

**FUNDAMENTAL SOLITONS IN COMPLEX TWO DIMENSIONAL  
LATTICES**

**M.Sc. Thesis by  
Mahmut BAĞCI**

**Department : Mathematical Engineering**

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M.Sc. Thesis by  
Mahmut BAĞCI  
509071006

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Thesis Supervisor : Ass. Prof. Dr. İlkey B. AKAR (ITU)  
Co-Supervisor : Assoc. Prof. Dr. Nalan ANTAR (ITU)  
Members of the Examining Committee : Prof. Dr. Emanullah HIZEL (ITU)  
: Prof. Dr. Hilmi DEMİRAY (ISIK U.)  
: Ass. Prof. Dr. Güler GAYGUSUZUOĞLU (NKU)

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## İKİ BOYUTLU KOMPLEKS KAFES (LATİS) İÇİNDE TEMEL SOLİTONLAR

Yüksek Lisans Tezi

Mahmut BAĞCI

509071006

Tezin Enstitüye Verildiği Tarih : 06 Mayıs 2010

Tezin Savunulduğu Tarih : 11 Haziran 2010

Danışmanı : Yrd. Doç. Dr. İlkey BAKIRTAŞ AKAR (İTÜ)

Eş Danışmanı : Doç. Dr. Nalan ANTAR (İTÜ)

Diğer Jüri Üyeleri : Prof. Dr. Emanullah HİZEL (İTÜ)

: Prof. Dr. Hilmi DEMİRAY (Işık Üniv.)

: Yrd. Doç. Dr. Güler GAYGUSUZUOĞLU (NKÜ)

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## **FOREWORD**

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# FUNDAMENTAL SOLITONS IN COMPLEX TWO DIMENSIONAL LATTICES

## SUMMARY

Nonlinear waves problems are of wide physical and mathematical interest and arise in a variety of scientific fields such as nonlinear optics, fluid dynamics, plasma physics, etc. The solutions of the governing nonlinear waves equations often exhibit important phenomena, such as stable localized waves (e.g., solitons) or self-similar structures and wave collapse (i.e., blowup).

In this study, band gap structures and nonlinear stability properties of localized optical solitons arise in the solution of the nonlinear Schrödinger (NLS) equation with an external potential (optical lattice). As the external potential, periodic (crystal) lattices, dislocated lattices and the non-periodic (quasicrystal) lattices are investigated in separate cases. Band gap structures of solitons are investigated by analytic approximations and numerical methods. Stability properties of solitons are investigated numerically and the results are shown to be in good agreement with Vakhitov-Kolokolov (VK) stability criterion.

In Section 1, the historical background of the nonlinear wave propagation and the solutions to nonlinear Schrödinger (NLS) equation are given. Basic definitions about lattices and the model that governs nonlinear wave propagation through a nonlinear optical lattice are also given in the same section. General information about periodic (crystal), non-periodic (quasicrystal or Penrose type) and defected potentials that have been used in the study is also given. The governing equation for the physical model that has been used in this study, is defined as the nonlinear Schrödinger (NLS) equation with an external potential in Eq.(1)

$$i\frac{\partial U}{\partial z} + \left( \frac{\partial^2}{\partial X^2} + \frac{\partial^2}{\partial Y^2} \right) U - VU + |U|^2 U = 0. \quad (1)$$

In Section 2, the solution of the model is given by perturbation and spectral methods when the external potential is taken to be periodic with the formulation below

$$V = V_0(\cos^2 X + \cos^2 Y). \quad (2)$$

Firstly, NLS equation with the external potential given in Eq.(2) (i.e., the model) is converted to 1D Mathieu's equation by separating the external potential in two parts as follows

$$\frac{\partial^2 u_1(X)}{\partial X^2} - V_0 \cos^2 X u_1(X) = -\mu_1 u_1(X). \quad (3)$$

After that, 1D Mathieu's equation is solved by perturbation and spectral methods respectively. Finally, results obtained by the use of perturbation and the spectral methods are compared.

In Section 3, solution of the NLS with an external potential is obtained by using a numerical algorithm named as the Spectral Renormalization Method (SR). At the beginning, the numerical method (SR) is explained and then, solution of the NLS equation with the periodic external potential (Eq.(2)) is computed by using SR Method. In a certain parameter regime for the potential depth ( $V_0$ ) and the eigenvalue ( $\mu$ ), the first nonlinear band gap is also obtained. In order to see the reliability of the numerical algorithm (SR), the first nonlinear band gap formation that is obtained by SR Method is compared with the ones that are obtained by the use of the perturbation and the spectral methods previously in Section 2.

Finally, the nonlinear stability properties of localized solitons in the first nonlinear band gap are investigated. Vakhitov-Kolokolov (VK) stability criterion is explained, which interprets the relation between the soliton power ( $P = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |U(x, y)|^2 dx dy$ ) and the nonlinear soliton stability. In order to use the VK stability criterion, the power versus the eigenvalue graph is depicted. The nonlinear stability is also investigated by directly computing the (Eq.(1)) over a long distance finite difference method was used on derivatives ( $u_{xx}$  and  $u_{yy}$ ) and fourth-order Runge Kutta method to advance in  $z$ . In order to check the drift instability, the location of the center of mass is also examined in long distance. It is observed that the nonlinear stability properties of the band gap solitons obtained by direct numerical simulations of the NLS equation with an external potential are consistent with the VK stability criterion.

In the Section 4, edge dislocated lattice solitons are investigated. Firstly, an optical lattice with an edge dislocation is formulated, contour image and diagonal cross section of such a lattice are plotted.

The nonlinear first band gap structure is obtained by SR method and soliton power ( $P$ ) versus the eigenvalue ( $\mu$ ) graph is plotted in order to investigate the nonlinear stability properties of the band gap solitons by using the VK stability criterion. The nonlinear stability is also investigated by directly computing the (Eq.(1)) over a long distance. The location of the center of mass is also examined.

In Section 5, as the external potential, quasicrystal (non-periodic) type lattices are considered. In particular, these lattices are called Penrose lattices namely, Penrose-5 and Penrose-7. Firstly, Penrose-5 and Penrose-7 potentials are formulated and contour images and diagonal cross sections of these lattices are plotted.

After that, first nonlinear band gap structures of Penrose-5 and Penrose-7 potentials are investigated by using the SR method. In order to analyze the band gap soliton properties, the mode profiles of Penrose-5 and Penrose-7 lattice solitons are plotted in same parameter regime,  $V_0$ ,  $\mu$ . By examining the graphs, the band gap solitons of Penrose-5 and Penrose-7 lattices are compared in the sense of amplitude and shape.

Soliton properties of Penrose-7 lattice are investigated when solitons are centered at lattice maxima and minima and the differences at the mode profiles are shown on figures.

Finally, the nonlinear stability properties of Penrose-7 solitons are investigated. In order to use VK stability criterion, a detailed soliton power analysis of Penrose-5 and Penrose-7 lattices are given. The nonlinear stability properties of Penrose-7 solitons are investigated by directly computing the (Eq.(1)) over a long distance when solitons are centered at lattice maxima and minima in separate cases. The location of the center of mass is also examined for each case.

In conclusion part, results of the study are expressed and possible future studies are discussed.



# İKİ BOYUTLU KOMPLEKS KAFES (LATİS) İÇİNDE TEMEL SOLİTONLAR

## ÖZET

Fizik ve Matematikte, lineer olmayan dalgalar, optik, akışkanlar dinamiği veya plazma fiziği gibi alanlarda geniş olarak yer tutan önemli bir konudur. Lineer olmayan dalga problemlerinin çözümleri farklı özellikler taşır. Bu çözümler stabil lokalize dalgalar (soliton) veya sönümlü dalgalar olabilir.

Bu çalışmada, kompleks kafes yapıları (latis) içindeki lokalize dalgaların (solitonların) birinci lineer olmayan bant yapıları ve lineer olmayan stabiliteyi incelenmiştir. Çalışmada, periyodik (kristal) potansiyel, düzensizlik (defect) içeren potansiyel ve periyodik olmayan (yarı kristal) potansiyel içindeki solitonlar ayrı ayrı incelenmiştir. Solitonların bant yapıları analitik ve sayısal yaklaşımlarla incelenmiştir. Solitonların stabilitesi incelenirken çeşitli sayısal algoritmalar kullanılmış ve bulunan sonuçların Vakhitov-Kolokolov (VK) stabilite kriterlerine uygunluğu gösterilmiştir.

Çalışmanın birinci bölümünde, öncelikle lineer olmayan dalgalarla ilgili tarihi bilgilere ve Nonlinear Schrödinger (NLS) denklemi çözümlerine yer verilmiş. Kristal, yarı kristal (kuazi kristal) ve düzensizlik içeren potansiyellerle ilgili gerekli bilgiler ve tanımlar verilmiştir. Sonrasında, fiziksel sistemin yönetici modeli, bir potansiyel içeren NLS denklemi olarak verilmiştir. Bu denklem

$$i\frac{\partial U}{\partial z} + \left( \frac{\partial^2}{\partial X^2} + \frac{\partial^2}{\partial Y^2} \right) U - VU + |U|^2 U = 0 \quad (1)$$

şeklindedir.

Çalışmanın ikinci bölümünde latıs yapısını ifade eden potansiyel,

$$V = V_0(\cos^2 X + \cos^2 Y) \quad (2)$$

şeklinde periyodik olarak alınmış, bu potansiyelin ayrılabilir tipte olmasından faydalanılarak, öncelikle NLS denklemi tek boyutlu Mathieu denklemine dönüştürülmüştür. Tek boyutlu Mathieu denklemi

$$\frac{\partial^2 u_1(X)}{\partial X^2} - V_0 \cos^2 X u_1(X) = -\mu_1 u_1(X) \quad (3)$$

şeklindedir. Sonrasında tek boyutlu Mathieu denklemi pertürbasyon yaklaşımı ve spektral yöntemlerle ayrı ayrı çözülmüştür. İkinci bölümün sonunda pertürbasyon ve spektral yöntemlerle elde edilen sonuçlar karşılaştırılmıştır.

Çalışmanın üçüncü bölümünde, Spektral Renormalizasyon (SR) yöntemi kullanılarak, bir potansiyel içeren NLS denkleminin lokalize çözümleri elde

edilmiştir. İlk olarak sayısal algoritmanın nasıl oluşturulduğu açıklanmış, sonrasında (2) denkleminde tanımlanan potansiyelin varlığında NLS denklemi SR yöntemiyle çözülmüştür. SR yöntemi kullanılarak (2) denkleminde verilen potansiyelin varlığında NLS denkleminin lineer olmayan bant yapısı incelenmiştir. SR yöntemiyle elde edilen sonuçlar, ikinci bölümde ele alınan pertürbasyon ve spektral yöntemin sonuçlarıyla karşılaştırılarak, yöntemin güvenilirliği test edilmiştir.

Üçüncü bölümün son kısmında, lokalize dalgaların (solitonların) lineer olmayan stabilitesi incelenmiştir. İlk olarak soliton kuvvetiyle ( $P = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |U(x, y)|^2 dx dy$ ) solitonun lineer olmayan stabilitesi arasındaki ilişkiyi ortaya koyan Vakhitov-Kolokolov (VK) stabilite kriterleri açıklanmıştır. VK stabilite kriterlerinin kullanılabilmesi için birinci lineer olmayan bant aralığındaki solitonların kuvvet-özdeğer grafiği çizilmiştir. Sonrasında, NLS denklemindeki türevler ( $u_{xx}$  and  $u_{yy}$ ) sonlu farklar yöntemiyle doğrudan çözülmüş ve solitonlar dördüncü dereceden Runge-Kutta yöntemiyle ilerletilerek soliton stabilitesinin sayısal analizi yapılmıştır. Kayma (lokalize olmuş solitonun, ilerlerken farklı bir latis hücresine kayması) nedeniyle oluşan stabilitedeki bozulmaları incelemek amacıyla, soliton ilerlerken ağırlık merkezinin yerindeki değişimler görüntülenmiştir. Üçüncü bölümün sonunda, sayısal yöntemle elde edilen stabilite analizlerinin VK stabilite kriterlerine uygunluğu gösterilmiştir.

Çalışmanın dördüncü bölümünde, düzensizlik içeren potansiyel içindeki solitonların özellikleri incelenmiştir. İlk olarak, düzensizlik içeren potansiyel tanımlanmış ve bu potansiyelin diyagonal kesitiyle üstten görünüşü çizilmiştir. Sonrasında, bu potansiyelin bant yapısı SR metoduyla incelenmiş ve VK stabilite kriterlerini kullanabilmek için birinci lineer olmayan bant içerisindeki solitonların kuvvet-özdeğer grafiği çizilmiştir. Ele alınan potansiyel içindeki solitonların lineer olmayan stabilite analizi ve ilerleyen solitonların ağırlık merkezi üçüncü bölümde verilen sayısal yöntemle incelenmiştir. Dördüncü bölümün sonunda, sayısal yöntemle elde edilen stabilite analizlerinin VK stabilite kriterlerine uygunluğu gösterilmiştir.

Çalışmanın son bölümünde, periyodik olmayan (yarı kristal) potansiyel içindeki solitonlar incelenmiştir. Periyodik olmayan potansiyeller Penrose-5 ve Penrose-7 potansiyelleri olarak alınmıştır. İlk olarak, Penrose-5 ve Penrose-7 potansiyelleri tanımlanmış ve bu potansiyellerin diyagonal kesitleriyle üstten görünüşleri çizilmiştir. Sonrasında, bu potansiyellerin bant yapıları SR yöntemiyle incelenmiştir. Penrose-5 ve Penrose-7 solitonlarının özelliklerini incelemek amacıyla  $V_0$  ve  $\mu$  parametreleri sabitlenerek  $x$  ve  $y$  eksenlerine göre solitonların kesitleri (mod profilleri) çizilmiştir. Bu kesitler kullanılarak Penrose-5 ve Penrose-7 solitonlarının genlikleri ve biçimleri karşılaştırılmıştır.

İkinci olarak, soliton çözümlerinin potansiyelin maksimum veya minimumunda elde edildiği durumlar için, Penrose-7 solitonlarının özellikleri incelenmiş ve bu solitonların mod profillerindeki farklılıklar gösterilmiştir.

Son olarak, Penrose-7 solitonlarının lineer olmayan stabilitesi, solitonun latisin maksimumunda veya minimumunda elde edilme durumuna göre iki farklı halde

incelenmiştir. VK stabilite kriterlerini kullanabilmek için Penrose-7 potansiyeli kullanıldığında elde edilen birinci lineer olmayan bant içerisindeki solitonlarının kuvvet-özdeğer grafiği çizilmiştir. Penrose-7 solitonlarının lineer olmayan stabilite analizi üçüncü bölümde verilen sayısal yöntemle incelenmiş ve bu stabilite analizinin VK stabilite kriterlerine uygunluğu gösterilmiştir.

Çalışmanın sonuç bölümünde, elde edilen sonuçlar ayrıntılı olarak açıklanmış ve sonraki çalışmalar için öneriler dile getirilmiştir.



## 1. INTRODUCTION

Nonlinear waves problems are of wide physical and mathematical interest and arise in a variety of scientific fields such as nonlinear optics, fluid dynamics, plasma physics, etc. The solutions of the governing nonlinear wave equations often exhibit important phenomena, such as stable localized waves (e.g., solitons) or self-similar structures and wave collapse (i.e., blowup) where the solution tends to infinity in finite time or finite propagation distance [1].

It has been proved that a large number of the nonlinear evolution equations have the soliton solutions by the numerical calculations and the theoretical analysis. Solitary waves have the striking property that they can keep the shape of wave stable after interaction. This is similar to the colliding property of particles. In 1965, Kruskal and Zabusky discovered that the pulselike solitary wave solution to the Korteweg-de Vries (KdV) equation had a property which was previously unknown: namely, that this solution interacted elastically with another such solution. They termed these solutions as *solitons*. The next principally important step in this direction was made by Zakharov and Shabat, who had demonstrated that the integrability is not a peculiarity specific to a single (KdV) equation, but is also featured by another equation which finds very important applications in physics, the nonlinear Schrödinger (NLS) equation ([2]-[5]).

The soliton theory is an interdisciplinary topic, where many ideas from mathematical physics, nonlinear optics, solid state physics and quantum theory are mutually benefited from each other. Solitons are localized nonlinear waves and their properties have provided a deep and fundamental understanding of complex nonlinear systems. In recent years there has been considerable interest in the study of solitons in systems with periodic potentials or lattices, in particular those that can be generated in nonlinear optical materials [6, 7].

In this study, we aim to investigate band gap structures and stability properties of localized solitons in various lattices (e.g., periodic, nonperiodic or defected lattices). Band gap structures and nonlinear stability properties of lattice solitons are investigated by analytical approximations and numerical methods. The numerical results that are obtained for nonlinear stability are supported with Vakhitov and Kolokolov (VK) stability criterion. In the study, solitons in periodic (crystal) lattices, in dislocated lattices and in non-periodic (quasicrystal) lattices are investigated in separate cases.

In this section, before investigating solitons in lattices, it is essential to give some definitions about lattices.

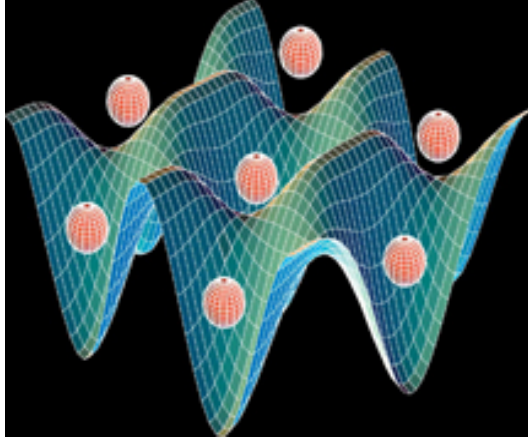
### 1.1. Definitions

A crystal or crystalline solid is a solid material, whose constituent atoms, molecules, or ions are arranged in an orderly repeating pattern extending in all three spatial dimensions [8]. Crystals have periodicity and translational symmetry that introduces interesting selection rules which can be applied usefully in the interpretation of experiments as well as theoretical modeling [9, 10].

In practice, the atomic arrangement in condensed matter is never perfect, but this aspect is neglected in crystallography and, crystals are described by reference to perfect infinite arrays of geometrical points called *lattices*. In a lattice, every point has identical surroundings; all lattice points are equivalent and the crystal lattice therefore exhibits perfect translational symmetry [10].

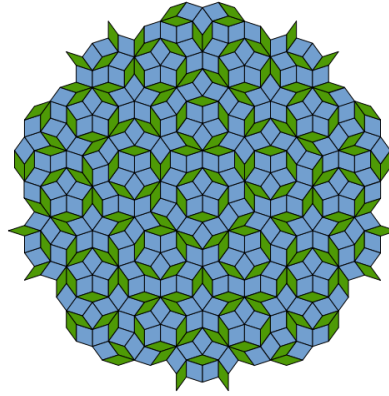
An optical lattice is formed by the interference of counter-propagating laser beams, creating a spatially periodic polarization pattern [11], simulation of an optical lattice is given in Fig.(1.1).

