ISTANBUL TECHNICAL UNIVERSITY ★ GRADUATE SCHOOL OF SCIENCE ENGINEERING AND TECHNOLOGY

QUASIMORPHISMS ON SYMPECTIC MANIFOLDS

M.Sc. THESIS

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Department of Mathematical Engineering

Mathematical Engineering Programme

JANUARY 2012

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SİMPLEKTİK MANİFOLDLAR ÜZERİNDE KUAZİMORFİZMALAR

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FOREWORD

I would like to express my appreciation to my advisor Dr. Ali Sait Demir for his guidance and support. I also would like to thank my family for their constant support.

December 2011 Baran Cem ZURNACI

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QUASIMORPHISMS ON SYMPLECTIC MANIFOLDS

SUMMARY

In this study, the main goal is to get an understanding of the concept of quasimorphisms. Quasimorphisms exist as the nontrivial elements of the second bounded cohomology group. The existence of quasimorphisms for certain groups of diffeomorphisms is an active research area among symplectic geometers. Main motivation of these studies focus on understanding the geometry of such groups. Basically, if there is a nontrivial quasimorphism on such diffeomorphism groups, then one concludes that the diameter of the group is infinite.

The first part of the study is devoted to bounded cohomology. Roughly, bounded cohomology is the cohomology of chain complexes where cochains are bounded functions into reals, integers etc. Since the seminal paper of Gromov, various studies have been made on this subject. In this thesis, we recall the definitions from the usual cohomology of spaces, following Hatcher. Next, we examine the bounded cohomology of discrete groups in more detail. As an example, we analyze the theorem of Trauber proving that the bounded cohomology groups of amenable groups are trivial, folowing Bucher.

The second bounded cohomology group is in the heart of this study. Namely, nontrivial quasimorphisms lie in the kernel of the comparison map between bounded and usual cohomology groups in the second degree. Roughly, quasimorphisms are homomorphisms up to an error. Since perfect groups do not admit nontrivial homomorphism into reals (or any abelian group), it is natural to seek nontrivial quasimorphisms defined for such groups. As an example, we include the result of Brooks in detail in which an infinite family of quasimorphisms are defined in the free group on two generators.

In the final chapter, we recall basics of symplectic topology. The study of classical mechanical systems such as the planetary system is based on configuration spaces of particles satisfying certain equations (Hamilton equations). The configuration spaces are symplectic manifolds and there have been large number of studies on symplectic manifolds in the last three decades. In this thesis, we first review basics of symplectic topology and focus on the group of Hamiltonian diffeomorphisms of symplectic manifolds. A classical result due to Banyaga shows that this group is perfect and hence, quasimorphisms on this group become important. Recently, Polterovich and his students and Gambaudo and Ghys construct quasimorphisms for certain symplectic manifolds. We finish the thesis by stating these results. Their proofs include spectral invariants of Floer theoretic backgrounds and are beyond the scope of this study.

SİMPLEKTİK MANİFOLDLAR ÜZERİNDE KUAZİMORFİZMALAR

ÖZET

Bu çalışmadaki temel amaç kuazimorfizmaların neler olduklarını anlamaktır. Homomorfizmaların daha genel bir durumu olan kuazimorfizmalar ikinci sınırlı kohomoloji gruplarının bariz olmayan elemanlarıdır, yani ikinci sınırlı kohomoloji grubu homomorfizma olmayan veya sınırlı olmayan kuazimorfizmalar tarafından oluşur. Difeomorfizma grupları için bu kuazimorfizmaların varlığı simplektik geometriciler arasında geniş bir çalışma alanıdır. Bu çalışmaların esas amacı bu grupların geometrilerini anlamak üzerine kuruludur. Temel olarak, eğer bu şekildeki difeomorfizma grupları üzerinde bariz olmayan kuazimorfizmalar var ise o zaman grubun çapı sonsuz olduğu sonucunu çıkarırız.

Tezin ilk bölümü standart ve sınırlı kohomoloji grupları üzerine kuruludur. İlk olarak kozincirlerimizi oluşturarak bu grupların temellerini atıyoruz. Standart kozincirler bir ayrık grubun kuvvetlerinden bir abelyen gruba giden fonksiyonların kümesi olarak tanımlanırken sınırlı kozincirler ise bir ayrık grubun kuvvetlerinden bir abelyen gruba giden sınırlı fonksiyonların kümesi olarak tanımlanır. Oluşturduğumuz kozincirler arasında kosınır fonksiyonlarımızı tanımlıyoruz. Kozincirleri birbirine bağlayan kosınır fonksiyonları ile kokompleksler elde ediyoruz. Standart kohomoloji grubunu elde etmek için standart kokompleks yardımıyla bir kosınır fonksiyonumuzun çekirdeğindeki elemanların kümesini aynı kosınır fonksiyonumuzun bir alt derecesindeki kosınır fonksiyonunun görüntüsü olan elemanların kümesine bölüyoruz. Sınırlı kohomoloji grubunu da elde etmek için aynı yöntem ile bu sefer sınırlı kokompleks yardımıylı sınırlı kohomoloji grubumuzu elde ediyoruz. Kabaca diyebiliriz ki sınırlı kohomoloji grubu bir ayrık gruptan abelyen bir gruba giden sınırlı fonksiyonların kümesidir. Standart kohomoloji grubunda ise böyle bir sınırlılık kısıtlaması olmadığı için sınırlı kohomoloji grubu standart kohomoloji gurubunun özel bir durumudur, yani alt kümesidir. Gromov'un çalışmalarıyla birlikte bu alanda bir çok çalışma da başlamış oldu. Bu tezde Hatcher'ı takip ederek standart kohomoloji uzaylarından tanımlara yer veriyoruz. Daha sonra da ayrık grupların sınırlı kohomoloji gruplarını detaylı bir şekilde inceliyoruz. Ayrıca, amenable grupların tanımını veriyoruz. Bir ayrık grubun amenable olması için bu ayrık grup üzerinde normla donatılmış reel değerli sınırlı Banah uzayının var olması gerektiğini ifade ediyoruz. Amenable gruplara örnek olarak ise sonlu grupları ve abelyen grupları gösterebiliriz. Amenable grupların sınırlı kohomoloji ile bağıntısını gösteren Trauber teoremini ispatlıyoruz. Bu teoremin yardımıyla eğer ayrık grubumuz amenable ise bu ayrık gruptan herhangi bir abelyen gruba giden fonksiyonların oluşturduğu sınırlı kohomoloji grubunun aslında sıfır grubundan başka birşey olmadığını görüyoruz ve bu teoremin standart kohomoloji grupları için geçerli olmadığını gösteren Bucher'ın örneğini analiz ediyoruz.

İkinci sınırlı kohomoloji grupları bu çalışmanın esas odak noktalarıdır. İkinci sınırlı kohomoloji grubu ile ikinci standart kohomoloji grubu arasında karşılaştırma fonksiyonu tanımlanır ve bariz olmayan kuazimorfizmalar bu karşılaştırma fonksiyonunun çekirdeğinin içinde kalırlar. Kabaca diyebiliriz ki kuazimorfizmalar homomorfizmaların hatalı halidir. Mükemmel gruplardan reel sayılara (veya herhangi bir abelyen gruba) tanımlı tüm homomorfizmalar sıfır homomorfizması olacağından bu gruplar üzerinde tanımlı kuazimorfizmaların varlığı önemli bir problem olur. Örnek olarak da iki üreteçli serbest gruplarda tanımlanan kuazimorfizmaların sonsuz bir ailesini tanımlayan Brooks'un sonucunu ayrıntılarıyla irdeliyoruz. Brooks'un bu örneğinde iki üreteçli serbest gruplar üzerinde tanımladığımız fonksiyonun homomorfizma olamayacağını ve buna ek olarak da bu fonksiyonun sınırlı da olamayacağını gösteriyoruz.

Son bölüme başlamadan önce diferensiyel geometriden temel kavramları hatırlıyoruz. İlk olarak, bir vektör uzayının kendisiyle kartezyen çarpımı üzerinde tensörün tanımını reel değerli bir fonksiyon yardımıyla tanımlıyoruz. Bu tensörlere örnek olarak fonksiyonelleri, iç çarpımı ve determinantı verebiliriz. Daha sonra da herhangi iki bileşeninin yerini değiştirdiğimiz zaman tensörün aldığı değerin işaretinin değişmesi durumunu bu tensörün değişken olması olarak tanımlıyoruz. Böyle değişken tensörlere örnek olarak ise determinantı verebiliriz fakat iç çarpımı veremeyiz. Tensörler üzerinde tensör çarpımı tanımladıktan sonra bu tensör çarpımın değişkenleştirerek wedge çarpımı tanımlıyoruz. Daha sonra ise diferensiyel formun tanımını veriyoruz. Türevlenebilir bir manifoldtan her noktasında değişken tensörlere sahip olan bu manifoldun teğet uzayına bir fonksiyonu diferensiyel form olarak tanımlıyoruz. Bir manifold üzerindeki reel değerli fonksiyonlar tanım gereğince 0-form oluştururlar ve bu fonksiyonların türevleri de manifoldun teğet uzayı üzerinde tanımlı reel değerli fonksiyonlar olarak tanımlanırlar. Daha sonra da geri çekme fonksiyonunun tanımını veriyoruz. Öklid uzayı üzerinde geri çekme fonksiyonunun aslında fonksiyonların bileşkesinden başka bir şey olmadığını gösteriyoruz. Böylece bu geri çekme fonksiyonu yardımıyla 1-formları 3 boyutlu öklid uzayındaki bir eğriden aslında birim kapalı aralığa geri çekerek bu 1-formun integralini hesaplayabiliyoruz ve benzer şekilde de 2-formları 3 boyutlu öklid uzayındaki bir yüzeyden geri çekerek bu 2-formun integralini hesaplayabiliyoruz. Daha sonra ise herhangi bir formun dış türevini tanımlıyoruz ve bu dış türevin formunun seçtiğimiz formdan derecesinin bir fazla olduğunu gösteriyoruz. Sonra da sırasıyla 3 boyutlu öklid uzayı üzerinde 0-form bir fonksiyonun türevinin 1-form olduğunu ve bunun da gradient vektörünü doğurduğunu, 1-form bir formun türevinin 2-form olduğunu ve bunun da rotasyon vektörünü doğurduğunu ve 2-form bir formun türevinin 3-form olduğunu ve bunun ise diverjans vektörünü doğurduğunu gösteriyoruz. Bunlara karşın 3 boyutlu öklid uzayı üzerinde 3-formların türevinin olmadığını çünkü 4-form bir formun 3 boyutlu bir manifoldta bulunamayacağını gösteriyoruz.

Diferensiyel geometriden bu temel kavramları hatırladıktan sonra ikinci bölüme simplektik topolojinin temellerini detaylandırarak başlıyoruz. Gezegen sistemi gibi klasik mekanik sistem çalışmaları Hamilton denklemlerini sağlayan parçacıkların konfigürasyon uzaylarına dayalıdır. Konfigürasyon uzayları simplektik manifoldlardır ve son otuz yılda simplektik manifoldlar üzerine bir çok çalışmalar yapılmaktadır. Bu tezde ilk olarak simplektik fonksiyonu tanımlayarak simplektik vektör uzayların tanımını veriyoruz. Simplektik vektör uzayının bazının çift sayıda elemandan oluşması sebebiyle de simplektik vektör uzaylarının çift boyutlu olması gerektiği gerçeğine varıyoruz. Çift boyutlu vektör uzayları arasındaki simplektomorfizmayı da daha önce tanımladığımız geri çekme fonksiyonu yardımıyla tanımlıyoruz. Bu tanım yardımıyla da çift boyutlu vektör uzaylarının aslında aynı boyutlu öklid uzayına simplektomorfik olduğununun farkına varıyoruz. Benzer şekilde simplektik manifoldu da bu sefer manifoldun teğet uzayı üzarindeki simplektik bir 2-form yardımıyla kuruyoruz. Bir manifoldun simplektik manifold olması için gerek ve yeter şartın bu manifold üzerinde kurulan formun türevinin sıfıra eşit olması ve bu formun dejenere olmayan bir form olması gerektiği tanımını veriyoruz. Örnek olarak ise çift boyutlu öklid uzayı, karmaşık sayılar uzayı ve küre yüzeyi örneklerini inceliyoruz. Aynı vektör uzayında olduğu gibi manifoldlar arasındaki simplektomorfizmayı da geri çekme fonksiyonu yardımıyla tanımlıyoruz ve yine benzer şekilde çift boyutlu simplektik manifoldların yerel olarak aynı boyutlu öklid uzayına simplektomorfik olduğunun farkına varıyoruz. Daha geometrisel olarak açıklamak gerekirse iki simplektik manifoldun birbirlerine simplektomorfik olabilmesi için gerek ve yeter şart aralarında hacim formu koruyan bir difeomorfizmanın var olmasıdır, yani örneğin birbirine difeomorfik bir küre ile bir elipsoidin simplektomorfik olabilmesi için hacim formlarının (yani, yüzey alanlarının) birbirine eşit olması gerekir. Bu temelleri attıktan sonra ise bir manifold üzerindeki Hamilton difeomorfizmaları grubu ve Simplektomorfizmalar grubuna odaklanıyoruz. Banyaga'nın klasik sonucu gösteriyor ki bu gruplar basit gruplardır ve dolayısıyla bu gruplardan reel sayılara bariz olmayan homomorfizma bulmak mümkün olmamaktadır. Bu yüzden bu gruplar üzerindeki kuazimorfizmalar bariz olmayan kuazimorfizmalar olmasından dolayı önem kazanır. Örnek ve son olarak ise disk ve torus üzerindeki simplektomorfizmalar grubundan reel sayılara bariz olmayan kuazimorfizmalar buluyoruz ve bunu da ilk bölümde incelediğimiz ikinci sınırlı kohomoloji gruplarıyla bağdaştırıyoruz.

