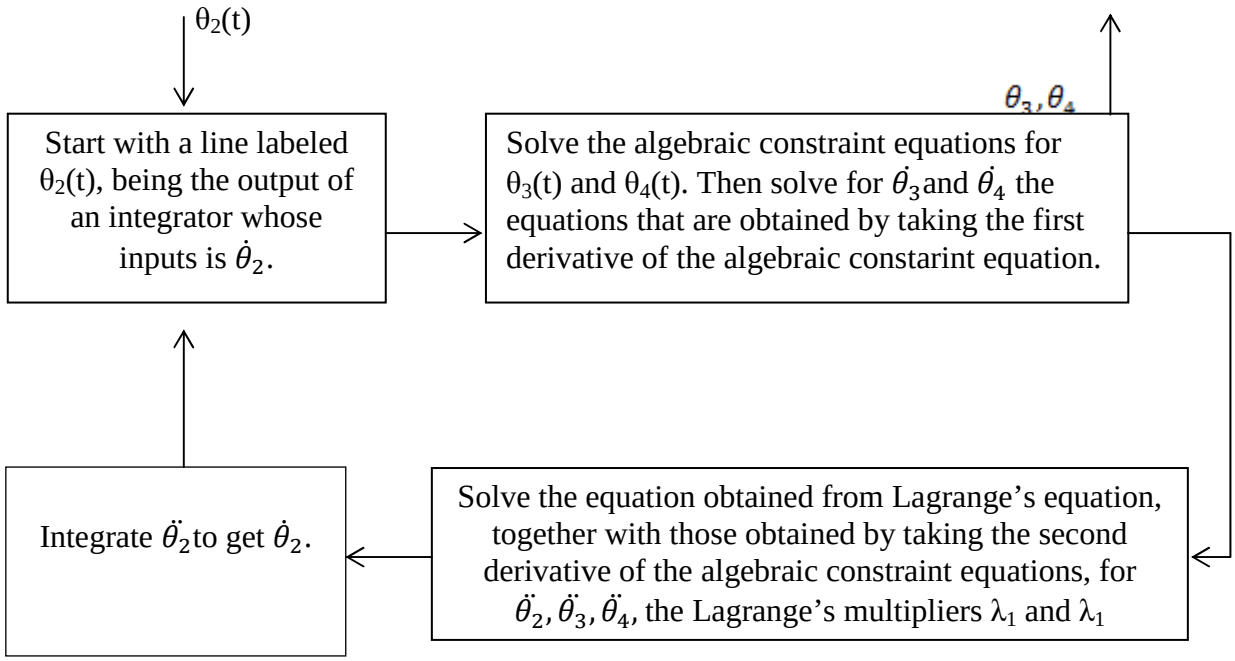

Figure 3.7 Flowchart for solving the differential-algebraic equations for the dynamics of a four-bar mechanism using Simulink.

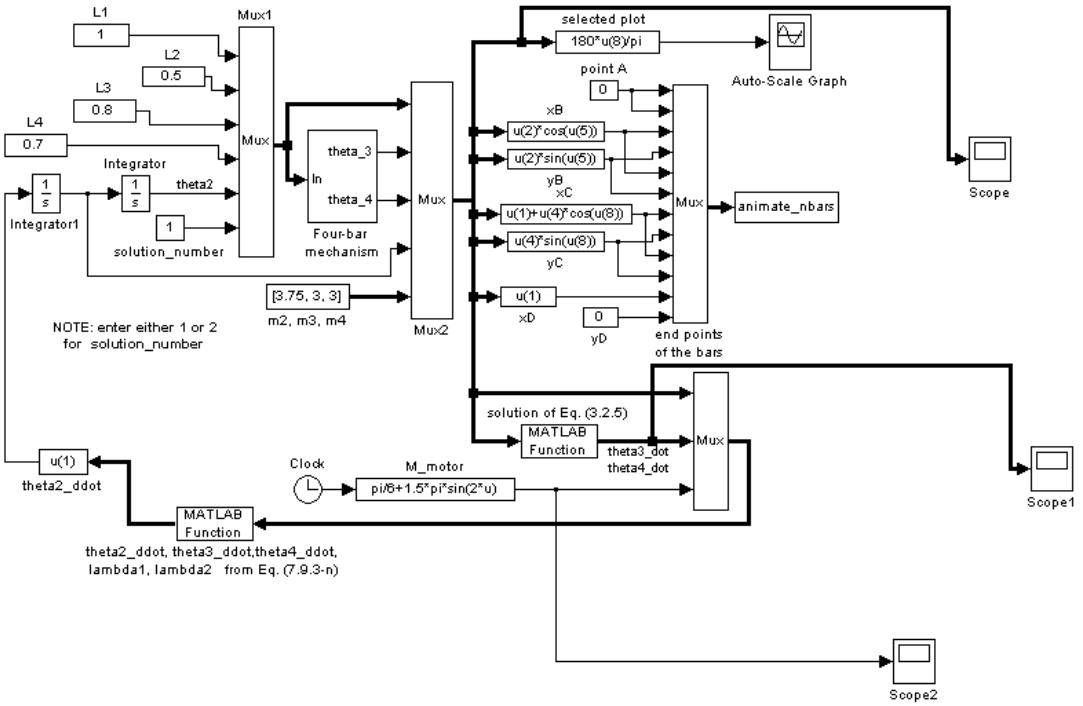

Figure 3.8 Simulink model of rigid four bar mechanism

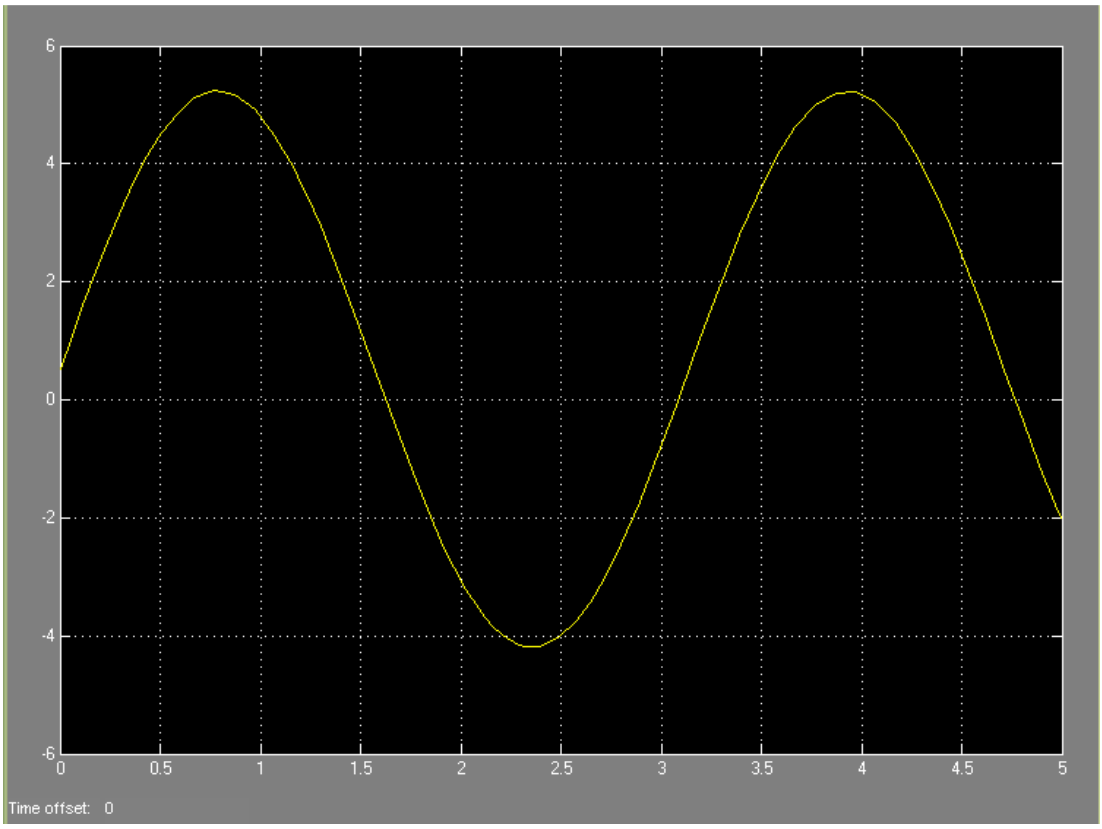

Figure 3.9 θ_2 graph

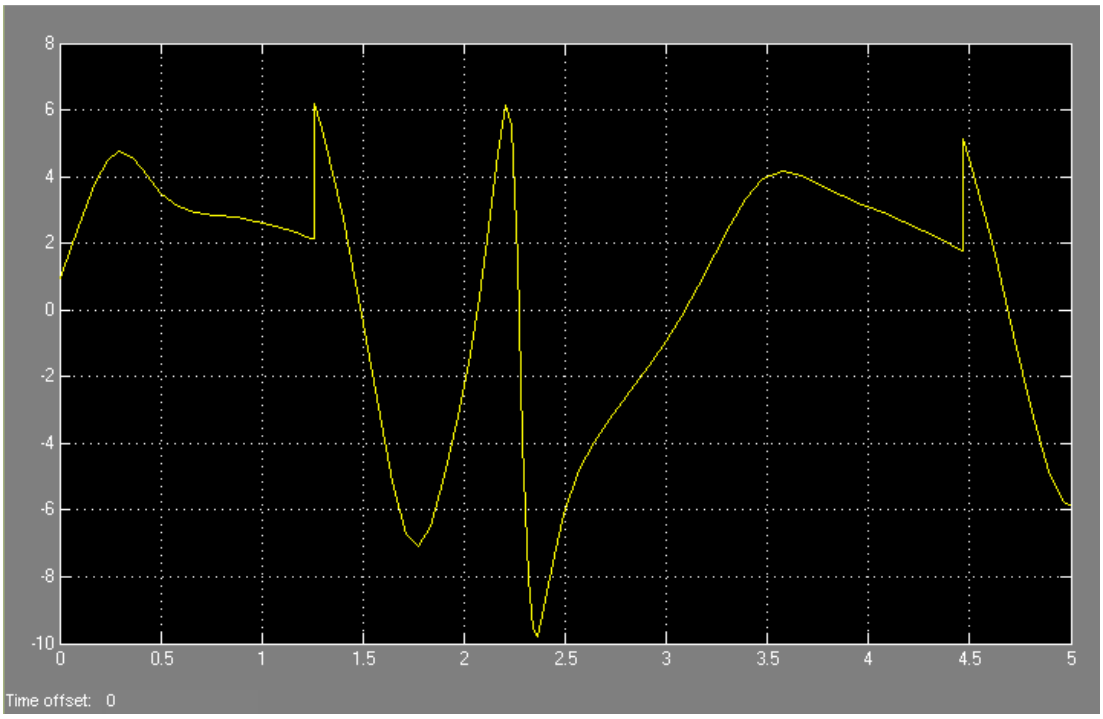

Figure 3.10 $\dot{\theta}_2$ graph

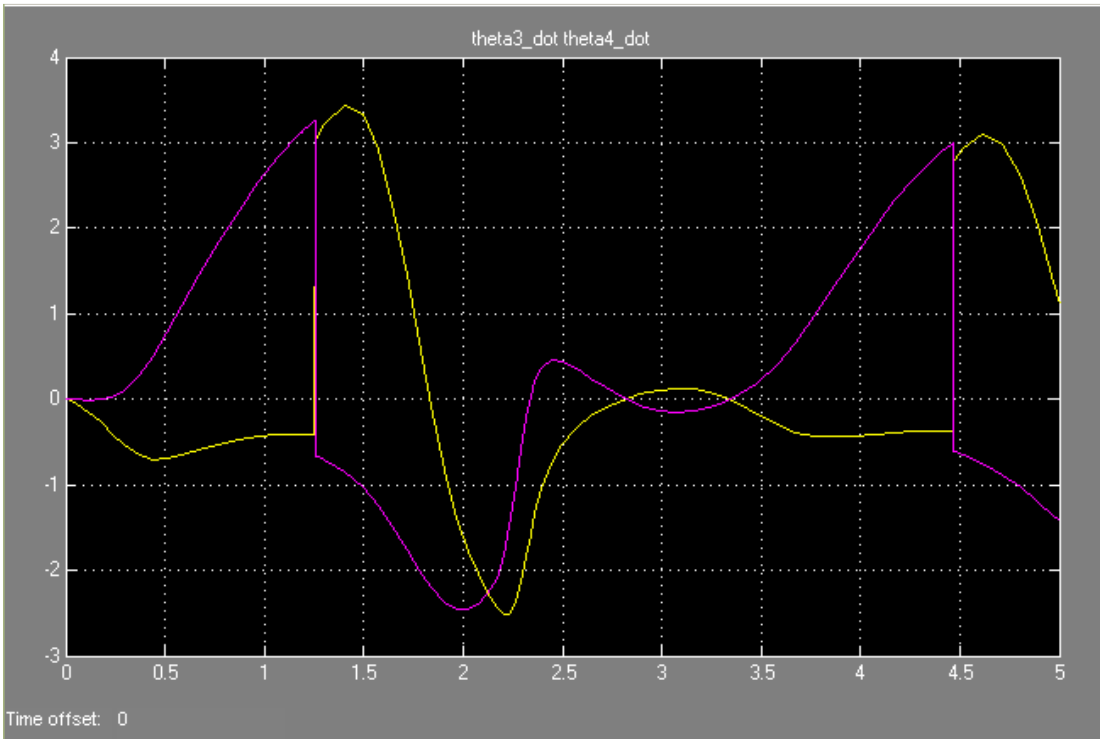

Figure 3.11 $\dot{\theta}_3$, $\dot{\theta}_4$ graph

4. DYNAMIC ANALYSIS OF ELASTIC FOUR-BAR MECHANISM

4.1 Dynamic Model of Flexible Coupler Link Four-Bar Mechanism

4.1.1 Lagrange's equation

In this section, we utilize the concepts introduced in the preceding section to develop Lagrange's equation of motion for a system of particles. The obtained Lagrange's equation, however, can be applied to the dynamics of rigid bodies as well, since a rigid body can be considered as a collection of a large number of particles.

In the following discussion, we assume that the system consists of np particles. The displacement r^i of the ith particle is assumed to depend on a set of generalized coordinates q_j, where j=1,2,…,n. Hence

$$r^i = r^i(q_1, q_2, \dots, q_n, t) \quad \textbf{(4.1)}$$

Differentiating Equation (E.1) with respect to time using the chain rule of differentiation yields

$$\dot{r}^i = \frac{\partial r^i}{\partial q_1}\dot{q}_1 + \frac{\partial r^i}{\partial q_2}\dot{q}_2 + \cdots + \frac{\partial r^i}{\partial q_n}\dot{q}_n + \frac{\partial r^i}{\partial t} = \sum_{j=1}^{\infty} \frac{\partial r^i}{\partial q_j}\dot{q}_J + \frac{\partial r^i}{\partial t} \quad \textbf{(4.2)}$$

The virtual displacement δr^i can be expressed in terms of the virtual change of the generalized coordinates as

$$\delta r^i = \sum_{j=1}^{n} \frac{\partial r^i}{\partial q_j}\delta q_j \quad \textbf{(4.3)}$$

The dynamic equilibrium of the particle can be defined using Newton's second law as

$$\dot{p}^i = F^i \quad \textbf{(4.4)}$$

Where p^i is the vector of linear momentum and F^i is the vector of the total forces acting on the particles i. the linear momentum vector p^i is defined as

$$p^i = m^i \dot{r}^i \quad \textbf{(4.5)}$$

Where m^i is the mass of the particle I which is assumed to be constant.

Consequently,

$$p^i = m^i \ddot{r}^i \quad \textbf{(4.6)}$$

Substituting **(4.6)** into **(4.5)** yields

$$m^i \ddot{r}^i - F^i = 0$$

Which yields

$$(m^i \ddot{r}^i - F^i)^T \delta r^i = 0 \quad \textbf{(4.7)}$$

By summing up these expressions for the particles and using the definition of δr^i given by **(4.3)**, one obtains

$$\sum_{i=1}^{n_p} \left(m^i \ddot{r}^i - F^i\right)^T \left(\sum_{j=1}^{n} \frac{\partial r^i}{\partial q_j} \delta q_j\right) = 0$$

Which can also be written as

$$\sum_{j=1}^{n} \sum_{i=1}^{n_p} \left(m^i \ddot{r}^i - F^i\right)^T \left(\frac{\partial r^i}{\partial q_j} \delta q_j\right) = 0 \quad \textbf{(4.8)}$$

Define the generalized force Q_j associated with the generalized coordinate q_j as

$$Q_j = \sum_{i=1}^{n_p} F^{iT} \frac{\partial r^i}{\partial q_j} \quad \textbf{(4.9)}$$

One can also show that

$$\sum_{i=1}^{n_p} \left(m^i \ddot{r}^{iT} \frac{\partial r^i}{\partial q_j}\right) = \sum_{i=1}^{n_p} \left[\frac{d}{dt}\left(m^i \dot{r}^{iT} \frac{\partial r^i}{\partial q_j}\right) - m^i \dot{r}^{iT} \frac{d}{dt}\left(\frac{\partial r^i}{\partial q_j}\right)\right] \quad \textbf{(4.10)}$$

It is, however, clear from Equation (4.2) that

$$\frac{\partial \dot{r}^i}{\partial \dot{q}_j} = \frac{\partial r^i}{\partial q_j} \quad \textbf{(4.11)}$$

Furthermore,

$$\frac{d}{dt}\left(\frac{\partial r^i}{\partial q_j}\right) = \sum_{k=1}^{n} \frac{\partial^2 r^i}{\partial q_j \partial q_k} \dot{q}_k + \frac{\partial^2 r^i}{\partial q_j \partial t} = \frac{\partial \dot{r}^i}{\partial q_j} \quad \textbf{(4.12)}$$

Substituting the results of **(4.11)** and **(4.12)** into **(4.10)** yields

$$\sum_{i=1}^{n_p} m^i \ddot{r}^{iT} \frac{\partial r^i}{\partial q_j} = \sum_{i=1}^{n_p} \frac{d}{dt}\left[\frac{\partial}{\partial \dot{q}_j}\left(\frac{1}{2} m^i \dot{r}^{iT} \dot{r}^i\right)\right] - \frac{\partial}{\partial q_j}\left(\frac{1}{2} m^i \dot{r}^{iT} \dot{r}^i\right)$$

$$= \sum_{i=1}^{n_p} \frac{d}{dt}\left(\frac{\partial T^i}{\partial \dot{q}_j}\right) - \frac{\partial T^i}{\partial q_j}$$

(4.13)

Where T^i is the kinetic energy of the particle i defined as

$$T^i = \frac{1}{2} m^i \dot{r}^{iT} \dot{r}^i \quad \textbf{(4.14)}$$

The kinetic energy of the system of particles is given by

$$T = \sum_{i=1}^{n_p} T^i \quad \textbf{(4.15)}$$

Therefore, **(4.13)** can be written in terms of the kinetic energy of the system of particles as

$$\sum_{i=1}^{n_p} m^i \ddot{r}^{iT} \frac{\partial r^i}{\partial q_j} = \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} \quad \textbf{(4.16)}$$

Substituting **(4.9)** and **(4.15)** into **(4.8)** yields

$$\sum_{j=1}^{n} \left[\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial \dot{q}_j} - Q_j\right] \delta q_j = 0 \quad \textbf{(4.17)}$$

Since the generalized coordinates $q_1, q_2, \dots, q_n$ are assumed to be linearly independent, **(4.17)** yields a set of n equations defined as

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} - Q_j = 0, \quad j = 1,2,\dots,n$$

Or

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} = Q_j, \qquad j = 1,2,\dots,n$$

(4.18)

Next if the system is such that a potential energy function U exists, the generalized active forces F, may be expressed as:

$$F_r = -\frac{\partial P}{\partial q_j}(j = 1, \dots, n)$$

(4.19)

By substituting **(3.30)** we have:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = Q_j, \qquad j = 1,2, \dots, n$$

(4.20)

Where L is defined as:

$$L \stackrel{\text{def}}{=} T - U \qquad \textbf{(4.21)}$$

And is called the Lagrangian.

The **(4.18)** and **(4.20)** are the common forms of Lagrange's equations. The principle advantage of these equations is that vector accelerations need not be computed; that is, the inertia forces are developed independently by differentiation of the kinetic energy function. Another advantage of Lagrange's equations is that nonworking forces are automatically eliminated from the equations through use of generalized active forces. As such, nonworking forces may be neglected at the onset of an analysis. Finally, Lagrange's equations produce exactly the same number of equations as the degrees of freedom.

Disadvantages of Lagrange's equations are that they are not readily applicable with nonholonomic systems and the scalar differentiations in **(4.18)** and **(4.20)** can be tedious and burdensome for large systems.

4.1.2 Dynamic equation of motion with Lagrange's equation

The kinetic energy of the four-bar mechanism shown in Figure 4.1 is given by:

$$T = \frac{1}{2} I_C \dot{\theta}_2^2 + \frac{1}{2} I_O \dot{\theta}_4^2 + \frac{1}{2}\int_0^{l_3} \rho \vec{\dot{r}}^{\,2} dx \qquad \textbf{(4.22)}$$

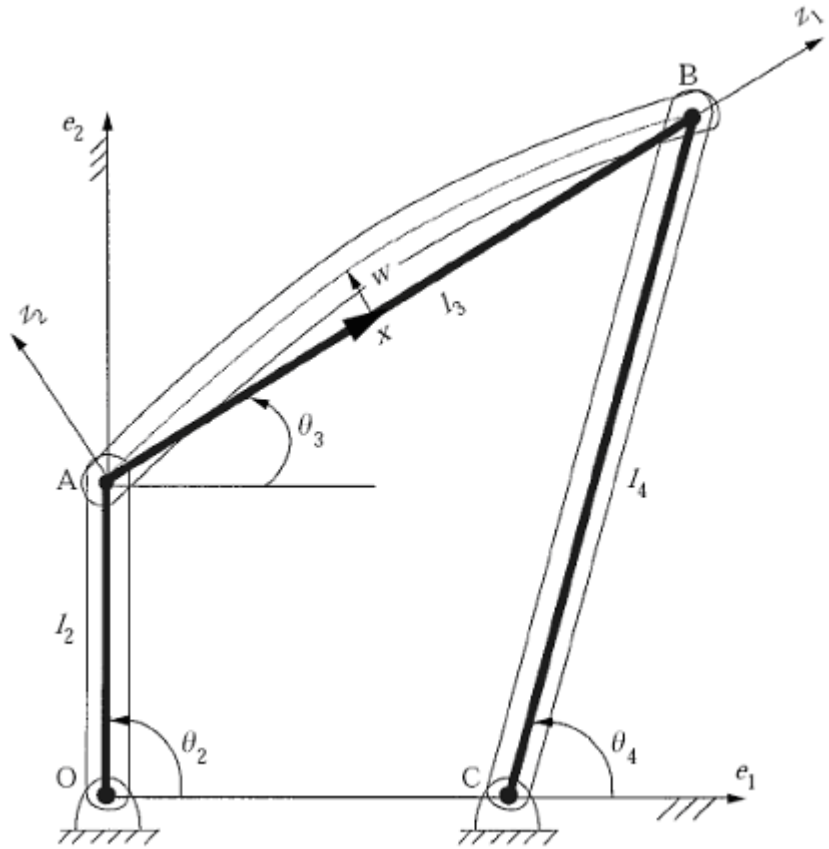

Figure 4.1 Sketch of the four-bar mechanism with elastic coupler link

where θ_2 and θ_4 are the input and output angles, respectively, I_C and I_O are the mass moments of inertia of the input and output links with respect to the joint axes, respectively, ρ is mass per unit length of the coupler link, and $\vec{r}$ is the position vector of a point on the coupler link given as:

$$\vec{r} = \vec{r}_2 + \vec{r}_3 + \vec{w} \tag{4.23}$$

where

$$\vec{r}_2 = l_2 \cos\theta_2 \vec{e}_1 + l_2 \sin\theta_2 \vec{e}_2 \tag{4.24}$$

$$\vec{r}_3 = x\cos\theta_3 \vec{e}_1 + x\sin\theta_3 \vec{e}_2 \tag{4.25}$$

$$\vec{w} = w\vec{z}_2 \tag{4.26}$$

where w represents the elastic deformation of the coupler link at an arbitrary position x. The unit vectors $\vec{z}_1$ and $\vec{z}_2$ can be written as

$$\vec{z}_1 = \cos\theta_3 \vec{e}_1 + \sin\theta_3 \vec{e}_2 \tag{4.27}$$

$$\vec{z}_2 = -\sin\theta_3 \vec{e}_1 + \cos\theta_3 \vec{e}_2 = \cos\theta_3 \vec{e}_2 - \sin\theta_3 \vec{e}_1 \tag{4.28}$$

Applying these to $\vec{r}$ equation:

$$\vec{r} = l_2 \cos\theta_2 \vec{e}_1 + l_2 \sin\theta_2 \vec{e}_2 + x\cos\theta_3 \vec{e}_1 + x\sin\theta_3 \vec{e}_2 + w\vec{z}_2 \quad \textbf{(4.29)}$$

$$\vec{r} = (l_2 \cos\theta_2 + x\cos\theta_3)\vec{e}_1 + (l_2 \sin\theta_2 + x\sin\theta_3)\vec{e}_2 + w\vec{z}_2 \quad \textbf{(4.30)}$$

$$\vec{r} = (l_2 \cos\theta_2 + x\cos\theta_3)\vec{e}_1 + (l_2 \sin\theta_2 + x\sin\theta_3)\vec{e}_2 + w(\cos\theta_3 \vec{e}_2 - \sin\theta_3 \vec{e}_1) \quad \textbf{(4.31)}$$

$$\vec{r} = (l_2 \cos\theta_2 + x\cos\theta_3 - w\sin\theta_3)\vec{e}_1 + (l_2 \sin\theta_2 + x\sin\theta_3 + w\cos\theta_3)\vec{e}_2 \quad \textbf{(4.32)}$$

The assumed modes method along with a constrained Lagrangian approach is used to obtain the discretized equations of motion. Let the deflection of the coupler link be written as

$$w(x,t) = \sum_{i=0}^{N} \psi_i(x) q_i(t) \quad \textbf{(4.33)}$$

where $q_i(t)$ are the modal co-ordinates and $\psi_i(x)$ are the mode shapes of a pinned-pinned beam given as

$$\psi_i(x) = \sin\left(\frac{i\pi}{l_3} x\right) \quad \textbf{(4.34)}$$

After inserting all above information into the $\vec{r}$ equation we get

$$\begin{aligned} \vec{r} = &\left(l_2 \cos\theta_2 + x\cos\theta_3 - \sin\theta_3 \sum_{i=0}^{N} \psi_i(x) q_i(t) \right) \vec{e}_1 \\ &+ \left(l_2 \sin\theta_2 + x\sin\theta_3 + \cos\theta_3 \sum_{i=0}^{N} \psi_i(x) q_i(t) \right) \vec{e}_2 \end{aligned} \quad \textbf{(4.35)}$$

It's derivative will be

$$\begin{aligned} \dot{\vec{r}} = &\left(-l_2\dot{\theta}_2 \sin\theta_2 - x\dot{\theta}_3 \sin\theta_3 - \sin\theta_3 \sum_{i=0}^{N} \psi_i(x)\dot{q}_i(t) - \dot{\theta}_3 \cos\theta_3 \sum_{i=0}^{N} \psi_i(x) q_i(t) \right) \vec{e}_1 \\ &+ \left(l_2\dot{\theta}_2 \cos\theta_2 + x\dot{\theta}_3 \cos\theta_3 + \cos\theta_3 \sum_{i=0}^{N} \psi_i(x)\dot{q}_i(t) - \dot{\theta}_3 \sin\theta_3 \sum_{i=0}^{N} \psi_i(x) q_i(t) \right) \vec{e}_2 \end{aligned} \quad \textbf{(4.36)}$$

For $\dot{\vec{r}}^{\,2}$ it should be more convenient to work with **(4.36)** divided into parts as for $\vec{e}_1$ and $\vec{e}_2$:

$$\left(-l_2\dot{\theta}_2\sin\theta_2 - x\dot{\theta}_3\sin\theta_3 - \sin\theta_3\sum_{i=0}^{N}\psi_i(x)\dot{q}_i(t) - \dot{\theta}_3\cos\theta_3\sum_{i=0}^{N}\psi_i(x)q_i(t)\right)^2 =$$

$$\left(l_2\dot{\theta}_2\sin\theta_2\right)^2 + \left(x\dot{\theta}_3\sin\theta_3\right)^2 + \left(\sin\theta_3\sum_{i=0}^{N}\psi_i(x)\dot{q}_i(t)\right)^2 + \left(\dot{\theta}_3\cos\theta_3\sum_{i=0}^{N}\psi_i(x)q_i(t)\right)^2$$

$$+2l_2x\dot{\theta}_3\dot{\theta}_2\sin\theta_2\sin\theta_3 + 2l_2\dot{\theta}_2\sin\theta_2\sin\theta_3\sum_{i=0}^{N}\psi_i(x)\dot{q}_i(t)$$

$$+2l_2\dot{\theta}_2\dot{\theta}_3\sin\theta_2\cos\theta_3\sum_{i=0}^{N}\psi_i(x)q_i(t) + 2x\dot{\theta}_3\sin\theta_3\sin\theta_3\sum_{i=0}^{N}\psi_i(x)\dot{q}_i(t)$$

$$+2x\dot{\theta}_3\dot{\theta}_3\sin\theta_3\cos\theta_3\sum_{i=0}^{N}\psi_i(x)q_i(t) + 2\dot{\theta}_3\cos\theta_3\sin\theta_3\sum_{i=0}^{N}\psi_i(x)\dot{q}_i(t)\sum_{i=0}^{N}\psi_i(x)q_i(t) \quad \textbf{(4.37)}$$

$$\left(l_2\dot{\theta}_2\cos\theta_2 + x\dot{\theta}_3\cos\theta_3 + \cos\theta_3\sum_{i=0}^{N}\psi_i(x)\dot{q}_i(t) - \dot{\theta}_3\sin\theta_3\sum_{i=0}^{N}\psi_i(x)q_i(t)\right)^2 =$$

$$\left(l_2\dot{\theta}_2\cos\theta_2\right)^2 + \left(x\dot{\theta}_3\cos\theta_3\right)^2 + \left(\cos\theta_3\sum_{i=0}^{N}\psi_i(x)\dot{q}_i(t)\right)^2 + \left(\dot{\theta}_3\sin\theta_3\sum_{i=0}^{N}\psi_i(x)q_i(t)\right)^2$$

$$+2l_2x\dot{\theta}_3\dot{\theta}_2\cos\theta_2\cos\theta_3 + 2l_2\dot{\theta}_2\cos\theta_2\cos\theta_3\sum_{i=0}^{N}\psi_i(x)\dot{q}_i(t)$$

$$+2l_2\dot{\theta}_2\dot{\theta}_3\cos\theta_2\sin\theta_3\sum_{i=0}^{N}\psi_i(x)q_i(t) + 2x\dot{\theta}_3\cos\theta_3\cos\theta_3\sum_{i=0}^{N}\psi_i(x)\dot{q}_i(t)$$

$$-2x\dot{\theta}_3\dot{\theta}_3\cos\theta_3\sin\theta_3\sum_{i=0}^{N}\psi_i(x)q_i(t) - 2\dot{\theta}_3\sin\theta_3\cos\theta_3\sum_{i=0}^{N}\psi_i(x)\dot{q}_i(t)\sum_{i=0}^{N}\psi_i(x)q_i(t) \quad \textbf{(4.38)}$$

Combining **(4.37)** and **(4.38)** we get

$$\dot{\vec{r}}^2 = \left(l_2\dot{\theta}_2\right)^2 + \left(x\dot{\theta}_3\right)^2 + \left(\sum_{i=0}^{N}\psi_i(x)\dot{q}_i(t)\right)^2 + \left(\dot{\theta}_3\sum_{i=0}^{N}\psi_i(x)q_i(t)\right)^2$$

$$+2l_2x\dot{\theta}_2\dot{\theta}_3\cos\left(\theta_2-\theta_3\right) + 2l_2\dot{\theta}_2\cos\left(\theta_2-\theta_3\right)\sum_{i=0}^{N}\psi_i(x)\dot{q}_i(t) \quad \textbf{(4.39)}$$

$$+2l_2\dot{\theta}_2\dot{\theta}_3\sin\left(\theta_2-\theta_3\right)\sum_{i=0}^{N}\psi_i(x)q_i(t) + 2x\dot{\theta}_3\sum_{i=0}^{N}\psi_i(x)\dot{q}_i(t)$$

Solving for $\frac{1}{2}\int_0^{l_3}\rho\dot{\vec{r}}^2 dx$ term in **(4.22)**

$$\begin{aligned}\frac{1}{2}\int_0^{l_3}\rho\dot{\vec{r}}^2 dx &= \frac{1}{2}\int_0^{l_3}\rho\left(l_2\dot{\theta}_2\right)^2 dx+\frac{1}{2}\int_0^{l_3}\rho\left(x\dot{\theta}_3\right)^2 dx\\ &+\frac{1}{2}\int_0^{l_3}\rho\left(\sum_{i=0}^{N}\psi_i(x)\dot{q}_i(t)\right)^2 dx+\frac{1}{2}\int_0^{l_3}\rho\left(\dot{\theta}_3\sum_{i=0}^{N}\psi_i(x)q_i(t)\right)^2 dx\\ &+\int_0^{l_3}\rho l_2 x\dot{\theta}_2\dot{\theta}_3\cos\left(\theta_2-\theta_3\right)dx+\int_0^{l_3}\rho l_2\dot{\theta}_2\cos\left(\theta_2-\theta_3\right)\sum_{i=0}^{N}\psi_i(x)\dot{q}_i(t)dx\\ &+\int_0^{l_3}\rho l_2\dot{\theta}_2\dot{\theta}_3\sin\left(\theta_2-\theta_3\right)\sum_{i=0}^{N}\psi_i(x)q_i(t)dx+\int_0^{l_3}\rho x\dot{\theta}_3\sum_{i=0}^{N}\psi_i(x)\dot{q}_i(t)dx\end{aligned} \tag{4.40}$$

$$\begin{aligned}\frac{1}{2}\int_0^{l_3}\rho\dot{\vec{r}}^2 dx &= \frac{1}{2}\rho\left(l_2\dot{\theta}_2\right)^2 l_3+\frac{1}{6}\rho l_3{}^3\dot{\theta}_3{}^2+\frac{1}{2}\int_0^{l_3}\rho\left(\sum_{i=0}^{N}\psi_i(x)\dot{q}_i(t)\right)^2 dx\\ &+\frac{1}{2}\dot{\theta}_3{}^2\int_0^{l_3}\rho\left(\sum_{i=0}^{N}\psi_i(x)q_i(t)\right)^2 dx+\frac{1}{2}\rho l_2 l_3{}^2\dot{\theta}_2\dot{\theta}_3\cos\left(\theta_2-\theta_3\right)\\ &+l_2\dot{\theta}_2\cos\left(\theta_2-\theta_3\right)\int_0^{l_3}\rho\left(\sum_{i=0}^{N}\psi_i(x)\dot{q}_i(t)\right)dx\\ &+l_2\dot{\theta}_2\dot{\theta}_3\sin\left(\theta_2-\theta_3\right)\int_0^{l_3}\rho\left(\sum_{i=0}^{N}\psi_i(x)q_i(t)\right)dx+\dot{\theta}_3\int_0^{l_3}\rho x\left(\sum_{i=0}^{N}\psi_i\left(x\right)\dot{q}_i\left(t\right)\right)dx\end{aligned} \tag{4.41}$$

So the final kinetic energy state is

$$\begin{aligned}T &= \frac{1}{2}I_C\dot{\theta}_2^2+\frac{1}{2}I_O\dot{\theta}_4^2+\frac{1}{2}\rho\left(l_2\dot{\theta}_2\right)^2 l_3+\frac{1}{6}\rho l_3{}^3\dot{\theta}_3{}^2+\frac{1}{2}\int_0^{l_3}\rho\left(\sum_{i=0}^{N}\psi_i(x)\dot{q}_i(t)\right)^2 dx\\ &+\frac{1}{2}\dot{\theta}_3{}^2\int_0^{l_3}\rho\left(\sum_{i=0}^{N}\psi_i(x)q_i(t)\right)^2 dx+\frac{1}{2}\rho l_2 l_3{}^2\dot{\theta}_2\dot{\theta}_3\cos\left(\theta_2-\theta_3\right)\\ &+l_2\dot{\theta}_2\cos\left(\theta_2-\theta_3\right)\int_0^{l_3}\rho\left(\sum_{i=0}^{N}\psi_i(x)\dot{q}_i(t)\right)dx\\ &+l_2\dot{\theta}_2\dot{\theta}_3\sin\left(\theta_2-\theta_3\right)\int_0^{l_3}\rho\left(\sum_{i=0}^{N}\psi_i(x)q_i(t)\right)dx+\dot{\theta}_3\int_0^{l_3}\rho x\left(\sum_{i=0}^{N}\psi_i\left(x\right)\dot{q}_i\left(t\right)\right)dx\end{aligned} \tag{4.42}$$

The strain energy is given by

$$U = \frac{1}{2}\int_0^{l_3} \frac{M^2}{EI}\,dx = \frac{1}{2}\int_0^{l_3} \frac{\left(EI\dfrac{\partial^2 w}{\partial x^2}\right)^2}{EI}\,dx$$
$$= \frac{1}{2}\int_0^{l_3} EI\left(\frac{\partial^2 w}{\partial x^2}\right)^2 dx = \frac{1}{2}\int_0^{l_3} EI\left(\frac{\partial^2\left(\sum_{i=0}^{N}\psi_i(x)q_i(t)\right)}{\partial x^2}\right)^2 dx \qquad \textbf{(4.43)}$$

where EI is the flexural rigidity of the coupler link.