Crystals have periodicity and translational symmetry, but there exist a certain class of structures called quasicrystals that do not possess periodicity nor translational symmetry, but have long-range order and rotational symmetry [9, 12].



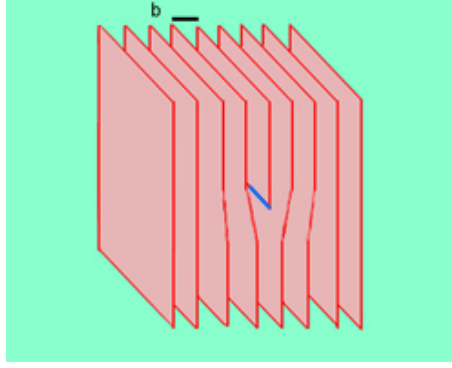
**Figure 1.1:** Simulation of an optical lattice potential.

A Penrose tiling is a nonperiodic (i.e., it lacks any translational symmetry) tiling generated by an aperiodic set of prototiles, it is a type of quasicrystal [13], simulation of a Penrose tiling is given in Fig.(1.2). Penrose lattices exhibit rotational symmetry which is inhibited in periodic crystals [9].



**Figure 1.2:** Simulation of a Penrose tiling.

Dislocations are linear defects around which some of the atoms of the crystal lattice are misaligned. Edge dislocations are caused by the termination of a plane of atoms in the middle of a crystal. In such a case, the adjacent planes are not straight, but instead, bend around the edge of the terminating plane so that the crystal structure is perfectly ordered on either side [14]. Schematic diagram of an edge dislocation is given in Fig.(1.3).



**Figure 1.3:** Schematic diagram of an edge dislocation, "b" describes the magnitude and direction of distortion to the lattice and blue line shows dislocation.

## 1.2. Lattice Solitons

In periodic lattices, solitons can form when their propagation constant, or eigenvalue, lies within certain regions, often called *gaps*. However, the external potential of complex systems can be much more general and physically richer than a periodic lattice. For example, atomic crystals can possess various irregularities, such as defects, and edge dislocations, as well as quasicrystal structures, which have long-range oriental order but no translational symmetry [15, 16].

In general, when the lattice's periodicity is slightly perturbed, the band gap structure and soliton properties become slightly perturbed as well, but otherwise solitons are expected to exist in much the same way as in the perfectly periodic case [17, 18]. Without the lattice potential, solitons would suffer collapse under small perturbations [19]. In this study we investigate solitons which form in optically generated lattices.

## 1.3. Model

The equation that governs the evolution of a laser beam propagating in a self-focusing, nonlinear, inhomogeneous Kerr medium (nonlinear Schrödinger (NLS) equation) with an external potential is taken as model in this study. Before explaining the model, it is essential to give some information about the NLS equation.

### 1.3.1. The NLS equation

(1+1)D Nonlinear Schrödinger (NLS) equation

$$iU_z + \lambda U_{xx} + \gamma |U|^2 U = 0 \quad (1.1)$$

arises in many physical problems, including nonlinear water waves and ocean waves, waves in plasmas, propagation of heat pulses in a solid, self trapping phenomena in nonlinear optics, nonlinear waves in a fluid filled elastic (viscoelastic) tube and various nonlinear instability phenomena in fluids and plasmas. This equation can also be used to investigate instability phenomena in many other physical systems.

In isotropic (Kerr) media, where the nonlinear response of the material depends cubically on the applied field, the dynamics of a quasi-monochromatic optical pulse is governed by the (2+1)D NLS equation

$$iU_z + \lambda(U_{xx} + U_{yy}) + \gamma |U|^2 U = 0 \quad (1.2)$$

In this model,  $U(x, y, z)$  is the amplitude of the envelope of the optical beam,  $z$  is the distance in the direction of propagation, and  $x$  and  $y$  are transverse spatial coordinates.

(1+1)D version of the NLS Eq.(1.1), is an integrable model and possesses both single soliton and multisoliton solutions [20]. On the contrary, the higher dimensional NLS models are no longer integrable. However they possess stationary solutions, which are unstable on propagation. Maybe the most fascinating issue related to the higher dimensional NLS is that, for a wide range of initial conditions, the system evolution shows collapse [21]. Wave collapse occurs where the solution tends to infinity in finite time (distance). Collapse was theoretically predicted for the (2+1)D NLS equation back in the 1960s ([19]).

It is known that there exist solutions of Eq.(1.2) which have a singularity in finite time and extremely sensitive to the addition of small perturbations to the equation and there has been much interest in the determination of the structure of this singularity [22, 23].

Therefore, an important challenge in nonlinear science is to find out mechanisms arresting wave collapse in NLS models.

One way of arresting collapse is adding a defocusing term to the classical (2+1)D NLS Eq.(1.2). Then the model equation becomes

$$U_z + \lambda(U_{xx} + U_{yy}) + \gamma|U|^2U - V(x, y)U = 0 \quad (1.3)$$

here the function  $V(x, y)$  acts as a defocusing mechanism.

Eq.(1.3) doesn't belong to a class of integrable nonlinear evolution equations even in (1+1)D. Thus, no linear techniques are available for solving this equation. Numerical solution to Eq.(1.3) is available by Fourier iteration methods ([24, 25]).

For nonlinear optic problems,  $V(x, y)$  appears in Eq.(1.3) is an optical potential or lattice which serves as an inhomogeneous environment for the propagating beam. Such an optical lattice can be created by interfering two laser beams or instead real crystals might be used.

Photonic crystals are periodic optical nanostructures that are designed to affect the motion of photons in a similar way that periodicity of a semiconductor crystal affects the motion of electrons. For example, opal is a well known photonic crystal. Photonic crystals occur in nature and in various forms have been studied scientifically for the last century.

Real crystals feature defects or irregularities in the ideal arrangements described above and it is these defects that critically determine many of the electrical and mechanical properties of real materials.

One dimensional photonic crystals are used in the form of thin-film optics with applications ranging from low and high reflection coatings on lenses and mirrors to colour changing paints and inks. Higher dimensional photonic crystals are of great interest for both fundamental and applied research, and the two dimensional ones are beginning to find commercial applications.

The first commercial products involving two-dimensionally periodic photonic crystals are already available in the form of photonic-crystal fibers, which use

a microscale structure to confine light with radically different characteristics compared to conventional optical fiber for applications in nonlinear devices and guiding exotic wavelengths.

Photonic crystals are composed of periodic dielectric or metallo-dielectric nanostructures that affect the propagation of electromagnetic waves (EM) in the same way as the periodic potential in a semiconductor crystal affects the electron motion by defining allowed and forbidden electronic energy bands. Photons, (behaving as waves) propagate through this structure - or not - depending on their wavelength. Wavelengths of light that are allowed to travel are known as modes, and groups of allowed modes form bands. Disallowed bands of wavelengths are called photonic band gaps. Localized solutions to Eq.(1.3) must lie inside a band gap, otherwise, the system will be in a periodic extended state (Bloch state) and nonlinearity cannot cause the wave function to localize.

In this thesis, considering various periodic and nonperiodic (quasicrystals and defective type) optical potentials (lattices), existing numerical solution techniques applied to a form of Eq.(1.3). In the study, the band gap boundaries and nonlinear stability properties are also analyzed in detail.

### 1.3.2. The NLS equation with an external potential

In this study, the NLS equation with an external potential will be considered as the governing equation by setting  $\gamma = \lambda = 1$  in Eq.(1.3)

$$i\frac{\partial U}{\partial z} + \left( \frac{\partial^2}{\partial X^2} + \frac{\partial^2}{\partial Y^2} \right) U - VU + |U|^2 U = 0. \quad (1.4)$$

In optics,  $U(x, y, z)$  corresponds to the complex-valued, slowly varying amplitude of the electric field in the  $xy$  plane propagating in the  $z$  direction and the cubic term in  $U$  originates from the nonlinear change of the refractive index, and the potential  $V(x, y)$  corresponds to a modulation of the linear refractive index of the medium. In this study, we consider potentials that can be written as the intensity of a sum of  $N$  phase-modulated plane waves which given as

$$V(x, y) = \frac{V_0}{N^2} \left| \sum_{n=0}^{N-1} e^{i\vec{k}_n \cdot \vec{x} + i\theta_n(x, y)} \right|^2 \quad (1.5)$$

where  $V_0 > 0$  is constant and corresponds to the peak depth of the potential, i.e.,  $V_0 = \max_{x,y} V(x, y)$ ,  $\vec{x} = (x, y)$ ,  $\vec{k}_n$  is a wave vector,  $\theta_n(x, y)$  is a phase function that characterizes edge irregularities [7].

NLS equation conserves the power quantity as

$$P = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |U(x, y)|^2 dx dy, \quad (\mathbf{1.6})$$

power plays an important role in determining the stability properties of localized modes (solitons).

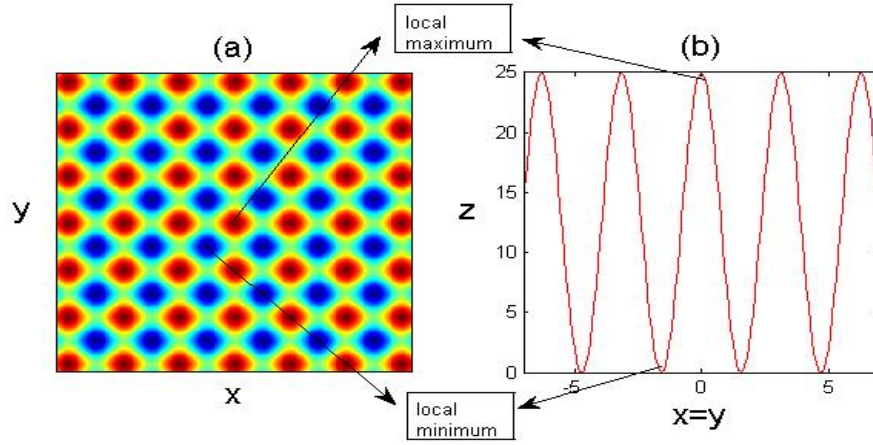
## 2. LINEAR SPECTRUM OF THE NLS EQUATION

### 2.1. Converting NLS Equation to Mathieu's Equation

In NLS equation, external potential  $V(x, y)$  represents the optically induced photonic lattice which serves as an inhomogeneous environment for the propagating beam. If we assume that both plane waves forming the lattice are mutually incoherent, the potential  $V$  takes the form [26],

$$V = V_0(\cos^2 X + \cos^2 Y) \quad (2.1)$$

here  $V_0$  is the depth of the potential. The contour image and the diagonal cross section of the potential are shown in Fig.(2.1).



**Figure 2.1:** (a) Contour image, (b) diagonal cross-section of the  $V = V_0(\cos^2 X + \cos^2 Y)$  potential with  $V_0 = 12.5$ .

As can be seen from Fig.(2.1), this potential is periodic and it has local minima and maxima.

In optical problems,  $V_0$  is generally negative. However, the sign of this potential depth can be converted from negative to positive by using some transformations in phase. Thus we take  $V_0 > 0$  in this study without loss of generality.

In order to find a stationary solution of Eq.(1.4), we should look solution of the form

$$U(X, Y, z) = \exp(-i\mu z)u(X, Y) \quad (2.2)$$

where  $u(X, Y)$  is a real valued function and  $\mu$  is the propagation constant (eigenvalue). Substituting Eq.(2.2) into Eq.(1.4), we get

$$\frac{\partial^2 u}{\partial X^2} + \frac{\partial^2 u}{\partial Y^2} - Vu + |u|^2 u = -\mu u \quad (2.3)$$

substituting Eq.(2.1) into Eq.(2.3) and neglecting the nonlinear term ( $|u|^2 u$ ), we get

$$\left( \frac{\partial^2}{\partial X^2} + \frac{\partial^2}{\partial Y^2} \right) u - V_0(\cos^2 X + \cos^2 Y)u = -\mu u. \quad (2.4)$$

The optical potential  $V(X, Y)$  is separable, which enables us to reduce the dimensionality of the system. By setting

$$u(X, Y) = u_1(X)u_2(Y),$$

we can split Eq.(2.4) into two 1D equations

$$\frac{\partial^2 u_1(X)}{\partial X^2} - V_0 \cos^2 X u_1(X) = -\mu_1 u_1(X) \quad (2.5)$$

$$\frac{\partial^2 u_2(Y)}{\partial Y^2} - V_0 \cos^2 Y u_2(Y) = -\mu_2 u_2(Y) \quad (2.6)$$

where

$$\mu = \mu_1 + \mu_2.$$

Eq.(2.5) and Eq.(2.6) are 1D Mathieu's equations whose properties are well known. We can write (2.5) as

$$\frac{\partial^2 u_1(X)}{\partial X^2} - \frac{V_0}{2}[1 + \cos(2X)]u_1(X) = -\mu_1 u_1(X). \quad (2.7)$$

## 2.2. General Solution of Mathieu's Equation by Perturbation Methods

General Solution of Mathieu's Equation by the use of Perturbation Methods is explained in detail by Grimshaw ([27]) as follows.

General Mathieu's equation is of the form

$$u'' + (\delta + \varepsilon \cos 2t)u = 0. \quad (2.8)$$

When  $\delta \approx n^2$ , where  $n = 0, 1, 2, \dots$ , and  $\varepsilon$  is small, the solutions of Eq.(2.8) will take the form

$$u(t) = e^{\mu t} q(t) \quad (2.9)$$

where  $q(t)$  is a periodic function of period  $\pi$  when  $n$  is an even integer and function of period  $2\pi$  when  $n$  is an odd integer. By substituting (2.9) into Eq.(2.8), we find that

$$q'' + 2\mu q' + \mu^2 q + (\delta + \varepsilon \cos 2t)q = 0. \quad (2.10)$$

The solutions of Eq.(2.10) have the required period  $\pi$ , or  $2\pi$ , and  $\mu$  will be determined as a function of  $\delta$  and  $\varepsilon$ .

Because the analyticity of the solutions of Eq.(2.10) with respect to the parameter  $\varepsilon$  can be assumed, we are allowed to seek solutions of Eq.(2.10) in the form

$$q = q_0(t) + \varepsilon q_1(t) + \varepsilon^2 q_2(t) + \dots, \quad \mu = \varepsilon \mu_1 + \varepsilon^2 \mu_2 + \dots, \quad \delta = n^2 + \varepsilon \delta_1 + \varepsilon^2 \delta_2 + \dots \quad (2.11)$$

Note that the leading order term for  $\mu$  (i.e.,  $\mu_0$ ) is zero. When the expansions (2.11) are substituted into the left-hand side of Eq.(2.10), the result will be a power series in  $\varepsilon$  and, clearly, since this power series must vanish, the coefficient of each power of  $\varepsilon$  must be zero. Applying this process to the coefficients of  $\varepsilon^m$  ( $m = 0, 1, 2, \dots$ ), we find that

$$\begin{aligned} q_0'' + n^2 q_0 &= 0, \\ q_1'' + n^2 q_1 &= f_1, \\ q_2'' + n^2 q_2 &= f_2, \\ &\vdots \\ q_m'' + n^2 q_m &= f_m, \end{aligned} \quad (2.12)$$

where the terms on the right-hand side of Eq.(2.12) are given by

$$\begin{aligned} f_1 &= -2\mu_1 q'_0 - (\delta_1 + \cos 2t)q_0, \\ f_2 &= -2\mu_1 q'_1 - (\delta_1 + \cos 2t)q_1 - 2\mu_2 q'_0 - \mu_1^2 q_0 - \delta_2 q_0. \end{aligned} \quad (2.13)$$

Eq.(2.12) and Eq.(2.13) show that  $f_1$  depend only on  $q_0$ ,  $f_2$  depends only on  $q_0$  and  $q_1$ , and it can be shown that  $f_m$  depends only on  $q_0, q_1, \dots, q_{m-1}$ . We will solve the problem for  $n = 0$ , from the above explanations we know that each term  $q_0, q_1, \dots, q_m$  is periodic, with period  $\pi$  (because  $n = 0$  is an even integer). By using this periodicity and following this process we will find the coefficients  $\mu_1, \mu_2, \dots, \mu_m$  in terms of  $\delta_1, \delta_2, \dots, \delta_m$ . Using Eq.(2.12),

$$q''_0 = 0,$$

so that,

$$q_0 = A_0 + B_0 t,$$

in order to  $q_0$  be periodic  $B_0$  must be zero, then

$$q_0 = A_0.$$

Using  $q_0$  in Eq.(2.13), we get

$$f_1 = -(\delta_1 + \cos 2t)A_0.$$

Next, by using  $q'_1 = f_1$ , we get

$$q_1 = A_1 + B_1 t - \frac{1}{2}\delta_1 A_0 t^2 + \frac{1}{4}A_0 \cos 2t,$$

in order to  $q_1$  be periodic  $\delta_1$  and  $B_1$  must be zero, then

$$q_1 = A_1 + \frac{1}{4}A_0 \cos 2t.$$

To find  $\mu_1$ , we must proceed to the next order. By using  $q_1$  in Eq.(2.13), we get

$$f_2 = -(\mu_1^2 + \delta_2 + \frac{1}{8})A_0 + \mu_1 A_0 \sin 2t - \frac{1}{8}A_0 \cos 4t - A_1 \cos 2t.$$

In this case the resonant term is the constant term and, removing its coefficient, we get

$$\mu_1^2 + \delta_2 + \frac{1}{8} = 0,$$

recalling the definitions in Eq.(2.11), this corresponds to

$$\mu^2 + \delta + \frac{1}{8}\varepsilon^2 = 0, \quad (2.14)$$

where the error term is order of  $\varepsilon^3$ . Thus, for  $\delta + \frac{1}{8}\varepsilon^2 > 0$ , the two solutions for  $\mu$  are both pure imaginary and correspond to stable behaviour. On the other hand, for  $\delta + \frac{1}{8}\varepsilon^2 < 0$ , the two solutions for  $\mu$  are both real-valued with  $\mu_1 > 0$  and  $\mu_2 = -\mu_1 < 0$ ; this corresponds to unstable behaviour. The stability boundary determined by taking  $\mu = 0$ ,

$$\delta = -\frac{1}{8}\varepsilon^2. \quad (2.15)$$

Rewriting Eq.(2.7) as follows

$$\frac{\partial^2 u_1(X)}{\partial X^2} + [\mu_1 - \frac{V_0}{2} - \frac{V_0}{2} \cos(2X)]u_1(X) = 0$$

then comparing this equation with general Mathieu's Eq.(2.8), we see that

$$\delta = \mu_1 - \frac{V_0}{2} \quad \text{and} \quad \varepsilon = -\frac{V_0}{2}$$

substituting  $\delta$  and  $\varepsilon$  in Eq.(2.15), we get

$$\mu_1 - \frac{V_0}{2} = -\frac{1}{8}\left(\frac{V_0}{2}\right)^2$$

and finally we obtain

$$\mu_1 = \frac{1}{2}V_0 - \frac{1}{32}V_0^2. \quad (2.16)$$

This result is in good agreement with Musslimani and Yang ([26]).