Sonuç olarak, eğer bir manifold üzerindeki simplektormorfizmalar grubunda bariz olmayan kuazimorfizmalar bulabiliyorsak o halde; daha önce de bahsettiğimiz üzere bu simplektomorfizmalar grubunun çapı sonsuz olur (bu da bize bu grubun geometrisi hakkında bilgi verir), komütatörler çarpımı biçiminde yazılabilen bir difeomorifizmayı oluşturan komütatörün boyu sonsuz olur ve bu simplektomorfizmalar grubunun ikinci sınırlı kohomoloji grubu ikinci standart kohomoloji grubundan farklı olur. Bu sonuçları ifade ederek tezi noktalıyoruz. Son zamanlarda Polterovich ve öğrencileri ile Gambaudo ve Ghys simplektik manifoldlar için bir çok kuazimorfizma örnekleri kuruyorlar. Onların yapmakta olduğu ispatlar bu tezin sınırları ötesindedir.

1. BOUNDED COHOMOLOGY

In this chapter, we recall basic definitions and results about homology and cohomology of spaces. Then, we introduce bounded cohomology of groups. We follow mainly Hatcher [10] and Bucher [4].

1.1 Singular Homology and Cohomology

We need to introduce some basic definitions to explain what homology is. Let us initially give a description of n-simplex. The n-simplex is the form of the n-dimensional triangle; i.e., 0-simplex is just a point in the line $\mathbb{R}$, 1-simplex is a line in the plane $\mathbb{R}^2$, 2-simplex is a triangular zone in $\mathbb{R}^3$, 3-simplex is a solid tetrahedron in $\mathbb{R}^4$ and etc. Clearly, we need $n+1$ points (corners) to obtain the n-simplex. These points are called the vertices of the simplex and denoted as v_i. Then, the n-simplex is denoted as $[v_0, \cdots, v_n]$ consisting of the terms of vertices. The standard n-simplex can be set as

$$\Delta^n = \left\{ (a_0, \cdots, a_n) \in \mathbb{R}^{n+1} \,\middle|\, \sum_{i=0}^{n} a_i = 1 \; and \; a_i \geq 0 \; for \; all \; i \right\}$$

This produces the standart n-simplex whose vertices are the unit vectors; i.e., $v_i = \left(0, \cdots, \underbrace{1}_{i.}, \cdots, 0\right)$. If we erase one of the vertices of an n-simplex, then the n vertices of erased n-simplex span an $(n-1)$-simplex and this new $(n-1)$-simplex is called a face of the n-simplex. Bringing all the possible faces of an n-simplex Δ^n together is the boundary of Δ^n, denoted as $\partial\Delta^n$.

1.1.1 Singular homology

Let Γ be a topological space. A singular n-simplex is a map $\sigma: \Delta^n \rightarrow \Gamma$. Let $C_n(\Gamma, A)$ be the free abelian group with basis the set of singular n-simplices in Γ. A typical element of $C_n(\Gamma, A)$ is a finite formal sum $\sum_i n_i \sigma_i$ for $n_i \in A = \mathbb{Z}, \mathbb{R}, \mathbb{Z}_2$ etc. and $\sigma_i: \Delta^n \rightarrow \Gamma$. This gives a sequence of homomorphisms of abelian groups

$$\cdots \to C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \to \cdots \to C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} 0$$

where the boundary map $\partial_n: C_n(\Gamma, A) \to C_{n-1}(\Gamma, A)$ is given by

$$\partial_n(\sigma) = \sum_i (-1)^i \sigma|_{[v_0,\cdots,\hat{v}_i,\cdots,v_n]}$$

The boundary map satisfies the relation $\partial_n \, \partial_{n+1} = 0$ and hence, we can define the singular homology group $H_n(\Gamma) = Ker \, \partial_n / Im \, \partial_{n+1}$.

1.1.2 Cohomology

To obtain cohomology groups, we replace the chain groups C_n by the dual groups $Hom(\Gamma^{n+1}, A)$ and the boundary maps ∂ by their dual maps δ.

Consider a general chain complex of free abelian groups

$$\cdots \to C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \to \cdots$$

We replace each chain group C_n by its dual cochain group $C^n = Hom(\Gamma^{n+1}, A)$ and each boundary map ∂ by its dual coboundary map $\delta = \partial^*: C^n \to C^{n+1}$. From the relation $\partial^2 = 0$, it follows that $\delta^2 = 0$. The cohomology group $H^n(\Gamma, A)$ is defined as the homology group $Ker \, \delta_n / Im \, \delta_{n-1}$ at C^n in the cochain complex

$$\cdots \leftarrow C^{n+1} \xleftarrow{\delta_n} C^n \xleftarrow{\delta_{n-1}} C^{n-1} \leftarrow \cdots$$

1.2 Bounded Cohomology

Firstly, we recall the coboundary map. Let Γ be a discrete group and A be an abelian group as we defined previously; i.e., $A = \mathbb{Z}, \mathbb{R}, \mathbb{Z}_2$ etc. Define the sets

$$C^n(\Gamma, A) = \{f: \Gamma^{n+1} \longrightarrow A\}$$
$$C_b^n(\Gamma, A) = \{f: \Gamma^{n+1} \longrightarrow A | f \text{ is bounded}\}$$

The coboundary map $\delta_n: C^n \to C^{n+1}$ is defined as

$$\delta_n f(\gamma_0, \cdots, \gamma_{n+1}) = \sum_{i=0}^{n+1} (-1)^i f(\gamma_0, \cdots, \hat{\gamma}_i, \cdots, \gamma_{n+1})$$

for $f \in C^n(\Gamma, A)$ and $\gamma_i \in \Gamma$ when $i = 0, \cdots, n+1$.

The elements of $C^n(\Gamma, A)$ are called cochains and also the elements of $C_b^n(\Gamma, A)$ are called bounded cochains. A cochain f is called cocycle if $\delta f = 0$. On the other hand, a cochain f is called coboundary if there exists g satisfying $\delta g = f$. Let f be a coboundary, then there exists g satisfying $\delta g = f$ and therefore, this implies that $\delta^2 g = \delta f = 0$. It means that if f is a coboundary, then it is always a cocycle.

Lemma: $\delta_n \delta_{n-1} = 0$.

Lemma: The cocomplex

$$0 \longrightarrow A \longrightarrow C_b^0(\Gamma, A) \longrightarrow C_b^1(\Gamma, A) \longrightarrow C_b^2(\Gamma, A) \longrightarrow \cdots$$

is exact.

Proof: We prove this lemma in three steps.

Exactness in A : Let us first analyze the short sequence of

$$0 \xrightarrow{0} A \xrightarrow{\lambda} C_b^0(\Gamma, A)$$

where 0 maps the group 0 to the zero of the group A and λ maps an element a of A to the constant function $f: \Gamma \to A$ with value a; i.e., for all γ in Γ, $f(\gamma) = a$. The image of 0 is definitely $\{0\}$ as well as the kernel of the map λ is $\{0\}$ since λ is an injective map. Owing to the equality $Ker\lambda = Im0$, this short sequence is exact.

Exactness in $C_b^0(\Gamma, A)$: Let us now analyze the sequence of

$$A \xrightarrow{\lambda} C_b^0(\Gamma, A) \xrightarrow{\delta_0} C_b^1(\Gamma, A)$$

Consider the $Ker\left(\delta_0 : C_b^0(\Gamma, A) \to C_b^1(\Gamma, A)\right) = \{f | \delta_0 f = 0\}$. If $\delta_0 f = 0$, then $\delta_0(f)(\gamma_0, \gamma_1) = f(\gamma_1) - f(\gamma_0) = 0 \Rightarrow f(\gamma_1) = f(\gamma_0)$ for every $\gamma_0, \gamma_1 \in \Gamma$. This means that chosen f is a constant function. Therefore, being the set of constant functions, the set $Ker(\delta_0) = \{f | \delta_0 f = 0\}$ is, by definition, equal to the image set of the map λ. Due to the equality $Ker\delta_0 = Im\lambda$, this sequence is exact, too.

Exactness in $C_b^n(\Gamma, A)$ for $n > 0$: To show that $Im\delta_{n-1} = Ker\delta_n$ in the sequence

$$C_b^{n-1}(\Gamma, A) \xrightarrow{\delta_{n-1}} C_b^n(\Gamma, A) \xrightarrow{\delta_n} C_b^{n+1}(\Gamma, A)$$

we firstly need to indicate that $Im\delta_{n-1} \subset Ker\delta_n$. Let f be an element of $Im\delta_{n-1}$. Then, there exists an element g in $C_b^{n-1}(\Gamma, A)$ with $\delta_{n-1} g = f$. It follows that

$\delta_n\delta_{n-1}g = \delta_n f = 0$. Since $\delta_n f = 0$, it means that f is the element of $Ker\delta_n$, which proves that $Im\delta_{n-1} \subset Ker\delta_n$.

Secondly, we need to demonstrate that $Ker\delta_n \subset Im\delta_{n-1}$. Let f be an element of $Ker\delta_n$, which means $f \in C_b^n(\Gamma, A)$ with $\delta_n f = 0$, and g be an element of $C_b^{n-1}(\Gamma, A)$. Define $g: \Gamma^n \to A$ by

$$g(\gamma_1, \cdots, \gamma_n) = f(1, \gamma_1, \cdots, \gamma_n)$$

for all $(\gamma_1, \cdots, \gamma_n)$ in Γ^n. Here, 1 is the identity element of the Γ. Then,

$$0 = \delta_n f(1, \gamma_0, \cdots, \gamma_n) = f(\gamma_0, \cdots, \gamma_n) - \sum_{i=0}^{n} (-1)^i (1, \gamma_0, \cdots, \hat{\gamma}_i, \cdots, \gamma_n)$$

and we thus have

$$\sum_{i=0}^{n} (-1)^i f(1, \gamma_0, \cdots, \hat{\gamma}_i, \cdots, \gamma_n) = f(\gamma_0, \cdots, \gamma_n)$$

Now, by considering the above equation and the definition of g, we have

$$\begin{aligned}\delta_{n-1}g(\gamma_0, \cdots, \gamma_n) &= \sum_{i=0}^{n} (-1)^i g(\gamma_0, \cdots, \hat{\gamma}_i, \cdots, \gamma_n) \\ &= \sum_{i=0}^{n} (-1)^i f(1, \gamma_0, \cdots, \hat{\gamma}_i, \cdots, \gamma_n) \\ &= f(\gamma_0, \cdots, \gamma_n)\end{aligned}$$

Therefore, we obtain $\delta_{n-1}g = f$, which means $f \in Im\delta_{n-1}$. Since we chose f from $Ker\delta_n$, it follows that $Ker\delta_n \subset Im\delta_{n-1}$. Finally, we obtain

$$Im\delta_{n-1} = Ker\delta_n$$

In these three steps, we conclude that the cocomplex

$$0 \longrightarrow A \longrightarrow C_b^0(\Gamma, A) \longrightarrow C_b^1(\Gamma, A) \longrightarrow C_b^2(\Gamma, A) \longrightarrow \cdots$$

is exact.

■

Define $C^n(\Gamma, A)^\Gamma$ and $C_b^n(\Gamma, A)^\Gamma$ as the subspaces of $C^n(\Gamma, A)$ and $C_b^n(\Gamma, A)$, respectively. These subspaces consist of Γ-invariant (bounded) cochains, that is, cochains $f: \Gamma^{n+1} \to A$ for which $\gamma \cdot f = f$, for every $\gamma \in \Gamma$. Here, $\gamma \cdot f: \Gamma^{n+1} \to A$ is defined as

$$\gamma \cdot f(\gamma_0, \cdots, \gamma_n) = f(\gamma^{-1}\gamma_0, \cdots, \gamma^{-1}\gamma_n)$$

for all $(\gamma_0, \cdots, \gamma_n) \in \Gamma^{n+1}$ and $\gamma \in \Gamma$.

Definition: The usual cohomology $H^*(\Gamma, A)$ of Γ is the cohomology of cocomplex

$$0 \longrightarrow C^0(\Gamma, A)^\Gamma \longrightarrow C^1(\Gamma, A)^\Gamma \longrightarrow C^2(\Gamma, A)^\Gamma \longrightarrow \cdots$$

The bounded cohomology $H_b^*(\Gamma, A)$ of Γ is the cohomology of the cocomplex

$$0 \longrightarrow C_b^0(\Gamma, A)^\Gamma \longrightarrow C_b^1(\Gamma, A)^\Gamma \longrightarrow C_b^2(\Gamma, A)^\Gamma \longrightarrow \cdots$$

The cohomology groups $H^*(\Gamma, A)$ and $H_b^*(\Gamma, A)$ can be written as

$$H^n(\Gamma, A) = \frac{Ker\big(\delta_n: C^n(\Gamma, A) \longrightarrow C^{n+1}(\Gamma, A)\big)}{Im\big(\delta_{n-1}: C^{n-1}(\Gamma, A) \longrightarrow C^n(\Gamma, A)\big)}$$

$$H_b^n(\Gamma, A) = \frac{Ker\Big(\delta_n: C_b^n(\Gamma, A) \longrightarrow C_b^{n+1}(\Gamma, A)\Big)}{Im\left(\delta_{n-1}: C_b^{n-1}(\Gamma, A) \longrightarrow C_b^n(\Gamma, A)\right)}$$

The inclusion of cocomplexes $C_b^*(\Gamma, A) \subset C^*(\Gamma, A)$ induces a comparison map

$$c_*: H_b^*(\Gamma, A) \longrightarrow H^*(\Gamma, A)$$

1.3 Amenable Groups

Definition: Let Γ be a discrete group. Γ is said to be amenable if there exists a left invariant mean on the Banach space $B(\Gamma)$ of real valued bounded functions on Γ equipped with the norm

$$\|f\|_\infty = sup\{f(\gamma) | \gamma \in \Gamma\}$$

In other words, Γ is said to be amenable if there exists a functional $m: B(\Gamma) \to \mathbb{R}$ satisfying the properties below:

1. $m(1_\Gamma) = 1$, where 1_Γ is the constant function with value 1,
2. $m(f) \geq 0$ if $f \geq 0$,
3. $m(L_\gamma f) = m(f)$, for all $f \in B(\Gamma)$ and $\gamma \in \Gamma$. Here, $L_\gamma: B(\Gamma) \to B(\Gamma)$ is the left shift operator defined as $L_\gamma f(\beta) = f(\gamma\beta)$ for all $\beta \in \Gamma$.