The variables θ_2, θ_3, and θ_4 are related by the following geometrical constraints:

$$\Phi = \begin{Bmatrix} -l_1 + l_2\cos\theta_2 + l_3\cos\theta_3 - l_4\cos\theta_4 \\ l_2\sin\theta_2 + l_3\sin\theta_3 - l_4\sin\theta_4 \end{Bmatrix} = 0 \qquad \textbf{(4.44)}$$

The constrained Lagrange's equations are

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\eta}_i}\right) - \frac{\partial L}{\partial \eta_i} + \left(\frac{\partial \vec{\Phi}}{\partial \eta_i}\right)^T \Lambda = F_i \qquad \textbf{(4.45)}$$

Where $L = T - U$, or

$$L = \frac{1}{2}I_C\dot{\theta}_2^2 + \frac{1}{2}I_O\dot{\theta}_4^2 + \frac{1}{2}\rho\left(l_2\dot{\theta}_2\right)^2 l_3 + \frac{1}{6}\rho l_3{}^3\dot{\theta}_3{}^2 + \frac{1}{2}\int_0^{l_3}\rho\left(\sum_{i=0}^{N}\psi_i(x)\dot{q}_i(t)\right)^2 dx$$
$$+\frac{1}{2}\dot{\theta}_3{}^2\int_0^{l_3}\rho\left(\sum_{i=0}^{N}\psi_i(x)q_i(t)\right)^2 dx + \frac{1}{2}\rho l_2 l_3{}^2\dot{\theta}_2\dot{\theta}_3\cos\left(\theta_2-\theta_3\right)$$
$$+l_2\dot{\theta}_2\cos\left(\theta_2-\theta_3\right)\int_0^{l_3}\rho\left(\sum_{i=0}^{N}\psi_i(x)\dot{q}_i(t)\right)dx \qquad \textbf{(4.46)}$$
$$+l_2\dot{\theta}_2\dot{\theta}_3\sin\left(\theta_2-\theta_3\right)\int_0^{l_3}\rho\left(\sum_{i=0}^{N}\psi_i(x)q_i(t)\right)dx$$
$$+\dot{\theta}_3\int_0^{l_3}\rho x\left(\sum_{i=0}^{N}\psi_i(x)\dot{q}_i(t)\right)dx - \frac{1}{2}\int_0^{l_3}EI\left(\frac{\partial^2 w}{\partial x^2}\right)^2 dx$$

and η_i are the generalized coordinates which includes rigid body coordinates as well as elastic modal coordinates. $\vec{\Phi}$ is the vector containing the constraint equations and $\vec{\Lambda}$ is the vector of Lagrange multipliers λ_1 and λ_2; and F_i are the generalized

forces. The force vector $\left(\partial\vec{\Phi}/\partial\eta_i\right)^T\Lambda$ is substituted in to replace the removed joint forces.

$$\left(\frac{\partial\Phi}{\partial\eta_i}\right)^T\Lambda=\left(\frac{\partial\begin{bmatrix}-l_1+l_2\cos\theta_2+l_3\cos\theta_3-l_4\cos\theta_4\\ l_2\sin\theta_2+l_3\sin\theta_3-l_4\sin\theta_4\end{bmatrix}}{\partial\eta_i}\right)^T\begin{bmatrix}\lambda_1\\ \lambda_2\end{bmatrix} \quad \textbf{(4.47)}$$

4.1.2.1 Dynamic equation of motion with Lagrange's equation for θ_2

Doing mathematical calculations for the constrained Lagrange's equations for θ_2

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\eta}_i}\right)-\frac{\partial L}{\partial\eta_i}+\left(\frac{\partial\Phi}{\partial\eta_i}\right)^T\Lambda=F_i$$

$\dot{\theta}_2$ derivative for $\frac{\partial L}{\partial\dot{\eta}_i}$:

$$\frac{\partial L}{\partial\dot{\theta}_2}=I_C\dot{\theta}_2+\rho l_2{}^2l_3\dot{\theta}_2+\frac{1}{2}\rho l_2l_3{}^2\dot{\theta}_3\cos\left(\theta_2-\theta_3\right)$$
$$+l_2\cos\left(\theta_2-\theta_3\right)\int_0^{l_3}\rho\left(\sum_{i=0}^{N}\psi_i(x)\dot{q}_i(t)\right)dx+l_2\dot{\theta}_3\sin\left(\theta_2-\theta_3\right)\int_0^{l_3}\rho\left(\sum_{i=0}^{N}\psi_i(x)q_i(t)\right)dx \quad \textbf{(4.48)}$$

Time derivative $\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\eta}_i}\right)$:

3rd term of **(4.48)**

$$\frac{d}{dt}\left(\frac{1}{2}\rho l_2l_3{}^2\dot{\theta}_3\cos\left(\theta_2-\theta_3\right)\right)=\frac{1}{2}\rho l_2l_3{}^2\ddot{\theta}_3\cos\left(\theta_2-\theta_3\right)-\left(\dot{\theta}_2-\dot{\theta}_3\right)\frac{1}{2}\rho l_2l_3{}^2\dot{\theta}_3\sin\left(\theta_2-\theta_3\right)$$

4th term of **(4.48)**

$$\frac{d}{dt}\left(l_2\cos\left(\theta_2-\theta_3\right)\int_0^{l_3}\rho\left(\sum_{i=0}^{N}\psi_i(x)\dot{q}_i(t)\right)dx\right)=-l_2\left(\dot{\theta}_2-\dot{\theta}_3\right)\sin\left(\theta_2-\theta_3\right)\int_0^{l_3}\rho\left(\sum_{i=0}^{N}\psi_i(x)\dot{q}_i(t)\right)dx$$
$$+l_2\cos\left(\theta_2-\theta_3\right)\int_0^{l_3}\rho\left(\sum_{i=0}^{N}\psi_i(x)\ddot{q}_i(t)\right)dx$$

5th term of **(4.48)**

$$\frac{d}{dt}\left(l_2\dot{\theta}_3 \sin\left(\theta_2-\theta_3\right)\int_0^{l_3}\rho\left(\sum_{i=0}^{N}\psi_i(x)q_i(t)\right)dx\right) = l_2\ddot{\theta}_3 \sin\left(\theta_2-\theta_3\right)\int_0^{l_3}\rho\left(\sum_{i=0}^{N}\psi_i(x)q_i(t)\right)dx$$
$$+l_2\dot{\theta}_3\left(\dot{\theta}_2-\dot{\theta}_3\right)\cos\left(\theta_2-\theta_3\right)\int_0^{l_3}\rho\left(\sum_{i=0}^{N}\psi_i(x)q_i(t)\right)dx + l_2\dot{\theta}_3 \sin\left(\theta_2-\theta_3\right)\int_0^{l_3}\rho\left(\sum_{i=0}^{N}\psi_i(x)\dot{q}_i(t)\right)dx$$

Combining all terms will give us

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}_2}\right) = I_C\ddot{\theta}_2 + \rho l_2{}^2 l_3\ddot{\theta}_2 + \frac{1}{2}\rho l_2 l_3{}^2\ddot{\theta}_3\cos\left(\theta_2-\theta_3\right) - \left(\dot{\theta}_2-\dot{\theta}_3\right)\frac{1}{2}\rho l_2 l_3{}^2\dot{\theta}_3\sin\left(\theta_2-\theta_3\right)$$
$$-l_2\left(\dot{\theta}_2-\dot{\theta}_3\right)\sin\left(\theta_2-\theta_3\right)\int_0^{l_3}\rho\left(\sum_{i=0}^{N}\psi_i(x)\dot{q}_i(t)\right)dx + l_2\cos\left(\theta_2-\theta_3\right)\int_0^{l_3}\rho\left(\sum_{i=0}^{N}\psi_i(x)\ddot{q}_i(t)\right)dx$$
$$+l_2\ddot{\theta}_3\sin\left(\theta_2-\theta_3\right)\int_0^{l_3}\rho\left(\sum_{i=0}^{N}\psi_i(x)q_i(t)\right)dx + l_2\dot{\theta}_3\left(\dot{\theta}_2-\dot{\theta}_3\right)\cos\left(\theta_2-\theta_3\right)\int_0^{l_3}\rho\left(\sum_{i=0}^{N}\psi_i(x)q_i(t)\right)dx$$
$$+l_2\dot{\theta}_3\sin\left(\theta_2-\theta_3\right)\int_0^{l_3}\rho\left(\sum_{i=0}^{N}\psi_i(x)\dot{q}_i(t)\right)dx$$

θ_2 derivative for $\frac{\partial L}{\partial\eta_i}$:

$$\frac{\partial L}{\partial\theta_2} = -\frac{1}{2}\rho l_2 l_3{}^2\dot{\theta}_2\dot{\theta}_3\sin\left(\theta_2-\theta_3\right) - l_2\dot{\theta}_2\sin\left(\theta_2-\theta_3\right)\int_0^{l_3}\rho\left(\sum_{i=0}^{N}\psi_i(x)\dot{q}_i(t)\right)dx$$
$$+l_2\dot{\theta}_2\dot{\theta}_3\cos\left(\theta_2-\theta_3\right)\int_0^{l_3}\rho\left(\sum_{i=0}^{N}\psi_i(x)q_i(t)\right)dx$$

For $\left(\frac{\partial\Phi}{\partial\eta_i}\right)^T\Lambda$:

$$\left(\frac{\partial\Phi}{\partial\theta_2}\right)^T\Lambda = \left(\frac{\partial\begin{bmatrix}-l_1+l_2\cos\theta_2+l_3\cos\theta_3-l_4\cos\theta_4\\ l_2\sin\theta_2+l_3\sin\theta_3-l_4\sin\theta_4\end{bmatrix}}{\partial\theta_2}\right)^T\begin{bmatrix}\lambda_1\\ \lambda_2\end{bmatrix}$$
$$=\begin{bmatrix}-l_2\sin\theta_2\\ l_2\cos\theta_2\end{bmatrix}^T\begin{bmatrix}\lambda_1\\ \lambda_2\end{bmatrix} = \begin{bmatrix}-l_2\sin\theta_2 & l_2\cos\theta_2\end{bmatrix}\begin{bmatrix}\lambda_1\\ \lambda_2\end{bmatrix}$$
$$=-l_2\sin\theta_2\lambda_1 + l_2\cos\theta_2\lambda_2$$

The final state of $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\eta}_i}\right)-\frac{\partial L}{\partial \eta_i}+\left(\frac{\partial \Phi}{\partial \eta_i}\right)^T \Lambda = F_i$ for θ_2:

$$I_C\ddot{\theta}_2+\rho l_2{}^2 l_3\ddot{\theta}_2+\frac{1}{2}\rho l_2 l_3{}^2\ddot{\theta}_3\cos(\theta_2-\theta_3)-(\dot{\theta}_2-\dot{\theta}_3)\frac{1}{2}\rho l_2 l_3{}^2\dot{\theta}_3\sin(\theta_2-\theta_3)$$
$$-l_2(\dot{\theta}_2-\dot{\theta}_3)\sin(\theta_2-\theta_3)\int_0^{l_3}\rho\left(\sum_{i=0}^{N}\psi_i(x)\dot{q}_i(t)\right)dx+l_2\cos(\theta_2-\theta_3)\int_0^{l_3}\rho\left(\sum_{i=0}^{N}\psi_i(x)\ddot{q}_i(t)\right)dx$$
$$+l_2\ddot{\theta}_3\sin(\theta_2-\theta_3)\int_0^{l_3}\rho\left(\sum_{i=0}^{N}\psi_i(x)q_i(t)\right)dx+l_2\dot{\theta}_3(\dot{\theta}_2-\dot{\theta}_3)\cos(\theta_2-\theta_3)\int_0^{l_3}\rho\left(\sum_{i=0}^{N}\psi_i(x)q_i(t)\right)dx$$
$$+l_2\dot{\theta}_3\sin(\theta_2-\theta_3)\int_0^{l_3}\rho\left(\sum_{i=0}^{N}\psi_i(x)\dot{q}_i(t)\right)dx+\frac{1}{2}\rho l_2 l_3{}^2\dot{\theta}_2\dot{\theta}_3\sin(\theta_2-\theta_3)$$
$$+l_2\dot{\theta}_2\sin(\theta_2-\theta_3)\int_0^{l_3}\rho\left(\sum_{i=0}^{N}\psi_i(x)\dot{q}_i(t)\right)dx-l_2\dot{\theta}_2\dot{\theta}_3\cos(\theta_2-\theta_3)\int_0^{l_3}\rho\left(\sum_{i=0}^{N}\psi_i(x)q_i(t)\right)dx$$
$$-l_2\sin\theta_2\lambda_1+l_2\cos\theta_2\lambda_2=\tau \qquad \textbf{(4.49)}$$

4.1.2.2Dynamic equation of motion with Lagrange's equation for θ_3

Doing mathematical calculations for the constrained Lagrange's equations for θ_3:

$\dot{\theta}_3$ derivative for $\frac{\partial L}{\partial \dot{\eta}_i}$:

$$\frac{\partial L}{\partial \dot{\theta}_3}=\frac{1}{3}\rho l_3{}^3\dot{\theta}_3+\dot{\theta}_3\int_0^{l_3}\rho\left(\sum_{i=0}^{N}\psi_i(x)q_i(t)\right)^2dx+\frac{1}{2}\rho l_2 l_3{}^2\dot{\theta}_2\cos(\theta_2-\theta_3)$$
$$+l_2\dot{\theta}_2\sin(\theta_2-\theta_3)\int_0^{l_3}\rho\left(\sum_{i=0}^{N}\psi_i(x)q_i(t)\right)dx+\int_0^{l_3}\rho x\left(\sum_{i=0}^{N}\psi_i(x)\dot{q}_i(t)\right)dx \qquad \textbf{(4.50)}$$

Time derivative $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\eta}_i}\right)$:

2nd term of **(4.50)**

$$\frac{d}{dt}\left(\dot{\theta}_3\int_0^{l_3}\rho\left(\sum_{i=0}^{N}\psi_i(x)q_i(t)\right)^2dx\right)=\ddot{\theta}_3\int_0^{l_3}\rho\left(\sum_{i=0}^{N}\psi_i(x)q_i(t)\right)^2dx+2\dot{\theta}_3\int_0^{l_3}\rho\sum_{i=0}^{N}\sum_{i=0}^{N}\psi_i(x)\psi_j(x)\dot{q}_i(t)\dot{q}_j(t)dx$$

3rd term of **(4.50)**

$$\frac{d}{dt}\left(\frac{1}{2}\rho l_2 l_3{}^2\dot{\theta}_2\cos(\theta_2-\theta_3)\right)=\frac{1}{2}\rho l_2 l_3{}^2\ddot{\theta}_2\cos(\theta_2-\theta_3)-\frac{1}{2}\rho l_2 l_3{}^2\dot{\theta}_2(\dot{\theta}_2-\dot{\theta}_3)\sin(\theta_2-\theta_3)$$

4th term of **(4.50)**

$$\frac{d}{dt}\left(l_2\dot{\theta}_2 \sin\left(\theta_2-\theta_3\right)\int_0^{l_3}\rho\left(\sum_{i=0}^{N}\psi_i(x)q_i(t)\right)dx\right)=$$

$$= l_2\ddot{\theta}_2 \sin\left(\theta_2-\theta_3\right)\int_0^{l_3}\rho\left(\sum_{i=0}^{N}\psi_i(x)q_i(t)\right)dx$$

$$+l_2\dot{\theta}_2\left(\dot{\theta}_2-\dot{\theta}_3\right)\cos\left(\theta_2-\theta_3\right)\int_0^{l_3}\rho\left(\sum_{i=0}^{N}\psi_i(x)q_i(t)\right)dx + l_2\dot{\theta}_2 \sin\left(\theta_2-\theta_3\right)\int_0^{l_3}\rho\left(\sum_{i=0}^{N}\psi_i(x)\dot{q}_i(t)\right)dx$$

Combining all terms will give us

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}_3}\right)=\frac{1}{3}\rho {l_3}^3\ddot{\theta}_3+\ddot{\theta}_3\int_0^{l_3}\rho\left(\sum_{i=0}^{N}\psi_i(x)q_i(t)\right)^2dx+2\dot{\theta}_3\int_0^{l_3}\rho\sum_{i=0}^{N}\sum_{i=0}^{N}\psi_i(x)\psi_j(x)q_i(t)\dot{q}_j(t)dx$$

$$+\frac{1}{2}\rho l_2{l_3}^2\ddot{\theta}_2\cos\left(\theta_2-\theta_3\right)-\frac{1}{2}\rho l_2{l_3}^2\dot{\theta}_2\left(\dot{\theta}_2-\dot{\theta}_3\right)\sin\left(\theta_2-\theta_3\right)$$

$$+l_2\ddot{\theta}_2 \sin\left(\theta_2-\theta_3\right)\int_0^{l_3}\rho\left(\sum_{i=0}^{N}\psi_i(x)q_i(t)\right)dx+l_2\dot{\theta}_2\left(\dot{\theta}_2-\dot{\theta}_3\right)\cos\left(\theta_2-\theta_3\right)\int_0^{l_3}\rho\left(\sum_{i=0}^{N}\psi_i(x)q_i(t)\right)dx$$

$$+l_2\dot{\theta}_2 \sin\left(\theta_2-\theta_3\right)\int_0^{l_3}\rho\left(\sum_{i=0}^{N}\psi_i(x)\dot{q}_i(t)\right)dx+\int_0^{l_3}\rho x\left(\sum_{i=0}^{N}\psi_i\left(x\right)\ddot{q}_i\left(t\right)\right)dx$$

θ_3 derivative for $\frac{\partial L}{\partial\eta_i}$:

$$\frac{\partial L}{\partial\theta_3}=\frac{1}{2}\rho l_2{l_3}^2\dot{\theta}_2\dot{\theta}_3\sin\left(\theta_2-\theta_3\right)+l_2\dot{\theta}_2 \sin\left(\theta_2-\theta_3\right)\int_0^{l_3}\rho\left(\sum_{i=0}^{N}\psi_i(x)\dot{q}_i(t)\right)dx$$

$$-l_2\dot{\theta}_2\dot{\theta}_3\cos\left(\theta_2-\theta_3\right)\int_0^{l_3}\rho\left(\sum_{i=0}^{N}\psi_i(x)q_i(t)\right)dx$$

For $\left(\frac{\partial\Phi}{\partial\eta_i}\right)^T\Lambda$:

$$\left(\frac{\partial\Phi}{\partial\theta_3}\right)^T\Lambda=\left(\frac{\partial\begin{bmatrix}-l_1+l_2\cos\theta_2+l_3\cos\theta_3-l_4\cos\theta_4\\ l_2\sin\theta_2+l_3\sin\theta_3-l_4\sin\theta_4\end{bmatrix}}{\partial\theta_3}\right)^T\begin{bmatrix}\lambda_1\\ \lambda_2\end{bmatrix}$$

$$=\begin{bmatrix}-l_3\sin\theta_3\\ l_3\cos\theta_3\end{bmatrix}^T\begin{bmatrix}\lambda_1\\ \lambda_2\end{bmatrix}=\begin{bmatrix}-l_3\sin\theta_3 & l_3\cos\theta_3\end{bmatrix}\begin{bmatrix}\lambda_1\\ \lambda_2\end{bmatrix}$$

$$=-l_3\sin\theta_3\lambda_1+l_3\cos\theta_3\lambda_2$$

The final state of $\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\eta}_i}\right)-\frac{\partial L}{\partial\eta_i}+\left(\frac{\partial\Phi}{\partial\eta_i}\right)^T\Lambda=F_i$ for θ_3:

$$\frac{1}{3}\rho {l_3}^3\ddot{\theta}_3+\ddot{\theta}_3\int_0^{l_3}\rho\left(\sum_{i=0}^{N}\psi_i(x)q_i(t)\right)^2dx+2\dot{\theta}_3\int_0^{l_3}\rho\sum_{i=0}^{N}\sum_{i=0}^{N}\psi_i(x)\psi_j(x)q_i(t)\dot{q}_j(t)dx$$
$$+\frac{1}{2}\rho l_2{l_3}^2\ddot{\theta}_2\cos(\theta_2-\theta_3)-\frac{1}{2}\rho l_2{l_3}^2\dot{\theta}_2(\dot{\theta}_2-\dot{\theta}_3)\sin(\theta_2-\theta_3)$$
$$+l_2\ddot{\theta}_2\sin(\theta_2-\theta_3)\int_0^{l_3}\rho\left(\sum_{i=0}^{N}\psi_i(x)q_i(t)\right)dx+l_2\dot{\theta}_2\sin(\theta_2-\theta_3)\int_0^{l_3}\rho\left(\sum_{i=0}^{N}\psi_i(x)\dot{q}_i(t)\right)dx$$
$$+l_2\dot{\theta}_2(\dot{\theta}_2-\dot{\theta}_3)\cos(\theta_2-\theta_3)\int_0^{l_3}\rho\left(\sum_{i=0}^{N}\psi_i(x)q_i(t)\right)dx+\int_0^{l_3}\rho x\left(\sum_{i=0}^{N}\psi_i(x)\ddot{q}_i(t)\right)dx$$
$$-\frac{1}{2}\rho l_2{l_3}^2\dot{\theta}_2\dot{\theta}_3\sin(\theta_2-\theta_3)-l_2\dot{\theta}_2\sin(\theta_2-\theta_3)\int_0^{l_3}\rho\left(\sum_{i=0}^{N}\psi_i(x)\dot{q}_i(t)\right)dx$$
$$+l_2\dot{\theta}_2\dot{\theta}_3\cos(\theta_2-\theta_3)\int_0^{l_3}\rho\left(\sum_{i=0}^{N}\psi_i(x)q_i(t)\right)dx-l_3\sin\theta_3\lambda_1+l_3\cos\theta_3\lambda_2=0$$

(4.51)

4.1.2.3Dynamic equation of motion with Lagrange's equation for θ_4

Doing mathematical calculations for the constrained Lagrange's equations for θ_4:

$\dot{\theta}_4$ derivative for $\frac{\partial L}{\partial\dot{\eta}_i}$ is $\frac{\partial L}{\partial\dot{\theta}_4}=I_O\dot{\theta}_4$

Time derivative $\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\eta}_i}\right)$ is $\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}_4}\right)=I_O\ddot{\theta}_4$

θ_4 derivative for $\frac{\partial L}{\partial\eta_i}$ is $\frac{\partial L}{\partial\theta_4}=0$

For $\left(\frac{\partial\Phi}{\partial\eta_i}\right)^T\Lambda$:

$$\left(\frac{\partial\Phi}{\partial\theta_4}\right)^T\Lambda=\left(\frac{\partial\begin{bmatrix}-l_1+l_2\cos\theta_2+l_3\cos\theta_3-l_4\cos\theta_4\\ l_2\sin\theta_2+l_3\sin\theta_3-l_4\sin\theta_4\end{bmatrix}}{\partial\theta_4}\right)^T\begin{bmatrix}\lambda_1\\ \lambda_2\end{bmatrix}$$
$$=\begin{bmatrix}l_4\sin\theta_4\\ -l_4\cos\theta_4\end{bmatrix}^T\begin{bmatrix}\lambda_1\\ \lambda_2\end{bmatrix}=\begin{bmatrix}l_4\sin\theta_4 & -l_4\cos\theta_4\end{bmatrix}\begin{bmatrix}\lambda_1\\ \lambda_2\end{bmatrix}$$
$$=l_4\sin\theta_4\lambda_1-l_4\cos\theta_4\lambda_2$$

The final state of $\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\eta}_i}\right)-\frac{\partial L}{\partial\eta_i}+\left(\frac{\partial\Phi}{\partial\eta_i}\right)^T\Lambda=F_i$ for θ_4:

$$I_O\ddot{\theta}_4+l_4\sin\theta_4\lambda_1-l_4\cos\theta_4\lambda_2=0$$

(4.52)

4.1.2.4Dynamic equation of motion with Lagrange's equation for w

Doing mathematical calculations for the constrained Lagrange's equations for w:

$\dot{w}$ derivative for $\frac{\partial L}{\partial \dot{\eta}_i}$:

$$\frac{\partial L}{\partial \dot{w}} = \int_0^{l_3} \rho \sum_{i=0}^{N} \psi_i(x)\psi_j(x)\dot{q}_i(t)dx + l_2\dot{\theta}_2 \cos(\theta_2 - \theta_3) \int_0^{l_3} \rho \left(\sum_{i=0}^{N} \psi_i(x) \right) dx + \dot{\theta}_3 \int_0^{l_3} \rho x \left(\sum_{i=0}^{N} \psi_i(x) \right) dx$$

Time derivative $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\eta}_i}\right)$:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{w}}\right) = \int_0^{l_3} \rho \sum_{i=0}^{N} \psi_i(x)\psi_j(x)\ddot{q}_i(t)dx + l_2\ddot{\theta}_2 \cos(\theta_2 - \theta_3) \int_0^{l_3} \rho \left(\sum_{i=0}^{N} \psi_i(x) \right) dx$$

$$-l_2\dot{\theta}_2 (\dot{\theta}_2 - \dot{\theta}_3) \sin(\theta_2 - \theta_3) \int_0^{l_3} \rho \left(\sum_{i=0}^{N} \psi_i(x) \right) dx + \ddot{\theta}_3 \int_0^{l_3} \rho x \left(\sum_{i=0}^{N} \psi_i(x) \right) dx$$

w derivative for $\frac{\partial L}{\partial \eta_i}$:

$$\frac{\partial L}{\partial w} = \dot{\theta}_3^2 \int_0^{l_3} \rho \sum_{i=0}^{N} \psi_i(x)\psi_j(x)q_i(t)dx + l_2\dot{\theta}_2\dot{\theta}_3 \sin(\theta_2 - \theta_3) \int_0^{l_3} \rho \left(\sum_{i=0}^{N} \psi_i(x)q_i(t) \right) dx$$

$$-\sum_{i=0}^{N} \int_0^{l_3} EI\psi_i''(x)\psi_j''(x)q_i(t)dx$$

For $\left(\frac{\partial \Phi}{\partial \eta_i}\right)^T \Lambda$:

$$\left(\frac{\partial \Phi}{\partial w}\right)^T \Lambda = \left(\frac{\partial \begin{bmatrix} -l_1 + l_2\cos\theta_2 + l_3\cos\theta_3 - l_4\cos\theta_4 \\ l_2\sin\theta_2 + l_3\sin\theta_3 - l_4\sin\theta_4 \end{bmatrix}}{\partial w} \right)^T \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} = 0$$

The final state of $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\eta}_i}\right) - \frac{\partial L}{\partial \eta_i} + \left(\frac{\partial \Phi}{\partial \eta_i}\right)^T \Lambda = F_i$ for w:

$$\int_0^{l_3}\rho\sum_{i=0}^{N}\psi_i(x)\psi_j(x)\ddot{q}_i(t)dx+l_2\ddot{\theta}_2\cos(\theta_2-\theta_3)\int_0^{l_3}\rho\left(\sum_{i=0}^{N}\psi_i(x)\right)dx$$
$$-l_2\dot{\theta}_2\left(\dot{\theta}_2-\dot{\theta}_3\right)\sin(\theta_2-\theta_3)\int_0^{l_3}\rho\left(\sum_{i=0}^{N}\psi_i(x)\right)dx+\ddot{\theta}_3\int_0^{l_3}\rho x\left(\sum_{i=0}^{N}\psi_i(x)\right)dx \quad \textbf{(4.53)}$$
$$-\dot{\theta}_3^{\,2}\int_0^{l_3}\rho\sum_{i=0}^{N}\psi_i(x)\psi_j(x)q_i(t)dx-l_2\dot{\theta}_2\dot{\theta}_3\sin(\theta_2-\theta_3)\int_0^{l_3}\rho\left(\sum_{i=0}^{N}\psi_i(x)q_i(t)\right)dx$$
$$+\sum_{i=0}^{N}\int_0^{l_3}EI\psi_i''(x)\psi_j''(x)q_i(t)dx=0$$