### 2.3. General Solution of Mathieu's Equation by Spectral Methods

General Mathieu's equation can be written

$$-u'' + 2q \cos(2x)u = \lambda u, \quad (2.17)$$

where  $q$  is a real parameter, and we look for periodic solutions on  $[-\pi, \pi]$ . For  $q = 0$  we have the linear pendulum equation, with eigenvalues  $\pi^2 n^2 / 4$  for  $n = 1, 2, 3, \dots$

To compute eigenvalues of the Mathieu equation by a spectral method, we should discretize Eq.(2.17) in a routine fashion. Transforming the equation from the domain  $[-\pi, \pi]$  to  $[0, 2\pi]$  leaves the eigenvalues unaltered, so the discretization

takes the form

$$L_N = -D_N^{(2)} + 2q \text{diag}(\cos(2x_1), \dots, \cos(2x_N)),$$

where  $D_N^{(2)}$  is the second-order Fourier differentiation matrix. The computation is straightforward in *program1*, and with  $N = 42$ , we get about 13 digits of accuracy [28].

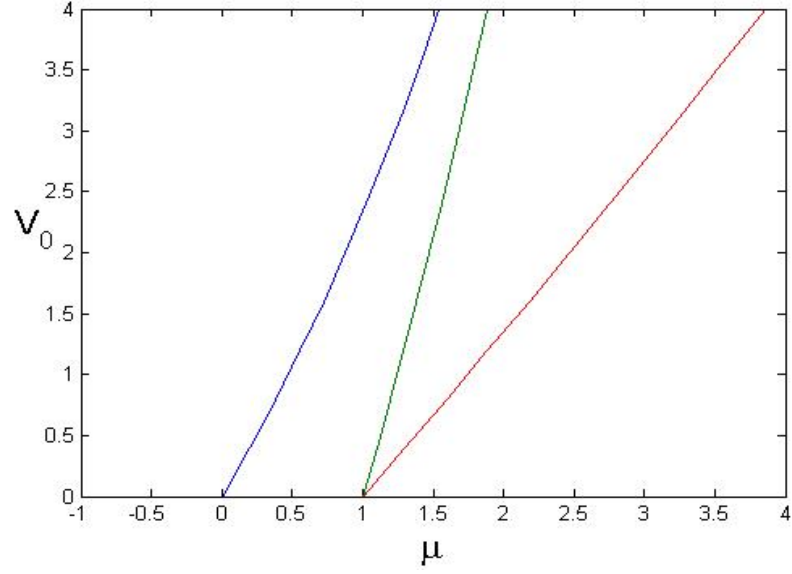
Program1

```
%Eigenvalues of Mathieu operator -u_xx + 2qcos(2x)u
clc; clear all;
N = 42; h = 2*pi/N; x = h*(1:N); L=10;
column = [-pi^2/(3*h^2)-1/6 ...
          -.5*(-1).^(1:N-1)./sin(h*(1:N-1)/2).^2];
D2 = toeplitz(column);
qq = 0:.2:5; data = [];
for q = qq;
    e = sort(eig(-D2 + 2*q*diag(cos(x).^2)))';
    data = [data; e(1:11)];
end
V0=2*qq;
plot(data, V0)
xlabel ('\mu', 'FontSize', 16)
ylabel ('V_0', 'FontSize', 16)
axis([-1 4 0 4])
```

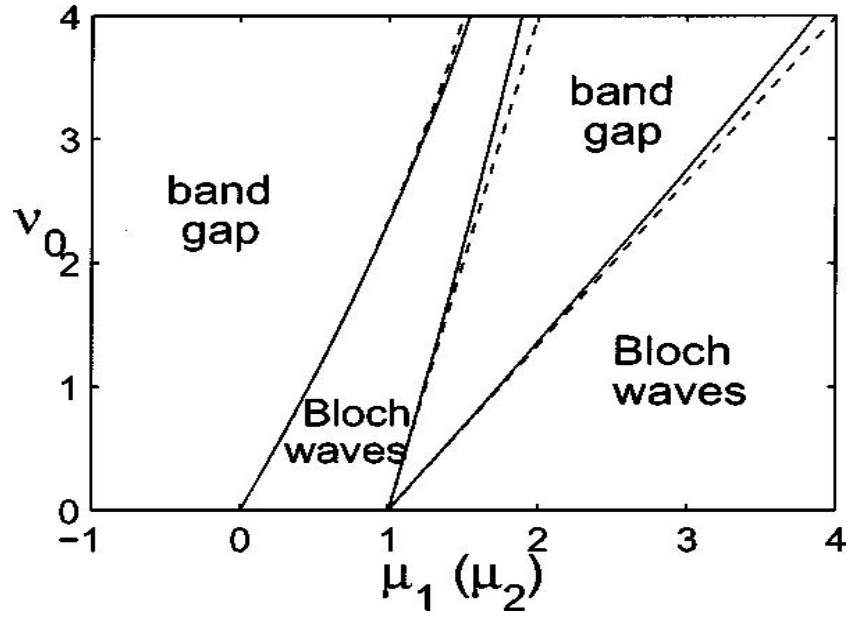
**Figure 2.2:** Numerical algorithm of Mathieu's equation solution with  $V = V_0(\cos^2 X + \cos^2 Y)$  potential by spectral methods.

As can be seen from Fig.(2.2), source code is programmed to solve 1D Mathieu's equation (i.e., Eq. (2.7)) with  $V = V_0(\cos^2 X + \cos^2 Y)$  potential. If we run *Program1* in MATLAB, we obtain the first nonlinear band gap information of 1D Mathieu's equation with  $V = V_0(\cos^2 X + \cos^2 Y)$  potential as in Fig.(2.3).

This band gap boundaries are also investigated by Musslimani and Yang ([26]). Their results are shown in Fig.(2.4). In Fig.(2.4), the first and the second nonlinear band gaps are depicted for nonlinear periodic problem. Bloch wave regions are the regions where there is no localized mode (soliton). By comparing Fig.(2.3) and Fig.(2.4), it can be seen that the results that are obtained by the use of the spectral method is in good agreement with Musslimani and Yang ([26]).



**Figure 2.3:** Band gap structure of 1D Mathieu's equation with  $V = V_0(\cos^2 X + \cos^2 Y)$  potential by spectral methods.



**Figure 2.4:** Band gap structure of the linear periodic problem of Eqs.(2.5) and (2.6). Solid curves show the numerically computed band gap boundaries whereas the dashed curves show analytic approximations for the same boundaries.



### 3. FUNDAMENTAL LATTICE SOLITONS AND THEIR STABILITY

In this section, solution to NLS equation with an external potential given in Eq.(1.4) will be obtained by a fixed point spectral computational method (spectral renormalization method) that is developed by Ablowitz and Musslimani ([24]). First we explain the spectral renormalization method and we obtain the soliton solutions to NLS equation, then we investigate nonlinear stability of this solution.

#### 3.1. Spectral Renormalization Method (SR)

A central issue for optical solitons of nonlinear guided waves is how to compute localized, i.e., soliton, solutions, which generally involve solving nonlinear ordinary or partial differential equations or difference equations. A method, introduced by Petviashvili ([25]), for constructing localized solutions of a nonlinear system is based on transforming to Fourier space and determining a convergence factor based upon the degree (homogeneity) of a single nonlinear term (e.g.,  $|U|^2U$ ). In this section, we explain spectral renormalization method scheme with which we can compute localized solutions in nonlinear waveguides.

We seek solution to NLS Eq.(1.4), as

$$U = e^{-i\mu z} f(x, y) \tag{3.1}$$

where  $f(x, y)$  is a real function. Substituting this solution suggestion in Eq.(1.4), we get

$$\mu e^{-i\mu z} f + (f_{xx} + f_{yy})e^{-i\mu z} + |f|^2 f e^{-i\mu z} - V(x, y)e^{-i\mu z} f = 0$$

simplifying this equation we obtain

$$\mu f + f_{xx} + f_{yy} + |f|^2 f - V(x, y)f = 0. \quad (3.2)$$

Applying Fourier Transform to the Eq.(3.2), we obtain the following equation

$$\mu \hat{f} - |\vec{k}|^2 \hat{f} + \mathcal{F}[|f|^2 f - V(x, y)f] = 0 \quad (3.3)$$

where  $\mathcal{F}$  shows the Fourier Transform ( $\mathcal{F}[f] = \hat{f}$ ) and  $\vec{k} = (k_x, k_y)$ ,  $|\vec{k}|^2 = k_x^2 + k_y^2$ . Adding and subtracting  $r \hat{f}$  (where  $r$  is a positive constant) to Eq.(3.3), we get

$$\mu \hat{f} - |\vec{k}|^2 \hat{f} + r \hat{f} - r \hat{f} + \mathcal{F}[|f|^2 f - V(x, y)f] = 0$$

this equation can be written as

$$(r + \mu) \hat{f} - (|\vec{k}|^2 + r) \hat{f} + \mathcal{F}[|f|^2 f - V(x, y)f] = 0$$

then,

$$\hat{f} = \frac{(r + \mu) \hat{f} + \mathcal{F}[|f|^2 f - V(x, y)f]}{|\vec{k}|^2 + r}. \quad (3.4)$$

In order to apply Fourier iteration method, we introduce a new variable

$$\hat{f} = \lambda \hat{w}$$

where  $\lambda$  is a parameter to be determined. Then Eq.(3.4) can be written as

$$\lambda \hat{w} = \frac{(r + \mu) \lambda \hat{w} + \mathcal{F}[|\lambda|^2 |w|^2 \lambda w - V(x, y) \lambda w]}{|\vec{k}|^2 + r}$$

simplifying this equation, we get

$$\hat{w} = \frac{(r + \mu) \hat{w} + |\lambda|^2 \mathcal{F}[|w|^2 w] - \mathcal{F}[V(x, y)w]}{|\vec{k}|^2 + r}. \quad (3.5)$$

In order to iterate Eq.(3.5), we should index and normalize it,

$$\hat{w}_{n+1} = \frac{(r + \mu) \hat{w}_n + |\lambda|^2 \mathcal{F}[|w_n|^2 w_n] - \mathcal{F}[V(x, y)w_n]}{|\vec{k}|^2 + r}$$

after obtaining  $\hat{w}_{n+1}$ , normalizing this indexed equation, we get

$$\begin{aligned} \int \hat{w} \hat{w}^* dk &= \int \frac{(r + \mu) \hat{w} \hat{w}^* + |\lambda|^2 \hat{w}^* \mathcal{F}[|w|^2 w] - \hat{w}^* \mathcal{F}[V(x, y)w]}{|\vec{k}|^2 + r} dk \\ \int |\hat{w}|^2 dk &= (r + \mu) \int \frac{|\hat{w}|^2}{|\vec{k}|^2 + r} dk + |\lambda|^2 \int \frac{\hat{w}^* \mathcal{F}[|w|^2 w]}{|\vec{k}|^2 + r} dk - \int \frac{\hat{w}^* \mathcal{F}[V(x, y)w]}{|\vec{k}|^2 + r} dk \end{aligned}$$

then,

$$|\lambda_n|^2 = \frac{\int |\hat{w}|^2 dk - (r + \mu) \int \frac{|\hat{w}|^2}{|\vec{k}|^2 + r} dk + \int \frac{\hat{w}^* \mathcal{F}[V(x, y)w]}{|\vec{k}|^2 + r} dk}{\int \frac{\hat{w}^* \mathcal{F}[|w|^2 w]}{|\vec{k}|^2 + r} dk}$$

$$|\lambda_n|^2 = \frac{\int (|\vec{k}|^2 - \mu) \frac{|\hat{w}|^2}{|\vec{k}|^2 + r} dk + \int \frac{\hat{w}^* \mathcal{F}[V(x, y)w]}{|\vec{k}|^2 + r} dk}{\int \frac{\hat{w}^* \mathcal{F}[|w|^2 w]}{|\vec{k}|^2 + r} dk}$$

this equation can be written as

$$|\lambda_n|^2 = \frac{\int \left[ (|\vec{k}|^2 - \mu) |\hat{w}|^2 + \hat{w}^* \mathcal{F}[V(x, y)w] \right] \frac{1}{|\vec{k}|^2 + r} dk}{\int \hat{w}^* \mathcal{F}[|w|^2 w] \frac{1}{|\vec{k}|^2 + r} dk} \quad (3.6)$$

where

$$SL = \int \left[ (|\vec{k}|^2 - \mu) |\hat{w}|^2 + \hat{w}^* \mathcal{F}[V(x, y)w] \right] dk$$

$$SR = \int \hat{w}^* \mathcal{F}[|w|^2 w] dk.$$

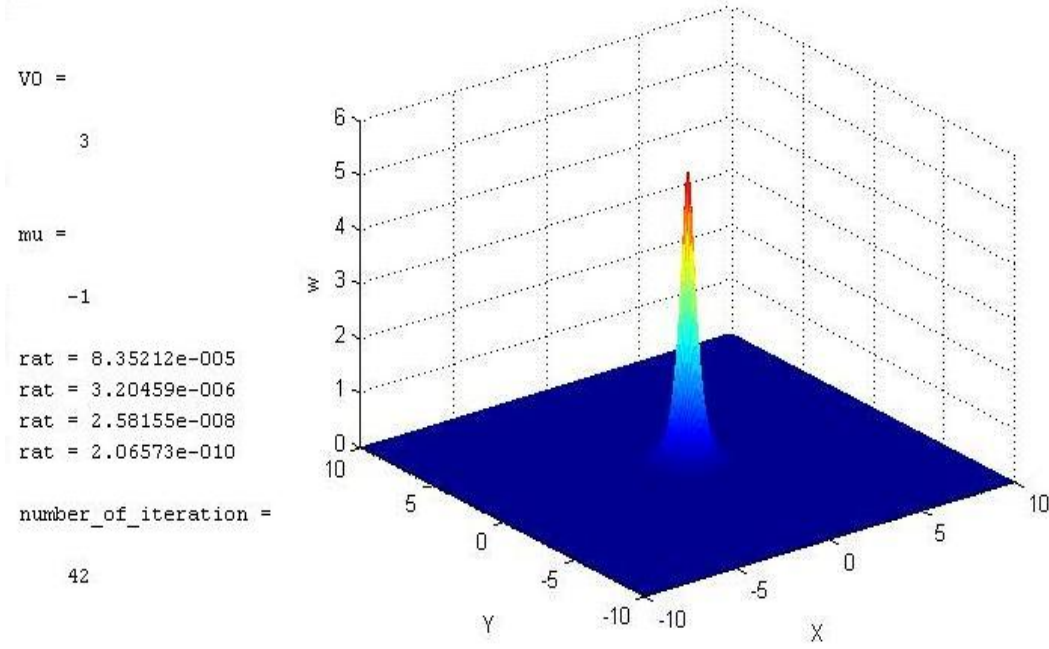
Here SL and SR are two scalar quantities that can be efficiently calculated by using Fast-Fourier Transforms in MATLAB, when potential  $V$  and initial condition  $w$  are given. Since  $SL=SR$  when  $w$  is solution of Eq.(3.2), we iterate Eq.(3.6) until we provide  $|\frac{SL}{SR} - 1| < 10^{-10}$  (error ratio). Typically, 20 – 40 steps (number of iteration) suffice for obtaining convergence about  $10^{-8}$ .

After programming SR algorithm, as the external potential, first we will consider the periodic potential  $V = V_0(\cos^2 X + \cos^2 Y)$ . The initial condition is taken as

$$w = e^{-[(x-x_0)^2 + (y-y_0)^2]} \quad (3.7)$$

which is a Gaussian type function. Here, the values of  $x_0$  and  $y_0$  define the location of the initial condition. In order to center the initial condition on the lattice maxima (the one appears at the center of the lattice), we should take  $x_0 = y_0 = 0$  and, to center the initial condition on a lattice minima (generally taken to be one of the closest minima to the central maximum), we should determine the location of the picked local minimum of the lattice and calculate those values. For the lattice assumed above, one of the local minimum close to the origin can be obtained by taking  $x_0 = y_0 = \pi/2$ .

By using the above potential and the initial condition, we obtain the soliton solution shown in Fig.(3.1) for the values of the potential depth  $V_0 = 3$  and the eigenvalue  $\mu = -1$ . Error ratios are shown as "rat" in the figure.



**Figure 3.1:** Error ratio and 3D view of the solution (fundamental soliton) according to  $X$  and  $Y$  with  $V_0 = 3$  and  $\mu = -1$ .

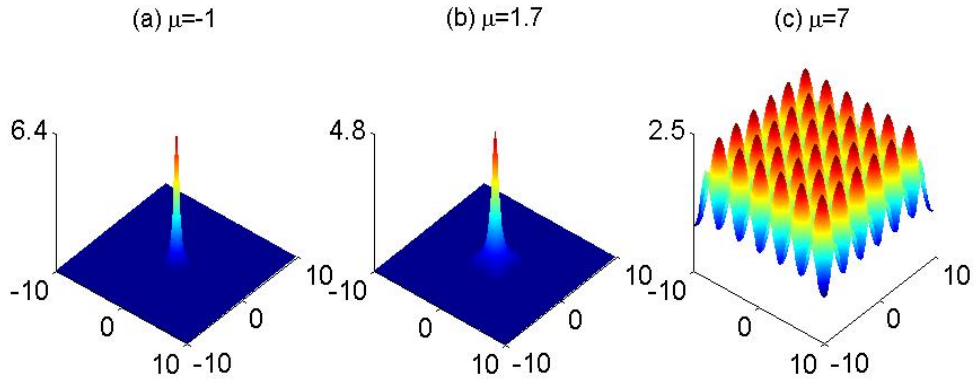
As can be seen from Fig.(3.1), 42 iterations provide  $10^{-10}$  convergence in error ratio.

### 3.2. Band Gap Structure

In this section, we investigate the first nonlinear band gap structure of  $V = V_0(\cos^2 X + \cos^2 Y)$  potential by SR method.

To determine the first nonlinear band gap formation of the lattice, we set the potential depth  $V_0$  to a fixed value (starting from 0 up to 4). For each value of  $V_0$ , by increasing the  $\mu$  values, we check both the convergence and the localization of the mode by SR method. When the mode becomes more extended, usually the convergence is slower and after a certain value of  $\mu$ , typically both the convergence can not be reached and the localization of the mode is lost.