Example: Finite groups are amenable.

Let us define the operator

$$m: B(\Gamma) \longrightarrow \mathbb{R}$$
$$f \longmapsto \frac{1}{|\Gamma|}\sum_{\gamma\in\Gamma} f(\gamma)$$

Now, let us check whether this operator satisfies three properties we mentioned above.

1. Let 1_Γ be the constant function with value 1 and $|\Gamma| = n \in \mathbb{Z}^+$ since $\Gamma = \{\gamma_1, \cdots, \gamma_n\}$ is finite. Then,

$$\begin{aligned} m(1_\Gamma) &= \frac{1}{|\Gamma|}\sum_{\gamma\in\Gamma} 1_\Gamma(\gamma) \\ &= \frac{1}{n}\sum_{i=1}^{n} 1_\Gamma(\gamma_i) \\ &= \frac{1}{n}\big(1_\Gamma(\gamma_1) + \cdots + 1_\Gamma(\gamma_n)\big) \\ &= \frac{1}{n}\Bigg(\underbrace{1 + \cdots + 1}_{n\ times}\Bigg) \\ &= \frac{1}{n}n = 1 \end{aligned}$$

   Thus, $m(1_\Gamma) = 1$.

2. Let $f \geq 0$. Then, $f(\gamma) \geq 0$ for all $\gamma \in \Gamma$ and hence, the finite sum $\sum_{\gamma\in\Gamma} f(\gamma) \geq 0$. Since $|\Gamma| = n \in \mathbb{Z}^+$, it follows that $|\Gamma| \geq 0$. Due to $\sum_{\gamma\in\Gamma} f(\gamma) \geq 0$ and $|\Gamma| \geq 0$, then

$$\frac{1}{|\Gamma|}\sum_{\gamma\in\Gamma} f(\gamma) \geq 0$$

   Therefore, it demonstrates that $m(f) \geq 0$.

3. Finally, we need to show that $m(L_\gamma f) = m(f)$ for $L_\gamma f(\beta) = f(\gamma\beta)$. Since $\gamma, \beta \in \Gamma$ and Γ is a group, then there always exists an element $\alpha \in \Gamma$ such that $\alpha = \gamma\beta$. Hence,

$$
\begin{aligned}
m(L_\gamma f) &= \frac{1}{|\Gamma|}\sum_{\beta\in\Gamma} L_\gamma f(\beta) \\
&= \frac{1}{|\Gamma|}\sum_{\beta\in\Gamma} f(\gamma\beta) \\
&= \frac{1}{|\Gamma|}\sum_{\alpha\in\Gamma} f(\alpha) \\
&= m(f)
\end{aligned}
$$

Therefore, we conclude by 1,2 and 3 that finite groups are amenable.

Example: Abelian groups are amenable.

Proposition (Trauber): If Γ is an amenable group, then for all $n > 0$

$$H_b^n(\Gamma, \mathbb{R}) = 0$$

Proof: Consider the cocomplex

$$\cdots \longrightarrow C_b^{n-1}(\Gamma,\mathbb{R})^\Gamma \xrightarrow{\delta_{n-1}} C_b^n(\Gamma,\mathbb{R})^\Gamma \xrightarrow{\delta_n} C_b^{n+1}(\Gamma,\mathbb{R})^\Gamma \longrightarrow \cdots$$

for $n > 0$. Let $f \in Ker\delta_n$; i.e., $f \in C_b^n(\Gamma,\mathbb{R})^\Gamma$ with $\delta_n f = 0$. In order to show $H_b^n(\Gamma,\mathbb{R}) = Ker\delta_n/Im\delta_{n-1} = 0$, it is needed to be shown that $f \in Im\delta_{n-1}$ with the existence of $g \in C_b^{n-1}(\Gamma,\mathbb{R})^\Gamma$ with $\delta_{n-1}g = f$. Let $m: B(\Gamma) \longrightarrow \mathbb{R}$ be a left invariant mean and define for every $(\gamma_1,\cdots,\gamma_n) \in \Gamma^n$

$$
\begin{gathered}
g: \Gamma^n \longrightarrow \mathbb{R} \\
(\gamma_1,\cdots,\gamma_n) \longmapsto m\big(f(*,\gamma_1,\cdots,\gamma_n)\big)
\end{gathered}
$$

where $f(*,\gamma_1,\cdots,\gamma_n)$ is the bounded function from Γ to $\mathbb{R}$ by only evaluating the first coordinate $*$. Consider for all $(*,\gamma_0,\cdots,\gamma_n)$,

$$0 = \delta_n f(*,\gamma_0,\cdots,\gamma_n) = f(\gamma_0,\cdots,\gamma_n) - \sum_{i=0}^{n}(-1)^i f(*,\gamma_0,\cdots,\hat{\gamma}_i,\cdots,\gamma_n)$$

which implies

$$\sum_{i=0}^{n}(-1)^i f(*,\gamma_0,\cdots,\hat{\gamma}_i,\cdots,\gamma_n) = f(\gamma_0,\cdots,\gamma_n)$$

Thus, by considering the definition of g, the properties of a functional and the above equation, we have

$$
\begin{aligned}
\delta_{n-1} g(\gamma_0, \cdots, \gamma_n) &= \sum_{i=0}^{n} (-1)^i g(\gamma_0, \cdots, \hat{\gamma}_i, \cdots, \gamma_n) \\
&= \sum_{i=0}^{n} (-1)^i m\big(f(*, \gamma_0, \cdots, \hat{\gamma}_i, \cdots, \gamma_n)\big) \\
&= m\left(\sum_{i=0}^{n} (-1)^i f(*, \gamma_0, \cdots, \hat{\gamma}_i, \cdots, \gamma_n)\right) \\
&= m\big(f(\gamma_0, \cdots, \gamma_n)\big) \\
&= f(\gamma_0, \cdots, \gamma_n)
\end{aligned}
$$

Therefore, $\delta_{n-1} g = f$.

If we show that g is Γ-invariant, the proof will be completed. By the definition of g, we have $g(\gamma\gamma_1, \cdots, \gamma\gamma_n) = m\big(f(*, \gamma\gamma_1, \cdots, \gamma\gamma_n)\big)$. Since f is Γ-invariant, we can write $m\big(f(*, \gamma\gamma_1, \cdots, \gamma\gamma_n)\big) = m\big(f(\gamma^{-1} *, \gamma_1, \cdots, \gamma_n)\big)$. Since f is again Γ-invariant and $f(*, \gamma_1, \cdots, \gamma_n)$ is the function only evaluating the first coordinate $*$, we can write $m\big(f(\gamma^{-1} *, \gamma_1, \cdots, \gamma_n)\big) = m\big(\gamma \cdot f(*, \gamma_1, \cdots, \gamma_n)\big)$. And finally since m is left invariant, $m\big(\gamma \cdot f(*, \gamma_1, \cdots, \gamma_n)\big) = m\big(f(*, \gamma_1, \cdots, \gamma_n)\big) = g(\gamma_1, \cdots, \gamma_n)$. Therefore, we obtain $g(\gamma\gamma_1, \cdots, \gamma\gamma_n) = g(\gamma_1, \cdots, \gamma_n)$ which means g is Γ-invariant.

As a result, we conclude that if Γ is an amenable group, then for all $n > 0$ $H_b^n(\Gamma, \mathbb{R}) = 0$.

■

This proposition is not always true for usual cohomology group. In order to show the contradiction, we can look at the example below.

Example: $H^1(\mathbb{Z}, \mathbb{R}) = \mathbb{R}$.

Let us consider the sequence

$$C^0(\mathbb{Z}, \mathbb{R}) \xrightarrow{\delta_0} C^1(\mathbb{Z}, \mathbb{R}) \xrightarrow{\delta_1} C^2(\mathbb{Z}, \mathbb{R})$$

where δ_0 and δ_1 are defined as

$$\delta_1 f(z_0, z_1, z_2) = f(z_1, z_2) - f(z_0, z_2) + f(z_0, z_1)$$

$$\delta_0 g(z_0, z_1) = g(z_1) - g(z_0)$$

for $f \in C^1(\mathbb{Z}, \mathbb{R})$; i.e., $f: \mathbb{Z}^2 \to \mathbb{R}$, and $g \in C^0(\mathbb{Z}, \mathbb{R})$; i.e., $g: \mathbb{Z} \to \mathbb{R}$, and $z_0, z_1, z_2 \in \mathbb{Z}$. Now, let us first find the set $Ker\delta_1$.

$$
\begin{aligned}
Ker\delta_1 &= \{f | \delta_1 f = 0\} \\
&= \{f | \delta_1 f(z_0, z_1, z_2) = f(z_1, z_2) - f(z_0, z_2) + f(z_0, z_1) = 0\} \\
&= \left\{ f | f(z_0, z_2) = \underbrace{f(z_1, z_2)}_{\in \mathbb{R}} + \underbrace{f(z_0, z_1)}_{\in \mathbb{R}} \right\} \\
&= \mathbb{R} \oplus \mathbb{R}
\end{aligned}
$$

Secondly, let us find the set $Im\delta_0$. For $g \in C^0(\mathbb{Z}, \mathbb{R})$,

$$
\begin{aligned}
Im\delta_0 &= \{f | \delta_0 g = f\} \\
&= \{f | \delta_0 g(z_0, z_1) = g(z_1) - g(z_0) = f(z_0, z_1)\} \\
&= \left\{ f | \overbrace{\underbrace{g(z_1)}_{\in \mathbb{R}} - \underbrace{g(z_0)}_{\in \mathbb{R}}}^{\in \mathbb{R}} = f(z_0, z_1) \right\} \\
&= \left\{ f | \underbrace{f(z_0, z_1)}_{\in \mathbb{R}} \right\} \\
&= \mathbb{R}
\end{aligned}
$$

Therefore,

$$
H^1(\mathbb{Z}, \mathbb{R}) = \frac{Ker\delta_1}{Im\delta_0} = \frac{\mathbb{R} \oplus \mathbb{R}}{\mathbb{R}} = \mathbb{R}
$$

Thus, it is shown that the previous proposition can be false on usual cohomology.

1.4 Low Degree

In low degree, it is better to use a different cocomplex which is the inhomogeneous cocomplex. Define the sets

$$
\begin{aligned}
\bar{C}^n(\Gamma, A) &= \{g: \Gamma^n \longrightarrow A\} \\
\bar{C}_b^n(\Gamma, A) &= \{g: \Gamma^n \longrightarrow A | g \text{ } is \text{ } bounded\}
\end{aligned}
$$

The inhomogeneous coboundary map $d_n: \bar{C}^n \to \bar{C}^{n+1}$ is defined as

$$
d_n g(\gamma_1, \cdots, \gamma_{n+1}) = g(\gamma_2, \cdots, \gamma_{n+1}) + \sum_{i=1}^{n} (-1)^i g(\gamma_1, \cdots, \gamma_i \gamma_{i+1}, \cdots, \gamma_{n+1})
$$

$$
+(-1)^{n+1} g(\gamma_1, \cdots, \gamma_n)
$$

With these definitions in mind, we still use the following sequence to define cohomology.

$$0 \xrightarrow{0} \bar{C}^0(\Gamma, A) \xrightarrow{d_0} \bar{C}^1(\Gamma, A) \xrightarrow{d_1} \bar{C}^2(\Gamma, A) \xrightarrow{d_2} \cdots$$

Now, we analyze first three degrees in detail.

Degree 0: $\bar{C}^0(\Gamma, A) = \{g: \Gamma^0 \to A\}$ which consists of constant functions on A and hence, is nothing but A itself. For $g \in \bar{C}^0(\Gamma, A)$ and $\gamma \in \Gamma$, we have

$$d_0 g(\gamma) = g - g = 0$$

This implies $Kerd_0 = A$. Clearly, $Im0 = 0$. Hence, $H^0(\Gamma, A) = Kerd_0/Im0 = A$.

Since constant functions are bounded, it follows $\bar{C}^0(\Gamma, A) = \bar{C}_b^0(\Gamma, A)$. Thus, $H_b^0(\Gamma, A) = H^0(\Gamma, A) = A$.