4.1.3 Dynamic analysis solutions for four-bar mechanism using Matlab Simulink

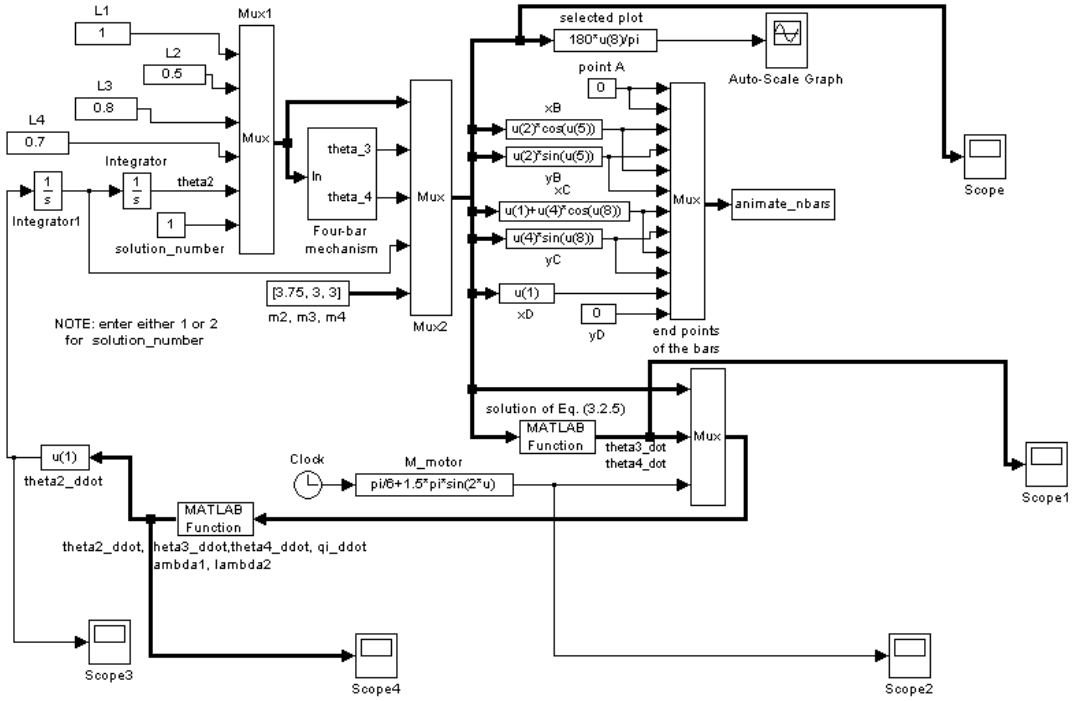

Figure 4.2 Simulink Block diagram of an elastic coupler link four-bar mechanism

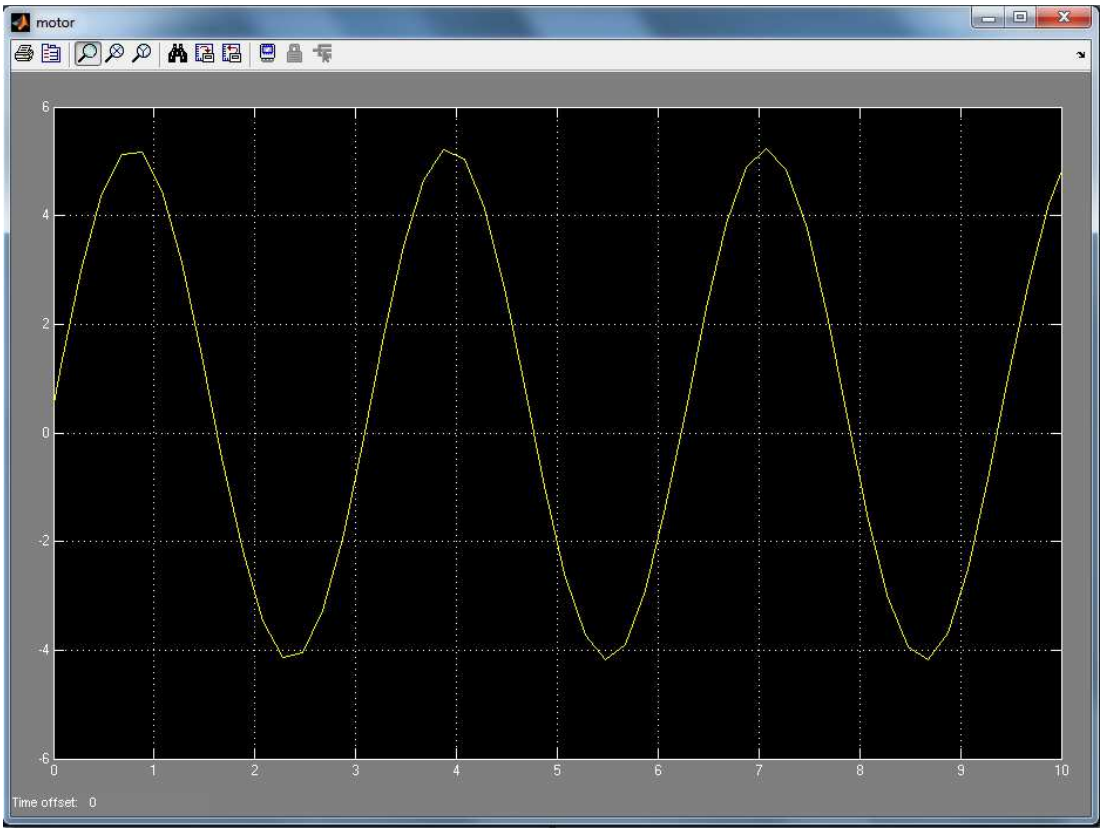

Figure 4.3 Driving Force of the System

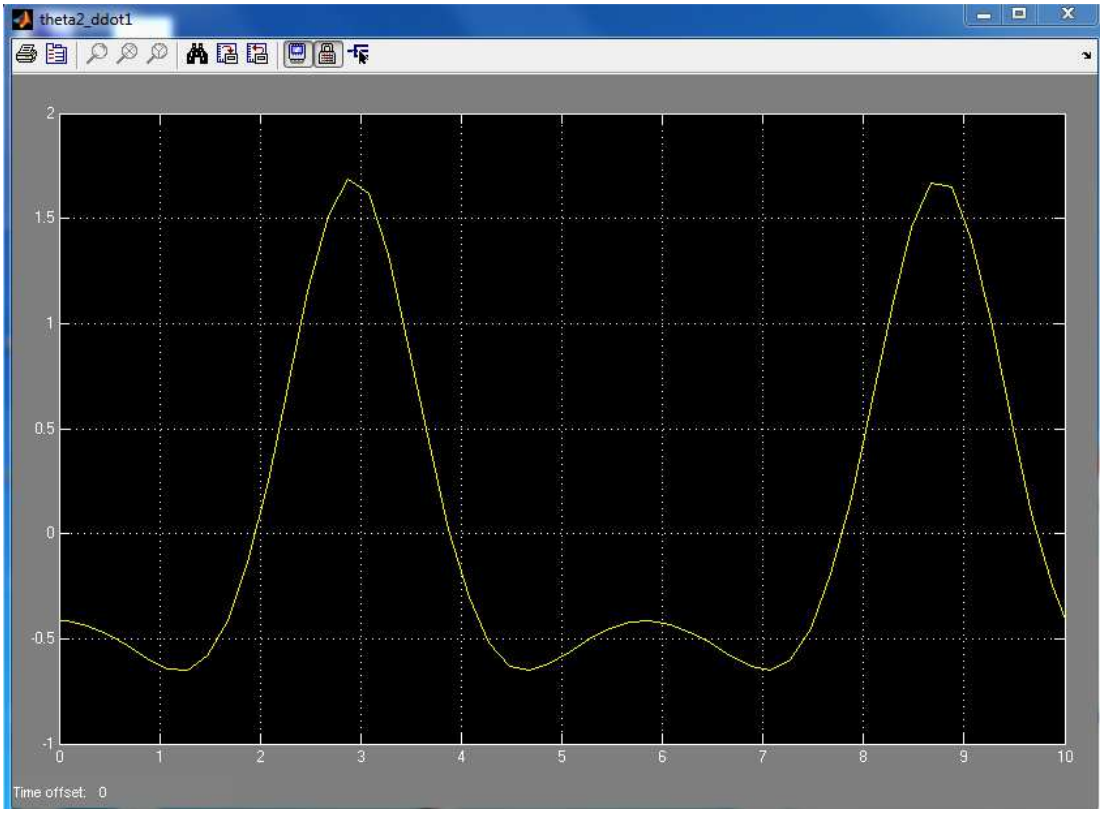

Figure 4.4 Crank link angular acceleration ($\ddot{\theta_2}$) time response

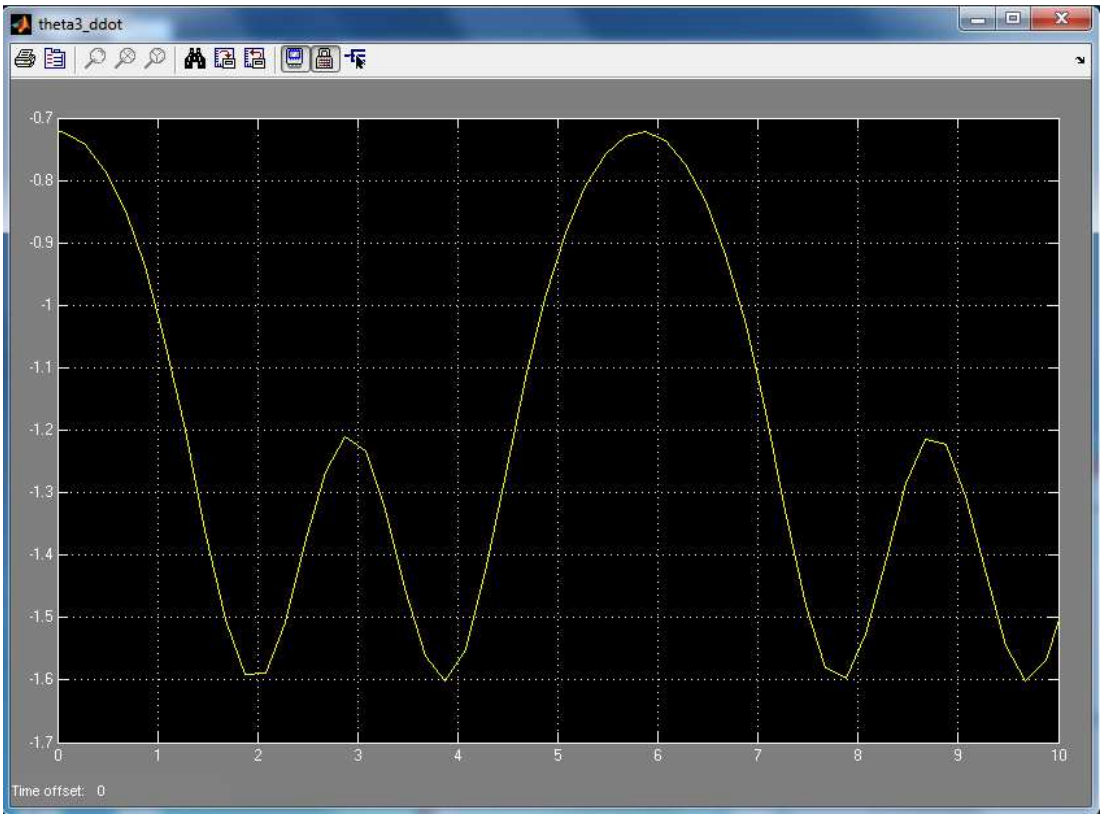

Figure 4.5 Coupler link angular acceleration time response

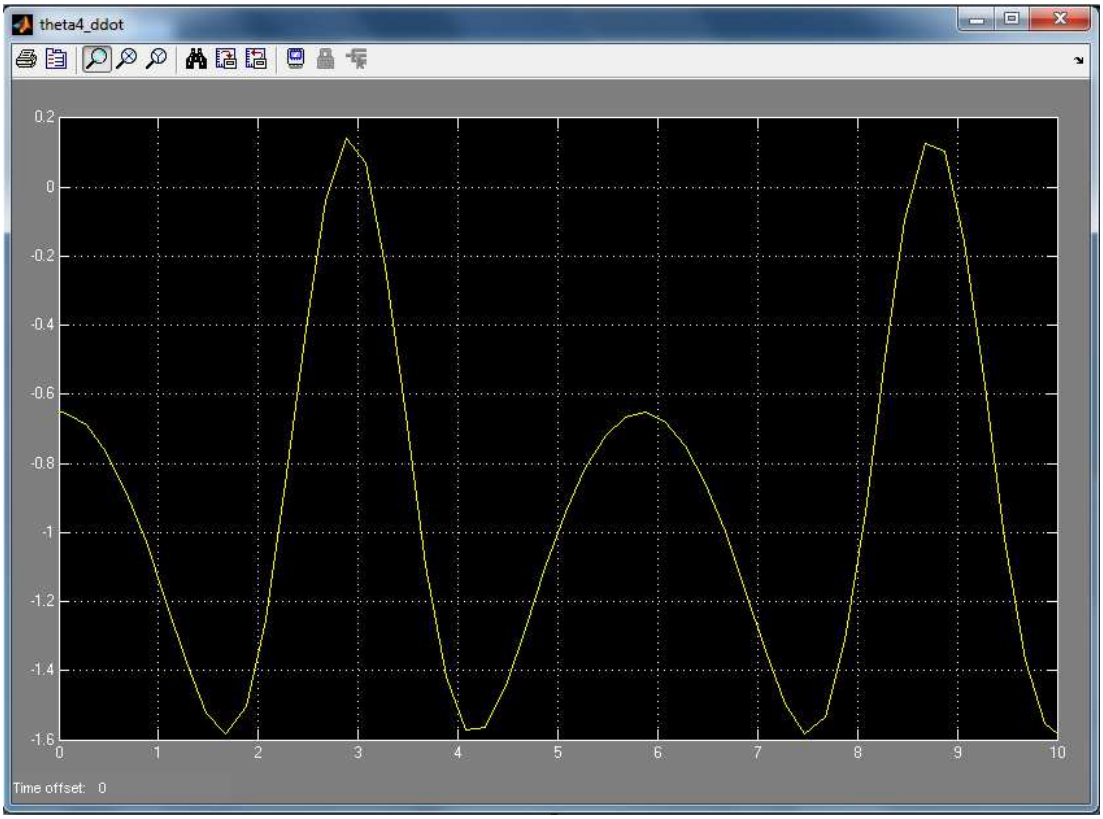

Figure 4.6 Output link angular acceleration time response

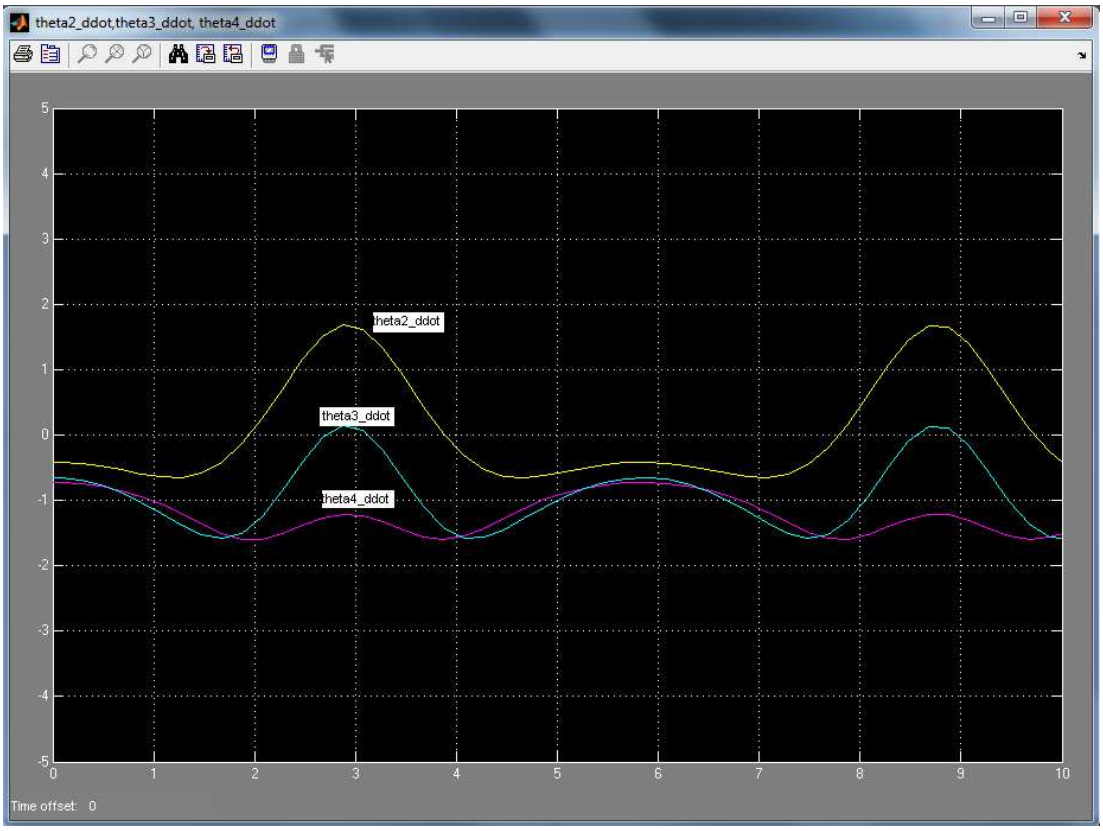

Figure 4.7 Comparison of the angular acceleration

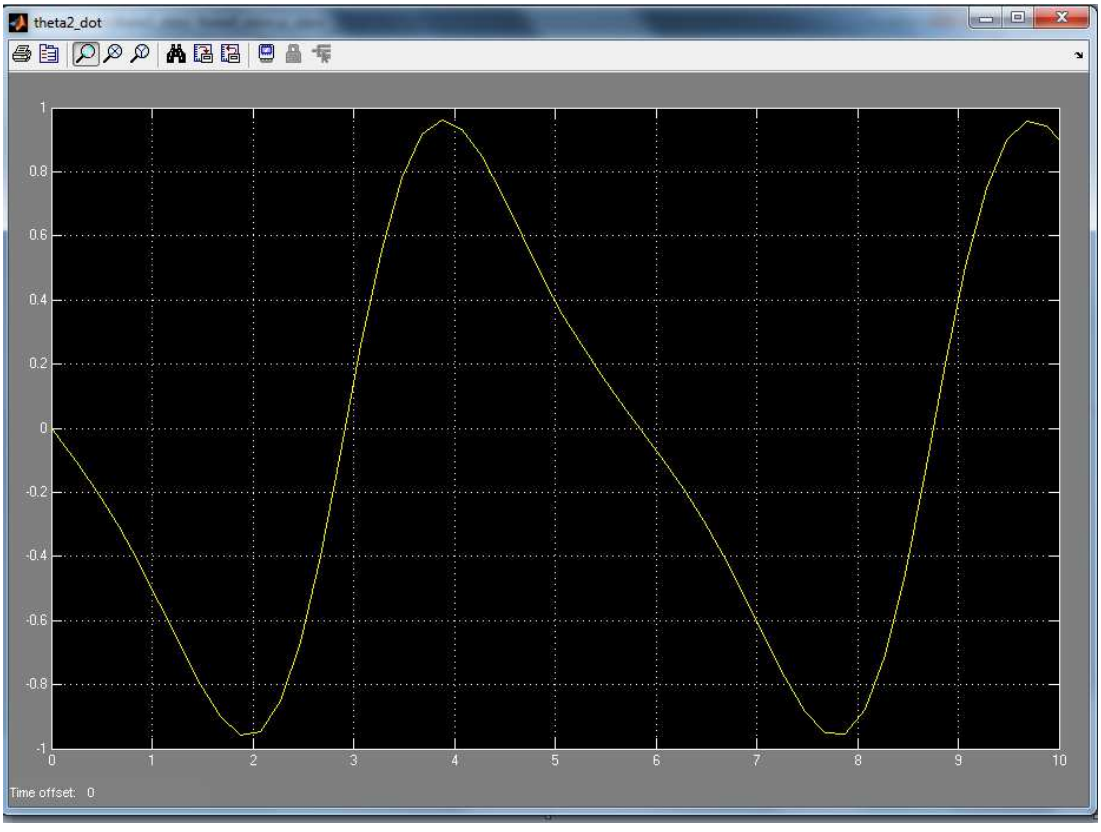

Figure 4.8 Crank link angular velocity time response

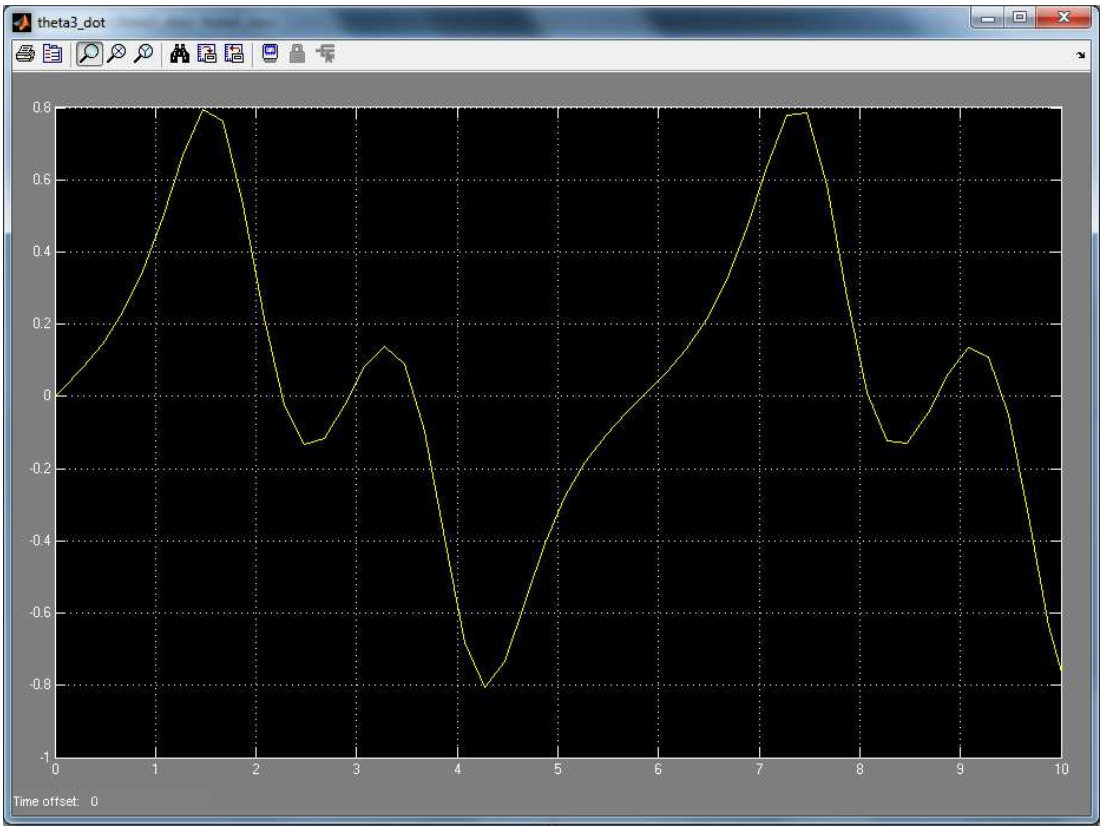

Figure 4.9 Coupler link angular velocity time response

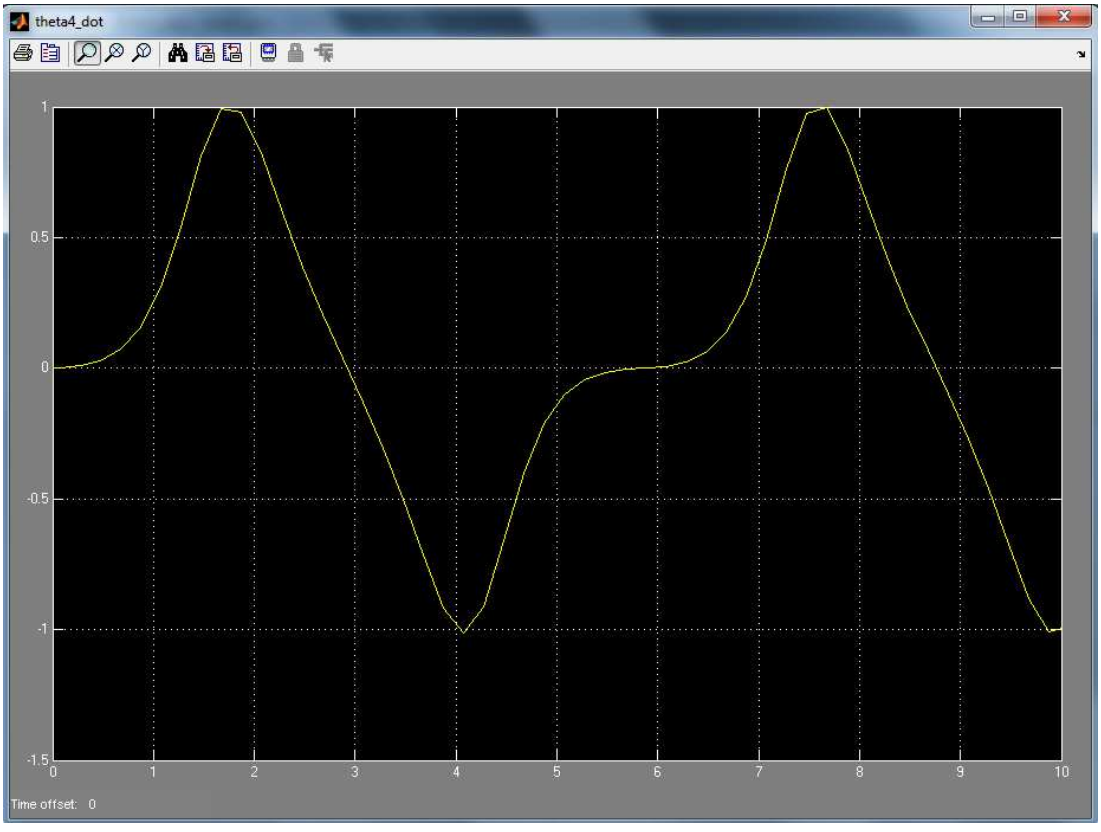

Figure 4.10 Output link angular velocity time response

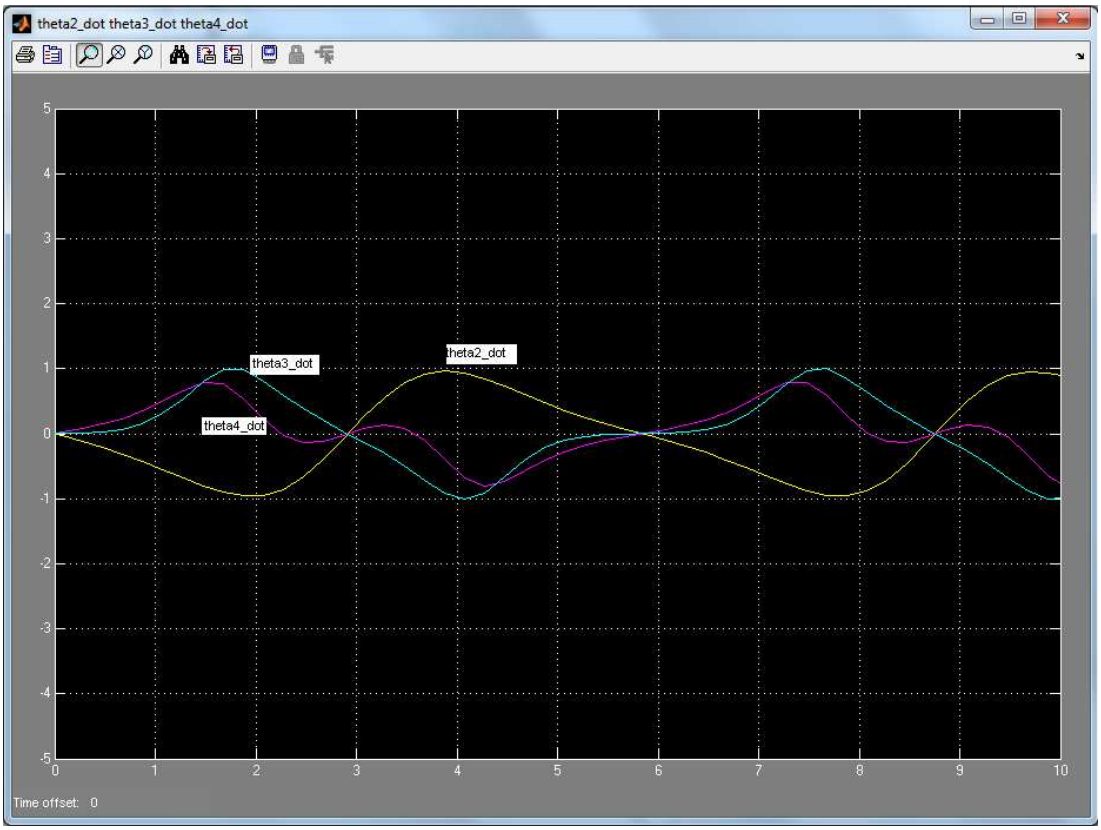

Figure 4.11 Comparison of the angular velocity

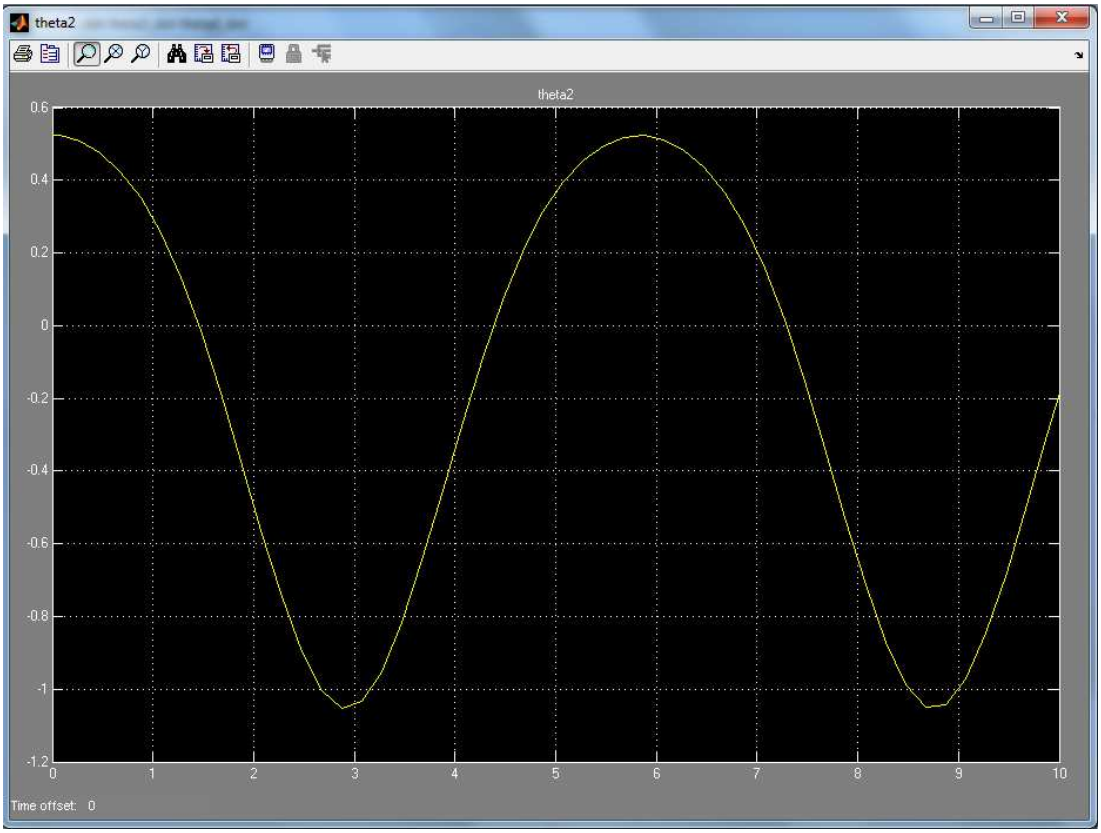

Figure 4.12 Crank link angle time response

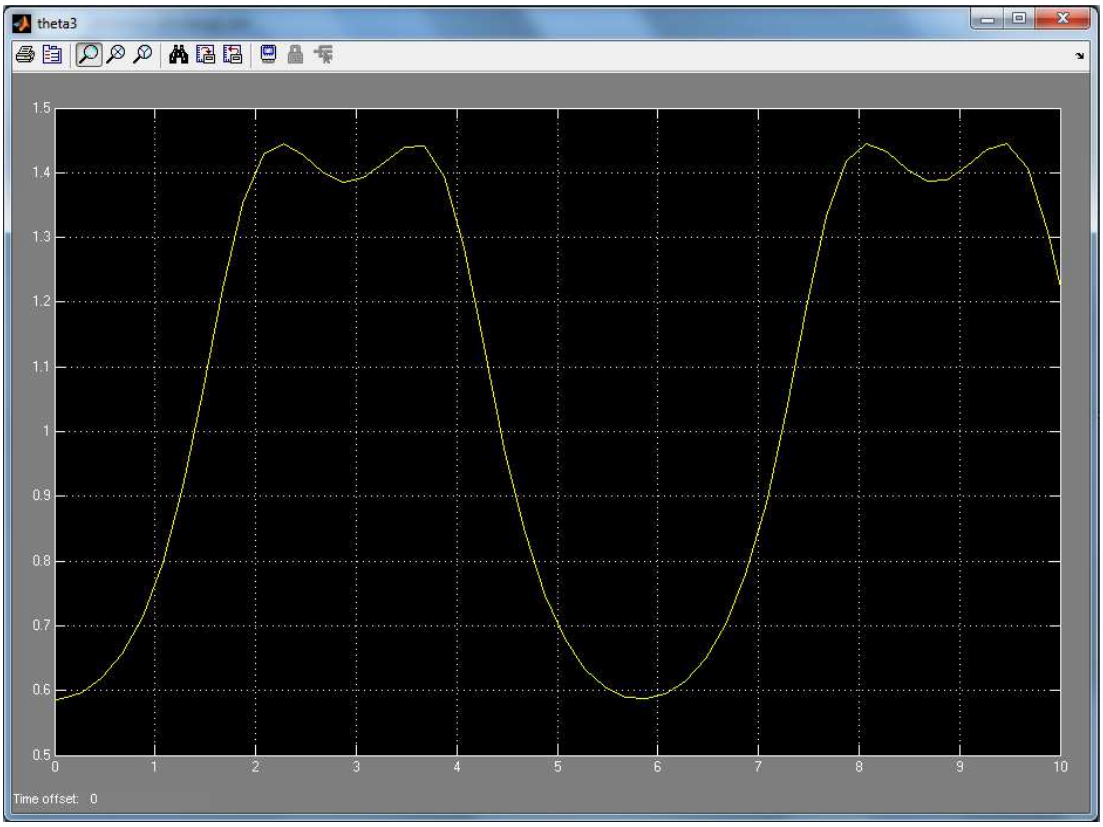

Figure 4.13 Coupler link angle time response

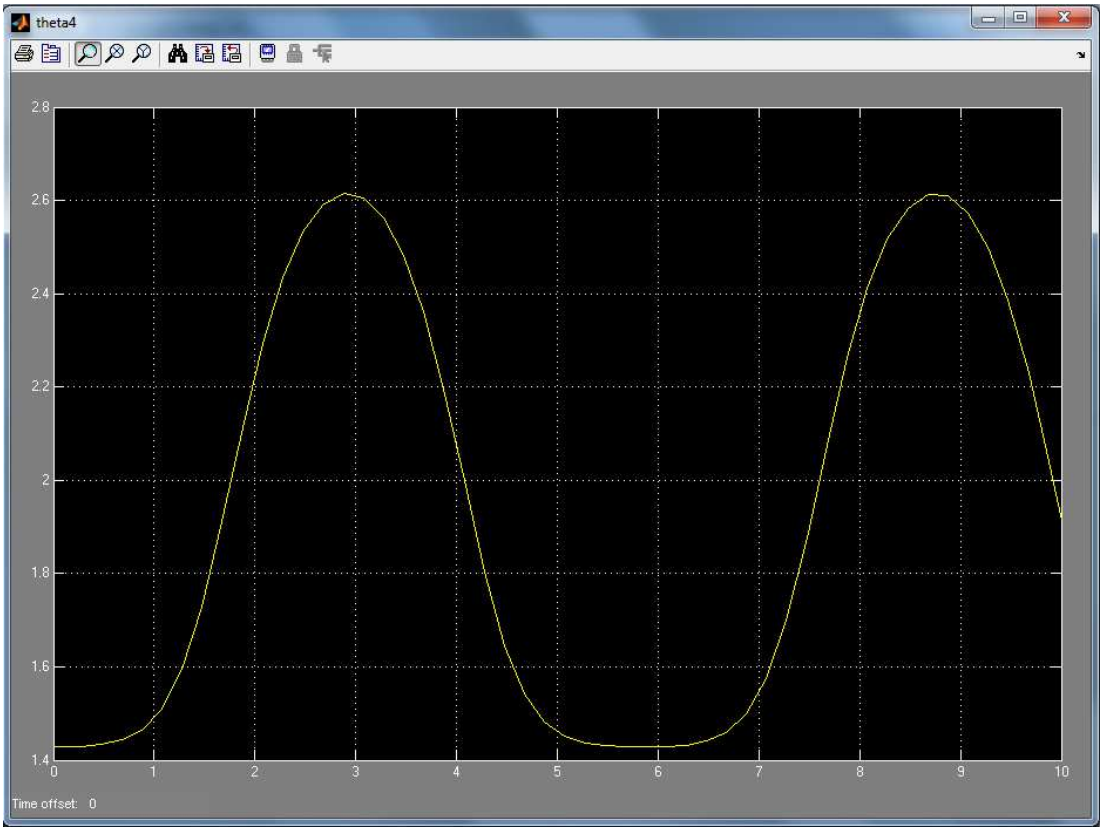

Figure 4.14 Output link angle time response

4.2 Dynamic Model of Flexible Output Link Four-Bar Mechanism

4.2.1 Dynamic equation of motion with Lagrange's equation

The kinetic energy of the four-bar mechanism shown in Figure 4.3 is given by:

$$T = \frac{1}{2} I_C \dot{\theta}_2^2 + \frac{1}{2} I_3 \dot{\theta}_3^2 + \frac{1}{2} \int_0^{l_4} \rho \dot{R}^2 dy \qquad \textbf{(4.54)}$$

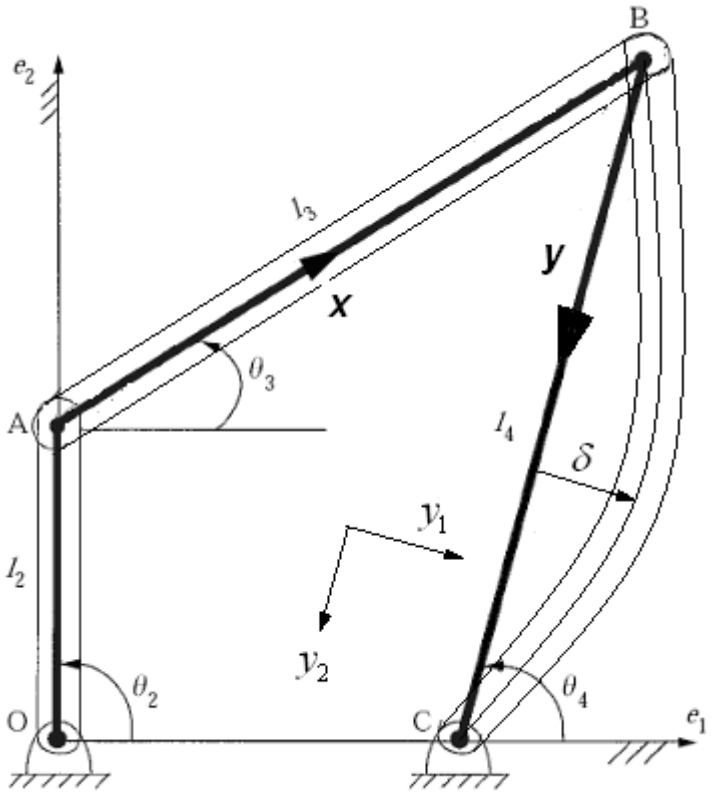

Figure 4.15 Sketch of the four-bar mechanism with elastic output link

$\vec{R}$ is the position vector of a point on the output link given as:

$$\vec{R} = \vec{r}_2 + \vec{r}_3' + \vec{r}_4 + \vec{\delta}$$

where

$$\vec{r}_2 = l_2 \cos\theta_2 \vec{e}_1 + l_2 \sin\theta_2 \vec{e}_2$$
$$\vec{r}_3' = l_3 \cos\theta_3 \vec{e}_1 + l_3 \sin\theta_3 \vec{e}_2$$
$$\vec{r}_4 = -y \cos\theta_4 \vec{e}_1 - y \sin\theta_4 \vec{e}_2$$
$$\vec{\delta} = \delta \vec{y}_1$$

where δ represents the elastic deformation of the output link at an arbitrary position y. The unit vectors $\vec{y}_1$ and $\vec{y}_2$ can be written as

$$\vec{y}_1 = \sin\theta_4\vec{e}_1 - \cos\theta_4\vec{e}_2$$

$$\vec{y}_2 = \cos\theta_4\vec{e}_1 + \sin\theta_4\vec{e}_2$$

Then the final position vector will be

$$\vec{R} = \left(l_2\cos\theta_2 + l_3\cos\theta_3 - y\cos\theta_4\right)\vec{e}_1 + \left(l_2\sin\theta_2 + l_3\sin\theta_3 - y\sin\theta_4\right)\vec{e}_2 + \delta\left(\sin\theta_4\vec{e}_1 - \cos\theta_4\vec{e}_2\right)$$

$$\begin{aligned}\vec{R} = &\left(l_2\cos\theta_2 + l_3\cos\theta_3 - y\cos\theta_4 + \delta\sin\theta_4\right)\vec{e}_1 \\ &+ \left(l_2\sin\theta_2 + l_3\sin\theta_3 - y\sin\theta_4 - \delta\cos\theta_4\right)\vec{e}_2\end{aligned} \qquad \textbf{(4.55)}$$

