ISTANBUL TECHNICAL UNIVERSITY ★ GRADUATE SCHOOL

COSMOLOGICAL INTERACTING MODELS VIA ENERGY-MOMENTUM SQUARED GRAVITY

M.Sc. THESIS

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Department of Physics Engineering

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(509201113)

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Thesis Advisor: Prof. Dr. Özgür AKARSU
Thesis Co-Advisor: Doç. Dr. A. Nihan KATIRCI

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İSTANBUL TEKNİK ÜNİVERSİTESİ ★ LİSANSÜSTÜ EĞİTİM ENSTİTÜSÜ

ENERJİ-MOMENTUM KARE KÜTLEÇEKİM KURAMIYLA KURULAN ETKİLEŞİM ÜZERİNE KOZMOLOJİK MODELLER

YÜKSEK LİSANS TEZİ

Bilal BULDUK
(509201113)

Fizik Mühendisliği Anabilim Dalı

Fizik Mühendisliği Programı

Tez Danışmanı: Prof. Dr. Özgür AKARSU
Eş Danışman: Doç. Dr. A. Nihan KATIRCI

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Bilal BULDUK, a M.Sc. student of ITU Graduate School student ID 509201113 successfully defended the thesis entitled "COSMOLOGICAL INTERACTING MODELS VIA ENERGY-MOMENTUM SQUARED GRAVITY ", which he prepared after fulfilling the requirements specified in the associated legislations, before the jury whose signatures are below.

Thesis Advisor :	**Prof. Dr. Özgür AKARSU** Istanbul Technical University	....
Co-advisor :	**Doç. Dr. A. Nihan KATIRCI** Doğuş University	....
Jury Members :	**Prof. Dr. Özgür AKARSU** Istanbul Technical University	....
	Doç. Dr. A. Nihan KATIRCI Doğuş University	....
	Prof. Dr. A. Savaş ARAPOĞLU Istanbul Technical University	....
	Prof. Dr. Neşe ÖZDEMİR Istanbul Technical University	....
	Prof. Dr. Cemsinan DELİDUMAN Mimar Sinan Fine Arts University	....

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To my mother, father, and siblings...

FOREWORD

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ABBREVIATIONS

1D	: One Dimensional
2D	: Two Dimensional
3D	: Three Dimensional
BAO	: Baryon Acoustic Oscillations
BBN	: Big Bang Nucleosynthesis
BOSS	: Baryon Oscillation Spectroscopic Survey
C-field	: Creation Field
CC	: Cosmic Chronometers
CDM	: Cold Dark Matter
CL	: Confidence Level
CMB	: Cosmic Microwave Background
DE	: Dark Energy
DM	: Dark Matter
EFE	: Einstein Field Equations
EH	: Einstein-Hilbert
EMPG	: Energy Momenrum Powered Gravity
EMSF	: Energy Momenrum Squared Field
EMSG	: Energy Momenrum Squared Gravity
EMT	: Energy-Momentum Tensor
EoS	: Equation of State
FLRW	: Friedmann–Lemaître–Robertson–Walker
GR	: General Relativity
IDE	: Interacting Dark Energy
ΛCDM	: Lambda Cold Dark Matter
PLK	: Planck Satellite
RW	: Robertson-Walker
RHS	: Right Hand Side
SN	: Supernovae
WIMPs	: Weakly Interacting Massive Particles

SYMBOLS

$(-,+,+,+)$	:	The signature of the metric
∂_μ	:	Partial Derivative
$\nabla_{\mu\nu}$	:	Covariant Derivative
$g_{\mu\nu}$	:	Metric Tensor
$\Gamma^{\gamma}_{\mu\nu}$	:	Christoffel Symbol
$R^{\sigma}_{\mu\nu\gamma}$	:	Riemann Tensor
$R_{\mu\nu}$	:	Ricci Tensor
R	:	Ricci Scalar
$T_{\mu\nu}$	:	Energy-Momentum Tensor
$G_{\mu\nu}$	:	Einstein Tensor
$\mathscr{L}_{\mathrm{m}}$	:	The matter Lagrangian
$\mathscr{Q}_\nu$	:	The interaction kernel
T	:	The trace of the energy momentum tensor
ρ	:	The energy density of the source
p	:	The pressure of the source
G	:	The Newton's gravitational constant
H	:	The Hubble parameter
w	:	The barotropic EoS parameter

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COSMOLOGICAL INTERACTING MODELS
VIA ENERGY-MOMENTUM SQUARED GRAVITY

SUMMARY

It was recently shown in the literature that gravity models that modify the material part of the standard Einstein-Hilbert action with $f(\mathscr{L}_{\rm m})$, $f(T)$, and $f(T_{\mu\nu}T^{\mu\nu})$ terms are equivalent to general relativity, encompassing non-minimal matter interactions between the material field and its accompanying partner, uniquely formed by the function f. In Energy-momentum squared gravity (EMSG), the "squared" terminology arises from the self contraction of EMT $f(T_{\mu\nu}T^{\mu\nu})$ added to Einstein Hilbert action, nontrivial interaction kernels have been obtained and these models diverge from phenomenological interacting models (constructed in ad hoc way); this is due to the function f and its variations with respect to both its argument and the metric, which intricately intertwine the interaction kernel $\mathscr{Q}(f, \delta f/\delta\mathbf{T^2}, \delta, \mathbf{T^2}/\delta g^{\mu\nu})$. This makes the interaction kernel as Equation of State (EoS) parameter dependent as well. Bianchi identity $\nabla^\mu G_{\mu\nu} = 0$ implies the conservation of total energy momentum tensor (EMT of the standard source plus its EMSF partner's), $\nabla^\mu(T_{\mu\nu} + T^{\rm EMSF}_{\mu\nu}) = 0$, leading cosmological models having an interaction between these sectors $\nabla^\mu T_{\mu\nu} = \mathscr{Q}_\nu$ and $\nabla^\mu T^{\rm EMSF}_{\mu\nu} = -\mathscr{Q}_\nu$ where $\mathscr{Q}_\nu \neq 0$.

In this thesis, different than the literature, still consistent with the Bianchi Identity, we focus on a scenario where the sector comprising conventional fluids (standard material fields) overall interacts minimally with the sector associated with their EMSF partners, i.e., satisfying $\nabla^\mu T_{\mu\nu} = 0 = \nabla^\mu T^{\rm EMSF}_{\mu\nu}$. Specifically, we consider the case characterized by $\mathscr{Q} = 0$. Accordingly, we will consider a two-fluid model (perfect fluids described by constant EoS parameters) leading to the following conservation equations, $\nabla^\mu\left(T_{\mu\nu,1} + T_{\mu\nu,2}\right) = 0$, and $\nabla^\mu\left(T^{\rm EMSF}_{\mu\nu,1} + T^{\rm EMSF}_{\mu\nu,2}\right) = 0$ where we name the partner arisen from EMSG corrections as "Energy Momentum Squared Field" (EMSF). We will explore this choice in detail within the framework of scale-independent EMSG which introduces a simple interaction kernel: a kernel linear in energy density. Then, we examine alternative cosmologies wherein the sector comprising conventional fluids minimally interacts with the sector associated with their EMSF partners, represented by $\nabla^\mu\left(T^1_{\mu\nu} + T^2_{\mu\nu}\right) = -\nabla^\mu\left(T^{\rm EMSF1}_{\mu\nu} + T^{\rm EMSF2}_{\mu\nu}\right) = \mathscr{Q}_\nu$ with $\mathscr{Q}_\nu = 0$, diverging from the more commonly studied scenarios in literature where $\mathscr{Q}_\nu \neq 0$. We also show that this model is reminiscent of the cosmological model with energy exchange studied by Barrow and Clifton in [Phys. Rev. D 73, 103520 (2006)] where the interaction term is taken ad hoc to be proportional to energy density, $\mathscr{Q}(H\rho)$. Unlike their model, the coefficients in our work are not arbitrary constants but are dependent on the species. Moreover, with an additional sector associated with the EMSF partners of the conventional fluids in the Friedmann equation, it is possible to negate one of the fluid's contributions in the Friedmann equation via its EMSF partner for a

specific choice of α and two sources may superpose in their energy densities in the Friedmann equation, resulting in a joint (degenerate) scale factor dependence even if $w_1 \neq w_2$ reproducing interesting cosmologies such as power-law universes where the scale factor of the universe grows as a EoS parameter dependent power of time in the presence of a perfect fluid and vacuum energy density/stiff fluid, de Sitter universe in the presence of a perfect fluid and vacuum energy density. In this thesis, we show a simple mathematical description of the exchange of energy between two standard fluids from matter modified theories within GR choosing the simplest case study, yet some non-trivial functions/behaviors are favored by observations to alleviate tensions, non-linear interactions and non-linear energy density contributions from matter-type modified theories which may work for the change of direction of energy transfer in dark sector are prospects for future research.

ENERJİ-MOMENTUM KARE KÜTLEÇEKİM KURAMIYLA KURULAN ETKİLEŞİM ÜZERİNE KOZMOLOJİK MODELLER

ÖZET

Maddenin Einstein genel görelilik teorisindeki formunu değiştirmek üzere kurulan $f(\mathscr{L}_{\mathrm{m}})$, $f(T)$ ve $f(T_{\mu\nu}T^{\mu\nu})$ teorilerinin alternatif gravitasyon kuramları olmadığı ve bu madde katkılarının partner olarak genel görelilikteki mevcut madde formları ile minimal olmayan etkileşim yaptıkları yakın zamanda gösterilmiştir. Madde formuna enerji-momentum tensörünün (EMT) kendi üzerine büzülmesi $T_{\mu\nu}T^{\mu\nu}$ ile elde edilen Enerji-momentum kare kütleçekim (EMSG) modeli çerçevesinde minimal olmayan etkileşimin çekirdek fonksiyonları ($\mathscr{Q}$ interaction kernelleri) elde edilebilmekte ve kozmolojik modeller, gözlemsel verilerle kozmolojinin kuramsal standart modeli (ΛCDM) arasında ortaya çıkan uyumsuzlukları ve yapısını bilmediğimiz karanlık madde/enerji modelleri/etkileşimleri vererek bazı imkanlar sunabilmektedir. Bu modelde etkileşim çekirdeği literatürde özel bir amaçla atama yolu ile önerilen etkileşim modellerinden farklı olarak $\mathscr{Q}(f, \delta f/\delta\mathbf{T^2}, \delta, \mathbf{T^2}/\delta g^{\mu\nu})$ fonksiyonuna, ve fonksiyonun metriğe göre varyasyonlarına bağlıdır, öyleki bu da konulan maddenin cinsine bağlı etkileşimler meydana getirmektedir.

En yalın şekilde ve literatürde yaygın olarak kullanılan etkileşim çekirdeği formu, bu modelde de ölçek bağımsız Enerji-momentum kare kütleçekim (EMSG) modelinde Einstein alan denklemine gelen enerji yoğunluğu terimi ile aynı mertebede gelmekte, ve standart madde formuyla evrenin genişleme hızı ile doğrusal, $\sim H\rho$ formunda, enerji alışverişi sağlamaktadır. Bianchi özdeşliği $\nabla^\mu G_{\mu\nu} = 0$ olduğu için bize Einstein alan denklemlerinin sağ tarafındaki toplam enerji-momentum tensörünün korunumunu verir $\nabla^\mu(T_{\mu\nu} + T_{\mu\nu}^{\mathrm{EMSF}}) = 0$, böylece bu da geometri ile bağlantısı genel görelilik (GR) çerçevesinde olan ve evrenin evrimine bu yönde katkı veren kaynakları barındıran madde sektörü ve geometri ile bağlantısı Enerji-momentum kare kütleçekim (EMSG) çerçevesinde olan ve madde sektöründeki kaynakların partner kaynaklarını barındıran partner sektörü arasında etkileşim veren $\mathscr{Q}_\nu \neq 0$, $\nabla^\mu T_{\mu\nu} = \mathscr{Q}_\nu$ ve $\nabla^\mu T_{\mu\nu}^{\mathrm{EMSF}} = -\mathscr{Q}_\nu$ kozmolojik modellerini vermektedir.

Bu tezde literatüre bir yenilik olarak, birden fazla kaynak olması durumunda, Bianchi özdeşliğinin kuramsal olarak yine mümkün kıldığı üzere, $\nabla^\mu G_{\mu\nu} = 0 \rightarrow \nabla^\mu(T_{\mu\nu} + T_{\mu\nu}^{\mathrm{EMSF}}) = 0$, madde sektörü ve partner (enerji momentum kare alanı olarak isimlendirilen) sektörü ayrı olarak korunmaktadır. Bir başka ifadeyle $\nabla^\mu \left(T_{\mu\nu}^1 + T_{\mu\nu}^2\right) = -\nabla^\mu \left(T_{\mu\nu}^{\mathrm{EMSF1}} + T_{\mu\nu}^{\mathrm{EMSF2}}\right) = \mathscr{Q}_\nu$, etkileşim çekirdeği $\mathscr{Q}_\nu$ için iki durum ortaya çıkmaktadır: i) literatürde daha önce incelenmiş olan ve madde sektörü ile partner sektörün minimal olmayan şekilde etkileşimine karşılık gelen durum $\mathscr{Q}_\nu \neq 0$ ve ii) $\mathscr{Q}_\nu = 0$ ile temsil edilen EMSF partnerleriyle bildiğimiz akışkanlardan oluşan sektörün minimum etkileşime girdiği (sadece kütleçekimle etkileştiği) kozmolojik çözümlerini bu tezde incelediğimiz durum.

Literatürde ilk olarak, bu modelin Barrow ve Clifton'ın [Phys. Rev. D 73, 103520 (2006)] çalışmasında fenomenolojik olarak yazdıkları kaynaklar arasında enerji alışverişi olan kozmolojik modelini kuramsal modifikasyonlarla ürettiğini; ancak çalışmamızda farklı madde formlarında etkileşim çekirdeklerinin farklı olması nedeniyle literatürdeki sabit ve rastgele katsayıların kullanımdan farklı olarak etkileşimi belirleyen bu katsayıların durum parametresine (w) bağlı geldiği gösterilmiştir. Kozmolojik çözümler hem sabit parametreler ile kurulmuş ve geleneksel formdaki Liénard denklemine çarpan olarak gelen bir denklem ile ortaya çıkan modifiye Liénard denklemi hem de enerji momentum korunumu denkleminden türetilmiş ikinci dereceden diferansiyel bir denklem formunda çözülmüş ve modeldeki EMSF alanının standart enerji terimine oranını belirleyen, serbest parametre α'nın belli değerleri için yeni bir kozmolojik çözüm ailesi ürettiği gösterilmiştir.

Çalışmamızda Liénard denkleminin geleneksel kısmına ek olarak bir çarpan terim daha elde edilmiştir ve bu terim ile α parametresinin serbest olarak bırakıldığı genel çözüm ailesine ek olarak belirli α değerlerinde ortaya çıkan ve çözüm ailesini daha da genişleten alt çözüm kümesi elde edilmiştir. Buradaki etkileşen kaynakların cinsine bağlı olarak değişebilen bu belirli α değerleri etkileşim içermeyen enerji momentum kare kütleçekim modelindeki α değerlerinden farklıdır, bunun sebebi ise kaynaklar arasındaki etkileşim sayesinde partner kaynağın evrenin evrimine katkısının değişmesidir. Serbest α ile ifade edilen genel kozmololjik çözümlerde kaynaklar birbirlerine etkileşimi/enerji akatarımını gerçekleştirirken standart GR davranışlarından hem çarpan hem de üstel olarak ayrılır, bu da kaynakların geometriyle bağlantılarının ve evrenin dinamiğine katkı paylarının etkileşimle birlikte değiştiğini açıkca göstermektedir. Örneğin, etkileşen vakum enerjisi ve herhangi bir w durum parametreli kaynak çifti durumunda w durum parametreli kaynağın enerji yoğunluğu standart GR davranışından $a^{-3(1+w)}$ daha hızlı bir şekilde seyrelir.

Ayrıca muhtemel kozmolojik senaryolar olarak belirli α değerlerinde, vakum enerjisi ve herhangi bir w durum parametreli kaynak için vakum enerjisinin/w kaynağının Friedmann denklemine/evrenin dinamiğine belli α değerlerinde katkı sağlamaması mümkündür, oysa ki böyle bir durum genel görelilikte mümkün olmamaktadır. Standart GR'daki sadece vakum enerjisi varlığında deneyimlenen de Sitter genişlemesi bu modelde herhangi bir w kaynağı varlığında ve belirli α değerinde deneyimlenebilir. Bu özel senaryoların meydana gelmesi için partnerleri ile birlikte etkileşen kaynakların enerji yoğunlukları Friedmann denkleminde birleşmekte, böylece $w_1 \neq w_2$ olduğunda bile ölçek faktörü enerji ilişkisi dejenere olmaktadır, bu da yine farklı iki kaynak varlığında tek kaynak gibi davranan ve davranışı kaynakların cinsine bağlı olan güç yasası (power law) modelleri, kozmolojileri üretmektedir. Bu çalışma sadece kozmolojik çözüm ailelerin verildiği, lakin farklı etkileşim çekirdekleri ile karanlık enerji/madde ve kozmolojik uyumsuzluklar nezdinde oldukça popüler olan etkileşim modellerini üretmek hususunda madde modifikasyonlarının ilginç kozmolojiler verebileceği konusunda fikir vermek üzere yapılmıştır. Doğrusal olmayan etkileşim çekirdekleri de doğrusal olmayan EMSG katkılarından elde edilebilir, gelecek çalışmalarda bu doğrusal olmayan etkileşim çekirdekleri Jacobian matris yaklaşımla madde modifiye modellere uygulanıp doğrusallaştırılmış aralıkta kaynakların davranışı ve evrenin evrimine katkısı incelenmekte ve ayrıca kernellere uygulanacak farklı kısıtlamalar ile matematiksel olarak mümkün olan alternatif etkileşim modelleri aranacaktır. Gözlemler kozmolojik uyumsuzlukları azaltmanın karanlık enerji ve karanlık madde arasında enerji transferinin yön değiştirdiği modeller (salınım yapan) ile mümkün olduğunu göstermiştir lakin teorik olarak bu karmaşık etkileşim çekirdeklerini elde etmek zordur. Doğrusal olmayan enerji momentum güçlendirilmiş ya da kare kuramlarla doğrusal olmayan enerji yoğunluğu katkılarının, karanlık enerji yoğunluğunun işaretinin değişmesi için işe yarayabileceğini de test etmekteyiz.

