**ISTANBUL TECHNICAL UNIVERSITY ★ GRADUATE SCHOOL OF SCIENCE ENGINEERING AND TECHNOLOGY**

# ALMOST $L$-STRUCTURES AND NEARLY-KAEHLERIAN STRUCTURES

**M.Sc. THESIS**

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**Department of Mathematical Engineering**

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İSTANBUL TEKNİK ÜNİVERSİTESİ ★ FEN BİLİMLERİ ENSTİTÜSÜ

# HEMEN HEMEN *L*-YAPILARI VE YAKLAŞIK-KAHLER YAPILAR

**YÜKSEK LİSANS TEZİ**

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**Mustafa Deniz TÜRKOĞLU**, a M.Sc. student of ITU Graduate School of Science Engineering and Technology 509081013 successfully defended the thesis entitled **"ALMOST *L*-STRUCTURES AND NEARLY-KAEHLERIAN STRUCTURES"**, which he prepared after fulfilling the requirements specified in the associated legislations, before the jury whose signatures are below.

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***To math, nothing could teach us,***
***the finite humans, better what infinity is!***

## FOREWORD

First of all, I am proud of studying both on modern mathematics and geometry in these territories where mathematics, more properly, where geometry which was the first form of mathematics came into life. Now my target is to come up with the most acclaimed geometry book ever after Euclid in the following days.

In fact this thesis belongs to all people whom I learned something or even just a word throughout my interactions so far, they molded my thoughts. In addition to this, I would like to thank to my beloved advisor Assoc. Prof. Dr. Fatma ÖZDEMİR. She supported me taking my first steps into the world of geometry so I could realize my first academical studies, as well as giving me strength in hard times to restart my master degree after a long break.

I also thank all my friends and best friends with whom we sometimes have diverging times and converging times on the linear life cycle, and I would like to thank my İstanbul and my family for being right beside me throughout these times. As I do not want to keep you apart from the mysterious world of geometry; I would like to thank one more person, my dearest grandmother in your presence. I will always cherish the memories of all the years we had together. I have always felt her love in all the years without her, and I will keep on holding on her memories in the future. Although she is now the infinity herself, our souls will eventually meet at a point of intersection...again.

"Let no one untrained in geometry enter." Motto over the entrance of Plato's Academy

June 2013 Mustafa Deniz TÜRKOĞLU

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## ABBREVIATIONS

| | |
|---|---|
| **co-vector** | **:** Covariant Vector |
| **$ELW_n$** | **:** Einstein L-Weyl Space |
| **$KW_n$** | **:** Kaehlerian Weyl Space |
| **$LW_n$** | **:** Almost L-Weyl Space |
| **$NKW_n$** | **:** Nearly-Kaehlerian Weyl Space |
| **$SKW_n$** | **:** Almost Semi-Kaehlerian Space |
| **$W_n$** | **:** n-dimensional Weyl Space |

## LIST OF SYMBOLS

| | | |
|---|---|---|
| $[jk,h]$ | : | Christoffel Symbol Of The First Kind |
| $\left\{ {i \atop jk} \right\}$ | : | Christoffel Symbol Of The Second Kind |
| $T_{li}{}^{h}$ | : | Tensor Field Of Type (1,2) |
| $S_{kj}^{\ \ t}$ | : | Torsion Tensor |
| $\Gamma^{i}_{(jk)}$ | : | The Symmetric Part Of The Connection Coefficient |
| $\Gamma^{i}_{[jk]}$ | : | The Anti-Symmetric Part Of The Connection Coefficient |
| $R_{kji}^{\ \ \ h}$ | : | The Curvature Tensor Of The Riemann Space |
| $R_{kjil}$ | : | The Covariant Curvature Tensor Of The Riemann Space |
| $R_{ji}$ | : | The Ricci Tensor Of The Riemann Space |
| $R$ | : | The Scalar Curvature Of The Riemann Space |
| $\nabla$ | : | The Riemannian Connection |
| $\nabla_k$ | : | Covariant Derivative Of The Riemannian Connection Respect To $k$ |
| $W_n(g,w)$ | : | n-dimensional Weyl Space With Metric Tensor g<br>and 1-form Complementary Co-Vector Field |
| $(0,1)$ | : | 0 Contravariant, 1 Covariant Form |
| $D$ | : | Torsion Free (Symmetric) Connection In Weyl Space |
| $D_k$ | : | Covariant Derivative Of Torsion Free Connection In Weyl Space<br>Respect To $k$ |
| $g$ | : | Metric Tensor |
| $g_{ij}$ | : | 0 Contravariant, 2 Covariant, (0,2) Form Metric Tensor With Weight 2 |
| $g_i^j$ | : | 1 Contravariant, 1 Covariant, (1,1) Form Metric Tensor |
| $g^{ij}$ | : | 2 Contravariant, 0 Covariant, (2,0) Form Metric Tensor With Weight $-2$ |
| $\bar{g}$ | : | Renormalized Metric Tensor |
| $w$ | : | 1-form Complementary Co-Vector Field |
| $w_k$ | : | Covariant Derivative Of 1-form Complementary Co-Vector Field<br>Respect To $k$ |
| $\bar{w}$ | : | Renormalized 1-form Complementary Co-Vector Field |
| $\Omega$ | : | Scalar Function on Weyl Space |
| $A$ | : | Quantity on Weyl Space |
| $p$ | : | Weight p |
| $\dot{\nabla}$ | : | The Prolonged Extended Derivative |
| $\dot{\nabla}_k$ | : | The Prolonged Extended Covariant Derivative Respect To $k$ |
| $W_{kji}{}^{h}$ | : | The Curvature Tensor Of Weyl Space |
| $W_{kjil}$ | : | The Covariant Curvature Tensor Of Weyl Space |
| $W_{ji}$ | : | The Ricci Tensor Of Weyl Space |
| $W$ | : | The Scalar Curvature Tensor Of Weyl Space |
| $\Gamma_{ji}^{\ l}$ | : | The Coefficient Of The Weyl Space |

| | | |
|---|---|---|
| $\delta_i^k$ | : | Kronecker Delta |
| $\partial_k$ | : | Partial Derivative Respect To $k$ |
| $W_{(ji)}$ | : | The Symmetric Part Of The Ricci Tensor Of The Weyl Space |
| $W_{[ji]}$ | : | The Anti-Symmetric Part Of The Ricci Tensor Of The Wey Space |
| $F_i^{\ j}$ | : | Type (1,1) Tensor With Weight 0 |
| $[\dot{\nabla}_j \dot{\nabla}_k]$ | : | The Lie Bracket Of The Prolonged Derivative |
| $N_{ij}^{\ \ k}$ | : | Nijhenuis Torsion Tensor |
| $\alpha$ | : | Scalar Function |
| $\beta$ | : | Scalar Function |
| $\theta$ | : | Scalar Function |
| $G_l^{\ j}$ | : | Generalized Einstein L-Weyl Tensor |

## ALMOST $L$-STRUCTURES AND NEARLY-KAEHLERIAN STRUCTURES

## SUMMARY

In this study, almost $L$-structures and nearly-Kaehlerian structures are investigated on Weyl spaces. After giving the definitions and theorems related to these structures, for an Einstein $L$-Weyl space to be an Einstein space, the necessary and sufficient condition is given.

In the first chapter, the basic notations and definitions of Riemannian space are given. And then, the definitions and properties of Weyl space have been studied. In addition, almost complex and almost Kaehlerian structures on Weyl space are defined. In the second chapter, we introduce almost $L$-structures and nearly-Kaehlerian structures and give some theorems concerning these structures which will be useful to find out generalized Einstein $L$-Weyl tensor. Next, we show that how all these structures are related to Einstein space. Also, we state the generalized Einstein $L$-Weyl tensor in terms of almost $L$-structures on $W_n$.

Let $W_n$ be $n$-dimensional space having a conformal metric tensor $g$ and let $D$ be torsion free (symmetric) connection. If $D$ satisfies the compatibility condition

$$D_k g_{ij} = 2 w_k g_{ij}$$

then $W_n$ is called a Weyl space, denoted by $W_n(g,w)$, where $w$ is a 1-form, named complementary co-vector field. Under the renormalization of the metric tensor $g$

$$\bar{g} = \Omega^2 g \ \ (\Omega > 0)$$

where $\Omega$ is a scalar function defined on $W_n$ and also $w$ is transformed by the law,

$$\bar{w} = w + d\ln\Omega$$

The prolonged (extended) covariant derivative of the satellite $A$ of weight $\{p\}$ is defined as

$$\dot{\nabla}_k A = D_k A - p\, w_k A$$

The curvature tensor, covariant curvature tensor, the Ricci curvature tensor and the scalar curvature of Weyl space are defined by, respectively,

$$\begin{aligned}
&W_{kji}{}^{h} = W_{kjil}\, g^{lh}\\
&W_{kjil} = W_{kji}{}^{m}\, g_{ml}\\
&W_{ji} = g^{kl}\, W_{kjil} = W_{kji}{}^{k}\\
&W = g^{ji}\, W_{ji}
\end{aligned}$$

The curvature tensor of Weyl space $W_n$ is defined

$$W_{kji}{}^{l} = R_{kji}{}^{l} - \delta^l_j w_{ki} + \delta^l_k w_{ji} + \delta^l_i (w_{jk} - w_{kj}) + g^{ls}(g_{ji} w_{ks} - g_{ki} w_{js})$$

where $w_{jk} = \nabla_j w_k + w_j w_k - \frac{1}{2} g_{jk} w_t w^t$ and $R_{kji}{}^{l}$ represents the Riemannian curvature tensor.

The curvature tensor, the covariant curvature tensor and the Ricci curvature tensor of $W_n(g,w)$ satisfy the following symmetry properties:

$$W_{kjim} = -W_{jkim}$$
$$W_{kjim} + W_{kjmi} = 2g_{im}(\nabla_j w_k - \nabla_k w_j) = 4g_{im}\nabla_{[j}w_{k]}$$

The Ricci curvature tensor of $W_n$ is calculated in terms of the Ricci curvature $R_{ji}$ of the Riemannian space as

$$W_{ji} = R_{ji} + (n-2)w_{ji} + (w_{ji} - w_{ij}) + g_{ji}g^{km}w_{km}$$

The first and the second Bianchi identities for $W_n$, respectively,

$$W_{kji}{}^{l} + W_{jik}{}^{l} + W_{ikj}{}^{l} = 0$$
$$\dot{\nabla}_m W_{kji}{}^{l} + \dot{\nabla}_k W_{jmi}{}^{l} + \dot{\nabla}_j W_{mki}{}^{l} = 0$$

Let $W_n$ be even dimensional ($n \geq 2$). A tensor $F_i{}^j$ of type (1,1) with weight $\{0\}$ is called an almost complex structure on $W_n$ if $F_i{}^j$ satisfies

$$F_i{}^j F_j{}^k = -\delta_i^k$$

and $W_n$ is called an almost complex space.

If $W_n$ admits an almost complex structure $F_i{}^j$ satisfying the condition

$$g_{ij} F_h{}^i F_k{}^j = g_{hk}$$

then $F_i{}^j$ is called an almost Hermitian structure with a Hermitian metric $g$ on $W_n$.

If almost Hermitian structure $F_i{}^j$ satisfies

$$\dot{\nabla}_k F_i{}^j = 0 \qquad \text{for all } i,j,k$$

then, it is called a Kaehlerian structure and then $W_n$ becomes a Kaehlerian space denoted by $KW_n$.

An almost Hermitian structure $F_i{}^j$ is called an almost Kaehlerian structure on $W_n$ (respectively, almost Kaehlerian Weyl space) if the tensor $F_i{}^j$ satisfies

$$F_{hij} = \dot{\nabla}_h F_{ij} + \dot{\nabla}_i F_{jh} + \dot{\nabla}_j F_{hi} = 0$$

An almost Hermitian structure $F_i{}^j$ is called an almost semi-Kaehlerian structure on $W_n$ (respectively, semi-Kaehlerian Weyl space) if the tensor $F_i{}^j$ satisfies

$$F_i \equiv \dot{\nabla}_j F_i{}^j = 0$$

An almost Hermitian structure $F_i{}^j$ of weight $\{0\}$ with a Hermitian metric $g$ on $W_n$ satisfying the relation

$$G_{ij}^k \equiv \dot{\nabla}_i F_j{}^k + \dot{\nabla}_j F_i{}^k = 0$$

is called a nearly-Kaehlerian structure or an almost Tachibana structure.

An almost Hermitian structure $F_i^{\ j}$ on $W_n$ satisfying

$$[\dot{\nabla}_j \dot{\nabla}_k] F_i^{\ h} \equiv (\dot{\nabla}_j \dot{\nabla}_k - \dot{\nabla}_k \dot{\nabla}_j) F_i^{\ h} = 0$$

is called an almost $L$-structure and a Weyl space admitting an almost $L$-structure is called almost $L$-Weyl space denoted by $LW_n$.