The assumed modes method along with a constrained Lagrangian approach is used to obtain the discretized equations of motion. Let the deflection of the output link be written as

$$\delta(y,t) = \sum_{i=0}^{N}\phi_i(y)q_i(t) \qquad \textbf{(4.56)}$$

where $q_i(t)$ are the modal co-ordinates and $\phi_i(y)$ are the mode shapes of a pinned-pinned beam given as

$$\phi_i(y) = \sin\left(\frac{i\pi}{l_4}y\right)$$

After inserting information of **(4.55)** into the **(4.56)** equation and taking its time derivative we get

$$\begin{aligned}\dot{\vec{R}} = &\left(-l_2\dot{\theta}_2\sin\theta_2 - l_3\dot{\theta}_3\sin\theta_3 + y\dot{\theta}_4\sin\theta_4 + \sin\theta_4\sum_{i=0}^{N}\phi_i(y)\dot{q}_i(t) + \dot{\theta}_4\cos\theta_4\sum_{i=0}^{N}\phi_i(y)q_i(t)\right)\vec{e}_1 \\ &+\left(l_2\dot{\theta}_2\cos\theta_2 + l_3\dot{\theta}_3\cos\theta_3 - y\dot{\theta}_4\cos\theta_4 - \cos\theta_4\sum_{i=0}^{N}\phi_i(y)\dot{q}_i(t) + \dot{\theta}_4\sin\theta_4\sum_{i=0}^{N}\phi_i(y)q_i(t)\right)\vec{e}_2\end{aligned} \qquad \textbf{(4.57)}$$

For $\dot{\vec{R}}^2$ it should be more convenient to work with **(4.57)** divided into parts:

$$\left(-l_2\dot{\theta}_2\sin\theta_2 - l_3\dot{\theta}_3\sin\theta_3 + y\dot{\theta}_4\sin\theta_4 + \sin\theta_4\sum_{i=0}^{N}\phi_i(y)\dot{q}_i(t) + \dot{\theta}_4\cos\theta_4\sum_{i=0}^{N}\phi_i(y)q_i(t)\right)^2 =$$

$$\left(l_2\dot{\theta}_2\sin\theta_2\right)^2 + \left(l_3\dot{\theta}_3\sin\theta_3\right)^2 + \left(y\dot{\theta}_4\sin\theta_4\right)^2 + \left(\sin\theta_4\sum_{i=0}^{N}\phi_i(y)\dot{q}_i(t)\right)^2 + \left(\dot{\theta}_4\cos\theta_4\sum_{i=0}^{N}\phi_i(y)q_i(t)\right)^2$$

$$+2l_2l_3\dot{\theta}_2\dot{\theta}_3\sin\theta_2\sin\theta_3 - 2l_2y\dot{\theta}_2\dot{\theta}_4\sin\theta_2\sin\theta_4 - 2l_2\dot{\theta}_2\sin\theta_2\sin\theta_4\sum_{i=0}^{N}\phi_i(y)\dot{q}_i(t)$$

$$-2l_2\dot{\theta}_2\dot{\theta}_4\sin\theta_2\cos\theta_4\sum_{i=0}^{N}\phi_i(y)q_i(t) - 2l_3y\dot{\theta}_3\dot{\theta}_4\sin\theta_3\sin\theta_4 - 2l_3\dot{\theta}_3\sin\theta_3\sin\theta_4\sum_{i=0}^{N}\phi_i(y)\dot{q}_i(t)$$

$$-2l_3\dot{\theta}_3\dot{\theta}_4\sin\theta_3\cos\theta_4\sum_{i=0}^{N}\phi_i(y)q_i(t) + 2y\dot{\theta}_4\sin\theta_4\sin\theta_4\sum_{i=0}^{N}\phi_i(y)\dot{q}_i(t)$$

$$+2y\dot{\theta}_4\dot{\theta}_4\sin\theta_4\cos\theta_4\sum_{i=0}^{N}\phi_i(y)q_i(t) + 2\dot{\theta}_4\sin\theta_4\cos\theta_4\sum_{i=0}^{N}\phi_i(y)\dot{q}_i(t)\sum_{i=0}^{N}\phi_i(y)q_i(t)$$

$$\left(l_2\dot{\theta}_2\cos\theta_2 + l_3\dot{\theta}_3\cos\theta_3 - y\dot{\theta}_4\cos\theta_4 - \cos\theta_4\sum_{i=0}^{N}\phi_i(y)\dot{q}_i(t) + \dot{\theta}_4\sin\theta_4\sum_{i=0}^{N}\phi_i(y)q_i(t)\right)^2 =$$

$$\left(l_2\dot{\theta}_2\cos\theta_2\right)^2 + \left(l_3\dot{\theta}_3\cos\theta_3\right)^2 + \left(y\dot{\theta}_4\cos\theta_4\right)^2 + \left(\cos\theta_4\sum_{i=0}^{N}\phi_i(y)\dot{q}_i(t)\right)^2 + \left(\dot{\theta}_4\sin\theta_4\sum_{i=0}^{N}\phi_i(y)q_i(t)\right)^2$$

$$+2l_2l_3\dot{\theta}_2\dot{\theta}_3\cos\theta_2\cos\theta_3 - 2l_2y\dot{\theta}_2\dot{\theta}_4\cos\theta_2\cos\theta_4 - 2l_2\dot{\theta}_2\cos\theta_2\cos\theta_4\sum_{i=0}^{N}\phi_i(y)\dot{q}_i(t)$$

$$+2l_2\dot{\theta}_2\dot{\theta}_4\cos\theta_2\sin\theta_4\sum_{i=0}^{N}\phi_i(y)q_i(t) - 2l_3y\dot{\theta}_3\dot{\theta}_4\cos\theta_3\cos\theta_4 - 2l_3\dot{\theta}_3\cos\theta_3\cos\theta_4\sum_{i=0}^{N}\phi_i(y)\dot{q}_i(t)$$

$$+2l_3\dot{\theta}_3\dot{\theta}_4\cos\theta_3\sin\theta_4\sum_{i=0}^{N}\phi_i(y)q_i(t) + 2y\dot{\theta}_4\cos\theta_4\cos\theta_4\sum_{i=0}^{N}\phi_i(y)\dot{q}_i(t)$$

$$-2y\dot{\theta}_4\dot{\theta}_4\cos\theta_4\sin\theta_4\sum_{i=0}^{N}\phi_i(y)q_i(t) - 2\dot{\theta}_4\sin\theta_4\cos\theta_4\sum_{i=0}^{N}\phi_i(y)\dot{q}_i(t)\sum_{i=0}^{N}\phi_i(y)q_i(t)$$

Combining two above equations will give us:

$$\dot{\bar{R}}^2 = \left(l_2\dot{\theta}_2\right)^2 + \left(l_3\dot{\theta}_3\right)^2 + \left(y\dot{\theta}_4\right)^2 + \left(\sum_{i=0}^{N}\phi_i(y)\dot{q}_i(t)\right)^2 + \left(\dot{\theta}_4\sum_{i=0}^{N}\phi_i(y)q_i(t)\right)^2$$

$$+2l_2l_3\dot{\theta}_2\dot{\theta}_3\cos\left(\theta_2-\theta_3\right) - 2l_2y\dot{\theta}_2\dot{\theta}_4\cos\left(\theta_2-\theta_4\right) - 2l_2\dot{\theta}_2\cos\left(\theta_2-\theta_4\right)\sum_{i=0}^{N}\phi_i(y)\dot{q}_i(t)$$

$$-2l_2\dot{\theta}_2\dot{\theta}_4\sin\left(\theta_2-\theta_4\right)\sum_{i=0}^{N}\phi_i(y)q_i(t) - 2l_3y\dot{\theta}_3\dot{\theta}_4\cos\left(\theta_3-\theta_4\right)$$

$$-2l_3\dot{\theta}_3\cos\left(\theta_3-\theta_4\right)\sum_{i=0}^{N}\phi_i(y)\dot{q}_i(t) - 2l_3\dot{\theta}_3\dot{\theta}_4\sin\left(\theta_3-\theta_4\right)\sum_{i=0}^{N}\phi_i(y)q_i(t) + 2y\dot{\theta}_4\sum_{i=0}^{N}\phi_i(y)\dot{q}_i(t)$$

Solving for $\frac{1}{2}\int_0^{l_4}\rho\dot{\vec{R}}^2 dy$ **(4.54)**

$$\frac{1}{2}\int_0^{l_4}\rho\dot{\vec{R}}^2 dy = \frac{1}{2}\int_0^{l_4}\rho\left(l_2\dot{\theta}_2\right)^2 dy + \frac{1}{2}\int_0^{l_4}\rho\left(l_3\dot{\theta}_3\right)^2 dy + \frac{1}{2}\int_0^{l_4}\rho\left(y\dot{\theta}_4\right)^2 dy + \frac{1}{2}\int_0^{l_4}\rho\left(\sum_{i=0}^{N}\phi_i(y)\dot{q}_i(t)\right)^2 dy$$
$$+\frac{1}{2}\int_0^{l_4}\rho\left(\dot{\theta}_4\sum_{i=0}^{N}\phi_i(y)q_i(t)\right)^2 dy + \int_0^{l_4}\rho l_2 l_3\dot{\theta}_2\dot{\theta}_3\cos\left(\theta_2-\theta_3\right)dy$$
$$-\int_0^{l_4}\rho l_2 y\dot{\theta}_2\dot{\theta}_4\cos\left(\theta_2-\theta_4\right)dy - \int_0^{l_4}\rho l_2\dot{\theta}_2\cos\left(\theta_2-\theta_4\right)\sum_{i=0}^{N}\phi_i(y)\dot{q}_i(t)dy$$
$$-\int_0^{l_4}\rho l_2\dot{\theta}_2\dot{\theta}_4\sin\left(\theta_2-\theta_4\right)\sum_{i=0}^{N}\phi_i(y)q_i(t)dy - \int_0^{l_4}\rho l_3 y\dot{\theta}_3\dot{\theta}_4\cos\left(\theta_3-\theta_4\right)dy$$
$$-\int_0^{l_4}\rho l_3\dot{\theta}_3\cos\left(\theta_3-\theta_4\right)\sum_{i=0}^{N}\phi_i(y)\dot{q}_i(t)dy - \int_0^{l_4}\rho l_3\dot{\theta}_3\dot{\theta}_4\sin\left(\theta_3-\theta_4\right)\sum_{i=0}^{N}\phi_i(y)q_i(t)dy + \int_0^{l_4}\rho y\dot{\theta}_4\sum_{i=0}^{N}\phi_i(y)\dot{q}_i(t)dy$$

$$\frac{1}{2}\int_0^{l_4}\rho\dot{\vec{R}}^2 dy = \frac{1}{2}\rho\left(l_2\dot{\theta}_2\right)^2 l_4 + \frac{1}{2}\rho\left(l_3\dot{\theta}_3\right)^2 l_4 + \frac{1}{6}l_4{}^3\dot{\theta}_4{}^2 + \frac{1}{2}\int_0^{l_4}\rho\left(\sum_{i=0}^{N}\phi_i(y)\dot{q}_i(t)\right)^2 dy + \frac{1}{2}\dot{\theta}_4{}^2\int_0^{l_4}\rho\left(\sum_{i=0}^{N}\phi_i(y)q_i(t)\right)^2 dy$$
$$+\rho l_2 l_3 l_4\dot{\theta}_2\dot{\theta}_3\cos\left(\theta_2-\theta_3\right) - \frac{1}{2}\rho l_2 l_4{}^2\dot{\theta}_2\dot{\theta}_4\cos\left(\theta_2-\theta_4\right) - l_2\dot{\theta}_2\cos\left(\theta_2-\theta_4\right)\int_0^{l_4}\rho\sum_{i=0}^{N}\phi_i(y)\dot{q}_i(t)dy$$
$$-l_2\dot{\theta}_2\dot{\theta}_4\sin\left(\theta_2-\theta_4\right)\int_0^{l_4}\rho\sum_{i=0}^{N}\phi_i(y)q_i(t)dy - \frac{1}{2}\rho l_3 l_4{}^2\dot{\theta}_3\dot{\theta}_4\cos\left(\theta_3-\theta_4\right)$$
$$-l_3\dot{\theta}_3\cos\left(\theta_3-\theta_4\right)\int_0^{l_4}\rho\sum_{i=0}^{N}\phi_i(y)\dot{q}_i(t)dy - l_3\dot{\theta}_3\dot{\theta}_4\sin\left(\theta_3-\theta_4\right)\int_0^{l_4}\rho\sum_{i=0}^{N}\phi_i(y)q_i(t)dy + \dot{\theta}_4\int_0^{l_4}\rho y\sum_{i=0}^{N}\phi_i(y)\dot{q}_i(t)dy$$

So the final kinetic energy state is

$$T = \frac{1}{2}I_C\dot{\theta}_2^2 + \frac{1}{2}\rho\left(l_2\dot{\theta}_2\right)^2 l_3 + \frac{1}{6}\rho l_3{}^3\dot{\theta}_3{}^2 + \frac{1}{2}\rho l_2 l_3{}^2\dot{\theta}_2\dot{\theta}_3\cos\left(\theta_2-\theta_3\right) + \frac{1}{2}\rho\left(l_2\dot{\theta}_2\right)^2 l_4$$
$$+\frac{1}{2}\rho\left(l_3\dot{\theta}_3\right)^2 l_4 + \frac{1}{6}l_4{}^3\dot{\theta}_4{}^2 + \frac{1}{2}\int_0^{l_4}\rho\left(\sum_{i=0}^{N}\phi_i(y)\dot{q}_i(t)\right)^2 dy + \frac{1}{2}\dot{\theta}_4{}^2\int_0^{l_4}\rho\left(\sum_{i=0}^{N}\phi_i(y)q_i(t)\right)^2 dy$$
$$+\rho l_2 l_3 l_4\dot{\theta}_2\dot{\theta}_3\cos\left(\theta_2-\theta_3\right) - \frac{1}{2}\rho l_2 l_4{}^2\dot{\theta}_2\dot{\theta}_4\cos\left(\theta_2-\theta_4\right)$$
$$-l_2\dot{\theta}_2\cos\left(\theta_2-\theta_4\right)\int_0^{l_4}\rho\sum_{i=0}^{N}\phi_i(y)\dot{q}_i(t)dy - l_2\dot{\theta}_2\dot{\theta}_4\sin\left(\theta_2-\theta_4\right)\int_0^{l_4}\rho\sum_{i=0}^{N}\phi_i(y)q_i(t)dy$$
$$-\frac{1}{2}\rho l_3 l_4{}^2\dot{\theta}_3\dot{\theta}_4\cos\left(\theta_3-\theta_4\right) - l_3\dot{\theta}_3\cos\left(\theta_3-\theta_4\right)\int_0^{l_4}\rho\sum_{i=0}^{N}\phi_i(y)\dot{q}_i(t)dy$$
$$-l_3\dot{\theta}_3\dot{\theta}_4\sin\left(\theta_3-\theta_4\right)\int_0^{l_4}\rho\sum_{i=0}^{N}\phi_i(y)q_i(t)dy + \dot{\theta}_4\int_0^{l_4}\rho y\sum_{i=0}^{N}\phi_i(y)\dot{q}_i(t)dy$$

The strain energy is given by

$$U=\frac{1}{2}\int_0^{l_4}\frac{{M_4}^2}{EI_4}dx=\frac{1}{2}\int_0^{l_4}\frac{\left(EI_4\dfrac{\partial^2\delta}{\partial y^2}\right)^2}{EI_4}dy=\frac{1}{2}\int_0^{l_4}EI_4\left(\frac{\partial^2\delta}{\partial y^2}\right)^2dy=\frac{1}{2}\int_0^{l_4}EI_4\left(\frac{\partial^2\left(\sum_{i=0}^{N}\phi_i(y)q_i(t)\right)}{\partial y^2}\right)^2dy$$

The constrained Lagrange's equations are

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\eta}_i}\right)-\frac{\partial L}{\partial\eta_i}+\left(\frac{\partial\vec{\Phi}}{\partial\eta_i}\right)^T\Lambda=F_i$$

Where $L=T-U$, or

$$\begin{aligned}
L=&\frac{1}{2}I_C\dot{\theta}_2^2+\frac{1}{2}\rho\left(l_2\dot{\theta}_2\right)^2l_3+\frac{1}{6}\rho{l_3}^3\dot{\theta}_3^{\,2}+\frac{1}{2}\rho l_2{l_3}^2\dot{\theta}_2\dot{\theta}_3\cos\left(\theta_2-\theta_3\right)+\frac{1}{2}\rho\left(l_2\dot{\theta}_2\right)^2l_4\\
&+\frac{1}{2}\rho\left(l_3\dot{\theta}_3\right)^2l_4+\frac{1}{6}{l_4}^3\dot{\theta}_4^{\,2}+\frac{1}{2}\int_0^{l_4}\rho\left(\sum_{i=0}^{N}\phi_i(y)\dot{q}_i(t)\right)^2dy+\frac{1}{2}\dot{\theta}_4^{\,2}\int_0^{l_4}\rho\left(\sum_{i=0}^{N}\phi_i(y)q_i(t)\right)^2dy\\
&+\rho l_2l_3l_4\dot{\theta}_2\dot{\theta}_3\cos\left(\theta_2-\theta_3\right)-\frac{1}{2}\rho l_2{l_4}^2\dot{\theta}_2\dot{\theta}_4\cos\left(\theta_2-\theta_4\right)\\
&-l_2\dot{\theta}_2\cos\left(\theta_2-\theta_4\right)\int_0^{l_4}\rho\sum_{i=0}^{N}\phi_i(y)\dot{q}_i(t)dy-l_2\dot{\theta}_2\dot{\theta}_4\sin\left(\theta_2-\theta_4\right)\int_0^{l_4}\rho\sum_{i=0}^{N}\phi_i(y)q_i(t)dy\\
&-\frac{1}{2}\rho l_3{l_4}^2\dot{\theta}_3\dot{\theta}_4\cos\left(\theta_3-\theta_4\right)-l_3\dot{\theta}_3\cos\left(\theta_3-\theta_4\right)\int_0^{l_4}\rho\sum_{i=0}^{N}\phi_i(y)\dot{q}_i(t)dy\\
&-l_3\dot{\theta}_3\dot{\theta}_4\sin\left(\theta_3-\theta_4\right)\int_0^{l_4}\rho\sum_{i=0}^{N}\phi_i(y)q_i(t)dy+\dot{\theta}_4\int_0^{l_4}\rho y\sum_{i=0}^{N}\phi_i(y)\dot{q}_i(t)dy-\frac{1}{2}\int_0^{l_4}EI_4\left(\frac{\partial^2\delta}{\partial y^2}\right)^2dy
\end{aligned}$$

(4.58)

and η_i are the generalized coordinates which includes rigid body coordinates as well as elastic modal coordinates. $\vec{\Phi}$ is the vector containing the constraint equations and $\vec{\Lambda}$ is the vector of Lagrange multipliers λ_1 and λ_2; and F_i are the generalized forces. The force vector $\left(\partial\vec{\Phi}/\partial\eta_i\right)^T\Lambda$ is substituted in to replace the removed joint forces.

$$\left(\frac{\partial\Phi}{\partial\eta_i}\right)^T\Lambda=\left(\frac{\partial\begin{bmatrix}-l_1+l_2\cos\theta_2+l_3\cos\theta_3-l_4\cos\theta_4\\ l_2\sin\theta_2+l_3\sin\theta_3-l_4\sin\theta_4\end{bmatrix}}{\partial\eta_i}\right)^T\begin{bmatrix}\lambda_1\\ \lambda_2\end{bmatrix} \qquad \textbf{(4.59)}$$

4.2.1.1 Dynamic equation of motion with Lagrange's equation for θ_2

Doing mathematical calculations for the constrained Lagrange's equations for θ_2

$\dot{\theta}_2$ derivative for $\frac{\partial L}{\partial \dot{\eta}_i}$:

$$\frac{\partial L}{\partial \dot{\theta}_2} = I_C\dot{\theta}_2 + \rho l_3 l_2^{\,2}\dot{\theta}_2 + \frac{1}{2}\rho l_2 l_3^{\,2}\dot{\theta}_3\cos(\theta_2-\theta_3) + \rho l_4 l_2^{\,2}\dot{\theta}_2$$
$$+\rho l_2 l_3 l_4\dot{\theta}_3\cos(\theta_2-\theta_3) - \frac{1}{2}\rho l_2 l_4^{\,2}\dot{\theta}_4\cos(\theta_2-\theta_4)$$
$$-l_2\cos(\theta_2-\theta_4)\int_0^{l_4}\rho\sum_{i=0}^{N}\phi_i(y)\dot{q}_i(t)dy - l_2\dot{\theta}_2\dot{\theta}_4\sin(\theta_2-\theta_4)\int_0^{l_4}\rho\sum_{i=0}^{N}\phi_i(y)q_i(t)dy$$

(4.60)

Time derivative $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\eta}_i}\right)$:

3rd term of **(4.60)**

$$\frac{d}{dt}\left(\frac{1}{2}\rho l_2 l_3^{\,2}\dot{\theta}_3\cos(\theta_2-\theta_3)\right) = \frac{1}{2}\rho l_2 l_3^{\,2}\ddot{\theta}_3\cos(\theta_2-\theta_3) - \frac{1}{2}\rho l_2 l_3^{\,2}\dot{\theta}_3\left(\dot{\theta}_2-\dot{\theta}_3\right)\sin(\theta_2-\theta_3)$$

5th term of **(4.60)**

$$\frac{d}{dt}\left(\rho l_2 l_3 l_4\dot{\theta}_3\cos(\theta_2-\theta_3)\right) = \rho l_2 l_3 l_4\ddot{\theta}_3\cos(\theta_2-\theta_3) - \rho l_2 l_3 l_4\dot{\theta}_3\left(\dot{\theta}_2-\dot{\theta}_3\right)\sin(\theta_2-\theta_3)$$

6th term of **(4.60)**

$$\frac{d}{dt}\left(-\frac{1}{2}\rho l_2 l_4^{\,2}\dot{\theta}_4\cos(\theta_2-\theta_4)\right) = -\frac{1}{2}\rho l_2 l_4^{\,2}\ddot{\theta}_4\cos(\theta_2-\theta_4) + \frac{1}{2}\rho l_2 l_4^{\,2}\dot{\theta}_4\left(\dot{\theta}_2-\dot{\theta}_4\right)\sin(\theta_2-\theta_4)$$

7th term of **(4.60)**

$$\frac{d}{dt}\left(-l_2\cos(\theta_2-\theta_4)\int_0^{l_4}\rho\sum_{i=0}^{N}\phi_i(y)\dot{q}_i(t)dy\right) = l_2\left(\dot{\theta}_2-\dot{\theta}_4\right)\sin(\theta_2-\theta_4)\int_0^{l_4}\rho\sum_{i=0}^{N}\phi_i(y)\dot{q}_i(t)dy$$
$$-l_2\cos(\theta_2-\theta_4)\int_0^{l_4}\rho\sum_{i=0}^{N}\phi_i(y)\ddot{q}_i(t)dy$$

8th term of **(4.60)**

$$\frac{d}{dt}\left(-l_2\dot{\theta}_2\dot{\theta}_4\sin(\theta_2-\theta_4)\int_0^{l_4}\rho\sum_{i=0}^{N}\phi_i(y)q_i(t)dy\right)=-l_2\ddot{\theta}_2\dot{\theta}_4\sin(\theta_2-\theta_4)\int_0^{l_4}\rho\sum_{i=0}^{N}\phi_i(y)q_i(t)dy$$

$$-l_2\dot{\theta}_2\ddot{\theta}_4\sin(\theta_2-\theta_4)\int_0^{l_4}\rho\sum_{i=0}^{N}\phi_i(y)q_i(t)dy$$

$$-l_2\dot{\theta}_2\dot{\theta}_4\left(\dot{\theta}_2-\dot{\theta}_4\right)\cos(\theta_2-\theta_4)\int_0^{l_4}\rho\sum_{i=0}^{N}\phi_i(y)q_i(t)dy$$

$$-l_2\dot{\theta}_2\dot{\theta}_4\sin(\theta_2-\theta_4)\int_0^{l_4}\rho\sum_{i=0}^{N}\phi_i(y)\dot{q}_i(t)dy$$

Combining all terms will give us

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}_2}\right)=I_C\ddot{\theta}_2+\rho l_3{l_2}^2\ddot{\theta}_2+\frac{1}{2}\rho l_2{l_3}^2\ddot{\theta}_3\cos(\theta_2-\theta_3)-\frac{1}{2}\rho l_2{l_3}^2\dot{\theta}_3\left(\dot{\theta}_2-\dot{\theta}_3\right)\sin(\theta_2-\theta_3)+\rho l_4{l_2}^2\ddot{\theta}_2$$

$$+\rho l_2l_3l_4\ddot{\theta}_3\cos(\theta_2-\theta_3)-\rho l_2l_3l_4\dot{\theta}_3\left(\dot{\theta}_2-\dot{\theta}_3\right)\sin(\theta_2-\theta_3)$$

$$-\frac{1}{2}\rho l_2{l_4}^2\ddot{\theta}_4\cos(\theta_2-\theta_4)+\frac{1}{2}\rho l_2{l_4}^2\dot{\theta}_4\left(\dot{\theta}_2-\dot{\theta}_4\right)\sin(\theta_2-\theta_4)$$

$$+l_2\left(\dot{\theta}_2-\dot{\theta}_4\right)\sin(\theta_2-\theta_4)\int_0^{l_4}\rho\sum_{i=0}^{N}\phi_i(y)\dot{q}_i(t)dy-l_2\cos(\theta_2-\theta_4)\int_0^{l_4}\rho\sum_{i=0}^{N}\phi_i(y)\ddot{q}_i(t)dy$$

$$-l_2\ddot{\theta}_2\dot{\theta}_4\sin(\theta_2-\theta_4)\int_0^{l_4}\rho\sum_{i=0}^{N}\phi_i(y)q_i(t)dy-l_2\dot{\theta}_2\ddot{\theta}_4\sin(\theta_2-\theta_4)\int_0^{l_4}\rho\sum_{i=0}^{N}\phi_i(y)q_i(t)dy$$

$$-l_2\dot{\theta}_2\dot{\theta}_4\left(\dot{\theta}_2-\dot{\theta}_4\right)\cos(\theta_2-\theta_4)\int_0^{l_4}\rho\sum_{i=0}^{N}\phi_i(y)q_i(t)dy-l_2\dot{\theta}_2\dot{\theta}_4\sin(\theta_2-\theta_4)\int_0^{l_4}\rho\sum_{i=0}^{N}\phi_i(y)\dot{q}_i(t)dy$$

θ_2 derivative for $\frac{\partial L}{\partial\eta_i}$:

$$\frac{\partial L}{\partial\theta_2}=-\frac{1}{2}\rho l_2{l_3}^2\dot{\theta}_2\dot{\theta}_3\left(\dot{\theta}_2-\dot{\theta}_3\right)\sin(\theta_2-\theta_3)-\rho l_2l_3l_4\dot{\theta}_2\dot{\theta}_3\left(\dot{\theta}_2-\dot{\theta}_3\right)\sin(\theta_2-\theta_3)$$

$$+\frac{1}{2}\rho l_2{l_4}^2\dot{\theta}_2\dot{\theta}_4\left(\dot{\theta}_2-\dot{\theta}_4\right)\sin(\theta_2-\theta_4)+l_2\dot{\theta}_2\left(\dot{\theta}_2-\dot{\theta}_4\right)\sin(\theta_2-\theta_4)\int_0^{l_4}\rho\sum_{i=0}^{N}\phi_i(y)\dot{q}_i(t)dy$$

$$-l_2\dot{\theta}_2\dot{\theta}_4\left(\dot{\theta}_2-\dot{\theta}_4\right)\cos(\theta_2-\theta_4)\int_0^{l_4}\rho\sum_{i=0}^{N}\phi_i(y)q_i(t)dy$$

For $\left(\frac{\partial \Phi}{\partial \eta_i}\right)^T \Lambda$:

$$\left(\frac{\partial \Phi}{\partial \theta_2}\right)^T \Lambda = \left(\frac{\partial \begin{bmatrix} -l_1 + l_2\cos\theta_2 + l_3\cos\theta_3 - l_4\cos\theta_4 \\ l_2\sin\theta_2 + l_3\sin\theta_3 - l_4\sin\theta_4 \end{bmatrix}}{\partial \theta_2}\right)^T \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix}$$

$$= \begin{bmatrix} -l_2\sin\theta_2 \\ l_2\cos\theta_2 \end{bmatrix}^T \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} = \begin{bmatrix} -l_2\sin\theta_2 & l_2\cos\theta_2 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix}$$

$$= -l_2\sin\theta_2\lambda_1 + l_2\cos\theta_2\lambda_2$$

The final state of $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\eta}_i}\right) - \frac{\partial L}{\partial \eta_i} + \left(\frac{\partial \Phi}{\partial \eta_i}\right)^T \Lambda = F_i$:

$$I_C\ddot{\theta}_2 + \rho l_3 l_2^{\,2}\ddot{\theta}_2 + \frac{1}{2}\rho l_2 l_3^{\,2}\ddot{\theta}_3\cos(\theta_2-\theta_3) - \frac{1}{2}\rho l_2 l_3^{\,2}\dot{\theta}_3(\dot{\theta}_2-\dot{\theta}_3)\sin(\theta_2-\theta_3) + \rho l_4 l_2^{\,2}\ddot{\theta}_2$$
$$+\rho l_2 l_3 l_4\ddot{\theta}_3\cos(\theta_2-\theta_3) - \rho l_2 l_3 l_4\dot{\theta}_3(\dot{\theta}_2-\dot{\theta}_3)\sin(\theta_2-\theta_3)$$
$$-\frac{1}{2}\rho l_2 l_4^{\,2}\ddot{\theta}_4\cos(\theta_2-\theta_4) + \frac{1}{2}\rho l_2 l_4^{\,2}\dot{\theta}_4(\dot{\theta}_2-\dot{\theta}_4)\sin(\theta_2-\theta_4)$$
$$+l_2(\dot{\theta}_2-\dot{\theta}_4)\sin(\theta_2-\theta_4)\int_0^{l_4}\rho\sum_{i=0}^{N}\phi_i(y)\dot{q}_i(t)dy - l_2\cos(\theta_2-\theta_4)\int_0^{l_4}\rho\sum_{i=0}^{N}\phi_i(y)\ddot{q}_i(t)dy$$
$$-l_2\ddot{\theta}_2\dot{\theta}_4\sin(\theta_2-\theta_4)\int_0^{l_4}\rho\sum_{i=0}^{N}\phi_i(y)q_i(t)dy - l_2\dot{\theta}_2\ddot{\theta}_4\sin(\theta_2-\theta_4)\int_0^{l_4}\rho\sum_{i=0}^{N}\phi_i(y)q_i(t)dy$$
$$-l_2\dot{\theta}_2\dot{\theta}_4(\dot{\theta}_2-\dot{\theta}_4)\cos(\theta_2-\theta_4)\int_0^{l_4}\rho\sum_{i=0}^{N}\phi_i(y)q_i(t)dy - l_2\dot{\theta}_2\dot{\theta}_4\sin(\theta_2-\theta_4)\int_0^{l_4}\rho\sum_{i=0}^{N}\phi_i(y)\dot{q}_i(t)dy$$
$$+\frac{1}{2}\rho l_2 l_3^{\,2}\dot{\theta}_2\dot{\theta}_3(\dot{\theta}_2-\dot{\theta}_3)\sin(\theta_2-\theta_3) + \rho l_2 l_3 l_4\dot{\theta}_2\dot{\theta}_3(\dot{\theta}_2-\dot{\theta}_3)\sin(\theta_2-\theta_3)$$
$$-\frac{1}{2}\rho l_2 l_4^{\,2}\dot{\theta}_2\dot{\theta}_4(\dot{\theta}_2-\dot{\theta}_4)\sin(\theta_2-\theta_4) - l_2\dot{\theta}_2(\dot{\theta}_2-\dot{\theta}_4)\sin(\theta_2-\theta_4)\int_0^{l_4}\rho\sum_{i=0}^{N}\phi_i(y)\dot{q}_i(t)dy$$
$$+l_2\dot{\theta}_2\dot{\theta}_4(\dot{\theta}_2-\dot{\theta}_4)\cos(\theta_2-\theta_4)\int_0^{l_4}\rho\sum_{i=0}^{N}\phi_i(y)q_i(t)dy - l_2\sin\theta_2\lambda_1 + l_2\cos\theta_2\lambda_2 = \tau$$

(4.61)

4.2.1.2 Dynamic equation of motion with Lagrange's equation for θ_3

Doing mathematical calculations for the constrained Lagrange's equations for θ_3:

$\dot{\theta}_3$ derivative for $\frac{\partial L}{\partial \dot{\eta}_i}$:

$$\frac{\partial L}{\partial \dot{\theta}_3} = \frac{1}{3}\rho l_3{}^3\dot{\theta}_3 + \frac{1}{2}\rho l_2 l_3{}^2\dot{\theta}_2\cos(\theta_2-\theta_3) + \rho l_3{}^2 l_4\dot{\theta}_3$$
$$+\rho l_2 l_3 l_4\dot{\theta}_2\cos(\theta_2-\theta_3) - \frac{1}{2}\rho l_3 l_4{}^2\dot{\theta}_4\cos(\theta_3-\theta_4) \qquad \textbf{(4.62)}$$
$$-l_3\cos(\theta_3-\theta_4)\int_0^{l_4}\rho\sum_{i=0}^{N}\phi_i(y)\dot{q}_i(t)dy - l_3\dot{\theta}_4\sin(\theta_3-\theta_4)\int_0^{l_4}\rho\sum_{i=0}^{N}\phi_i(y)q_i(t)dy$$