1. INTRODUCTION

Matter-type modified theories of gravity are constructed by either generalizing the matter Lagrangian density away from the linear function of $\mathscr{L}_{\mathrm{m}}$, viz., extending it to $f(\mathscr{L}_{\mathrm{m}})$ [1] or adding some analytic function of a scalar formed from the usual energy-momentum tensor (EMT), $T_{\mu\nu}$, of material stresses such as $f(g_{\mu\nu}T^{\mu\nu})$ [2] and $f(T_{\mu\nu}T^{\mu\nu})$ [3]–[6] into the Einstein-Hilbert (EH) action of GR. While these models are still GR, alas they are somewhat different from the usual dynamical dark energy/modified gravity models in the sense that they adds further dynamical degrees of freedom to the Lagrangian density, often in the form of scalar fields. One can construct models in which the components that we do not know their content and nature, such as cold dark matter and relativistic relics, may interact with *energy-momentum squared field* (EMSF) partner, whereas usual sources such as baryons and photons interact minimally with their EMSF partners and adhere to the local energy-momentum conservation. Therefore the contributions coming from this field/interaction can be interpreted as a DE component. A recent study [7] has shown that such generalizations are indeed equivalent to general relativity (GR) with non-minimal matter interactions.

Notably, the accompanying partner, $T_{\mu\nu}^{\mathrm{EMSF}}$ has a unique form constructed via $f(\mathbf{T}^2)$ where $\mathbf{T}^2 = T_{\mu\nu}T^{\mu\nu}$. In literature, due to the non-minimal nature of the interaction between the material field and its EMSF partner, both individually violate the local energy-momentum conservation law, leading to $\nabla^{\mu}T_{\mu\nu} = -Q_{\nu} = -\nabla^{\mu}T_{\mu\nu}^{\mathrm{EMSF}}$, where Q_{ν} is the interaction kernel governing the rate of energy transfer between the material field and its EMSF partner. On the other hand, in this study, we opt to examine an alternative scenario, applicable if there are at least two fluids, in which the sector comprising conventional fluids interacts minimally with the sector associated with their EMSF partners. Specifically, we consider the case characterized by $\nabla^{\mu}\left(T_{\mu\nu}^{1} + T_{\mu\nu}^{2}\right) = -\nabla^{\mu}\left(T_{\mu\nu}^{\mathrm{EMSF1}} + T_{\mu\nu}^{\mathrm{EMSF2}}\right) = \mathscr{Q}_{\nu}$ with $\mathscr{Q}_{\nu} = 0$.

We consider a case study, in favor of mathematical simplicity, *scale-independent* EMSG whose function leading the energy exchange between two fluids at rates proportional to a linear combination of their densities and the expansion rate of the universe. In this model, the EMSF enters with the same power as the usual terms from Einstein-Hilbert part of the action, viz., $f(\mathbf{T^2}) = \alpha\sqrt{\mathbf{T^2}}$. The continuity equations, $\dot{\rho}_1 + 3H(w_1+1)\rho_1 = \beta H\rho_1 + \zeta H\rho_2$ and $\dot{\rho}_2 + 3H(w_2+1)\rho_2 = -\beta H\rho_1 - \zeta H\rho_2$ are reminiscent of the cosmological model with energy exchange studied by Barrow and Clifton in [8]. However, unlike their model where the coefficients are α and β as in [8], the coefficients ζ and β in our work are not arbitrary constants but depend on the species, i.e., the EoS parameter.

At the standard GR limit—for the case that there is no EMSF partner for each conventional fluid—phenomenological models such as [8] have been studied where the interaction kernel becomes an unknown degree of freedom, rendering the interaction correction indefinite. Attempts initially driven by theoretical curiosity rather than motivation from cosmological tensions of that time, led Barrow and Clifton to adjust it manually so that it could manifest in a variety of cosmic phenomena. Examples include the decay of the vacuum energy [9], interactions between some scalar field [10], and the interaction between matter and radiation [11], all studied within the context of GR. However, in matter modified theories, by redefining $\mathscr{L}_{\rm m}$ as $\mathscr{L}_{\rm m} + f$, the matter modified models diverge from phenomenological interacting models; this is due to the function f and its variations with respect to both its argument and the metric, which intricately intertwine the interaction kernel, for instance $\mathscr{Q}(f, \delta f/\delta\mathbf{T^2}, \delta, \mathbf{T^2}/\delta g^{\mu\nu})$, in EMSG.

The tensions in the cosmological parameters have been recently quite severe and challenging with the growing sensitivity of the experimental data. This has led to a literature of interacting dark energy (IDE) models which provides a natural mechanism to alleviate the coincidence problem and can also relieve the observational tensions under the ΛCDM model. A very well known parametric form of the interaction function Q, namely, $Q = \zeta H\rho_{\rm vac}$, where ζ is the coupling parameter between the dark components, has been constrained as $\zeta < 0$ viz. the energy flows from dark matter (DM) to dark energy (DE), see also [12] for a review of the DM and DE interactions.

H_0 tension can be addressed in [13]–[16], alas worsening of others such as the S_8 tension. In [17]–[30], both H_0 and S_8 tensions have been alleviated simultaneously, see also [31] for a review of the current tensions and anomalies in cosmology.

On the other hand, the canonical/simple extensions of ΛCDM in addressing the observational tensions simultaneously has failed indicating that the deviations from the standard ΛCDM model necessary for the resolution of its challenges may be highly non-trivial from the fundamental physics point of view, such as an oscillatory EoS parameter that can even cross the phantom divide line [32]–[45], and/or an oscillatory density [32,40,43]–[51], a density that attains negative values in the past accompanied by a singular EoS parameter [32,45,52]–[72]. Implemented Binned and Gaussian model-independent reconstructions for the interaction kernel alongside the equation of state [using data from BAOs, Pantheon+ and Cosmic Chronometers] studied in [32] reveals non-trivial interaction kernels, such as some oscillatory dynamics in the interaction kernel, explained by a direction change in the energy transfer between DE and DM and a possible signature change in DE energy density from early times to late times. This shows that some non-trivial functions/behaviors are required to alleviate tensions, non-linear interactions and non-linear energy density contributions from matter-type modified theories may work for the change of sign in the dark energy density with continuity equations $\dot{\rho}_1 + 3H(w_1+1)\rho_1 = +\mathscr{Q}$ and $\dot{\rho}_2 + 3H(w_2+1)\rho_2 = -\mathscr{Q}$ where $\mathscr{Q} = \beta(\rho_1,\rho_2)H\rho_1 + \zeta(\rho_1,\rho_2)H\rho_2$. However, we are interested in investigating a simple mathematical description of the exchange of energy between two standard fluids from matter modified theories within GR choosing the simplest case study.

Thanks to EMSF sector, we experience different features of this model. With EMSF partners, the model turns into a model with four sources and we first give general cosmological solutions in Sec. 4.3, and then we discuss that for some spesific value of α, which quantifies the extent of the EMSF field relative to the usual source, this additional sector leads a degeneracy between density parameters leading a decrease in one parameter of the model, reproducing interesting cosmologies in Sec. 4.4. In these cosmologies, one of the fluid's contributions can be negated by its EMSF partner, or the two sources may superpose in their energy densities in the Friedmann equation, resulting in a joint (degenerate) scale factor dependence even if $w_1 \neq w_2$. Examples are power-law universes where the scale factor of the universe grows as a EoS parameter dependent power of time in the presence of a perfect fluid and vacuum energy density/stiff fluid, de Sitter universe in the presence of a perfect fluid and vacuum energy density, or stiff fluid dominated-like universe in the presence of a perfect fluid and stiff fluid.

2. FUNDAMENTALS OF COSMOLOGY

2.1 Geometry of Space-time

In cosmology, mathematical objects like tensors, vectors, and scalars have a crucial position via variables in describing the quantities belonging to world of physical beings ("der realen Welt" in Hartmann's ontology) like electromagnetic fields, spin angular momentum of fermions and geometry of the Universe so that these objects redound us to model the fundamental equations of the dynamics of both general relativity and cosmological models. Actually, vectors and scalars are interpreted in the scope of tensors, but they possess different ranks. Tensors roughly encodes information about the behaviour of physical quantity with geometrical instruments. There exist two types of tensors: i) covariant tensor denoted with lower indices (A_ν), ii) contravariant tensor denoted with upper indices (A^ν). For any random two coordinate systems (x and x'), transformation of covariant vectors and contravariant vectors can be handled from one to another with

$$A_\mu(x') = \frac{\partial x^\nu}{\partial x'^\mu} A_\nu(x), \quad A^\mu(x') = \frac{\partial x'^\mu}{\partial x^\nu} A^\nu(x), \tag{2.1}$$

where μ and ν runs from 0 to 3 (i.e. $\mu, \nu = 0, 1, 2, 3$) for covariant vectors and contravariant vectors, respectively.

There exist a useful notation while dealing with summation of elements of tensors known as Einstein summation. It has essentially three rules: i) repeated indices are implicitly summed over, ii) each index can appear at most twice in any term, and iii) each term must contain identical non-repeated indices. For example, we can simply expose it on two multiplied rank-1 tensors as

$$\sum_{i=0}^{3} A^i B_i = A^i B_i = A^0 B_0 + A^1 B_1 + A^2 B_2 + A^3 B_3 \tag{2.2}$$

For any observer placed in space-time, the indisputable argument to compromise is the infinitesimal line element which reveals the information about causal structure of space-time and physical events. The equivalence of infinitesimal space-time interval can be expressed as

$$ds^2 = ds'^2, \tag{2.3}$$

where ds and ds' are the infinitesimal intervals referring the same quantity. It is defined with a metric by observer-dependent to describe the geometry of the Universe. The most known metric is 4-dimensional Cartesian metric, which named after René Descartes and can be given as

$$ds^2 = -c^2 dt^2 + \frac{1}{\left[1 + \frac{k}{4}(x^2 + y^2 + z^2)\right]^2} \left(dx^2 + dy^2 + dz^2\right), \tag{2.4}$$

where c is the speed of light in vacuum, t is the cosmic time, x, y and z are the comoving spatial coordinates, and k is the curvature of space. The curvatue can be positive ($k > 0$), negative ($k < 0$), or zero ($k = 0$), referring to closed, open, or flat Universes, respectively.

The cosmological principle states that the Universe is uniform in comparably large scales, viz. larger than 100 Mpc. It comes up with two affairs: being homogeneous, which is absence of superior points, and being isotropic, which is indistinguishability of all directions from each other.

To define the geometry of our known visible Universe, we should follow the cosmological principle so that the metric corresponds a homogeneous and isotropic Universe. From the first observation by Slipher [73] to recent observations [74,75], the Universe is expanding at an accelerated pace in the simplest sense. Hence, considering them we utilize the most common metric known as the Friedmann-Lemaître-Robertson-Walker (FLRW) metric, which is a generalization of the metric used in Einstein's theory of general relativity. Without any assumption on the curvature k of the Universe, the FLRW metric acquires the subsequent form

$$ds^2 = -dt^2 + a^2(t) \frac{dx^2 + dy^2 + dz^2}{\left[1 + k\left(x^2 + y^2 + z^2\right)/4\right]^2}, \tag{2.5}$$

which can be also represented by polar coordinates as

$$ds^2 = -dt^2 + a^2(t) \left[\frac{dr^2}{1 - kr^2} + r^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right)\right], \tag{2.6}$$

where c is taken as unity for the sake of the naturalness property and $a(t)$ is the scale factor of the Universe, standing for the parameter of variation of the size of the Universe with time. As the scale factor $a(t)$ increases with time, time itself appears to pass more slowly for distant objects. Furthermore, the metric can be written in tensor form

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu, \tag{2.7}$$

where μ, ν counts 0 to 3 with $dx^0 = dt$ representing time-like coordinates and dx^i representing space-like coordinates with $i : \{1,2,3\}$, and the metric tensor is

$$g_{\mu\nu} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & \frac{a^2}{1-kr^2} & 0 & 0 \\ 0 & 0 & a^2 r^2 & 0 \\ 0 & 0 & 0 & a^2 r^2 \sin^2\theta \end{bmatrix}. \tag{2.8}$$

In the course of investigating the evolution of the Universe in cosmology, we utilize General Relativity with some basic mathematical tools to analyze the behavior of space-time shaped by massive objects (like galaxy clusters) on the cosmic time scale, and to define the paths (geodesic) that massive and massless test particles (photons) can take freely in space without external force.

The Christoffel symbol, $\Gamma^\alpha_{\beta\gamma}$, is defined in terms of the metric tensor and its derivatives in the form of a (1,2) type mixed tensor and describes changes in the metric in space-time, so that we can explain how any tensor is moved from one point to another in curved space The Christoffel symbols are given by

$$\Gamma^\alpha_{\beta\gamma} = \frac{1}{2} g^{\alpha\sigma} \left(\frac{\partial g_{\sigma\beta}}{\partial x^\gamma} + \frac{\partial g_{\sigma\gamma}}{\partial x^\beta} - \frac{\partial g_{\beta\gamma}}{\partial x^\sigma} \right). \tag{2.9}$$

The Riemann tensor is a tensor with four-index, which comprises Christoffel symbol and its derivatives, and it describes completely the curvature of any space with affine connection. The Riemann tensor appears as

$$R^\rho_{\sigma\mu\nu} = \Gamma^\rho_{\nu\sigma,\mu} - \Gamma^\rho_{\mu\sigma,\nu} + \Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\nu\sigma} - \Gamma^\rho_{\nu\lambda}\Gamma^\lambda_{\mu\sigma}, \tag{2.10}$$

also it can be written as a (0,4) type covariant tensor by using index lowering with metric tensor

$$R_{\rho\sigma\mu\nu} = g_{\rho\lambda} R^\lambda_{\sigma\mu\nu}. \tag{2.11}$$

Riemann tensor has elegant symmetries which are skew symmetry and interchange symmetry, can be achieved respectively as

$$R_{\rho\sigma\mu\nu} = -R_{\sigma\rho\mu\nu} = -R_{\rho\sigma\nu\mu}, \tag{2.12}$$

$$R_{\rho\sigma\mu\nu} = R_{\mu\nu\rho\sigma}, \tag{2.13}$$

so that they reveal some relations, which are known as Bianchi identities, as the following form

$$R_{\rho\sigma\mu\nu} + R_{\rho\nu\sigma\mu} + R_{\rho\mu\nu\sigma} = 0, \tag{2.14}$$

$$R^{\lambda}_{\rho\sigma\mu;\nu} + R^{\lambda}_{\rho\nu\sigma;\mu} + R^{\lambda}_{\rho\mu\nu;\sigma} = 0. \tag{2.15}$$

The Ricci tensor is a symmetric $(0,2)$ covariant tensor encoding the curvature of spacetime. It can be achieved by contracting indices of Riemann tensor:

$$R_{\mu\nu} = R^{\alpha}_{\mu\alpha\nu} = \Gamma^{\alpha}_{\mu\nu,\alpha} - \Gamma^{\alpha}_{\mu\alpha,\nu} + \Gamma^{\alpha}_{\beta\alpha}\Gamma^{\beta}_{\mu\nu} - \Gamma^{\alpha}_{\beta\nu}\Gamma^{\beta}_{\mu\alpha}. \tag{2.16}$$

It is obvious to assert that a scalar can be derived by utilizing the Ricci tensor. This scalar is known as Ricci scalar which reveals the characteristic properties of the intrinsic curvature of a n-dimensional spacetime. We obtain the Ricci scalar with the multiplication of inverse metric tensor and the Ricci scalar as the following

$$R = R^{\mu}_{\mu} = g^{\mu\nu} R_{\mu\nu} \tag{2.17}$$

where R is the Ricci scalar and R^{μ}_{μ} is the trace of the Ricci tensor, $R_{\mu\nu}$.

2.2 Standard Cosmological Model: Λ-Cold Dark Matter (ΛCDM) Model

The Universe roughly can be expressed into two parts: *i*) the geometrical content and *ii*) the energy content. It is simply argued that any physical system can be described by an action and the dynamics of the system with equations of system which is derived by variation of the action of the system. The action is constructed by an integration of a Lagrangian density over n-dimensional system. The simplest method to achieve a Lagrangian of a system is to utilize scalars since they are encrypted variables. Hence, we follow the literature by calling the geometry content with the Ricci scalar R and the energy content with $\mathscr{L}_{\mathrm{m}}$ which is Lagrangian density of material field and has two opportunity to set the scalar: the energy density of a source ρ and the pressure of the source p. Our observed space-time is four-dimensional, and the recent observations reveals evidences of cold dark matter (CDM) with anisotropies in Cosmic Micowave Background (CMB) and broken hierarchy of galaxy formation [76,77] and accelerating expansion addressing to the presence of dark energy (DE) with the photometric and spectral observations of 10 type Ia supernovae (SNe Ia) in the low redshift [74]. Considering them, we construct the action of the standard model of the modern cosmology, which known as ΛCDM where Λ refers to cosmolocigal constant/ dark energy and CDM refers to cold dark matter, as the following:

$$S=\int d^4x\sqrt{-g}\left[\frac{1}{2\kappa}(R-2\Lambda)+\mathscr{L}_{\mathrm{m}}\right], \tag{2.18}$$

where $\kappa=8\pi G$ with G referring to Newton's gravitational constant, R is the Ricci scalar, Λ is the cosmological constant and $\mathscr{L}_{\mathrm{m}}$ is the Lagrangian density of the material field. The variation of (2.18) with respect to the inverse metric tensor $g^{\mu\nu}$

$$\frac{\delta S}{\delta g^{\mu\nu}}=\int d^4x\left[\frac{1}{2\kappa}\frac{\delta\left[\sqrt{-g}(R-2\Lambda)\right]}{\delta g^{\mu\nu}}+\frac{\delta\left(\sqrt{-g}\mathscr{L}_{\mathrm{m}}\right)}{\delta g^{\mu\nu}}\right]=0, \tag{2.19}$$

where we used the following relation revealing the variation of metric determinant g:

$$\delta\left(\sqrt{-g}\right)=-\frac{1}{2}\sqrt{-g}g_{\mu\nu}\delta g^{\mu\nu}. \tag{2.20}$$

The variation in (2.19) reveals the Einstein Field Equations (EFE) of the ΛCDM model as

$$G_{\mu\nu}=R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R=\kappa T_{\mu\nu}+\Lambda g_{\mu\nu}, \tag{2.21}$$

where $G_{\mu\nu}$ is the Einstein tensor and the EMT is defined as

$$T_{\mu\nu}=-\frac{2}{\sqrt{-g}}\frac{\delta\left(\sqrt{-g}\mathscr{L}_{\mathrm{m}}\right)}{\delta g^{\mu\nu}}=g_{\mu\nu}\mathscr{L}_{\mathrm{m}}-2\frac{\partial\mathscr{L}_{\mathrm{m}}}{\partial g^{\mu\nu}}. \tag{2.22}$$

where we can choose whether $\mathscr{L}_{\mathrm{m}}=p$ or $\mathscr{L}_{\mathrm{m}}=-\rho$, which gives the EMT of the perfect fluid the same as defined as

$$T_{\mu\nu}=(\rho+p)u_\mu u_\nu+pg_{\mu\nu}, \tag{2.23}$$

which is a $(0,2)$ covariant symmetric tensor, $u^\mu=(1,0,0,0)$ is four-velocity of the source in comoving coordinates, p is pressure and ρ is the energy density of the source.