**Theorem 2.1.** A nearly-Kaehlerian structure is an almost semi-Kaehlerian Weyl structure.

**Theorem 2.2.** A nearly-Kaehlerian Weyl space is a Kaehlerian-Weyl space if the structure is integrable.

**Theorem 2.3.** An almost Hermitian structure $F_i^{\ j}$ is an almost $L$-structure on $LW_n$ if and only if

$$F_i^{\ a} W_{kja}^{\ \ \ h} = F_a^{\ h} W_{kji}^{\ \ \ a}$$

**Theorem 2.4.** On an almost Hermitian Weyl space, the condition in **Theorem 2.3.** is equivalent to

$$W_{kijh} = F_j^{\ l} F_h^{\ t} W_{kilt}$$

**Theorem 2.5.** On an almost $L$-Weyl space, the following conditions hold;

$$\textbf{i)}\ W_{jm} = \frac{1}{2} F^{ks} F_m^{\ i} W_{ksij} + F^{ks} F_{jm} (w_{k,s} - w_{s,k})$$

$$\textbf{ii)}\ Q \equiv W + \frac{1}{2} F^{sk} F^{ij} W_{skij} = 0$$

$$\textbf{iii)}\ W = F^{ji} F_m^{\ k} W_{kji}^{\ \ \ m}$$

**Theorem 2.7.** For a nearly-Kaehlerian structure $F_i^{\ j}$ on $NKW_n$,

$$Q \geq 0$$

where $Q$ is given by **Theorem 2.5.**, and the equality holds if and only the structure $F_i^{\ j}$ is Kaehlerian.

**Theorem 2.8.** If an almost $L$-structure $F_i^{\ j}$ on $LW_n$ is almost semi-Kaehlerian and satisfies either

$$\dot{\nabla}^k G_{hi}^{\ \ j} = 0 \quad or \quad \dot{\nabla}^k F_{hij} = 0$$

where $\dot{\nabla}^k = g^{jk} \dot{\nabla}_j$, then the structure $F_i^{\ j}$ is Kaehlerian.

**Theorem 3.1.** For an Einstein $L$-Weyl space $(ELW)$ to be an Einstein space, the necessary and sufficient condition is

$$\frac{(n-2)}{2} [\nabla_j w_i + \nabla_i w_j + 2 w_j w_i] = \theta g_{ji}$$

where $\theta$ is a scalar function. Also we obtain generalized Einstein $L$-Weyl tensor as

$$G_l^{\ j} = F^{jm} F^{kt} W_{klmt} - \frac{1}{4} \delta_l^j F^{mt} F_k^{\ h} W_{mth}^{\ \ \ \ k} - 2 g^{jk} \nabla_{[l} w_{k]}$$

**HEMEN HEMEN *L*-YAPILARI VE
YAKLAŞIK-KAHLER YAPILAR**


**ÖZET**

Bu tezde, Weyl uzayı üzerinde hemen-hemen $L$-yapıları ve yaklaşık Kahler yapılar incelenmiştir. Bu yapılarla ilgili tanımlar ve teoremler verildikten sonra, genelleştirilmiş Einstein $L$-Weyl tensörü bulunmuştur.

Birinci bölümde, Rieman uzayı için temel notasyonlar ve ifadeler verilmiştir. Sonrasında ise, Weyl uzayı için tanımlar ve eğrilik tensörünün özellikleri gösterilmiştir. Bunlara ek olarak, Weyl uzayında hemen-hemen kompleks ve yaklaşık Kahler yapılar tanımlanmıştır. İkinci bölümde, hemen-hemen $L$-yapılar ve yaklaşık-Kahler yapılar verilerek, genelleştirilmiş Einstein $L$-Weyl tensörünün bulunmasında faydalı olacak teoremlere yer verilmiştir. Son bölümde ise, tüm bu yapıların Einstein uzayıyla nasıl bağlantılı olduğundan bahsedilmiştir. Ayrıca Einstein $L$-Weyl uzayının, Einstein uzayı olabilmesi için gerek ve yeter koşul ispatlandıktan sonra genelleştirilmiş Einstein $L$-Weyl tensörü bulunmuştur.

$W_n$, konformal $g$ metriğine sahip $n$ boyutlu bir uzay ve $D$ burulmasız (simetrik) bir konneksiyon olmak üzere, eğer $D$ aşağıda belirtilen uygunluk koşulunu sağlarsa, $W_n$'e Weyl uzayı denir ve $W_n(g,w)$ ile gösterilir:

$$D_k g_{ij} = 2 w_k g_{ij}$$

Burada $w$, tamamlayıcı eş-vektör alanı diye isimlendirilen 1 formdur.

$g$ metrik tensörünün normalizasyonu ile

$$\bar{g} = \Omega^2 g \;\; (\Omega > 0)$$

ifadesi bulunur. Burada $\Omega$, $W_n$ üzerinde tanımlı skaler bir fonksiyondur. Ayrıca $w$ belirtilen bu kurala göre

$$\bar{w} = w + d \ln \Omega$$

şeklinde değişir.

$\{p\}$ ağırlıklı $A$ uydusunun genişletilmiş türevi aşağıda belirtildiği gibi tanımlanmıştır.

$$\dot{\nabla}_k A = D_k A - p\, w_k A$$

Weyl uzayında eğrilik tensörü, kovaryant eğrilik tensörü, Ricci eğrilik tensörü ve skaler eğrilik tensörü sırasıyla

$$\begin{aligned} W_{kji}{}^{h} &= W_{kjil}\, g^{lh} \\ W_{kjil} &= W_{kji}{}^{m}\, g_{ml} \\ W_{ji} &= g^{kl}\, W_{kjil} = W_{kji}{}^{k} \\ W &= g^{ji}\, W_{ji} \end{aligned}$$

olarak tanımlanmıştır.

$W_n$ Weyl uzayının eğrilik tensörü

$$W_{kji}{}^{l} = R_{kji}{}^{l} - \delta_j^l w_{ki} + \delta_k^l w_{ji} + \delta_i^l (w_{jk} - w_{kj}) + g^{ls}(g_{ji}w_{ks} - g_{ki}w_{js}) \tag{1}$$

şeklindedir. Burada $w_{jk}$ ifadesi $w_{jk} = \nabla_j w_k + w_j w_k - \frac{1}{2} g_{jk} w_t w^t$ ve $R_{kji}{}^{l}$ Rieman eğrilik tensörünü belirtir.

$W_n(g,w)$'nin eğrilik, kovaryant eğrilik ve Ricci eğrilik tensörleri aşağıda belirtilen simetri özelliklerini sağlarlar.

$$\begin{aligned} &W_{kjim} = -W_{jkim} \\ &W_{kjim} + W_{kjmi} = 2g_{im}(\nabla_j w_k - \nabla_k w_j) = 4g_{im}\nabla_{[j}w_{k]} \end{aligned}$$

$W_n$'de Ricci eğrilik tensörü, Rieman uzayının $R_{ji}$ Ricci eğrilik tensörü cinsinden,

$$W_{ji} = R_{ji} + (n-2)w_{ji} + (w_{ji} - w_{ij}) + g_{ji}g^{km}w_{km}$$

ile ifade edilir.

$W_n$'de birinci ve ikinci Bianchi özdeşlikleri sırasıyla,

$$\begin{aligned} &W_{kji}{}^{l} + W_{jik}{}^{l} + W_{ikj}{}^{l} = 0 \\ &\dot{\nabla}_m W_{kji}{}^{l} + \dot{\nabla}_k W_{jmi}{}^{l} + \dot{\nabla}_j W_{mki}{}^{l} = 0 \end{aligned}$$

ile verilir.

$(n \geq 2)$ için $W_n$ çift boyutlu bir uzay olsun. $\{0\}$ ağırlıklı (1,1) tipinde $F_i{}^{j}$ tensörün $W_n$ üzerinde hemen-hemen kompleks yapı olması için

$$F_i{}^{j} F_j{}^{k} = -\,\delta_i^k$$

ifadesini sağlaması gerekir. Bu durumda $W_n$'e hemen-hemen kompleks uzay denir.

Eğer $W_n$, hemen-hemen kompleks yapıyı sağlayan bir $F_i{}^{j}$ tensörüne sahip ve

$$g_{ij} F_h{}^{i} F_k{}^{j} = g_{hk}$$

ifadesini sağlıyorsa, $F_i{}^{j}$ tensörüne, $W_n$ üzerinde Hermitik $g$ metriğine sahip hemen-hemen Hermite yapı denir.

Eğer $F_i{}^{j}$ hemen-hemen Hermitsel yapısı

$$\dot{\nabla}_k F_i{}^{j} = 0 \qquad \text{for all } \; i,j,k$$

ifadesini sağlarsa, bu yapıya Kahler yapı denir ve $W_n$, $KW_n$ şeklinde ifade edilen Kahler uzaya dönüşür.

$F_i{}^{j}$ hemen-hemen Hermite yapısının, $W_n$ üzerinde hemen-hemen Kahler yapı olarak adlandırılabilmesi için $F_i{}^{j}$ tensörünün

$$F_{hij} = \dot{\nabla}_h F_{ij} + \dot{\nabla}_i F_{jh} + \dot{\nabla}_j F_{hi} = 0$$

denklemini sağlaması gerekir.

$F_i^{\ j}$ hemen-hemen Hermite yapısının, $W_n$ üzerinde hemen-hemen yarı-Kahler yapı olması için $F_i^{\ j}$ tensörü

$$F_i \equiv \dot{\nabla}_j F_i^{\ j} = 0$$

ifadesini sağlar.

$W_n$ üzerinde $g$ hermite metriği ile verilen $\{0\}$ ağırlıklı hemen-hemen $F_i^{\ j}$ hermite yapısı

$$G_{ij}^{\ \ k} \equiv \dot{\nabla}_i F_j^{\ k} + \dot{\nabla}_j F_i^{\ k} = 0$$

ifadesini sağlar ve bu yapıya yaklaşık-Kahler yapı veya hemen-hemen Tachibana yapısı denilerek, $NKW_n$ ile gösterilir.

$W_n$ üzerinde hemen-hemen Hermite yapı olan $F_i^{\ j}$

$$[\dot{\nabla}_j \dot{\nabla}_k] F_i^{\ h} \equiv (\dot{\nabla}_j \dot{\nabla}_k - \dot{\nabla}_k \dot{\nabla}_j) F_i^{\ h} = 0$$

denklemini sağlarsa bu yapıya hemen-hemen $L$-yapısı denir ve bu yapıya sahip Weyl uzayına da hemen-hemen $L$-Weyl uzayı denir ve $LW_n$ şeklinde gösterilir.

**Teorem 2.1.** Yaklaşık-Kahler yapı ($NKW_n$ uzayı), hemen-hemen yarı-Kahler Weyl ($SKW_n$ uzayı) yapısıdır.

**Teorem 2.2.** Eğer yapı integre edilebiliyorsa, yaklaşık-Kahler Weyl uzayı Kahler Weyl uzayıdır.

**Teorem 2.3.** $F_i^{\ j}$ hemen-hemen hermite yapısının, $LW_n$ üzerinde hemen-hemen $L$-yapısı olması için gerek ve yeter şart

$$F_i^{\ a} W_{kja}^{\ \ \ h} = F_a^{\ h} W_{kji}^{\ \ \ a}$$

olmasıdır.

**Teorem 2.4.** Hemen-hemen Hermite Weyl uzayında, **Teorem 2.3.**'deki ifade aşağıda verilen ifadeye denktir.

$$W_{kijh} = F_j^{\ l} F_h^{\ t} W_{kilt}$$

**Teorem 2.5.** Hemen-hemen $L$-Weyl uzayında,

$$\textbf{i)}\ W_{jm} = \frac{1}{2} F^{ks} F_m^{\ i} W_{ksij} + F^{ks} F_{jm} (w_{k,s} - w_{s,k})$$

$$\textbf{ii)}\ Q \equiv W + \frac{1}{2} F^{sk} F^{ij} W_{skij} = 0$$

$$\textbf{iii)}\ W = F^{ji} F_m^{\ k} W_{kji}^{\ \ \ m}$$

verilen koşullar sağlanır.

**Teorem 2.7.** **Teorem 2.5.**'de verilen $Q$ için, $NKW_n$ üzerinde yaklaşık-Kahler $F_i^{\ j}$ yapısı,

$$Q \geq 0$$

eşitliğini sağlaması için gerek ve yeter koşul $F_i^{\ j}$ yapısının Kahler olmasıdır.

**Teorem 2.8.** Eğer $LW_n$ üzerinde hemen-hemen $F_i^{\ j}$ $L$-yapısı hemen-hemen yarı-Kahler'dir ve

$$\dot{\nabla}^k G_{hi}^{\ \ j} = 0$$

ya da

$$\dot{\nabla}^k F_{hij} = 0$$

koşullarından birini sağlar. Burada $\dot{\nabla}^k = g^{jk}\dot{\nabla}_j$ şeklinde tanımlıdır.

**Teorem 3.1.** Einstein $L$-Weyl uzayının $ELW$, Einstein uzayı olabilmesi için gerek ve yeter koşul

$$\frac{(n-2)}{2}[\nabla_j w_i + \nabla_i w_j + 2w_j w_i] = \theta g_{ji}$$

olmasıdır. Burada $\theta$, $W_n$ 'de tanımlı skaler bir fonksiyondur.