Degree 1: Since $d_0 = 0$, we have $Imd_0 = 0$. To determine the $Kerd_1$, take an element $g \in \bar{C}^1(\Gamma, A)$ with $d_1 g = 0$. Hence, we obtain that

$$d_1 g(\gamma_1, \gamma_2) = g(\gamma_2) - g(\gamma_1\gamma_2) + g(\gamma_1) = 0$$

for every $\gamma_1, \gamma_2 \in \Gamma$. This implies $g(\gamma_1\gamma_2) = g(\gamma_1) + g(\gamma_2)$, which means g is a homomorphism from Γ to A. Hence, $Kerd_1 = Hom(\Gamma, A)$. Thus, $H^1(\Gamma, A) = Kerd_1/Imd_0 = Hom(\Gamma, A)$.

A homomorphism from Γ to $\mathbb{R}$ or $\mathbb{Z}$ must be unbounded and hence, for any group Γ, we have

$$H_b^1(\Gamma, \mathbb{R}) = H_b^1(\Gamma, \mathbb{Z}) = 0$$

Degree 2: Consider the case $A = \mathbb{R}$.

Definition: Let Γ be a group. The space of quasimorphisms of Γ is defined as

$$QM(\Gamma) = \{f: \Gamma \to \mathbb{R} | \exists D > 0 \; s.t. |f(\gamma_1) + f(\gamma_2) - f(\gamma_1\gamma_2)| < D, \forall \gamma_1, \gamma_2 \in \Gamma\}$$

Define a map from $QM(\Gamma)$ into the kernel of comparison map $Ker\left(c_2: H_b^2(\Gamma, \mathbb{R}) \to H^2(\Gamma, \mathbb{R})\right)$ by sending a quasimorphism f in $QM(\Gamma)$ to the cohomology class $[df]$. This map is surjective and its kernel consists of homomorphisms $f \in Hom(\Gamma, \mathbb{R})$ or bounded maps $f \in B(\Gamma, \mathbb{R})$. Therefore, we have an isomorphism

$$QM(\Gamma)/\big(Hom(\Gamma,\mathbb{R})\oplus B(\Gamma,\mathbb{R})\big)\cong Ker\Big(c_2: H_b^2(\Gamma,\mathbb{R})\to H^2(\Gamma,\mathbb{R})\Big)$$

Example (Brooks): Quasimorphisms in the free group F_2

Let us anayze the example of Brooks [3]. Let $F_2=\langle a,b\rangle$ be the free group on the two generators a and b. Select a word ω in a,b,a^{-1},b^{-1}. Define $f_\omega: F_2\to\mathbb{R}$ as

$$f_\omega(\gamma)=number\ of\ \omega\ occuring\ in\ \gamma-number\ of\ \omega^{-1}\ occuring\ in\ \gamma$$

for all $\gamma\in F_2$. If ω is the empty word or one of a,b,a^{-1},b^{-1}, then f_ω is a homomorphism. But if the number of letters a,b,a^{-1},b^{-1} used to obtain ω is greater or equal to 2, which is $length(\omega)\geq 2$, then f_ω can not be a homomorphism. For given $\gamma_1,\gamma_2\in F_2$, we have

$$|df_\omega(\gamma_1,\gamma_2)|=|f_\omega(\gamma_1)-f_\omega(\gamma_1\gamma_2)+f_\omega(\gamma_2)|\leq length(\omega)$$

since $|f_\omega(\gamma_1)-f_\omega(\gamma_1\gamma_2)+f_\omega(\gamma_2)|=1$, which is a case that one of ω or ω^{-1} additionally occurs in $\gamma_1\gamma_2$ (in which ω or ω^{-1} starts in the end of γ_1 and ends in the begining of γ_2), otherwise $|f_\omega(\gamma_1)-f_\omega(\gamma_1\gamma_2)+f_\omega(\gamma_2)|=0$, and in both cases $|f_\omega(\gamma_1)-f_\omega(\gamma_1\gamma_2)+f_\omega(\gamma_2)|\leq length(\omega)$. Therefore, f_ω is a quasimorphism.

Let us now show that f_ω is a nontrivial quasimorphism when $length(\omega)\geq 2$; i.e., f_ω belongs to $QM(F_2)$ but does not belong to $Hom(F_2,\mathbb{R})\oplus B(F_2,\mathbb{R})$. We see that $Hom(F_2,\mathbb{R})$ is two dimensional, generated by f_a and f_b. If f_ω did belong to $Hom(F_2,\mathbb{R})\oplus B(F_2,\mathbb{R})$, then there would exist $\alpha,\beta\in\mathbb{R}$ such that $f_\omega-\alpha f_a-\beta f_b$ is a bounded function on F_2. We need to follow two steps below.

Firstly, suppose ω is not a power of a or b. For any integer k, we have

$$f_\omega(a^k)-\alpha f_a(a^k)-\beta f_b(a^k)=0-\alpha k-0=-\alpha k$$

α must be equal to 0 in order $f_\omega-\alpha f_a-\beta f_b$ could be bounded independently of k. Also, we thus have

$$f_\omega(b^k)-\beta f_b(b^k)=0-\beta k=-\beta k$$

β must be equal to 0 in order $f_\omega-\beta f_b$ could be bounded independently of k. Hence, we find that f_ω must be bounded.

Secondly, suppose $\omega=a^m$ or $\omega=b^m$ for integers $|m|\geq 2$. Let $\omega=a^m$. For any integer k, we have

$$f_\omega(b^k)-\alpha f_a(b^k)-\beta f_b(b^k)=0-0-\beta k=-\beta k$$

and hence, $\beta = 0$.

$$f_\omega((ab)^k) - \alpha f_a((ab)^k) = 0 - \alpha k = -\alpha k$$

and hence, $\alpha = 0$. And now let $\omega = b^m$. For any integer k, we have

$$f_\omega(a^k) - \alpha f_a(a^k) - \beta f_b(a^k) = 0 - \alpha k - 0 = -\alpha k$$

and hence, $\alpha = 0$.

$$f_\omega((ab)^k) - \beta f_b((ab)^k) = 0 - \beta k = -\beta k$$

and hence, $\beta = 0$. It means that we again find f_ω must be bounded.

Now, it remains to indicate that f_ω is unbounded for ω is not an empty word. We first recognize that ω^{-1} does not occur in ω^k for an integer $k \geq 2$. If it did occur in ω^2, then ω would have to be of the form $\omega = vz$ with $\omega^{-1} = zv$. Then, since $\omega^{-1} = z^{-1}v^{-1} = zv$, we had $v = z = id$ as well as $\omega = id$; i.e., ω is an empty word. If ω is cyclically reduced (i.e., any power of the word can not be reduced), then for any integer k, we have

$$|f_\omega(\omega^k)| \geq |k|$$

which means that f_ω is unbounded. If ω is not cyclically reduced, then it has one of the form aza^{-1}, bzb^{-1}, $a^{-1}za$ or $b^{-1}zb$ where z is a reduced word. Suppose $\omega = aza^{-1}$ or $a^{-1}za$, then we have $|f_\omega((\omega b)^k)| \geq |k|$. Suppose $\omega = bzb^{-1}$ or $b^{-1}zb$, then we have $|f_\omega((\omega a)^k)| \geq |k|$. Hence, we conclude that f_ω is unbounded.

As a result, we find that the following group is nontrivial.

$$H_b^2(F_2, \mathbb{R}) \cong Ker\left(c_2 \colon H_b^2(F_2, \mathbb{R}) \to H^2(F_2, \mathbb{R})\right) \cong QM(F_2)/Hom(F_2, \mathbb{R}) \oplus B(F_2, \mathbb{R})$$

2. SYMPLECTIC MANIFOLDS

Detailed exposition on symplectic topology can be found in McDuff [11] and Silva [14]. We start with recalling basic facts about differential forms and their exterior algebra from Guillemin and Pollack [8].

2.1 Exterior Algebra

Let V be a real vector space. A p-tensor on V is a real valued function T on the cartesian product

$$V^p = \underbrace{V \times \cdots \times V}_{p\ times}$$

which is linear in each variable.

1-tensors are just linear functionals. Dot product on $\mathbb{R}^k$ is a 2-tensor. Determinant is a k-tensor on $\mathbb{R}^k$. Since sums and scalar multiples of multilinear functions are multilinear, the collection of all p-tensors is a vector space $\mathfrak{I}^p(V^*)$. Note that we have $\mathfrak{I}^p(V^*) = V^*$.

Let T be a p-tensor and S be a q-tensor. Then,

$$T \otimes S(v_1, \cdots, v_p, v_{p+1}, \cdots, v_{p+q}) = T(v_1, \cdots, v_p) \cdot S(v_{p+1}, \cdots, v_{p+q})$$

defines the tensor product $T \otimes S$ of T and S. In general, tensor product is not commutative but it is associative and distributive over addition.

Theorem: Let $\{\phi_1, \cdots, \phi_k\}$ be a basis for V^*. Then, the p-tensors $\left\{\phi_{i_1} \otimes \cdots \otimes \phi_{i_p} | 1 \leq i_1, \cdots, i_p \leq k\right\}$ form a basis for $\mathfrak{I}^p(V^*)$. As a result, $dim\mathfrak{I}^p(V^*) = k^p$.

Proof: Let $I = (i_1, \cdots, i_p)$ where each component is between 1 and k, and let

$$\phi_I = \phi_{i_1} \otimes \cdots \otimes \phi_{i_p}$$

Let $\{v_1, \cdots, v_k\}$ be the dual basis in V and v_I denote $\left(v_{i_1}, \cdots, v_{i_p}\right)$. Suppose there is another integer sequence J. We see that if $I = J$, then $\phi_I\left(v_J\right) = 1$ and otherwise $\phi_I\left(v_J\right) = 0$. By multilinearity, for two p-tensors T and S, we have

$$T = S \Leftrightarrow T\left(v_J\right) = S\left(v_J\right)$$

whatever J is. Hence, for $T\left(v_J\right) = S\left(v_J\right)$, we obtain that

$$S = \sum_I T\,(v_I)\phi_I$$

has to be equal to T. Thus, $\{\phi_I\}$ span $\mathfrak{I}^p(V^*)$ and the ϕ_I are also independent. To see, if

$$S = \sum_I a_I\, \phi_I$$

then for each J

$$0 = S\left(v_J\right) = a_J$$

Therefore, we conclude that the p-tensors $\phi_I = \phi_{i_1} \otimes \cdots \otimes \phi_{i_p}$ form a basis for $\mathfrak{I}^p(V^*)$.

■

Definition: A tensor T is alternating if

$$T\left(v_1, \cdots, v_i, \cdots, v_j, \cdots, v_p\right) = -T\left(v_1, \cdots, v_j, \cdots, v_i, \cdots, v_p\right)$$

To exemplify, 1-tensors are alternating, determinant is alternating but the dot product is not.

Let S_p denote the group of permutations of the numbers 1 to p. A permutation $\sigma \in S_p$ is called even or odd if it can be written as a product of an even or odd number of simple transpositions.

If T is any p-tensor, then

$$Alt(T) = \frac{1}{p!}\sum_{\sigma \in S_p} (-1)^\sigma T^\sigma$$

is always altenating where $T^\sigma\left(v_1, \cdots, v_p\right) = T\left(v_{\sigma(1)}, \cdots, v_{\sigma(p)}\right)$.

Note that if T is alternating, then $Alt(T) = T$.

The alternating p-tensors form a vector subspace $\Lambda^p(V^*)$ of $\mathfrak{I}^p(V^*)$ as sums and scalar multiples of alternating functions are alternating. However, this is not true for the tensor product.

If $T \in \Lambda^p(V^*)$ and $S \in \Lambda^q(V^*)$, then the wedge product

$$T \wedge S = Alt(T \otimes S) \in \Lambda^{p+q}(V^*)$$

is alternating. With this definition, wedge product is distributive over addition and scalar multiplication.

Theorem: Wedge product is associative; i.e.,

$$(T \wedge S) \wedge R = T \wedge (S \wedge R)$$

Let $\{\phi_1, \cdots, \phi_k\}$ be a basis for V^*, then for any p-tensor T, we can write

$$T = \sum t_{i_1,\cdots,i_p} \phi_{i_1} \otimes \cdots \otimes \phi_{i_p}$$

where the sum ranges over all index sequences $(i_1, \cdots, i_p)$ for which each index is between 1 and k. If T is alternating, then

$$T = Alt(T) = \sum t_{i_1,\cdots,i_p} Alt\left(\phi_{i_1} \otimes \cdots \otimes \phi_{i_p}\right) = \sum t_{i_1,\cdots,i_p} \phi_{i_1} \wedge \cdots \wedge \phi_{i_p}$$

Hence, the alternating tensors $\phi_{i_1} \wedge \cdots \phi_{i_p}$ denoted by ϕ_I where $I = (i_1, \cdots, i_p)$ span $\Lambda^p(V^*)$ (but not independent).

For the wedge product of two linear functionals on V, we have

$$\phi \wedge \psi = \frac{1}{2}(\phi \otimes \psi - \psi \otimes \phi) = -\psi \wedge \phi$$

implying that $\wedge$ is anticommutative on $\Lambda^1(V^*)$. In general, $T \wedge S = (-1)^{pq} S \wedge T$ whenever $T \in \Lambda^p(V^*)$ and $S \in \Lambda^q(V^*)$.

Theorem: If $\{\phi_1, \cdots, \phi_k\}$ is a basis for V^*, then

$$\left\{\phi_I = \phi_{i_1} \wedge \cdots \phi_{i_p} | 1 \leq i_1, \cdots, i_p \leq k\right\}$$

is a basis for $\Lambda^p(V^*)$. As a result,

$$dim \Lambda^p(V^*) = \binom{k}{p} = \frac{k!}{p!\,(k-p)!}$$

2.2 Differential Forms

Let M be a smooth manifold. A p-form on M is a function ω that assigns to each point $x \in M$ an alternating p-tensor $\omega(p) = \omega_p$ on the tangent space of M at x; $\omega(x) \in \Lambda^p(T_x M^*)$.