If we establish time-time components and space-space components i.e. $(0,0)$ and (i,j) of the equation (2.21), we obtain the Friedmann equation and pressure equation respectively:

$$3H^2 + \frac{3k}{a^2} = \Lambda + \kappa\rho, \tag{2.24}$$

$$-2\dot{H} - 3H^2 - \frac{k}{a^2} = -\Lambda + \kappa p, \tag{2.25}$$

where $H = \frac{\dot{a}}{a}$ is the Hubble parameter and dot sign over the scale factor represents the derivative with respect to cosmic time i.e. $\dot{a} = \frac{da}{dt}$, k is the spatial curvature, Λ is cosmological constant/ dark energy.

As we mentioned in (2.14) and (2.15), the Bianchi identifies exhibit some novel properties due to skew-symmetric Riemann tensor. Hence, we express the relation (2.15) with covariant derivative operator ∇^μ, which is known as the contracted Bianchi identity, i.e. $\nabla^\mu G_{\mu\nu} = 0$. Thence, we obtain the conservation of EMT by applying the covariant derivative to the RHS of EFE via the contracted Bianchi identity:

$$\nabla^\mu T_{\mu\nu} = 0, \tag{2.26}$$

where $\nabla^\mu T^{\rm DE}_{\mu\nu} = 0$ can be described with $T^{\rm DE}_{\mu\nu} = \Lambda g_{\mu\nu}$ since $\nabla^\mu g_{\mu\nu} = 0$. It argues the conservation equation of sources separately

$$\dot{\rho} + 3H(1+w)\rho = 0, \tag{2.27}$$

which can be also derived by using the combination of the Friedmann equations (2.24) and (2.25), is linearly dependent to the two equations. This simple first order differential equation displays a familiar solution

$$\rho \propto a^{-3(1+w)}, \tag{2.28}$$

where w is barotropic equation of state (EoS), $w = \frac{p}{\rho}$, which is a constant and it yields energy density evolution as $\rho_{\rm m} = \rho_{\rm m0}a^{-3}$ for matter $w = 0$, $\rho_{\rm r} = \rho_{\rm r0}a^{-4}$ for radiation $w = \frac{1}{3}$ and $\rho_\Lambda = \Lambda$ for dark energy $w = -1$. In order to have flat geometry of space, viz. setting $k = 0$ in (2.24) without cosmological constant, the cosmic density of the Universe is required to be a special value i.e. $\rho_{\rm crit} = 3H^2$ which is called as critical density, and its present-day value is denoted as $\rho_{\rm crit0} = 3H_0^2$. Hence, by regarding the critical density it can be stated that the Universe is spatially closed when $\rho_{\text{total}} > \rho_{\text{crit}}$ and open when $\rho_{\text{total}} < \rho_{\text{crit}}$ where ρ_{total} is the total energy density of the Universe. Using the relation between scale factor a and redshift z as $a = (1+z)^{-1}$, the Friedmann equation is as follows:

$$\frac{H^2}{H_0^2} = +\Omega_{\Lambda 0} + \Omega_{\text{m}0}(1+z)^3 + \Omega_{\text{r}0}(1+z)^4 \tag{2.29}$$

where H_0 is the present-day value of the Hubble parameter, and

$$\Omega_{\Lambda 0} = \frac{\Lambda}{3H_0^2} \quad , \quad \Omega_{\text{m}0} = \frac{\rho_{\text{m}0}}{3H_0^2}, \quad \text{and} \quad \Omega_{\text{r}0} = \frac{\rho_{\text{r}0}}{3H_0^2}, \tag{2.30}$$

are the present-day density parameters of matter including the cosmological constant and matter including baryon and CDM, radiation respectively.

There exist some advantages of using redshift both observationally and theoretically such as discarding singularities like the Big Bang with $z \to \infty$ instead of $a = 0$. Also, the present-day values can be achieved by setting $z = 0$ with $H(z=0) = H_0$, which reveals the completeness relation:

$$1 = \Omega_{\text{m}0} + \Omega_{\text{r}0} + \Omega_{\Lambda 0}. \tag{2.31}$$

The recent observations [75,78,79] displays the energy content of our Universe as $\Omega_{\text{m}0} = \sim 0.3$, $\Omega_{\Lambda 0} \sim 0.7$ and $\Omega_{\text{r}0} \sim 10^{-4}$. The dominant part of our observable universe comes from dark sector, i.e. dark matter and dark energy; hence, by considering remarkably small contribution of radiation in late time provides a solution of Friedmann equation with scale factor as a function of cosmic time

$$a(t) = \left(\frac{\Omega_{\text{m}0}}{\Omega_\Lambda}\right)^{1/3} \sinh^{2/3}(t/t_\Lambda), \tag{2.32}$$

where $t_\Lambda = 2/\left(3H_0\sqrt{\Omega_\Lambda}\right)$. The ΛCDM model is plausible model since it explains the observations of the late-time accelerated cosmic expansion and present data [74,75,77]–[80] with high-precision by utilizing the least six parameters on the linearized level; however, it encounters some tensions like H_0 (Hubble constant) [81], S_8 (growth parameter) [82] and etc. [31,83]. Tensions in the cosmological parameters have been recently quite severe and challenging with the growing sensitivity of the experimental data. This motivated the scientific community to test some variations on properties of the dark sector such as equation of state (EoS) parameters, interactions with gravity/visible and gravity/dark sectors, etc. against observational evidences.

3. ENERGY-MOMENTUM SQUARED GRAVITY

Generalisation the form of the matter Lagrangian in a nonlinear way, to some analytic function of $T_{\mu\nu}T^{\mu\nu}$ where $T^{\mu\nu}$ is the energy-momentum tensor of the matter stresses is added to Einstein- Hilbert Lagrangian $F(R, T_{\mu\nu}T^{\mu\nu}) = \frac{R}{2\kappa} + f(T_{\mu\nu}T^{\mu\nu})$ and we obtain energy-momentum squared gravity (EMSG) model. Despite modifications of the gravity theory, $F(R)$ theories, have been more common in literature, EMSG is a matter modified model, equivalent to general relativity with a non minimal interaction between the standard source and its EMSG-based partner.

3.1 EMSG from non-minimal matter interaction perspective

We start with the action

$$S = \int \mathrm{d}^4x\sqrt{-g}\left[\frac{1}{2\kappa}R + \mathscr{L}_{\mathrm{m}} + f(\mathbf{T^2})\right], \tag{3.1}$$

constructed by the addition of the term $f(\mathbf{T^2})$, where $\mathbf{T^2} = T_{\mu\nu}T^{\mu\nu}$, to the Lagrangian density $\mathscr{L}_{\mathrm{m}}$ corresponding to the conventional material field described by the EMT $T_{\mu\nu}$, in the usual Einstein-Hilbert action. Here, g is the determinant of the metric tensor $g_{\mu\nu}$, viz., $g = \det(g_{\mu\nu})$, $R = g^{\mu\nu}R_{\mu\nu}$ is the Ricci curvature scalar with $R_{\mu\nu}$ being the Ricci curvature tensor and $\kappa = 8\pi G$ with G is the Newton's gravitational constant. We adopt units where $\hbar = c = 1$ throughout this thesis.

To derive equations of motion, namely Einstein Field Equations (EFEs) in the context of GR, we vary the action (3.1) with respect to inverse metric tensor, $g^{\mu\nu}$, which reads

$$\begin{aligned}\delta S = \int d^4x \Bigg[&\frac{1}{2\kappa}\frac{\delta\left(\sqrt{-g}R\right)}{\delta g^{\mu\nu}} + \frac{\delta\left(\sqrt{-g}\mathscr{L}_{\mathrm{m}}\right)}{\delta g^{\mu\nu}} \\ &+ f\left(T_{\sigma\varepsilon}T^{\sigma\varepsilon}\right)\frac{\delta\left(\sqrt{-g}\right)}{\delta g^{\mu\nu}} + \frac{\partial f}{\partial\left(T_{\alpha\beta}T^{\alpha\beta}\right)}\frac{\delta\left(T_{\sigma\varepsilon}T^{\sigma\varepsilon}\right)}{\delta g^{\mu\nu}}\sqrt{-g}\Bigg]\delta g^{\mu\nu}.\end{aligned} \tag{3.2}$$

Assuming that the matter Lagrangian density, $\mathscr{L}_\mathrm{m}$, depends solely on the metric tensor and not on its derivatives, we define the conventional EMT as follows:

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta(\sqrt{-g}\mathscr{L}_\mathrm{m})}{\delta g^{\mu\nu}} = \mathscr{L}_\mathrm{m} g_{\mu\nu} - 2\frac{\partial \mathscr{L}_\mathrm{m}}{\partial g^{\mu\nu}}. \tag{3.3}$$

where independent of the choice of matter Lagrangian, $\mathscr{L}_\mathrm{m} = p$ or $\mathscr{L}_\mathrm{m} = -\rho$, the EMT of the perfect fluid reads

$$T_{\mu\nu} = (\rho + p)u_\mu u_\nu + p g_{\mu\nu}, \tag{3.4}$$

we consider $\mathscr{L}_\mathrm{m} = p$.

Consequently, with the variation of the action given in (3.1) the field equations are given by

$$G_{\mu\nu} = \kappa T_{\mu\nu} + \kappa T^{\mathrm{mod}}_{\mu\nu}, \tag{3.5}$$

where $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}R g_{\mu\nu}$ is the Einstein tensor and $T^{\mathrm{mod}}_{\mu\nu}$ denotes the new terms coming from the metric variation of $\sqrt{-g}f$ and reads

$$T^{\mathrm{mod}}_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta(\sqrt{-g}f)}{\delta g^{\mu\nu}} = f g_{\mu\nu} - 2 f_{\mathbf{T^2}}\Theta_{\mu\nu}. \tag{3.6}$$

Here

$$f_{\mathbf{T^2}} \equiv \frac{\delta f}{\delta \mathbf{T^2}} \quad , \quad \Theta_{\mu\nu} \equiv \frac{\delta \mathbf{T^2}}{\delta g^{\mu\nu}}, \tag{3.7}$$

To derive the new tensor explicitly, we follow the next step: the self contraction of EMT viz. $T_{\sigma\varepsilon}T^{\sigma\varepsilon} = g^{\sigma\lambda}g^{\varepsilon\gamma}T_{\lambda\gamma}T^{\sigma\varepsilon}$ since we examine the symmetry property of the new scalar $T^2 = T_{\sigma\varepsilon}T^{\sigma\varepsilon}$ by introducing new indices with metric tensors in addition to regarding variation of the scalar T^2. Hence, by putting this self contraction in new tensor $\Theta_{\mu\nu}$, we obtain

$$\frac{\delta\left(T_{\sigma\varepsilon}T^{\sigma\varepsilon}\right)}{\delta g^{\mu\nu}} = \left(g^{\varepsilon\gamma}\frac{\delta g^{\sigma\lambda}}{\delta g^{\mu\nu}} + g^{\sigma\lambda}\frac{\delta g^{\varepsilon\gamma}}{\delta g^{\mu\nu}}\right)T_{\lambda\gamma}T_{\sigma\varepsilon} + g^{\sigma\lambda}g^{\varepsilon\gamma}\left(T_{\sigma\varepsilon}\frac{\delta T_{\lambda\gamma}}{\delta g^{\mu\nu}} + T_{\lambda\gamma}\frac{\delta T_{\sigma\varepsilon}}{\delta g^{\mu\nu}}\right). \tag{3.8}$$

which requires a variation of EMT given in (3.3) with respect to metric tensor as follows

$$\frac{\delta T_{\sigma\varepsilon}}{\delta g^{\mu\nu}} = -g_{\sigma\mu}g_{\varepsilon\nu}\mathscr{L}_\mathrm{m} + g_{\sigma\varepsilon}\frac{\delta \mathscr{L}_\mathrm{m}}{\delta g^{\mu\nu}} - 2\frac{\delta^2 \mathscr{L}_\mathrm{m}}{\delta g^{\mu\nu} g^{\sigma\varepsilon}}, \tag{3.9}$$

where we substituted the relation between metric tensor and inverse metric tensor

$$\delta g_{\sigma\varepsilon} = -g_{\sigma\mu}g_{\varepsilon\nu}\delta g^{\mu\nu}. \tag{3.10}$$

and the choice of matter Lagrangian changes the form of $\Theta_{\mu\nu}$. Now, we acquire the explicit form of the variation the scalar T^2 by putting (3.9) and (3.10) in (3.8)

$$\frac{\delta\left(T_{\sigma\varepsilon}T^{\sigma\varepsilon}\right)}{\delta g^{\mu\nu}} = 2T_{\mu}^{\gamma}T_{\nu\gamma} - 2T_{\mu\nu}\mathscr{L}_{\mathrm{m}} + 2T\frac{\delta\mathscr{L}_{\mathrm{m}}}{\delta g^{\mu\nu}} - 4T^{\sigma\varepsilon}\frac{\delta^2\mathscr{L}_{\mathrm{m}}}{\delta g^{\mu\nu}\delta g^{\sigma\varepsilon}}. \tag{3.11}$$

Thence, by means of Eq. (3.11) and (3.3), the new tensor is yielded

$$\Theta_{\mu\nu} = -2\mathscr{L}_{\mathrm{m}}\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\right) - TT_{\mu\nu} + 2T_{\mu}^{\gamma}T_{\nu\gamma} - 4T^{\sigma\varepsilon}\frac{\partial^2\mathscr{L}_{\mathrm{m}}}{\partial g^{\mu\nu}\partial g^{\sigma\varepsilon}}, \tag{3.12}$$

with $T = g_{\mu\nu}T^{\mu\nu}$ being the trace of EMT. The last term of (3.12), the second derivative of $\mathscr{L}_{\mathrm{m}}$ with respect to the inverse metric tensor—namely the first metric derivative of $T_{\mu\nu}$—generally arises in matter-modified gravity models. In the literature, this term has been conventionally assumed to be zero for a perfect fluid, even though this is not generally true. Besides, choosing $\mathscr{L}_{\mathrm{m}} = -\rho$ or $\mathscr{L}_{\mathrm{m}} = p$ in Eq. (3.3) for the perfect fluid EMT combines the metric derivatives of ρ (energy density) and p (pressure) within this term [84]–[86]. And, for a perfect fluid, the choice of $\mathscr{L}_{\mathrm{m}} = p$ results in a divergence of this term when $p = 0$. For a detailed discussion within the thermodynamics framework, readers are referred to Ref. [7]. Although this term has traditionally been assumed to be zero, even when it is not, models incorporating the second metric derivative of the matter Lagrangian density have not encountered any inconsistencies in studies to date. This point has significantly impacted, and even altered, our fundamental understanding of EMSG as well as other matter-type modified gravity theories. It is possible to keep this term in the calculation of the new tensor given in (3.12); however, as a degree of freedom of our system we prefer to set it equal to zero in our subsequent discussion. These models are actually equivalent to GR, as the material part of the Lagrangian density in these models can be redefined as

$$\mathscr{L}_{\mathrm{m}}^{\mathrm{tot}} = \mathscr{L}_{\mathrm{m}} + f \tag{3.13}$$

and then, following the definition stated in Eq. (3.3), we can write the total EMT as follows:

$$T_{\mu\nu}^{\mathrm{tot}} = -\frac{2}{\sqrt{-g}}\frac{\delta(\sqrt{-g}\mathscr{L}_{\mathrm{m}}^{\mathrm{tot}})}{\delta g^{\mu\nu}} = T_{\mu\nu} + T_{\mu\nu}^{\mathrm{mod}}. \tag{3.14}$$

Consequently, these models describe a non-minimally interacting two-fluid, the conventional EMT, $T_{\mu\nu}$, associated with the matter Lagrangian density $\mathscr{L}_{\rm m}$ is accompanied by a modification field $T^{\rm mod}_{\mu\nu}$, in GR at the action level. Taking the covariant derivative of Eq. (3.5) with the twice contracted Bianchi identity, $\nabla^{\mu}G_{\mu\nu}=0$, we obtain

$$\nabla^{\mu}(T_{\mu\nu}+T^{\rm mod}_{\mu\nu})=0, \tag{3.15}$$

which does not necessarily imply conservation of each sector separately, $\nabla^{\mu}T_{\mu\nu}=0$ or $\nabla^{\mu}T^{\rm mod}_{\mu\nu}=0$. Instead, such models provide a non-minimal interaction between the material field and its counterpart, the modification field, which can be represented as:

$$\nabla^{\mu}T_{\mu\nu}=-\mathscr{Q}_{\nu} \quad \text{and} \quad \nabla^{\mu}T^{\rm mod}_{\mu\nu}=\mathscr{Q}_{\nu}, \tag{3.16}$$

where $\mathscr{Q}\nu$ is the energy transfer rate four-vector acting as the interaction kernel. The interaction kernel, however, has a specific form:

$$\mathscr{Q}_{\nu}=\nabla_{\nu}f-2\,\Theta_{\mu\nu}\nabla^{\mu}f_{\mathbf{T^2}}-2f_{\mathbf{T^2}}\nabla^{\mu}\Theta_{\mu\nu}, \tag{3.17}$$

in which the function f and the new tensor $\Theta_{\mu\nu}$ are involved in Eqs. (3.6) and (3.17) together. These then participate in the EoS of the second source described by $T^{\rm mod}_{\mu\nu}$ and in the interaction kernel $\mathscr{Q}_{\nu}$ [7]. The function f, controlling the Lorentz scalar $\mathbf{T}^2$ contribution, thus formulates the interaction kernel.