Einstein $L$-Weyl uzayında,

$$G_l{}^j = W_l{}^j - \frac{1}{2}\delta_l^j W - 2g^{jk}\nabla_{[l}w_{k]}$$

denklemiyle verilen tensöre genelleştirilmiş Einstein $L$-Weyl tensörü denir.

## 1. INTRODUCTION

In an $n$-dimensional Riemannian space, for any given coordinates $x^i$ $(i = 1, \cdots, n)$, the infinite small distance between very close points $x^i$ and $x^i + dx^i$ is defined by

$$ds^2 = g_{ij} dx^i dx^j, \quad (i, j = 1, 2, \cdots, n) \tag{1.1}$$

In the equation (1.1), the coefficient $g_{ij}$ is coordinate function of $x^i$ and said to be Riemannian metric tensor. In addition, the space and the geometry which are characterized by such a metric, are called Riemannian space and the Riemannian geometry respectively [1-7].

If a vector $A$ is parallel transported along a curve between very close points such as $x^i$ and $x^i + dx^i$, the components of the vector change as follows;

$$dA^i = -\Gamma^{\ i}_{jk} A^j dx^k \tag{1.2}$$

where $\Gamma^{\ i}_{jk}$ represent the connection coefficients.

**Definition 1.1.** Let $M_n$ be a Riemannian space defined by the metric tensor $g_{ij}$, there exists only one connection which is compatible with the metric tensor $g_{ij}$ of $M_n$.

Assume $\Gamma^{\ i}_{jk}$ and $\Gamma'^{\ \gamma}_{\alpha\beta}$ be the coordinate functions of $x$ and $x'$, respectively, so the following statement is provided [4];

$$\Gamma'^{\ \gamma}_{\alpha\beta} \frac{\partial x^i}{\partial x'^\gamma} = \frac{\partial^2 x^i}{\partial x'^\alpha \partial x'^\beta} + \Gamma^{\ i}_{jk} \frac{\partial x^j}{\partial x'^\alpha} \frac{\partial x^k}{\partial x'^\beta} \tag{1.3}$$

which means $\Gamma^{\ i}_{jk}$ is not a tensor.

Furthermore, if the connection $\Gamma^{\ i}_{jk}$ is Riemannian (i.e. Levi-Civita), then we have

$$g^{ih} g_{jh} = \delta^i_j, \qquad \delta^i_j = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases} \tag{1.4}$$

$$\Gamma^{\ i}_{jk} = \left\{ \begin{matrix} i \\ jk \end{matrix} \right\} = g^{ij}[jk, h], \qquad [jk, h] = \frac{1}{2}\left(\frac{\partial g_{jh}}{\partial x^k} + \frac{\partial g_{kh}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^h}\right) \tag{1.5}$$

Here, we call $[jk, h]$ and $\left\{ \begin{smallmatrix} i \\ jk \end{smallmatrix} \right\}$ as Christoffel symbols of the first kind and the second kind, respectively. It is seen that Christoffel symbols are symmetric respect to the lower two indices.

In general, if connection is different from Levi-Civita, the connection coefficient $\Gamma_{jk}^{\,i}$ is composed of its symmetric and anti-symmetric parts on $n$-dimensional Riemannnian space $M_n$. The symmetric and anti-symmetric part of the connection coefficients $\Gamma_{jk}^{\,i}$ are defined by, respectively,

$$\Gamma_{(jk)}^{\,i} = \frac{1}{2}(\Gamma_{jk}^{\,i} + \Gamma_{kj}^{\,i}) \tag{1.6}$$

$$\Gamma_{[jk]}^{\,i} = \frac{1}{2}(\Gamma_{jk}^{\,i} - \Gamma_{kj}^{\,i}) \tag{1.7}$$

where $\Gamma_{(jk)}^{\,i}$ is a connection coefficient and $\Gamma_{[jk]}^{\,i}$ is a tensor which is called the torsion tensor of the connection. Recall that, since $\Gamma_{(jk)}^{\,i}$ in (1.3) is not a tensor, $\Gamma_{[jk]}^{\,i}$ is a tensor. By (1.6) and (1.7), we write

$$\Gamma_{jk}^{\,i} = \Gamma_{(jk)}^{\,i} + \Gamma_{[jk]}^{\,i} \tag{1.8}$$

In Riemannian space, $\Gamma_{jk}^{\,i}$ is symmetric and the torsion tensor $\Gamma_{[jk]}^{\,i}$ equals to zero.

Let $v^i$, $v_i$ and $T_{ji}{}^h$ be the components of a contravariant vector field, a covariant vector field and a tensor field, in order. The covariant derivative of these quantities with respect to the Riemannian connection $\nabla$, are defined by, respectively [4,14]

$$\nabla_j v^i = \frac{\partial v^i}{\partial x^j} + v^h \Gamma_{hj}^{\,i} \tag{1.9}$$

$$\nabla_j v_i = \frac{\partial v_i}{\partial x^j} - v_k \Gamma_{ij}^{\,k} \tag{1.10}$$

$$\nabla_k T_{ji}{}^h = \frac{\partial T_{ji}{}^a}{\partial x^k} + T_{ij}{}^a \Gamma_{ka}^{\,h} - T_{ai}{}^h \Gamma_{kj}^{\,a} - T_{ja}{}^h \Gamma_{ki}^{\,a} \tag{1.11}$$

The covariant derivative of the metric tensor $g_{ij}$ respect to Levi-Civita connection is

$$\nabla_k g_{ij} = \frac{\partial g_{ij}}{\partial x^k} - \begin{Bmatrix} a \\ kj \end{Bmatrix} g_{ai} - \begin{Bmatrix} a \\ ki \end{Bmatrix} g_{aj} = 0 \tag{1.12}$$

In general, the curvature tensor of a space $M_n$ is defined by

$$R_{kji}{}^h = \partial_k \Gamma_{ji}^{\,h} - \partial_j \Gamma_{ki}^{\,h} + \Gamma_{kt}^{\,h} \Gamma_{ji}^{\,t} - \Gamma_{jt}^{\,h} \Gamma_{ki}^{\,t}, \quad (\partial_k = \frac{\partial}{\partial x^k}) \tag{1.13}$$

If we take the coefficients $\Gamma_{ji}^{\,h}$ as Christoffel symbols of second kind, then the curvature tensor of the Riemannian space admitting the metric tensor $g_{ij}$ turns into [14]

$$R_{kji}{}^h = \partial_k \begin{Bmatrix} h \\ ji \end{Bmatrix} - \partial_j \begin{Bmatrix} h \\ ki \end{Bmatrix} + \begin{Bmatrix} h \\ ka \end{Bmatrix} \begin{Bmatrix} a \\ ji \end{Bmatrix} - \begin{Bmatrix} h \\ ja \end{Bmatrix} \begin{Bmatrix} a \\ ki \end{Bmatrix} \tag{1.14}$$

Let $v^i$, $v_i$, $f$, $T_{li}{}^h$ and $S_{kj}^{\,t}$ be the components of a contravariant vector field, a covariant vector field, a scalar function, a tensor field of type $(1,2)$ and a torsion tensor, on $M_n$

respectively. The covariant tensor $R_{kji}{}^{h}$ satisfies the Ricci identities

$$\nabla_k\nabla_j v^i - \nabla_j\nabla_k v^i = R_{kjh}{}^{i} v^h - 2S_{kj}{}^{t}\nabla_t v^i \tag{1.15}$$

$$\nabla_k\nabla_j v_i - \nabla_j\nabla_k v_i = -R_{kji}{}^{h} v_h - 2S_{kj}{}^{t}\nabla_t v_i \tag{1.16}$$

$$\nabla_k\nabla_j f - \nabla_j\nabla_k f = -2S_{kj}{}^{t}\nabla_t f \tag{1.17}$$

$$\nabla_k\nabla_j T_{li}{}^{h} - \nabla_j\nabla_k T_{li}{}^{h} = R_{kjt}{}^{h} T_{li}{}^{t} - R_{kjl}{}^{t} T_{ti}{}^{h} - R_{kji}{}^{t} T_{lt}{}^{h} - 2S_{kj}{}^{t}\nabla_t T_{li}{}^{h} \tag{1.18}$$

In particular, if $M_n$ is the Riemannian space, then torsion tensor $S_{kj}{}^{t} = 0$. Thus, we reach the Ricci identities for Riemannian case.

The covariant curvature tensor of $R_{kjih}$ is defined as

$$R_{kjih} = R_{kji}{}^{a} g_{ah} \tag{1.19}$$

From (1.14), it can be seen that the Riemannian curvature satisfies the following properties

$$\begin{aligned}
&(\mathbf{1})\ R_{kjih} + R_{jikh} + R_{ikjh} = 0\\
&(\mathbf{2})\ R_{kjih} = -R_{jkih}\\
&(\mathbf{3})\ R_{kjih} = -R_{kjhi}\\
&(\mathbf{4})\ R_{kjih} = R_{ihkj}\\
&(\mathbf{5})\ R_{kkih} = -R_{kjhh} = 0
\end{aligned} \tag{1.20}$$

The first property is called the first Bianchi identity.
Also, the covariant derivative of the curvature tensor satisfies

$$\nabla_l R_{kji}{}^{h} + \nabla_k R_{jli}{}^{h} + \nabla_j R_{lki}{}^{h} = 0 \tag{1.21}$$

which is called the second Bianchi identity.

In the equation of (1.14), contracting indices $h$ and $k$, then we reach the Ricci curvature tensor

$$R_{ji} = R_{aji}{}^{a} \tag{1.22}$$

Also, we have

$$R_{ji} = R_{aji}{}^{a} = g^{ab} R_{ajib} = g^{ba} R_{ibaj} = g^{ba} R_{bija} = R_{ij} \tag{1.23}$$

which shows that the Ricci curvature tensor is symmetric. By using Ricci curvature tensor, the scalar curvature of the Riemannian space is defined by [13,14]

$$R = g^{ji} R_{ji} \tag{1.24}$$

## 1.1 Literature Review

In literature, many authors have been studied Weyl spaces and Kaehlerian Weyl spaces. The starting point of Weyl was to state a unified theory for the electromagnetic and gravitational interactions. But the electric charge conservation is violated in this unified theory. In spite of unacceptable theory for physics, it presents some special properties for differential geometry. It relates spaces by conformal transformations and display some conserved quantities during transformations. The change in curvature and curvature related geometric quantities by introducing some structures on these type of spaces have been widely studied [1-25].

When we look at Einstein's geometrization of gravity, we see the usage of the metric tensor $g_{ij}$ and the symmetric linear connection $\Gamma_{jk}^{i}$ instead of the Newtonian gravitational potential and the Newtonian gravitational force. In essence, the general theory of relativity the gravitational field is represented by the metric tensor and indirectly by the curvature of spacetime.

In Einstein's theory of relativity non-vacuum solutions admit conserved energy-momentum tensors. Then, electromagnetic field as an example of conserved source is included in Einstein's theory of relativity [2].

In differential geometric context, differential geometric structures such as almost complex, Kaehlerian and $L$-Structures may introduce new interesting properties to the geometrical quantities and therefore, they can be used to classify these spaces [15-25].

## 1.2 Purpose of Thesis

In this thesis, we define almost $L$-structures and Nearly-Kaehlerian structures on Weyl space and examine curvature and curvature related properties, we state and prove theorems concerning these extended structures. Then, we introduce a generalized Einstein $L$-Weyl tensor and prove the necessary and the sufficient condition of Einstein $L$-Weyl to be an Einstein space.

## 1.3 Preliminaries

In this section, we give some preliminary concepts related to Weyl spaces. First, we define the Weyl space and obtain the curvature, the covariant curvature, the Ricci curvature tensor and scalar curvature of Weyl space in terms of Riemannian tensor and the metric tensor. Also, we mention about the Bianchi identities and give some symmetry properties of the curvature tensors of Weyl space. Also, curvature similarities between Riemannian and Weyl space are emphasized.

Let $W_n$ be $n$-dimensional space having a conformal metric tensor $g$ and let $D$ be torsion free (symmetric) connection.

**Definition 1.3.** If $D$ satisfies the compatibility condition

$$D_k g_{ij} = 2\, w_k\, g_{ij} \tag{1.25}$$

then, $W_n$ is called a Weyl space and denoted by $W_n(g,w)$. Here, $w$ is a 1-form, named complementary co-vector field.

Let us consider renormalization transformation of the metric tensor $g$

$$\bar{g} = \Omega^2 g \ \ (\Omega > 0) \tag{1.26}$$

where $\Omega$ is a scalar function defined on $W_n$ [4,8,9].

We examine the transformation of the complementary vector $\bar{w}$ to get the covariant derivative of $\bar{g}$ as

$$D_k\bar{g} = 2\,\bar{w}_k\,\bar{g} \tag{1.27}$$

If we use (1.26) in (1.25), to obtain

$$D_k\bar{g} = 2\,\bar{w}_k\,\bar{g} \tag{1.28}$$

and we find

$$D_k(\Omega^2 g) = 2\,\bar{w}_k\,\Omega^2 g \tag{1.29}$$

If we take the derivative of (1.29) with respect to $D_k$, then we obtain

$$2\Omega\Omega_k\, g + \Omega^2\, 2 w_k\, g - 2\bar{w}_k\,\Omega^2\, g = 0 \tag{1.30}$$

and re-arranging (1.30), then we get

$$\bar{w}_k = w_k + \frac{\Omega_k}{\Omega} \tag{1.31}$$

So, $w$ is transformed by the law,

$$\bar{w} = w + d\ln\Omega \tag{1.32}$$

**Definition 1.4.** A quantity $A$ defined on $W_n(g,w)$ is called a satellite of $g$ of weight $\{p\}$, if it admits a transformation of the form

$$\bar{A} = \Omega^p A \tag{1.33}$$

under the renormalization (1.26).