In Fig.(3.2), 3D views of solitons for three different propagation constants (eigenvalues)  $\mu$  namely, in (a) for  $\mu = -1$  which lies in the band gap and therefore a very localized structure, in (b) for  $\mu = 1.7$  which is close to the band gap boundary and therefore a structure which is started to extend, and in (c) an extended state which is called a Bloch wave are shown.



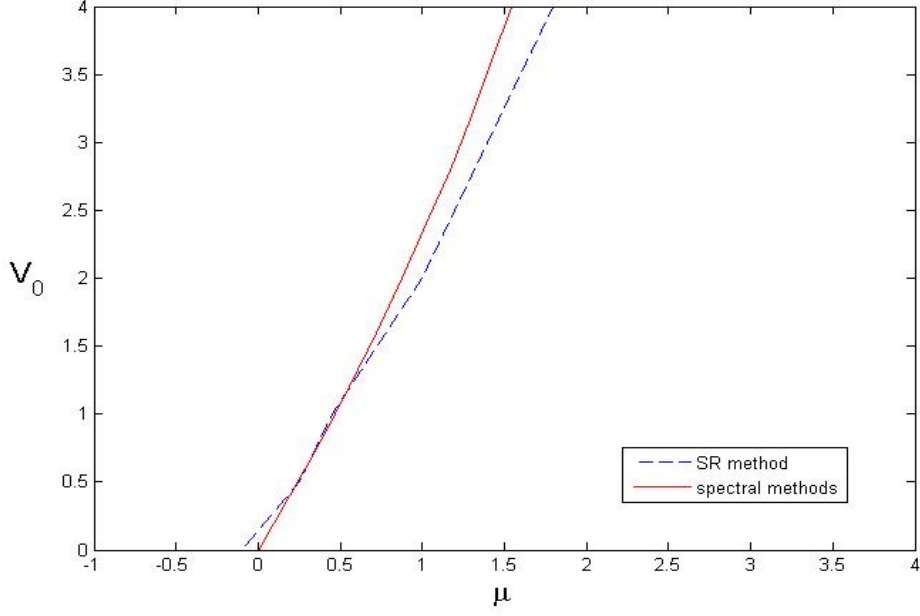
**Figure 3.2:** 3D view of (a) Localized soliton inside the band gap with  $\mu = -1$ , (b) Soliton started to extend (close to the band gap boundary) with  $\mu = 1.7$ , (c) Bloch wave (outside of the band gap) with  $\mu = 7$ , for  $V = V_0(\cos^2 X + \cos^2 Y)$  potential all with  $V_0 = 4$ .

By keeping both the localization and the convergence criterions, we locate the boundary of the first nonlinear band gap of the related lattice shown in Fig.(3.3). In the same figure, we demonstrate a comparison of the first band gap boundary of the linear periodic problem obtained by spectral methods and the SR method. As can be seen in the figure, the band gap boundaries of both cases are close to each other.

### 3.3. Power and Stability Analysis

In this section, we investigate the nonlinear stability of the fundamental solitons in the first nonlinear band gap. Power plays an important role in determining stability of the solitons. The power is defined as

$$P = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |u(x, y)|^2 dx dy \quad (3.8)$$



**Figure 3.3:** Band-gap formation for NLS equation with  $V = V_0(\cos^2 X + \cos^2 Y)$  by SR method (dashed line) and by spectral method (solid line).

where  $u(x, y)$  is the solution of the NLS equation with an external potential (lattice).

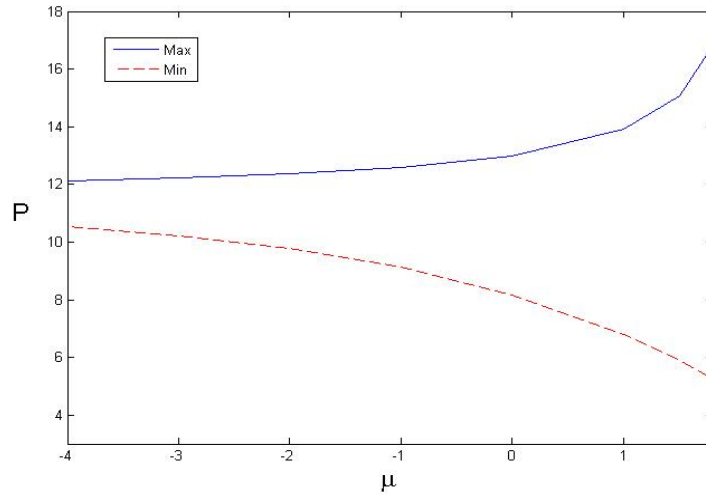
An important analytic result on soliton stability was obtained by Vakhitov and Kolokolov ([29]). They proved, by use of the linearized perturbation equation, that a necessary condition for the nonlinear stability of the soliton  $u(x; \mu)$  is

$$\frac{dP}{d\mu} < 0, \quad (3.9)$$

i.e., the soliton is stable only if its power decreases with increasing propagation constant  $\mu$ . This condition is called the slope condition. In [30] and [31], Weinstein and Rose proved that the necessary conditions for nonlinear stability are the slope condition and spectral condition. A necessary condition for collapse in the 2D cubic NLS equation is that the power of the beam exceeds the critical power  $P_c \approx 11.7$ , [32].

In order to understand the relation between power and eigenvalue (propagation constant) better, we investigate soliton power ( $P$ ) versus the eigenvalue ( $\mu$ ), shown in Fig.(3.4).

In Fig.(3.4), the solid curve shows the soliton power versus the eigenvalue when the initial condition is centered at the lattice maxima ( $x_0 = y_0 = 0$ ), and the dashed curve shows the soliton power versus the eigenvalue when the initial condition is centered at the lattice minima ( $x_0 = y_0 = \pi/2$ ).



**Figure 3.4:** Soliton power as a function of eigenvalue for  $V = V_0(\cos^2 X + \cos^2 Y)$  lattice with potential depth  $V_0 = 4$ .

As can be seen from Fig.(3.4), soliton power decreases with the increasing propagation constant (eigenvalue) when soliton is centered at the lattice minima. In other words, soliton power provides slope condition when soliton is centered at the lattice minima. Conversely, soliton power increases with the increasing propagation constant when soliton is centered at the lattice maxima.

The fundamental solitons of NLS equation can become unstable in two ways : Focusing instability or drift instability [33].

(a) If the slope condition (3.9) is not satisfied, this leads to a focusing instability.

(b) The spectral condition is associated with the eigenvalue problem (see [33]). If the spectral condition is violated, it leads to a drift instability, i.e., the fundamental soliton moves from the lattice maximum towards a nearby lattice minimum. At this point we define center of mass of a perturbed soliton.

Let us define the center of mass of a soliton as

$$CM = \frac{1}{P} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (x + iy) |u|^2 dx dy \quad (3.10)$$

where the center of mass in  $x$  and  $y$  coordinates are defined as,

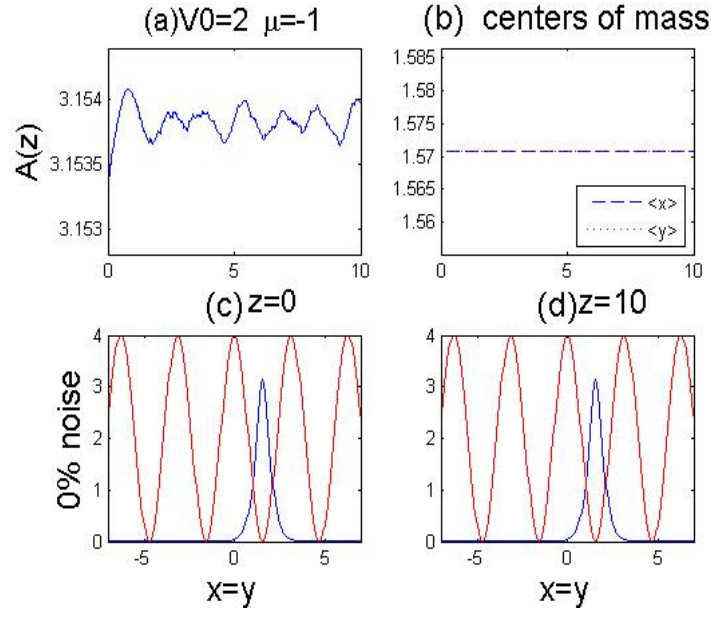
$$\langle x \rangle := \text{real}(CM), \quad \langle y \rangle := \text{imag}(CM). \quad (3.11)$$

To numerically investigate the nonlinear stability, we directly compute Eq.(1.4) over a long distance. Finite difference method was used on derivatives  $u_{xx}$  and  $u_{yy}$ , and fourth-order Runge-Kutta method to advance in  $z$ , for both periodic and nonperiodic (Penrose) potentials. The initial conditions were taken to be a fundamental soliton with 1% random noise in the amplitude and phase [34].

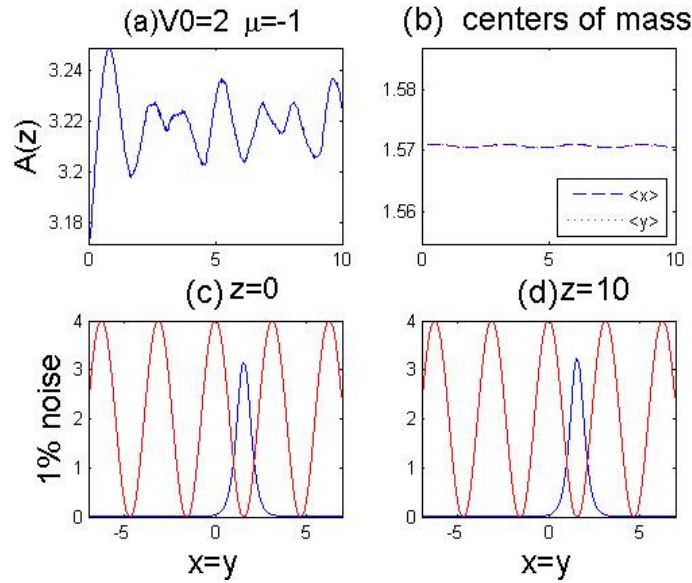
Now, we investigate the nonlinear stability properties of the solitons obtained for  $V = V_0(\cos^2 X + \cos^2 Y)$  lattice. As can be seen from Fig.(3.4), solitons centered at lattice minima provide stability conditions (i.e., slope condition and critical power) for  $V = V_0(\cos^2 X + \cos^2 Y)$  lattice. In both Fig.(3.5) and Fig.(3.6), evolution of the fundamental solitons at lattice minima are plotted for the potential depth  $V_0 = 2$  and  $\mu = -1$ . In Fig.(3.5), fundamental soliton (which is obtained by SR method) is taken as the initial condition without noise and, in Fig.(3.6) fundamental soliton is perturbed with 1% random noise in the amplitude and phase.

As can be seen from Fig.(3.5) and Fig.(3.6), peak amplitudes of the fundamental solitons ( $A(z) = \max_{x,y} |\tilde{u}(x, y, z)|$ ) oscillate with the propagation distance  $z$  and the center of mass in the  $x$  and  $y$  axis nearly stay at the same place. This suggests that the fundamental solitons for  $V = V_0(\cos^2 X + \cos^2 Y)$  lattice are nonlinearly stable in this parameter regime.

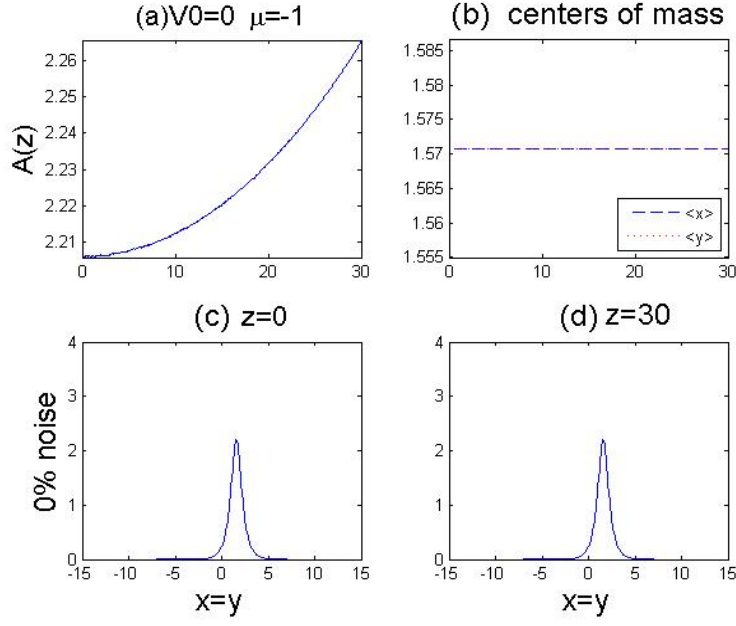
In Fig.(3.7) and Fig.(3.8), we plot the stability graphs of a lattice free soliton which is obtained by taking  $V_0 = 0$ . This case corresponds to a standard (2+1)D NLS equation that defined in Eq.(1.2) with  $\gamma = \lambda = 1$ . In Fig.(3.7), as it is expected, if the lattice is removed from the medium, a rapid collapse occurs even without a perturbation. If 1% perturbation is added to the initial condition, the soliton solution collapses even more rapidly that can be seen from Fig.(3.8).



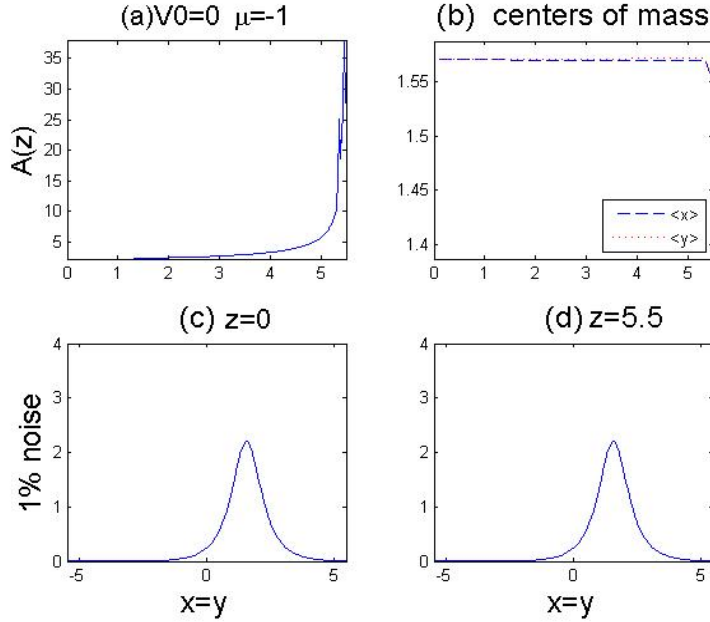
**Figure 3.5:** Evolution of soliton situated at  $V = V_0(\cos^2 X + \cos^2 Y)$  potential with  $V_0 = 2$  and  $\mu = -1$ . (a) Peak amplitude  $A(z)$  as a function of the propagation distance. The initial condition is taken as the fundamental soliton, (b) Center of mass in the  $x$  and  $y$  coordinates, (c) Cross section along the diagonal axis of a fundamental soliton at minimum superimposed on the potential at  $z = 0$ , (d) Cross section along the diagonal axis of the fundamental soliton at the minimum superimposed on the potential after the propagation ( $z = 10$ ).



**Figure 3.6:** Same as Fig.(3.5), added 1% noise to fundamental soliton in amplitude and phase.



**Figure 3.7:** Evolution of soliton situated at lattice free medium ( $V_0 = 0$ ) with  $\mu = -1$ . (a) Peak amplitude  $A(z)$  as a function of the propagation distance. The initial condition is taken as the fundamental soliton, (b) Center of mass in the  $x$  and  $y$  coordinates, (c) Cross section along the diagonal axis of a fundamental soliton at  $z = 0$ , (d) Cross section along the diagonal axis of the fundamental soliton after the propagation ( $z = 30$ ).



**Figure 3.8:** Same as Fig.(3.7), added 1% noise to fundamental soliton in amplitude and phase.

In [26], the effect of the potential depth ( $V_0$ ) is investigated for the periodic potential  $V = V_0(\cos^2 X + \cos^2 Y)$  and it is found that, deeper potential (increased values of  $V_0$ ) does improve the nonlinear stability of the fundamental solitons.



## 4. SOLITONS IN TWO-DIMENSIONAL LATTICES POSSESSING DISLOCATIONS

### 4.1. Band Gap Structure

In this section, we investigate the first nonlinear band gap structure of NLS equation with a lattice possessing an edge dislocation. Dislocation is a type of defect which cause an irregularity in crystal structure, (i.e., see [7]).

$$V(x, y) = \frac{V_0}{25} [2\cos(k_x x + \theta(x, y)) + 2\cos(k_y y) + 1]^2 \quad (4.1)$$

where edge dislocation ( $\theta$ ) defined as

$$\theta(x, y) = \frac{3\pi}{2} - \tan^{-1}\left(\frac{y}{x}\right) \quad (4.2)$$

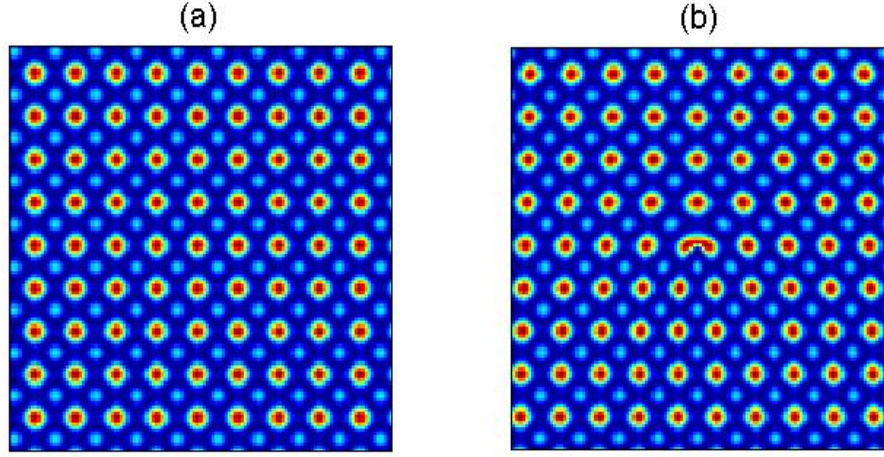
$\theta(x, y)$  is a phase function through which irregularities are introduced.

Two lattice images with the edge dislocation and without the dislocation for a bounded region are demonstrated in Fig.(4.1) and the diagonal cross-sections of those two lattices are shown in Fig.(4.2).