Time derivative $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\eta}_i}\right)$:

2nd term of **(4.62)**

$$\frac{d}{dt}\left(\frac{1}{2}\rho l_2 l_3{}^2\dot{\theta}_2\cos(\theta_2-\theta_3)\right) = \frac{1}{2}\rho l_2 l_3{}^2\ddot{\theta}_2\cos(\theta_2-\theta_3) - \frac{1}{2}\rho l_2 l_3{}^2\dot{\theta}_2\left(\dot{\theta}_2-\dot{\theta}_3\right)\sin(\theta_2-\theta_3)$$

4th term of **(4.62)**

$$\frac{d}{dt}\left(\rho l_2 l_3 l_4\dot{\theta}_2\cos(\theta_2-\theta_3)\right) = \rho l_2 l_3 l_4\ddot{\theta}_2\cos(\theta_2-\theta_3) - \rho l_2 l_3 l_4\dot{\theta}_2\left(\dot{\theta}_2-\dot{\theta}_3\right)\sin(\theta_2-\theta_3)$$

5th term of **(4.62)**

$$\frac{d}{dt}\left(-\frac{1}{2}\rho l_3 l_4{}^2\dot{\theta}_4\cos(\theta_3-\theta_4)\right) = -\frac{1}{2}\rho l_3 l_4{}^2\ddot{\theta}_4\cos(\theta_3-\theta_4) + \frac{1}{2}\rho l_3 l_4{}^2\dot{\theta}_4\left(\dot{\theta}_3-\dot{\theta}_4\right)\sin(\theta_3-\theta_4)$$

6th term of **(4.62)**

$$\frac{d}{dt}\left(-l_3\cos(\theta_3-\theta_4)\int_0^{l_4}\rho\sum_{i=0}^{N}\phi_i(y)\dot{q}_i(t)dy\right) = l_3\left(\dot{\theta}_3-\dot{\theta}_4\right)\sin(\theta_3-\theta_4)\int_0^{l_4}\rho\sum_{i=0}^{N}\phi_i(y)\dot{q}_i(t)dy$$
$$-l_3\cos(\theta_3-\theta_4)\int_0^{l_4}\rho\sum_{i=0}^{N}\phi_i(y)\ddot{q}_i(t)dy$$

7th term of **(4.62)**

$$\frac{d}{dt}\left(-l_3\dot{\theta}_4\sin(\theta_3-\theta_4)\int_0^{l_4}\rho\sum_{i=0}^{N}\phi_i(y)q_i(t)dy\right) = -l_3\dot{\theta}_4\left(\dot{\theta}_3-\dot{\theta}_4\right)\cos(\theta_3-\theta_4)\int_0^{l_4}\rho\sum_{i=0}^{N}\phi_i(y)q_i(t)dy$$
$$-l_3\dot{\theta}_4\sin(\theta_3-\theta_4)\int_0^{l_4}\rho\sum_{i=0}^{N}\phi_i(y)\dot{q}_i(t)dy$$

Combining all terms will give us

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_3}\right)=\frac{1}{3}\rho l_3{}^3\ddot{\theta}_3+\frac{1}{2}\rho l_2 l_3{}^2\ddot{\theta}_2\cos(\theta_2-\theta_3)-\frac{1}{2}\rho l_2 l_3{}^2\dot{\theta}_2(\dot{\theta}_2-\dot{\theta}_3)\sin(\theta_2-\theta_3)+\rho l_3{}^2 l_4\ddot{\theta}_3$$
$$+\rho l_2 l_3 l_4\ddot{\theta}_2\cos(\theta_2-\theta_3)-\rho l_2 l_3 l_4\dot{\theta}_2(\dot{\theta}_2-\dot{\theta}_3)\sin(\theta_2-\theta_3)$$
$$-\frac{1}{2}\rho l_3 l_4{}^2\ddot{\theta}_4\cos(\theta_3-\theta_4)+\frac{1}{2}\rho l_3 l_4{}^2\dot{\theta}_4(\dot{\theta}_3-\dot{\theta}_4)\sin(\theta_3-\theta_4)$$
$$+l_3(\dot{\theta}_3-\dot{\theta}_4)\sin(\theta_3-\theta_4)\int_0^{l_4}\rho\sum_{i=0}^{N}\phi_i(y)\dot{q}_i(t)dy-l_3\cos(\theta_3-\theta_4)\int_0^{l_4}\rho\sum_{i=0}^{N}\phi_i(y)\ddot{q}_i(t)dy$$
$$-l_3\dot{\theta}_4(\dot{\theta}_3-\dot{\theta}_4)\cos(\theta_3-\theta_4)\int_0^{l_4}\rho\sum_{i=0}^{N}\phi_i(y)q_i(t)dy-l_3\dot{\theta}_4\sin(\theta_3-\theta_4)\int_0^{l_4}\rho\sum_{i=0}^{N}\phi_i(y)\dot{q}_i(t)dy$$

θ_3 derivative for $\frac{\partial L}{\partial \eta_i}$:

$$\frac{\partial L}{\partial \theta_3}=-\frac{1}{2}\rho l_2 l_3{}^2\dot{\theta}_2\dot{\theta}_3(\dot{\theta}_2-\dot{\theta}_3)\sin(\theta_2-\theta_3)-\rho l_2 l_3 l_4\dot{\theta}_2\dot{\theta}_3(\dot{\theta}_2-\dot{\theta}_3)\sin(\theta_2-\theta_3)$$
$$+\frac{1}{2}\rho l_3 l_4{}^2\dot{\theta}_3\dot{\theta}_4(\dot{\theta}_3-\dot{\theta}_4)\sin(\theta_3-\theta_4)+l_3\dot{\theta}_3(\dot{\theta}_3-\dot{\theta}_4)\sin(\theta_3-\theta_4)\int_0^{l_4}\rho\sum_{i=0}^{N}\phi_i(y)\dot{q}_i(t)dy$$
$$-l_3\dot{\theta}_3\dot{\theta}_4(\dot{\theta}_3-\dot{\theta}_4)\cos(\theta_3-\theta_4)\int_0^{l_4}\rho\sum_{i=0}^{N}\phi_i(y)q_i(t)dy$$

For $\left(\frac{\partial \Phi}{\partial \eta_i}\right)^T\Lambda$:

$$\left(\frac{\partial \Phi}{\partial \theta_3}\right)^T\Lambda=\left(\frac{\partial\begin{bmatrix}-l_1+l_2\cos\theta_2+l_3\cos\theta_3-l_4\cos\theta_4\\ l_2\sin\theta_2+l_3\sin\theta_3-l_4\sin\theta_4\end{bmatrix}}{\partial\theta_3}\right)^T\begin{bmatrix}\lambda_1\\ \lambda_2\end{bmatrix}$$
$$=\begin{bmatrix}-l_3\sin\theta_3\\ l_3\cos\theta_3\end{bmatrix}^T\begin{bmatrix}\lambda_1\\ \lambda_2\end{bmatrix}=\begin{bmatrix}-l_3\sin\theta_3 & l_3\cos\theta_3\end{bmatrix}\begin{bmatrix}\lambda_1\\ \lambda_2\end{bmatrix}$$
$$=-l_3\sin\theta_3\lambda_1+l_3\cos\theta_3\lambda_2$$

The final state of $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\eta}_i}\right)-\frac{\partial L}{\partial \eta_i}+\left(\frac{\partial \Phi}{\partial \eta_i}\right)^T\Lambda=F_i$:

$$\frac{1}{3}\rho l_3{}^3\ddot{\theta}_3+\frac{1}{2}\rho l_2 l_3{}^2\ddot{\theta}_2\cos(\theta_2-\theta_3)-\frac{1}{2}\rho l_2 l_3{}^2\dot{\theta}_2(\dot{\theta}_2-\dot{\theta}_3)\sin(\theta_2-\theta_3)+\rho l_3{}^2 l_4\ddot{\theta}_3$$
$$+\rho l_2 l_3 l_4\ddot{\theta}_2\cos(\theta_2-\theta_3)-\rho l_2 l_3 l_4\dot{\theta}_2(\dot{\theta}_2-\dot{\theta}_3)\sin(\theta_2-\theta_3)$$
$$-\frac{1}{2}\rho l_3 l_4{}^2\ddot{\theta}_4\cos(\theta_3-\theta_4)+\frac{1}{2}\rho l_3 l_4{}^2\dot{\theta}_4(\dot{\theta}_3-\dot{\theta}_4)\sin(\theta_3-\theta_4)$$
$$+l_3(\dot{\theta}_3-\dot{\theta}_4)\sin(\theta_3-\theta_4)\int_0^{l_4}\rho\sum_{i=0}^{N}\phi_i(y)\dot{q}_i(t)dy-l_3\cos(\theta_3-\theta_4)\int_0^{l_4}\rho\sum_{i=0}^{N}\phi_i(y)\ddot{q}_i(t)dy$$
$$-l_3\dot{\theta}_4(\dot{\theta}_3-\dot{\theta}_4)\cos(\theta_3-\theta_4)\int_0^{l_4}\rho\sum_{i=0}^{N}\phi_i(y)q_i(t)dy-l_3\dot{\theta}_4\sin(\theta_3-\theta_4)\int_0^{l_4}\rho\sum_{i=0}^{N}\phi_i(y)\dot{q}_i(t)dy$$
$$+\frac{1}{2}\rho l_2 l_3{}^2\dot{\theta}_2\dot{\theta}_3(\dot{\theta}_2-\dot{\theta}_3)\sin(\theta_2-\theta_3)+\rho l_2 l_3 l_4\dot{\theta}_2\dot{\theta}_3(\dot{\theta}_2-\dot{\theta}_3)\sin(\theta_2-\theta_3)$$
$$-\frac{1}{2}\rho l_3 l_4{}^2\dot{\theta}_3\dot{\theta}_4(\dot{\theta}_3-\dot{\theta}_4)\sin(\theta_3-\theta_4)-l_3\dot{\theta}_3(\dot{\theta}_3-\dot{\theta}_4)\sin(\theta_3-\theta_4)\int_0^{l_4}\rho\sum_{i=0}^{N}\phi_i(y)\dot{q}_i(t)dy$$
$$+l_3\dot{\theta}_3\dot{\theta}_4(\dot{\theta}_3-\dot{\theta}_4)\cos(\theta_3-\theta_4)\int_0^{l_4}\rho\sum_{i=0}^{N}\phi_i(y)q_i(t)dy-l_3\sin\theta_3\lambda_1+l_3\cos\theta_3\lambda_2=0 \qquad \textbf{(4.63)}$$

4.2.1.3 Dynamic equation of motion with Lagrange's equation for θ_4

Doing mathematical calculations for the constrained Lagrange's equations for θ_4:

$\dot{\theta}_4$ derivative for $\frac{\partial L}{\partial\dot{\eta}_i}$:

$$\frac{\partial L}{\partial\dot{\theta}_4}=\frac{1}{3}l_4{}^3\dot{\theta}_4+\dot{\theta}_4\int_0^{l_4}\rho\left(\sum_{i=0}^{N}\phi_i(y)q_i(t)\right)^2dy-\frac{1}{2}\rho l_2 l_4{}^2\dot{\theta}_2\cos(\theta_2-\theta_4)$$
$$-l_2\dot{\theta}_2\sin(\theta_2-\theta_4)\int_0^{l_4}\rho\sum_{i=0}^{N}\phi_i(y)q_i(t)dy-\frac{1}{2}\rho l_3 l_4{}^2\dot{\theta}_3\cos(\theta_3-\theta_4) \qquad \textbf{(4.64)}$$
$$-l_3\dot{\theta}_3\sin(\theta_3-\theta_4)\int_0^{l_4}\rho\sum_{i=0}^{N}\phi_i(y)q_i(t)dy+\int_0^{l_4}\rho y\sum_{i=0}^{N}\phi_i(y)\dot{q}_i(t)dy$$

Time derivative $\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\eta}_i}\right)$:

2nd term of **(4.64)**

$$\frac{d}{dt}\left(\dot{\theta}_4\int_0^{l_4}\rho\left(\sum_{i=0}^{N}\phi_i(y)q_i(t)\right)^2dy\right)=\ddot{\theta}_4\int_0^{l_4}\rho\left(\sum_{i=0}^{N}\phi_i(y)q_i(t)\right)^2dy+2\dot{\theta}_4\int_0^{l_4}\rho\sum_{i=0}^{N}\dot{q}_i(t)\left(\sum_{i=0}^{N}\phi_i(y)q_i(t)\right)^2dy$$

3rd term of **(4.64)**

$$\frac{d}{dt}\left(-\frac{1}{2}\rho l_2 l_4{}^2\dot{\theta}_2\cos(\theta_2-\theta_4)\right)=-\frac{1}{2}\rho l_2 l_4{}^2\ddot{\theta}_2\cos(\theta_2-\theta_4)+\frac{1}{2}\rho l_2 l_4{}^2\dot{\theta}_2(\dot{\theta}_2-\dot{\theta}_4)\sin(\theta_2-\theta_4)$$

4th term of **(4.64)**

$$\frac{d}{dt}\left(-l_2\dot{\theta}_2\sin(\theta_2-\theta_4)\int_0^{l_4}\rho\sum_{i=0}^{N}\phi_i(y)q_i(t)dy\right)=-l_2\ddot{\theta}_2\sin(\theta_2-\theta_4)\int_0^{l_4}\rho\sum_{i=0}^{N}\phi_i(y)q_i(t)dy$$
$$-l_2\dot{\theta}_2(\dot{\theta}_2-\dot{\theta}_4)\cos(\theta_2-\theta_4)\int_0^{l_4}\rho\sum_{i=0}^{N}\phi_i(y)q_i(t)dy-l_2\dot{\theta}_2\sin(\theta_2-\theta_4)\int_0^{l_4}\rho\sum_{i=0}^{N}\phi_i(y)\dot{q}_i(t)dy$$

5th term of **(4.64)**

$$\frac{d}{dt}\left(-\frac{1}{2}\rho l_3 l_4{}^2\dot{\theta}_3\cos(\theta_3-\theta_4)\right)=-\frac{1}{2}\rho l_3 l_4{}^2\ddot{\theta}_3\cos(\theta_3-\theta_4)+\frac{1}{2}\rho l_3 l_4{}^2\dot{\theta}_3(\dot{\theta}_3-\dot{\theta}_4)\sin(\theta_3-\theta_4)$$

6th term of **(4.64)**

$$\frac{d}{dt}\left(-l_3\dot{\theta}_3\sin(\theta_3-\theta_4)\int_0^{l_4}\rho\sum_{i=0}^{N}\phi_i(y)q_i(t)dy\right)=-l_3\ddot{\theta}_3\sin(\theta_3-\theta_4)\int_0^{l_4}\rho\sum_{i=0}^{N}\phi_i(y)q_i(t)dy$$
$$-l_3\dot{\theta}_3(\dot{\theta}_3-\dot{\theta}_4)\cos(\theta_3-\theta_4)\int_0^{l_4}\rho\sum_{i=0}^{N}\phi_i(y)q_i(t)dy-l_3\dot{\theta}_3\sin(\theta_3-\theta_4)\int_0^{l_4}\rho\sum_{i=0}^{N}\phi_i(y)\dot{q}_i(t)dy$$

Combining all terms will give us

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}_4}\right)=\frac{1}{3}l_4{}^3\ddot{\theta}_4+\ddot{\theta}_4\int_0^{l_4}\rho\left(\sum_{i=0}^{N}\phi_i(y)q_i(t)\right)^2dy+2\dot{\theta}_4\int_0^{l_4}\rho\sum_{i=0}^{N}\dot{q}_i(t)\left(\sum_{i=0}^{N}\phi_i(y)q_i(t)\right)^2dy$$
$$-\frac{1}{2}\rho l_2 l_4{}^2\ddot{\theta}_2\cos(\theta_2-\theta_4)+\frac{1}{2}\rho l_2 l_4{}^2\dot{\theta}_2(\dot{\theta}_2-\dot{\theta}_4)\sin(\theta_2-\theta_4)-l_2\ddot{\theta}_2\sin(\theta_2-\theta_4)\int_0^{l_4}\rho\sum_{i=0}^{N}\phi_i(y)q_i(t)dy$$
$$-l_2\dot{\theta}_2(\dot{\theta}_2-\dot{\theta}_4)\cos(\theta_2-\theta_4)\int_0^{l_4}\rho\sum_{i=0}^{N}\phi_i(y)q_i(t)dy-l_2\dot{\theta}_2\sin(\theta_2-\theta_4)\int_0^{l_4}\rho\sum_{i=0}^{N}\phi_i(y)\dot{q}_i(t)dy$$
$$-\frac{1}{2}\rho l_3 l_4{}^2\ddot{\theta}_3\cos(\theta_3-\theta_4)+\frac{1}{2}\rho l_3 l_4{}^2\dot{\theta}_3(\dot{\theta}_3-\dot{\theta}_4)\sin(\theta_3-\theta_4)$$
$$-l_3\ddot{\theta}_3\sin(\theta_3-\theta_4)\int_0^{l_4}\rho\sum_{i=0}^{N}\phi_i(y)q_i(t)dy-l_3\dot{\theta}_3(\dot{\theta}_3-\dot{\theta}_4)\cos(\theta_3-\theta_4)\int_0^{l_4}\rho\sum_{i=0}^{N}\phi_i(y)q_i(t)dy$$
$$-l_3\dot{\theta}_3\sin(\theta_3-\theta_4)\int_0^{l_4}\rho\sum_{i=0}^{N}\phi_i(y)\dot{q}_i(t)dy+\int_0^{l_4}\rho y\sum_{i=0}^{N}\phi_i(y)\ddot{q}_i(t)dy$$

θ_4 derivative for $\frac{\partial L}{\partial\eta_i}$:

$$\frac{\partial L}{\partial \theta_4} = \frac{1}{2}\rho l_2 l_4{}^2 \dot{\theta}_2 \dot{\theta}_4 \left(\dot{\theta}_2 - \dot{\theta}_4\right)\sin\left(\theta_2 - \theta_4\right) + l_2 \dot{\theta}_2 \left(\dot{\theta}_2 - \dot{\theta}_4\right)\sin\left(\theta_2 - \theta_4\right)\int_0^{l_4} \rho \sum_{i=0}^{N} \phi_i(y)\dot{q}_i(t)dy$$

$$-l_2 \dot{\theta}_2 \dot{\theta}_4 \left(\dot{\theta}_2 - \dot{\theta}_4\right)\cos\left(\theta_2 - \theta_4\right)\int_0^{l_4} \rho \sum_{i=0}^{N} \phi_i(y) q_i(t)dy + \frac{1}{2}\rho l_3 l_4{}^2 \dot{\theta}_3 \dot{\theta}_4 \left(\dot{\theta}_3 - \dot{\theta}_4\right)\sin\left(\theta_3 - \theta_4\right)$$

$$+l_3 \dot{\theta}_3 \left(\dot{\theta}_3 - \dot{\theta}_4\right)\sin\left(\theta_3 - \theta_4\right)\int_0^{l_4} \rho \sum_{i=0}^{N} \phi_i(y)\dot{q}_i(t)dy - l_3 \dot{\theta}_3 \dot{\theta}_4 \left(\dot{\theta}_3 - \dot{\theta}_4\right)\cos\left(\theta_3 - \theta_4\right)\int_0^{l_4} \rho \sum_{i=0}^{N} \phi_i(y) q_i(t)dy$$

For $\left(\frac{\partial \Phi}{\partial \eta_i}\right)^T \Lambda$:

$$\left(\frac{\partial \Phi}{\partial \theta_4}\right)^T \Lambda = \left(\frac{\partial \begin{bmatrix} -l_1 + l_2\cos\theta_2 + l_3\cos\theta_3 - l_4\cos\theta_4 \\ l_2\sin\theta_2 + l_3\sin\theta_3 - l_4\sin\theta_4 \end{bmatrix}}{\partial \theta_4}\right)^T \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix}$$

$$= \begin{bmatrix} l_4\sin\theta_4 \\ -l_4\cos\theta_4 \end{bmatrix}^T \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} = \begin{bmatrix} l_4\sin\theta_4 & -l_4\cos\theta_4 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix}$$

$$= l_4\sin\theta_4\lambda_1 - l_4\cos\theta_4\lambda_2$$

The final state of $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\eta}_i}\right) - \frac{\partial L}{\partial \eta_i} + \left(\frac{\partial \Phi}{\partial \eta_i}\right)^T \Lambda = F_i$:

$$\frac{1}{3}{l_4}^3\ddot{\theta}_4+\ddot{\theta}_4\int_0^{l_4}\rho\left(\sum_{i=0}^{N}\phi_i(y)q_i(t)\right)^2dy+2\dot{\theta}_4\int_0^{l_4}\rho\sum_{i=0}^{N}\dot{q}_i(t)\left(\sum_{i=0}^{N}\phi_i(y)q_i(t)\right)^2dy$$

$$-\frac{1}{2}\rho l_2{l_4}^2\ddot{\theta}_2\cos(\theta_2-\theta_4)+\frac{1}{2}\rho l_2{l_4}^2\dot{\theta}_2\left(\dot{\theta}_2-\dot{\theta}_4\right)\sin(\theta_2-\theta_4)$$

$$-l_2\ddot{\theta}_2\sin(\theta_2-\theta_4)\int_0^{l_4}\rho\sum_{i=0}^{N}\phi_i(y)q_i(t)dy-l_2\dot{\theta}_2\left(\dot{\theta}_2-\dot{\theta}_4\right)\cos(\theta_2-\theta_4)\int_0^{l_4}\rho\sum_{i=0}^{N}\phi_i(y)q_i(t)dy$$

$$-l_2\dot{\theta}_2\sin(\theta_2-\theta_4)\int_0^{l_4}\rho\sum_{i=0}^{N}\phi_i(y)\dot{q}_i(t)dy-\frac{1}{2}\rho l_3{l_4}^2\ddot{\theta}_3\cos(\theta_3-\theta_4)$$

$$+\frac{1}{2}\rho l_3{l_4}^2\dot{\theta}_3\left(\dot{\theta}_3-\dot{\theta}_4\right)\sin(\theta_3-\theta_4)-l_3\ddot{\theta}_3\sin(\theta_3-\theta_4)\int_0^{l_4}\rho\sum_{i=0}^{N}\phi_i(y)q_i(t)dy$$

$$-l_3\dot{\theta}_3\left(\dot{\theta}_3-\dot{\theta}_4\right)\cos(\theta_3-\theta_4)\int_0^{l_4}\rho\sum_{i=0}^{N}\phi_i(y)q_i(t)dy-l_3\dot{\theta}_3\sin(\theta_3-\theta_4)\int_0^{l_4}\rho\sum_{i=0}^{N}\phi_i(y)\dot{q}_i(t)dy$$

$$+\int_0^{l_4}\rho y\sum_{i=0}^{N}\phi_i(y)\ddot{q}_i(t)dy-\frac{1}{2}\rho l_2{l_4}^2\dot{\theta}_2\dot{\theta}_4\left(\dot{\theta}_2-\dot{\theta}_4\right)\sin(\theta_2-\theta_4)$$

$$-l_2\dot{\theta}_2\left(\dot{\theta}_2-\dot{\theta}_4\right)\sin(\theta_2-\theta_4)\int_0^{l_4}\rho\sum_{i=0}^{N}\phi_i(y)\dot{q}_i(t)dy$$

$$+l_2\dot{\theta}_2\dot{\theta}_4\left(\dot{\theta}_2-\dot{\theta}_4\right)\cos(\theta_2-\theta_4)\int_0^{l_4}\rho\sum_{i=0}^{N}\phi_i(y)q_i(t)dy-\frac{1}{2}\rho l_3{l_4}^2\dot{\theta}_3\dot{\theta}_4\left(\dot{\theta}_3-\dot{\theta}_4\right)\sin(\theta_3-\theta_4)$$

$$-l_3\dot{\theta}_3\left(\dot{\theta}_3-\dot{\theta}_4\right)\sin(\theta_3-\theta_4)\int_0^{l_4}\rho\sum_{i=0}^{N}\phi_i(y)\dot{q}_i(t)dy$$

$$+l_3\dot{\theta}_3\dot{\theta}_4\left(\dot{\theta}_3-\dot{\theta}_4\right)\cos(\theta_3-\theta_4)\int_0^{l_4}\rho\sum_{i=0}^{N}\phi_i(y)q_i(t)dy+l_4\sin\theta_4\lambda_1-l_4\cos\theta_4\lambda_2=0$$

(4.65)

4.2.1.4 Dynamic equation of motion with Lagrange's equation for δ

Doing mathematical calculations for the constrained Lagrange's equations forδ:

δderivative for $\frac{\partial L}{\partial\dot{\eta}_i}$:

$$\frac{\partial L}{\partial\dot{\delta}}=\int_0^{l_4}\rho\sum_{i=1}^{N}\varphi_i(y)\varphi_j(y)\dot{q}_i(t)dy-l_2\dot{\theta}_2cos(\theta_2-\theta_4)\int_0^{l_4}\rho\sum_{i=1}^{N}\varphi_i(y)dy$$

$$+l_3\dot{\theta}_3cos(\theta_3-\theta_4)\int_0^{l_4}\rho\sum_{i=1}^{N}\varphi_i(y)dy+\dot{\theta}_4\int_0^{l_4}\rho y\sum_{i=1}^{N}\varphi_i(y)dy$$

Time derivative $\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\eta}_i}\right)$:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\delta}}\right)=\int_0^{l_4}\rho\sum_{i=1}^{N}\varphi_i(y)\varphi_j(y)\ddot{q}_i(t)dy-l_2\ddot{\theta}_2cos(\theta_2-\theta_4)\int_0^{l_4}\rho\sum_{i=1}^{N}\varphi_i(y)dy$$

$$+l_2\dot{\theta}_2(\dot{\theta}_2-\dot{\theta}_4)sin(\theta_2-\theta_4)\int_0^{l_4}\rho\sum_{i=1}^{N}\varphi_i(y)dy$$

$$+l_3\ddot{\theta}_3cos(\theta_3-\theta_4)\int_0^{l_4}\rho\sum_{i=1}^{N}\varphi_i(y)dy$$

$$-l_3\dot{\theta}_3(\dot{\theta}_3-\dot{\theta}_4)sin(\theta_3-\theta_4)\int_0^{l_4}\rho\sum_{i=1}^{N}\varphi_i(y)dy+\ddot{\theta}_4\int_0^{l_4}\rho y\sum_{i=1}^{N}\varphi_i(y)dy$$

δ derivative for $\frac{\partial L}{\partial \eta_i}$:

$$\frac{\partial L}{\partial \delta}=\dot{\theta}_4^2\int_0^{l_4}\rho\sum_{i=1}^{N}\varphi_i(y)\varphi_j(y)q_i(t)dy$$

$$-l_2\dot{\theta}_2\dot{\theta}_4sin(\theta_2-\theta_4)\int_0^{l_4}\rho\sum_{i=1}^{N}\varphi_i(y)q_i(t)dy-\sum_{j=1}^{N}\int_0^{l_4}EI\;\ddot{\varphi}_i(y)\ddot{\varphi}_j(y)q_i(t)dy$$

For $\left(\frac{\partial \Phi}{\partial \eta_i}\right)^T\Lambda$:

$$\left(\frac{\partial \Phi}{\partial w}\right)^T\Lambda=\left(\frac{\partial\begin{bmatrix}-l_1+l_2\cos\theta_2+l_3\cos\theta_3-l_4\cos\theta_4\\ l_2\sin\theta_2+l_3\sin\theta_3-l_4\sin\theta_4\end{bmatrix}}{\partial w}\right)^T\begin{bmatrix}\lambda_1\\ \lambda_2\end{bmatrix}=0$$

The final state of $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\eta}_i}\right)-\frac{\partial L}{\partial \eta_i}+\left(\frac{\partial \Phi}{\partial \eta_i}\right)^T\Lambda=F_i$ for δ:

$$\int_0^{l_4}\rho\sum_{i=1}^{N}\varphi_i(y)\varphi_j(y)\dot{q}_i(t)dy-l_2\dot{\theta}_2cos(\theta_2-\theta_4)\int_0^{l_4}\rho\sum_{i=1}^{N}\varphi_i(y)dy$$

$$+l_3\dot{\theta}_3cos(\theta_3-\theta_4)\int_0^{l_4}\rho\sum_{i=1}^{N}\varphi_i(y)dy+\dot{\theta}_4\int_0^{l_4}\rho y\sum_{i=1}^{N}\varphi_i(y)dy$$

$$+\int_0^{l_4}\rho\sum_{i=1}^{N}\varphi_i(y)\varphi_j(y)\ddot{q}_i(t)dy-l_2\ddot{\theta}_2cos(\theta_2-\theta_4)\int_0^{l_4}\rho\sum_{i=1}^{N}\varphi_i(y)dy$$

$$+l_2\dot{\theta}_2(\dot{\theta}_2-\dot{\theta}_4)sin(\theta_2-\theta_4)\int_0^{l_4}\rho\sum_{i=1}^{N}\varphi_i(y)dy$$

$$+l_3\ddot{\theta}_3cos(\theta_3-\theta_4)\int_0^{l_4}\rho\sum_{i=1}^{N}\varphi_i(y)dy$$

$$-l_3\dot{\theta}_3(\dot{\theta}_3-\dot{\theta}_4)sin(\theta_3-\theta_4)\int_0^{l_4}\rho\sum_{i=1}^{N}\varphi_i(y)dy+\ddot{\theta}_4\int_0^{l_4}\rho y\sum_{i=1}^{N}\varphi_i(y)dy$$

(4.66)