Two p-forms ω_1 and ω_2 may be added point by point to create a new p-form $\omega_1 + \omega_2$:

$$(\omega_1 + \omega_2)(x) = \omega_1(x) + \omega_2(x)$$

Wedge product is given by $(\omega \wedge \theta)(x)$=$\omega(x) \wedge \theta(x)$.

0-forms are arbitrary real valued functions on M.

If $\phi: M \to \mathbb{R}$ is a smooth function, then $d\phi_x: T_x M \to \mathbb{R}$ is a linear map at each point x. Therefore, $x \to d\phi_x$ defines a 1-form $d\phi$ on X, called the differential of ϕ.

If $x_1, \cdots, x_k$ are coordinate functions on $\mathbb{R}^k$, then $dx_1, \cdots, dx_k$ are 1-forms on $\mathbb{R}^k$. For any $z \in \mathbb{R}^k$, $T_z\mathbb{R}^k = \mathbb{R}^k$. We have $dx_i(z)(a_1, \cdots, a_k) = a_i$. Therefore, $dx_1(z), \cdots, dx_k(z)$ form the standard basis for $(\mathbb{R}^k)^*$. If $I = (i_1, \cdots, i_p)$ is a strictly increasing index sequence, then let $dx_I = dx_{i_1} \wedge \cdots \wedge dx_p$ be a p-form on $\mathbb{R}^k$. If U is an open subset of $\mathbb{R}^k$ and $f_i: U \to \mathbb{R}^k$ are the coordinate functions, then any p-form on U may be uniquely expressed as a sum $\sum_I f_I dx_I$.

In particular, if ϕ is a smooth function on $\mathbb{R}^k$, then

$$d\phi = \sum \frac{\partial \phi}{\partial x_i} dx_i$$

2.2.1 Pull-back Form

If $f: X \to Y$ is a smooth map and ω is a p-form on Y, then $f^*\omega$ is a p-form on X defined as follows: Let $f(x) = y$. Then, f induces a derivative map on tangent spaces $df_x: T_x X \to T_y Y$. Since $\omega(y)$ is an alternating p-tensor on $T_y Y$, its pull back to $T_x X$ is described through the transpose $(df_x)^*$. We define

$$f^*\omega(x) = (df_x)^*\omega(y)$$

When ω is a 0-form, then $f^*\omega = \omega \circ f$. The following properties of pull back is familiar:

$$f^*(\omega_1 + \omega_2) = f^*\omega_1 + f^*\omega_2$$
$$f^*(\omega \wedge \theta) = (f^*\omega) \wedge (f^*\theta)$$
$$(f \circ h)^*\omega = h^*(f^*\omega)$$

2.2.2 f^* on Euclidean space

Let $U \subset \mathbb{R}^k$ and $V \subset \mathbb{R}^l$ be open subsets and $f: V \to U$ be a smooth map with $f(x_1, \cdots, x_l) = \big(f_1(x_1, \cdots, x_l), \cdots, f_k(x_1, \cdots, x_l)\big)$ where each f_i is a smooth function on V; $f_i: V \to \mathbb{R}$.

For a point $y \in V$, df_y is represented by the matrix

$$\left(\frac{\partial f_i}{\partial y_i}(y)\right)$$

and $\left(df_y\right)^*$ is the transpose of this matrix. Therefore,

$$f^*dx_i = \sum_{j=1}^{l} \frac{\partial f_i}{\partial y_j} dy_j = df_i$$

Since an arbitrary form ω on U can be written uniquely as $\omega = \sum_I a_I dx_I$, f^* can be understood through 0-forms and 1-forms dx_i.

By the above listed properties of f^*, we have

$$f^*(\omega) = \sum_I (f^*a_I) df_I$$

Note that $f^*a_I = a_I \circ f$.

2.2.3 Volume form

Suppose $f: V \xrightarrow{\simeq} U$ is a diffeomorphism of two open sets in $\mathbb{R}^k$ and $\omega = dx_1 \wedge \cdots \wedge dx_k$ which is called volume form on U. Then,

$$f^*\omega(y) = \left(df_y\right)^* dx_1(x) \wedge \cdots \wedge \left(df_y\right)^* dx_k(x)$$
$$= det\big(df_y\big) dy_1(y) \wedge \cdots \wedge dy_k(y)$$

which yields the Change of Variables formula of differential Calculus.

Theorem: Assume $f: V \xrightarrow{\simeq} U$ is a diffeomorphism of open sets in $\mathbb{R}^k$ and that a is an integrable function on U. Then,

$$\int_U a dx_1 \cdots dx_k = \int_V (a \circ f)|det(df)| dy_1 \cdots dy_k$$

Note that if f is orientation preserving, then $det(df) > 0$. Thus, the above theorem can be stated in the form $\int_U \omega = \int_V f^*\omega$ if f is an orientation preserving diffeomorphism.

Let us now look at some examples below.

Example: (Line Integral)

Let ω be a smooth 1-form on $\mathbb{R}^3$ as

$$\omega = f_1 dx_1 + f_2 dx_2 + f_3 dx_3$$

and let $\gamma: I \xrightarrow{\simeq} \mathbb{R}^3$ be a simple curve; i.e., a diffeomorphism of the interval $I = [0,1]$ onto a compact one-manifold $C = \gamma(I)$ with boundary. Then,

$$\int_C \omega = \int_I \gamma^*\omega$$

If we say $\gamma(t) = \big(\gamma_1(t), \gamma_2(t), \gamma_3(t)\big)$, then

$$\gamma^* dx_i = d\gamma_i = \frac{d\gamma_i}{dt} dt$$

for $i = 1,2,3$.

Thus, we have

$$\int_C \omega = \sum_{i=1}^{3} \int_0^1 f_i[\gamma(t)] \frac{d\gamma_i}{dt}(t) dt$$

If we define $\vec{F} = (f_1, f_2, f_3)$ as a vector field in $\mathbb{R}^3$, then the right side is called the line integral of $\vec{F}$ over C, and is denoted as $\oint \vec{F}\, d\gamma$.

Example: Let ω be a compactly supported 2-form on $\mathbb{R}^3$ as

$$\omega = f_1 dx_2 \wedge dx_3 + f_2 dx_3 \wedge dx_1 + f_3 dx_1 \wedge dx_2$$

and S be a surface in $\mathbb{R}^3$, for simplicity, the graph of a function

$$G: \mathbb{R}^2 \longrightarrow \mathbb{R}$$
$$(x_1, x_2) \longmapsto x_3$$

Let us now find $\int_S \omega$.

We can choose the parametrization

$$h: \mathbb{R}^2 \longrightarrow S$$
$$(x_1, x_2) \longmapsto \big(x_1, x_2, G(x_1, x_2)\big)$$

We compute $h^* dx_1 \wedge dx_2$. Clearly,

$$h^* dx_1 \wedge dx_2 = dx_1 \wedge dx_2$$

We now compute $h^* dx_2 \wedge dx_3$.

$$\begin{aligned} h^* dx_2 \wedge dx_3 &= dx_2 \wedge dG \\ &= dx_2 \wedge \left(\frac{\partial G}{\partial x_1} dx_1 + \frac{\partial G}{\partial x_2} dx_2\right) \\ &= -\frac{\partial G}{\partial x_1} dx_1 \wedge dx_2 \end{aligned}$$

And we also compute $h^* dx_3 \wedge dx_1$.

$$\begin{aligned} h^* dx_3 \wedge dx_1 &= dG \wedge dx_1 \\ &= \left(\frac{\partial G}{\partial x_1} dx_1 + \frac{\partial G}{\partial x_2} dx_2\right) \wedge dx_1 \\ &= -\frac{\partial G}{\partial x_2} dx_1 \wedge dx_2 \end{aligned}$$

Thus, we have

$$\int_S \omega = \int_{\mathbb{R}^2} (n_1 f_1 + n_2 f_2 + n_3 f_3) dx_1 dx_2$$

where

$$\vec{n} = (n_1, n_2, n_3) = \left(-\frac{\partial G}{\partial x_1}, -\frac{\partial G}{\partial x_2}, 1\right)$$

We see that for any point $x = \big(x_1, x_2, G(x_1, x_2)\big)$ in the surface S, the vector $\vec{n}(x)$ is normal to S. If we let $\vec{u} = \vec{n}/|\vec{n}|$ be the unit normal vector and $\vec{F} = (f_1, f_2, f_3)$ and we define a smooth 2-form $dA = |\vec{n}| dx_1 \wedge dx_2$, then we can rewrite the integral as

$$\int_S \omega = \int_{\mathbb{R}^2} \big(\vec{F} \cdot \vec{u}\big) dA$$

The form dA is called the area form of the surface S.

2.3 Exterior Derivative

For a smooth p-form $\omega = \sum a_I dx_I$, define its exterior derivative $d\omega = \sum da_I \wedge dx_I$, which is a $(p+1)$-form.

Theorem:

1. $d(\omega_1 + \omega_2) = d\omega_1 + d\omega_2$
2. $d(\omega \wedge \theta) = (d\omega) \wedge \theta + (-1)^p \omega \wedge d\theta$
3. $d(d\omega) = 0$

As a consequence, if $g: V \overset{\simeq}{\rightarrow} U$ is a diffeomorphism of open sets of $\mathbb{R}^k$, then for any form ω on U

$$d(g^*\omega) = g^*(d\omega)$$

This result also holds for smooth maps of manifolds with boundary.

Example: Let us completely calculate the operator d in $\mathbb{R}^3$.

0**-forms:** If f is a function on $\mathbb{R}^3$, then we obtain

$$df = g_1 dx_1 + g_2 dx_2 + g_3 dx_3$$

Here, the right hand side of the above equation is the gradient vector field of f, denoted as $\overrightarrow{grad}(f)$.

1**-forms:** Let ω be a 1-form as

$$\omega = f_1 dx_1 + f_2 dx_2 + f_3 dx_3$$

Then, we have

$$\begin{aligned} d\omega &= df_1 \wedge dx_1 + df_2 \wedge dx_2 + df_3 \wedge dx_3 \\ &= g_1 dx_2 \wedge dx_3 + g_2 dx_3 \wedge dx_1 + g_3 dx_1 \wedge dx_2 \end{aligned}$$

where

$$g_1 = \frac{\partial f_3}{\partial x_2} - \frac{\partial f_2}{\partial x_3}\, , g_2 = \frac{\partial f_1}{\partial x_3} - \frac{\partial f_3}{\partial x_1}\, , g_3 = \frac{\partial f_2}{\partial x_1} - \frac{\partial f_1}{\partial x_2}$$

If we define the vector fields $\vec{F} = (f_1, f_2, f_3)$ and $\vec{G} = (g_1, g_2, g_3)$, then $\vec{G} = curl\vec{F}$.

2-**forms:** Let ω be a 2-form as

$$\omega = f_1 dx_2 \wedge dx_3 + f_2 dx_3 \wedge dx_1 + f_3 dx_1 \wedge dx_2$$

Then, we have

$$\begin{aligned} d\omega &= \left(\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} + \frac{\partial f_3}{\partial x_3}\right) dx_1 \wedge dx_2 \wedge dx_3 \\ &= \left(div\vec{F}\right) dx_1 \wedge dx_2 \wedge dx_3 \end{aligned}$$

3-**forms:** $d(3 - form) = 0$ on $\mathbb{R}^3$.

Exterior derivative of a 3-form is a 4-form, but there is no 4-form on $\mathbb{R}^3$. This is why $d(3 - form) = 0$.

2.3.1 DeRham cohomology groups

A p-form ω on X is closed if $d\omega = 0$ and exact if $\omega = d\theta$ for some $(p-1)$-form θ on X. Exact forms are closed since $d^2 = 0$. But not every closed form is exact.

Two closed p-forms ω and ω' are cohomologous $(\omega \sim \omega')$ if $\omega - \omega' = d\theta$. This is an equivalence relation and the set of equivalence classes is denoted $H^p(X)$: the p^{th} deRham cohomology group of X.

If $f: X \to Y$ is a smooth map, then f^* defines a map $f^*: H^p(Y) \to H^p(X)$ on cohomology groups.

2.4 Symplectic Linear Algebra

Let V be an m-dimensional vector space over $\mathbb{R}$ and $\Omega: V \times V \to \mathbb{R}$ be a linear map. Ω is called skew-symmetric if $\Omega(u, v) = -\Omega(v, u)$ for all $u, v \in V$.