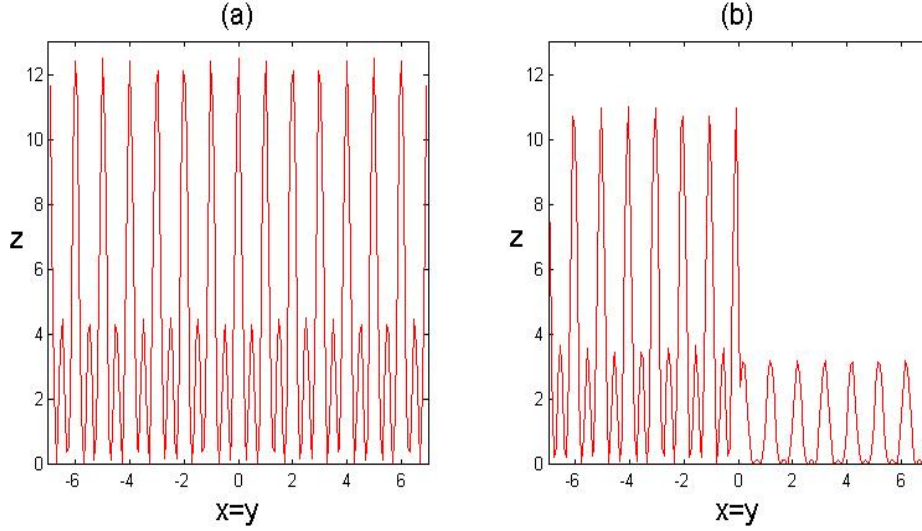
As can be seen from Fig.(4.2)(b), the density of lattice sites changes vertically across the lattice with edge dislocation.

To determine the first nonlinear band gap formation NLS equation with the lattice given in Eq.(4.1), we set the potential depth  $V_0$  to a fixed value (starting from 0 up to 30). For each value of  $V_0$ , by increasing the  $\mu$  values, we check both the convergence and the localization of the mode by SR method. When the mode becomes more extended, usually the convergence is slower and after a certain value of  $\mu$ , typically both the convergence can not be reached and the localization of the mode is lost. In this way we locate the boundary of the first nonlinear band gap of the dislocated lattice.

The first nonlinear band gap boundaries for NLS equation with an edge dislocation and without dislocation are depicted in Fig.(4.3).

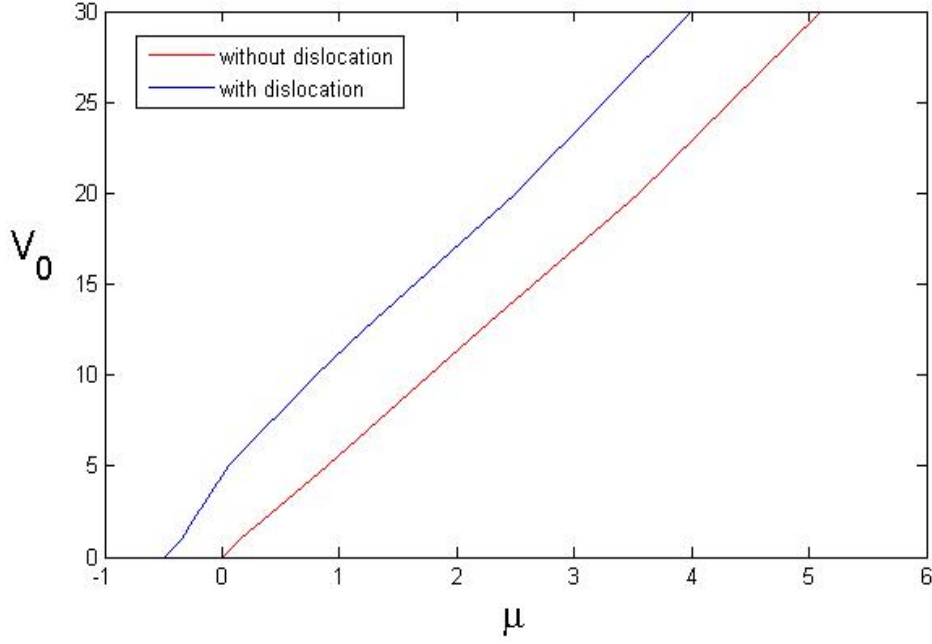


**Figure 4.1:** Contour images of the lattices (a) Without dislocation ( $\theta = 0$ ), (b) With an edge dislocation which defined in Eq.(4.2), where  $k_x = k_y = 2\pi$  and  $V_0 = 12.5$ .



**Figure 4.2:** Diagonal cross-sections of the lattices (a) Without dislocation ( $\theta = 0$ ), (b) With an edge dislocation which defined in Eq.(4.2), where  $k_x = k_y = 2\pi$  and  $V_0 = 12.5$ .

As can be seen from Fig.(4.3), the first nonlinear band gap for dislocated lattice solitons is narrower than that of lattice solitons without dislocation.



**Figure 4.3:** Band gap formations for NLS equation with and without the dislocation.

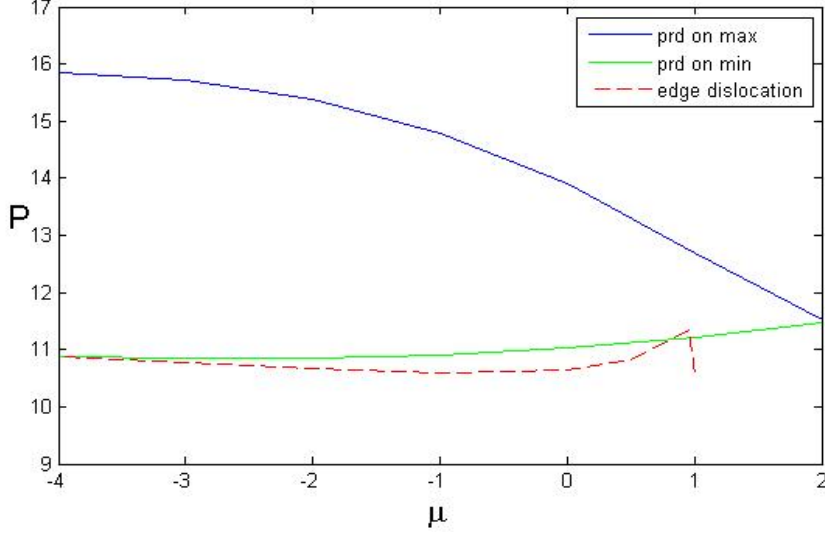
## 4.2. Soliton Power of Dislocated Lattice

Soliton power of dislocated lattice is investigated in two separate cases that are the solitons centered at lattice maxima and the solitons centered at lattice minima.

In Fig.(4.4), we plot the soliton power  $P$  as a function of the eigenvalue  $\mu$ , with and without dislocation, for the lattice defined in Eq.(4.1).

In Fig.(4.4), soliton power versus the eigenvalue graphs are depicted: i) for the lattice with the edge dislocation ( $\theta \neq 0$ ) when soliton is centered above the phase dislocation, ii) the lattice without dislocation (periodic lattice) when soliton is centered at the lattice maxima and iii) the lattice without dislocation (periodic lattice) when soliton is centered at the lattice minima.

As can be seen from the figure, edge dislocation reduces the gap size (i.e.,  $\mu_{max} \approx 0.95$ ), where the lattice without dislocation has a larger gap size (i.e.,  $\mu_{max} \approx 2$ ). The numerical method converges to a localized state when  $\mu < \mu_{max}$ , and the numerical method converges to an extended state when  $\mu > \mu_{max}$ .  $\mu_{max}$  should be taken as a critical eigenvalue for this edge dislocated lattice.



**Figure 4.4:** Power analysis of dislocated lattice which given in Eq.(4.1) with  $V_0 = 12.5$  and  $k = 2\pi$ .

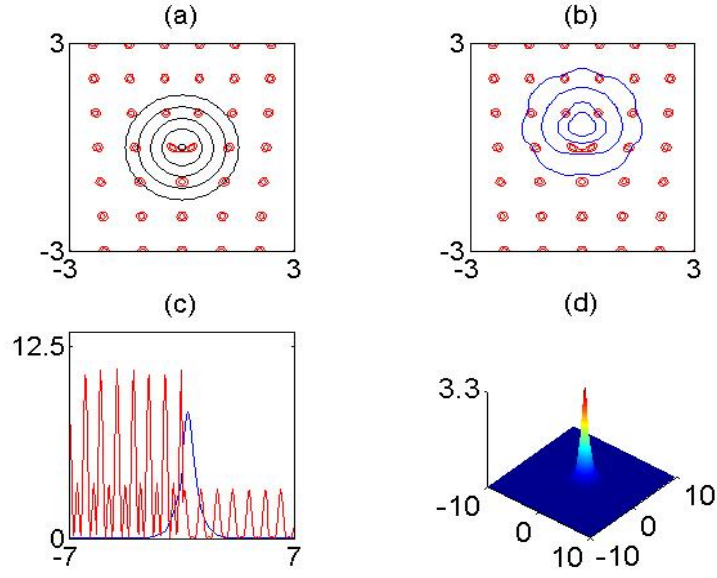
### 4.3. Properties of Dislocated Lattice Solitons

In order to investigate properties of dislocated lattice solitons, we locate the initial condition above the phase dislocation, in between neighboring local maxima of the lattice. Therefore, the initial condition looks like centered at lattice minima, as it can be seen from Fig.(4.5) (c).

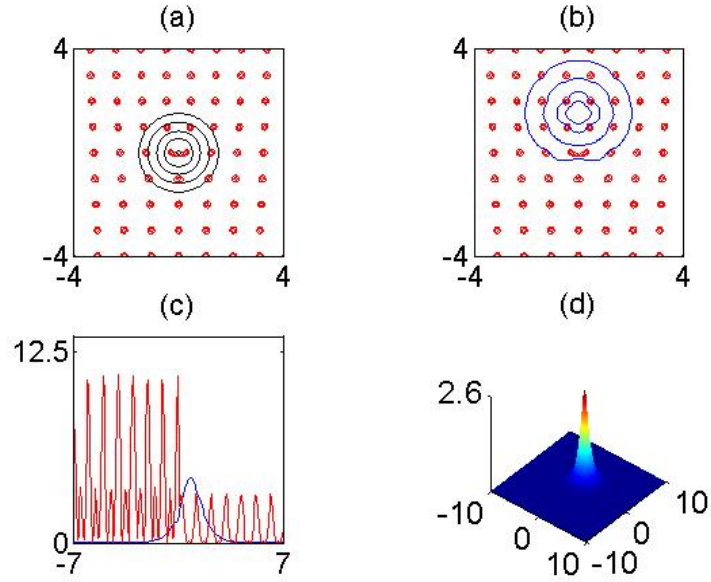
In Fig.(4.5), we take eigenvalue in the gap (i.e.,  $\mu = 0.5 < \mu_{max}$ ) and as expected in Section 4.2, the soliton converges to a localized state (see Fig.(4.5) (d)). As can be seen from the figure, the starting point of the computational method is around the origin (see Fig.(4.5) (a)) and during the iterations, the solution moves upward along the  $y$  axis, until convergence is reached (see Fig.(4.5) (b)). This result is also supported by Ablowitz *et al.* ([7]).

In order to see the difference between  $\mu < \mu_{max}$  and  $\mu > \mu_{max}$ , we take eigenvalue out of the gap (i.e.,  $\mu = 1 > \mu_{max}$ ), and repeat same procedure in Fig.(4.6).

As can be seen from the figure, the starting point of the computational method is around the origin (see Fig.(4.6) (a)), the solution is converged one cell above (see Fig.(4.6) (b)) and, as expected and the soliton is more extended (compare Fig.(4.5) (d) and Fig.(4.6) (d)).

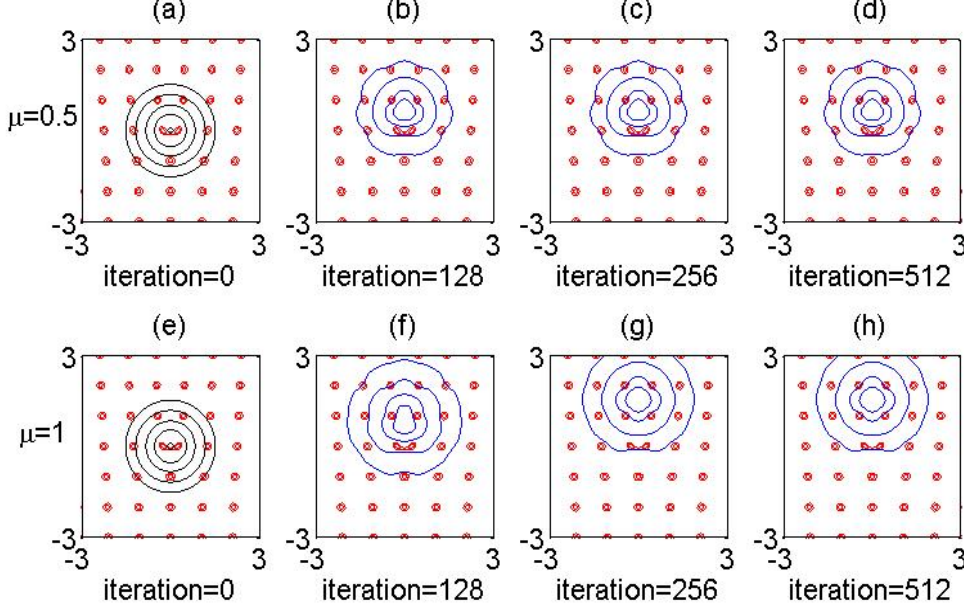


**Figure 4.5:** Contour plot of the (a) Initial condition superimposed on dislocated lattice, (b) Soliton superimposed on dislocated lattice, (c) Diagonal cross section of soliton superimposed on dislocated lattice, (d) 3D view of dislocated lattice soliton, when the soliton is centered at lattice minima with  $\mu = 0.5$  and  $V_0 = 12.5$ .



**Figure 4.6:** Same as Fig.(4.5) with  $\mu = 1$  and  $V_0 = 12.5$ .

As explained above, dislocated lattice solitons moves upward along the  $y$  axis during the iterations. In order to see this fact, we plot the same soliton with different iteration numbers when  $\mu = 0.5$  (i.e.,  $\mu < \mu_{max}$ ) and  $\mu = 1$  (i.e.,  $\mu > \mu_{max}$ ) in Fig.(4.7).



**Figure 4.7:** Contour plot of the solitons, all with  $V_0 = 12.5$  and  $k = 2\pi$ , Top: (a) The initial condition, (b) Soliton with 128 iteration, (c) 256 iteration, (d) 512 iteration with  $\mu = 0.5$ ; Bottom: (e) The initial condition, (f) Soliton with 128 iteration, (g) 256 iteration, (h) 512 iteration with  $\mu = 1$ ; when the initial condition is centered at dislocated lattice minima.

As can be seen from Fig.(4.7), if  $\mu = 0.5$ , the soliton converges in the first cell between local maxima and stays at the same location during the iterations (up to 512), if  $\mu = 1$  the soliton converges in the first cell between local maxima in 128 iterations, after that, if the iteration number is increased up to 256 the soliton keeps on moving upward along the  $y$  axis and finally soliton converges one cell above between local maxima. Therefore, the soliton for  $\mu = 1$  is not considered to be a gap soliton since, there is two-cell shift during the iterations.  $\mu = 0.95$  is the value of the propagation constant that should be considered as the threshold for the band gap and the Bloch wave region.

After 256 iterations, in both cases, iteration number doesn't change the place of convergence. This result is supported by Ablowitz *et al.* ([7]).

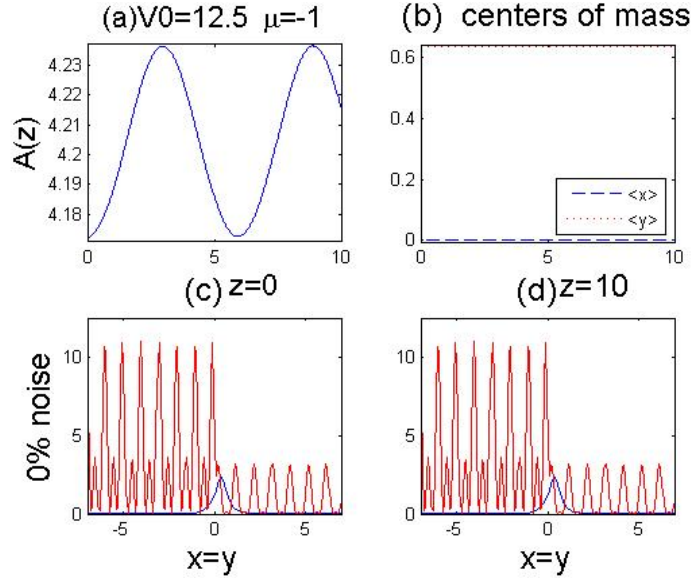
#### 4.4. Stability Properties of Dislocated Lattice Solitons

As explained in previous section, soliton power plays an important role in determining stability properties of fundamental solitons. We investigated soliton power of dislocated lattice solitons in Fig.(4.4). In the light of this power analysis, we now investigate stability properties of dislocated lattice solitons when the initial condition is centered near the edge dislocation.

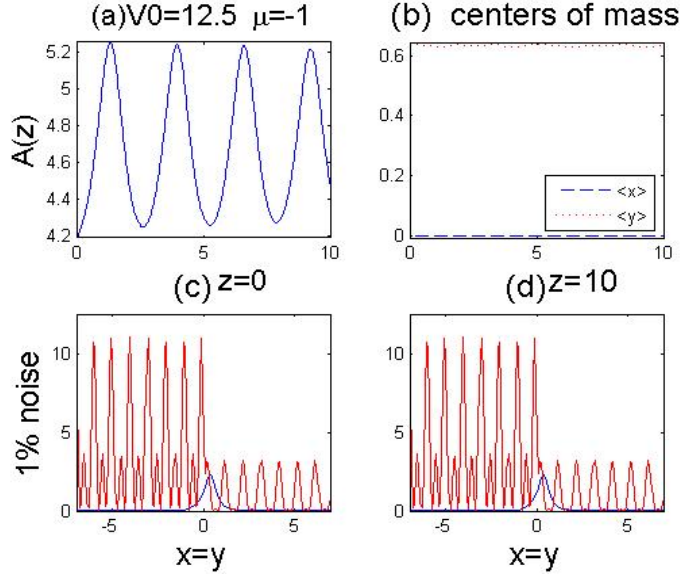
Firstly, we take propagation constant smaller than critical eigenvalue (i.e.,  $\mu < \mu_{max} = 0.95$ ). As can be seen from Fig.(4.4), the dislocated lattice soliton with  $\mu = -1$  provide slope condition and soliton power doesn't exceed the critical power ( $P_c \approx 11.7$ ). Therefore, the soliton is expected to be stable in this parameter regime. In both Fig.(4.8) and Fig.(4.9), evolution of the fundamental soliton near the edge dislocation are plotted for the potential depth  $V_0 = 12.5$  and  $\mu = -1$ . In Fig.(4.8), fundamental soliton (which obtained by SR method) taken as initial condition without noise and, in Fig.(4.9) fundamental soliton is perturbed with 1% random noise in the amplitude and phase.