We note that the weight of the metric tensor $g_{ij}$ is $\{2\}$ by the rule of (1.26).

**Definition 1.5.** The prolonged (extended) covariant derivative of the satellite $A$ of weight $\{p\}$ is defined by [4,8,9]

$$\dot{\nabla}_k A = D_k A - p w_k A \tag{1.34}$$

Taking prolonged covariant derivative of the metric tensor $g_{ij}$, we have

$$\dot{\nabla}_k g_{ij} = D_k g_{ij} - 2w_k g_{ij} \tag{1.35}$$

From (1.25), it's seen that

$$\dot{\nabla}_k g_{ij} = D_k g_{ij} - D_k g_{ij} \tag{1.36}$$

Thus we have

$$\dot{\nabla}_k g_{ij} = 0 \tag{1.37}$$

The curvature tensor, covariant curvature tensor, the Ricci curvature tensor and the scalar curvature of Weyl space are defined by, respectively,

$$(\dot{\nabla}_i\dot{\nabla}_j - \dot{\nabla}_j\dot{\nabla}_i)\, v_k = -W_{kji}{}^{t}\, v_t \tag{1.38}$$

$$W_{kji}{}^{h} = W_{kjil}\, g^{lh} \tag{1.39}$$

$$W_{kjil} = W_{kji}{}^{m}\, g_{ml} \tag{1.40}$$

$$W_{ji} = g^{kl} W_{kjil} = W_{kji}{}^{k} \tag{1.41}$$

$$W = g^{ji} W_{ji} \tag{1.42}$$

From (1.38), the curvature tensor of $W_n$ is obtained

$$W_{kji}{}^{l} = \partial_k \Gamma_{ji}^{\ l} - \partial_j \Gamma_{ki}^{\ l} - \Gamma_{ki}^{\ t} \Gamma_{jt}^{\ l} + \Gamma_{ji}^{\ t} \Gamma_{kt}^{\ l} \tag{1.43}$$

where $\Gamma_{ji}^{\ l}$ are the coefficients of the Weyl connection

$$\Gamma_{ji}^{\ l} = \begin{Bmatrix} l \\ ji \end{Bmatrix} - (w_j \delta_i^l + w_i \delta_j^l - g_{ij} w^l) \tag{1.44}$$

In (1.44), $\begin{Bmatrix} l \\ ji \end{Bmatrix}$ are the coefficients of the metric connection $\nabla$.

Substituting (1.44) in (1.43), the curvature tensor of $W_n$ is obtained

$$\begin{aligned} W_{kji}{}^{l} = \partial_k \Big[ \begin{Bmatrix} l \\ ji \end{Bmatrix} - (w_j \delta_i^l + w_i \delta_j^l - g_{ji} w^l) \Big] - \partial_j \Big[ \begin{Bmatrix} l \\ ki \end{Bmatrix} - (w_k \delta_i^l + w_i \delta_k^l - g_{ki} w^l) \Big] \\ + \Big[ \begin{Bmatrix} t \\ ji \end{Bmatrix} - (w_i \delta_j^t + w_j \delta_i^t - g_{ji} w^t) \Big] \Big[ \begin{Bmatrix} l \\ kt \end{Bmatrix} - (w_k \delta_t^l + w_t \delta_k^l - g_{kt} w^l) \Big] \\ - \Big[ \begin{Bmatrix} t \\ ki \end{Bmatrix} - (w_k \delta_i^t + w_i \delta_k^t - g_{ki} w^t) \Big] \Big[ \begin{Bmatrix} l \\ jt \end{Bmatrix} - (w_j \delta_t^l + w_t \delta_j^l - g_{jt} w^l) \Big] \end{aligned} \tag{1.45}$$

and using Riemannian curvature (1.14),

$$\begin{aligned} W_{kji}{}^{l} = R_{kji}{}^{l} - \partial_k (w_i \delta_j^l + w_j \delta_i^l - g_{ji} w^l) + \partial_j (w_k \delta_i^l + w_i \delta_k^l - g_{ki} w^l) \\ - \begin{Bmatrix} t \\ ji \end{Bmatrix} (w_k \delta_t^l + w_t \delta_k^l - g_{kt} w^l) - \begin{Bmatrix} l \\ kt \end{Bmatrix} (w_i \delta_j^t + w_j \delta_i^t - g_{ji} w^t) \\ + (w_k \delta_t^l + w_t \delta_k^l - g_{kt} w^l)(w_i \delta_j^t + w_j \delta_i^t - g_{ji} w^t) \\ + \begin{Bmatrix} t \\ ki \end{Bmatrix} (w_j \delta_t^l + w_i \delta_t^j - g_{jt} w^l) + \begin{Bmatrix} l \\ jt \end{Bmatrix} (w_k \delta_i^t + w_i \delta_k^t - g_{ki} w^t) \\ - (w_j \delta_t^l + w_i \delta_t^j - g_{jt} w^l)(w_k \delta_i^t + w_i \delta_k^t - g_{ki} w^t) \end{aligned} \tag{1.46}$$

So, using the properties of the Riemannian curvature tensor and simplying the (1.46), we reach

$$W_{kji}{}^{l} = R_{kji}{}^{l} - \delta_j^l w_{ki} + \delta_k^l w_{ji} + \delta_i^l (w_{jk} - w_{kj}) + g^{ls} (g_{ji} w_{ks} - g_{ki} w_{js}) \tag{1.47}$$

where

$$w_{jk} = \nabla_j w_k + w_j w_k - \frac{1}{2} g_{jk} w_t w^t \tag{1.48}$$

and $R_{kji}{}^{l}$ represents the Riemannian curvature tensor.

Multiplying (1.47) by metric tensor $g_{lm}$ and using (1.40), we find

$$\begin{aligned} W_{kjim} = R_{kji}{}^{l} g_{lm} - \delta_j^l w_{ki} g_{lm} + \delta_k^l w_{ji} g_{lm} + \delta_i^l (w_{jk} - w_{kj}) g_{lm} \\ + g^{ls} (g_{ji} w_{ks} - g_{ki} w_{js}) g_{lm} \end{aligned} \tag{1.49}$$

From (1.19) and (1.4), we get

$$W_{kjim} = R_{kjim} - g_{jm}w_{ki} + g_{km}w_{ji} + g_{im}(w_{jk} - w_{kj}) \\ +g^{ls}g_{lm}(g_{ji}w_{ks} - g_{ki}w_{js}) \tag{1.50}$$

So, the covariant curvature tensor of $W_n$ is found as

$$W_{kjim} = R_{kjim} - g_{jm}w_{ki} + g_{mk}w_{ji} + g_{im}(w_{jk} - w_{kj}) + (g_{ji}w_{km} - g_{ki}w_{jm}) \tag{1.51}$$

We observe that the curvature tensor, the covariant curvature tensor and the Ricci curvature tensor of $W_n(g,w)$ satisfy the following symmetry properties:

$$W_{kjim} = -W_{jkim} \tag{1.52}$$

$$W_{kjim} + W_{kjmi} = 2g_{im}(\nabla_j w_k - \nabla_k w_j) = 4g_{im}\nabla_{[j}w_{k]} \tag{1.53}$$

In (1.51), by interchanging the indices $k$ and $j$, the covariant curvature tensor $W_{jkim}$ is obtained

$$W_{jkim} = R_{jkim} - g_{km}w_{ji} + g_{mj}w_{ki} + g_{im}(w_{kj} - w_{jk}) + (g_{ki}w_{jm} - g_{ji}w_{km}) \tag{1.54}$$

Since the metric tensor $g_{ij}$ is symmetric and using (1.20), we get

$$W_{jkim} = -[-R_{jkim} + g_{mk}w_{ji} - g_{jm}w_{ki} + g_{im}(-w_{kj} + w_{jk}) + (-g_{ki}w_{jm} + g_{ji}w_{km})] \tag{1.55}$$

by rearranging the (1.55), we find

$$W_{jkim} = -[R_{kjim} - g_{jm}w_{ki} + g_{mk}w_{ji} + g_{im}(w_{jk} - w_{kj}) + (g_{ji}w_{km} - g_{ki}w_{jm})] \tag{1.56}$$

so, we see that

$$W_{kjim} = -W_{jkim} \tag{1.57}$$

Now, using (1.56) we calculate the sum of the covariant tensors $W_{kjim}$ and $W_{kjmi}$

$$W_{kjim} + W_{kjmi} = R_{kjim} + R_{kjmi} + 2g_{im}(w_{jk} - w_{kj}) \tag{1.58}$$

By the symmetry property of covariant curvature tensor of Riemannian space (1.20), we have

$$W_{kjim} + W_{kjmi} = 2g_{im}(w_{jk} - w_{kj}) \tag{1.59}$$

where $w_{jk}$ is defined in (1.48).

If we substituting (1.48) in (1.59), then (1.59) reduces to

$$W_{kjim} + W_{kjmi} = 2g_{im}(\nabla_j w_k - \nabla_k w_j) = 4g_{im}\nabla_{[j}w_{k]} \quad \textbf{(1.60)}$$

Let's now consider the covariant curvature tensor $W_{kjim}$ on Weyl space. Multiplying the (1.51), by the metric tensor $g^{km}$, we obtain

$$W_{ji} = g^{km}[R_{kjim} - g_{jm}w_{ki} + g_{mk}w_{ji} + g_{im}(w_{jk} - w_{kj}) + (g_{ji}w_{km} - g_{ki}w_{jm})] \quad \textbf{(1.61)}$$

by the rule of the (1.41).

From (1.23), we find

$$\begin{aligned} W_{ji} = R_{ji} - g^{km}g_{mj}w_{ki} + g^{km}g_{mk}w_{ji} + g^{km}g_{mi}(w_{jk} - w_{kj}) \\ +g^{km}g_{ji}w_{km} - g^{mk}g_{ki}w_{jm} \end{aligned} \quad \textbf{(1.62)}$$

From (1.4), (1.62) can be written as

$$W_{ji} = R_{ji} - \delta_j^k w_{ki} + \delta_m^m w_{ji} + \delta_i^k(w_{jk} - w_{kj}) + g^{km}g_{ji}w_{km} - \delta_i^m w_{jm} \quad \textbf{(1.63)}$$

Now, re-arranging (1.63), we find that the Ricci curvature tensor of $W_n$ is calculated in terms of the Ricci curvature of the Riemannian space $R_{ji}$ as

$$W_{ji} = R_{ji} + (n-2)w_{ji} + (w_{ji} - w_{ij}) + g_{ji}g^{km}w_{km} \quad \textbf{(1.64)}$$

where $w_{ji}$ is defined in (1.48).

The Ricci curvature tensor in (1.63) is not necessarily symmetric on $W_n$ since the Weyl connection $D$ is non-metric. The Ricci curvature tensor $W_{ji}$ of $W_n$ can be decomposed symmetric and anti-symmetric parts as

$$W_{ji} = W_{(ji)} + W_{[ji]} \quad \textbf{(1.65)}$$

Here $W_{(ji)}$ represents the symmetric part of $W_{ji}$ and $W_{[ji]}$ represents the anti-symmetric part of $W_{ji}$

$$W_{(ji)} = \frac{1}{2}(W_{ji} + W_{ij}) \quad \textbf{(1.66)}$$

$$W_{[ji]} = \frac{1}{2}(W_{ji} - W_{ij}) = n\nabla_{[j}w_{i]} \quad \textbf{(1.67)}$$

By using (1.63) the symmetric and anti-symmetric parts of the Ricci curvature tensor of $W_{ji}$ are given as follows

$$W_{(ji)} = R_{ji} + \frac{(n-2)}{2}(w_{ji} + w_{ij}) + g_{ji}g^{kt}w_{kt} \tag{1.68}$$

or

$$W_{(ji)} = R_{ji} + (n-2)[\nabla_j w_i + \nabla_i w_j + 2w_j w_i] - g_{ji}(2w^t w_t - g^{kt}\nabla_k w_t) \tag{1.69}$$

and

$$W_{[ji]} = n\nabla_{[j}w_{i]} \tag{1.70}$$

Also, multiplying (1.64) by $g^{ji}$ and using (1.24), we find the scalar curvature of $W_n$

$$W = R + 2(n-1)\nabla_j w^j - (n-1)(n-2)w_j w^j \tag{1.71}$$

in terms of the Riemannian scalar curvature $R$ and the complementary co-vector $w$.