4.2.2 Dynamic analysis solutions for four-bar mechanism using Matlab Simulink

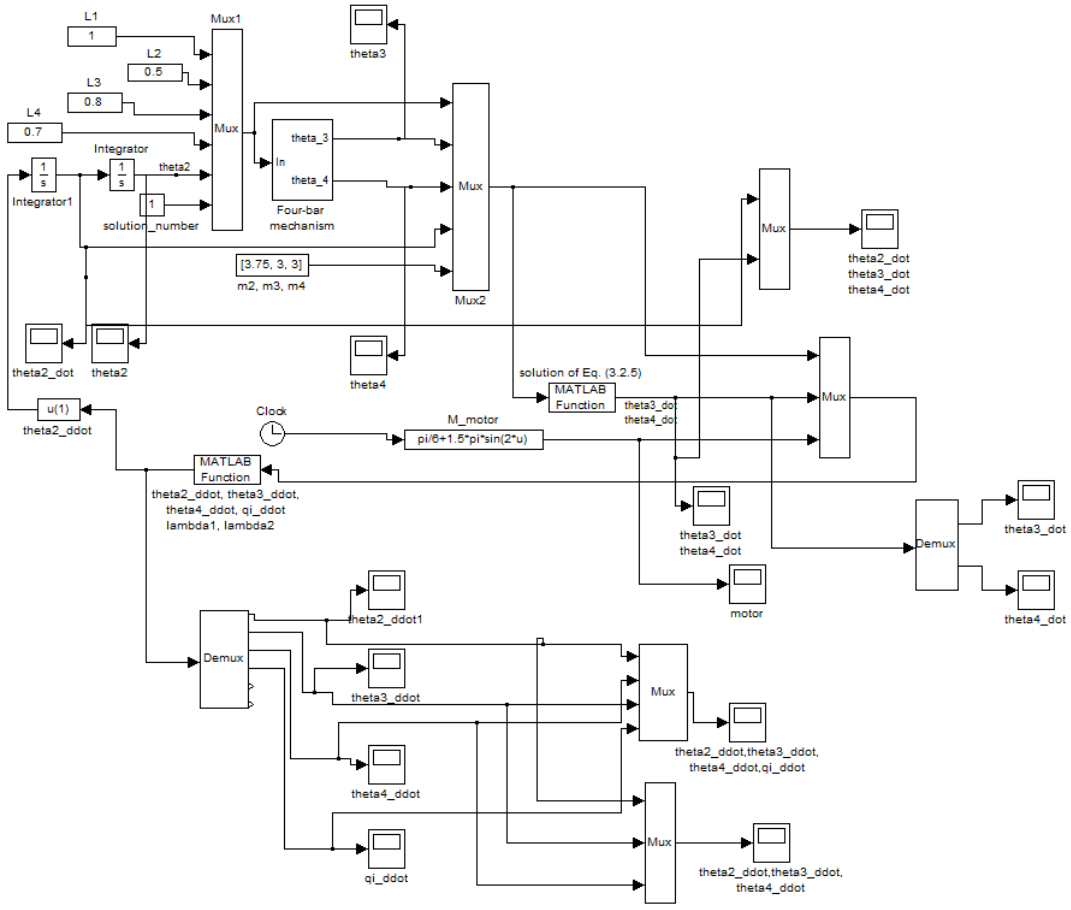

Figure 4.16 Simulink Block diagram of an elastic output link four bar mechanism

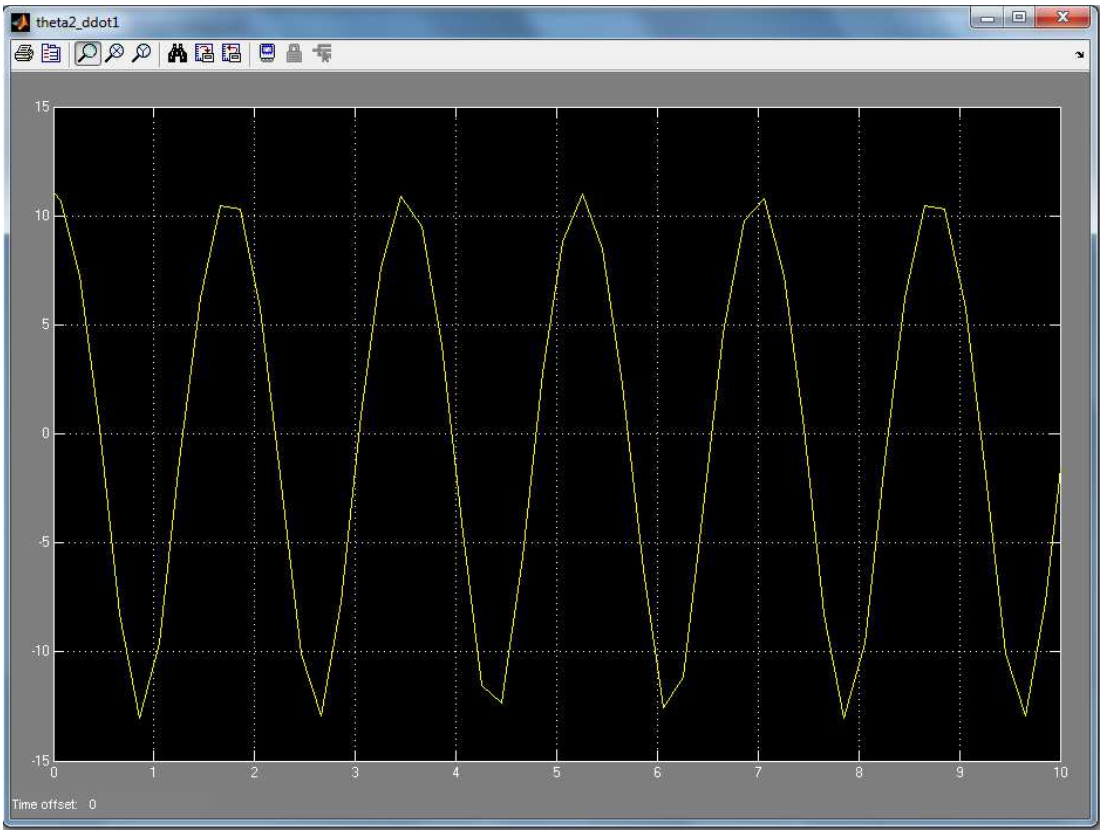

Figure 4.17 Crank angular acceleration time response

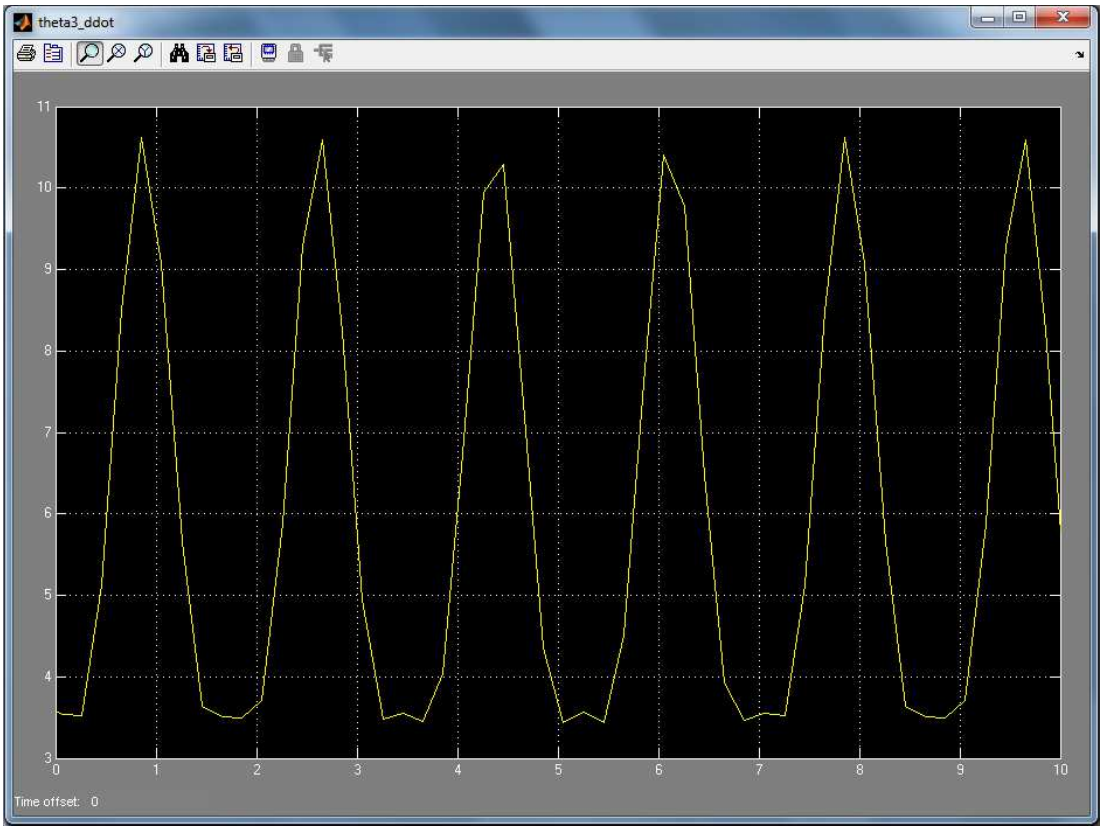

Figure 4.18 Coupler angular acceleration time response

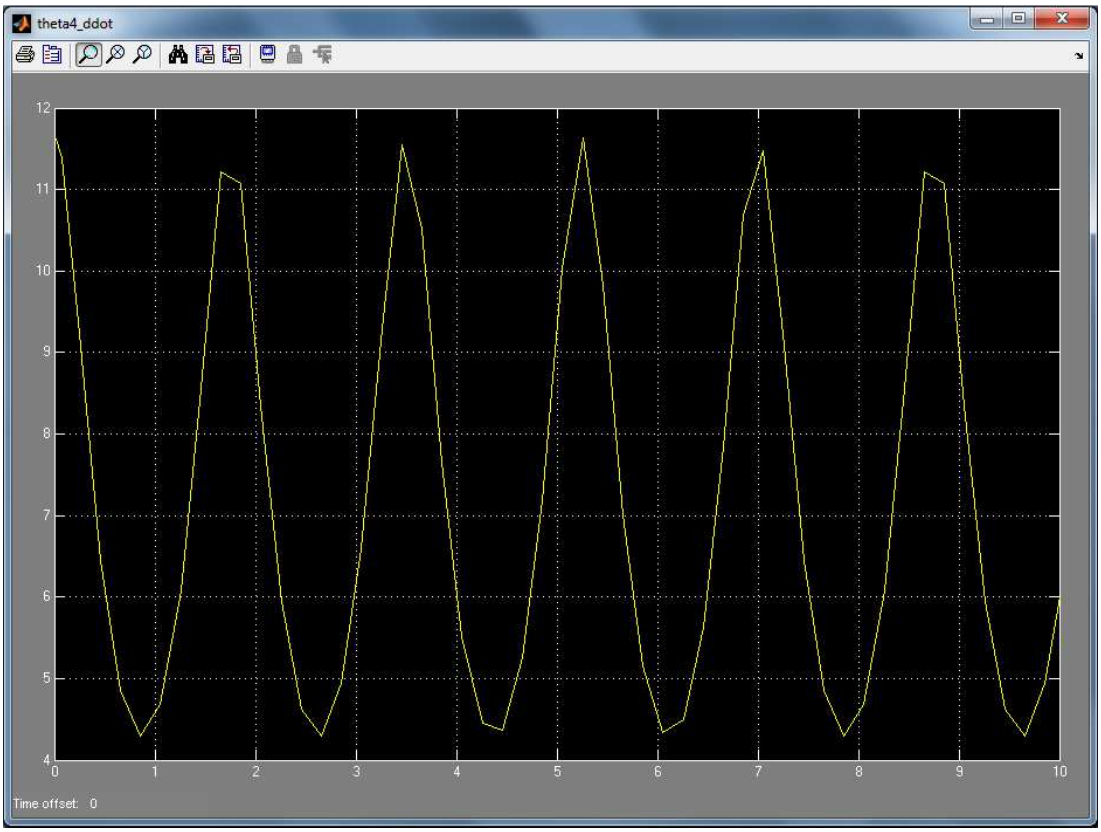

Figure 4.19 Output angular acceleration time response

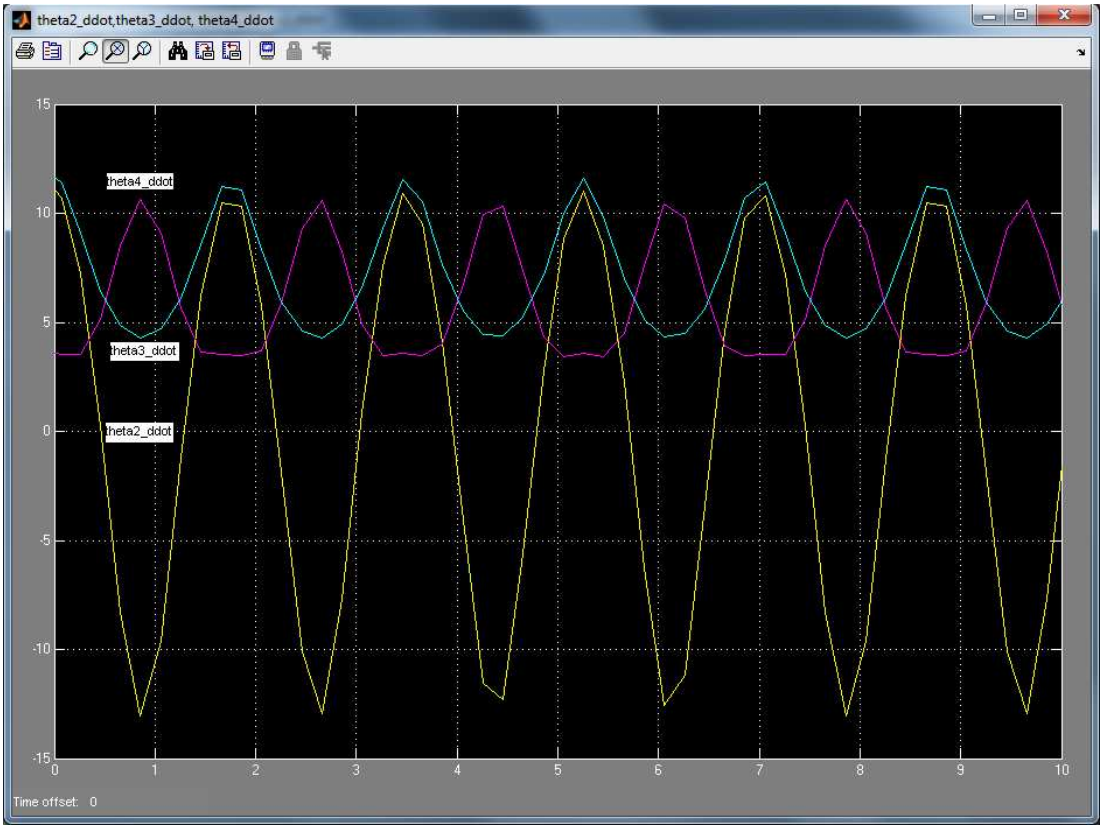

Figure 4.20 Comparison of the angular acceleration

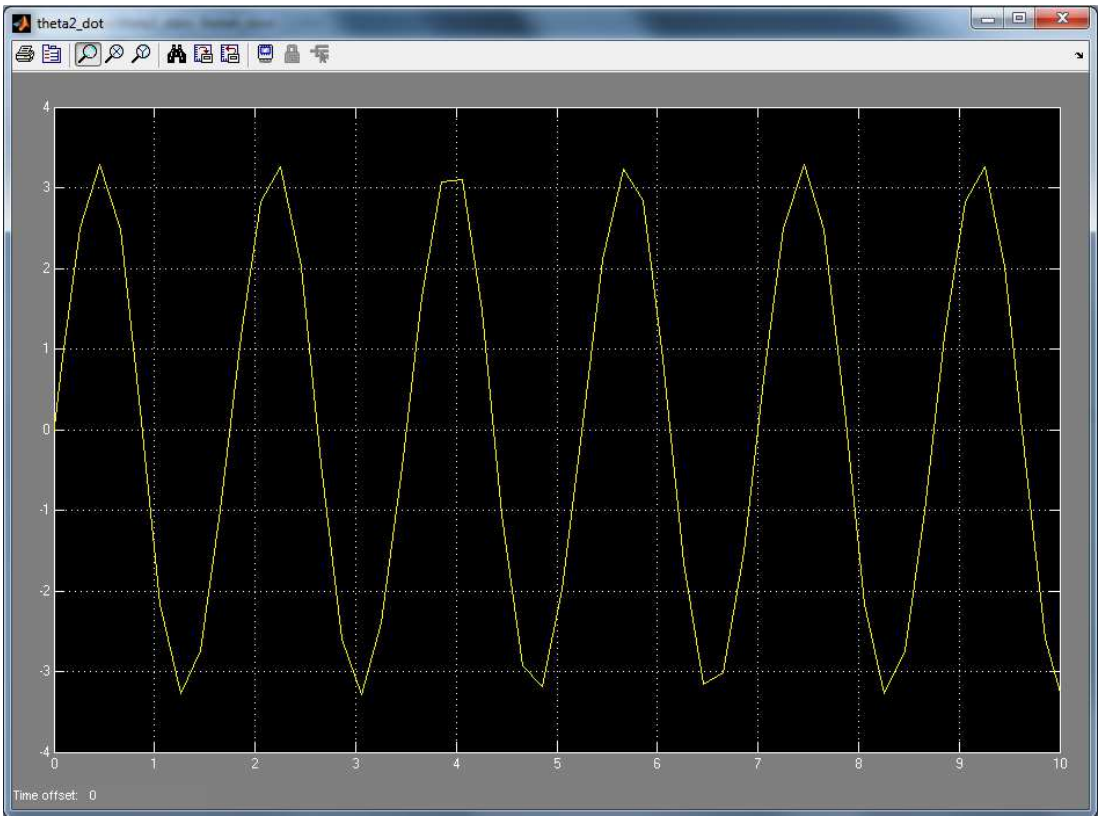

Figure 4.21 Crank angular velocity

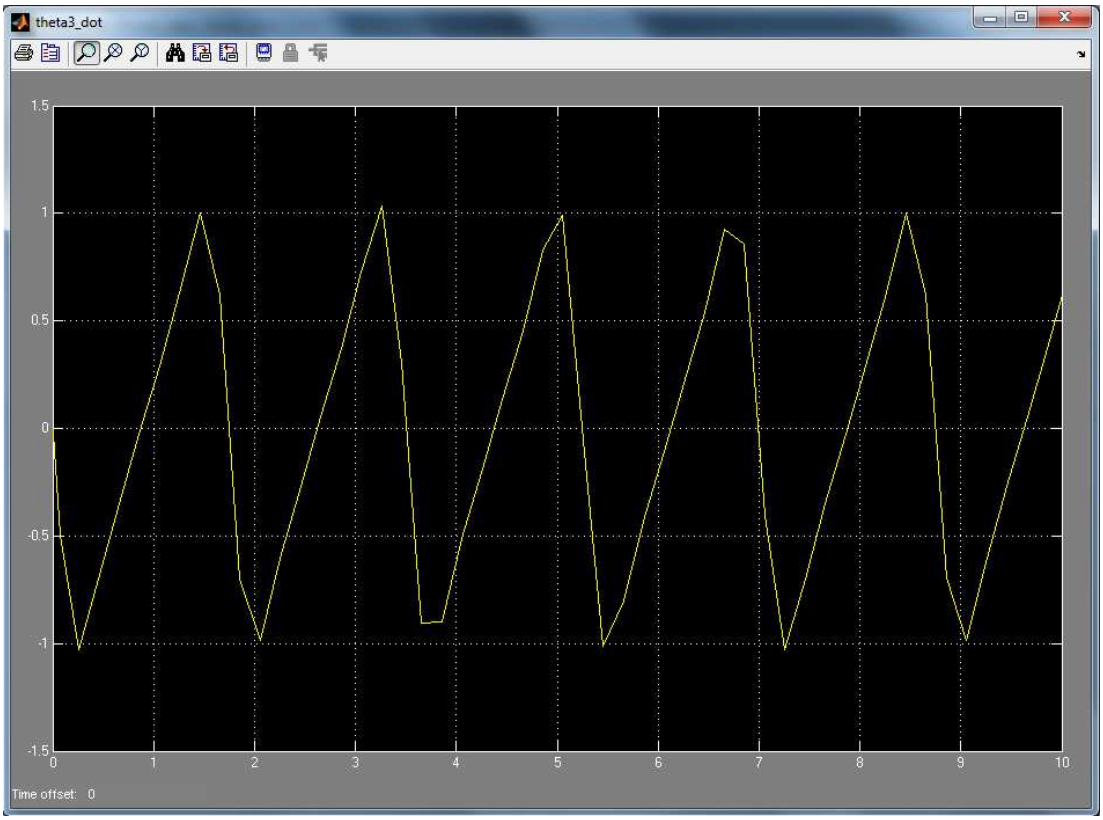

Figure 4.22 Coupler angular velocity

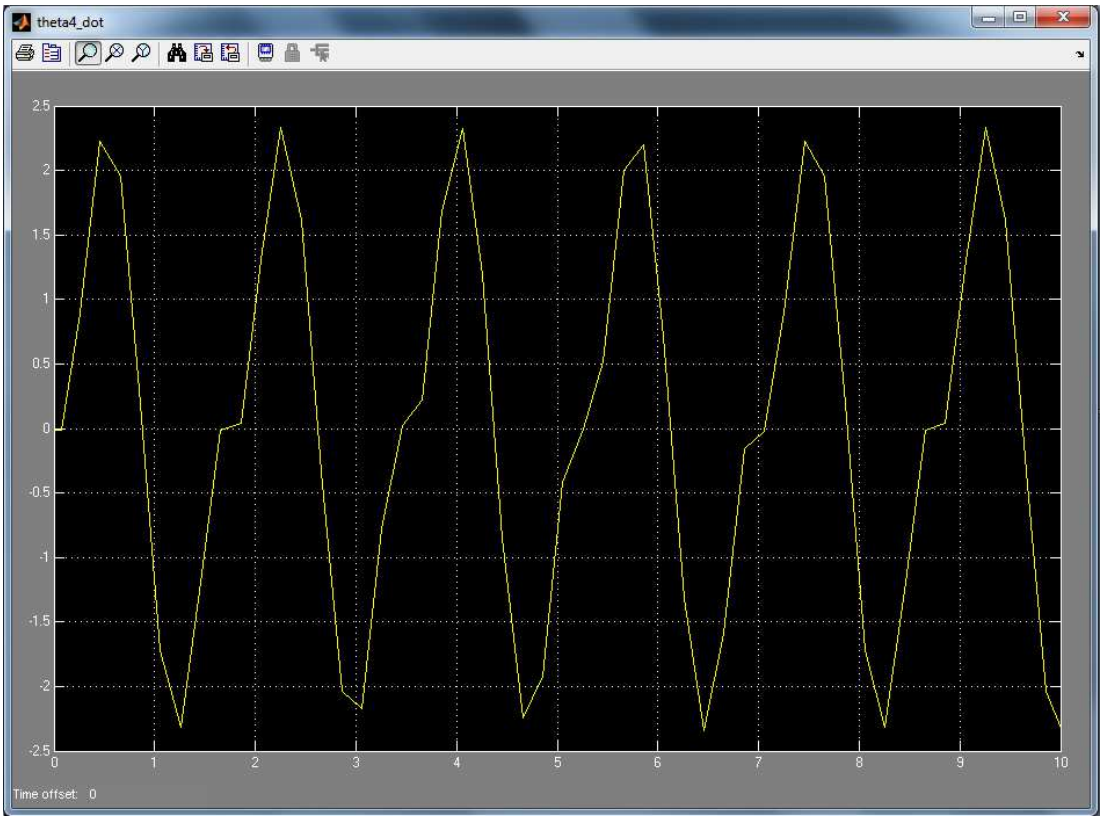

Figure 4.23 Output angular velocity

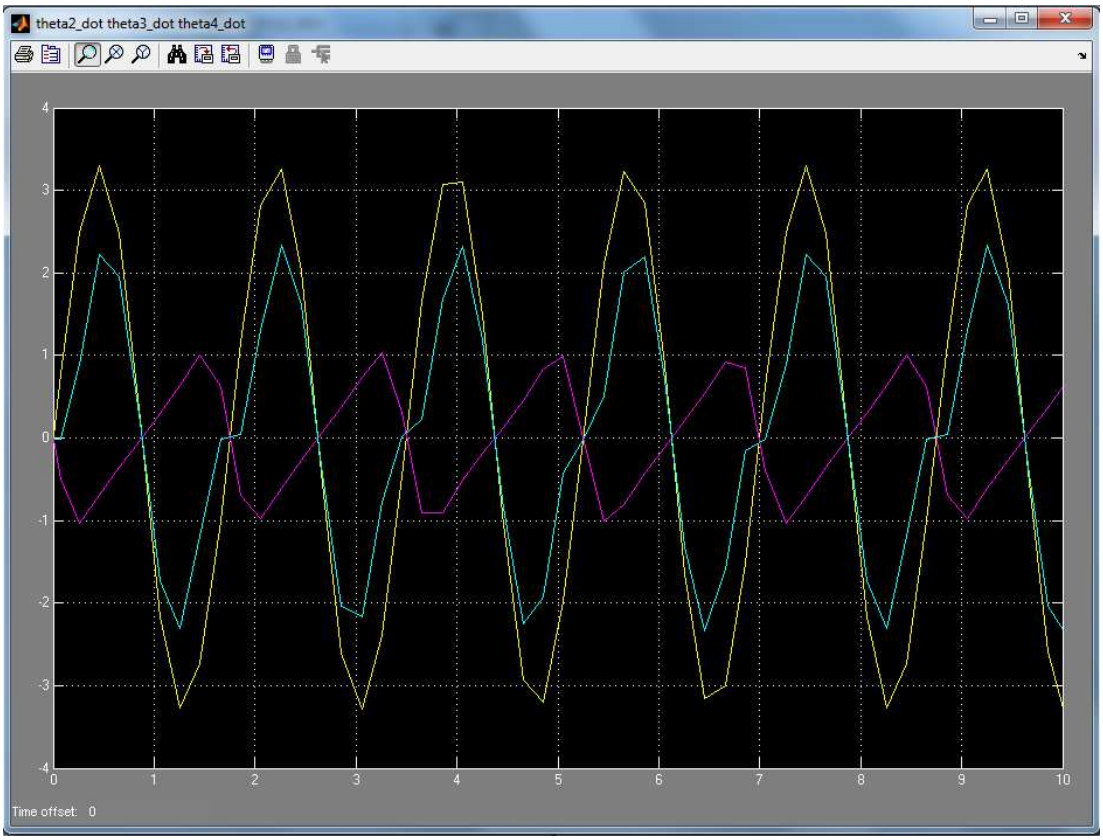

Figure 4.24 Comparison of the angular velocity

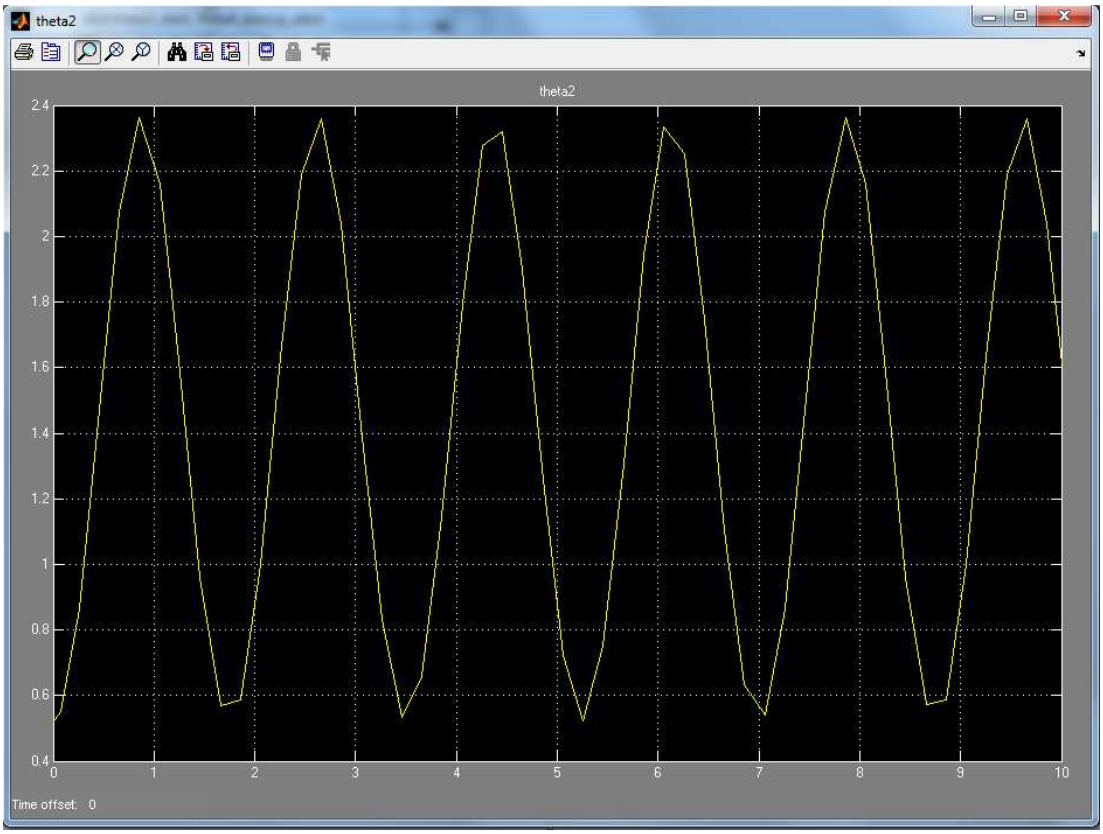

Figure 4.25 Crank angle time response

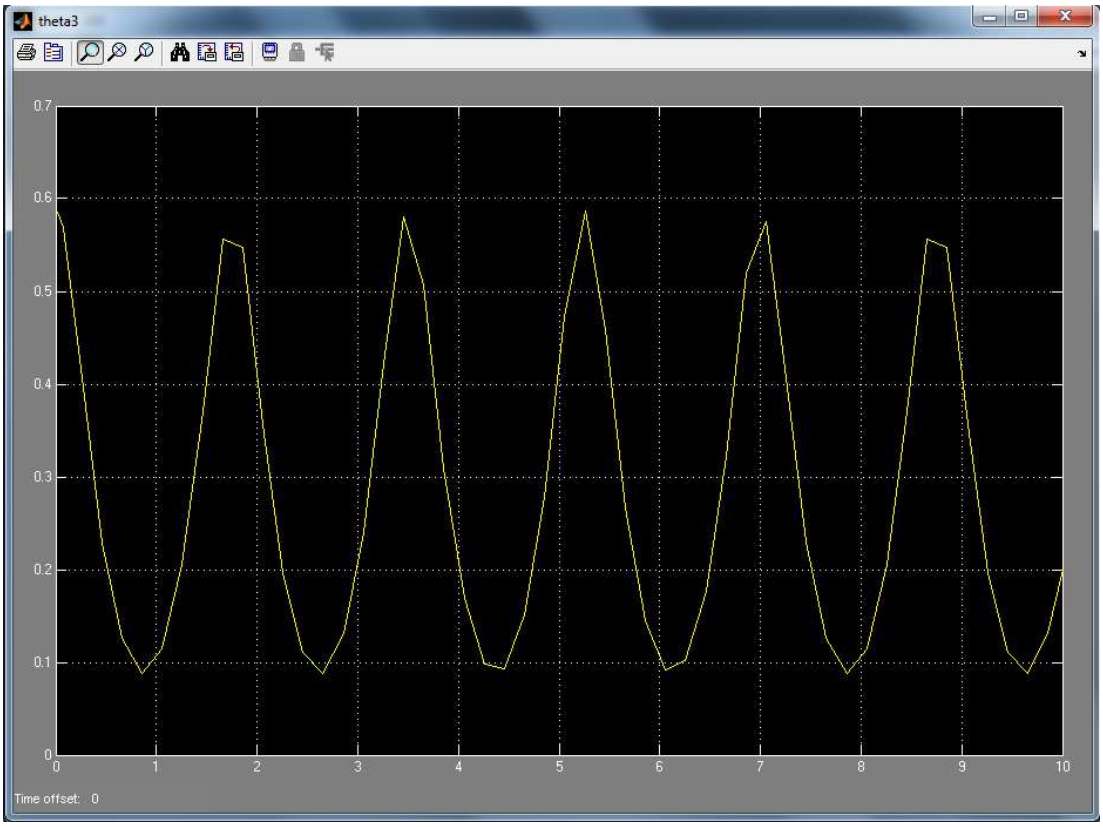

Figure 4.26 Coupler angle time response

Figure 4.27 Output angle time response

4.3 Dynamic Model of Flexible Coupler - Output Link Four-Bar Mechanism

4.3.1 Dynamic equations of motion with Lagrange's equation

The kinetic energy of the four-bar mechanism shown in Figure 4.3 is given by:

$$T = I_C\dot{\theta}_2^2 + \frac{1}{2}I_O\dot{\theta}_4^2 + \frac{1}{2}\int_0^{l_3}\rho\dot{\vec{r}}^{\,2}dx + \frac{1}{2}I_3\dot{\theta}_3^2 + \frac{1}{2}\int_0^{l_4}\rho\dot{\vec{R}}^{2}dy \qquad \textbf{(4.67)}$$

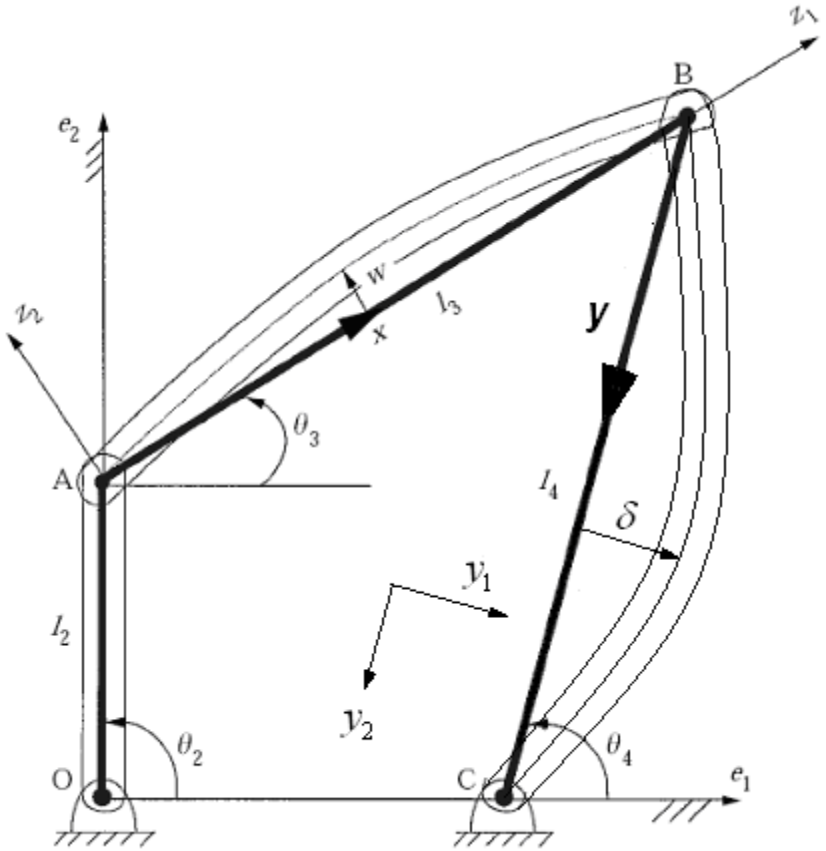

Figure 4.28 Sketch of the four-bar mechanism with elastic couler - output link

$\vec{r}$ is the position vector of a point on the coupler link given as

$$\vec{r} = \vec{r}_2 + \vec{r}_3$$

$$\vec{r}_2 = l_2 \cos\theta_2 \vec{e}_1 + l_2 \sin\theta_2 \vec{e}_2$$

$$\vec{r}_3 = x \cos\theta_3 \vec{e}_1 + x \sin\theta_3 \vec{e}_2$$

$$\vec{r} = l_2 \cos\theta_2 \vec{e}_1 + l_2 \sin\theta_2 \vec{e}_2 + x \cos\theta_3 \vec{e}_1 + x \sin\theta_3 \vec{e}_2$$

$$\vec{r} = \left(l_2 \cos\theta_2 + x \cos\theta_3\right)\vec{e}_1 + \left(l_2 \sin\theta_2 + x \sin\theta_3\right)\vec{e}_2$$

It's derivative will be

$$\dot{\vec{r}} = \left(-l_2\dot{\theta}_2 \sin\theta_2 - x\dot{\theta}_3 \sin\theta_3\right)\vec{e}_1 + \left(l_2\dot{\theta}_2 \cos\theta_2 + x\dot{\theta}_3 \cos\theta_3\right)\vec{e}_2 \qquad \textbf{(4.68)}$$

For $\dot{\vec{r}}^2$ it should be more convenient to work with **(4.68)** divided into parts:

$$\left(-l_2\dot{\theta}_2 \sin\theta_2 - x\dot{\theta}_3 \sin\theta_3\right)^2 = \left(l_2\dot{\theta}_2 \sin\theta_2\right)^2 + \left(x\dot{\theta}_3 \sin\theta_3\right)^2 + 2l_2 x\dot{\theta}_2\dot{\theta}_3 \sin\theta_2 \sin\theta_3 \qquad \textbf{(4.69)}$$

$$\left(l_2\dot{\theta}_2 \cos\theta_2 + x\dot{\theta}_3 \cos\theta_3\right)^2 = \left(l_2\dot{\theta}_2 \cos\theta_2\right)^2 + \left(x\dot{\theta}_3 \cos\theta_3\right)^2 + 2l_2 x\dot{\theta}_2\dot{\theta}_3 \cos\theta_2 \cos\theta_3 \qquad \textbf{(4.70)}$$

Combining equation **(4.69)** and **(4.70)** we get

$$\dot{\vec{r}}^2 = \left(l_2\dot{\theta}_2\right)^2 + \left(x\dot{\theta}_3\right)^2 + 2l_2x\dot{\theta}_2\dot{\theta}_3\cos\left(\theta_2-\theta_3\right)$$

Solving for $\frac{1}{2}\int_0^{l_3}\rho\dot{\vec{r}}^2dx$ in **(4.67)**

$$\begin{aligned}\frac{1}{2}\int_0^{l_3}\rho\dot{\vec{r}}^2dx &= \frac{1}{2}\int_0^{l_3}\rho\left(l_2\dot{\theta}_2\right)^2dx + \frac{1}{2}\int_0^{l_3}\rho\left(x\dot{\theta}_3\right)^2dx + \frac{1}{2}\int_0^{l_3}\rho 2l_2x\dot{\theta}_2\dot{\theta}_3\cos\left(\theta_2-\theta_3\right)dx \\ &= \frac{1}{2}\rho\left(l_2\dot{\theta}_2\right)^2 l_3 + \frac{1}{6}\rho l_3{}^3\dot{\theta}_3{}^2 + \frac{1}{2}\rho l_2l_3{}^2\dot{\theta}_2\dot{\theta}_3\cos\left(\theta_2-\theta_3\right)\end{aligned} \quad \textbf{(4.71)}$$

$\vec{R}$ is the position vector of a point on the output link given as:

$$\vec{R} = \vec{r}_2 + \vec{r}_3' + \vec{r}_4 + \vec{\delta}$$

where

$$\vec{r}_2 = l_2\cos\theta_2\vec{e}_1 + l_2\sin\theta_2\vec{e}_2$$
$$\vec{r}_3' = l_3\cos\theta_3\vec{e}_1 + l_3\sin\theta_3\vec{e}_2$$
$$\vec{r}_4 = -y\cos\theta_4\vec{e}_1 - y\sin\theta_4\vec{e}_2$$
$$\vec{\delta} = \delta\vec{y}_1$$

where δ represents the elastic deformation of the output link at an arbitrary position y. The unit vectors $\vec{y}_1$ and $\vec{y}_2$ can be written as

$$\vec{y}_1 = \sin\theta_4\vec{e}_1 - \cos\theta_4\vec{e}_2$$

$$\vec{y}_2 = \cos\theta_4\vec{e}_1 + \sin\theta_4\vec{e}_2$$

Then the final position vector will be

$$\vec{R} = \left(l_2\cos\theta_2 + l_3\cos\theta_3 - y\cos\theta_4\right)\vec{e}_1 + \left(l_2\sin\theta_2 + l_3\sin\theta_3 - y\sin\theta_4\right)\vec{e}_2 + \delta\left(\sin\theta_4\vec{e}_1 - \cos\theta_4\vec{e}_2\right)$$

$$\begin{aligned}\vec{R} &= \left(l_2\cos\theta_2 + l_3\cos\theta_3 - y\cos\theta_4 + \delta\sin\theta_4\right)\vec{e}_1 \\ &\quad + \left(l_2\sin\theta_2 + l_3\sin\theta_3 - y\sin\theta_4 - \delta\cos\theta_4\right)\vec{e}_2\end{aligned} \quad \textbf{(4.72)}$$