As can be seen from Fig.(4.8) and Fig.(4.9), peak amplitudes of the fundamental solitons ( $A(z) = \max_{x,y} |\tilde{u}(x, y, z)|$ ) oscillate with the propagation distance  $z$  and the center of mass in the  $x$  and  $y$  axis nearly stay at the same place. This suggests that the fundamental solitons for the edge dislocated lattice are nonlinearly stable in this parameter regime.

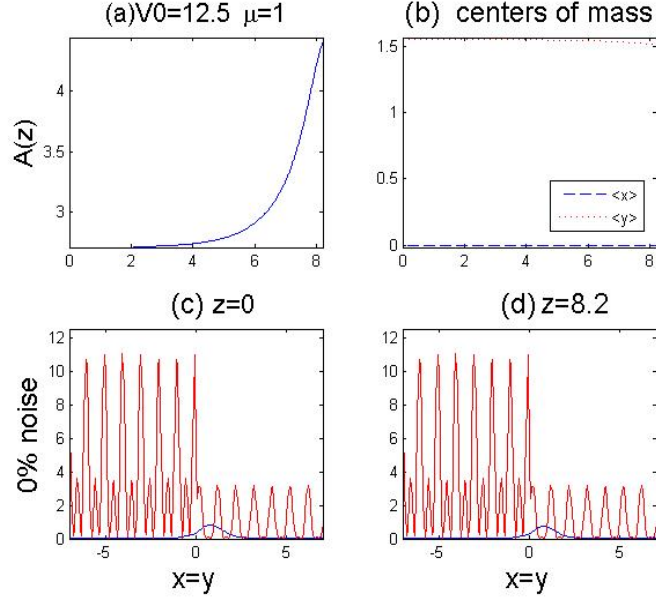
Now, we take propagation constant bigger than critical eigenvalue (i.e.,  $\mu > \mu_{max} = 0.95$ ) and investigate stability properties of dislocated lattice soliton. As can be seen from Fig.(4.4), the power of dislocated lattice soliton with  $\mu = 1$  doesn't exceed the critical power ( $P_c \approx 11.7$ ), but the eigenvalue exceeds critical eigenvalue (i.e.,  $\mu > 0.95$ ) and it doesn't provide slope condition. Therefore the soliton is expected to be unstable in this parameter regime. In Fig.(4.10), evolution of the fundamental soliton near the edge dislocation is plotted for the potential depth  $V_0 = 12.5$  and  $\mu = 1$ .



**Figure 4.8:** Evolution of soliton situated at dislocated lattice with  $V_0 = 12.5$  and  $\mu = -1$ . (a) Peak amplitude  $A(z)$  as a function of the propagation distance. The initial condition is taken as the fundamental soliton, (b) Center of mass in the  $x$  and  $y$  coordinates, (c) Cross section along the diagonal axis of a fundamental soliton near the dislocation superimposed on the dislocated potential at  $z = 0$ , (d) Cross section along the diagonal axis of the fundamental soliton near the dislocation superimposed on the dislocated potential after the propagation ( $z = 10$ ).



**Figure 4.9:** Same as Fig.(4.8), added 1% noise to fundamental soliton in amplitude and phase.



**Figure 4.10:** Same as Fig.(4.8) with  $V_0 = 12.5$  and  $\mu = 1$ .

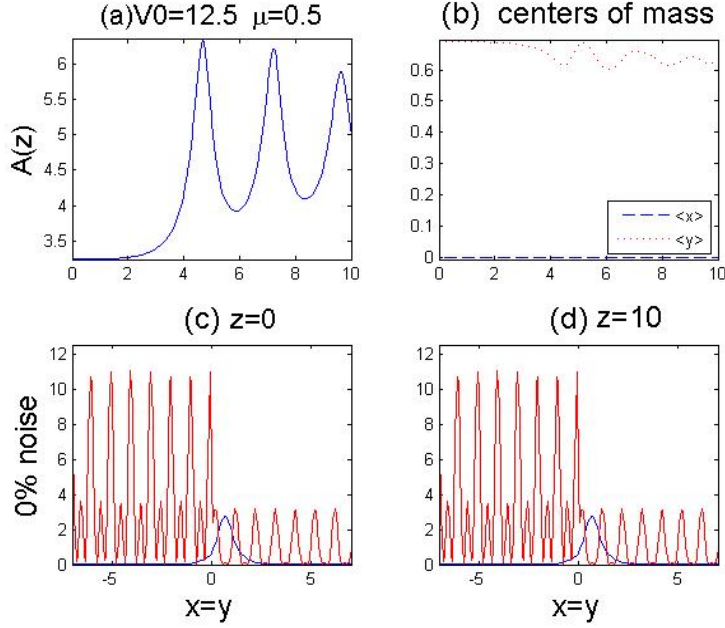
As can be seen from Fig.(4.10), the peak amplitude of the fundamental soliton  $A(z) = \max_{x,y} |\tilde{u}(x, y, z)|$  increases with the propagation distance  $z$ . The fundamental soliton is unstable in this parameter regime.

Finally, we take propagation constant ( $\mu = 0.5$ ) smaller than critical eigenvalue (i.e.,  $\mu$  is in the band gap) when the initial condition is centered near the edge dislocation. As can be seen from Fig.(4.4), the power doesn't exceed the critical collapse value ( $P_c = 11.72$ ), but the slope condition (Eq.(3.9)) is not satisfied when  $\mu = 0.5$  and this leads to a focusing instability that shown in Fig.(4.11).

As can be seen from Fig.(4.11), the peak amplitude of the fundamental soliton  $A(z) = \max_{x,y} |\tilde{u}(x, y, z)|$  increases with the propagation distance  $z$ . The fundamental soliton is unstable in this parameter regime.

Now, we increase the depth ( $V_0$ ) of the lattice and repeat the same procedure with the same propagation constant ( $\mu = 0.5$ ) for dislocated lattice soliton. The nonlinear stability properties of dislocated lattice solitons with  $V_0 = 40$  and  $\mu = 0.5$  are investigated in Fig.(4.12) and Fig.(4.13). In Fig.(4.12), fundamental soliton (which obtained by SR method) taken as initial condition without noise

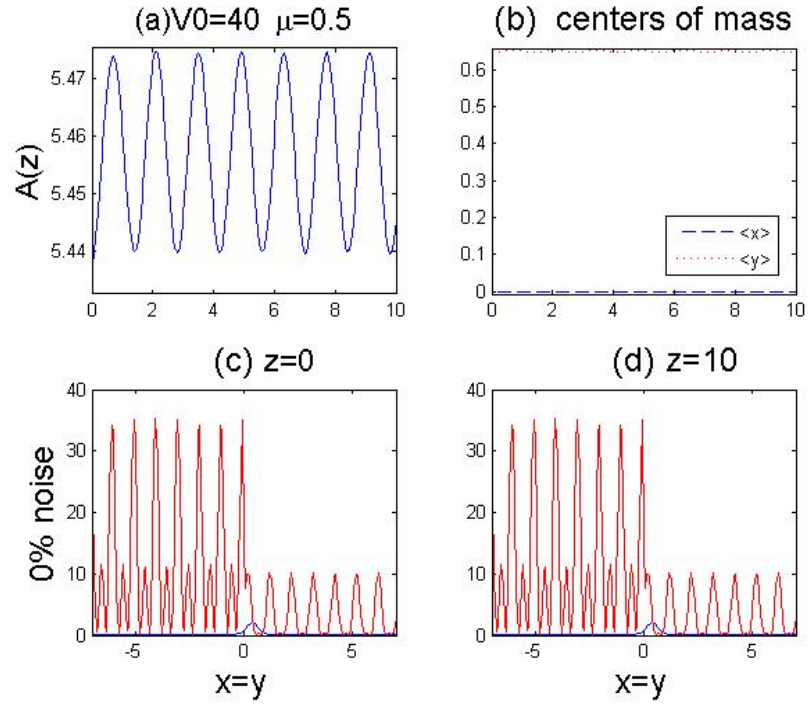
and, in Fig.(4.13) fundamental soliton is perturbed with 1% random noise in the amplitude and phase.



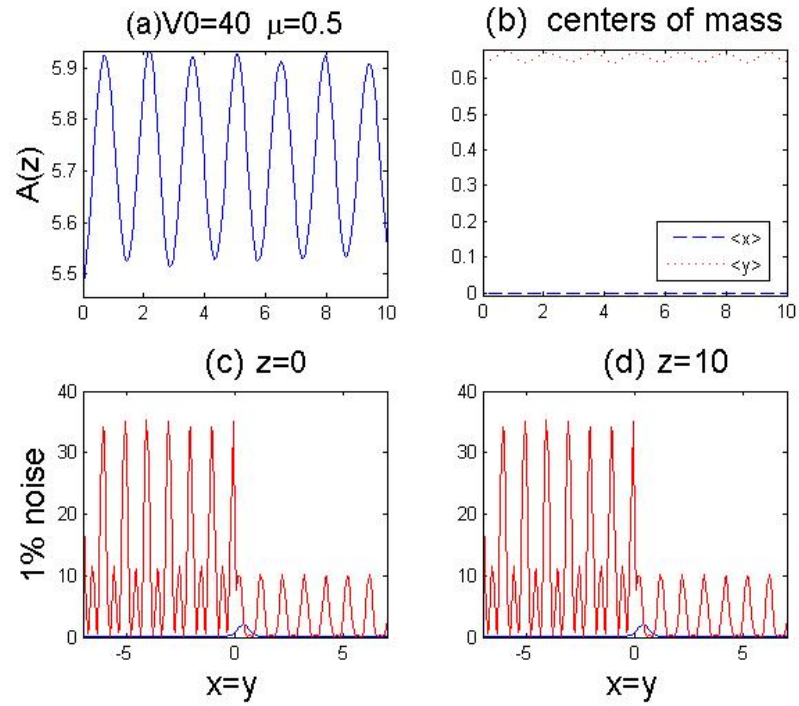
**Figure 4.11:** Evolution of soliton situated at dislocated lattice with  $V_0 = 12.5$  and  $\mu = 0.5$ . (a) Peak amplitude  $A(z)$  as a function of the propagation distance. The initial condition is taken as the fundamental soliton, (b) Center of mass in the  $x$  and  $y$  coordinates, (c) Cross section along the diagonal axis of a fundamental soliton near the dislocation superimposed on the dislocated potential at  $z = 0$ , (d) Cross section along the diagonal axis of the fundamental soliton near the dislocation superimposed on the dislocated potential after the propagation ( $z = 10$ ).

As can be seen from Fig.(4.12) and Fig.(4.13), peak amplitudes of the fundamental solitons ( $A(z) = \max_{x,y} |\tilde{u}(x, y, z)|$ ) oscillate with the propagation distance  $z$  and the center of mass in the  $x$  and  $y$  axis nearly stay at the same place. This suggests that the fundamental solitons for the edge dislocated lattice are nonlinearly stable in this parameter regime.

As a result, deeper potential (increased values of  $V_0$ ) does improve the nonlinear stability of the fundamental solitons in dislocated lattice. This fact is also shown by Musslimani and Yang [26] for the periodic lattice  $V = V_0(\cos^2 X + \cos^2 Y)$ .



**Figure 4.12:** Same as Fig.(4.11) with  $V_0 = 40$  and  $\mu = 0.5$ .



**Figure 4.13:** Same as Fig.(4.12), added 1% noise to fundamental soliton in amplitude and phase.



## 5. PROPERTIES OF QUASICRYSTAL LATTICE SOLITONS

In this section, we investigate 2D solitons on quasicrystal lattices. In particular, the quasicrystal with  $N = 5$  is often called the Penrose tiling, the quasicrystal with  $N = 7$  is also a Penrose type potential in the formulation that given in Eq.(1.5).

### 5.1. Penrose Type Potentials

In this study, the formulation of lattices without defect ( $\theta = 0$ ), given as follows

$$V_N(x, y) = \frac{V_0}{N^2} \left| \sum_{n=0}^{N-1} e^{i(k_x x + k_y y)} \right|^2. \quad (5.1)$$

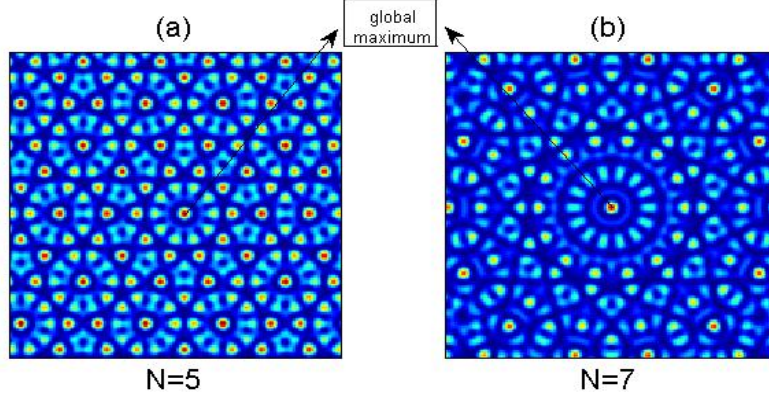
The fundamental lattice solitons and their nonlinear stability properties for  $N = 4$  corresponds to a periodic lattice and for  $N = 5$  corresponds to Penrose tiling have been analyzed by Ablowitz *et al.* ([34]) in detail. In this part of the study, we will consider two Penrose type quasicrystals corresponding to  $N = 5$  and  $N = 7$  in Eq.(5.1).

If we take  $N = 5$  in this formulation, we get Penrose-5 potential and similarly by taking  $N = 7$ , we get Penrose-7 potential, where  $k_x = k_y = 2\pi$ . Solitons in Penrose quasicrystal lattices, which were generated by the method of optical induction, were predicted and observed by Freedman *et al.* ([35]).

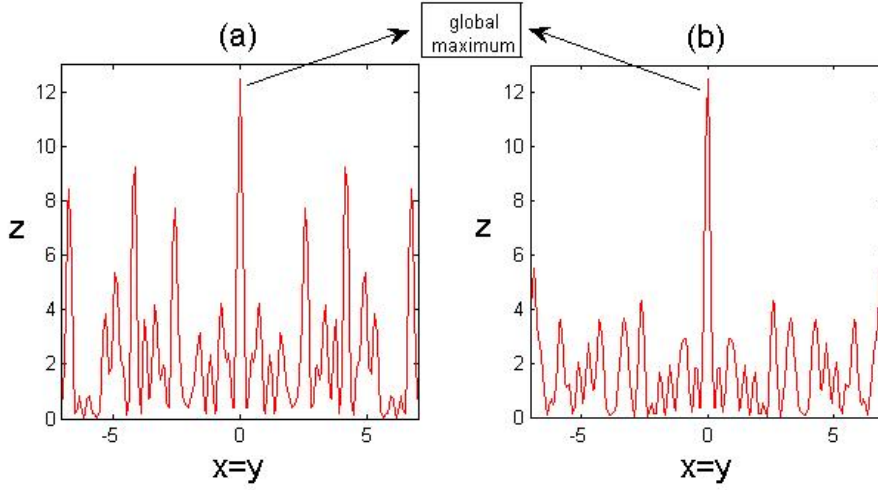
In Fig.(5.1), contour images of Penrose-5 and Penrose-7 lattices are plotted for a bounded region.

As can be seen from Fig.(5.1), Penrose-5 and Penrose-7 lattices are non-periodic.

If we plot diagonal cross section of the Penrose-5 and Penrose-7 lattices, we can see maxima and minima for each lattices (see Fig.(5.2)).



**Figure 5.1:** Contour images of Penrose-5 and Penrose-7 lattices, all with  $V_0 = 12.5$  and  $k = 2\pi$ , (a) for  $N=5$ , (b)  $N=7$ .



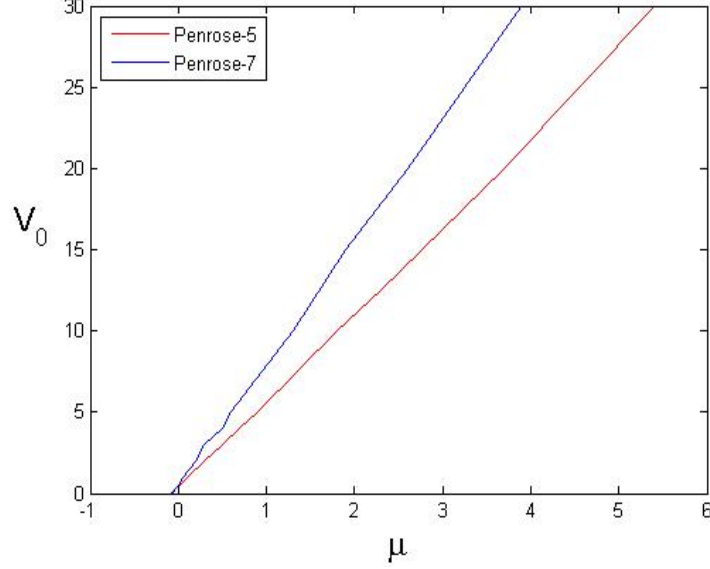
**Figure 5.2:** Diagonal cross-sections of Penrose-5 and Penrose-7 lattices, all with  $V_0 = 12.5$  and  $k = 2\pi$ , (a) for  $N=5$ , (b)  $N=7$ .

As can be seen from Fig.(5.2), there is one global maximum and many local minima for both non-periodic lattices ( $N = 5$ ,  $N = 7$ ).

## 5.2. Band Gap Structure of Penrose-5 and Penrose-7 Lattices

To compare the first nonlinear band gap formation of Penrose-5 and Penrose-7 lattices, we set the potential depth  $V_0$  to a fixed value (starting from 0 up to 30). For each value of  $V_0$ , by increasing the  $\mu$  values, we check both the convergence and the localization of the mode by SR method. When the mode becomes more

extended, usually the convergence is slower and after a certain value of  $\mu$ , typically both the convergence can not be reached and the localization of the mode is lost. In this way we locate the boundary of the first nonlinear band gap of the related lattices. The first nonlinear band gap boundaries for Penrose-5 and Penrose-7 potentials are depicted in the Fig.(5.3).



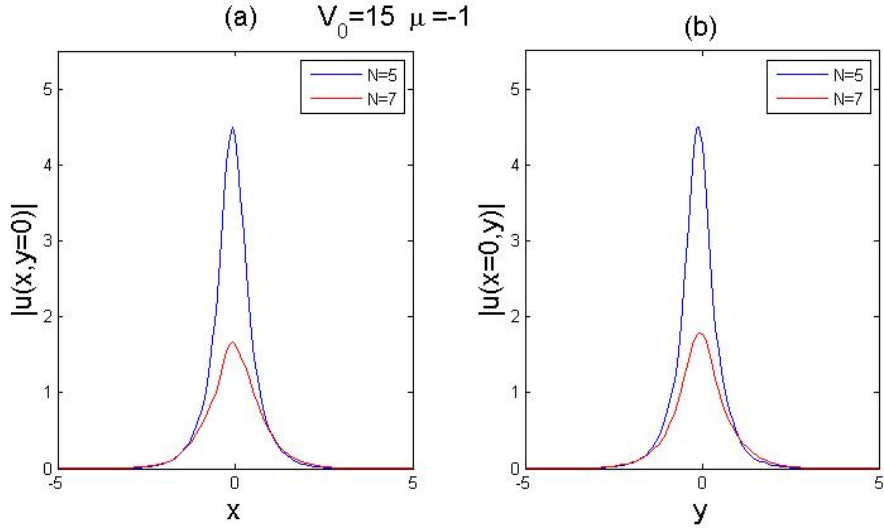
**Figure 5.3:** Band gap formation for NLS equation with external potentials, Penrose-5, Penrose-7 by SR method.