The undefined equation of state of the source generated by $T^{\rm mod}_{\mu\nu}$, along with the choice of f, provides flexibility in determining $\Theta_{\mu\nu}$ as a degree of freedom. To avoid the metric derivatives of ρ and p together, and to describe dust effectively for the choice of $\mathscr{L}_{\rm m}=p$, it is preferable to omit the second metric derivative of $\mathscr{L}_{\rm m}$, as has been done in the literature thus far. Consequently, the energy-momentum squared field (EMSF) was defined in [7] to denote both the new tensor

$$\Theta^{\rm EMSF}_{\mu\nu}=-2\mathscr{L}_{\rm m}\left(T_{\mu\nu}-\frac{1}{2}g_{\mu\nu}T\right)-TT_{\mu\nu}+2T^{\lambda}_{\mu}T_{\nu\lambda}, \tag{3.18}$$

and the corresponding EMT definition of EMSF as given in Eq. (3.6):

$$T^{\rm EMSF}_{\mu\nu}=f\,g_{\mu\nu}-2f_{\mathbf{T^2}}\Theta^{\rm EMSF}_{\mu\nu}, \tag{3.19}$$

which in turn implies

$$\nabla^{\mu}(T_{\mu\nu} + T^{\mathrm{EMSF}}_{\mu\nu}) = 0. \tag{3.20}$$

In [7], it was revealed that by omitting the last term of Eq. (3.12), nontrivial interaction kernels have been obtained with a covariant formulation through various choices for the function f. The models studied in the literature thus far have not encountered inconsistencies as a result of this omission. However, this leads to the conclusion that the EMSF interaction lacks a well-defined Lagrangian formulation. This conclusion holds true for all matter-type modified gravity theories by replacing $f(\mathbf{T}^2)$ with, for instance, $f(\mathscr{L}_{\mathrm{m}})$ as in [1] or $f(T)$ as in [2]. It is also valid for matter-curvature coupling theories, such as $f(R_{\mu\nu}T^{\mu\nu})$ [87,88] and $f(G_{\mu\nu}T^{\mu\nu})$ [89], where this term is assumed to be zero. Regardless of the matter sector, if there is a matter-curvature coupling, it has been declared in [90] that these theories are plagued by ghost states at cosmological energy levels.

4. COSMOLOGIES WITH ENERGY EXCHANGE

This chapter is based on the ongoing joint work in collaboration with Ö. Akarsu and N. Katırcı.

4.1 Minimal overall interaction between two sectors

In this study, we focus on a scenario where the sector comprising conventional fluids (standard material fields) overall interacts minimally with the sector associated with their EMSF partners, i.e., satisfying $\nabla^{\mu} T_{\mu\nu} = 0 = \nabla^{\mu} T_{\mu\nu}^{\text{EMSF}}$. Specifically, we consider the case characterized by

$$\mathscr{Q} = 0. \tag{4.1}$$

It is important to note that this condition does not preclude the possibility of non-minimal interactions within each individual sector; interactions can still occur non-minimally within the domain of the conventional fluids and, separately, within the group of their EMSF counterparts. Accordingly, we will consider a two-fluid model (perfect fluids described by constant EoS parameters) with $\mathscr{Q} = 0$, leading to the following conservation equations:

$$\nabla^{\mu} \left(T_{\mu\nu,1} + T_{\mu\nu,2} \right) = 0, \quad \nabla^{\mu} \left(T_{\mu\nu,1}^{\text{EMSF}} + T_{\mu\nu,2}^{\text{EMSF}} \right) = 0. \tag{4.2}$$

We will explore this choice in detail within the framework of scale-independent EMSG [91], which introduces a simple interaction kernel widely considered in the literature: a kernel linear in energy density. To follow the study conveyed along the thesis, we present the scheme of the study in Fig 4.1.

4.2 A Case Study: Scale-independent EMSG

We consider the spatially maximally symmetric spacetime metric, i.e., the FRW metric, with flat space-like sections, expressed as $\mathrm{d}s^2 = -\mathrm{d}t^2 + a^2 \mathrm{d}\vec{x}^2$, where the scale factor $a = a(t)$ is a function of cosmic time t only. We adopt $\mathscr{L}_{\mathrm{m}} = p$ and consider co-moving coordinates, wherein the perfect fluid's energy density (ρ) and

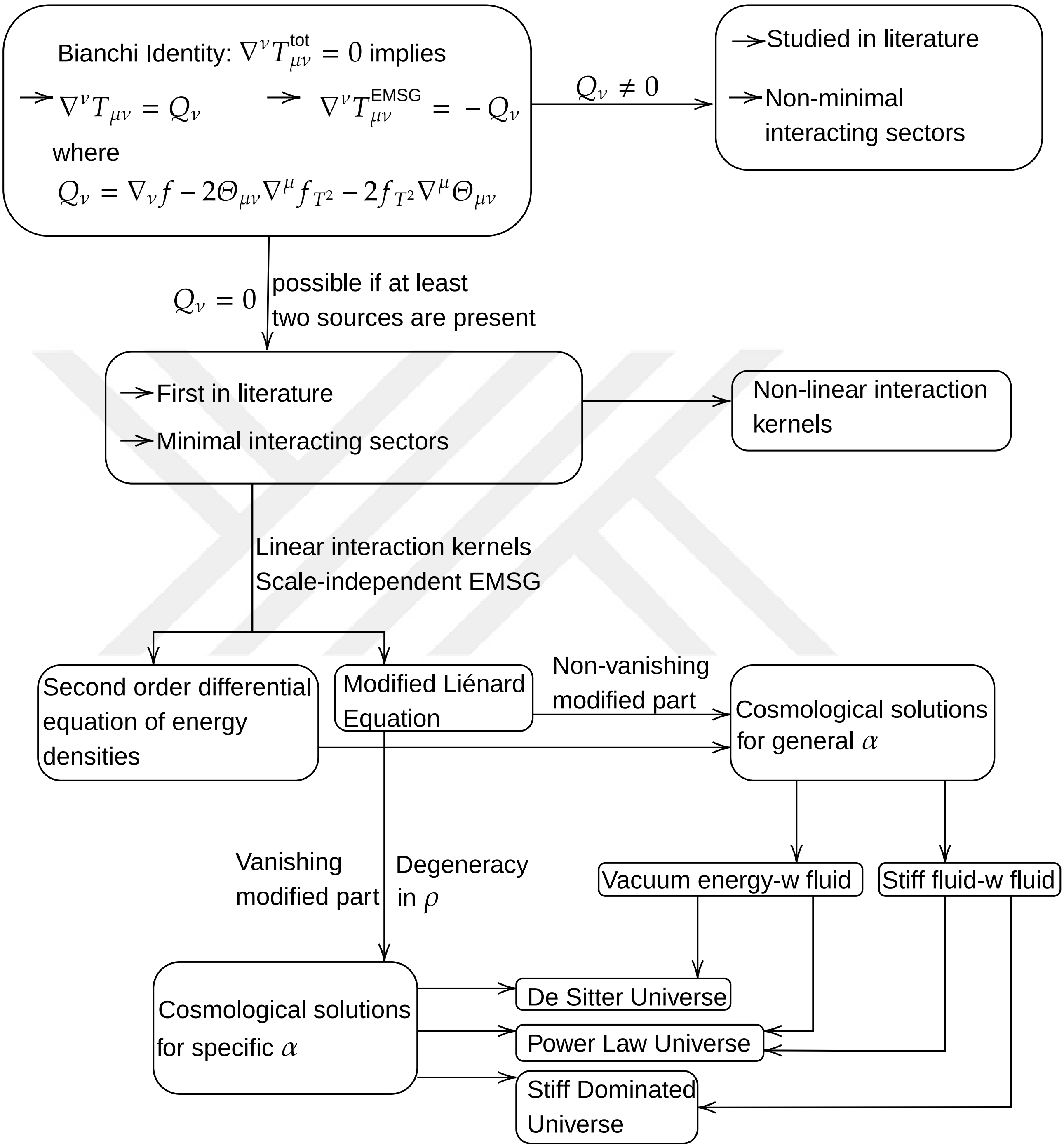

Figure 4.1 : The scheme of the thesis

thermodynamic pressure (p) are measured by an observer moving with the fluid, and the fluid's four-velocity u^μ satisfies the condition $u_\mu u^\mu = -1$. Accordingly, the EMT for perfect fluids takes the form

$$T_{\mu\nu} = (\rho + p)u_\mu u_\nu + p g_{\mu\nu}, \tag{4.3}$$

where $\rho > 0$. Substituting Eq. (4.3) into Eq. (3.18) using a barotropic EoS parameter, $w = p/\rho =$ const., we obtain

$$\Theta_{\mu\nu}^{\text{EMSF}} = -\rho^2(1+w)(1+3w)u_\mu u_\nu, \tag{4.4}$$

and the self-contraction of the EMT as

$$T_{\mu\nu}T^{\mu\nu} = \rho^2(3w^2+1). \tag{4.5}$$

Using Eq. (4.4) in Eq. (3.17), the interaction kernel for a perfect fluid is derived as

$$\mathscr{Q}_\nu = \nabla_\nu f + 2(3w^2+4w+1)\left[\rho^2 u_\mu u_\nu \nabla^\mu f_{\mathbf{T}^2} + f_{\mathbf{T}^2}\left(2\rho\nabla^\mu\rho\, u_\mu u_\nu + \rho^2\nabla^\mu u_\mu u_\nu\right)\right], \tag{4.6}$$

which contributes to the continuity equation for $\nu = 0$. The interaction kernel can then be expressed as a function of ρ:

$$\mathscr{Q} = \frac{3w^2+4w+1}{1+3w^2}\left[\left(f''\rho + \frac{4w}{3w^2+4w+1}f'\right)\dot{\rho} + 3H\rho f'\right], \tag{4.7}$$

where $'$ denotes the derivative with respect to ρ and $H = \dot{a}/a$ is the Hubble parameter. For linear interaction kernels, i.e., $\mathscr{Q}(\rho)$, f should be linear in ρ as in the scale-independent EMSG [91]

$$f = \alpha\sqrt{T_{\mu\nu}T^{\mu\nu}} = \alpha\sqrt{1+3w^2}\rho, \tag{4.8}$$

leading to $f' = \alpha\sqrt{1+3w^2} =$ const. leading $f'' = 0$.

Total conservation as per Eq. (3.20) leads to

$$\begin{aligned}&\dot{\rho}_1 + 3H(w_1+1)\rho_1 + \dot{\rho}_2 + 3H(w_2+1)\rho_2\\&+\alpha\left[A_1\dot{\rho}_1 + 3H(w_1+1)B_1\rho_1 + A_2\dot{\rho}_2 + 3H(w_2+1)B_2\rho_2\right] = 0.\end{aligned} \tag{4.9}$$

The EMSF contribution, represented by the terms following α, equals to the interaction kernel as given in Eq. (4.7):

$$\alpha\left[A_1\dot{\rho}_1 + 3H(w_1+1)B_1\rho_1 + A_2\dot{\rho}_2 + 3H(w_2+1)B_2\rho_2\right] = \mathscr{Q}, \tag{4.10}$$

where A and B denote the dimensionless, species-dependent constant coefficients. For instance, for fluid with EoS parameter w_1, they are defined as:

$$A_1 = \frac{4w_1}{\sqrt{3w_1^2+1}}, \quad B_1 = \frac{3w_1+1}{\sqrt{3w_1^2+1}}. \tag{4.11}$$

Similar definitions apply to fluid having w_2, *mutatis mutandis*. This notation simplifies the equations, and we will use it in subsequent sections.

By combining Eq. (4.9) with Eq. (4.10), we derive

$$\dot{\rho}_1 + 3H(w_1+1)\rho_1 + \dot{\rho}_2 + 3H(w_2+1)\rho_2 = -\mathscr{Q}, \tag{4.12}$$

thereby fulfilling the conditions of Eq. (4.1) and, consequently, Eq. (4.2). For a comparative study and further insights into non-minimal interactions between the material field and its EMSF partner, particularly in the context of scale-independent energy-momentum squared gravity (EMSG), refer to [91,92].

We observe that Eq. (4.1) is fulfilled in (4.10) either when $\alpha = 0$, which corresponds to a return to GR with minimal interaction between sources, or when the multiplier of α is zero. The latter scenario leads to the implication, as derived from (4.10), that

$$A_1\dot{\rho}_1 + 3H(w_1+1)B_1\rho_1 + A_2\dot{\rho}_2 + 3H(w_2+1)B_2\rho_2 = 0, \tag{4.13}$$

as long as $\alpha \neq 0$ signifying the presence of EMSF partners. Concurrently, in the sector comprising conventional (standard) fluids, we arrive at

$$\dot{\rho}_1 + 3H(w_1+1)\rho_1 + \dot{\rho}_2 + 3H(w_2+1)\rho_2 = 0, \tag{4.14}$$

as indicated by Eq. (4.12) considering (4.1).

We demonstrate that the $Q(\dot{\rho}, H\rho)$ interaction transforms into $Q(H\rho)$ with the redefinition of constants in front of $H\rho$. Specifically, the multipliers of $\dot{\rho}$ can be incorporated into the $H\rho$ terms, thereby rendering the interaction kernel unique and a function of the Hubble parameter times energy density, viz. $Q(H\rho)$. As seen from Eq. (4.7), the most general form of Kernel $Q(\dot{\rho}, H\rho)$ is achieved by EMSF. One can question that the kernel $Q(\dot{\rho}, H\rho)$ [93]–[95] can be studied, is indeed transformed into $Q(H\rho)$, after some manipulations, we conclude a general statement as $Q(\dot{\rho}, H\rho)$

kernel can be reduced $Q(H\rho)$ as first defined as this kernel in [8],

$$\begin{aligned} \dot{\rho}_1 + 3H(1+w_1)\rho_1 &= a\dot{\rho}_1 + b\dot{\rho}_2 + cH\rho_1 + dH\rho_2, \\ \dot{\rho}_2 + 3H(1+w_2)\rho_2 &= -\big(a\dot{\rho}_1 + b\dot{\rho}_2 + cH\rho_1 + dH\rho_2\big), \end{aligned} \tag{4.15}$$

after some manipulations, leading the well known kernel

$$\begin{aligned} \dot{\rho}_1 + 3H(1+w_1)\rho_1 &= \gamma H\rho_1 + \Gamma H\rho_2, \\ \dot{\rho}_2 + 3H(1+w_2)\rho_2 &= -\gamma H\rho_1 - \Gamma H\rho_2, \end{aligned} \tag{4.16}$$

where $\gamma = \frac{c-3(1+w_1)a}{1+b-a}$ and $\Gamma = \frac{d-3(1+w_2)b}{1+b-a}$.

We will multiply (4.13) with $-\frac{1}{A_1}$, then we obtain

$$-\dot{\rho}_1 - 3H(w_1+1)\frac{B_1}{A_1}\rho_1 - \frac{A_2}{A_1}\dot{\rho}_2 - 3H(w_2+1)\frac{B_2}{A_1}\rho_2 = 0. \tag{4.17}$$

After summing it with the equation (4.14), it becomes

$$3H(w_1+1)\frac{A_1-B_1}{A_1}\rho_1 + \frac{A_1-A_2}{A_1}\dot{\rho}_2 + 3H(w_2+1)\frac{A_1-B_2}{A_1}\rho_2 = 0. \tag{4.18}$$

By rearranging the equation (4.18) and multiplying it with $\frac{A_1}{A_1-A_2}$ reveals the following

$$\dot{\rho}_2 = -3H(w_1+1)\frac{A_1-B_1}{A_1-A_2}\rho_1 - 3H(w_2+1)\frac{A_1-B_2}{A_1-A_2}\rho_2. \tag{4.19}$$

Now, summing both sides with $3H(w_2+1)\rho_2$ gives a conventional interacting source form as:

$$\dot{\rho}_2 + 3H(w_2+1)\rho_2 = -\beta H\rho_1 - \zeta H\rho_2, \tag{4.20}$$

where

$$\beta = \frac{3(w_1+1)(A_1-B_1)}{A_1-A_2}, \quad \zeta = \frac{3(w_2+1)(A_2-B_2)}{A_1-A_2}. \tag{4.21}$$

By the same way we follow the derivation for the other source ρ_1. Now, we multiply the equation (4.13) with $-\frac{1}{A_2}$

$$-\frac{A_1}{A_2}\dot{\rho}_1 - 3H(w_1+1)\frac{B_1}{A_2}\rho_1 - \dot{\rho}_2 - 3H(w_2+1)\frac{B_2}{A_2}\rho_2 = 0. \tag{4.22}$$

The summation of it with the equation (4.14) yields the following

$$\frac{A_2-A_1}{A_2}\dot{\rho}_1 + 3H(w_1+1)\frac{A_2-B_1}{A_2}\rho_1 + 3H(w_2+1)\frac{A_2-B_2}{A_2}\rho_2 = 0, \tag{4.23}$$

and multiplying the result with $\frac{A_2}{A_1-A_2}$ brings

$$-\dot{\rho}_1 + 3H(w_1+1)\frac{A_2-B_1}{A_1-A_2}\rho_1 + 3H(w_2+1)\frac{A_2-B_2}{A_1-A_2}\rho_2 = 0. \tag{4.24}$$

Finally, we rearrange it and sum with $3H(w_1+1)\rho_1$ so that we gather the interacting form of the source ρ_1 as

$$\dot{\rho}_1 + 3H(w_1+1)\rho_1 = \beta H\rho_1 + \zeta H\rho_2. \tag{4.25}$$

As can be seen from above after some manipulations, the system of equations comprising Eq. (4.13) and Eq. (4.14) can be equivalently reformulated as follows:

$$\dot{\rho}_1 + 3H(w_1+1)\rho_1 = \beta(w_1,w_2)H\rho_1 + \zeta(w_1,w_2)H\rho_2, \tag{4.26}$$

$$\dot{\rho}_2 + 3H(w_2+1)\rho_2 = -\beta(w_1,w_2)H\rho_1 - \zeta(w_1,w_2)H\rho_2, \tag{4.27}$$

where the coefficients $\beta(w_1,w_2)$ and $\zeta(w_1,w_2)$ are species-dependent constants, defined as:

$$\begin{aligned}\beta(w_1,w_2) &= 3(w_1+1)\frac{A_1(w_1)-B_1(w_1)}{A_1(w_1)-A_2(w_2)},\\ \zeta(w_1,w_2) &= 3(w_2+1)\frac{A_2(w_2)-B_2(w_2)}{A_1(w_1)-A_2(w_2)},\end{aligned} \tag{4.28}$$

which can be explicitly expressed in terms of the EoS parameters of the sources, w_1 and w_2:

$$\begin{aligned}\beta(w_1,w_2) &= \frac{3(w_1^2-1)\sqrt{3w_2^2+1}}{4\left[w_1\sqrt{3w_2^2+1}-w_2\sqrt{3w_1^2+1}\right]},\\ \zeta(w_1,w_2) &= \frac{3(w_2^2-1)\sqrt{3w_1^2+1}}{4\left[w_1\sqrt{3w_2^2+1}-w_2\sqrt{3w_1^2+1}\right]}.\end{aligned} \tag{4.29}$$