The assumed modes method along with a constrained Lagrangian approach is used to obtain the discretized equations of motion. Let the deflection of the output link be written as

$$\delta(y,t) = \sum_{i=0}^{N}\phi_i(y)q_i(t) \quad \textbf{(4.73)}$$

where $q_i(t)$ are the modal co-ordinates and $\phi_i(y)$ are the mode shapes of a pinned-pinned beam given as

$$\phi_i(y) = \sin\left(\frac{i\pi}{l_4} y\right)$$

After inserting information of equation **(4.73)** into the equation **(4.72)** equation and taking its time derivative we get

$$\dot{\vec{R}} = \left(-l_2\dot{\theta}_2 \sin\theta_2 - l_3\dot{\theta}_3 \sin\theta_3 + y\dot{\theta}_4 \sin\theta_4 + \sin\theta_4 \sum_{i=0}^{N} \phi_i(y)\dot{q}_i(t) + \dot{\theta}_4 \cos\theta_4 \sum_{i=0}^{N} \phi_i(y) q_i(t)\right)\vec{e}_1$$
$$+\left(l_2\dot{\theta}_2 \cos\theta_2 + l_3\dot{\theta}_3 \cos\theta_3 - y\dot{\theta}_4 \cos\theta_4 - \cos\theta_4 \sum_{i=0}^{N} \phi_i(y)\dot{q}_i(t) + \dot{\theta}_4 \sin\theta_4 \sum_{i=0}^{N} \phi_i(y) q_i(t)\right)\vec{e}_2 \quad \textbf{(4.74)}$$

For $\dot{\vec{R}}^2$ it should be more convenient to work with equation (4.40) divided into parts:

$$\left(-l_2\dot{\theta}_2 \sin\theta_2 - l_3\dot{\theta}_3 \sin\theta_3 + y\dot{\theta}_4 \sin\theta_4 + \sin\theta_4 \sum_{i=0}^{N} \phi_i(y)\dot{q}_i(t) + \dot{\theta}_4 \cos\theta_4 \sum_{i=0}^{N} \phi_i(y) q_i(t)\right)^2 =$$
$$\left(l_2\dot{\theta}_2 \sin\theta_2\right)^2 + \left(l_3\dot{\theta}_3 \sin\theta_3\right)^2 + \left(y\dot{\theta}_4 \sin\theta_4\right)^2 + \left(\sin\theta_4 \sum_{i=0}^{N} \phi_i(y)\dot{q}_i(t)\right)^2 + \left(\dot{\theta}_4 \cos\theta_4 \sum_{i=0}^{N} \phi_i(y) q_i(t)\right)^2$$
$$+2l_2 l_3 \dot{\theta}_2\dot{\theta}_3 \sin\theta_2 \sin\theta_3 - 2l_2 y\dot{\theta}_2\dot{\theta}_4 \sin\theta_2 \sin\theta_4 - 2l_2\dot{\theta}_2 \sin\theta_2 \sin\theta_4 \sum_{i=0}^{N} \phi_i(y)\dot{q}_i(t)$$
$$-2l_2\dot{\theta}_2\dot{\theta}_4 \sin\theta_2 \cos\theta_4 \sum_{i=0}^{N} \phi_i(y) q_i(t) - 2l_3 y\dot{\theta}_3\dot{\theta}_4 \sin\theta_3 \sin\theta_4 - 2l_3\dot{\theta}_3 \sin\theta_3 \sin\theta_4 \sum_{i=0}^{N} \phi_i(y)\dot{q}_i(t)$$
$$-2l_3\dot{\theta}_3\dot{\theta}_4 \sin\theta_3 \cos\theta_4 \sum_{i=0}^{N} \phi_i(y) q_i(t) + 2y\dot{\theta}_4 \sin\theta_4 \sin\theta_4 \sum_{i=0}^{N} \phi_i(y)\dot{q}_i(t)$$
$$+2y\dot{\theta}_4\dot{\theta}_4 \sin\theta_4 \cos\theta_4 \sum_{i=0}^{N} \phi_i(y) q_i(t) + 2\dot{\theta}_4 \sin\theta_4 \cos\theta_4 \sum_{i=0}^{N} \phi_i(y)\dot{q}_i(t) \sum_{i=0}^{N} \phi_i(y) q_i(t)$$

$$\left(l_2\dot{\theta}_2 \cos\theta_2 + l_3\dot{\theta}_3 \cos\theta_3 - y\dot{\theta}_4 \cos\theta_4 - \cos\theta_4 \sum_{i=0}^{N} \phi_i(y)\dot{q}_i(t) + \dot{\theta}_4 \sin\theta_4 \sum_{i=0}^{N} \phi_i(y) q_i(t)\right)^2 =$$
$$\left(l_2\dot{\theta}_2 \cos\theta_2\right)^2 + \left(l_3\dot{\theta}_3 \cos\theta_3\right)^2 + \left(y\dot{\theta}_4 \cos\theta_4\right)^2 + \left(\cos\theta_4 \sum_{i=0}^{N} \phi_i(y)\dot{q}_i(t)\right)^2 + \left(\dot{\theta}_4 \sin\theta_4 \sum_{i=0}^{N} \phi_i(y) q_i(t)\right)^2$$
$$+2l_2 l_3 \dot{\theta}_2\dot{\theta}_3 \cos\theta_2 \cos\theta_3 - 2l_2 y\dot{\theta}_2\dot{\theta}_4 \cos\theta_2 \cos\theta_4 - 2l_2\dot{\theta}_2 \cos\theta_2 \cos\theta_4 \sum_{i=0}^{N} \phi_i(y)\dot{q}_i(t)$$
$$+2l_2\dot{\theta}_2\dot{\theta}_4 \cos\theta_2 \sin\theta_4 \sum_{i=0}^{N} \phi_i(y) q_i(t) - 2l_3 y\dot{\theta}_3\dot{\theta}_4 \cos\theta_3 \cos\theta_4 - 2l_3\dot{\theta}_3 \cos\theta_3 \cos\theta_4 \sum_{i=0}^{N} \phi_i(y)\dot{q}_i(t)$$
$$+2l_3\dot{\theta}_3\dot{\theta}_4 \cos\theta_3 \sin\theta_4 \sum_{i=0}^{N} \phi_i(y) q_i(t) + 2y\dot{\theta}_4 \cos\theta_4 \cos\theta_4 \sum_{i=0}^{N} \phi_i(y)\dot{q}_i(t)$$
$$-2y\dot{\theta}_4\dot{\theta}_4 \cos\theta_4 \sin\theta_4 \sum_{i=0}^{N} \phi_i(y) q_i(t) - 2\dot{\theta}_4 \sin\theta_4 \cos\theta_4 \sum_{i=0}^{N} \phi_i(y)\dot{q}_i(t) \sum_{i=0}^{N} \phi_i(y) q_i(t)$$

Combining two above equations will give us:

$$\dot{\vec{R}}^2 = \left(l_2\dot{\theta}_2\right)^2 + \left(l_3\dot{\theta}_3\right)^2 + \left(y\dot{\theta}_4\right)^2 + \left(\sum_{i=0}^{N}\phi_i(y)\dot{q}_i(t)\right)^2 + \left(\dot{\theta}_4\sum_{i=0}^{N}\phi_i(y)q_i(t)\right)^2$$
$$+2l_2l_3\dot{\theta}_2\dot{\theta}_3\cos\left(\theta_2-\theta_3\right) - 2l_2y\dot{\theta}_2\dot{\theta}_4\cos\left(\theta_2-\theta_4\right) - 2l_2\dot{\theta}_2\cos\left(\theta_2-\theta_4\right)\sum_{i=0}^{N}\phi_i(y)\dot{q}_i(t)$$
$$-2l_2\dot{\theta}_2\dot{\theta}_4\sin\left(\theta_2-\theta_4\right)\sum_{i=0}^{N}\phi_i(y)q_i(t) - 2l_3y\dot{\theta}_3\dot{\theta}_4\cos\left(\theta_3-\theta_4\right)$$
$$-2l_3\dot{\theta}_3\cos\left(\theta_3-\theta_4\right)\sum_{i=0}^{N}\phi_i(y)\dot{q}_i(t) - 2l_3\dot{\theta}_3\dot{\theta}_4\sin\left(\theta_3-\theta_4\right)\sum_{i=0}^{N}\phi_i(y)q_i(t) + 2y\dot{\theta}_4\sum_{i=0}^{N}\phi_i(y)\dot{q}_i(t)$$

Solving for $\frac{1}{2}\int_0^{l_4}\rho\dot{\vec{R}}^2dy$ in **(4.67)**

$$\frac{1}{2}\int_0^{l_4}\rho\dot{\vec{R}}^2dy = \frac{1}{2}\int_0^{l_4}\rho\left(l_2\dot{\theta}_2\right)^2dy + \frac{1}{2}\int_0^{l_4}\rho\left(l_3\dot{\theta}_3\right)^2dy + \frac{1}{2}\int_0^{l_4}\rho\left(y\dot{\theta}_4\right)^2dy + \frac{1}{2}\int_0^{l_4}\rho\left(\sum_{i=0}^{N}\phi_i(y)\dot{q}_i(t)\right)^2dy$$
$$+\frac{1}{2}\int_0^{l_4}\rho\left(\dot{\theta}_4\sum_{i=0}^{N}\phi_i(y)q_i(t)\right)^2dy + \int_0^{l_4}\rho l_2l_3\dot{\theta}_2\dot{\theta}_3\cos\left(\theta_2-\theta_3\right)dy$$
$$-\int_0^{l_4}\rho l_2y\dot{\theta}_2\dot{\theta}_4\cos\left(\theta_2-\theta_4\right)dy - \int_0^{l_4}\rho l_2\dot{\theta}_2\cos\left(\theta_2-\theta_4\right)\sum_{i=0}^{N}\phi_i(y)\dot{q}_i(t)dy$$
$$-\int_0^{l_4}\rho l_2\dot{\theta}_2\dot{\theta}_4\sin\left(\theta_2-\theta_4\right)\sum_{i=0}^{N}\phi_i(y)q_i(t)dy - \int_0^{l_4}\rho l_3y\dot{\theta}_3\dot{\theta}_4\cos\left(\theta_3-\theta_4\right)dy$$
$$-\int_0^{l_4}\rho l_3\dot{\theta}_3\cos\left(\theta_3-\theta_4\right)\sum_{i=0}^{N}\phi_i(y)\dot{q}_i(t)dy - \int_0^{l_4}\rho l_3\dot{\theta}_3\dot{\theta}_4\sin\left(\theta_3-\theta_4\right)\sum_{i=0}^{N}\phi_i(y)q_i(t)dy + \int_0^{l_4}\rho y\dot{\theta}_4\sum_{i=0}^{N}\phi_i(y)\dot{q}_i(t)dy$$

$$\frac{1}{2}\int_0^{l_4}\rho\dot{\vec{R}}^2dy = \frac{1}{2}\rho\left(l_2\dot{\theta}_2\right)^2l_4 + \frac{1}{2}\rho\left(l_3\dot{\theta}_3\right)^2l_4 + \frac{1}{6}{l_4}^3{\dot{\theta}_4}^2 + \frac{1}{2}\int_0^{l_4}\rho\left(\sum_{i=0}^{N}\phi_i(y)\dot{q}_i(t)\right)^2dy + \frac{1}{2}{\dot{\theta}_4}^2\int_0^{l_4}\rho\left(\sum_{i=0}^{N}\phi_i(y)q_i(t)\right)^2dy$$
$$+\rho l_2l_3l_4\dot{\theta}_2\dot{\theta}_3\cos\left(\theta_2-\theta_3\right) - \frac{1}{2}\rho l_2{l_4}^2\dot{\theta}_2\dot{\theta}_4\cos\left(\theta_2-\theta_4\right) - l_2\dot{\theta}_2\cos\left(\theta_2-\theta_4\right)\int_0^{l_4}\rho\sum_{i=0}^{N}\phi_i(y)\dot{q}_i(t)dy$$
$$-l_2\dot{\theta}_2\dot{\theta}_4\sin\left(\theta_2-\theta_4\right)\int_0^{l_4}\rho\sum_{i=0}^{N}\phi_i(y)q_i(t)dy - \frac{1}{2}\rho l_3{l_4}^2\dot{\theta}_3\dot{\theta}_4\cos\left(\theta_3-\theta_4\right)$$
$$-l_3\dot{\theta}_3\cos\left(\theta_3-\theta_4\right)\int_0^{l_4}\rho\sum_{i=0}^{N}\phi_i(y)\dot{q}_i(t)dy - l_3\dot{\theta}_3\dot{\theta}_4\sin\left(\theta_3-\theta_4\right)\int_0^{l_4}\rho\sum_{i=0}^{N}\phi_i(y)q_i(t)dy + \dot{\theta}_4\int_0^{l_4}\rho y\sum_{i=0}^{N}\phi_i(y)\dot{q}_i(t)dy$$

The constrained Lagrange's equations are

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\eta}_i}\right) - \frac{\partial L}{\partial\eta_i} + \left(\frac{\partial\vec{\Phi}}{\partial\eta_i}\right)^T\Lambda = F_i$$

Where $L = T - U$, or

$$L = \frac{1}{2} I_C \dot{\theta}_2^2 + \frac{1}{2}\rho\left(l_2\dot{\theta}_2\right)^2 l_3 + \frac{1}{6}\rho {l_3}^3 \dot{\theta}_3^{\,2} + \frac{1}{2}\int_0^{l_3} \rho\left(\sum_{i=0}^{N}\psi_i(x)\dot{q}_i(t)\right)^2 dx$$

$$+\frac{1}{2}\dot{\theta}_3^{\,2}\int_0^{l_3} \rho\left(\sum_{i=0}^{N}\psi_i(x) q_i(t)\right)^2 dx + \frac{1}{2}\rho l_2 {l_3}^2 \dot{\theta}_2\dot{\theta}_3\cos\left(\theta_2-\theta_3\right)$$

$$+l_2\dot{\theta}_2\cos\left(\theta_2-\theta_3\right)\int_0^{l_3}\rho\left(\sum_{i=0}^{N}\psi_i(x)\dot{q}_i(t)\right)dx + l_2\dot{\theta}_2\dot{\theta}_3\sin\left(\theta_2-\theta_3\right)\int_0^{l_3}\rho\left(\sum_{i=0}^{N}\psi_i(x) q_i(t)\right)dx$$

$$+\dot{\theta}_3\int_0^{l_3}\rho x\left(\sum_{i=0}^{N}\psi_i(x)\dot{q}_i(t)\right)dx + \frac{1}{2}\rho\left(l_2\dot{\theta}_2\right)^2 l_4 + \frac{1}{2}\rho\left(l_3\dot{\theta}_3\right)^2 l_4$$

$$+\frac{1}{6}{l_4}^3\dot{\theta}_4^{\,2} + \frac{1}{2}\int_0^{l_4}\rho\left(\sum_{i=0}^{N}\phi_i(y)\dot{q}_i(t)\right)^2 dy + \frac{1}{2}\dot{\theta}_4^{\,2}\int_0^{l_4}\rho\left(\sum_{i=0}^{N}\phi_i(y) q_i(t)\right)^2 dy$$

$$+\rho l_2 l_3 l_4\dot{\theta}_2\dot{\theta}_3\cos\left(\theta_2-\theta_3\right) - \frac{1}{2}\rho l_2 {l_4}^2\dot{\theta}_2\dot{\theta}_4\cos\left(\theta_2-\theta_4\right)$$

$$-l_2\dot{\theta}_2\cos\left(\theta_2-\theta_4\right)\int_0^{l_4}\rho\sum_{i=0}^{N}\phi_i(y)\dot{q}_i(t)dy - l_2\dot{\theta}_2\dot{\theta}_4\sin\left(\theta_2-\theta_4\right)\int_0^{l_4}\rho\sum_{i=0}^{N}\phi_i(y) q_i(t)dy$$

$$-\frac{1}{2}\rho l_3 {l_4}^2\dot{\theta}_3\dot{\theta}_4\cos\left(\theta_3-\theta_4\right) - l_3\dot{\theta}_3\cos\left(\theta_3-\theta_4\right)\int_0^{l_4}\rho\sum_{i=0}^{N}\phi_i(y)\dot{q}_i(t)dy$$

$$-l_3\dot{\theta}_3\dot{\theta}_4\sin\left(\theta_3-\theta_4\right)\int_0^{l_4}\rho\sum_{i=0}^{N}\phi_i(y) q_i(t)dy + \dot{\theta}_4\int_0^{l_4}\rho y\sum_{i=0}^{N}\phi_i(y)\dot{q}_i(t)dy$$

$$-\frac{1}{2}\int_0^{l_3} EI_3\left(\frac{\partial^2 w}{\partial x^2}\right)^2 dx - \frac{1}{2}\int_0^{l_4} EI_4\left(\frac{\partial^2 \delta}{\partial y^2}\right)^2 dy \qquad \textbf{(4.75)}$$

and η_i are the generalized coordinates which includes rigid body coordinates as well as elastic modal coordinates. $\vec{\Phi}$ is the vector containing the constraint equations and $\vec{\Lambda}$ is the vector of Lagrange multipliers λ_1 and λ_2; and F_i are the generalized forces. The force vector $\left(\partial\vec{\Phi}/\partial\eta_i\right)^T\Lambda$ is substituted in to replace the removed joint forces.

$$\left(\frac{\partial\Phi}{\partial\eta_i}\right)^T\Lambda = \left(\frac{\partial\begin{bmatrix} -l_1 + l_2\cos\theta_2 + l_3\cos\theta_3 - l_4\cos\theta_4 \\ l_2\sin\theta_2 + l_3\sin\theta_3 - l_4\sin\theta_4 \end{bmatrix}}{\partial\eta_i}\right)^T\begin{bmatrix}\lambda_1\\ \lambda_2\end{bmatrix} \qquad \textbf{(4.76)}$$

4.3.1.1 Dynamic equation of motion with Lagrange's equation for θ_2

For generalized coordinate θ_2 **(4.39)** and **(4.61)** are effective for the mechanism with flexible coupler and output links:

$$I_C\ddot{\theta}_2+\rho l_2{}^2 l_3\ddot{\theta}_2+\frac{1}{2}\rho l_2 l_3{}^2\ddot{\theta}_3\cos(\theta_2-\theta_3)-(\dot{\theta}_2-\dot{\theta}_3)\frac{1}{2}\rho l_2 l_3{}^2\dot{\theta}_3\sin(\theta_2-\theta_3)$$
$$-l_2(\dot{\theta}_2-\dot{\theta}_3)\sin(\theta_2-\theta_3)\int_0^{l_3}\rho\left(\sum_{i=0}^{N}\psi_i(x)\dot{q}_i(t)\right)dx+l_2\cos(\theta_2-\theta_3)\int_0^{l_3}\rho\left(\sum_{i=0}^{N}\psi_i(x)\ddot{q}_i(t)\right)dx$$
$$+l_2\ddot{\theta}_3\sin(\theta_2-\theta_3)\int_0^{l_3}\rho\left(\sum_{i=0}^{N}\psi_i(x)q_i(t)\right)dx+l_2\dot{\theta}_3(\dot{\theta}_2-\dot{\theta}_3)\cos(\theta_2-\theta_3)\int_0^{l_3}\rho\left(\sum_{i=0}^{N}\psi_i(x)q_i(t)\right)dx$$
$$+l_2\dot{\theta}_3\sin(\theta_2-\theta_3)\int_0^{l_3}\rho\left(\sum_{i=0}^{N}\psi_i(x)\dot{q}_i(t)\right)dx+\frac{1}{2}\rho l_2 l_3{}^2\dot{\theta}_2\dot{\theta}_3\sin(\theta_2-\theta_3)$$
$$+l_2\dot{\theta}_2\sin(\theta_2-\theta_3)\int_0^{l_3}\rho\left(\sum_{i=0}^{N}\psi_i(x)\dot{q}_i(t)\right)dx-l_2\dot{\theta}_2\dot{\theta}_3\cos(\theta_2-\theta_3)\int_0^{l_3}\rho\left(\sum_{i=0}^{N}\psi_i(x)q_i(t)\right)dx$$
$$-l_2\sin\theta_2\lambda_1+l_2\cos\theta_2\lambda_2=\tau$$

$$I_C\ddot{\theta}_2+\rho l_3 l_2{}^2\ddot{\theta}_2+\frac{1}{2}\rho l_2 l_3{}^2\ddot{\theta}_3\cos(\theta_2-\theta_3)-\frac{1}{2}\rho l_2 l_3{}^2\dot{\theta}_3(\dot{\theta}_2-\dot{\theta}_3)\sin(\theta_2-\theta_3)+\rho l_4 l_2{}^2\ddot{\theta}_2$$
$$+\rho l_2 l_3 l_4\ddot{\theta}_3\cos(\theta_2-\theta_3)-\rho l_2 l_3 l_4\dot{\theta}_3(\dot{\theta}_2-\dot{\theta}_3)\sin(\theta_2-\theta_3)$$
$$-\frac{1}{2}\rho l_2 l_4{}^2\ddot{\theta}_4\cos(\theta_2-\theta_4)+\frac{1}{2}\rho l_2 l_4{}^2\dot{\theta}_4(\dot{\theta}_2-\dot{\theta}_4)\sin(\theta_2-\theta_4)$$
$$+l_2(\dot{\theta}_2-\dot{\theta}_4)\sin(\theta_2-\theta_4)\int_0^{l_4}\rho\sum_{i=0}^{N}\phi_i(y)\dot{q}_i(t)dy-l_2\cos(\theta_2-\theta_4)\int_0^{l_4}\rho\sum_{i=0}^{N}\phi_i(y)\ddot{q}_i(t)dy$$
$$-l_2\ddot{\theta}_2\dot{\theta}_4\sin(\theta_2-\theta_4)\int_0^{l_4}\rho\sum_{i=0}^{N}\phi_i(y)q_i(t)dy-l_2\dot{\theta}_2\ddot{\theta}_4\sin(\theta_2-\theta_4)\int_0^{l_4}\rho\sum_{i=0}^{N}\phi_i(y)q_i(t)dy$$
$$-l_2\dot{\theta}_2\dot{\theta}_4(\dot{\theta}_2-\dot{\theta}_4)\cos(\theta_2-\theta_4)\int_0^{l_4}\rho\sum_{i=0}^{N}\phi_i(y)q_i(t)dy-l_2\dot{\theta}_2\dot{\theta}_4\sin(\theta_2-\theta_4)\int_0^{l_4}\rho\sum_{i=0}^{N}\phi_i(y)\dot{q}_i(t)dy$$
$$+\frac{1}{2}\rho l_2 l_3{}^2\dot{\theta}_2\dot{\theta}_3(\dot{\theta}_2-\dot{\theta}_3)\sin(\theta_2-\theta_3)+\rho l_2 l_3 l_4\dot{\theta}_2\dot{\theta}_3(\dot{\theta}_2-\dot{\theta}_3)\sin(\theta_2-\theta_3)$$
$$-\frac{1}{2}\rho l_2 l_4{}^2\dot{\theta}_2\dot{\theta}_4(\dot{\theta}_2-\dot{\theta}_4)\sin(\theta_2-\theta_4)-l_2\dot{\theta}_2(\dot{\theta}_2-\dot{\theta}_4)\sin(\theta_2-\theta_4)\int_0^{l_4}\rho\sum_{i=0}^{N}\phi_i(y)\dot{q}_i(t)dy$$
$$+l_2\dot{\theta}_2\dot{\theta}_4(\dot{\theta}_2-\dot{\theta}_4)\cos(\theta_2-\theta_4)\int_0^{l_4}\rho\sum_{i=0}^{N}\phi_i(y)q_i(t)dy-l_2\sin\theta_2\lambda_1+l_2\cos\theta_2\lambda_2=\tau$$

4.3.1.2 Dynamic equation of motion with Lagrange's equation for θ_3

For generalized coordinate θ_3 **(4.51)** and **(4.63)** are effective for the mechanism with flexible coupler and output links:

$$\frac{1}{3}\rho l_3^3\ddot{\theta}_3+\ddot{\theta}_3\int_0^{l_3}\rho\left(\sum_{i=0}^{N}\psi_i(x)q_i(t)\right)^2dx+2\dot{\theta}_3\int_0^{l_3}\rho\sum_{i=0}^{N}\sum_{i=0}^{N}\psi_i(x)\psi_j(x)q_i(t)\dot{q}_j(t)dx$$
$$+\frac{1}{2}\rho l_2 l_3^2\ddot{\theta}_2\cos(\theta_2-\theta_3)-\frac{1}{2}\rho l_2 l_3^2\dot{\theta}_2(\dot{\theta}_2-\dot{\theta}_3)\sin(\theta_2-\theta_3)$$
$$+l_2\ddot{\theta}_2\sin(\theta_2-\theta_3)\int_0^{l_3}\rho\left(\sum_{i=0}^{N}\psi_i(x)q_i(t)\right)dx+l_2\dot{\theta}_2\sin(\theta_2-\theta_3)\int_0^{l_3}\rho\left(\sum_{i=0}^{N}\psi_i(x)\dot{q}_i(t)\right)dx$$
$$+l_2\dot{\theta}_2(\dot{\theta}_2-\dot{\theta}_3)\cos(\theta_2-\theta_3)\int_0^{l_3}\rho\left(\sum_{i=0}^{N}\psi_i(x)q_i(t)\right)dx+\int_0^{l_3}\rho x\left(\sum_{i=0}^{N}\psi_i(x)\ddot{q}_i(t)\right)dx$$
$$-\frac{1}{2}\rho l_2 l_3^2\dot{\theta}_2\dot{\theta}_3\sin(\theta_2-\theta_3)-l_2\dot{\theta}_2\sin(\theta_2-\theta_3)\int_0^{l_3}\rho\left(\sum_{i=0}^{N}\psi_i(x)\dot{q}_i(t)\right)dx$$
$$+l_2\dot{\theta}_2\dot{\theta}_3\cos(\theta_2-\theta_3)\int_0^{l_3}\rho\left(\sum_{i=0}^{N}\psi_i(x)q_i(t)\right)dx-l_3\sin\theta_3\lambda_1+l_3\cos\theta_3\lambda_2=0$$

$$\frac{1}{3}\rho l_3^3\ddot{\theta}_3+\frac{1}{2}\rho l_2 l_3^2\ddot{\theta}_2\cos(\theta_2-\theta_3)-\frac{1}{2}\rho l_2 l_3^2\dot{\theta}_2(\dot{\theta}_2-\dot{\theta}_3)\sin(\theta_2-\theta_3)+\rho l_3^2 l_4\ddot{\theta}_3$$
$$+\rho l_2 l_3 l_4\ddot{\theta}_2\cos(\theta_2-\theta_3)-\rho l_2 l_3 l_4\dot{\theta}_2(\dot{\theta}_2-\dot{\theta}_3)\sin(\theta_2-\theta_3)$$
$$-\frac{1}{2}\rho l_3 l_4^2\ddot{\theta}_4\cos(\theta_3-\theta_4)+\frac{1}{2}\rho l_3 l_4^2\dot{\theta}_4(\dot{\theta}_3-\dot{\theta}_4)\sin(\theta_3-\theta_4)$$
$$+l_3(\dot{\theta}_3-\dot{\theta}_4)\sin(\theta_3-\theta_4)\int_0^{l_4}\rho\sum_{i=0}^{N}\phi_i(y)\dot{q}_i(t)dy-l_3\cos(\theta_3-\theta_4)\int_0^{l_4}\rho\sum_{i=0}^{N}\phi_i(y)\ddot{q}_i(t)dy$$
$$-l_3\dot{\theta}_4(\dot{\theta}_3-\dot{\theta}_4)\cos(\theta_3-\theta_4)\int_0^{l_4}\rho\sum_{i=0}^{N}\phi_i(y)q_i(t)dy-l_3\dot{\theta}_4\sin(\theta_3-\theta_4)\int_0^{l_4}\rho\sum_{i=0}^{N}\phi_i(y)\dot{q}_i(t)dy$$
$$+\frac{1}{2}\rho l_2 l_3^2\dot{\theta}_2\dot{\theta}_3(\dot{\theta}_2-\dot{\theta}_3)\sin(\theta_2-\theta_3)+\rho l_2 l_3 l_4\dot{\theta}_2\dot{\theta}_3(\dot{\theta}_2-\dot{\theta}_3)\sin(\theta_2-\theta_3)$$
$$-\frac{1}{2}\rho l_3 l_4^2\dot{\theta}_3\dot{\theta}_4(\dot{\theta}_3-\dot{\theta}_4)\sin(\theta_3-\theta_4)-l_3\dot{\theta}_3(\dot{\theta}_3-\dot{\theta}_4)\sin(\theta_3-\theta_4)\int_0^{l_4}\rho\sum_{i=0}^{N}\phi_i(y)\dot{q}_i(t)dy$$
$$+l_3\dot{\theta}_3\dot{\theta}_4(\dot{\theta}_3-\dot{\theta}_4)\cos(\theta_3-\theta_4)\int_0^{l_4}\rho\sum_{i=0}^{N}\phi_i(y)q_i(t)dy-l_3\sin\theta_3\lambda_1+l_3\cos\theta_3\lambda_2=0$$

4.3.1.3 Dynamic equation of motion with Lagrange's equation for θ_4

For generalized coordinate θ_4 **(4.52)** and **(4.65)** are effective for the mechanism with flexible coupler and output links:

$$I_O\ddot{\theta}_4 + l_4\sin\theta_4\lambda_1 - l_4\cos\theta_4\lambda_2 = 0$$

$$\frac{1}{3}{l_4}^3\ddot{\theta}_4 + \ddot{\theta}_4\int_0^{l_4}\rho\left(\sum_{i=0}^{N}\phi_i(y)q_i(t)\right)^2 dy + 2\dot{\theta}_4\int_0^{l_4}\rho\sum_{i=0}^{N}\dot{q}_i(t)\left(\sum_{i=0}^{N}\phi_i(y)q_i(t)\right)^2 dy$$

$$-\frac{1}{2}\rho l_2{l_4}^2\ddot{\theta}_2\cos(\theta_2-\theta_4) + \frac{1}{2}\rho l_2{l_4}^2\dot{\theta}_2\left(\dot{\theta}_2-\dot{\theta}_4\right)\sin(\theta_2-\theta_4)$$

$$-l_2\ddot{\theta}_2\sin(\theta_2-\theta_4)\int_0^{l_4}\rho\sum_{i=0}^{N}\phi_i(y)q_i(t)dy - l_2\dot{\theta}_2\left(\dot{\theta}_2-\dot{\theta}_4\right)\cos(\theta_2-\theta_4)\int_0^{l_4}\rho\sum_{i=0}^{N}\phi_i(y)q_i(t)dy$$

$$-l_2\dot{\theta}_2\sin(\theta_2-\theta_4)\int_0^{l_4}\rho\sum_{i=0}^{N}\phi_i(y)\dot{q}_i(t)dy - \frac{1}{2}\rho l_3{l_4}^2\ddot{\theta}_3\cos(\theta_3-\theta_4)$$

$$+\frac{1}{2}\rho l_3{l_4}^2\dot{\theta}_3\left(\dot{\theta}_3-\dot{\theta}_4\right)\sin(\theta_3-\theta_4) - l_3\ddot{\theta}_3\sin(\theta_3-\theta_4)\int_0^{l_4}\rho\sum_{i=0}^{N}\phi_i(y)q_i(t)dy$$

$$-l_3\dot{\theta}_3\left(\dot{\theta}_3-\dot{\theta}_4\right)\cos(\theta_3-\theta_4)\int_0^{l_4}\rho\sum_{i=0}^{N}\phi_i(y)q_i(t)dy - l_3\dot{\theta}_3\sin(\theta_3-\theta_4)\int_0^{l_4}\rho\sum_{i=0}^{N}\phi_i(y)\dot{q}_i(t)dy$$

$$+\int_0^{l_4}\rho y\sum_{i=0}^{N}\phi_i(y)\ddot{q}_i(t)dy - \frac{1}{2}\rho l_2{l_4}^2\dot{\theta}_2\dot{\theta}_4\left(\dot{\theta}_2-\dot{\theta}_4\right)\sin(\theta_2-\theta_4)$$

$$-l_2\dot{\theta}_2\left(\dot{\theta}_2-\dot{\theta}_4\right)\sin(\theta_2-\theta_4)\int_0^{l_4}\rho\sum_{i=0}^{N}\phi_i(y)\dot{q}_i(t)dy$$

$$+l_2\dot{\theta}_2\dot{\theta}_4\left(\dot{\theta}_2-\dot{\theta}_4\right)\cos(\theta_2-\theta_4)\int_0^{l_4}\rho\sum_{i=0}^{N}\phi_i(y)q_i(t)dy - \frac{1}{2}\rho l_3{l_4}^2\dot{\theta}_3\dot{\theta}_4\left(\dot{\theta}_3-\dot{\theta}_4\right)\sin(\theta_3-\theta_4)$$

$$-l_3\dot{\theta}_3\left(\dot{\theta}_3-\dot{\theta}_4\right)\sin(\theta_3-\theta_4)\int_0^{l_4}\rho\sum_{i=0}^{N}\phi_i(y)\dot{q}_i(t)dy$$

$$+l_3\dot{\theta}_3\dot{\theta}_4\left(\dot{\theta}_3-\dot{\theta}_4\right)\cos(\theta_3-\theta_4)\int_0^{l_4}\rho\sum_{i=0}^{N}\phi_i(y)q_i(t)dy + l_4\sin\theta_4\lambda_1 - l_4\cos\theta_4\lambda_2 = 0$$