The following identities are known as the first and the second Bianchi identities for $W_n$, respectively, [4,8,9]

$$W_{kji}{}^{l} + W_{jik}{}^{l} + W_{ikj}{}^{l} = 0 \tag{1.72}$$

$$\dot{\nabla}_m W_{kji}{}^{l} + \dot{\nabla}_k W_{jmi}{}^{l} + \dot{\nabla}_j W_{mki}{}^{l} = 0 \tag{1.73}$$

The first and the second Bianchi identities can be also written as follows

$$W_{kjim} + W_{jikm} + W_{ikjm} = 0 \tag{1.74}$$

$$\dot{\nabla}_m W_{kjin} + \dot{\nabla}_k W_{jmin} + \dot{\nabla}_j W_{mkin} = 0 \tag{1.75}$$

By using (1.51), we observe that the difference of the covariant tensors $W_{kjim}$ and $W_{imkj}$ as

$$\begin{aligned} W_{kjim} - W_{imkj} = {} & g_{mk}(w_{ji} - w_{ij}) + g_{mj}(w_{ik} - w_{ki}) + g_{ji}(w_{km} - w_{mk}) \\ & + g_{ik}(w_{mj} - w_{jm}) + g_{im}(w_{jk} - w_{kj}) + g_{kj}(w_{mi} - w_{im}) \end{aligned} \tag{1.76}$$

Substituting (1.48) in (1.76) and after some calculations, we find

$$\begin{aligned} W_{kjim} - W_{imkj} = {} & g_{mk}(\nabla_j w_i - \nabla_i w_j) + g_{mj}(\nabla_i w_k - \nabla_k w_i) + g_{ji}(\nabla_k w_m - \nabla_m w_k) \\ & + g_{ik}(\nabla_m w_j - \nabla_j w_m) + g_{im}(\nabla_j w_k - \nabla_k w_j) + g_{kj}(\nabla_m w_i - \nabla_i w_m) \end{aligned} \tag{1.77}$$

So, the covariant curvature tensor $W_{kjim}$ satisfies

$$W_{kjim} - W_{imkj} = g_{mk}(w_{i,j} - w_{j,i}) + g_{mj}(w_{k,i} - w_{i,k}) + g_{ji}(w_{m,k} - w_{k,m}) \\ + g_{ik}(w_{j,m} - w_{m,j}) + g_{im}(w_{k,j} - w_{j,k}) + g_{kj}(w_{i,m} - w_{m,i}) \qquad \textbf{(1.78)}$$

where $\dot{\nabla}_m w_j = \nabla_m w_j = w_{j,m}$.

# 2. STRUCTURES ON WEYL SPACES

## 2.1 Almost Complex And Almost Kaehlerian Structures On Weyl Space

In this chapter, we start with some definitions and properties of almost complex and almost Hermitian structures on Weyl spaces.

**Definition 2.1.** Let $W_n$ be even dimensional ($n \geq 2$). A tensor $F_i^{\ j}$ of type (1,1) with weight $\{0\}$ is called an almost complex structure on $W_n$ if $F_i^{\ j}$ satisfies

$$F_i^{\ j} F_j^{\ k} = -\delta_i^k \tag{2.1}$$

and $W_n$ is called an almost complex space [15,16].

**Definition 2.2.** If $W_n$ admits an almost complex structure $F_i^{\ j}$ satisfying the condition

$$g_{ij} F_h^{\ i} F_k^{\ j} = g_{hk} \tag{2.2}$$

then $F_i^{\ j}$ is called an almost Hermitian structure with a Hermitian metric $g$ on $W_n$.

**Definition 2.3.** If almost Hermitian structure $F_i^{\ j}$ satisfies

$$\dot{\nabla}_k F_i^{\ j} = 0 \qquad \text{for all } \ i, j, k \tag{2.3}$$

then, it is called a Kaehlerian structure and then $W_n$ becomes a Kaehlerian space denoted by $KW_n$.

**Definition 2.4.** An almost Hermitian structure $F_i^{\ j}$ is called an almost Kaehlerian structure on $W_n$ (respectively, almost Kaehlerian Weyl space) if the tensor $F_i^{\ j}$ satisfies

$$F_{hij} = \dot{\nabla}_h F_{ij} + \dot{\nabla}_i F_{jh} + \dot{\nabla}_j F_{hi} = 0 \tag{2.4}$$

**Definition 1.10.** An almost Hermitian structure $F_i^{\ j}$ is called an almost semi-Kaehlerian structure on $W_n$ (respectively, semi-Kaehlerian Weyl space $SKW_n$) if the tensor $F_i^{\ j}$ satisfies

$$F_i \equiv \dot{\nabla}_j F_i^{\ j} = 0 \tag{2.5}$$

The tensors $F_{ij}$ and $F^{ij}$, respectively, are defined on $W_n$ by

$$F_{ij} = F_i^{\ k} g_{kj} \tag{2.6}$$

$$F^{ij} = g^{ih} F_h^{\ j} \tag{2.7}$$

From (2.6) and (2.7) it can be seen that

$$F_{ij} = -F_{ji} \tag{2.8}$$

$$F^{ij} = -F^{ji} \tag{2.9}$$

From (1.26) and (2.1), it can be seen that the weight of $F_{ij}$ is $\{2\}$. Since, the weight of $g^{ih}$ is $\{-2\}$, the weight of $F^{ij}$ is $\{-2\}$, respectively [15].

### 2.2 Almost $L$-Structures And Nearly-Kaehlerian Structures On Weyl Spaces

We now introduce an almost $L$-structure and a nearly-Kaehlerian structure on $W_n$ and define the Nijhenuis torsion tensor on $W_n$ for an almost $L$-structure to be integrable. We state and prove some theorems concerning these structures which show the relation between almost Hermitian and almost $L$-structures and how they are related to each other. Results of these theorems will be very helpful to construct the generalized Einstein $L$-Weyl tensor to be introduced in the next chapter.

**Definition 2.1.** An almost Hermitian structure $F_i^{\ j}$ with a Hermitian metric $g$ on $W_n$ satisfying the relation

$$G_{ij}^{k} \equiv \dot{\nabla}_i F_j^{\ k} + \dot{\nabla}_j F_i^{\ k} = 0 \tag{2.10}$$

is called a nearly-Kaehlerian structure or an almost Tachibana structure (respectively, nearly-Kaehlerian Weyl space denoted by $NKW_n$).

**Definition 2.2.** An almost Hermitian structure $F_i^{\ j}$ on $W_n$ satisfying

$$[\dot{\nabla}_j \dot{\nabla}_k] F_i^{\ h} \equiv (\dot{\nabla}_j \dot{\nabla}_k - \dot{\nabla}_k \dot{\nabla}_j) F_i^{\ h} = 0 \tag{2.11}$$

is called an almost $L$-structure and a Weyl space admitting an almost $L$-structure is called almost $L$-Weyl space denoted by $LW_n$.

**Corollary 2.1.** A Kaehlerian structure of $W_n$ is an almost $L$-structure.

**Proof.** Since an almost Hermitian structure $F_i^{\ j}$ on $KW_n$ satisfies

$$\dot{\nabla}_k F_i^{\ j} = 0 \quad \text{for all } \ i, j, k \tag{2.12}$$

then using (2.10), the Kaehlerian structure (respectively, the space $KW_n$) is an almost $L$-structure (respectively, the space $LW_n$).

The Nijhenuis torsion tensor on $W_n$ is defined by

$$N_{ij}{}^{k} = F_i{}^{h}(\dot{\nabla}_h F_j{}^{k} - \dot{\nabla}_j F_h{}^{k}) - F_j{}^{h}(\dot{\nabla}_h F_i{}^{k} - \dot{\nabla}_i F_h{}^{k}) \tag{2.13}$$

and it is said that an almost complex structure is integrable on $W_n$ if it has no torsion, i.e. $N_{ij}{}^{k} = 0$ [15].

**Remark 2.1.** If we take the complementary vector field $w$ to be locally gradient, then, (2.13) reduces Riemannian space [14].

**Theorem 2.1.** A nearly-Kaehlerian structure (respectively, the space $NKW_n$) is an almost semi-Kaehlerian Weyl structure (respectively, the space $SKW_n$).

**Proof.** In (2.10), contracting indices with respect to $i$ and $k$

$$\dot{\nabla}_i F_j{}^{i} = -\dot{\nabla}_j F_i{}^{i} \tag{2.14}$$

From (2.6)

$$F_i{}^{m} = F_{ij}\, g^{jm} \tag{2.15}$$

and contracting indices with respect to $m$ and $i$, we obtain

$$F_i{}^{i} = F_{ij}\, g^{ij} = 0 \tag{2.16}$$

Then, from (2.14) it follows that

$$\dot{\nabla}_i F_j{}^{i} = 0 \tag{2.17}$$

which shows that the structure $F_i{}^{j}$ is an almost semi-Kaehlerian Weyl structure.

**Theorem 2.2.** A nearly-Kaehlerian Weyl space is a Kaehlerian-Weyl space if the structure is integrable.

**Proof.** Since Nijhenuis torsion tensor on $W_n$ is

$$N_{ij}{}^{k} = F_i{}^{h}(\dot{\nabla}_h F_j{}^{k} - \dot{\nabla}_j F_h{}^{k}) - F_j{}^{h}(\dot{\nabla}_h F_i{}^{k} - \dot{\nabla}_i F_h{}^{k}) \tag{2.18}$$

and using the defining condition (2.10) of the nearly-Kaehlerian structure we obtain

$$\dot{\nabla}_h F_j{}^{k} = -\dot{\nabla}_j F_h{}^{k} \tag{2.19}$$

and

$$\dot{\nabla}_h F_i{}^{k} = -\dot{\nabla}_i F_h{}^{k} \tag{2.20}$$

Replacing (2.19) and (2.20) in (2.18) find

$$N_{ij}{}^{k} = -2F_i{}^{h}\dot{\nabla}_j F_h{}^{k} + 2F_j{}^{h}\dot{\nabla}_i F_h{}^{k} \tag{2.21}$$

Since

$$F_j{}^{h}F_h{}^{k} = -\delta_j^k \tag{2.22}$$

by taking the prolonged covariant derivative of (2.22), we get

$$(\dot{\nabla}_i F_h{}^{k})F_j{}^{h} = -(\dot{\nabla}_i F_j{}^{h})F_h{}^{k} \tag{2.23}$$

Similarly, the prolonged covariant derivative of

$$F_i{}^{h}F_h{}^{k} = -\delta_i^k \tag{2.24}$$

yields

$$(\dot{\nabla}_j F_h{}^{k})F_i{}^{h} = -(\dot{\nabla}_j F_i{}^{h})F_h{}^{k} \tag{2.25}$$

and using (2.10) we obtain

$$(\dot{\nabla}_j F_h{}^{k})F_i{}^{h} = -(\dot{\nabla}_j F_i{}^{h})F_h{}^{k} = (\dot{\nabla}_i F_j{}^{h})F_h{}^{k} \tag{2.26}$$

On the other hand, substituting (2.23) and (2.26) in (2.21), we reach

$$N_{ij}{}^{k} = -4(\dot{\nabla}_i F_j{}^{h})F_h{}^{k} \tag{2.27}$$

By assumption, since the nearly-Kaehlerian structure is integrable, the Nijhenuis torsion tensor $N_{ij}{}^{k}$ is zero.
Hence, we obtain

$$(\dot{\nabla}_i F_j{}^{h})F_h{}^{k} = 0 \tag{2.28}$$

If we multiply (2.28) by $F_k{}^{m}$ and use (2.1), thus we obtain

$$\dot{\nabla}_i F_j{}^{m} = 0 \qquad \text{for all} \qquad i, j, m, \tag{2.29}$$

which gives that the structure $F_i{}^{j}$ is Kaehlerian.

Now, we can express the defining condition (2.11) of an almost $L$-structure on $LW_n$ in terms of the Weyl and Ricci curvature tensors.

**Theorem 2.3.** An almost Hermitian structure $F_i{}^{j}$ is an almost $L$-structure on $LW_n$ if and only if

$$F_i{}^{a}W_{kja}{}^{h} = F_a{}^{h}W_{kji}{}^{a} \tag{2.30}$$

**Proof.** Since the almost Hermitian structure $F_i^{\ j}$ is a tensor of weight $\{0\}$, we have

$$\dot{\nabla}_k\dot{\nabla}_j F_i^{\ h} - \dot{\nabla}_j\dot{\nabla}_k F_i^{\ h} = \nabla_k\nabla_j F_i^{\ h} - \nabla_j\nabla_k F_i^{\ h} \qquad \textbf{(2.31)}$$

also the Ricci identity on $W_n$, we have by [14]

$$(\dot{\nabla}_k\dot{\nabla}_j - \dot{\nabla}_j\dot{\nabla}_k)F_i^{\ h} = F_i^{\ a}W_{kja}^{\ \ \ h} - F_a^{\ h}W_{kji}^{\ \ \ a} \qquad \textbf{(2.32)}$$

If an almost Hermitian structure $F_i^{\ j}$ is an almost $L$-structure on $LW_n$, then from (2.11), we obtain

$$(\dot{\nabla}_k\dot{\nabla}_j - \dot{\nabla}_j\dot{\nabla}_k)F_i^{\ h} = F_i^{\ a}W_{kja}^{\ \ \ h} - F_a^{\ h}W_{kji}^{\ \ \ a} = 0 \qquad \textbf{(2.33)}$$

which follows that

$$F_i^{\ a}W_{kja}^{\ \ \ h} = F_a^{\ h}W_{kji}^{\ \ \ a} \qquad \textbf{(2.34)}$$

Conversely, suppose (2.30) holds. Then we have

$$F_i^{\ a}W_{kja}^{\ \ \ h} - F_a^{\ h}W_{kji}^{\ \ \ a} = \dot{\nabla}_k\dot{\nabla}_j F_i^{\ h} - \dot{\nabla}_j\dot{\nabla}_k F_i^{\ h} = 0 \qquad \textbf{(2.35)}$$

which shows that the structure $F_i^{\ j}$ is an almost $L$-structure on $LW_n$.