Theorem: For any skew-symmetric bilinear map Ω on V, there is a basis $\{u_1, \cdots, u_k, e_1, \cdots, e_n, f_1, \cdots, f_n\}$ of V such that

$$\begin{aligned} &\Omega(u_i, v) = 0 \; for\ all\ i\ and\ v \in V \\ &\Omega(e_i, e_j) = 0 = \Omega(f_i, f_j) \; for\ all\ i, j \\ &\Omega(e_i, f_j) = \delta_{ij} \; for\ all\ i, j \end{aligned}$$

Proof: Let $U = \{u \in V | \Omega(u, v) = 0 \; for\ all\ v \in V\}$ and let $u_1, \cdots, u_k$ be a basis of U. Choose a complementary space W to U in V; i.e., $V = U \oplus W$. For any nonzero $e_1 \in W$, there is $f_1 \in W$ such that $\Omega(e_1, f_1) \neq 0$. Assume $\Omega(e_1, f_1) = 1$ and let

$$W_1 = span\ of\ e_1, f_1$$
$$W_1^{\Omega} = \{w \in W | \Omega(w, v) = 0\ for\ all\ v \in W_1\}$$

With this setting, if $v \in W_1 \cap W_1^{\Omega}$, then $v = ae_1 + bf_1$. Then, we have

$$\begin{aligned} 0 = \Omega(v, e_1) &= \Omega(ae_1 + bf_1, e_1) \\ &= a\Omega(e_1, e_1) + b\Omega(f_1, e_1) \\ &= a0 + b(-1) = -b \end{aligned}$$

implies $b = 0$. Likewise,

$$\begin{aligned} 0 = \Omega(v, f_1) &= \Omega(ae_1 + bf_1, f_1) \\ &= a\Omega(e_1, f_1) + b\Omega(f_1, f_1) \\ &= a1 + b0 = a \end{aligned}$$

implies $a = 0$. Since $v = 0$, we obtain $W_1 \cap W_1^{\Omega} = \{0\}$.

To show that $W = W_1 \oplus W_1^{\Omega}$, suppose $v \in W$ has $\Omega(v, e_1) = c$ and $\Omega(v, f_1) = d$. Then,

$$v = (-cf_1 + de_1) + (v + cf_1 - de_1)$$

It is clear that $-cf_1 + de_1 \in W_1$.

Claim: $v + cf_1 - de_1 \in W_1^{\Omega}$.

Suppose $gf_1 + he_1 \in W_1$. Then,

$$\begin{aligned} \Omega(v + cf_1 - de_1, gf_1 + he_1) = g\Omega(v, f_1) + h\Omega(v, e_1) \\ +cg\Omega(f_1, f_1) + ch\Omega(f_1, e_1) \\ -dg\Omega(e_1, f_1) - dh\Omega(e_1, e_1) \\ = gd + hc + cg0 - ch - dg - d0 = 0 \end{aligned}$$

Hence, $v + cf_1 - de_1 \in W_1^{\Omega}$, and we thus show that $W = W_1 \oplus W_1^{\Omega}$.

Continuing this Gram-Schmidt process by letting $0 \neq e_2 \in W_1^{\Omega}$, there is $f_2 \in W_1^{\Omega}$ such that $\Omega(e_2, f_2) \neq 0$. Assume $\Omega(e_2, f_2) = 1$ and $W_2 = span\ of\ e_2, f_2$ and so on.

Hence, one obtains

$$V = U \oplus W_1 \oplus W_2 \oplus \cdots \oplus W_n$$

as desired.

■

2.4.1 Symplectic vector space

Let V be an m-dimensional vector space over $\mathbb{R}$ and $\Omega: V \times V \to \mathbb{R}$ be a bilinear map. Define a map $\tilde{\Omega}: V \to V^*$ by $\tilde{\Omega}(v)(u) = \Omega(v, u)$ which is linear. Note that the kernel of $\tilde{\Omega}$ is the subspace U above.

Definition: A skew-symmetric bilinear map Ω is symplectic if $\tilde{\Omega}$ is bijective; in other words, if $U = \{0\}$. In this case, Ω is called a linear symplectic structure on V and (V, Ω) is called a symplectic vector space. If (V, Ω) is a sypmlectic vector space, then $dimV = 2n$ is even. (V, Ω) has a basis $e_1, \cdots, e_n, f_1, \cdots, f_n$ satisfying

$$\Omega(e_i, f_j) = \delta_{ij} \; and \; \Omega(e_i, e_j) = 0 = \Omega(f_i, f_j)$$

Once such a symplectic basis is given, Ω can be written in the matrix notation as

$$\Omega(u, v) = [u] \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} [v]^T$$

The prototype of a symplectic vector space is $(\mathbb{R}^{2n}, \Omega_0)$ with Ω_0 such that the basis

$$e_1 = (1,0, \cdots ,0), \cdots , e_n = \left(0,0, \cdots ,0, \overbrace{1}^{n.}, 0, \cdots ,0\right)$$

$$f_1 = \left(0,0, \cdots ,0, \underbrace{1}_{n+1.}, 0, \cdots ,0\right), \cdots , f_n = (0, \cdots ,0,1)$$

is a symplectic basis.

Definition: A symplectomorphism φ between symplectic vector spaces (V, Ω) and (V', Ω') is a linear isomorphism $\varphi: V \to V'$ such that $\varphi^*\Omega' = \Omega$. $\left((\varphi^*\Omega')(u, v) = \Omega'(\varphi(u), \varphi(v)) = \Omega(u, v)\right)$

In this case, (V, Ω) and (V', Ω') are called symplectomorphic.

Being symplectomorphic is an equivalence relation. By the above theorem, every $2n$-dimensional symplectic vector space is symplectomorphic to $(\mathbb{R}^{2n}, \Omega_0)$.

2.5 Symplectic Manifolds

Let ω be a de-Rham 2-form on a manifold M; that is, for each $p \in M$, the map $\omega_p: T_pM \times T_pM \to \mathbb{R}$ is skew-symmetric bilinear on the tangent space to M at p and ω_p varies smoothly in p. It is closed if $d\omega = 0$.

Definition: The 2-form ω is symplectic if ω is closed and ω_p is symplectic for all $p \in M$.

Note that if ω is symplectic, then $dimT_pM = dimM$ must be even.

Definition: A symplectic manifold is a pair (M, ω) where M is a manifold and ω is a symplectic form.

Example: Let $M = \mathbb{R}^{2n}$ with linear coordinate $x_1, \cdots, x_n, y_1, \cdots, y_n$. Set the form

$$\omega_0 = \sum_{i=1}^{n} dx_i \wedge dy_i$$

Since

$$\begin{aligned} d\omega_0 &= d\left(\sum_{i=1}^{n} dx_i \wedge dy_i\right) \\ &= d(dx_1 \wedge dy_1) + d(dx_2 \wedge dy_2) + \cdots + d(dx_n \wedge dy_n) \\ &= d^2x_1 \wedge dy_1 + (-1)^2 d^2y_1 \wedge dx_1 + \cdots + d^2x_n \wedge dy_n + (-1)^2 d^2y_n \wedge dx_n \\ &= 0 \wedge dy_1 + 0 \wedge dx_1 + \cdots + 0 \wedge dy_n + 0 \wedge dx_n = 0 \end{aligned}$$

we obtain that the form ω_0 is closed, and since

$$(\omega_0)^n = n!\, dx_1 \wedge \cdots \wedge dx_n \wedge dy_1 \wedge \cdots \wedge dy_n = n!\, Vol(\mathbb{R}^{2n}) \neq 0$$

it follows that ω_0 is symplectic.

Example: Let $M = \mathbb{C}^n$ with linear coordinate $z_1, \cdots, z_n$. Set the form

$$\omega_0 = \frac{i}{2}\sum_{k=1}^{n} dz_k \wedge d\overline{z_k}$$

If we identify $\mathbb{C}^n$ with $\mathbb{R}^{2n}$ by $z_k = x_k + iy_k$ for $k = 1, \cdots, n$, we see that this form is the standard form of $\mathbb{R}^{2n}$ described in the previous example.

Example: Let $M = S^2$ regarded as the set of the unit vectors in $\mathbb{R}^3$. Let T_pS^2 be the tangent plane of S^2 and u, v be the vectors of T_pS^2. Then, set the form

$$\omega_p(u, v) = \langle p, u \times v \rangle$$

This form is closed since its exterior derivative would be a 3-form which must be 0 since S^2 is a 2-dimensional manifold. Due to the fact that when $u \neq 0$, we choose $v = u \times p$ so that $\omega_p(u, v) \neq 0$, the form ω_p is symplectic.

Definition: Let (M_1, ω_1) and (M_2, ω_2) be $2n$-dimensional symplectic manifolds and let $g: M_1 \xrightarrow{\simeq} M_2$ be a diffeomorphism. Then, g is a symplectomorphism if $g^*\omega_2 = \omega_1$.

$\left((g^*\omega_2)_p(u, v) = (\omega_2)_{g(p)}\left(dg_p(u), dg_p(v)\right) \ where \ u, v \in T_pM_1\right)$

As for the symplectic vector spaces, $(\mathbb{R}^{2n}, \Omega_0)$ is the local protype of symplectic manifolds.

Theorem (Darboux): Let (M, ω) be a $2n$-dimensional symplectic manifold and let p be any point in M. Then, there is a coordinate chart $(U, x_1, \cdots, x_n, y_1, \cdots, y_n)$ centered at p such that on U

$$\omega = \sum_{i=1}^{n} dx_i \wedge dy_i$$

Such a chart $(U, x_1, \cdots, x_n, y_1, \cdots, y_n)$ is called a Darboux chart.

Example (Cotangent Bundle): Let X be an n-dimensional manifold. $M = T^*X$ is its cotangent bundle. Recall that if the manifold structure on X is described by coordinate chart $(U, x_1, \cdots, x_n)$ with $x_i: U \to \mathbb{R}$, then at any $x \in U$, the differentials $(dx_1)_x, \cdots, (dx_n)_x$ form a basis of T_x^*X, the cotangent space at $x \in X$. The cotangent bundle is the $2n$-dimensional smooth manifold $T^*X = \{(x, v) | v \in T_x^*X\}$. If $\xi \in T_x^*X$, then $\xi = \sum_{i=1}^{n} \xi_i (dx_i)_x$ for some coefficients $\xi_1, \cdots, \xi_n$. This induces a map

$$\begin{aligned} T^*U &\longrightarrow \mathbb{R}^{2n} \\ (x, \xi) &\longmapsto (x_1, \cdots, x_n, \xi_1, \cdots, \xi_n) \end{aligned}$$

giving the smooth structure to the $2n$-dimensional manifold T^*X. With this setting, define a 2-form ω on T^*U by

$$\omega = \sum_{i=1}^{n} dx_i \wedge d\xi_i$$

Note that for $\alpha = \sum_{i=1}^{n} \xi_i dx_i$ and $\omega = -d\alpha$, α is called the tautological 1-form and ω is the canonical symplectic form on the cotangent bundle. This makes every cotangent bundle into a symplectic manifold.

2.6 Lagrangian Submanifolds

Let X and M be manifolds with $dimX < dimM$. A map $i: X \to M$ is an immersion if $di_p: T_pX \to T_{i(p)}M$ is injective for any point $p \in X$.

An embedding is an immersion which is a homeomorphism onto its image.

A closed embedding is a proper injective immersion.

Definition: A submanifold of M is a manifold X with a closed embedding $i: X \hookrightarrow M$.

Definition: Let (M, ω) be a $2n$-dimensional symplectic manifold. A submanifold Y of M is a lagrangian submanifold if at each $p \in Y$, T_pY is a lagrangian subspace of T_pM; i.e.,

$$\omega_p|_{T_pY} \equiv 0 \; and \; dimT_pY = \frac{1}{2}dimT_pM$$

Example: The zero section $X_0 = \{(x,\xi) | \xi = 0 \in T_x^*X\}$ is an n-dimensional submanifold of T^*X, diffeomorphic to X. Note that $\omega = \sum_{i=1}^{n} dx_i \wedge d\xi_i \equiv 0$ on X_0. Therefore, it is a lagrangian submanifold of the cotangent bundle.

In other words, every manifold is a lagrangian submanifold of its cotangent bundle.

Example (Product Manifolds): Let (M_1, ω_1) and (M_2, ω_2) be $2n$-dimensional symplectic manifolds. Let $\varphi: M_1 \xrightarrow{\simeq} M_2$ be a diffeomorphism. Let us find that in which case φ is a symplectomorphism; i.e., $\varphi^*\omega_2 = \omega_1$.

Define the projection maps

$$pr_i: M_1 \times M_2 \longrightarrow M_i$$

where $pr_1(p_1, p_2) = p_1$ and $pr_2(p_1, p_2) = p_2$ for $i = 1,2$. Then, we obtain that $\omega = (pr_1)^*\omega_1 + (pr_2)^*\omega_2$ is a 2-form on $M_1 \times M_2$. Since

$$d\omega = (pr_1)^* \underbrace{d\omega_1}_{0} + (pr_2)^* \underbrace{d\omega_2}_{0} = 0$$

the form ω is closed, and since

$$\omega^{2n} = \binom{2n}{n} ((pr_1)^*\omega_1)^n \wedge ((pr_2)^*\omega_2)^n \neq 0$$

ω is symplectic.

Given nonzero real numbers λ_1 and λ_2, then $\lambda_1(pr_1)^*\omega_1 + \lambda_2(pr_2)^*\omega_2$ is also a symplectic form on $M_1 \times M_2$. Specifically, if we choose $\lambda_1 = 1$ and $\lambda_2 = -1$, we then obtain the twisted product form on $M_1 \times M_2$ as follows:

$$\widetilde{\omega} = (pr_1)^*\omega_1 - (pr_2)^*\omega_2$$

The graph of a diffeomorphism φ is the $2n$-dimensional submanifold of $M_1 \times M_2$, denoted as Γ_φ:

$$\Gamma_\varphi = Graph\,\varphi = \{(p, \varphi(p)) | p \in M_1\}$$

This submanifold is an embedded image of M_1 in $M_1 \times M_2$. The embedding map is

$$\begin{aligned} \gamma: M_1 &\longrightarrow M_1 \times M_2 \\ p &\longmapsto (p, \varphi(p)) \end{aligned}$$

Theorem: A diffeomorphism φ is symplectomorphism if and only if Γ_φ is a lagrangian submanifold of $(M_1 \times M_2, \widetilde{\omega})$.