It is noteworthy that, as indicated by Eqs. (4.26) and (4.27), the dynamics of the sector comprising conventional fluids in our scenario are reminiscent of the model studied by Barrow and Clifton in [8]. However, unlike their model where the coefficients are α and β as in [8], the coefficients ζ and β in our work are not arbitrary constants but depend on the species, i.e., the EoS parameter. Additionally, our model includes a sector associated with the EMSF partners of the conventional fluids. The Friedmann and pressure equations in our framework are expressed as

$$3H^2 = \kappa\rho_1 + \kappa\alpha\frac{4w_1}{\sqrt{3w_1^2+1}}\rho_1 + \kappa\rho_2 + \kappa\alpha\frac{4w_2}{\sqrt{3w_2^2+1}}\rho_2, \tag{4.30}$$

$$-2\dot{H} - 3H^2 = \kappa\rho_1 w_1 + \kappa\alpha\sqrt{3w_1^2+1}\,\rho_1 + \kappa\rho_2 w_2 + \kappa\alpha\sqrt{3w_2^2+1}\,\rho_2, \tag{4.31}$$

satisfying the separated continuity equations given in Eqs. (4.13)-(4.14) (or equivalently (4.26)-(4.27)) can be used for $\alpha \neq 0$. Dividing Eq. (4.31) by Eq. (4.30) reveals that the EMSF partner acts as a distinct source, with its EoS parameter governed by the standard material source's EoS parameter, as outlined in

$$w_{1(2)}^{\mathrm{EMSF}} = \frac{3w_{1(2)}^2 + 1}{4w_{1(2)}}, \tag{4.32}$$

where the EMSF partner of dust ($w = 0$) replicates the Creation field (C-field) proposed by Hoyle in [96], which has no energy density but does have pressure (P_{c}), effectively yielding $w_{\mathrm{C-field}} = P_{\mathrm{c}}/0 = \infty$. Radiation ($w = \frac{1}{3}$) has a stiff fluid-like EMSF partner, while cosmic strings ($w = -\frac{1}{3}$) have a vacuum energy-like EMSF partner. Both vacuum energy ($w = -1$) and stiff fluid ($w = 1$) preserve their identities, with vacuum energy having a vacuum energy EMSF partner and stiff fluid having a stiff EMSF partner.

We now proceed to examine the evolution of energy densities of both the material source and its accompanying partner. Eqs. (4.26) and (4.27) provide insights into the evolution of energy densities as functions of the scale factor, and we will now proceed with their exact solutions.

We will consider the Eqs.(4.26) and (4.27), and we use chain rule as $\frac{d}{dt} = \frac{dlna}{dt}\frac{d}{dlna} = H\frac{d}{dlna}$. Then, the system equations appears as

$$H\rho_1' + 3H(w_1 + 1)\rho_1 = \beta H\rho_1 + \zeta H\rho_2, \tag{4.33}$$

$$H\rho_2' + 3H(w_2 + 1)\rho_2 = -\beta H\rho_1 - \zeta H\rho_2. \tag{4.34}$$

We remove H and arrange them, thus we get the following forms

$$\rho_1' + 3(w_1 + 1)\rho_1 = \beta\rho_1 + \zeta\rho_2, \tag{4.35}$$

$$\rho_2' + 3(w_2 + 1)\rho_2 = -\beta\rho_1 - \zeta\rho_2. \tag{4.36}$$

Now, we take derivative of Eqs. (4.35) and (4.36) with respect to lna

$$\rho_1'' + 3(w_1 + 1)\rho_1' = \beta\rho_1' + \zeta\rho_2', \tag{4.37}$$

$$\rho_2'' + 3(w_2 + 1)\rho_2' = -\beta\rho_1' - \zeta\rho_2', \tag{4.38}$$

and by putting above definitions given in (4.35) and (4.36) into (4.38) we obtain

$$\rho_2'' + [3(w_2+1)+\zeta]\rho_2' = -\beta[(-3(w_1+1)+\beta)\rho_1 + \zeta\rho_2]. \tag{4.39}$$

We also multiply (4.36) with $-\beta + 3(w_1+1)$

$$\begin{aligned}[-\beta+3(w_1+1)]\rho_2' =& -\beta[-\beta+3(w_1+1)]\rho_1 \\ & -(\zeta+3(w_2+1)[-\beta+3(w_1+1)]\rho_2,\end{aligned} \tag{4.40}$$

summing it with the equation (4.39) reveals the well-known second order differential equation as:

$$\frac{\rho_2''}{\rho_2} + C\frac{\rho_2'}{\rho_2} + 2D = 0. \tag{4.41}$$

By following the similar steps, we can easily obtain the same equation for the other source ρ_1.

As can be seen from above after some manipulations and isolating sources as per [8] in Eqs. (4.26) and (4.27), we arrive at the following differential equations:

$$\frac{\rho_1''}{\rho_1} + C(w_1,w_2)\frac{\rho_1'}{\rho_1} + 2D(w_1,w_2) = 0, \tag{4.42}$$

and

$$\frac{\rho_2''}{\rho_2} + C(w_1,w_2)\frac{\rho_2'}{\rho_2} + 2D(w_1,w_2) = 0, \tag{4.43}$$

where $'$ denotes the derivative with respect to $\ln a$, and $C(w_1,w_2)$ and $D(w_1,w_2)$ are defined as:

$$C(w_1,w_2) = \zeta(w_1,w_2) - \beta(w_1,w_2) + 3(w_2+1) + 3(w_1+1), \tag{4.44}$$

$$D(w_1,w_2) = \frac{3}{2}\left[\zeta(w_1,w_2)(w_1+1) - \beta(w_1,w_2)(w_2+1+3(w_1+1)(w_2+1)\right]. \tag{4.45}$$

For instance, considering a standard source such as vacuum energy $w_2 = -1$ results in $\zeta(w_1,-1) = 0$, and consequently, $D(w_1,-1) = 0$, leading to $\rho \propto a^{-C(w_1,-1)}$. The energy densities are thus obtained as

$$\rho_1 = d_\pm a^{-\gamma_\pm(w_1,w_2)} \quad , \quad \rho_2 = f_\pm a^{-\gamma_\pm(w_1,w_2)}, \tag{4.46}$$

where $\gamma_\pm(w_1,w_2)$ is defined as:

$$\gamma_\pm(w_1,w_2) = \frac{1}{2}\left[C(w_1,w_2) \pm \sqrt{C^2(w_1,w_2) - 8D(w_1,w_2)}\right]. \tag{4.47}$$

These solutions are derived by applying Eqs. (4.46) separately in Eqs. (4.26) and (4.27). Substituting Eqs. (4.46) and (4.47) into Eq. (4.30), we arrive at the Friedmann equation, where fluids are superposed, and their coupling constants ($d_\pm$ and $f_\pm$) are intertwined, leading to the relation

$$d_\pm = \frac{q_\pm(w_1, w_2)}{-\beta(w_1, w_2)} f_\pm, \tag{4.48}$$

with $q_\pm(w_1, w_2)$ given by

$$q_\pm(w_1, w_2) = \gamma_\pm(w_1, w_2) - 3(w_2 + 1) - \zeta(w_1, w_2). \tag{4.49}$$

Incorporating them into the Friedmann equation (4.30), it is expressed as follows:

$$\begin{aligned} 3H^2 =& \kappa\left[d_+ a^{-\gamma_+(w_1,w_2)} + d_- a^{-\gamma_-(w_1,w_2)}\right] \\ &+ \kappa\alpha \frac{4w_1}{\sqrt{3w_1^2+1}} \left[d_+ a^{-\gamma_+(w_1,w_2)} + d_- a^{-\gamma_-(w_1,w_2)}\right] \\ &+ \kappa\left[f_+ a^{-\gamma_+(w_1,w_2)} + f_- a^{-\gamma_-(w_1,w_2)}\right] \\ &+ \kappa\alpha \frac{4w_2}{\sqrt{3w_2^2+1}} \left[f_+ a^{-\gamma_+(w_1,w_2)} + f_- a^{-\gamma_-(w_1,w_2)}\right], \end{aligned} \tag{4.50}$$

with today's values $\rho_{10} = d_+ + d_-$ and $\rho_{20} = f_+ + f_-$, where the subscript zero denotes today's value with $a(t = t_0) = 1$.

In this model, the Friedmann equation is more usefully expressed in terms of density parameters (Ω) and the Hubble constant H_0, the present value of H. Thus, we define the present time values of the density parameters as $\Omega_{i0} = \rho_{i0}/\rho_{c0}$, where $\rho_{c0} = 3H_0^2/\kappa$ is the present time value of the critical energy density. The explicit definitions of $\Omega_i(w_1, w_2; a)$ for today are:

$$\Omega_1(w_1, w_2; 1) = \frac{\kappa}{3H_0^2}\left[d_+ + d_-\right] = \Omega_{10}, \tag{4.51}$$

$$\Omega_2(w_1, w_2; 1) = \frac{\kappa}{3H_0^2}\left[f_+ + f_-\right] = \Omega_{20}, \tag{4.52}$$

and respectively their accompanying partners are:

$$\Omega_1^{\mathrm{EMSF}}(\alpha, w_1, w_2; 1) = \alpha \frac{4w_1}{\sqrt{3w_1^2+1}} \Omega_{10} = \Omega_{10}^{\mathrm{EMSF}}(\alpha, w_1), \tag{4.53}$$

$$\Omega_2^{\mathrm{EMSF}}(\alpha, w_1, w_2; 1) = \alpha \frac{4w_2}{\sqrt{3w_2^2+1}} \Omega_{20} = \Omega_{20}^{\mathrm{EMSF}}(\alpha, w_2). \tag{4.54}$$

To derive exact form of the Friedmann equation given in (4.50), we write the coefficients d_+, d_-, f_+ and f_- explicitly. Hence, we consider the explicit forms of them with the relation given in (4.49)

$$-\rho_{10} = \frac{q_+}{\beta} f_+ + \frac{q_-}{\beta} f_-, \tag{4.55}$$

$$\rho_{20} = f_+ + f_-. \tag{4.56}$$

We multiply ρ_{20} with $-\frac{q_\mp}{\beta}$, then we gather

$$-\frac{q_\mp}{\beta}\rho_{20} = -\frac{q_\mp}{\beta} f_+ - \frac{q_\mp}{\beta} f_-, \tag{4.57}$$

and summing it with the equation (4.55) yields an explicit relation for f_+ and f_-:

$$\begin{aligned}\frac{\beta}{q_+ - q_-}\left(-\frac{q_-}{\beta}\rho_{20} - \rho_{10}\right) &= f_+,\\ \frac{\beta}{q_+ - q_-}\left(\frac{q_+}{\beta}\rho_{20} + \rho_{10}\right) &= f_-.\end{aligned} \tag{4.58}$$

Now, by using obtained relations above in (4.49) we can easily obtain explicit forms of d_+ and d_-, which can be presented as finally

$$\begin{aligned}d_+ &= \frac{q_+}{-\beta}\frac{\beta}{q_+ - q_-}\left(-\frac{q_-}{\beta}\rho_{20} - \rho_{10}\right),\\ d_- &= \frac{q_-}{-\beta}\frac{\beta}{q_+ - q_-}\left(\frac{q_+}{\beta}\rho_{20} + \rho_{10}\right).\end{aligned} \tag{4.59}$$

Since we utilize density parameters Ω instead of densities ρ, we should re-express them by multiplying Eqs.(4.58) and (4.59) with $\frac{1}{3H_0^2}$; thence, we reach the following explicit relations:

$$\begin{aligned}f_+ &= \frac{1}{q_+ - q_-}\left(-q_-\rho_{20} - \beta\rho_{10}\right),\\ f_- &= \frac{1}{q_+ - q_-}\left(q_+\rho_{20} + \beta\rho_{10}\right),\\ d_+ &= \frac{q_+}{-\beta}\frac{1}{q_+ - q_-}\left(-q_-\rho_{20} - \beta\rho_{10}\right),\\ d_- &= \frac{q_-}{-\beta}\frac{1}{q_+ - q_-}\left(q_+\rho_{20} + \beta\rho_{10}\right).\end{aligned} \tag{4.60}$$

We put them in (4.50) and multiply them with α parts; however, instead using free parameter α explicitly we use the following relations derived in (4.53) and (4.54)

$$\alpha = \frac{\sqrt{3w_{1(2)}+1}}{4w_{1(2)}}\frac{\Omega_{10(20)}^{\mathrm{EMSF}}}{\Omega_{10(20)}}. \tag{4.61}$$

Rearranging Eq. (4.50) with Eqs. (4.60) and (4.61) and two free parameters Ω_{10} and $\Omega_{10}^{\rm EMSF}$, the Friedmann equation, which allows us to distinguish the contributions of sources and EMSF partners to the expansion, becomes

$$
\begin{aligned}
\frac{H^2}{H_0^2} =&\Omega_{10}\Bigg[\frac{q_+}{q_+-q_-}a^{-\gamma_+}-\frac{q_-}{q_+-q_-}a^{-\gamma_-}+\frac{q_+q_-}{\beta\,(q_+-q_-)}\\
&\times\frac{1-\Omega_{10}-\Omega_{10}^{\rm EMSF}}{\Omega_{10}\left[1+\frac{\Omega_{10}^{\rm EMSF}}{\Omega_{10}}\frac{w_2}{w_1}\frac{\sqrt{3w_1^2+1}}{\sqrt{3w_2^2+1}}\right]}\left(a^{-\gamma_+}-a^{-\gamma_-}\right)\Bigg]\\
&+\Omega_{10}^{\rm EMSF}\Bigg[\frac{q_+}{q_+-q_-}a^{-\gamma_+}-\frac{q_-}{q_+-q_-}a^{-\gamma_-}+\frac{q_+q_-}{\beta\,(q_+-q_-)}\\
&\times\frac{1-\Omega_{10}-\Omega_{10}^{\rm EMSF}}{\Omega_{10}\left[1+\frac{\Omega_{10}^{\rm EMSF}}{\Omega_{10}}\frac{w_2}{w_1}\frac{\sqrt{3w_1^2+1}}{\sqrt{3w_2^2+1}}\right]}\left(a^{-\gamma_+}-a^{-\gamma_-}\right)\Bigg]\\
&+\Omega_{20}\Bigg[\frac{q_+}{q_+-q_-}a^{-\gamma_-}-\frac{q_-}{q_+-q_-}a^{-\gamma_+}+\frac{\beta}{q_+-q_-}\\
&\times\frac{\Omega_{10}\left[1+\frac{\Omega_{10}^{\rm EMSF}}{\Omega_{10}}\frac{w_2}{w_1}\frac{\sqrt{3w_1^2+1}}{\sqrt{3w_2^2+1}}\right]}{1-\Omega_{10}-\Omega_{10}^{\rm EMSF}}\left(a^{-\gamma_-}-a^{-\gamma_+}\right)\Bigg]\\
&+\Omega_{20}^{\rm EMSF}\Bigg[\frac{q_+}{q_+-q_-}a^{-\gamma_-}-\frac{q_-}{q_+-q_-}a^{-\gamma_+}+\frac{\beta}{q_+-q_-}\\
&\times\frac{\Omega_{10}\left[1+\frac{\Omega_{10}^{\rm EMSF}}{\Omega_{10}}\frac{w_2}{w_1}\frac{\sqrt{3w_1^2+1}}{\sqrt{3w_2^2+1}}\right]}{1-\Omega_{10}-\Omega_{10}^{\rm EMSF}}\left(a^{-\gamma_-}-a^{-\gamma_+}\right)\Bigg],
\end{aligned}
\tag{4.62}
$$

where the consistency relation for today, $a=1$, is also used, and it is defined as:

$$1 =\Omega_{10}+\Omega_{10}^{\rm EMSF}(\alpha,w_1)+\Omega_{20}+\Omega_{20}^{\rm EMSF}(\alpha,w_2). \tag{4.63}$$

4.3 Cosmological solutions

In the subsequent analysis, we will focus on w_1 and w_2, along with their EMSF partners, which have EoS parameters $w^{\rm EMSF}=\frac{3w^2+1}{4w}$ as delineated in Eq. (4.32). The species-dependent parameters—$\beta(w_1,w_2)$, $\zeta(w_1,w_2)$, $C(w_1,w_2)$, and $D(w_1,w_2)$—are denoted with a subscript following the same analogy as the energy densities $\rho_{\rm vac}$, ρ_w. We will demonstrate that these parameters lead to a scenario where both w and $w^{\rm EMSF}$ effectively become equal, namely $w_{\rm v,eff}=w_{\rm v,eff}^{\rm EMSF}$.