**Theorem 2.4.** On an almost Hermitian Weyl space, the condition (2.30) is equivalent to

$$W_{kijh} = F_j^{\ l}F_h^{\ t}W_{kilt} \qquad \textbf{(2.36)}$$

**Proof.** Suppose (2.30) holds. Multiplying (2.30) by $F_m^{\ i}$ and using (2.1), we have

$$\delta_m^a W_{kja}^{\ \ \ h} = -F_a^{\ h}F_m^{\ i}W_{kji}^{\ \ \ a} \qquad \textbf{(2.37)}$$

So, re-arranging the equation (2.37), we get

$$W_{kjm}^{\ \ \ h} = -F_a^{\ h}F_m^{\ i}W_{kji}^{\ \ \ a} \qquad \textbf{(2.38)}$$

First, multiplying (2.38) by $g_{ht}$

$$W_{kjmt} = -F_{at}F_m^{\ i}W_{kji}^{\ \ \ a} \qquad \textbf{(2.39)}$$

and then using (2.8), we obtain

$$W_{kjmt} = F_{ta}F_m^{\ i}W_{kji}^{\ \ \ a} \qquad \textbf{(2.40)}$$

Also, applying (2.6) to (2.40) for the metric tensor $g_{pa}$, we find

$$W_{kjmt} = F_t^{\ p}F_m^{\ i}W_{kji}^{\ \ \ a}g_{pa} \qquad \textbf{(2.41)}$$

Since the metric tensor is symmetric, we have

$$W_{kjmt} = F_t{}^p F_m{}^i W_{kjip} \qquad \textbf{(2.42)}$$

which shows (2.36) is valid

$$W_{kijh} = F_j{}^l F_h{}^t W_{kilt} \qquad \textbf{(2.43)}$$

Conversely, suppose that (2.36) holds. Then, multiplying (2.36) by $F_u{}^m$, we have

$$F_u{}^m W_{kjmt} = F_u{}^m F_m{}^i F_t{}^p W_{kjip} \qquad \textbf{(2.44)}$$

By (2.1), we get

$$F_u{}^m W_{kjmt} = -\delta_u^i F_t{}^p W_{kjip} \qquad \textbf{(2.45)}$$

which (2.45) turns into

$$F_u{}^m W_{kjmt} = -F_t{}^p W_{kjup} \qquad \textbf{(2.46)}$$

Multiplying (2.46) by $g^{tn}$ and by the rules of (1.39) and (2.7), we obtain

$$F_u{}^m W_{kjm}{}^n = -F^{np} W_{kjup} \qquad \textbf{(2.47)}$$

Now, applying (2.7) and (2.9) to $F^{np}$ again, we get

$$F_u{}^m W_{kjm}{}^n = g^{pm} F_m{}^n W_{kjup} \qquad \textbf{(2.48)}$$

From (1.39), we reach

$$F_u{}^m W_{kjm}{}^n = F_m{}^n W_{kju}{}^m \qquad \textbf{(2.49)}$$

**Theorem 2.5.** On an almost $L$-Weyl space, the following conditions hold

$$\textbf{i)}\ W_{jm} = \frac{1}{2} F^{ks} F_m{}^i W_{ksij} + F^{ks} F_{jm} (w_{k,s} - w_{s,k}) \qquad \textbf{(2.50)}$$

$$\textbf{ii)}\ Q \equiv W + \frac{1}{2} F^{sk} F^{ij} W_{skij} = 0 \qquad \textbf{(2.51)}$$

$$\textbf{iii)}\ W = F^{ji} F_m{}^k W_{kji}{}^m \qquad \textbf{(2.52)}$$

**Proof.**

**i)** According to **Theorem 2.4.**, we write

$$W_{kjmt} = F_m{}^i F_t{}^s W_{kjis} \qquad \textbf{(2.53)}$$

By multiplying (2.53) by $g^{kt}$, we get

$$W_{jm} = F^{ks} F_m{}^i W_{kjis} \qquad \textbf{(2.54)}$$

here we use the results of (1.41) and (2.7).

Now, from (1.52), we rewrite (2.54) as

$$W_{jm} = -F^{ks}F_m{}^i W_{jkis} \tag{2.55}$$

Applying (1.53) to $W_{jkis}$, we reach

$$W_{jm} = -F^{ks}F_m{}^i(-W_{jksi} + 2g_{si}(w_{j,k} - w_{k,j})) \tag{2.56}$$

By re-arranging the (2.56), we have

$$W_{jm} = F^{ks}F_m{}^i W_{jksi} - F^{ks}F_m{}^i 2g_{si}(w_{j,k} - w_{k,j}) \tag{2.57}$$

From (1.78), we write

$$\begin{aligned} W_{sijk} - W_{jksi} &= g_{ki}(w_{s,j} - w_{j,s}) + g_{ks}(w_{j,i} - w_{i,j}) + g_{ji}(w_{k,s} - w_{s,k}) \\ &\quad + g_{js}(w_{i,k} - w_{k,i}) + g_{jk}(w_{s,i} - w_{i,s}) + g_{si}(w_{k,j} - w_{j,k}) \end{aligned} \tag{2.58}$$

Taking (2.58) into (2.57), we have

$$\begin{aligned} W_{jm} = F^{ks}F_m{}^i[&W_{sijk} - g_{ki}(w_{s,j} - w_{j,s}) + g_{ks}(w_{j,i} - w_{i,j}) + g_{ji}(w_{k,s} - w_{s,k}) \\ &+ g_{js}(w_{i,k} - w_{k,i}) + g_{jk}(w_{s,i} - w_{i,s}) + g_{si}(w_{k,j} - w_{j,k})] \\ &- F^{ks}F_m{}^i 2g_{si}(w_{j,k} - w_{k,j}) \end{aligned} \tag{2.59}$$

Simplifying the equation (2.59), then we get

$$\begin{aligned} W_{jm} = F^{ks}F_m{}^i W_{sijk} - F^{ks}F_m{}^i[&g_{ki}(w_{s,j} - w_{j,s}) + g_{ks}(w_{j,i} - w_{i,j}) + g_{ji}(w_{k,s} - w_{s,k}) \\ &+ g_{js}(w_{i,k} - w_{k,i}) + g_{jk}(w_{s,i} - w_{i,s}) + 3g_{si}(w_{k,j} - w_{j,k})] \end{aligned} \tag{2.60}$$

Also, from (1.74), we write the first Bianchi for $W_{isjk}$ as

$$W_{isjk} + W_{sjik} + W_{jisk} = 0 \tag{2.61}$$

From (2.61)

$$W_{sijk} = W_{sjik} + W_{jisk} \tag{2.62}$$

Taking account (2.62) into (2.60) and arranging the (2.60), we obtain

$$\begin{aligned} W_{jm} = F^{ks}F_m{}^i W_{sjik} + F^{ks}F_m{}^i W_{jisk} - F^{ks}F_m{}^i[&g_{ki}(w_{s,j} - w_{j,s}) + g_{ks}(w_{j,i} - w_{i,j}) \\ + g_{ji}(w_{k,s} - w_{s,k}) + g_{js}(w_{i,k} - w_{k,i}) &+ g_{jk}(w_{s,i} - w_{i,s}) + 3g_{si}(w_{k,j} - w_{j,k})] \end{aligned} \tag{2.63}$$

By interchanging the indices $k$ and $s$ for $W_{sjik}$ in (2.63), we have

$$W_{jm} = F^{sk}F_m{}^iW_{kjis} + F^{ks}F_m{}^iW_{jisk} - F^{ks}F_m{}^i[g_{ki}(w_{s,j} - w_{j,s}) + g_{ks}(w_{j,i} - w_{i,j}) + g_{ji}(w_{k,s} - w_{s,k}) + g_{js}(w_{i,k} - w_{k,i}) + g_{jk}(w_{s,i} - w_{i,s}) + 3g_{si}(w_{k,j} - w_{j,k})] \quad \textbf{(2.64)}$$

Using (2.9) in (2.64), we get

$$W_{jm} = -F^{ks}F_m{}^iW_{kjis} + F^{ks}F_m{}^iW_{jisk} - F^{ks}F_m{}^i[g_{ki}(w_{s,j} - w_{j,s}) + g_{ks}(w_{j,i} - w_{i,j}) + g_{ji}(w_{k,s} - w_{s,k}) + g_{js}(w_{i,k} - w_{k,i}) + g_{jk}(w_{s,i} - w_{i,s}) + 3g_{si}(w_{k,j} - w_{j,k})] \quad \textbf{(2.65)}$$

and using (2.54), (2.65) becomes

$$W_{jm} = -W_{jm} + F^{ks}F_m{}^iW_{jisk} - F^{ks}F_m{}^i[g_{ki}(w_{s,j} - w_{j,s}) + g_{ks}(w_{j,i} - w_{i,j}) + g_{ji}(w_{k,s} - w_{s,k}) + g_{js}(w_{i,k} - w_{k,i}) + g_{jk}(w_{s,i} - w_{i,s}) + 3g_{si}(w_{k,j} - w_{j,k})] \quad \textbf{(2.66)}$$

Substituting (1.52) in (2.66) and re-arranging (2.66), we reach

$$2W_{jm} = -F^{ks}F_m{}^iW_{ijsk} - F^{ks}F_m{}^i[g_{ki}(w_{s,j} - w_{j,s}) + g_{ks}(w_{j,i} - w_{i,j}) + g_{ji}(w_{k,s} - w_{s,k}) + g_{js}(w_{i,k} - w_{k,i}) + g_{jk}(w_{s,i} - w_{i,s}) + 3g_{si}(w_{k,j} - w_{j,k})] \quad \textbf{(2.67)}$$

From (1.78), we have

$$W_{ijsk} - W_{skij} = g_{kj}(w_{i,s} - w_{s,i}) + g_{ki}(w_{s,j} - w_{j,s}) + g_{js}(w_{k,i} - w_{i,k}) + g_{si}(w_{j,k} - w_{k,j}) + g_{sk}(w_{i,j} - w_{j,i}) + g_{ij}(w_{k,s} - w_{s,k}) \quad \textbf{(2.68)}$$

If we substitute (2.68) into (2.67), we obtain

$$2W_{jm} = -F^{ks}F_m{}^i[W_{skij} + g_{kj}(w_{i,s} - w_{s,i}) + g_{ki}(w_{s,j} - w_{j,s}) + g_{js}(w_{k,i} - w_{i,k}) + g_{si}(w_{j,k} - w_{k,j}) + g_{sk}(w_{i,j} - w_{j,i}) + g_{ij}(w_{k,s} - w_{s,k})] - F^{ks}F_m{}^i[g_{ki}(w_{s,j} - w_{j,s}) + g_{ks}(w_{j,i} - w_{i,j}) + g_{ji}(w_{k,s} - w_{s,k}) + g_{js}(w_{i,k} - w_{k,i}) + g_{jk}(w_{s,i} - w_{i,s}) + 3g_{si}(w_{k,j} - w_{j,k})] \quad \textbf{(2.69)}$$

After some calculations, we obtain

$$2W_{jm} = -F^{ks}F_m{}^iW_{skij} - F^{ks}F_m{}^i[2g_{ki}(w_{s,j} - w_{j,s}) + 2g_{ji}(w_{k,s} - w_{s,k}) + 2g_{si}(w_{k,j} - w_{j,k})] \quad \textbf{(2.70)}$$

By simplifying (2.70), we find

$$W_{jm} = -\frac{1}{2}F^{ks}F_m{}^iW_{skij} - F^{ks}F_m{}^i[g_{ki}(w_{s,j} - w_{j,s}) + g_{ji}(w_{k,s} - w_{s,k}) + g_{si}(w_{k,j} - w_{j,k})] \quad \textbf{(2.71)}$$

or

$$W_{jm} = -\frac{1}{2}F^{ks}F_m{}^i W_{skij} - F^{ks}F_m{}^i g_{ki}(w_{s,j} - w_{j,s})$$
$$-F^{ks}F_m{}^i g_{ji}(w_{k,s} - w_{s,k}) - F^{ks}F_m{}^i g_{si}(w_{k,j} - w_{j,k}) \quad \textbf{(2.72)}$$

From (2.6) and (2.1), we obtain

$$W_{jm} = -\frac{1}{2}F^{ks}F_m{}^i W_{skij} + \delta_m^s(w_{s,j} - w_{j,s})$$
$$-F^{ks}F_{mj}(w_{k,s} - w_{s,k}) - \delta_m^k(w_{k,j} - w_{j,k}) \quad \textbf{(2.73)}$$

After some calculations,

$$W_{jm} = -\frac{1}{2}F^{ks}F_m{}^i W_{skij} - F^{ks}F_{mj}(w_{k,s} - w_{s,k}) \quad \textbf{(2.74)}$$