Proof: The graph Γ_φ is lagrangian if and only if $\gamma^*\widetilde{\omega} = 0$.

$$\begin{aligned} \gamma^*\widetilde{\omega} &= \gamma^*(pr_1)^*\omega_1 - \gamma^*(pr_2)^*\omega_2 \\ &= \left(\underbrace{pr_1 \circ \gamma}_{id_{M_1}}\right)^* \omega_1 - \left(\underbrace{pr_2 \circ \gamma}_{\varphi}\right)^* \omega_2 \end{aligned}$$

Thus, we obtain that

$$\gamma^*\widetilde{\omega} = 0 \Leftrightarrow \omega_1 - \varphi^*\omega_2 = 0 \Leftrightarrow \varphi^*\omega_2 = \omega_1$$

■

2.7 Isotopies and Vector Fields

Let M be a manifold and $\rho: M \times \mathbb{R} \to M$ be a map with $\rho(p,t) = \rho_t(p)$.

Definition: The map ρ is an isotopy if each $\rho_t: M \xrightarrow{\simeq} M$ is a diffeomorphism and $\rho_0 = id_M$.

If we are given an isotopy ρ, then we obtain a time-dependent vector field; that is, a family of vector fields $v_t, t \in \mathbb{R}$, which satisfy at $p \in M$

$$v_t(p) = \frac{d}{ds}\rho_s(q)|_{s=t}$$

where $q = \rho_t^{-1}(p)$; i.e.,

$$\frac{d\rho_t}{dt} v_t \circ \rho_t$$

If we are given a time-dependent vector fields v_t and M is compact or the v_t's are compactly supported, then there exists an isotopy ρ satisfying the previous equation.

Let M be compact. Then, there exists one to one correspondence

$$\{isotopies\ of\ M\} \leftrightarrow \{time - dependent\ vector\ fields\ on\ M\}$$

$$\underbrace{\rho_t, t}_{\in \mathbb{R}} \leftrightarrow \underbrace{v_t, t}_{\in \mathbb{R}}$$

Definition: If $v_t = v$ is independent of t, the associated isotopy is called the exponential map or flow of v and denoted $exp\ tv$; i.e.,

$$\left\{exp\ tv: M \xrightarrow{\simeq} M \middle| t \in \mathbb{R}\right\}$$

is the unique smooth family of diffeomorphisms satisfying

$$exp\ tv|_{t=0} = id_M$$

$$\frac{d}{dt}(exp\ tv)(p) = v\big(exp\ tv(p)\big)$$

Definition: The Lie derivative is the operator defined as

$$\mathcal{L}_v: \Omega^k(M) \longrightarrow \Omega^k(M)$$

$$\omega \longmapsto \frac{d}{dt}(exp\ tv)^*\omega|_{t=0}$$

If a vector field v_t is time dependent, then its flow; i.e., the corresponding isotopy ρ, locally exists. There is a one-parameter family of local diffeomorphisms ρ_t within the neighborhood of any point p and for small t as

$$\frac{d\rho_t}{dt} = v_t \circ \rho_t$$

$$\rho_0 = id$$

Thus, we define the Lie derivative by v_t is

$$\mathcal{L}_{v_t}: \Omega^k(M) \longrightarrow \Omega^k(M)$$

$$\omega \longmapsto \frac{d}{dt}(\rho_t)^*\omega|_{t=0}$$

Theorem (Cartan formula): $\mathcal{L}_v\omega = \iota_v d\omega + d\iota_v\omega$

Theorem: For a smooth family $\omega_t, t \in \mathbb{R}$ of d-forms, we have

$$\frac{d}{dt}(\rho_t)^*\omega_t = (\rho_t)^*\left(\mathcal{L}_{v_t}\omega_t + \frac{d\omega_t}{dt}\right)$$

Proof: If $f = (x, y)$ is a real valued function, we have by the chain rule

$$\frac{d}{dt}f(t,t) = \frac{d}{dx}f(x,t)|_{x=t} + \frac{d}{dy}f(t,y)|_{y=t}$$

Thus,

$$\begin{aligned}\frac{d}{dt}(\rho_t)^*\omega_t &= \frac{d}{dx}(\rho_x)^*\omega_t|_{x=t} + \frac{d}{dy}(\rho_t)^*\omega_y|_{y=t}\\ &= (\rho_x)^*\mathcal{L}_{v_x}\omega_t|_{x=t} + (\rho_t)^*\frac{d\omega_y}{dy}|_{y=t}\\ &= (\rho_t)^*\left(\mathcal{L}_{v_t}\omega_t + \frac{d\omega_t}{dt}\right)\end{aligned}$$

■

2.7.1 Tubular neighborhood theorem

Let M be an n-dimensional manifold and X be a k-dimensional submanifold with $k < n$ and inclusion map

$$i: X \hookrightarrow M$$

At each point x in X, the tangent space to X can be seen as a subspace of the tangent space to M by linear inclusion $di_x: T_xX \hookrightarrow T_xM$. Here, we denote $x = i(x)$. The normal space to X at x is $(n-k)$-dimensional vector space $N_xX = T_xM/T_xX$. The normal bundle set of X is

$$NX = \{(x, v) | x \in X, v \in N_xX\}$$

The set NX is an n-dimensional manifold since it has the structure of a vector bundle over X of rank $n - k$ under the naturel projection. The zero section of NX

$$\begin{aligned} i_0: X &\hookrightarrow NX\\ x &\longmapsto (x, 0)\end{aligned}$$

is an embedding map into X as a closed submanifold of NX. If the intersection of N_xX and a neighborhood $\mathcal{U}_0$ of the zero section X in NX is convex with each fiber, then $\mathcal{U}_0$ is called convex.

Theorem (Tubular Neighborhood Theorem): The diagram

$$\begin{array}{ccc} NX \supseteq \mathcal{U}_0 & \overset{\varphi}{\to} & \mathcal{U} \subseteq M \\ i_0 \nwarrow & & \nearrow i \\ & X & \end{array}$$

is commutative where $\mathcal{U}_0$ is a convex neighborhood of X in NX, $\mathcal{U}$ is a neighborhood of X in M and $\varphi: \mathcal{U}_0 \overset{\simeq}{\to} \mathcal{U}$ is a diffeomorphism.

2.7.2 Tangent space to the group of symplectomorphisms

The symplectomorphisms of a symplectic manifold (M, ω) form the group

$$Symp(M, \omega) = \left\{ f: M \overset{\simeq}{\to} M \middle| f^*\omega = \omega \right\}$$

We use notions from C^1-topology in order to understand what $T_{id}\big(Symp(M, \omega)\big)$ is and what a neighborhood of id in $Symp(M, \omega)$ looks like:

C^1-topology

Let X and Y be two manifolds.

Definition: A sequence of map f_i's converges in the C^0-topology to $f: X \to Y$ if and only if $f_i: X \to Y$ converges uniformly on compact sets.

Definition: A sequence of C^1 maps $f_i: X \to Y$ converges in the C^1-topology to $f: X \to Y$ if and only if both itself and the sequence $df_i: TX \to TY$ converges uniformly on compact sets.

Let (M, ω) be a compact symplectic manifold and take an element $f \in Symp(M, \omega)$. Then, both $Graph\, f$ and $Graph\, id = \Delta$ are lagrangian subspaces of $(M \times M, pr_1^*\omega - pr_2^*\omega)$. Here, $pr_i: M \times M \to M$ are the projections to each factors for $i = 1{,}2$.

By the tubular neighborhood theorem, we can say there exists a neighborhood $\mathcal{U}$ of Δ such that $\Delta \simeq M$ in $(M \times M, pr_1^*\omega - pr_2^*\omega)$ which is symplectomorphic to a neighborhood $\mathcal{U}_0$ of M in (T^*M, ω_0). Let φ be the symplectomorphism from $\mathcal{U}$ to

$\mathcal{U}_0$. Let f be sufficiently C^1-close to id; that is, be in some sufficiently small neighborhood of id in the C^1-topology. Then,

1. We assume that $Graph\, f \subseteq \mathcal{U}$. Then let $j: M \hookrightarrow \mathcal{U}$ the embedding as $Graph\, f$ and $i: M \hookrightarrow \mathcal{U}$ the embedding as $Graph\, id = \Delta$.
2. The map j is sufficiently C^1-close to id.
3. By the tubular neighborhood theorem, $\mathcal{U} \simeq \mathcal{U}_0 \subseteq T^*M$, thus the j and i we mentioned in 1. induce $j_0: M \hookrightarrow \mathcal{U}$ is an embedding where $j_0 = \varphi \circ j$ and $i_0: M \hookrightarrow \mathcal{U}$ is an embedding as 0-section. Therefore, we obtain

$$\begin{array}{ccc} \mathcal{U} & \overset{\varphi}{\to} & \mathcal{U}_0 \\ i \nwarrow & & \nearrow i_0 \\ & M & \end{array}$$

and

$$\begin{array}{ccc} \mathcal{U} & \overset{\varphi}{\to} & \mathcal{U}_0 \\ j \nwarrow & & \nearrow j_0 \\ & M & \end{array}$$

where

$$\begin{aligned} i(p) &= (p, p) \\ i_0(p) &= (p, 0) \\ j(p) &= \big(p, f(p)\big) \\ j_0(p) &= \varphi\big(p, f(p)\big) \end{aligned}$$

for $p \in M$.

4. The map j_0 is sufficiently C^1-close to i_0. This implies that the image set $j_0(M)$ intersects each T_p^*M at one point μ_p which depends smoothly on p.
5. The image of j_0 is the image of a smooth section $\mu: M \to T^*M$; i.e., a 1-form $\mu = j_0 \circ (\pi \circ j_0)^{-1}$.

Therefore, we conclude that

$$Graph\, f \simeq \{(p, \mu_p) | p \in M, \mu_p \in T_p^*M\}$$

Theorem: If μ is a 1-form sufficiently C^1-close to the zero 1-form, then for some diffeomorphisms $f: M \overset{\simeq}{\to} M$, we have

$$\{(p,\mu_p)|p \in M, \mu_p \in T_p^*M\} \simeq Graph\, f$$

Conclusion: A C^1-neighborhood of zero in the vector space of closed 1-forms on M is homeomorphic to a small C^1-neighborhood of id in $Symp(M,\omega)$. Thus,

$$T_{id}\big(Symp(M,\omega)\big) = \{\mu \in \Omega^1(M)|d\mu = 0\}$$

$T_{id}Symp(M,\omega)$ contains the space of exact 1-forms

$$\{\mu = dh|h \in C^\infty(M)\} \simeq {C^\infty(M)}\Big/{locally\ constant\ functions}$$

2.8 Hamiltonian and Symplectic Vector Fields

Let (M,ω) be a symplectic manifold and $H: M \to \mathbb{R}$ be a smooth function. The differential of H is 1-form. Since (M,ω) is a symplectic and so by nondegeneracy of it, there is a unique vector field X_H on M such that $\iota_{X_H}\omega = dH$. Suppose M is compact or X_H is complete and let ρ_t be the one-parameter family of diffeomorphisms for $t \in \mathbb{R}$ from M to M generated by X_H such that

$$\rho_0 = id_M$$

$$\frac{d\rho_t}{dt} \circ \rho_t^{-1} = X_H$$

Claim: Each diffeomorphism ρ_t preserves ω; that is, for all t,

$$\rho_t^*\omega = \omega$$

Proof: For all t, we have

$$\frac{d}{dt}\rho_t^*\omega = \rho_t^*\mathcal{L}_{X_H}\omega = \rho_t^*\big(d\iota_{X_H}\omega + \iota_{X_H}d\omega\big) = 0$$

since $\iota_{X_H}\omega = dH$ and $d\omega = 0$. Thus, $\rho_t^*\omega = \omega$.

■

Hence, every function on (M,ω) gives a family of symplectomorphisms.

Definition: A vector field X_H as above is called hamiltonian vector field with hamiltonian function H.

Example: Let M be the sphere S^2, $H(\theta,h) = h$ be the height function and $\omega = d\theta \wedge dh$. Thus, for the symplectic manifold $(M,\omega) = (S^2, d\theta \wedge dh)$, we have

$$\iota_{X_H}(d\theta \wedge dh) = dh \Leftrightarrow X_H = \frac{\partial}{\partial\theta}$$

Therefore, $\rho_t(\theta, h) = (\theta + t, h)$ which is rotation about the vertical axis and the height function of H is thus preserved by this motion.

Remark: If X_H is hamiltonian, then

$$\mathcal{L}_{X_H}H = \iota_{X_H}dH = \iota_{X_H}\iota_{X_H}\omega = 0$$

Therefore, hamiltonian vector fields preserve their hamiltonian functions and each integral curve $\{\rho_t(x) | t \in \mathbb{R}\}$ of X_H has to be contained in a level set of H:

$$H(x) = (\rho_t^* H)x = H\big(\rho_t(x)\big)$$

for all t.

Definition: Let X be a vector field on M preserving ω; i.e., $\mathcal{L}_X\omega = 0$. Then, the vector field X is called a symplectic vector field. Hence,

$$X\ is\ symplectic \Leftrightarrow \iota_X\omega\ is\ closed$$
$$X\ is\ hamiltonian \Leftrightarrow \iota_X\omega\ is\ exact$$

Every symplectic vector field on every contractible open set is hamiltonian. Additionally, if $H^1_{deRham}(M) = 0$, then every symplectic vector field is hamiltonian.