As can be seen from Fig.(5.3), the first nonlinear band gap for Penrose-7 lattice solitons is narrower than that of the Penrose-5 lattice solitons.

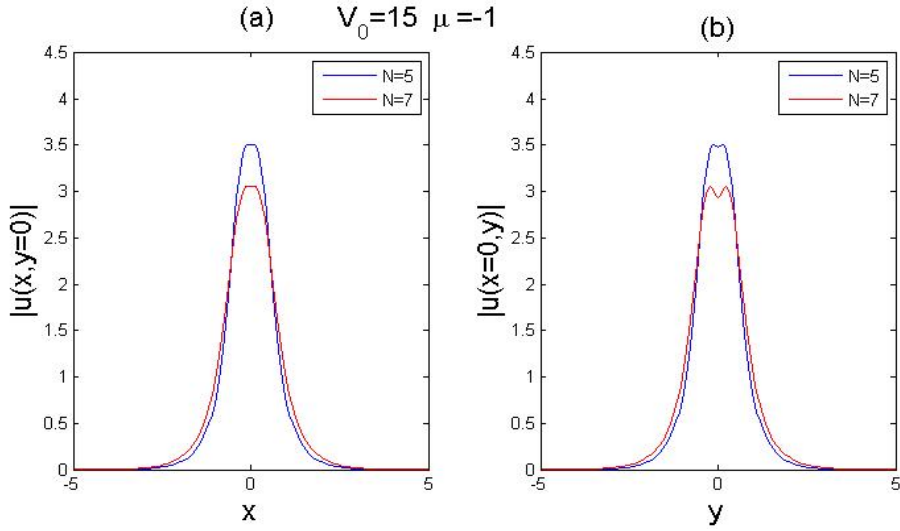
To compare the mode profiles of Penrose-5 and Penrose-7 lattices, we plotted the on axis mode profiles for  $N = 5$  and  $N = 7$  with same parameters ( $V_0$  and  $\mu$ ) by SR method.

As can be seen from Fig.(5.4) and Fig.(5.5), as  $N$  increases the amplitudes of the fundamental solitons decrease. Also, we can see the dimples for solitons which are centered at lattice maxima in Fig.(5.5) (b).

When we increased the depth of potential ( $V_0$ ) from 12.5 to 100, interestingly the fundamental mode of the Penrose-7 lattice approaches the fundamental mode of the penrose-5 lattice (see Fig.(5.6)).

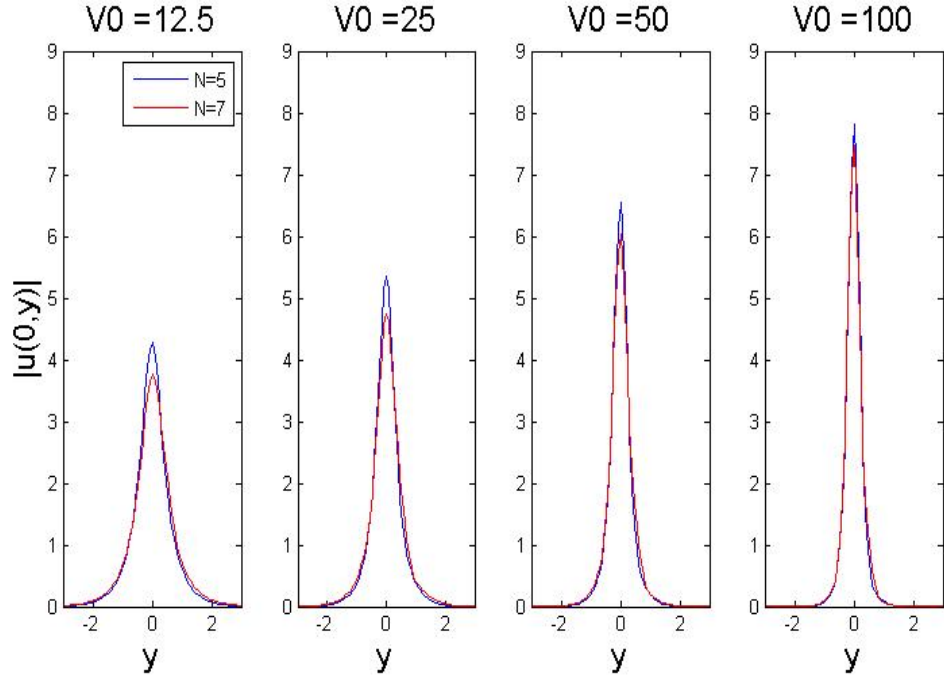


**Figure 5.4:** On axis mode profiles for  $N = 5$  and  $N = 7$ , (a) Initial condition on Min, along the x-axis, (b) Initial condition on Min, along the y-axis for  $V_0 = 15$  and  $\mu = -1$ .



**Figure 5.5:** On axis mode profiles for  $N = 5$  and  $N = 7$ , (a) Initial condition on Max, along the x-axis, (b) Initial condition on Max, along the y-axis for  $V_0 = 15$  and  $\mu = -1$ .

The mode profiles of Penrose-5 and Penrose-7 solitons show that these modes have similar properties. In next sections, we will investigate properties of Penrose-7 solitons and compare them with Penrose-5 solitons which are investigated by Ablowitz *et al.* ([7]).



**Figure 5.6:** The mode profiles for Penrose-5 and Penrose-7 potentials.

### 5.3. Properties of Penrose-7 Solitons

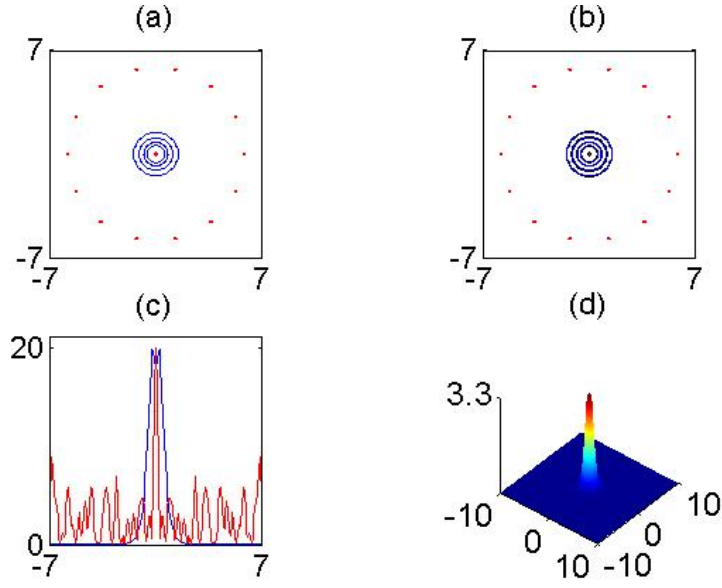
Soliton properties of Penrose-7 lattice are investigated in separate cases for the solitons centered at lattice maxima and solitons centered at lattice minima.

#### 5.3.1 Solitons centered at lattice maxima

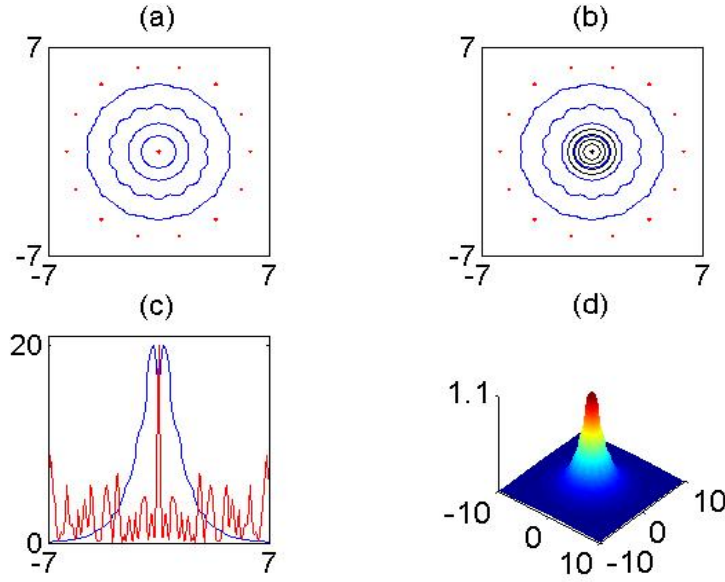
As can be seen from Fig.(5.2), the origin is the global maximum of Penrose-7 lattice. If we locate the initial condition at the origin, then the fundamental soliton obtained by the use of SR method is shown in Fig.(5.7).

Penrose-7 solitons which are centered on lattice maxima have dimples. The dimples are more pronounced for large values of  $\mu$  (i.e., near the gap edge), in order to see this fact compare Fig.(5.7) and (5.8). Penrose-7 solitons which are centered around local minima do not have a dimple. This fact is also true for penrose-5 solitons which have been investigated by Ablowitz *et al.* ([7]).

Also, it can be seen that, Penrose-7 solitons converge at lattice maxima when the initial condition is centered at lattice maxima (see Fig.(5.8) (c)).



**Figure 5.7:** Contour plot of the (a) Penrose-7 soliton, (b) Initial condition and Penrose-7 soliton on top of each other, (c) Diagonal cross section of Penrose-7 soliton superimposed on Penrose-7 lattice, (d) 3D view of Penrose-7 soliton's intensity showing the dimple, when the soliton is centered at the lattice maxima with  $\mu = -1$  and  $V_0 = 20$ .

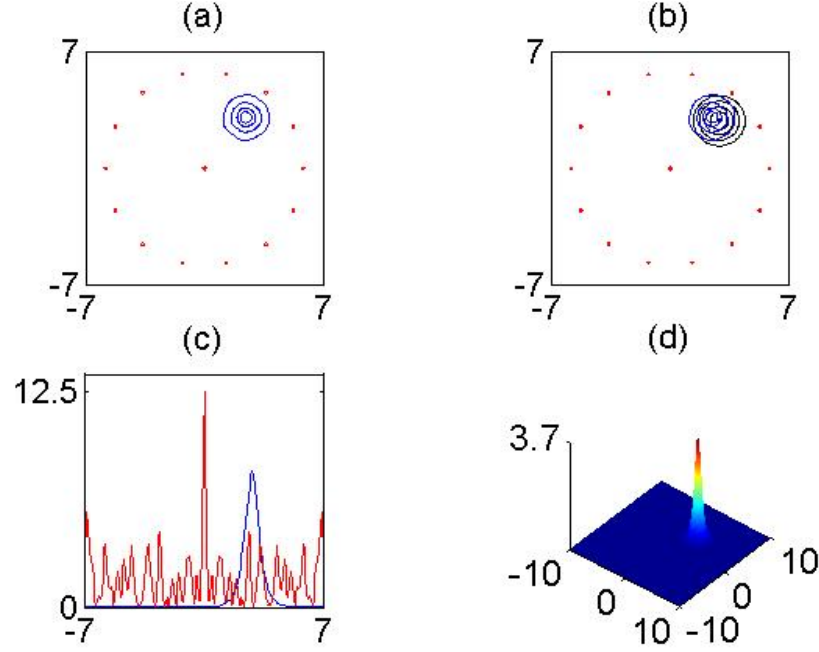


**Figure 5.8:** Same as Fig.(5.7) with  $\mu = 2.5$ .

### 5.3.2 Solitons centered at lattice minima

We now investigate solitons centered at lattice minima for Penrose-7 potential. In order to center the soliton to lattice minima, we determine any local minima of Penrose-7 potential close to the lattice center and then we pinpoint the initial condition to this location.

As explained above, solitons centered around local minima do not have dimples (see Fig.(5.9) and Fig.(5.10)).

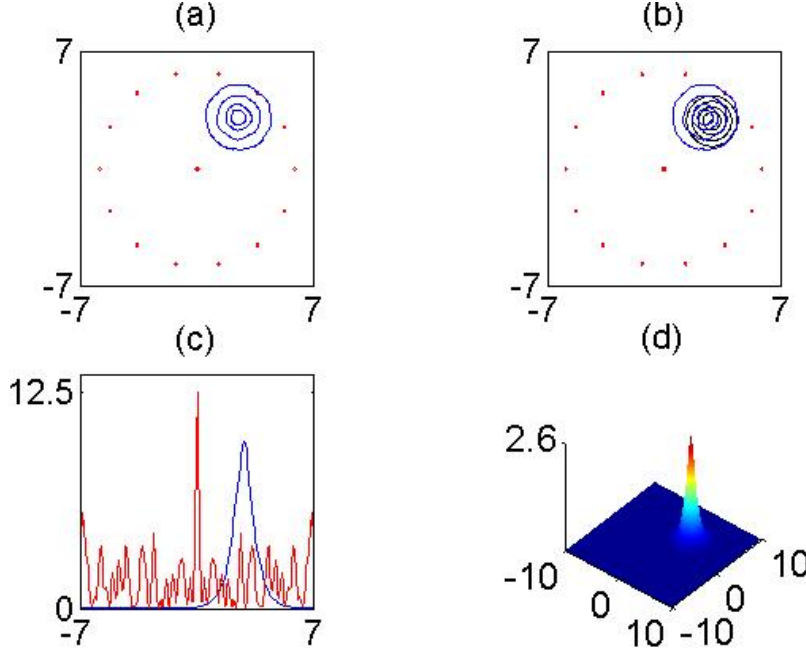


**Figure 5.9:** Contour plot of the (a) Penrose-7 soliton, (b) Initial condition and Penrose-7 soliton on top of each other, (c) Diagonal cross section of Penrose-7 soliton superimposed on Penrose-7 lattice, (d) 3D view of Penrose-7 soliton, when the soliton is centered at the lattice minima with  $\mu = -1$  and  $V_0 = 12.5$ .

As can be seen from Fig.(5.9) (b) and Fig.(5.10) (b), Penrose-7 solitons converge a bit upward of the place that the initial condition is centered.

### 5.4. Stability Properties of Penrose-7 Solitons

Now we address the critical question of nonlinear stability of the fundamental Penrose-7 solitons. Soliton power plays an important role in determining the stability properties of the fundamental solitons. As explained in Section 3, soliton



**Figure 5.10:** Same as Fig.(5.9) with  $\mu = 0.5$ .

power defined as

$$P = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |u(x, y)|^2 dx dy. \quad (5.2)$$

and slope condition defined as

$$\frac{dP}{d\mu} < 0 \quad (5.3)$$

i.e., the soliton is stable only if its power decreases with increasing propagation constant  $\mu$ .

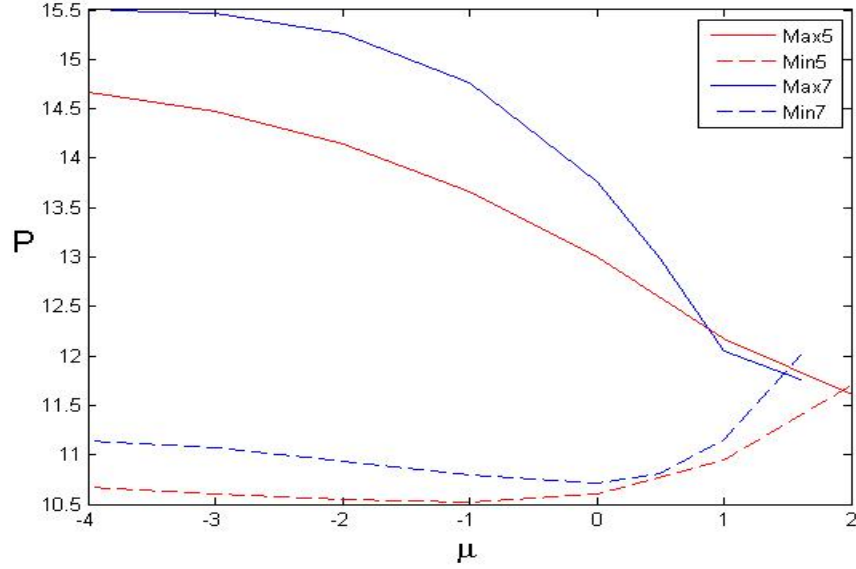
Also it is important to remember that, a necessary condition for collapse in the 2D cubic NLS equation is that the power of the beam exceeds the critical power  $P_c \approx 11.7$ .

After remembering these conditions, we now investigate soliton powers of Penrose-5 and Penrose-7 potentials in Fig.(5.11).

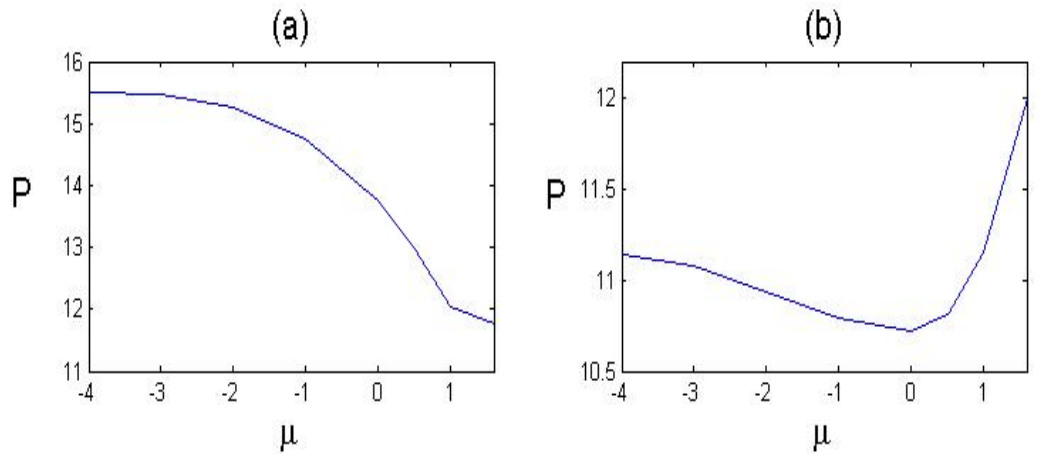
Before beginning stability analysis, we should give a detailed soliton power analysis of penrose-7 lattice (see Fig.(5.12)).