4.3.1 w-fluid and vacuum energy $(w=-1)$ pair

We consider a standard source with a general EoS parameter $w_1 = w = \text{const}$ (except vacuum energy described by $w = -1$), and a standard vacuum energy $w_2 = -1$. Their respective EMSF partners are described as follows:

$$w_1 = w \quad , w_1^{\text{EMSF}} = \frac{3w^2+1}{4w} \quad , w_2 = w_2^{\text{EMSF}} = -1. \tag{4.64}$$

For pairs comprising a source described by w and vacuum energy, Eq. (4.29) yields $\zeta_{\text{vac}}(w) = 0$, leading to $D_{\text{vac}}(w) = 0$ [refer to Eq. (4.45)]. Consequently, Eqs. (4.26) and (4.27) become:

$$\dot{\rho}_w + 3H(w+1)\rho_w = \beta_{\text{vac}}(w)H\rho_w, \tag{4.65}$$

$$\dot{\rho}_{\text{vac}} = -\beta_{\text{vac}}(w)H\rho_w, \tag{4.66}$$

where $\beta_{\text{vac}}(w)$ is defined as:

$$\beta_{\text{vac}}(w) = 3w - \frac{3}{2}\sqrt{3w^2+1}. \tag{4.67}$$

Defining w_{eff} for both the material field and its EMSF partner, the energy densities are given by:

$$\rho_w = \rho_{w0} a^{-3[w_{\text{v,eff}}(w)+1]}, \tag{4.68}$$

$$\rho_{\text{vac}} = \rho_{\text{vac}0} - \frac{w - w_{\text{v,eff}}(w)}{1 + w_{\text{v,eff}}(w)} \rho_{w0} \left(1 - a^{-3[w_{\text{v,eff}}(w)+1]}\right), \tag{4.69}$$

where

$$w_{\text{v,eff}}(w) = \frac{1}{2}\sqrt{3w^2+1}. \tag{4.70}$$

Eq. (4.50) reveals that both the source and its partner evolve identically, and the effective EoS parameters are equivalent:

$$w_{\text{v,eff}}(w) = w_{\text{v,eff}}^{\text{EMSF}}(w) = \frac{1}{2}\sqrt{3w^2+1}. \tag{4.71}$$

For instance, considering a pair of radiation ($w = 1/3$) and vacuum energy ($w = -1$), we find that $w_{\text{v,eff}}(w) = w_{\text{v,eff}}^{\text{EMSF}}(w) = 1/\sqrt{3}$. We note that $\beta < 0$ within the EoS parameter region of $-1 < w < 1$. This indicates an energy transfer from the fluid described by w to the vacuum energy, implying that the fluid dilutes faster than in a non-interacting GR scenario (i.e., in the absence of EMSF partners) as the universe expands. For the stiff fluid ($w = 1$) and vacuum energy ($w = -1$) pair, $\beta_{\text{vac}} = \zeta_{\text{vac}} = 0$, suggesting that the stiff fluid (or Zeldovich fluid, characterized as the most rigid EoS compatible with relativity theory [97]) and vacuum energy interact only gravitationally. Consequently, our solution aligns with those in non-interacting GR, as thoroughly explored in Sections V.C, VI.C, and VII.B of Ref. [98], and is applicable in our model as well.

Friedmann equation (4.62) in this case reads as follows:

$$\begin{aligned}\frac{H^2}{H_0^2} =&\, \Omega_{w0} a^{-3(1+w_{\text{v,eff}}(w))} + \Omega_{w0}^{\text{EMSF}} a^{-3(1+w_{\text{v,eff}}(w))} \\ &+ \Omega_{\text{vac0}} \left[1 - \left[\frac{w - w_{\text{v,eff}}(w)}{1 + w_{\text{v,eff}}(w)} \right] \frac{\Omega_{w0} \left[1 - \frac{\Omega_{w0}^{\text{EMSF}}}{\Omega_{w0}} \frac{w_{\text{v,eff}}(w)}{w} \right]}{1 - \Omega_{w0}^{\text{EMSF}} - \Omega_{w0}} \left(1 - a^{-3(1+w_{\text{v,eff}}(w))} \right) \right] \\ &+ \Omega_{\text{vac0}}^{\text{EMSF}} \left[1 - \left[\frac{w - w_{\text{v,eff}}(w)}{1 + w_{\text{v,eff}}(w)} \right] \frac{\Omega_{w0} \left[1 - \frac{\Omega_{w0}^{\text{EMSF}}}{\Omega_{w0}} \frac{w_{\text{v,eff}}(w)}{w} \right]}{1 - \Omega_{w0}^{\text{EMSF}} - \Omega_{w0}} \left(1 - a^{-3(1+w_{\text{v,eff}}(w))} \right) \right],\end{aligned} \tag{4.72}$$

where the energy density parameters are defined as in Eqs. (4.51) to (4.54), fulfilling the relation

$$1 = \Omega_{w0} + \Omega_{w0}^{\text{EMSF}} + \Omega_{\text{vac0}} + \Omega_{\text{vac0}}^{\text{EMSF}}. \tag{4.73}$$

To derive Hubble parameter H explicitly, we use following definitions:

$$\begin{aligned}\chi_+ =& \left[\left[\frac{w - w_{\text{v,eff}}(w)}{1 + w_{\text{v,eff}}(w)} \right] \frac{\Omega_{w0} \left[1 - \frac{\Omega_{w0}^{\text{EMSF}}}{\Omega_{w0}} \frac{w_{\text{v,eff}}(w)}{w} \right]}{1 - \Omega_{w0}^{\text{EMSF}} - \Omega_{w0}} \left[\Omega_{\text{vac0}} + \Omega_{\text{vac0}}^{\text{EMSF}} \right] \right. \\ &\left. + \Omega_{w0} + \Omega_{w0}^{\text{EMSF}} \right] H_0^2, \\ \chi_- =& \left[1 - \left[\frac{w - w_{\text{v,eff}}(w)}{1 + w_{\text{v,eff}}(w)} \right] \frac{\Omega_{w0} \left[1 - \frac{\Omega_{w0}^{\text{EMSF}}}{\Omega_{w0}} \frac{w_{\text{v,eff}}(w)}{w} \right]}{1 - \Omega_{w0}^{\text{EMSF}} - \Omega_{w0}} \right] \left[\Omega_{\text{vac0}} + \Omega_{\text{vac0}}^{\text{EMSF}} \right] H_0^2.\end{aligned} \tag{4.74}$$

Now, by using the definition $H = \frac{\dot{a}}{a}$ we rearrange (4.72)

$$H^2 = \frac{\dot{a}^2}{a^2} = \chi_- + \chi_+ a^{-3(1+w_{\text{v,eff}}(w))}, \tag{4.75}$$

and also we take square-root of both sides of the equation (4.75) and rearranging the result yields:

$$\frac{da}{a\sqrt{\chi_- + \chi_+ a^{-3(1+w_{\mathrm{v,eff}}(w))}}} = dt. \tag{4.76}$$

By taking integration of both sides, we gather the following relation:

$$\frac{2}{3(1+w_{\mathrm{v,eff}}(w))\sqrt{-\chi_-}}\mathrm{arccot}\left(\frac{1}{\sqrt{-\chi_-}}\sqrt{\chi_- + \chi_+ a^{-3(1+w_{\mathrm{v,eff}}(w))}}\right) = t + c, \tag{4.77}$$

and we assume $\chi_- > 0$; then, we apply inverse function to both sides of the equation (4.77) with $\mathrm{arccot}^{-1}(x) = \cot(x)$

$$\frac{1}{i\sqrt{\chi_-}}\sqrt{\chi_- + \chi_+ a^{-3(1+w_{\mathrm{v,eff}}(w))}} = \cot\left(\frac{3i(1+w_{\mathrm{v,eff}}(w))\sqrt{\chi_-}}{2}(t+c)\right). \tag{4.78}$$

Finally, we utilize the identity $\cot(ix) = -i\coth(x)$ and rearrange it

$$\sqrt{\chi_- + \chi_+ a^{-3(1+w_{\mathrm{v,eff}}(w))}} = \sqrt{\chi_-}\coth\left(\frac{3(1+w_{\mathrm{v,eff}}(w))\sqrt{\chi_-}}{2}(t+c)\right). \tag{4.79}$$

Hence, by remembering the equation (4.75), the corresponding Hubble parameter is expressed as:

$$H(t) = \mp\sqrt{\chi_-}\coth\left[\frac{3(1+w_{\mathrm{v,eff}}(w))}{2}\sqrt{\chi_-}t\right]. \tag{4.80}$$

To derive the scale factor of the system of vacuum energy-w fluid pair, there exist two possible ways: i) integrating (4.80), and ii) rearranging (4.79).

We use (4.80) with the explicit relation of Hubble parameter $H = \frac{da/dt}{a}$. After integrating both sides, we obtain the following relation:

$$\ln(a) = \frac{2}{3(1+w_{\mathrm{v,eff}}(w))}\ln\left(\sinh\left(\frac{3(1+w_{\mathrm{v,eff}}(w))\sqrt{\chi_-}}{2}(t+c)\right)\right) + C. \tag{4.81}$$

Employing exponential to both sides of the eqıation (4.81), we gather a relation for the scale factor as the following

$$a = C'\sinh^{\frac{2}{3(1+w_{\mathrm{v,eff}}(w))}}\left(\frac{3(1+w_{\mathrm{v,eff}}(w))\sqrt{\chi_-}}{2}(t+c)\right), \tag{4.82}$$

where we define $C' = e^C$ as a_0 present-day value of the scale factor.

For the other way, we continue with (4.79). We take square of both sides and leave the scale factor alone

$$a^{-3(1+w_{\mathrm{v,eff}}(w))} = \frac{\chi_-}{\chi_+}\left(\coth^2\left(\frac{3(1+w_{\mathrm{v,eff}}(w))\sqrt{\chi_-}}{2}(t+c)\right) - 1\right). \tag{4.83}$$

Using the following trigonometrical identity $\coth^2(x) - 1 = \mathrm{cosech}^2(x)$; thence, we reach

$$\begin{aligned} a^{-3(1+w_{\mathrm{v,eff}}(w))} =& \frac{\chi_-}{\chi_+}\mathrm{cosech}^2\left(\frac{3(1+w_{\mathrm{v,eff}}(w))\sqrt{\chi_-}}{2}(t+c)\right) \\ =& \frac{\chi_-}{\chi_+}\mathrm{sech}^{-2}\left(\frac{3(1+w_{\mathrm{v,eff}}(w))\sqrt{\chi_-}}{2}(t+c)\right). \end{aligned} \tag{4.84}$$

By rearranging it, the analytical expression for the scale factor is:

$$a(t) = a_0\left(\frac{\chi_+}{\chi_-}\right)^{\mp\frac{1}{3(1+w_{\mathrm{v,eff}}(w))}} \sinh^{\mp\frac{2}{3(1+w_{\mathrm{v,eff}}(w))}}\left[\frac{3(1+w_{\mathrm{v,eff}}(w))}{2}\sqrt{\chi_-}t\right], \tag{4.85}$$

where

$$a_0 = \left(\frac{\chi_+}{\chi_-}\right)^{\pm\frac{1}{3(1+w_{\mathrm{v,eff}}(w))}} \sinh^{\pm\frac{2}{3(1+w_{\mathrm{v,eff}}(w))}}\left[\frac{3(1+w_{\mathrm{v,eff}}(w))}{2}\sqrt{\chi_-}t_0\right]. \tag{4.86}$$

For the pair comprising dark matter ($w=0$) and vacuum energy, we ascertain that $w_{\mathrm{v,eff}}(w) = w_{\mathrm{v,eff}}^{\mathrm{EMSF}}(w) = 1/2$, with $\beta = -3/2$. In this scenario, ρ_{dm} is found to be proportional to $a^{-9/2}$, and the interaction kernel is given as $Q = -\frac{3}{2}H\rho_{\mathrm{dm}}$, leading to a rate of dilution for dark matter faster than its standard behavior of $\rho_{\mathrm{dm}} \propto a^{-3}$, and even faster than radiation. However, this interaction kernel appears to be non-viable, as it markedly differs from the widely used kernels in interacting dark energy models (IDE), such as $Q = -\frac{3}{2}H\rho_{\mathrm{vac}}$ (see, e.g, Refs. [13]–[16,99]–[101]), which are employed to address the H_0 (and potentially S_8) tension. The critical distinction is that in our model, Q is proportional to ρ_{dm}, whereas the common approach in IDE models involves Q being proportional to ρ_{vac}. This leads to a significant distinction, as vacuum energy is generally expected to be approximately constant, while dark matter is typically not anticipated to deviate significantly from its standard behavior of $\rho_{\mathrm{dm}} \propto a^{-3}$. Our study, however, does not aim to propose specific promising interaction kernels in GR with an EMSF sector to alleviate cosmological tensions. Instead, it illustrates that specific forms of interaction kernels can naturally emerge depending on the choice of EMSG theories and the types of fluids involved. The EMSG framework offers a pathway for deriving non-trivial dark energy-dark matter interaction kernels, as indicated by model-independent reconstructions from recent observational data [32]. In particular, Ref. [32] suggests the possibility of oscillatory dynamics in the interaction kernel between dark energy and dark matter, potentially leading to a sign change in the direction of energy transfer between them and a transition from a negative DE energy density in early times to a positive one in later times. For further reading and references on non-trivial interaction kernels that contrast with widely used monotonically decreasing or increasing parameterizations, we refer the reader to Ref. [32] and references therein.

4.3.2 w-fluid–stiff fluid ($w = 1$) pair

We consider a standard source with a general EoS parameter $w_1 = w =$ const. (excluding the stiff fluid described by $w = 1$) and a stiff fluid ($w_2 = 1$). Their respective EMSF partners are described as follows:

$$w_1 = w\,,\; w_1^{\rm EMSF} = \frac{3w^2+1}{4w}\,,\; w_2 = w_2^{\rm EMSF} = 1. \tag{4.87}$$

For pairs comprising a source described by w and a stiff fluid, Eq. (4.29) yields $\zeta_{\rm s}(w) = 0$. Consequently, Eqs. (4.26) and (4.27) become:

$$\rho_w = \rho_{w0} a^{-3(w+1)+\beta_{\rm s}(w)}, \tag{4.88}$$

$$\rho_{\rm s} = \rho_{\rm s0} a^{-6} + \frac{\beta_{\rm s}(w)}{3(1+w)-\beta_{\rm s}(w)} \rho_{w0} \left(a^{-6} - a^{-\frac{3}{2}\left(2-\sqrt{3w^2+1}\right)}\right), \tag{4.89}$$

where the corresponding $\beta_{\rm s}(w)$ value is given by:

$$\beta_{\rm s}(w) = \frac{3}{2}\left(2w + \sqrt{3w^2+1}\right). \tag{4.90}$$

Eq. (4.50) reveals that both the source and its partner evolve identically, with equivalent effective EoS parameters:

$$w_{\rm s,eff}(w) = w_{\rm s,eff}^{\rm EMSF}(w) = -\frac{1}{2}\sqrt{3w^2+1}. \tag{4.91}$$

The Friedmann equation (4.62) for this case reads:

$$\begin{aligned}\frac{H^2}{H_0^2} =& \,\Omega_{w0} a^{-3\left(1+w_{\rm s,eff}(w)\right)} + \Omega_{w0}^{\rm EMSF} a^{-3\left(1+w_{\rm s,eff}(w)\right)} \\ &+ \Omega_{\rm s0} a^{-6} \left[1 - \left[\frac{w - w_{\rm s,eff}(w)}{1 - w_{\rm s,eff}(w)}\right] \frac{\Omega_{w0}\left[1 - \frac{\Omega_{w0}^{\rm EMSF}}{\Omega_{w0}} \frac{w_{\rm s,eff}(w)}{w}\right]}{1 - \Omega_{w0} - \Omega_{w0}^{\rm EMSF}} \left(1 - a^{3\left(1-w_{\rm s,eff}(w)\right)}\right)\right] \\ &+ \Omega_{\rm s0}^{\rm EMSF} a^{-6} \left[1 - \left[\frac{w - w_{\rm s,eff}(w)}{1 - w_{\rm s,eff}(w)}\right] \frac{\Omega_{w0}\left[1 - \frac{\Omega_{w0}^{\rm EMSF}}{\Omega_{w0}} \frac{w_{\rm s,eff}(w)}{w}\right]}{1 - \Omega_{w0} - \Omega_{w0}^{\rm EMSF}} \left(1 - a^{3\left(1-w_{\rm s,eff}(w)\right)}\right)\right],\end{aligned} \tag{4.92}$$

where the energy density parameters are defined as in Eqs. (4.51) to (4.54), fulfilling the relation

$$1 = \Omega_{w0} + \Omega_{w0}^{\mathrm{EMSF}} + \Omega_{\mathrm{s}0} + \Omega_{\mathrm{s}0}^{\mathrm{EMSF}}. \tag{4.93}$$

Both dark matter ($\rho_{\mathrm{dm}} \propto a^{-3/2}$) and radiation ($\rho_{\mathrm{r}} \propto a^{-3+\sqrt{3}}$), when partnered with stiff fluid, dilute slower than they would in a non-interacting GR scenario (i.e., in the absence of EMSF partners) as the universe expands. For the pair consisting of vacuum energy ($w = -1$) and stiff fluid ($w = 1$), we find that $\beta_{\mathrm{s}} = 0$. This suggests that the stiff (Zeldovich) fluid and vacuum energy interact minimally, i.e., solely gravitationally.

An intriguing aspect arises when considering the pairing of phantom or quintessence dark energy sources with stiff fluid. For the quintessential dark energy ($w > -1$) and stiff fluid pair, we determine that $\beta_{\mathrm{s}}(w) > 0$. This implies that the interaction flow direction is from the stiff fluid to quintessence dark energy. As a result, the energy density of quintessence dilutes at a slower rate than observed in standard GR without an EMSF partner, as the universe expands. Conversely, in the case of phantom dark energy ($w < -1$) paired with stiff fluid, we find $\beta_{\mathrm{s}}(w) < 0$. In this scenario, energy is transferred from phantom dark energy to stiff fluid. As the universe expands, the rate at which the energy density of phantom dark energy increases is slower compared to the behavior of a standard phantom source in GR without an EMSF partner.