Example: Consider the 2-torus. On $(M, \omega) = (\mathbb{T}^2, d\theta_1 \wedge d\theta_2)$, the vector fields

$$X_1 = \frac{\partial}{\partial\theta_1}\ and\ X_2 = \frac{\partial}{\partial\theta_2}$$

are symplectic but not hamiltonian. $H^1(\mathbb{T}^2, \mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}$ and two generators of this group are $[d\theta_1] = \iota_{X_2}\omega \neq 0$ and $[d\theta_2] = \iota_{X_1}\omega \neq 0$.

2.9 The Group of Symplectomorphisms

Let (M^{2n}, ω) be a symplectic manifold. Consider the group of symplectomorphisms

$$Symp(M, \omega) = \{\phi \in Diff^{\infty}(M) | \phi^*\omega = \omega\}$$

where $Diff^{\infty}(M)$ is the group of C^{∞}-diffeomorphisms on M.

$Symp(M, \omega)$ is by definition equipped with C^{∞}-topology and is observed by Weinstein [15], it is locally path connected.

Let $Symp_0(M,\omega)$ denote the path component of id_M in $Symp(M,\omega)$ where $id_M\colon M \to M$ is the identity function with $id_M(p) = p$. For any $\psi \in Symp_0(M,\omega)$, let $\psi_t \in Symp(M,\omega)$ for all $t \in [0,1]$ such that $\psi_0 = id_M$ and $\psi_1 = \psi$. There exists a unique family of vector fields (corresponding to ψ_t)

$$X_t\colon M \longrightarrow TM \; such\ that \; \frac{d}{dt}\psi_t = X_t \circ \psi_t$$

The vector fields X_t are called symplectic since they satisfy $\mathcal{L}_{X_t}\omega = 0$, where $\mathcal{L}_{X_t}\omega$ denotes the Lie derivative of the form ω along the vector field X_t. By Cartan's formula

$$\mathcal{L}_{X_t}\omega = i_{X_t}(d\omega) + d\big(i_{X_t}\omega\big)$$

Hence, X_t is a symplectic vector field if and only if $i_{X_t}\omega$ is closed for all t. If moreover $i_{X_t}\omega$ is exact, in other words $i_{X_t}\omega = dH_t$, $H_t\colon M \to \mathbb{R}$ a family of smooth functions, then X_t are called a Hamiltonian diffeomorphism and H_1 is a Hamiltonian for ψ. The Hamiltonian diffeomorphisms form a group as a subgroup in the identity component of the group of symplectomorphisms. Thus, we have a sequence of groups and inclusions:

$$Ham(M) \hookrightarrow Symp_0(M) \hookrightarrow Symp(M) \hookrightarrow Diff^{\infty}(M)$$

Theorem (Moser): Any path ω_t, $t \in [0,1]$ of cohomologous symplectic forms on a closed manifold M is induced by an isotopy $\phi_t\colon M \to M$; i.e., $\phi_t^*(\omega_t) = \omega_0$, $\phi_t = id$.

The teorem says that $Symp(M)$ and $Ham(M)$ depend only on the diffeomorphism class of ω. Therefore, when ω_t varies along a path of cohomologous forms, the topological or algebraic properties of these groups do not change.

2.9.1 Flux homomorphism

Let ω_t, $t \in [0,1]$ be a loop of symplectomorphisms on a smooth symplectic manifold (M,ω). Then, the flux homomorphism $\widetilde{Flux}\colon \widetilde{Symp}_0(M) \to H^1(M,\mathbb{R})$ is given by

$$\widetilde{Flux}(\{\omega_t\}) = \int_0^1 \big[i_{X_t}\omega\big]dt \in H^1(M,\mathbb{R})$$

where X_t is defined by $\frac{d}{dt}\psi_t = X_t \circ \psi_t$. Here, $\widetilde{Symp}_0(M)$ denotes the universal cover of $Symp_0(M)$. Recall that the universal cover of a space is just the set of equivalence classes of paths in that space with fixed ends. The notation $\{\omega_t\}$ denotes the equivalence class of homotopic isotopies that have fixed ends $\omega_0 = id\,, \omega_1 = \omega$.

Let us now express the the theorem of Banyaga [2].

Theorem: Let $\psi \in Symp_0(M)$. Then, ψ is a Hamiltonian symplectomorphism if and only if there exists a symplectic isotopy $\psi_t \in Symp_0(M)$ such that $\omega_0 = id\,, \omega_1 = \omega\,, Flux(\{\psi_t\}) = 0$. Moreover, if $Flux(\{\psi_t\}) = 0$, then $\{\psi_t\}$ is isotopic with fixed ends to a Hamiltonian isotopy.

Flux in the theorem is the homomorphism that $\widetilde{Flux}$ induces on $Symp_0(M)$. Namely, if

$$\Gamma = \widetilde{Flux}\big(\pi_1(Symp_0(M).)\big) \subset H^1(M, \mathbb{R})$$

is the flux group, then

$$Flux: Symp_0(M) \longrightarrow {H^1(M, \mathbb{R})}/{\Gamma}$$

The above theorem says that the group of Hamiltonian diffeomorphisms is exactly the kernel of the flux homomorphism.

2.9.2 Calabi homomorphism

Let (M, ω) be a noncompact symplectic manifold, $\widetilde{Ham}_c(M)$ be the universal cover of the compactly supported Hamiltonian diffeomorphisms. Then, the Calabi homomorphism $\widetilde{Cal}: \widetilde{Ham}_c(M) \to \mathbb{R}$ is defined by

$$\{\phi_t\} \longmapsto \int_0^1 \int_M H_t \omega^n dt$$

where H_t is given by $i_{X_t}\omega = dH_t$ and $\frac{d}{dt}\phi_t = X_t \circ \phi_t$.

If one considers Hamiltonian diffeomorphisms supported in an open set U, then a local version of the Calabi homomorphism for compact manifolds may be defined; i.e.,

$$\widetilde{Cal}_U: \widetilde{Ham}_U(M)_c \longrightarrow \mathbb{R}$$

As in the flux homomorphism, the map induces homomorphisms for the underlying groups.

Namely, if $\Lambda = Cal\left(\pi_1\left(Ham_c(M)\right)\right)$, then

$$Cal: Ham_c(M) \longrightarrow {}^{\mathbb{R}}/_{\Lambda}$$

can be defined.

2.10 Calabi Quasimorphisms

Recall that a group G is called simple if it has no normal subgroups. G is perfect if G is equal to its commutator group; i.e., $G = [G, G]$. A classical result due to Banyaga is as follows.

Theorem (Banyaga): If M is closed, then the group $Ham(M)$ is simple (and hence perfect).

The theorem implies that there is no nontrivial homomorphism $h: Ham(M) \to \mathbb{R}$. However, recent studies show that in some cases the group $Ham(M)$ admits quasimorphisms.

An open set U is called if displacable if there exists $f \in Ham(M, \omega)$ such that $f(U) \cap \overline{U} = \emptyset$.

Definition: Let $\mu: \widetilde{Ham}(M, \omega) \to \mathbb{R}$ be a homogeneous quasimorphism such that

1. If $U \subset M$ is an open displacable set, then

$$\mu|_{\widetilde{Ham}(U)} = \widetilde{Cal}_U$$

2. Let $f, g \in \widetilde{Ham}(M, \omega)$ represent paths in $Ham(M, \omega)$ generated by normalized Hamiltonians F and G, respectively. Then,

$$vol(M) \cdot \int_0^1 \min_M (F_t - G_t)\, dt \leq \mu(g) - \mu(f) \leq vol(M) \cdot \int_0^1 \max_M (F_t - G_t)\, dt$$

Then, μ is called a Calabi quasimorphism.

So far, $\mathbb{C}P^n$, S^2, $S^2 \times S^2$ and $\Sigma_{g>1}$ surface of genus g are shown to admit such Calabi quasimorphisms [13].

3. CONCLUSION

In this chapter, we conclude the study with the application of Polterovich [13] and the examples of Gambauda & Ghys [5].

3.1 Application (Polterovich)

Let G be a group and $[G, G]$ its commutator subgroup. Every element $h \in [G, G]$ can be written as a product of simple commutators $fgf^{-1}g^{-1} = h$. The commutator norm $\|h\|$ is by definition the minimal number of simple commutators needed in order to represent h. For a homogeneous quasimorphism $\mu: G \to \mathbb{R}$, Barg [1] shows

$$\|h\| \geq const(\mu) \cdot \mu(h) \quad , h \in [G, G]$$

Therefore, if μ does not vanish on $[G, G]$, then the diameter of the group $[G, G]$ with respect to commutator norm is infinite.

Next, consider an open covering $\{u_1, \cdots, u_m\}$ for a closed connected symplectic manifold (M, ω). Banyaga [2] shows that any Hamiltonian diffeomorphism f can be written as a product of diffeomorphisms g_i such that each g_i is supported in U_i and moreover $g_i \in Ker(Cal)$. If $l(f)$ is the minimal number of g_i's needed in order to represent f, then the existence of a Calabi quasimorphism gives a lower bound for $l(f)$. See Polterovich [13].

Example (Gambaudo & Ghys): Following the above mentioned results of Polterovich, Gambaudo and Ghys [5] construct Calabi quasimorphisms on Σ_g, S^2 and D^2 in a more elementary setting based on the classical notion of rotation number.

Definition: Let $Homeo_+(S^1)$ be the group of orientation preserving homeomorphisms of S^1. For $f \in Homeo_+(S^1)$, choose a lift $\tilde{f}$ of f which is a homeomorphism of $\mathbb{R}$ such that it satisfies $\tilde{f}(x+m) = \tilde{f}(x) + m$ for every real number x and every integer m. Poincaré's rotation number is the map $\rho: Homeo_+(S^1) \to \mathbb{R}/_{\mathbb{Z}}$ defined as

$$\rho(f) = \lim_{n \to \infty} \frac{\tilde{f}^n(x_0) - x_0}{n}$$

where x_0 is a fixed base point in $\mathbb{R}$. Poincaré proved that the limit exists and is independent of the point x_0. Its lift to universal covers $\tilde{\rho}: \widetilde{Homeo}_+(S^1) \to \mathbb{R}$ with

$\tilde{\rho}(id) = 0$ is called the translation number. $\tilde{\rho}$ is the unique homogeneous quasimorphism defined on $\widetilde{Homeo}_+(S^1)$ (Ghys [6]). As an example, let f be a rotation by an angle θ. Then, $\tilde{f}(x) = x + \theta$ and hence, $\rho(f) = \theta$.

Example: (The disc D^2)

Let $G = Diff_0^\infty(D^2, \partial D^2, \omega_0)$ denote the group of C^∞-diffeomorphisms of D^2 preserving the area form $\omega_0 = dx \wedge dy$ which are identity near the boundary.

Let $g \in G$ and choose an isotopy $(g_t)_{t\in[0,1]}$ between $g_0 = id$ and $g_1 = g$. Denote by $dg_t(x)$ the differential at $x \in D^2$. $dg_t(x)$ can be thought as an element of $SL(2,\mathbb{R})$. The first column of $dg_t(x)$ is a nonzero vector in $\mathbb{R}^2$ denoted by $v_t(x)$. If $Ang_g(x)$ is the variation of the angle of this curve $v_t(x)$ of nonzero vectors when t varies from 0 to 1, then define

$$r(g) = \int_{D^2} Ang_g(x) d\omega_0$$

Claim: r is a quasimorphism.

$$\left|Ang_{gh}(x) - Ang_h(x) - Ang_g\big(h(x)\big)\right| < \frac{1}{2}$$

since for a curve $A_t \in SL(2,\mathbb{R})$, the variations of the arguments of $A_t(u)$ and $A_t(v)$ differ by at most one half turn for any two nonzero vectors $u, v \in \mathbb{R}^2$.

Definition: Homogenizing r defines the Ruelle's quasimorphism

$$\mathfrak{R}_{D^2}(g) = \lim_{p\to\infty} \frac{1}{p} r(g^p)$$

Let Rot_θ denote the rotation around the origin by θ. For a function $f: [0,1] \to \mathbb{R}$ which is zero near 1 and constant near 0, define an area preserving diffeomorphism F_f of the disc by

$$F_f(x) = Rot_{f(\|x\|)}(x)$$

Then,

$$\mathfrak{R}_{D^2}\big(F_f\big) = \frac{1}{\pi}\int_0^1 f(t) 2\pi t dt = 2\int_0^1 tf(t)dt \neq 0$$

Example: (The torus $\mathbb{T}^2$)

The above construction can also be adapted to $G = Symp(\mathbb{T}^2, \omega)$ where ω is the usual area form on $\mathbb{T}^2$. G contains translations $\mathbb{T}^2 \subset Symp(\mathbb{T}^2, \omega)$ as a subgroup (Banyaga) which is a homotopy equivalence.

Let $g \in G$ and $(g_t)_{t\in[0,1]}$ be an isotopy between g and id. Its lift $(\widetilde{g_t})$ to $\mathbb{R}^2$ gives an isotopy in $\mathbb{R}^2$. Since the tangent bundle of the torus is trivialized, for each point x, the curve $dg_t(x)$ is a curve in $SL(2,\mathbb{R})$. Any loop in $Symp(\mathbb{T}^2, \omega)$ is homotopic to a loop in translation subgroup, hence to a loop with constant differential. Thus, $Ang_g(x)$ is well defined and $\mathfrak{R}_{\mathbb{T}^2}$ is a quasimorphism on $Symp(\mathbb{T}^2, \omega)$.

Theorem: Both quasimorphisms $\mathfrak{R}_{\mathbb{T}^2}$ and $\mathfrak{R}_{D^2}$ defined above are Calabi quasimorphisms.

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