4.3.2 Dynamic Analysis solutions for four-bar mechanism using Matlab Simulink

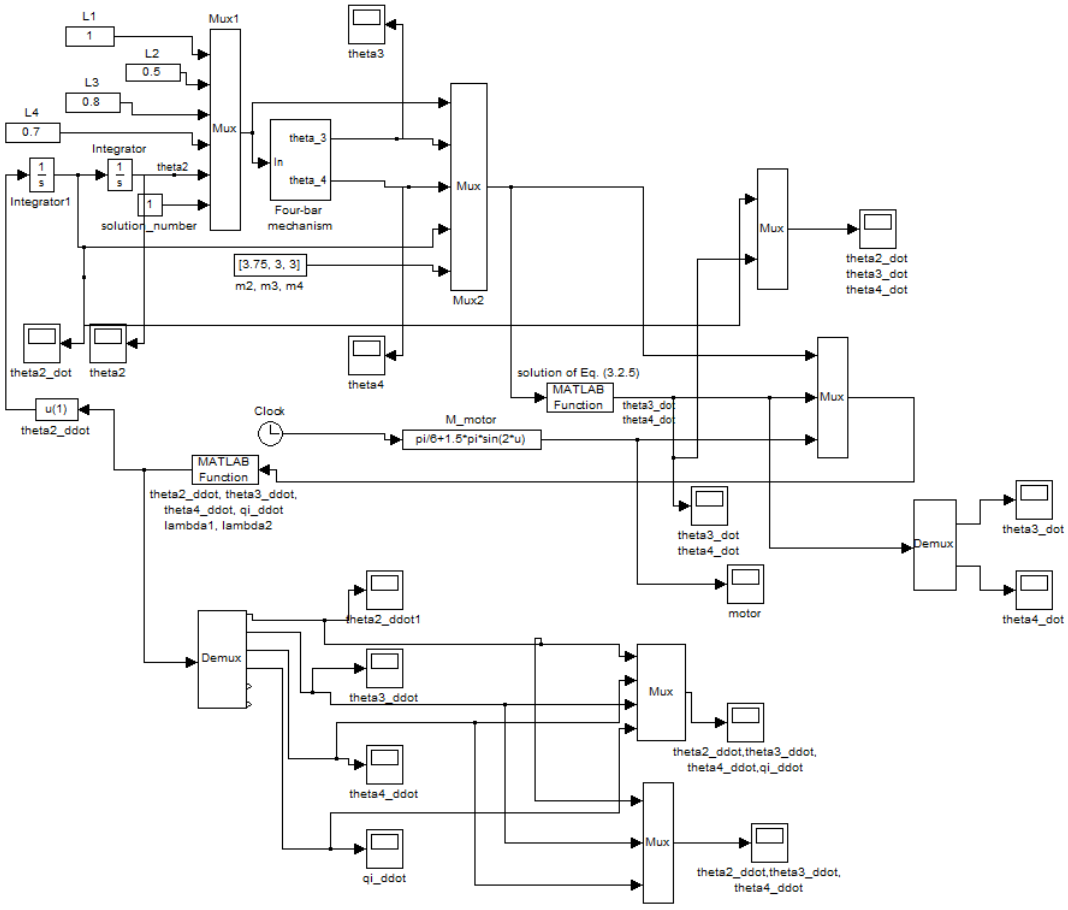

Figure 4.29 Simulink Block diagram of an elastic output link four bar mechanism

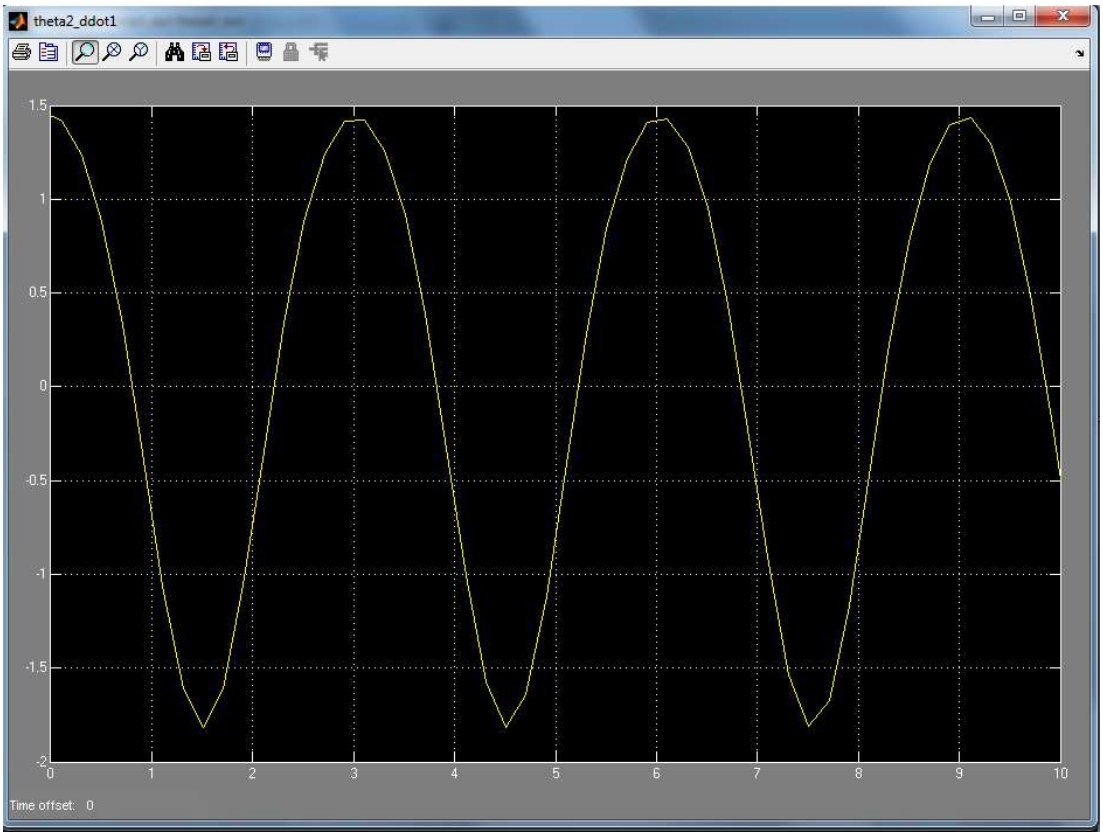

Figure 4.30 Crank angular acceleration time response

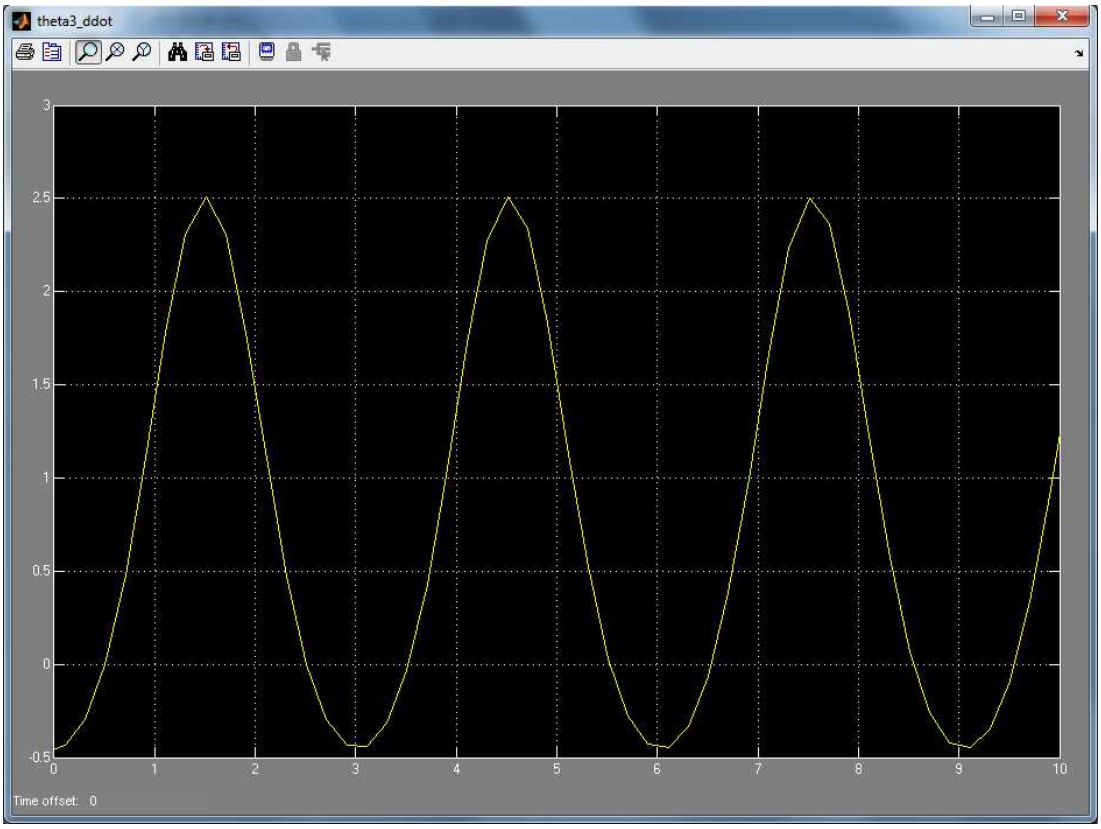

Figure 4.31 Coupler angular acceleration time response

Figure 4.32 Output angular acceleration time response

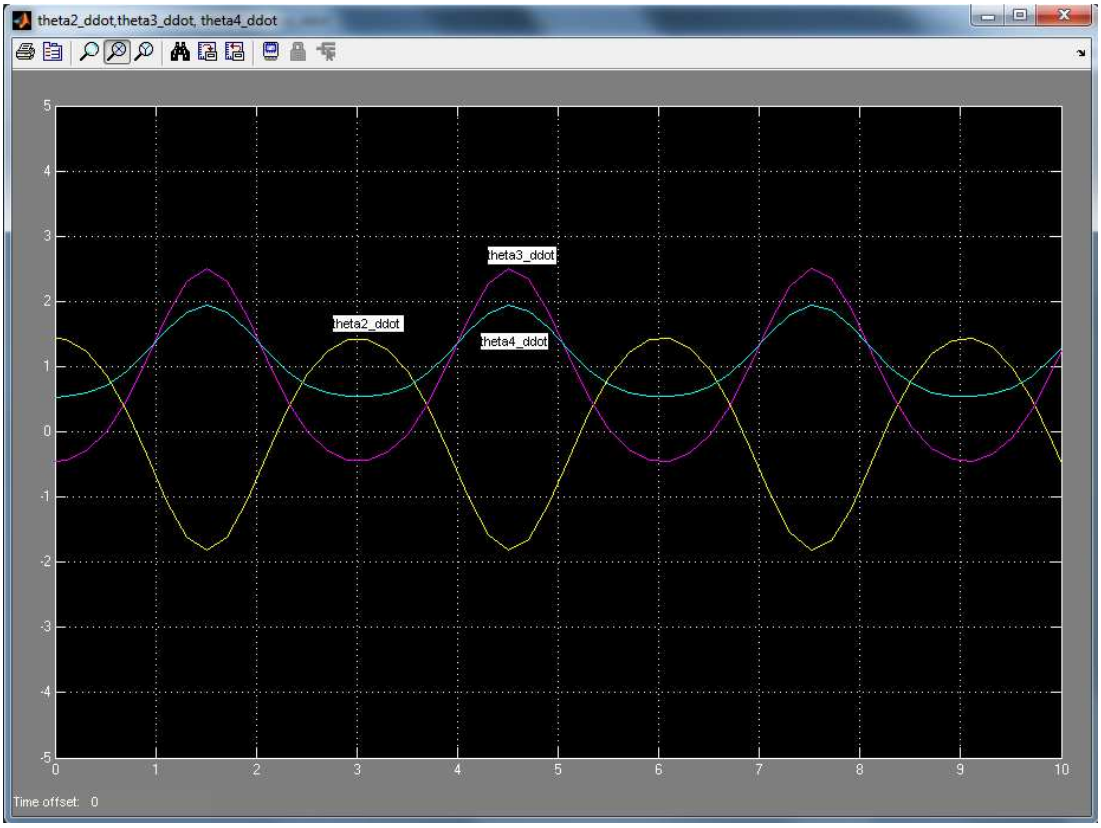

Figure 4.33 Comparison of the angular acceleration

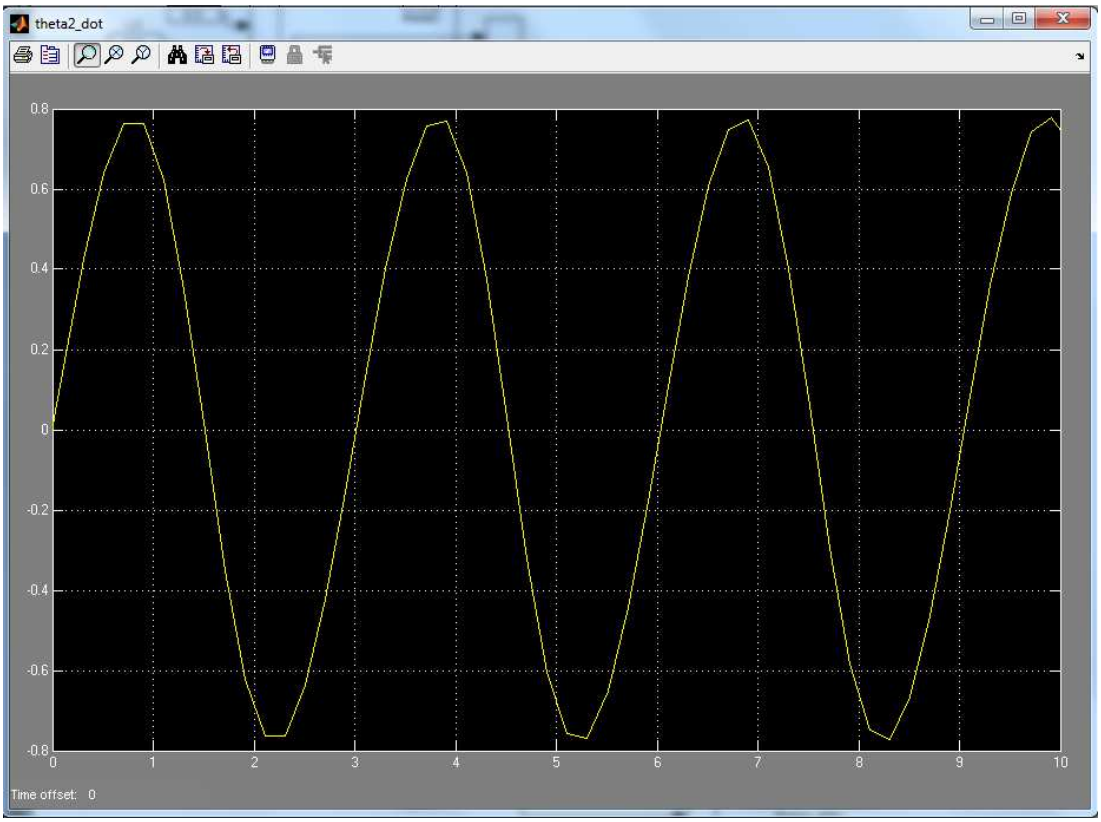

Figure 4.34 Crank angular velocity time response

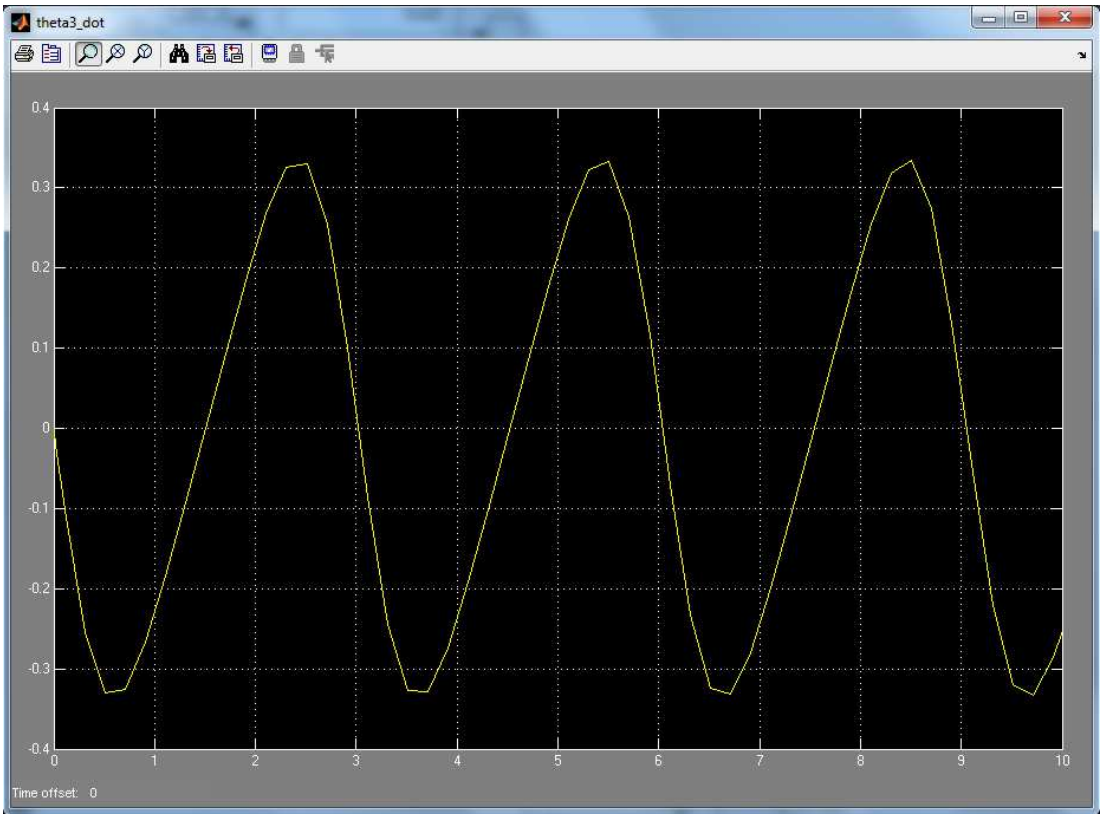

Figure 4.35 Coupler angular velocity time response

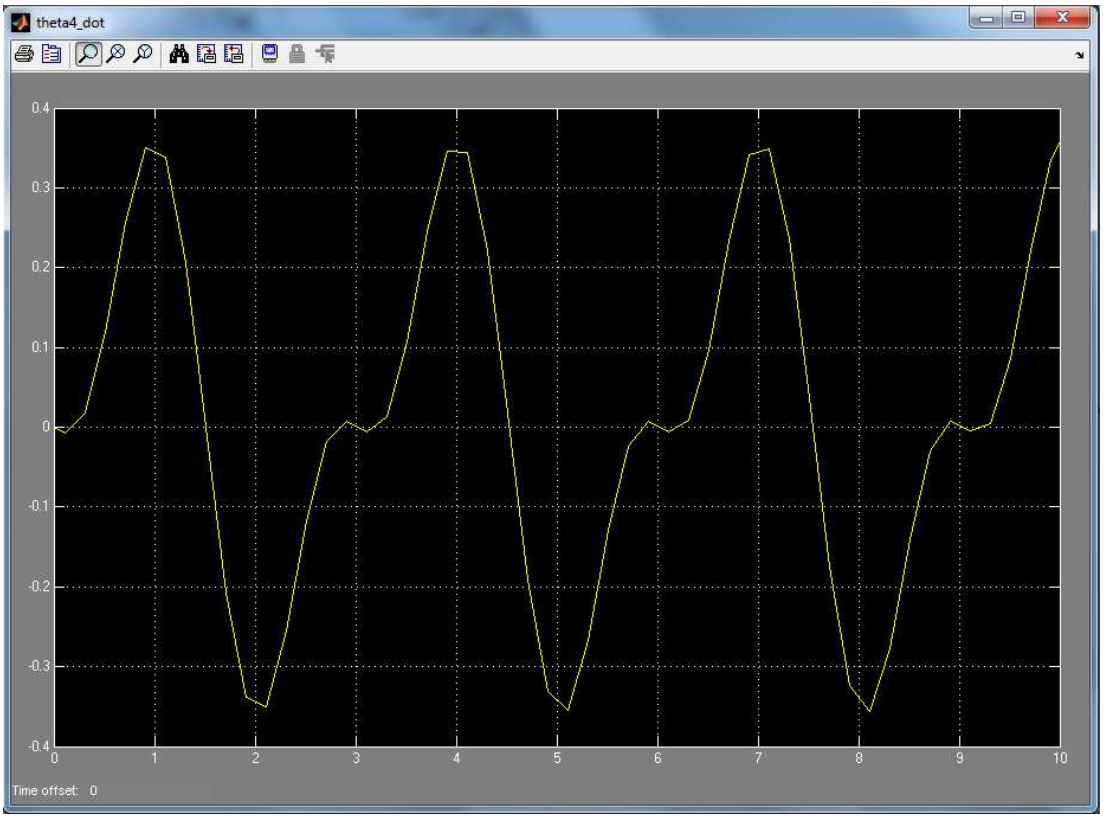

Figure 4.36 Output angular velocity time response

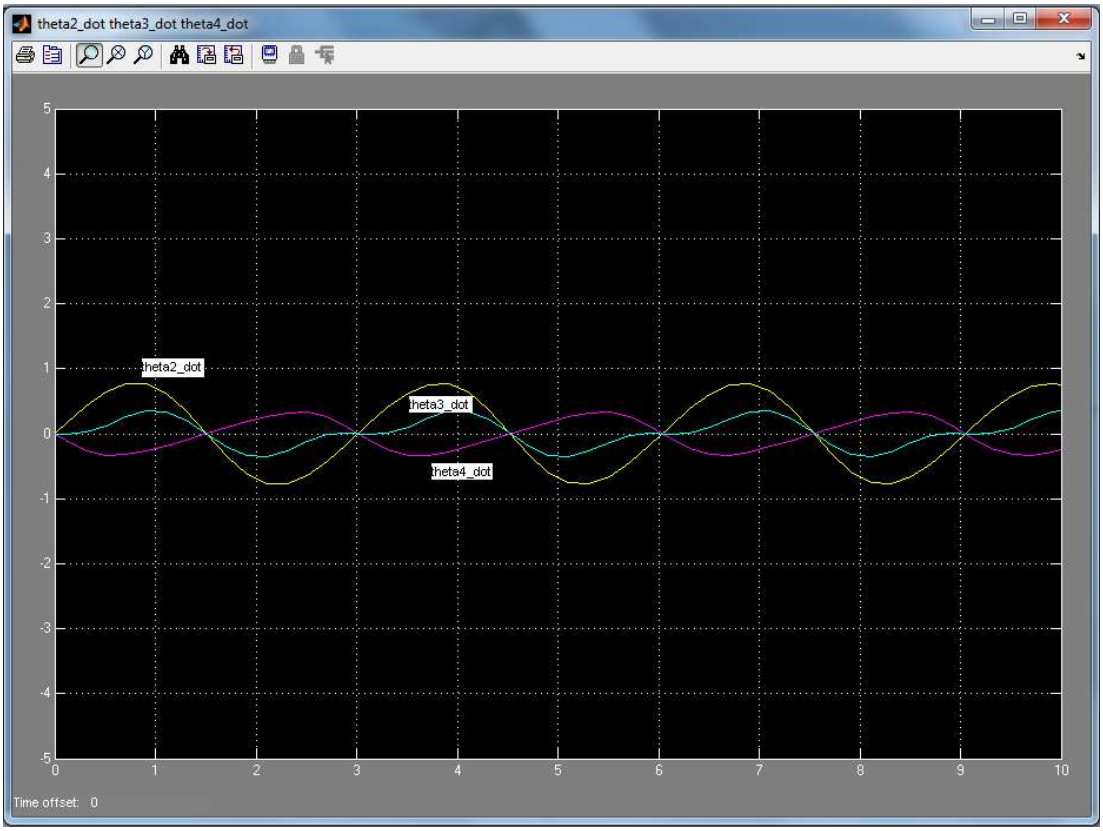

Figure 4.37 Comparison of the angular velocity

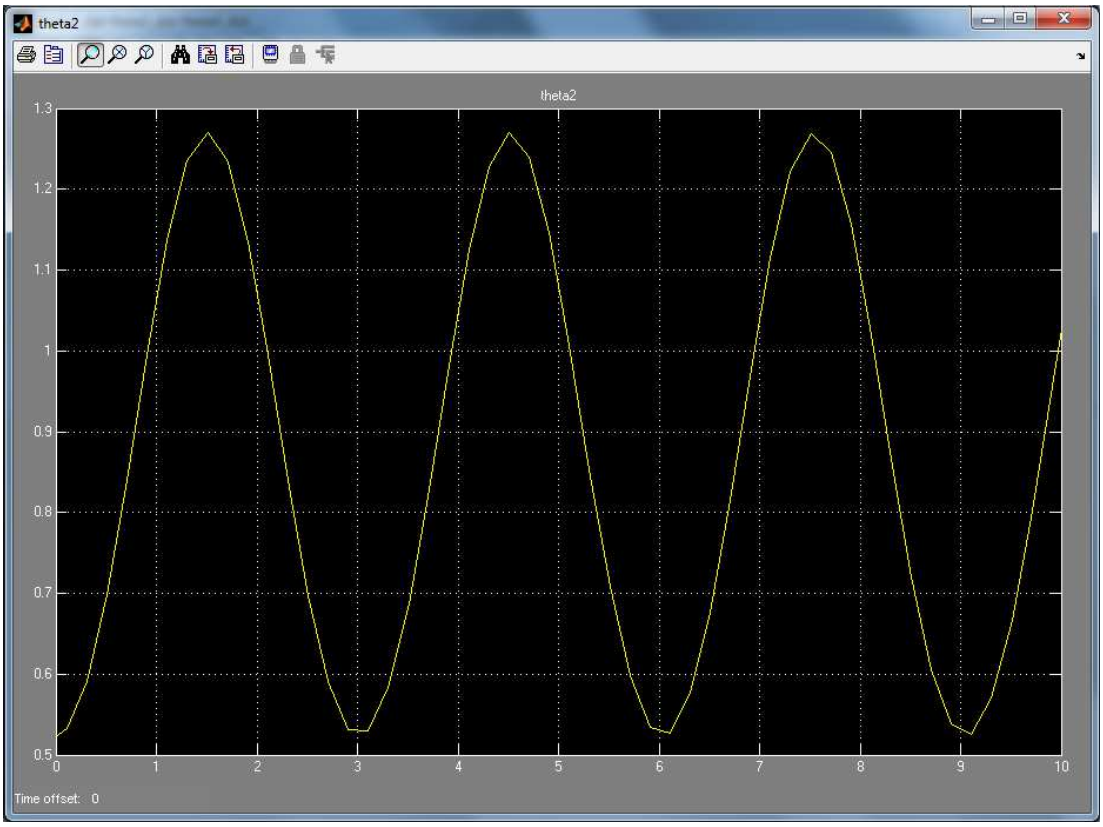

Figure 4.38 Crank angle time response

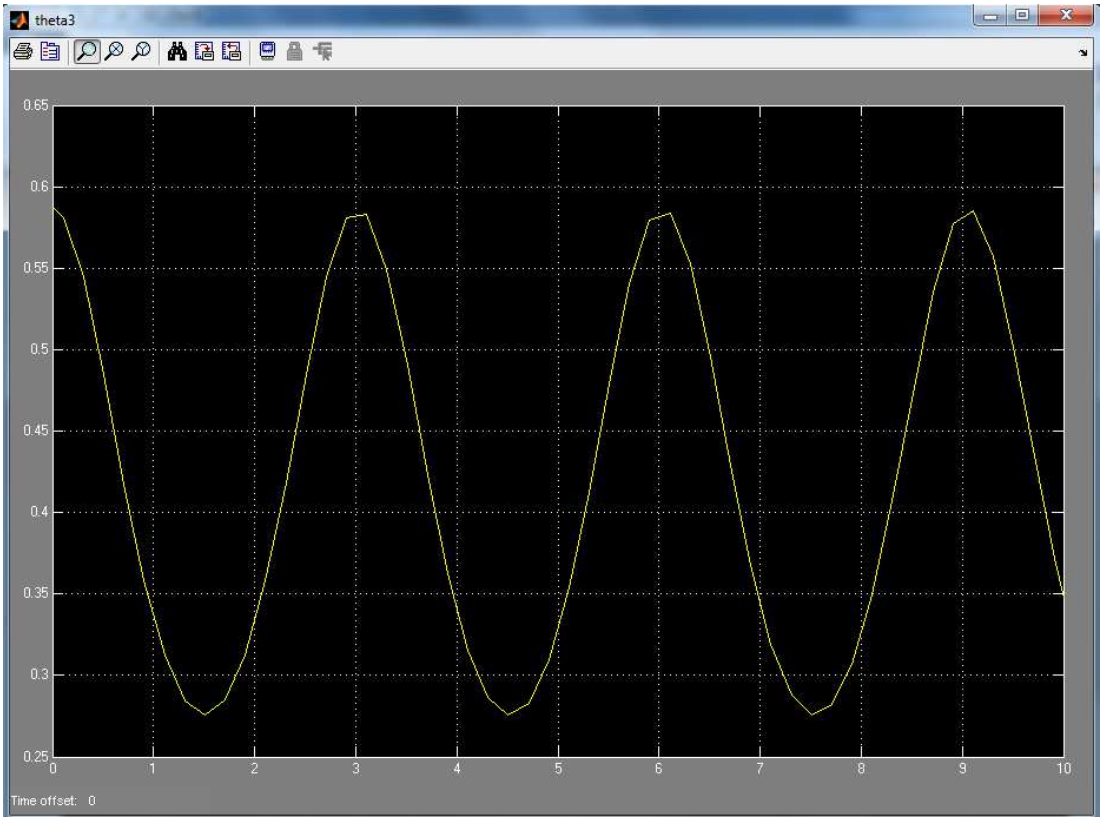

Figure 4.39 Coupler angle time response

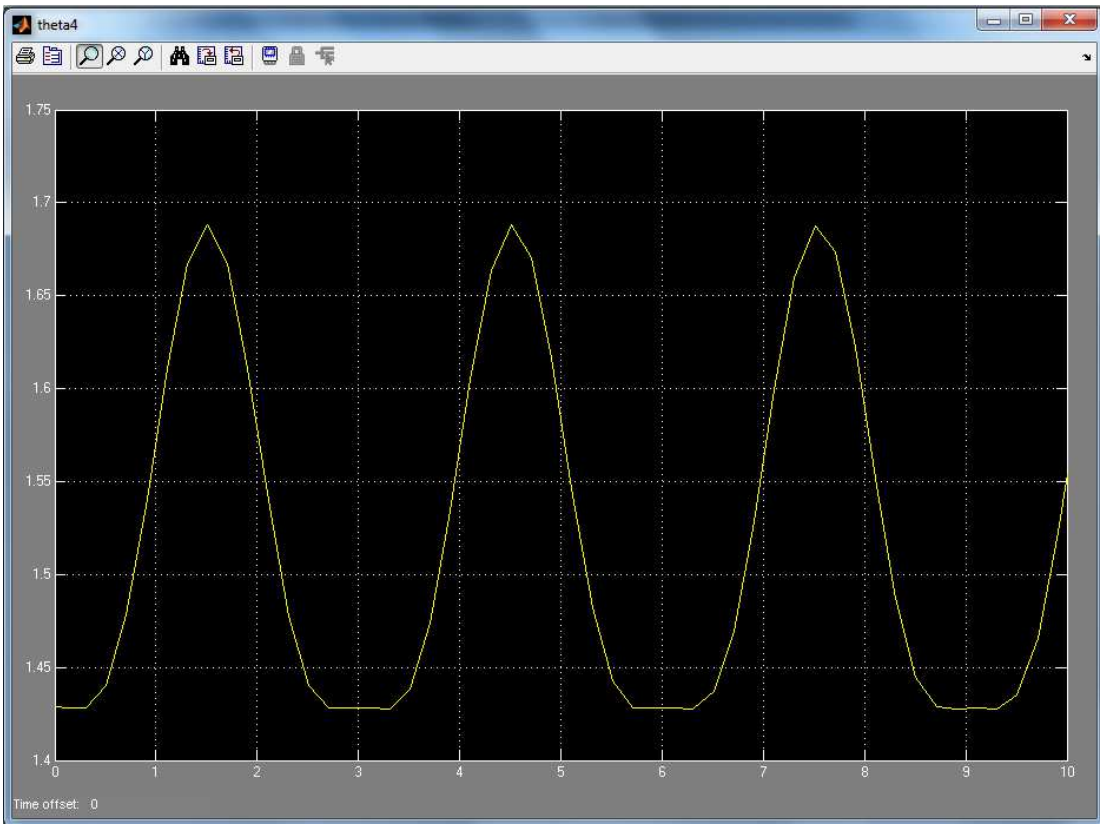

Figure 4.40 Output angle time response

5. CONCLUSION AND RECOMMENDATIONS

Kinematics and dynamics modelling for a planar four-bar mechanism is studied.

5.1 Application of The Work

As a first and second part, analyses are based on the assumption that the links behave as rigid bodies.The fully coupled non-linear equations of motion are obtained through a constrained Lagrangian approach. Resulting differential algebraic equations are solved numerically to obtain the system response. A linearized dynamic model is developed. Rigid body analysiswhich presumes low operating speeds becomes insufficient for describing the performance ofsuch highspeedmachinery. In mechanical systems operating at high speeds, some oscillatory elastic motion is inevitable.

As a third part, analyses are based on the assumption that the coupler and output links are elastic. That case, elastic coupler link and elastic output link are modeled dynamically. Governing non-linear equations are obtained through a constrained Lagrangian approach. The equations are fully coupled such that the mutual dependence between rigid body and elastic motions is preserved. Using the equations vibration effect on the angles of the links is investigated. It is seen from the equations that elastic links effect the motion of the planar four-bar mechanism so much.

For the case of flexible coupler-output links it is clear that all procedure is same as for other cases. Besides this the position vectors for flexible cpuler and output links are the exact ones written for the other two cases separately. Thus the resulting differential equations can be written by just combining respective equations.

5.2 Conclusions

Dynamic modelling for a flexible four-bar mechanism is studied. The fully coupled non-linear equations of motion are obtained through a constrained Lagrangian approach. Resulting differential-algebraic equations are solved numerically to obtain the system response.

Mathematical modelling of flexible mechanisms is complicated.The rigid body motion involves changing geometries resulting in varying system parameters, and these influence the elastic deformations.The elastic deformations themselves in turn influence the rigid body motion.Though the importance of including the effect of elastic motion in dynamic modelling has been recognized for quite some time, most models were based on a prescribed rigid body motion.

The assumption here is that the elastic deformations do not have significant effect on rigid body motion.In other words, there is only a one-way coupling between elastic deformations and rigid body motion.For lightweight and high-speed mechanisms where there are no large flywheel type inertia elements to maintain the rigid body motion independent of the elastic deformations, this modelling approach results in gross errors.

Therefore, in these cases, a fully coupled formulation is essential. There are two distinct modelling approaches to fully coupled formulations. The first one is a finite element formulation where both rigid and elastic degrees of freedoms are considered as generalized co-ordinates. The second approach uses the Lagrange multiplier technique to incorporate joint constraints into the equations of motion, and results in a mixed set ofdifferential and algebraic equations.

Traditionally, the dynamic analysis have been based on the assumption that the links behave as rigid bodies. In mechanical systems operating at high speeds some oscillatory elastic motion is inevitable. This motion becomes a major concern when performance requirements are such that high precision is important. Because of the trend toward increasing operating speeds and reducing weights in modern machinery it may be inaccurate to treat certain links in such systems as rigid bodies.

Although dynamic analysis of flexible mechanisms has been the subject ofnumerous investigations, the control of such systems has not received much attention.Sung and Chen proposed an optimal control scheme to suppress the vibrations of a four-bar linkage with a flexible output link.They used piezoceramic sensors and actuators which are located on the flexible link.The effect of elastic deformations as well as control action on the rigid body motion was not considered since the authors used a model with only one-way coupling.

The effect of flexibility is significant not only on the joint angular velocities, but also on the joint angles.Clearly, in this case, a rigid model or a flexible model without full coupling will yield gross errors with respect to rigid body motions since the effect of elastic deformation will not be seen in rigid body co-ordinates.

The dimension and stiffness of linkages have great effects on the dynamic response characteristics.Their effects can be reflected in changing the quantity of the inertial forces.

Increasing the crank length may increase the displacements of the coupler and rocker, then the inertial force as well as the dynamic response will be increased. Increasing the stiffness of the crank,the vibratory displacement at the crank coupler joint will be forced to attenuate,therefore the free development of the vibratory responses of the coupler and rocker will be greatly restricted.Therefore, increasing the crank's stiffness may reduce the dynamic response of linkages.

When thecoupler and rocker have nearly the same geometric shape, the distributions of the inertial force incoupler and rocker are proportional. In this case, the effects of internal resonance are not so violent,and the dynamic responses of linkages are not intense as well.

When the geometric shapes of the coupler and rocker are quite different, the effects of the inertial force will concentrate mainly on one flexible member (coupler or rocker). The violent dynamic responses produced in one flexible member, through the internal resonance, will cause another flexible member to produce a relatively higher dynamic response.

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CURRICULUM VITA

Candidate's full name	**:** Elif SAKAR KILINÇ
Place and date of birth	**:** BALIKESİR, 08.07.1984
Permanent Address	**:** Funda 07 04 D13 Blok Kat:15 Daire:66 Bahçeşehir / İSTANBUL
Universities and Colleges attended	**:** Kocaeli University, Mechatronic Engineering