4.3.3 (w_1, w_2) fluid pair

For a general fluid pair (w_1, w_2), unless one of them is vacuum energy, both powers of scale factors in solution (4.62) [see also (4.47)] are non-zero. It is possible to integrate this to obtain a solution in terms of cosmic time, t, using the Gaussian hyper-geometric function, $2F_1(a,b;c;z)$:

$${}_2F_1\left(1, 1+\frac{\gamma_-}{2(\gamma_+-\gamma_-)}; 1+\frac{\gamma_+}{2(\gamma_+-\gamma_-)}; -\frac{a^{\gamma_+-\gamma_-}\chi_-}{\chi_+}\right)\frac{2\sqrt{\chi_- a^{-\gamma_-}+\chi_+ a^{-\gamma_+}}}{H_0\gamma_+\chi_+ a^{-\gamma_+}} = t + c, \tag{4.94}$$

where c is an integration constant. The $\chi_\pm$ coefficients are composite, combining all coefficients relevant to the scale factor for the case at hand:

$$\chi_\pm = \pm\frac{1}{q_+ - q_-}\Bigg[(q_\pm-\beta)\Omega_{10} + \frac{q_\mp(q_\pm-\beta)}{\beta}\Omega_{20} + \left(q_\pm - \beta\frac{w_2\sqrt{3w_1^2+1}}{w_1\sqrt{3w_2^2+1}}\right)\Omega_{10}^{\mathrm{EMSF}} + q_\mp\left(\frac{q_\pm}{\beta}\frac{w_1\sqrt{3w_2^2+1}}{w_2\sqrt{3w_1^2+1}} - 1\right)\Omega_{20}^{\mathrm{EMSF}}\Bigg]. \tag{4.95}$$

4.4 Cosmological Solutions for specific values of α

It has been previously emphasized that the primary distinction of the EMSG model under discussion in this work, as compared to the phenomenological interaction model examined in [8], lies in the fact that the coefficients in our model are not arbitrary constants; instead, they depend on the species, specifically, w. Moving forward, we aim to establish a comparison between our model and the one studied in [8] by deriving the Liénard equation and exploring its adaptation within the EMSG framework. Utilizing the separated continuity equations (4.26)-(4.27) and the Friedmann equation (4.30), we can eliminate the energy densities to obtain a single non-autonomous master equation for the Hubble expansion, the modified Liénard equation for our model, presented as follows:

$$\left[\ddot{H}+H\dot{H}C(w_1,w_2)+H^3D(w_1,w_2)\right]\left[3(w_1-w_2)+E(w_1,w_2)\right]=0, \tag{4.96}$$

where $C(w_1,w_2)$ and $D(w_1,w_2)$ are specified in (4.44)-(4.45). The new term $E(w_1,w_2)$, arising from EMSF, thereby determined by α, is expressed as

$$E(w_1,w_2)=3\alpha^2A_1A_2(w_1-w_2)+\alpha(A_1-A_2)\left[\zeta(1+\alpha A_1)-\beta(1+\alpha A_2)\right]. \tag{4.97}$$

The Liénard equation, as explored in Refs. [102] and [103], represents a special case of more general differential equations. Chimento's analysis involves the differential equation $\ddot{y}+\alpha f(y)\dot{y}+\beta f(y)\int f(y)dy+\gamma f(y)=0$, where $y=y(x)$ and $f=f(y)=by^n+k$, with an overdot indicating differentiation with respect to the variable x [103]. In this framework, α, β, and γ are constants. In the specific case where $\gamma=0$, $k=0$, and $y=H$ (with the variable chosen as cosmic time, $x=t$), this equation aligns with our model. Similarly, Harko et al. explored the differential equation $\ddot{x}(t)+f(x)\dot{x}(t)+g(x)=0$, with an overdot denoting differentiation with respect to t [102]. Their equation corresponds to our model for a particular form of $f(x)=ax+b$ when $b=0$ and $C_1=0$, and with $g(x)$ defined as $\frac{1}{2}a^2kx^3$, where the variable is the Hubble parameter, $x=H$.

In our model, the first bracket of (A.14) becomes null for the general solution presented in (4.62), making it unnecessary for the second bracket to be zero. However, setting this second term to zero, as in

$$3(w_1 - w_2) + E(w_1, w_2) = 0, \tag{4.98}$$

and substitution of (4.97) into (4.98) results in a quadratic equation regarding the EMSF field strength parameter:

$$\alpha^2 s(w_1, w_2) + \alpha p(w_1, w_2) + 3(w_1 - w_2) = 0, \tag{4.99}$$

where functions $s(w_1, w_2)$ and $p(w_1, w_2)$ are defined as:

$$s(w_1, w_2) = \frac{4(w_2 - w_1)(3w_1 w_2 - 1)}{\sqrt{(3w_1^2 + 1)(3w_2^2 + 1)}}, \tag{4.100}$$

$$p(w_1, w_2) = \frac{3w_2^2 - 4w_1 w_2 + 1}{\sqrt{1 + 3w_2^2}} - \frac{3w_1^2 - 4w_1 w_2 + 1}{\sqrt{1 + 3w_1^2}}. \tag{4.101}$$

The solutions to this equation, which are the roots of α, are given as:

$$\alpha_\pm = -\frac{p(w_1, w_2) \pm \sqrt{p(w_1, w_2)^2 - 4s(w_1, w_2)(w_1 - w_2)}}{2s(w_1, w_2)}. \tag{4.102}$$

It has been shown in Eqs. (4.53)-(4.54) that α quantifies the extent of the EMSF field relative to the usual source, irrespective of the presence of an explicit non-minimal interaction between the usual sector and its EMSF counterpart. For a more focused discussion on density parameters, we can bypass the use of $\alpha_\pm$. Instead, by employing Eqs. (4.53)-(4.54), we re-formulate the relationship stated in Eq. (4.102) in terms of present-day density parameters:

$$\left(\frac{\Omega^{\mathrm{EMSF}}_{10(20)}}{\Omega_{10(20)}}\right)_\pm = -\frac{p \pm \sqrt{p^2 - 4s(w_1 - w_2)}}{2s} \frac{4w_{1(2)}}{\sqrt{3w_{1(2)}^2 + 1}}. \tag{4.103}$$

This specific relationship between Ω^{EMSF} and Ω leads to a reduction in the number of parameters free parameters. Otherwise, the amount of EMSF field relative to the usual source would remain as a free parameter as studied in solutions given in Section 4.3.

We can follow an alternative approach for explicit explanation about the linear dependence between Ω^{EMSF} and Ω with matrix notation of continuity equation. It has been shown above, a novel term $E(w_1, w_2)$ arises in Liénard equation and from setting it to zero gives an additional equation (4.98) that relates Ω^{EMSF} and Ω (4.103), then we reproduce solutions with this particular relation. Here will discuss this extra equation and the fact that linear dependence between the density parameters following the matrix notation of continuity equations. First, we will write Eq. (4.9) in matrix notation as

$$\dot{\vec{\rho}}^{\,T}\vec{x}_A + \vec{\rho}^{\,T}\vec{x}_B = 0, \tag{4.104}$$

where

$$\vec{x}_A = \begin{bmatrix}(1+\alpha A_1)\\(1+\alpha A_2)\end{bmatrix}, \vec{x}_B = \begin{bmatrix}3H(1+w_1)(1+\alpha B_1)\\3H(1+w_2)(1+\alpha B_2)\end{bmatrix}, \tag{4.105}$$

With the vectors $\vec{x}_A$ and $\vec{x}_B$, we define two matrices:

$$\mathbf{X} = \begin{bmatrix}\vec{x}_B & \vec{x}_A\end{bmatrix} = \begin{bmatrix}3H(1+w_1)(1+\alpha B_1) & (1+\alpha A_1)\\3H(1+w_2)(1+\alpha B_2) & (1+\alpha A_2)\end{bmatrix}, \tag{4.106}$$

$$\mathbf{W} = \begin{bmatrix}\vec{\rho}^{T}\\ \dot{\vec{\rho}}^{T}\end{bmatrix} = \begin{bmatrix}\rho_1 & \rho_2\\ \dot{\rho}_1 & \dot{\rho}_2\end{bmatrix}. \tag{4.107}$$

In this configuration, $\mathbf{X}$ represents a matrix correlating the coefficients adjacent to energy densities and their derivatives, while $\mathbf{W}$ is the Wronskian matrix, which governs the linear dependence of the energy densities. A determinant of $\mathbf{W}$ equating to zero, $\mathrm{Det}\,\mathbf{W} = 0$, implies $\rho_1 = \rho_2$, while $\mathrm{Det}\,\mathbf{W} \neq 0$ signifies the presence of two distinct sources, with $w_1 \neq w_2$. To investigate the linear dependence of the functions $w_1(\rho_1, p_1)$ and $w_2(\rho_2, p_2)$, we derive an EMSF-modified Wronskian matrix, denoted as $\mathbf{L}$:

$$\mathbf{L} = \mathbf{WX} = \begin{bmatrix}\vec{\rho}^{T}\vec{x}_B & \vec{\rho}^{T}\vec{x}_A\\ \dot{\vec{\rho}}^{T}\vec{x}_B & \dot{\vec{\rho}}^{T}\vec{x}_A\end{bmatrix}. \tag{4.108}$$

For the matrix $\mathbf{L}$ to indicate linear dependence, its trace, $\mathrm{Tr}[\mathbf{L}] = \mathrm{Tr}[\mathbf{WX}]$, must equal zero. Additionally, if $\mathbf{L}$ is singular, meaning it is non-invertible, as shown by:

$$\mathrm{Det}\,\mathbf{L} = \mathrm{Det}\,\mathbf{W}\,\mathrm{Det}\,\mathbf{X} = 0, \tag{4.109}$$

then this implies linear dependence. In the thesis by Barrow and Clifton [8], a determinant of $\mathbf{X}$ equating to zero, $\mathrm{Det}\,\mathbf{X} = 0$, implies the existence of only one species, i.e., $w_1 = w_2$. Conversely, in our model, the condition:

$$\mathrm{Det}\,\mathbf{X} = 3(w_1 - w_2) + E(w_1, w_2) = 0, \tag{4.110}$$

can hold true even when $w_1 \neq w_2$. Here, one may see that $\mathrm{Det}\,\mathbf{X}$ is the second multiplier of the modified master equation given in (A.14). Thus, solutions satisfying $\mathrm{Det}\,\mathbf{X} \neq 0$

can be classified as 'general α solutions', and those with $\mathrm{Det}\,\mathbf{X} = 0$ can be regarded as 'specific α solutions' whose values of α found in (4.102). In this context, α quantifies the extent of the EMSF field relative to the usual source and solutions discussed below this section.

In what follows, we will proceed to show that the solutions given for general α values for $(w, \pm 1)$ fluid pairs, studied in Sections 4.3.1 and 4.3.2, these solutions turn into power-law/exponential solutions for specific values of α when substituting the relation between Ω^{EMSF} and Ω from Equation (4.103). This specific relation leads to the negation of the EMSF partner's contribution to the Friedmann equation of the usual source.

4.4.1 Power-law universes in the presence of a perfect fluid and vacuum energy density

We now consider a pair of a usual source described by a constant EoS parameter $w_1 = w = \text{const.}$ and vacuum energy ($w_2 = -1$). Utilizing Eq. (4.103), for the case α_-, along with definitions (4.100) and (4.101), we derive the following relationships between the present-day density parameters of the usual source and vacuum energy and their EMSF counterparts:

$$\left(\frac{\Omega_{w0}^{\mathrm{EMSF}}}{\Omega_{w0}}\right)_- = \frac{w}{w_{\mathrm{v,eff}}(w)} \quad \text{and} \quad \left(\frac{\Omega_{\mathrm{vac0}}^{\mathrm{EMSF}}}{\Omega_{\mathrm{vac0}}}\right)_- = -1, \tag{4.111}$$

where the latter implies that the contributions of the vacuum energy and its EMSF partner cancel out each other in the Friedmann equation. By incorporating these relations from (4.111) into the Friedmann equation (4.72), we obtain:

$$\frac{H^2}{H_0^2} = \Omega_{w0} \frac{w_{\mathrm{v,eff}}(w) + w}{w_{\mathrm{v,eff}}(w)} a^{-3\left(1 + w_{\mathrm{v,eff}}(w)\right)}. \tag{4.112}$$

We use the explicit form $H^2 = \frac{\dot{a}^2}{a^2}$ and take square-root of both sides. After rearranging it, we obtain the following differential form:

$$\frac{da}{a\sqrt{H_0^2 \Omega_{w0} \frac{w_{\mathrm{v,eff}}(w)+w}{w_{\mathrm{v,eff}}(w)} a^{-3-3w_{\mathrm{v,eff}}(w)}}} = dt. \tag{4.113}$$

We integrate it and by rearranging the result we have the following from:

$$a^{\frac{3+3w_{\mathrm{v,eff}}(w)}{2}} = \frac{3 + 3w_{\mathrm{v,eff}}(w)}{2} \sqrt{H_0^2 \Omega_{w0} \frac{w_{\mathrm{v,eff}}(w) + w}{w_{\mathrm{v,eff}}(w)}} (t + c). \tag{4.114}$$

By setting the integration constant as $c=0$ and organise the result so that we leave the scale factor alone. Then, the scale factor factor is given by

$$a(t)=\left[\frac{3+3w_{\text{v,eff}}(w)}{2}\sqrt{H_0^2\Omega_{w0}\frac{w_{\text{v,eff}}(w)+w}{w_{\text{v,eff}}(w)}}\right]^{\frac{2}{3+3w_{\text{v,eff}}(w)}} t^{\frac{2}{3+3w_{\text{v,eff}}(w)}}. \tag{4.115}$$

Correspondingly, the Hubble parameter is expressed as:

$$H(t)=\frac{2}{3(1+\frac{1}{2}\sqrt{3w^2+1})}\frac{1}{t}, \tag{4.116}$$

and deceleration parameter $q=-1-\frac{\dot{H}}{H^2}$ can be written as

$$q=\frac{1}{2}+\frac{3}{4}\sqrt{3w^2+1}, \tag{4.117}$$

which is always positive, irrespective of w, implying decelerated expansion in the presence of usual source described by w and vacuum energy density. This is quite unpredictable, since vacuum energy dominates universe leading acceleration in expansion in the presence of radiation and/or matter in GR without an EMSF partner.

4.4.2 De Sitter universe in the presence of a perfect fluid and vacuum energy density

We again consider a pair of a usual source described by a constant EoS parameter $w_1=w=$ const. and vacuum energy ($w_2=-1$) which satisfies (4.71). Utilizing Eq. (4.103), but now for the case α_+, along with definitions (4.100) and (4.101), we derive the following relationships between the present-day density parameters of the usual source and vacuum energy and their EMSF counterparts:

$$\left(\frac{\Omega_{w0}^{\text{EMSF}}}{\Omega_{w0}}\right)_+=-\frac{4w}{1+3w} \text{ and } \left(\frac{\Omega_{\text{vac0}}^{\text{EMSF}}}{\Omega_{\text{vac0}}}\right)_+=\frac{4w_{\text{v,eff}}(w)}{1+3w}. \tag{4.118}$$

Substituting (4.118) in Friedmann equation (4.72), we obtain Hubble parameter as a constant

$$H=\pm H_0\sqrt{\left(1+\frac{4w_{\text{v,eff}}}{1+3w}\right)\left(\Omega_{\text{v}0}-\frac{w-w_{\text{v,eff}}}{1+w_{\text{v,eff}}}\Omega_{w0}\right)}, \tag{4.119}$$

where the square root expression equals to unity from (4.73), Eq. (4.119) reads

$$H=\pm H_0 \quad \text{leading} \quad a(t)=e^{H(t-t_0)}. \tag{4.120}$$

Notably, this results in a de Sitter expansion, despite the universe containing a perfect fluid with a constant EoS parameter other than -1, along with vacuum energy. This contrasts with the standard de Sitter universe in the framework of GR, where the universe contains only a positive cosmological constant/vacuum energy density.

4.4.3 Stiff fluid dominated-like universe in the presence of a perfect fluid and stiff fluid

In a cosmological scenario in the presence of usual source described by a constant EoS parameter $w_1 = w = \text{const.}$ and stiff fluid ($w_2 = 1$), utilizing Eq. (4.103) along with definitions (4.100) and (4.101) for the case α_-, we obtain the following relations of the present-day density parameters of the usual source and stiff fluid and their EMSF counterparts:

$$\left(\frac{\Omega_{w0}^{\text{EMSF}}}{\Omega_{w0}}\right)_- = \frac{4w}{1-3w} \text{ with } \left(\frac{\Omega_{\text{s}0}^{\text{EMSF}}}{\Omega_{\text{s}0}}\right)_- = -\frac{4w_{\text{s,eff}}(w)}{1-3w}. \tag{4.121}$$

Substituting these relations into Friedmann equation (4.92), we obtain

$$\frac{H^2}{H_0^2} = \left[\frac{1+w}{1-3w}\Omega_{w0} + \left(1 - \frac{4w_{\text{s,eff}}}{1-3w}\right)\Omega_{\text{s}0}\right] a^{-6}, \tag{4.122}$$

from (4.93), the expression in square brackets equals to unity leading

$$H^2 = H_0^2 a^{-6} \quad \text{and} \quad a(t) = t^{\frac{1}{3}}. \tag{4.123}$$

Corresponding Hubble parameter is written as

$$H(t) = \frac{1}{3t}. \tag{4.124}$$

4.4.4 Power-law universes in the presence of a perfect fluid and stiff fluid

For the same content considered in Section 4.4.3—a usual source described by constant EoS parameter $w_1 = w = \text{const.}$ and a stiff fluid ($w_2 = 1$)—, we may obtain power-law universes with w-dependent power as well for the case α_+. Considering Eq. (4.103) with definitions (4.100) and (4.101), we obtain the following relations of the present-day density parameters of the usual source and stiff fluid and their EMSF counterparts:

$$\left(\frac{\Omega_{w0}^{\text{EMSF}}}{\Omega_{w0}}\right)_+ = \frac{w}{w_{\text{s,eff}}(w)} \quad \text{with} \quad \left(\frac{\Omega_{\text{s}0}^{\text{EMSF}}}{\Omega_{\text{s}0}}\right)_+ = -1. \tag{4.125}$$

Using these relations given in (4.125) in the Friedmann equation (4.92), we obtain

$$\frac{H^2}{H_0^2} = \Omega_{w0}\left[\frac{w_{\mathrm{s,eff}}(w)+w}{w_{\mathrm{s,eff}}(w)}\right] a^{-3\left[1+w_{\mathrm{s,eff}}(w)\right]}. \tag{4.126}$$

Hence, the analytical expression for the scale factor is as follows:

$$a(t) = \left[\frac{3+3w_{\mathrm{s,eff}}(w)}{2}\sqrt{\frac{(w_{\mathrm{s,eff}}(w)+w)\Omega_{w0}}{w_{\mathrm{s,eff}}(w)}}\right]^{\frac{2}{3+3w_{\mathrm{s,eff}}(w)}} t^{\frac{2}{3+3w_{\mathrm{s,eff}}(w)}}, \tag{4.127}$$

and the corresponding Hubble parameter

$$H(t) = \frac{2}{3(1-\frac{1}{2}\sqrt{3w^2+1})}\frac{1}{t}, \tag{4.128}$$

and deceleration parameter are expressed as

$$q = \frac{1}{2} - \frac{3}{4}\sqrt{3w^2+1}, \tag{4.129}$$

which is always negative, $-1 \leq q \leq -1/4$, giving accelerated expansion irrespective of w in the presence of usual source described by an EoS in the range of $-1 \leq w \leq 1$. For this choice of α_{+}, non minimal interaction between dust and stiff fluid or between radiation and stiff fluid, if interaction between EMSF counterparts is not allowed, leads accelerated expansion whereas each decelerates the expansion individually. Notably, we show that the universe will enter a super-accelerated expansion phase ($q < -1$) and ends in a finite time with big rip singularity in the presence of a source having $w > 1$, though it contradicts with special relativity, in addition to phantom DE source ($w < -1$).