By using (1.52) and (2.8), (2.74) turns into

$$W_{jm} = \frac{1}{2}F^{ks}F_m{}^i W_{ksij} + F^{ks}F_{jm}(w_{k,s} - w_{s,k}) \quad \textbf{(2.75)}$$

**ii)** Multiplying (2.71) by $g^{jm}$, and using (1.42), we get

$$W = -\frac{1}{2}g^{jm}F_m{}^i F^{ks}W_{skij} - g^{jm}F^{ks}F_m{}^i[g_{ki}(w_{s,j} - w_{j,s})$$
$$+g_{ij}(w_{k,s} - w_{s,k}) + g_{si}(w_{k,j} - w_{j,k})] \quad \textbf{(2.76)}$$

By (2.7) and we rewrite (2.76) as

$$Q \equiv W + \frac{1}{2}F^{ji}F^{ks}W_{skij} + F^{ks}F^{ji}[g_{ki}(w_{s,j} - w_{j,s})$$
$$+g_{ij}(w_{k,s} - w_{s,k}) + g_{si}(w_{k,j} - w_{j,k})] \quad \textbf{(2.77)}$$

and

$$Q \equiv W + \frac{1}{2}F^{ji}F^{ks}W_{skij} + F^{ks}F^{ji}g_{ki}(w_{s,j} - w_{j,s})$$
$$+F^{ks}F^{ji}g_{ij}(w_{k,s} - w_{s,k}) + F^{ks}F^{ji}g_{si}(w_{k,j} - w_{j,k}) \quad \textbf{(2.78)}$$

Since $F^{ks}F^{ji}g_{ij} = 0$, (2.78) turns into

$$Q \equiv W + \frac{1}{2}F^{ji}F^{ks}W_{skij} + F^{ks}F^{ji}g_{ki}(w_{s,j} - w_{j,s}) + F^{ks}F^{ji}g_{si}(w_{k,j} - w_{j,k}) \quad \textbf{(2.79)}$$

Now, interchanging the indices $k$ and $s$ in (2.79), we get

$$Q \equiv W + \frac{1}{2}F^{ji}F^{ks}W_{skij} + F^{sk}F^{ji}g_{si}(w_{k,j} - w_{j,k}) + F^{ks}F^{ji}g_{si}(w_{k,j} - w_{j,k}) \quad \textbf{(2.80)}$$

Using the (2.9) in (2.80), we obtain

$$Q \equiv W + \frac{1}{2}F^{ji}F^{ks}W_{skij} - F^{ks}F^{ji}g_{si}(w_{k,j} - w_{j,k}) + F^{ks}F^{ji}g_{si}(w_{k,j} - w_{j,k}) \tag{2.81}$$

which yields

$$Q \equiv W + \frac{1}{2}F^{ji}F^{ks}W_{skij} \tag{2.82}$$

Using the (2.9) in (2.82), hence we find

$$Q \equiv W + \frac{1}{2}F^{sk}F^{ij}W_{skij} = 0 \tag{2.83}$$

**iii)** According to **Theorem 2.4.**, we write (2.36) as

$$W_{kjmt} = F_m{}^i F_t{}^s W_{kjis} \tag{2.84}$$

Multiplying (2.84) by $g^{kt}$ and using (1.41) and (2.7), we obtain

$$W_{jm} = F^{ks}F_m{}^i W_{kjis} \tag{2.85}$$

Contracting (2.85) by $g^{jm}$ and using (1.42) and (2.7), we reach

$$W = F^{ks}F^{ji}W_{kjis} \tag{2.86}$$

and then

$$W = -F^{sk}F^{ji}W_{kjis} \tag{2.87}$$

Substituting (2.7) in (2.87), we get

$$W = -g^{sm}F_m{}^k F^{ji}W_{kjis} \tag{2.88}$$

From (1.39), we reach

$$W = -F_m{}^k F^{ji}W_{kji}{}^m \tag{2.89}$$

Applying (2.9) in (2.89), we get

$$W = F^{ij}F_m{}^k W_{kji}{}^m \tag{2.90}$$

We quote the following theorem from [15].

**Theorem 2.6.** If an almost semi-Kaehlerian structure $F_i^j$ satisfies

$$\dot{\nabla}_k F_{ij}\dot{\nabla}^i F^{jk} = a\dot{\nabla}_k F_{ij}\dot{\nabla}^k F^{ij} \tag{2.91}$$

where $a$ is a non-zero constant, then $Q$ defined by (2.51)

$$Q \equiv W + \frac{1}{2}F^{ij}F^{kl}W_{ijkl} + 2g^{kj}(\nabla_j w_k - \nabla_k w_j) \tag{2.92}$$

is non-negative (non-positive) according as $a$ is positive (negative). Further, the structure $F_i{}^j$ is Kaehlerian on $W_n$ if and only if $Q = 0$.

In [15], it is shown that

$$Q = a(\dot{\nabla}_k F_{ij})(\dot{\nabla}^k F^{ij}) \tag{2.93}$$

where $\dot{\nabla}^k = g^{jk}\dot{\nabla}_j$.

In particular, $Q = 0$ if and only if the structure is Kaehlerian on $W_n$.

We now state the following theorem which gives a relation between a nearly-Kaehlerian structure on $NKW_n$ and a Kaehlerian structure on $KW_n$.

**Theorem 2.7.** For a nearly-Kaehlerian structure $F_i{}^j$ on $NKW_n$

$$Q \geq 0 \tag{2.94}$$

where $Q$ is given by (2.51), and the equality holds if and only if the structure $F_i{}^j$ is Kaehlerian.

**Proof.** Since a nearly-Kaehlerian structure $F_i{}^j$ is an almost semi-Kaehlerian structure, multiplying (2.10) by $g^{jm}$, we obtain

$$\dot{\nabla}_i F^{mk} + \dot{\nabla}^m F_i{}^k = 0 \tag{2.95}$$

and also multiplying (2.95) by $g^{in}$, we get

$$\dot{\nabla}^n F^{mk} = -\dot{\nabla}^m F^{nk} \tag{2.96}$$

From equations (2.11), (2.91) and (2.96), we have

$$\dot{\nabla}_k F_{ij}(\dot{\nabla}^i F^{jk}) = (\dot{\nabla}_k F_{ij})(-\dot{\nabla}^i F^{kj}) = (\dot{\nabla}_k F_{ij})(\dot{\nabla}^k F^{ij}) \tag{2.97}$$

from which it follows that $a = 1$.

Also, we observe that the equality in (2.94) holds if and only if the structure $F_i{}^j$ is Kaehlerian.

**Theorem 2.8.** If an almost $L$-structure $F_i{}^j$ on $LW_n$ is almost semi-Kaehlerian and satisfies either

$$\dot{\nabla}^k G_{hi}{}^j = 0 \tag{2.98}$$

or

$$\dot{\nabla}^k F_{hij} = 0 \tag{2.99}$$

where $\dot{\nabla}^k = g^{jk}\dot{\nabla}_j$, $F_{hij}$ and $G_{hi}{}^j$ are defined in (2.4) and (2.10) respectively, then the structure $F_i{}^j$ is Kaehlerian.

**Proof.** From the defining condition (2.11) of an almost $L$-structure, it follows that

$$\dot{\nabla}_k \dot{\nabla}_h F_i{}^j = \dot{\nabla}_h \dot{\nabla}_k F_i{}^j \tag{2.100}$$

Multiplying (2.100) by $g^{ik}$ gives

$$\dot{\nabla}^i \dot{\nabla}_h F_i{}^j = \dot{\nabla}_h \dot{\nabla}_k F^{kj} = -\dot{\nabla}_h \dot{\nabla}_k F^{jk} = -\dot{\nabla}_h \dot{\nabla}_k g^{ij} F_i{}^k = -g^{ij} \dot{\nabla}_h \dot{\nabla}_k F_i{}^k \tag{2.101}$$

Since $F_i{}^j$ is an almost semi-Kaehlerian structure, we have

$$\dot{\nabla}_k F_i{}^k = 0 \tag{2.102}$$

then, (2.101) reduces to

$$\dot{\nabla}^i \dot{\nabla}_h F_i{}^j = 0 \tag{2.103}$$

Now, suppose (2.98) holds. Multiplying (2.98) by $\delta_k^i$, we obtain

$$\delta_k^i \dot{\nabla}^k G_{hi}{}^j = \delta_k^i \dot{\nabla}^k (\dot{\nabla}_h F_i{}^j + \dot{\nabla}_i F_h{}^j) = 0 \tag{2.104}$$

and using (2.103), (2.104) gives

$$\dot{\nabla}^i \dot{\nabla}_i F_h{}^j = 0 \tag{2.105}$$

On the other hand, in [15] it is proved that an almost Hermitian structure $F_i{}^j$ satisfies

$$F^{ij} \dot{\nabla}^k \dot{\nabla}_k F_{ij} = 0 \tag{2.106}$$

the structure is Kaehlerian. Thus, from (2.105) and (2.106), it is obtained that the structure $F_h{}^j$ is Kaehlerian.

Now, suppose (2.99) holds. Taking the prolonged covariant derivative of (2.99), we have

$$\dot{\nabla}^k F_{hij} = \dot{\nabla}^k \dot{\nabla}_h F_{ij} + \dot{\nabla}^k \dot{\nabla}_i F_{jh} + \dot{\nabla}^k \dot{\nabla}_j F_{hi} = 0 \tag{2.107}$$

Multiplying (2.107) by $\delta_k^i$, and using (2.6), respectively, we obtain

$$\dot{\nabla}^i \dot{\nabla}_h F_{ij} + \dot{\nabla}^i \dot{\nabla}_i F_{jh} + \dot{\nabla}^i \dot{\nabla}_j F_{hi} = 0 \tag{2.108}$$

and

$$g_{jm}\dot{\nabla}^i\dot{\nabla}_h F_i{}^m + \dot{\nabla}^i\dot{\nabla}_i F_{jh} - g_{hm}\dot{\nabla}^i\dot{\nabla}_j F_i{}^m = 0 \tag{2.109}$$

By means of (2.103), the equation (2.109) yields

$$\dot{\nabla}^i\dot{\nabla}_i F_{jh} = 0 \tag{2.110}$$

From (2.105), and (2.110), it follows that the structure is Kaehlerian.

# 3. EINSTEIN $L$-WEYL SPACES

## 3.1 Einstein $L$-Weyl Spaces

In this section, we give the definition of an Einstein $L$-Weyl space and show its relation with the symmetric part of the Ricci curvature tensor of Weyl space. Then, we consider an Einstein space and state that the necessary and the sufficient condition for an Einstein $L$-Weyl space to be an Einstein space.

We call a Weyl space an Einstein $L$-Weyl space, denote by $ELW$, if the symmetric part of the Ricci curvature tensor of $W_n$ is proportional to the metric tensor $g$

$$W_{(kj)} = \lambda g_{kj} \quad \textbf{(3.1)}$$

where $\lambda$ is a scalar function defined on $W_n$.

From (1.64), we have

$$W_{ji} = R_{ji} + (n-2)w_{ji} + (w_{ji} - w_{ij}) + g_{ji}g^{km}w_{km} \quad \textbf{(3.2)}$$

On the other hand, since the Riemannian space is an $L$-space, its Ricci curvature tensor has the following property [17]

$$R_{ji} = F_j{}^h F_i{}^k R_{hk} \quad \textbf{(3.3)}$$

Hence, if we write (3.2) and the Ricci curvature tensor $W_{ij}$ in terms of (3.3), we find

$$W_{ji} = F_i{}^h F_j{}^k R_{hk} + (n-2)w_{ij} + (w_{ij} - w_{ji}) + g_{ij}g^{km}w_{km} \quad \textbf{(3.4)}$$

and

$$W_{ij} = F_j{}^h F_i{}^k R_{hk} + (n-2)w_{ji} + (w_{ji} - w_{ij}) + g_{ji}g^{km}w_{km} \quad \textbf{(3.5)}$$

From (3.5) and (3.4), we conclude

$$W_{ji} + W_{ij} = 2F_j{}^h F_i{}^k R_{hk} + (n-2)(w_{ji} + w_{ij}) + 2g_{ji}g^{km}w_{km} \quad \textbf{(3.6)}$$

and

$$W_{(ji)} = F_j{}^h F_i{}^k R_{hk} + \frac{(n-2)}{2}(w_{ji} + w_{ij}) + g_{ji} g^{km} w_{km} \qquad \textbf{(3.7)}$$

Substituting (1.48) into (3.7), we get

$$W_{(ji)} = F_j{}^h F_i{}^k R_{hk} + \frac{(n-2)}{2}[\nabla_j w_i + w_j w_i - \frac{1}{2} g_{ji} w_t w^t + \nabla_i w_j + w_i w_j - \frac{1}{2} g_{ij} w_t w^t]$$
$$+ g_{ji} g^{km}(\nabla_k w_m + w_k w_m - \frac{1}{2} g_{km} w_t w^t) \qquad \textbf{(3.8)}$$

and after some simplification, we obtain

$$W_{(ji)} = F_j{}^h F_i{}^k R_{hk} + \frac{(n-2)}{2}[\nabla_j w_i + \nabla_i w_j + 2 w_j w_i - g_{ji} w_t w^t$$
$$+ g_{ji}(\nabla_k w_k + w_k w_k - \frac{n}{2} w_t w^t) \qquad \textbf{(3.9)}$$

Finally, the symmetric part of the Ricci curvature tensor of $(ELW)$ can be written as

$$W_{(ji)} = F_j{}^h F_i{}^k R_{hk} + \frac{(n-2)}{2}[\nabla_j w_i + \nabla_i w_j + 2 w_j w_i] + g_{ji}(\nabla_k w_k + (2-n) w_t w^t) \; \textbf{(3.10)}$$

**Theorem 3.1.** For an Einstein $L$-Weyl space $(ELW)$ to be an Einstein space, the necessary and sufficient condition is

$$\frac{(n-2)}{2}[\nabla_j w_i + \nabla_i w_j + 2 w_j w_i] = \theta g_{ji} \qquad \textbf{(3.11)}$$

where $\theta$ is a scalar function on $W_n$.