5. CONCLUSIONS

It was revealed in [7], all matter-type modified gravity theories including $f(\mathscr{L}_{\rm m})$ [1], $f(T)$ [2] and $f(\mathbf{T^2})$ [3]–[6] in the action are equivalent to GR with non-minimal interactions in which the choices for the function f can yield nontrivial interaction kernels with a covariant formulation. The material part of the Lagrangian density is redefined as $\mathscr{L}_{\rm m}^{\rm tot}=\mathscr{L}_{\rm m}+f$, therefore in these models, the twice contracted Bianchi identity, $\nabla^{\mu}G_{\mu\nu}=0$ implies $\nabla^{\mu}(T_{\mu\nu}+T_{\mu\nu}^{\rm mod})=0$, which does not necessarily imply separate sector conservation, $\nabla^{\mu}T_{\mu\nu}=0$ or $\nabla^{\mu}T_{\mu\nu}^{\rm mod}=0$. Instead, such models provide a non-minimal interaction between the material field and its counterpart, the modification field, which can be represented as $\nabla^{\mu}T_{\mu\nu}=-\mathscr{Q}_{\nu}$ and $\nabla^{\mu}T_{\mu\nu}^{\rm mod}=\mathscr{Q}_{\nu}$ where $\mathscr{Q}_{\nu}(f,f_{\mathbf{T^2}},\Theta_{\mu\nu})$ is the energy transfer rate four-vector acting as the interaction kernel. Notably, both the function f and the new tensor $\Theta_{\mu\nu}$ take part in the EoS of the second source described by $T_{\mu\nu}^{\rm mod}$ and in the interaction kernel $\mathscr{Q}_{\nu}$ [7]. The function f controlling the Lorentz scalar $\mathbf{T^2}$ contribution, then formulates the interaction kernel.

Indeed, the model has been reanalyzed in the search for the correct calculation of the second derivative of $\mathscr{L}_{\rm m}$ with respect to the inverse metric tensor within the definition of $\Theta_{\mu\nu}$ and this term can be taken zero as degrees of freedom and $T_{\mu\nu}^{\rm mod}$ turns into $T_{\mu\nu}^{\rm EMSF}$ with this missing term, yet it lacks a well-defined Lagrangian formulation. In this model, interaction of EMSF partners determines the form of interaction of usual fluids, as species dependent way. On the other hand, in GR without EMSF partners, the interaction between sources is not clearly defined, making it uncertain which source will engage with another, different parameterizations for the interaction kernel have been widely studied [8,12,104,105].

In this thesis, alternatively we have examined the scenario in which the sector comprising conventional (standard) fluids interacts minimally with the sector associated with their EMSF partners, thereby we are considering the case that is characterized by $\mathscr{Q}=0$. We show how $\mathscr{Q}=0$ affects the model on a case study:

the *scale-independent* EMSG with $f(\mathbf{T^2}) = \alpha\sqrt{\mathbf{T^2}}$, in which the energy-momentum squared field (EMSF) partner enters with the same power of energy density as that of the conventional fluid in the Einstein field equations, see [91,92,106] where EMSF-usual field interaction takes place for comparison.

For $\mathscr{Q} = 0$, non-minimal interactions happen within each individual sector separately. After a series of manipulations, Eqs. (4.26)-(4.27) becomes equivalent to the conservation equations designed in *Cosmologies with energy exchange* paper [8], yet the interaction rates, β and ζ are species dependent due to the contribution of EMSF partners while the coefficients α and β in their work are arbitrary constants as there is no EMSF counterpart. On top of that, this additional sector associated with the EMSF partners of the conventional (standard) fluids different than [8], contributes to the Friedmann and pressure equation in our construction. We then arrive at the Friedmann equation, where fluids are superposed, and their coupling constants are intertwined. Both vacuum energy ($w = -1$) and stiff fluid ($w = 1$) preserve their identities, with vacuum energy having a vacuum energy EMSF partner and stiff fluid having a stiff EMSF partner in the presence of usual source described by w.

From a comparison between our model and the one studied in [8], we have explored the modified Liénard equation within GR by EMSF sector: **(i)** coefficients of $H\dot{H}$ and H^3 terms in the first bracket of Liénard equation has arbitrary parameters in [8] whereas they are specified by EMSF sector, **(ii)** The first bracket becomes null for the general solution presented in (4.62) and in [8], yet there arises a new term from EMSF, $E(w_1, w_2)$ brings in a quadratic equation which can be also set to zero along with the first bracket, on the other hand second bracket can not be zero since $w_1 \neq w_2$ in [8]. An algebraic equation of the second degree in α, which quantifies the extent of the EMSF field relative to the usual source, irrespective of the presence of an explicit non-minimal interaction between the usual sector and its EMSF counterpart, gives a spesific relation between Ω_{EMSF} and Ω then we have bypassed the use of $\alpha_{\pm}$ and re-formulate the relationship stated in Eq. (4.102) in terms of present-day density parameters in (4.103).

We have first given general solutions leaving the second bracket nonzero (without relating Ω_{EMSF} with Ω), and then we have investigated solutions with some specific relationships between Ω_{EMSF} and Ω fixing the second bracket to zero as well. For instance, for some specific amount of EMSF partner that equal to the contribution of the usual source to Friedmann equation, we do not see the source in Friedmann equation, as the feature of this interaction model. By courtesy of this feature, we are able to reproduce power-law universes where the scale factor of the universe grows as a EoS parameter dependent power of time in the presence of a perfect fluid and vacuum energy density/stiff fluid, de Sitter universe in the presence of a perfect fluid and vacuum energy density, stiff fluid dominated-like universe in the presence of a perfect fluid and stiff fluid.

This case study aimed to illustrate how the EMSG framework may provide a pathway for deriving non-trivial dark energy-dark matter interaction kernels, as indicated by model-independent reconstructions from recent observational data [32], and specific promising interaction kernels in GR with an EMSF sector can be designated to alleviate cosmological tensions.

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APPENDICES

APPENDIX A : Derivation of Modified Liénard Equation

APPENDIX A : Derivation of Modified Liénard Equation

Here, we derive modified Liénard equation by utilizing the separated continuity equations (4.26)-(4.27) and the Friedmann equation (4.30), we can eliminate the energy densities to obtain a single non-autonomous master equation for the Hubble expansion, the modified Liénard equation for our model. The mathematical approach can be described that renaming the α labels of the derivative forms of the Friedmann equation (viz., α for no derivation applied form of (4.30), α^* and $\overline{\alpha}$ for first and second derivatives applied form of (4.30), respectively). While calculating, we applied this trick and we fixed them again α at the last step (viz., $\{\alpha, \alpha^*, \overline{\alpha}\} \rightarrow \{\alpha\}$), revealing an invisible equation in the form of the curly parentheses of (A.14). Now, for the sake of simplicity, we name $(1+\alpha A_1) = \alpha_1$, $(1+\alpha^* A_2) = \alpha_2^*$, $w_1 + 1 = W_1$, etc.; then, the derivations of the Friedmann Equation can be written as

$$6H\dot{H} = \kappa\alpha_1^*\dot{\rho}_1 + \kappa\alpha_2^*\dot{\rho}_2, \tag{A.1}$$

$$6(\dot{H}^2 + H\ddot{H}) = \kappa\overline{\alpha}_1\ddot{\rho}_1 + \kappa\overline{\alpha}_2\ddot{\rho}_2. \tag{A.2}$$

We take a derivative of (4.26) with respect to cosmic time t and multiply it with $\overline{\alpha}_1$

$$\overline{\alpha}_1\ddot{\rho}_1 + 3\overline{\alpha}_1 W_1(\dot{H}\rho_1 + H\dot{\rho}_1) = \overline{\alpha}_1\beta(\dot{\rho}_1 H + \rho_1\dot{H}) + \overline{\alpha}_1\zeta(\dot{\rho}_2 H + \rho_2\dot{H}). \tag{A.3}$$

We follow the same steps for the interacting conservation equation of the other source ρ_2, but we multiply the derivative of (4.27) with $\overline{\alpha}_2$ for once

$$\overline{\alpha}_2\ddot{\rho}_2 + 3\overline{\alpha}_2 W_2(\dot{H}\rho_2 + H\dot{\rho}_2) = -\overline{\alpha}_2\beta(\dot{\rho}_1 H + \rho_1\dot{H}) - \overline{\alpha}_2\zeta(\dot{\rho}_2 H + \rho_2\dot{H}), \tag{A.4}$$

and the summation of (A.3) and (A.4) yields the following

$$\begin{aligned}\overline{\alpha}_1\ddot{\rho}_1 + \overline{\alpha}_2\ddot{\rho}_2 = &-\dot{H}(\rho_1[3\overline{\alpha}_1 W_1 - \overline{\alpha}_1\beta + \overline{\alpha}_2\beta] + \rho_2[3\overline{\alpha}_2 W_2 - \overline{\alpha}_1\zeta + \overline{\alpha}_2\zeta)]) \\ &- H(\dot{\rho}_1[3\overline{\alpha}_1 W_1 - \overline{\alpha}_1\beta + \overline{\alpha}_2\beta] + \dot{\rho}_2[3\overline{\alpha}_2 W_2 - \overline{\alpha}_1\zeta + \overline{\alpha}_2\zeta]).\end{aligned} \tag{A.5}$$

By utilizing (4.26) and (4.27) in it, we reach a bit complicated equation as:

$$\begin{aligned}\overline{\alpha}_1\ddot{\rho}_1 + \overline{\alpha}_2\ddot{\rho}_2 = &-\dot{H}(\rho_1[3\overline{\alpha}_1 W_1 - \overline{\alpha}_1\beta + \overline{\alpha}_2\beta] + \rho_2[3\overline{\alpha}_2 W_2 - \overline{\alpha}_1\zeta + \overline{\alpha}_2\zeta)]) \\ &-H([-3HW_1\rho_1 + \beta\rho_1 H + \zeta\rho_2 H][3\overline{\alpha}_1 W_1 - \overline{\alpha}_1\beta + \overline{\alpha}_2\beta] \\ &\quad + [-3HW_2\rho_2 - \beta\rho_1 H - \zeta\rho_2 H][3\overline{\alpha}_2 W_2 - \overline{\alpha}_1\zeta + \overline{\alpha}_2\zeta]).\end{aligned} \tag{A.6}$$

Now, we put it in the equation (A.2), thence we reach the following

$$\begin{aligned}6(\dot{H}^2 + H\ddot{H}) = &-\kappa\dot{H}(\rho_1[3\overline{\alpha}_1 W_1 - \overline{\alpha}_1\beta + \overline{\alpha}_2\beta] + \rho_2[3\overline{\alpha}_2 W_2 - \overline{\alpha}_1\zeta + \overline{\alpha}_2\zeta)]) \\ &-\kappa H^2([-3W_1\rho_1 + \beta\rho_1 + \zeta\rho_2][3\overline{\alpha}_1 W_1 - \overline{\alpha}_1\beta + \overline{\alpha}_2\beta] \\ &+ [-3W_2\rho_2 - \beta\rho_1 - \zeta\rho_2][3\overline{\alpha}_2 W_2 - \overline{\alpha}_1\zeta + \overline{\alpha}_2\zeta]),\end{aligned} \tag{A.7}$$

and also using (4.26) and (4.27) in the equation (A.1) yields

$$6H\dot{H} = \kappa\alpha_1^*[-3HW_1\rho_1 + \beta\rho_1 H + \zeta\rho_2 H] + \kappa\alpha_2^*[-3HW_2\rho_2 - \beta\rho_1 H - \zeta\rho_2 H]. \tag{A.8}$$

Besides, we also utilize (A.7) in the equation (A.8) so we derive a combined relation revealing the second derivative of Hubble parameter $\ddot{H}$ as:

$$\begin{aligned} 6H\ddot{H} = & -\kappa\dot{H}(\alpha_1^*[-3W_1\rho_1+\beta\rho_1+\zeta\rho_2]+\alpha_2^*[-3W_2\rho_2-\beta\rho_1-\zeta\rho_2]) \\ & -\kappa\dot{H}(\rho_1[3\overline{\alpha}_1W_1-\overline{\alpha}_1\beta+\overline{\alpha}_2\beta]+\rho_2[3\overline{\alpha}_2W_2-\overline{\alpha}_1\zeta+\overline{\alpha}_2\zeta]) \\ & -\kappa H^2([-3W_1\rho_1+\beta\rho_1+\zeta\rho_2][3\overline{\alpha}_1W_1-\overline{\alpha}_1\beta+\overline{\alpha}_2\beta] \\ & +[-3W_2\rho_2-\beta\rho_1-\zeta\rho_2][3\overline{\alpha}_2W_2-\overline{\alpha}_1\zeta+\overline{\alpha}_2\zeta]). \end{aligned} \qquad (\text{A.9})$$

Again, using derivatives of (4.26) and (4.27) with appropriate multiplications in (A.8) and (A.9), then we get the following intermediate results

$$6\dot{H} = \kappa\left[\alpha_1^*(-3W_1\rho_1+\beta\rho_1+\zeta\rho_2)+\alpha_2^*(-3W_2\rho_2-\beta\rho_1-\zeta\rho_2)\right], \qquad (\text{A.10})$$

and

$$\begin{aligned} & H^2\Big[\kappa\overline{\alpha}_1(-9W_1^2\rho_1+3\zeta W_1\rho_2+6\beta W_1\rho_1+3W_2\zeta\rho_2+\beta\zeta\rho_1-\zeta\beta\rho_2+\zeta^2\rho_2-\beta^2\rho_1) \\ & -\kappa\overline{\alpha}_2(9W_2^2\rho_2+6\zeta W_2\rho_2+3\beta W_2\rho_1+3W_1\beta\rho_1+\beta\zeta\rho_1-\zeta\beta\rho_2+\zeta^2\rho_2-\beta^2\rho_1)\Big] \\ & +\dot{H}\left[\kappa\overline{\alpha}_1(3W_1\rho_1-\zeta\rho_2-\beta\rho_1)+\kappa\overline{\alpha}_2(3W_2\rho_2+\zeta\rho_2+\beta\rho_1)\right]+6(\dot{H}^2+H\ddot{H})=0. \end{aligned} \qquad (\text{A.11})$$

Also using (4.30) and (A.10) with appropriate multiplications in (A.11), we obtain the following

$$\begin{aligned} & 2H\ddot{H}\Big[(\alpha_1^*-\alpha_2^*)(\alpha_1\zeta-\alpha_2\beta)+3\alpha_1^*\alpha_2W_1-3\alpha_1\alpha_2^*W_2\Big] \\ & +H^2\dot{H}\Big[3(\overline{\alpha}_2\alpha_1^*-\overline{\alpha}_1\alpha_2^*)(W_1(\zeta+3W_2)-\beta W_2) \\ & +2(\zeta-\beta)(\alpha_1\zeta-\alpha_2\beta)(\overline{\alpha}_1-\overline{\alpha}_2) \\ & +6W_2(\alpha_2\overline{\alpha}_2\beta+\alpha_1\zeta(\overline{\alpha}_1-2\overline{\alpha}_2)) \\ & +6W_1(\alpha_1\overline{\alpha}_1\zeta+\alpha_2\beta(\overline{\alpha}_2-2\overline{\alpha}_1)) \\ & +18\alpha_2\overline{\alpha}_1W_1^2-18\alpha_1\overline{\alpha}_2W_2^2\Big] \\ & +2\dot{H}^2\Big[3\alpha_2W_1(\alpha_1^*-\overline{\alpha}_1)+3\alpha_1W_2(\overline{\alpha}_2-\alpha_2^*) \\ & +(\alpha_2\beta-\alpha_1\zeta)(\overline{\alpha}_1-\alpha_1^*-\overline{\alpha}_2+\alpha_2^*)\Big] \\ & +3H^4\Big[((\overline{\alpha}_1-\overline{\alpha}_2)(\alpha_1^*\zeta-\beta\alpha_2^*)+3\overline{\alpha}_1\alpha_2^*W_1-3\alpha_1^*\overline{\alpha}_2W_2) \\ & \times(W_1(\zeta+3W_2)-\beta W_2)\Big]=0. \end{aligned} \qquad (\text{A.12})$$

Now, we can change back $\{\alpha,\alpha^*,\overline{\alpha}\}\rightarrow\{\alpha\}$ (viz., $\alpha_1^*=\overline{\alpha}_1=\alpha_1$ and $\alpha_2^*=\overline{\alpha}_2=\alpha_2$); then, we obtain

$$\begin{aligned} & \left[2\ddot{H}+2H\dot{H}(-\beta+\zeta+3W_1+3W_2)+3H^3(W_1(\zeta+3W_2)-\beta W_2)\right] \\ & \times[(\alpha_1-\alpha_2)(\alpha_1\zeta-\alpha_2\beta)+3\alpha_1\alpha_2(W_1-W_2)]=0. \end{aligned} \qquad (\text{A.13})$$

Substituting explicit forms of definitions of α_1, α_2, W_1, W_2, β, and ζ, we reach the modified Liénard equation for our model, presented as follows:

$$\left[\ddot{H}+H\dot{H}C(w_1,w_2)+H^3D(w_1,w_2)\right]\left[3(w_1-w_2)+E(w_1,w_2)\right]=0, \qquad (\text{A.14})$$

where $C(w_1,w_2)$ and $D(w_1,w_2)$ are specified in (4.44)-(4.45).

CURRICULUM VITAE

Name SURNAME: Bilal BULDUK

EDUCATION:

- **B.Sc.:** 2020, Middle East Technical University, Faculty of Arts and Science, Department of Physics

PROFESSIONAL EXPERIENCE AND REWARDS:

- 2022 Research Assistant at Gebze Technical University.

PUBLICATIONS, PRESENTATIONS AND PATENTS ON THE THESIS:

- **Bulduk B.** Sui generis interaction kernels: A Case Study with scale-independent EMSG. 3rd *International Graduate Research Symposium*, May 8 – 10, 2024 Istanbul, Turkey.