**Proof.** Let the space be an Einstein $L$-Weyl, then we have

$$W_{(ij)} = \lambda g_{ij} \qquad \textbf{(3.12)}$$

If the Riemannian space is Einstein i.e., $R_{ij} = \beta g_{ij}$, then (3.10) reduces to

$$\lambda g_{ji} = \beta g_{ij} + \frac{(n-2)}{2}[\nabla_j w_i + \nabla_i w_j + 2 w_j w_i] + g_{ji}[\nabla_k w_k + (2-n) w_t w^t] \qquad \textbf{(3.13)}$$

with $\beta$ is a scalar function of coordinates.

Since the last term in the right-handside is scalar multiple of the metric tensor, say $\alpha g_{ji}$, (3.13) can be written as

$$(\lambda - \beta - \alpha) g_{ji} = \frac{(n-2)}{2}[\nabla_j w_i + \nabla_i w_j + 2 w_j w_i] \qquad \textbf{(3.14)}$$

or, in another form

$$\frac{(n-2)}{2}[\nabla_j w_i + \nabla_i w_j + 2 w_j w_i] = \theta g_{ji} \qquad \textbf{(3.15)}$$

where $\theta = (\lambda - \beta - \alpha)$.

Conversely, by using (3.11) and the Riemannian space is an Einstein

$$R_{ij} = \beta\, g_{ij} \tag{3.16}$$

we obtain that $W_n$ is an Einstein $L$-Weyl space.
Also, it follows from (3.10) that

**Corollary 3.1.** Every 2-dimensional Einstein $L$-Weyl space $(ELW)$ is an Einstein space.

### 3.2 Generalized Einstein $L$-Weyl Tensor

Weyl spaces with semi-symmetric connection and generalized Einstein tensor in Weyl spaces are studied in the literature [19,25]. In these spaces, a covariantly constant tensor is obtained such that its generalized derivative with respect to the connection of the space vanishes, and then, it is called generalized Einstein tensor.

In this section we obtain Einstein $L$-Weyl tensor by using Bianchi identities, and properties of curvature tensors of almost $L$-Weyl spaces. Einstein tensor in Weyl space is defined in [20,21]

$$G_l{}^j = W_l{}^j - \frac{1}{2}\delta_l^j W - 2g^{jk}\nabla_{[l}w_{k]} \tag{3.17}$$

Here $W_l{}^k = W_{lm}\, g^{mk}$ is the Ricci curvature tensor, $W$ is the scalar curvature of the Weyl space, and $w$ is the complementary vector. We prove that the tensor $G_l{}^j$ is covariantly constant with respect to the prolonged covariant derivative and called Einstein tensor.

Using the covariantly constant property of the metric tensor $g$, the first and the second Bianchi identities (1.72) and (1.73) can be reduced

$$W_{kjim} + W_{jikm} + W_{ikjm} = 0 \tag{3.18}$$

$$\dot{\nabla}_l W_{kjim} + \dot{\nabla}_j W_{lkim} + \dot{\nabla}_k W_{jlim} = 0 \tag{3.19}$$

On the other hand, the Ricci formula for $F_i{}^h$ is [14]

$$(\dot{\nabla}_k\dot{\nabla}_j - \dot{\nabla}_j\dot{\nabla}_k)F_i{}^h = F_i{}^a W_{kja}{}^h - F_a{}^h W_{kji}{}^a \tag{3.20}$$

Since the Weyl connection is symmetric and $F_i{}^h$ is zero weight then we have

$$(\dot{\nabla}_k\dot{\nabla}_j - \dot{\nabla}_j\dot{\nabla}_k)F_i{}^h = (\nabla_k\nabla_j - \nabla_j\nabla_k)F_i{}^h \tag{3.21}$$

Since the structure $F_i^{\ h}$ is an almost $L$-structure

$$(\dot{\nabla}_k \dot{\nabla}_j - \dot{\nabla}_j \dot{\nabla}_k) F_i^{\ h} = 0 \tag{3.22}$$

and then

$$F_i^{\ a} W_{kja}^{\ \ \ h} = F_a^{\ h} W_{kji}^{\ \ \ a} \tag{3.23}$$

Multiplying (3.23) by $F_m^{\ i}$, we get

$$W_{kjm}^{\ \ \ h} = -F_a^{\ h} F_m^{\ i} W_{kji}^{\ \ \ a} \tag{3.24}$$

and multiplying (3.24) by $g_{ht}$ and using (2.6), we reach

$$W_{kjmt} = -F_{at} F_m^{\ i} W_{kji}^{\ \ \ a} \tag{3.25}$$

Substituting (2.6), (2.8) and (1.40) in (3.25), we find

$$W_{kjmt} = F_t^{\ a} F_m^{\ i} W_{kjia} \tag{3.26}$$

Contracting (3.26) by $g^{kt}$ and using (2.7), we obtain

$$W_{jm} = F_m^{\ l} F^{ka} W_{kjla} \tag{3.27}$$

Furthermore, multiplying (3.27) by $g^{mt}$, (3.27) reduces

$$W_j^{\ t} = F^{tl} F^{ka} W_{kjla} \tag{3.28}$$

Let us multiply the first Bianchi identity (1.72) by $F_l^{\ k}$, then we have

$$F_l^{\ k} (W_{ijk}^{\ \ \ l} + W_{jki}^{\ \ \ l} + W_{kij}^{\ \ \ l}) = 0 \tag{3.29}$$

By using (2.36), (1.52) in (3.29), we have

$$F_k^{\ l} W_{ijl}^{\ \ \ k} - F_i^{\ l} W_{kjl}^{\ \ \ k} + F_j^{\ l} W_{kil}^{\ \ \ k} = 0 \tag{3.30}$$

Simplifying (3.30), we get

$$F_k^{\ l} W_{ijl}^{\ \ \ k} - F_i^{\ l} W_{jl} + F_j^{\ l} W_{il} = 0 \tag{3.31}$$

Also multiplying (3.31) by $F^{ij}$, we get

$$F^{ij} F_k^{\ l} W_{ijl}^{\ \ \ k} - F^{ij} F_i^{\ l} W_{jl} + F^{ij} F_j^{\ l} W_{il} = 0 \tag{3.32}$$

Using (2.7) and re-arranging the equation (3.32), we find

$$F^{ij}F_k{}^l W_{ijl}{}^k - g^{lj}W_{jl} + g^{li}W_{il} = 0 \tag{3.33}$$

By (1.42), (3.33) gives rise to the relation

$$F^{ij}F_k{}^l W_{ijl}{}^k - 2W = 0 \tag{3.34}$$

so we get

$$F^{ij}F_k{}^l W_{ijl}{}^k = 2W \tag{3.35}$$

Multiplying (3.18) by $F^{ik}$, we write

$$F^{ik}(W_{ijkl} + W_{jkil} + W_{kijl}) = 0 \tag{3.36}$$

By the symmetry rule of (1.52), we get

$$F^{ik}W_{ijkl} - F^{ik}W_{kjil} + F^{ik}W_{kijl} = 0 \tag{3.37}$$

Interchanging the indices $i$ and $k$, we rewrite the equation (3.37) as

$$F^{ki}W_{kjil} - F^{ik}W_{kjil} + F^{ik}W_{kijl} = 0 \tag{3.38}$$

From (2.9) and re-arranging (3.38), we reach

$$2F^{ki}W_{kjil} - F^{ki}W_{kijl} = 0 \tag{3.39}$$

Thus we get

$$F^{ki}(2W_{kjil} - W_{kijl}) = 0 \tag{3.40}$$

Let us now derive the generalized Einstein equation for the $L$-Weyl space. Contracting (3.19) by $g^{km}$, we get

$$\dot{\nabla}_l W_{ji} - \dot{\nabla}_j W_{li} + g^{km}\dot{\nabla}_k W_{jlim} = 0 \tag{3.41}$$

then, contracting (3.41) by $g^{ji}$, we obtain

$$\dot{\nabla}_l W - \dot{\nabla}_j W_l{}^j + g^{km}g^{ji}\dot{\nabla}_k(-W_{jlmi} + 2g_{im}(\nabla_l w_j - \nabla_j w_l)) = 0 \tag{3.42}$$

Re-arranging (3.42)

$$\dot{\nabla}_l W - \dot{\nabla}_j W_l{}^j + g^{km}\dot{\nabla}_k(-W_{lm}) + 2g^{km}\dot{\nabla}_k(\nabla_l w_m - \nabla_m w_l) = 0 \tag{3.43}$$

After some calculations, we reach

$$\dot{\nabla}_j W - 2\dot{\nabla}_j W_l{}^j + 2\dot{\nabla}_j(\nabla_l w^j - g^{jm}\nabla_m w_l) = 0 \tag{3.44}$$

and we have

$$\dot{\nabla}_j[W_l{}^j - \frac{1}{2}\delta_l^j W - (\nabla_l w^j - g^{jm}\nabla_m w_l)] = 0 \tag{3.45}$$

In (3.45), we call the tensor

$$G_l{}^j = W_l{}^j - \frac{1}{2}\delta_l^j W - 2g^{jk}\nabla_{[l}w_{k]} \tag{3.46}$$

as the generalized Einstein $L$-Weyl tensor.

From (3.45), $G_l{}^j$ is covariantly conserved, that is,

$$\dot{\nabla}_j G_l{}^j = 0 \tag{3.47}$$

We now express (3.45) in terms of almost $L$-structure. By using (3.28) and (3.35), generalized Einstein equation (3.45) takes the form

$$\dot{\nabla}_j(F^{jm}F^{kt}W_{klmt} - \frac{1}{4}\delta_l^j F^{mt}F_k{}^h W_{mth}{}^k - 2g^{jk}\nabla_{[l}w_{k]}) = 0 \tag{3.48}$$

In almost $L$-Weyl space, the generalized Einstein $L$-Weyl tensor becomes

$$G_l{}^j = F^{jm}F^{kt}W_{klmt} - \frac{1}{4}\delta_l^j F^{mt}F_k{}^h W_{mth}{}^k - 2g^{jk}\nabla_{[l}w_{k]} \tag{3.49}$$

Then, we have proved the following theorem.

**Theorem 3.2.** The generalized Einstein tensor in $L$-Weyl space is given by (3.49).

**Remark.** If we take the complementary vector field as locally covariant, then (3.49) reduces to

$$G_l{}^j = F^{jm}F^{kt}W_{klmt} - \frac{1}{4}\delta_l^j F^{mt}F_k{}^h W_{mth}{}^k \tag{3.50}$$

# 4. CONCLUSIONS AND RECOMMENDATIONS

In differential geometry spaces can be classified with respect to structures defined on them. Weyl spaces are similar to Riemannian space that admit symmetric connection but the covariant derivative of the metric tensor is not zero in Weyl spaces.

In this thesis, we introduce almost $L$-structures and nearly-Kaehlerian structures on Weyl spaces $W_n$ to examine curvature properties of Weyl spaces having these structures. We also define Einstein $L$-Weyl space and we give a necessary and sufficient condition for an Einstein $L$-Weyl space $(ELW)$ to be Einstein space. Moreover, we define the generalized Einstein tensor in $L$-Weyl spaces and express it in terms of almost $L$-structures.

Then, almost complex and almost Kaehlerian structures on Weyl space are defined. We construct almost $L$-structures and nearly-Kaehlerian structures. We prove the integrability condition of Kaehlerian Weyl space and proved that nearly-Kaehlerian Weyl space is a Kaehlerian-Weyl space if the structure is integrable. In addition, we give some theorems which are used to find out generalized Einstein $L$-Weyl tensor.

In the conclusion, we reveal how all these structures are related with Einstein space. By means of the theorems, we also state the generalized Einstein $L$-Weyl tensor in the terms of curvature tensor, covariant curvature tensor and $F_i^{\ j}$ tensors which are defined.

Moreover, if we take our Weyl connection as semi-symmetric, then it reduces more richer spaces then we found. Also it gives new curvature properties and theorems to construct new structures and classify them.

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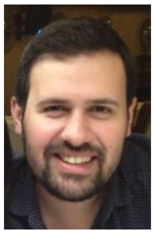
## CURRICULUM VITAE



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## PUBLICATIONS/PRESENTATIONS ON THE THESIS

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