To study the nonlinear stability, we directly compute Eq.(1.4) over a long distance, (finite difference method was used on derivatives  $u_{xx}$  and  $u_{yy}$ , and

fourth-order Runge-Kutta method to advance in  $z$ ) for Penrose-7 potential. The initial conditions were taken to be a fundamental soliton with 1% random noise in the amplitude and phase. For Penrose-7 lattice, the nonlinear stability of the fundamental solitons located on the lattice maxima and minima are investigated in separate cases.



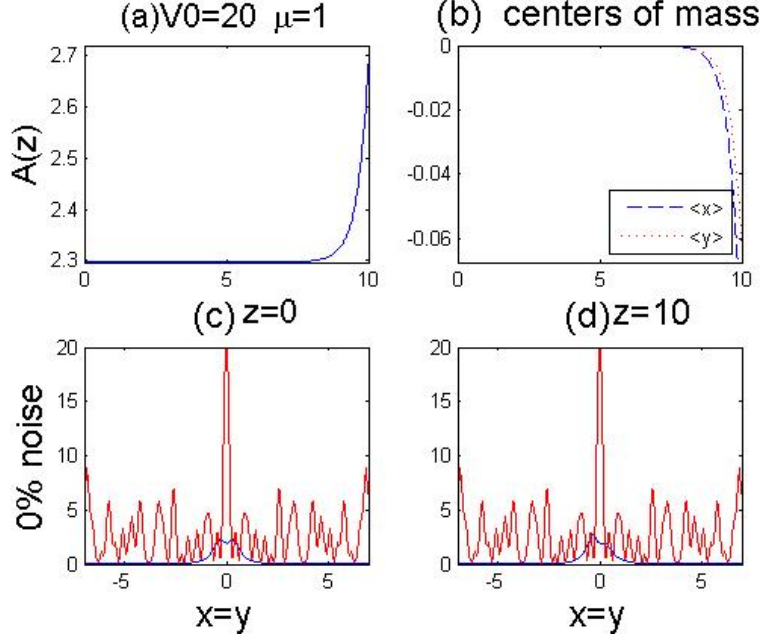
**Figure 5.11:** Soliton power as a function of the eigenvalue  $\mu$  within the semi-infinite band gap for Penrose-5 and Penrose-7 lattices, for solitons at lattice minima and lattice maxima. All lattices share a common peak depth  $V_0 = 12.5$ .



**Figure 5.12:** The power versus  $\mu$  for Penrose-7 potential, (a) Soliton located on the lattice maxima, (b) Soliton located on the lattice minima, for both cases peak depth  $V_0 = 12.5$ .

#### 5.4.1 Solitons centered at lattice maxima

Firstly, we locate the initial condition to the lattice maxima (the origin). In Fig.(5.13), evolution of the fundamental soliton at the lattice maxima are plotted for the potential depth  $V_0 = 20$  and  $\mu = 1$ .

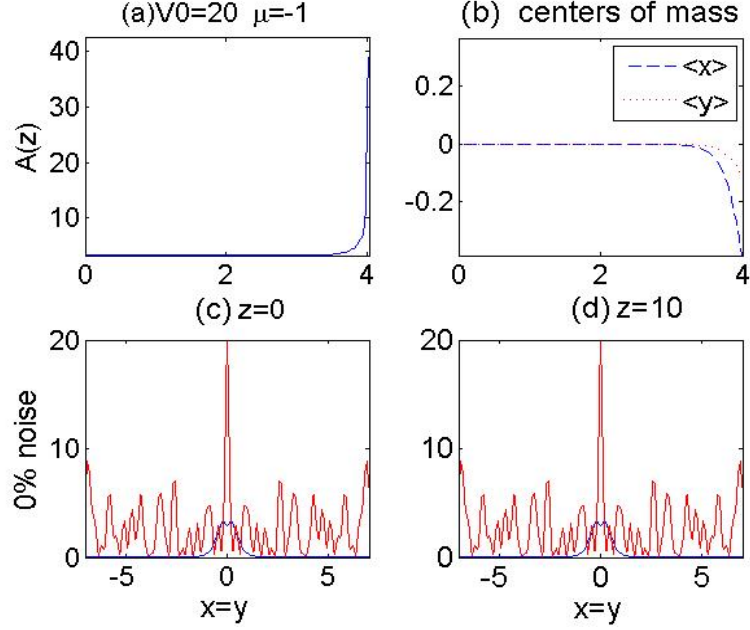


**Figure 5.13:** Evolution of soliton situated at the Penrose-7 potential with  $V_0 = 20$  and  $\mu = 1$ . (a) Peak amplitude  $A(z)$  as a function of the propagation distance. The initial condition is taken as the fundamental soliton, (b) Center of mass in the  $x$  and  $y$  coordinates, (c) Cross section along the diagonal axis of a fundamental soliton at the maximum superimposed on the Penrose-7 potential at  $z = 0$ , (d) Cross section along the diagonal axis of the fundamental soliton at the maximum superimposed on the Penrose-7 potential after the propagation ( $z = 10$ ).

As can be seen from Fig.(5.13), the peak amplitude of the fundamental soliton  $A(z) = \max_{x,y} |\tilde{u}(x,y,z)|$  increases with the propagation distance  $z$ . The fundamental soliton is unstable in this parameter regime.

In Fig.(5.14), we repeat the same stability analysis with different parameters ( $V_0 = 20$  and  $\mu = -1$ ), when the initial condition is centered at lattice maxima.

In Fig.(5.13) ( $\mu = 1$ ) and Fig.(5.14) ( $\mu = -1$ ) evolution of the fundamental soliton at the lattice maxima are computed for a range of values of  $\mu$  ( $-4 \leq \mu \leq 2$ ), power versus  $\mu$  is depicted for both cases in Fig.(5.12). As can be seen from Fig.(5.12) (a), the slope condition (Eq.(5.3)) is satisfied for both cases ( $\mu = -1$



**Figure 5.14:** Same as Fig.(5.13) with  $\mu = -1$ .

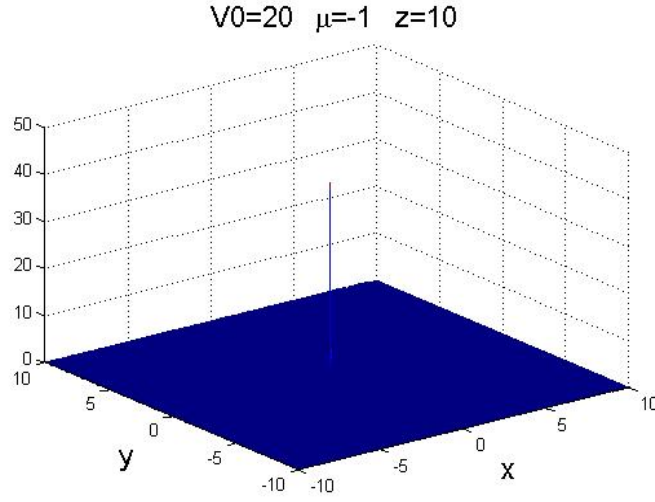
and  $\mu = 1$ ), but the power exceeds the critical collapse value ( $P_c = 11.72$ ) and collapse appears to be eventually attained.

Although we see the soliton peak  $A(z)$  about 40 in Fig.(5.14) (a), we don't see this in Fig.(5.14) (d), because the fundamental soliton moves from the potential maximum (i.e., centers of mass moves largely, see Fig.(5.14) (b)). If we plot 3D view of the fundamental soliton at  $z = 10$ , we can see this peak (see Fig.(5.15)).

#### 5.4.2 Solitons centered at lattice minima

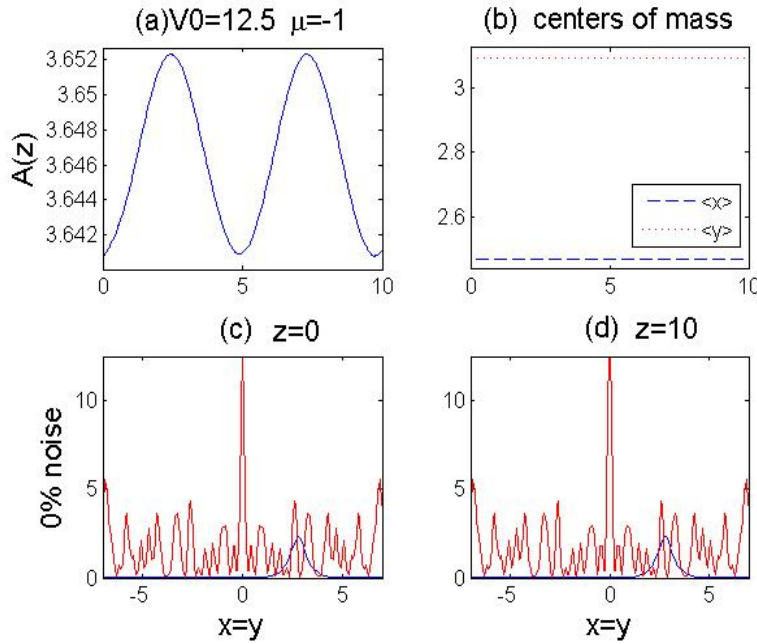
We now investigate solitons centered at the lattice minima for Penrose-7 potential. In both Fig.(5.16) and Fig.(5.17), evolution of the fundamental solitons at the lattice minima are plotted for the potential depth  $V_0 = 12.5$  and  $\mu = -1$ . In Fig.(5.16), fundamental soliton (which obtained by SR method) taken as initial condition without noise and, in Fig.(5.17) fundamental soliton is perturbed with 1% random noise in the amplitude and phase.

As can be seen from Fig.(5.16) and Fig.(5.17), peak amplitudes of the fundamental solitons  $A(z) = \max_{x,y} |\tilde{u}(x,y,z)|$  oscillate with the propagation distance  $z$  and the center of mass in the  $x$  and  $y$  axis nearly stay at the same

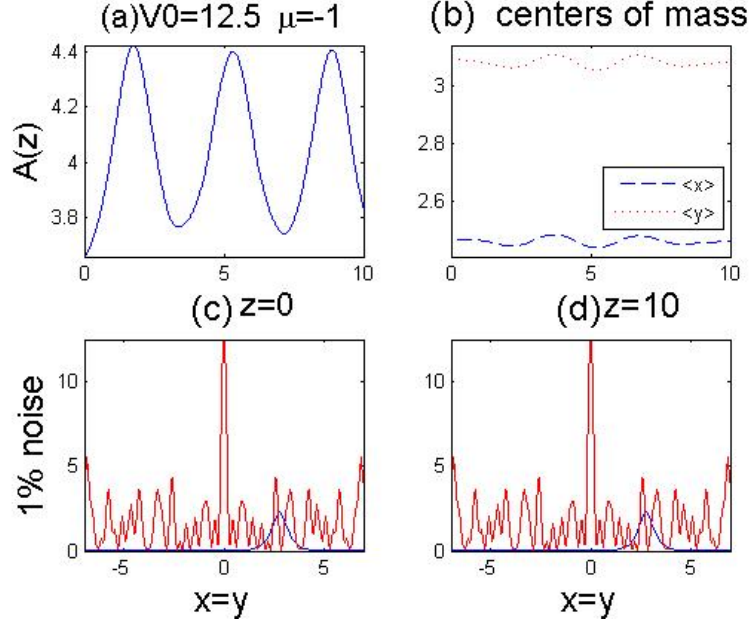


**Figure 5.15:** 3D view of the fundamental soliton at the maximum superimposed on the Penrose-7 potential after the propagation ( $z = 10$ ) with  $V_0 = 20$  and  $\mu = -1$ .

place. This suggests that the fundamental solitons for both cases are nonlinearly stable in this parameter regime for Penrose-7 potential.



**Figure 5.16:** Evolution of soliton situated at the Penrose-7 potential with  $V_0 = 12.5$  and  $\mu = -1$ . (a) Peak amplitude  $A(z)$  as a function of the propagation distance. The initial condition is taken as the fundamental soliton, (b) Center of mass in the  $x$  and  $y$  coordinates, (c) Cross section along the diagonal axis of a fundamental soliton at the minimum superimposed on the Penrose-7 potential at  $z = 0$ , (d) Cross section along the diagonal axis of the fundamental soliton at the minimum superimposed on the Penrose-7 potential after the propagation ( $z = 10$ ).



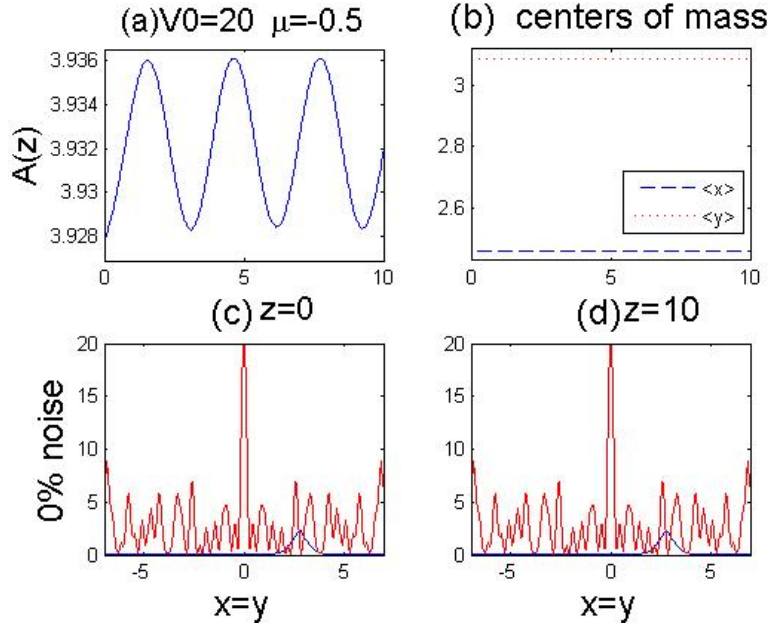
**Figure 5.17:** Same as Fig.(5.16), added 1% noise to fundamental soliton in amplitude and phase.

In Fig.(5.18) and Fig.(5.19), we repeat the same procedure for solitons which are centered at the lattice minima. In both Fig.(5.18) and Fig.(5.19), evolution of the fundamental solitons at the lattice minima are plotted for the potential depth  $V_0 = 20$  and  $\mu = -0.5$ . In Fig.(5.18), fundamental soliton (which obtained by SR method) taken as initial condition without noise and, in Fig.(5.19), fundamental soliton is perturbed with 1% random noise in the amplitude and phase.

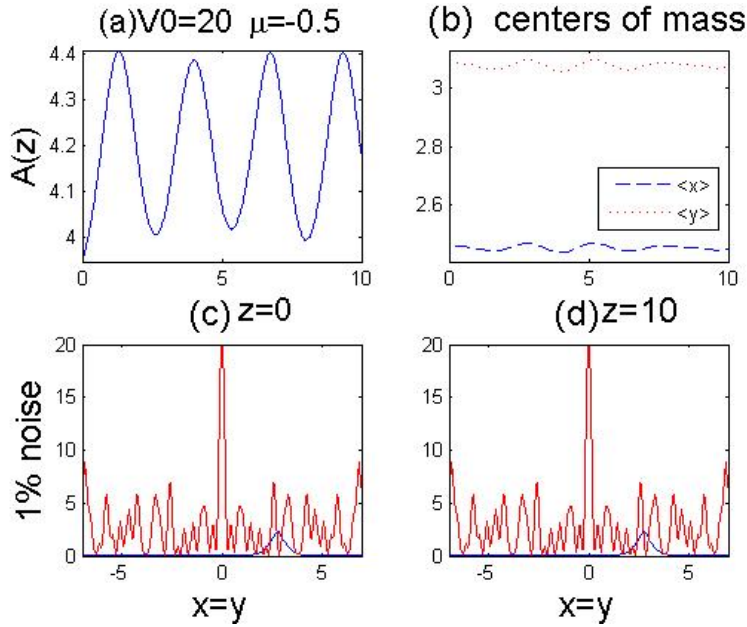
As can be seen from Fig.(5.18) and Fig.(5.19), peak amplitudes of the fundamental solitons  $A(z) = \max_{x,y} |\tilde{u}(x, y, z)|$  oscillate with the propagation distance  $z$  and the center of mass in the  $x$  and  $y$  axis nearly stay at the same place. This suggests that the fundamental solitons for both cases are nonlinearly stable in this parameter regime for Penrose-7 potential.

Finally, we give an example of unstable Penrose-7 soliton, when the initial condition is centered at lattice minima (see Fig.(5.20)).

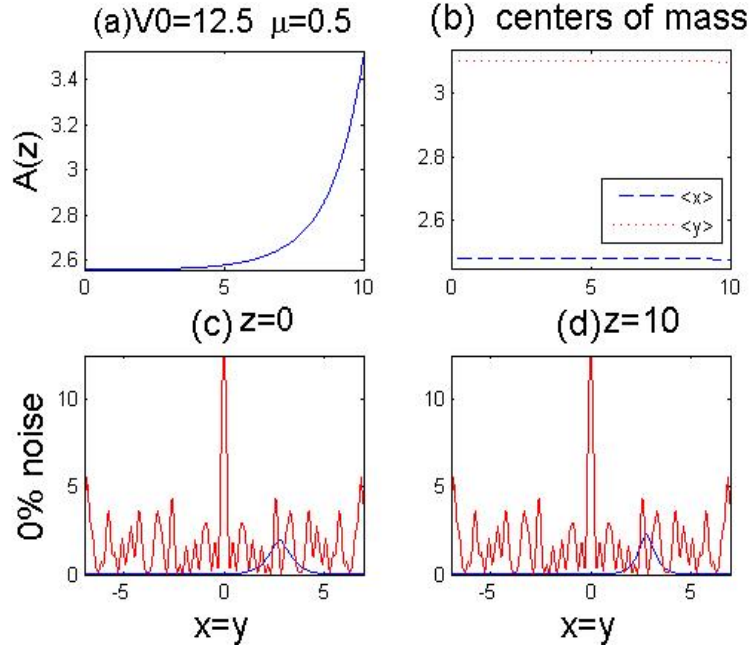
In Fig.(5.20), the soliton is centered at local minima and computed for  $\mu = 0.5$ , as can be seen from Fig.(5.12)(b), the power doesn't exceed the critical collapse value ( $P_c = 11.72$ ), but the slope condition (Eq.(3.9)) is not satisfied and this leads to a focusing instability.



**Figure 5.18:** Evolution of soliton situated at the Penrose-7 potential with  $V_0 = 20$  and  $\mu = -0.5$ . (a) Peak amplitude  $A(z)$  as a function of the propagation distance. The initial condition is taken as the fundamental soliton, (b) Center of mass in the  $x$  and  $y$  coordinates, (c) Cross section along the diagonal axis of a fundamental soliton at the minimum superimposed on the Penrose-7 potential at  $z = 0$ , (d) Cross section along the diagonal axis of the fundamental soliton at the minimum superimposed on the Penrose-7 potential after the propagation ( $z = 10$ ).



**Figure 5.19:** Same as Fig.(5.18), added 1% noise to fundamental soliton in amplitude and phase.

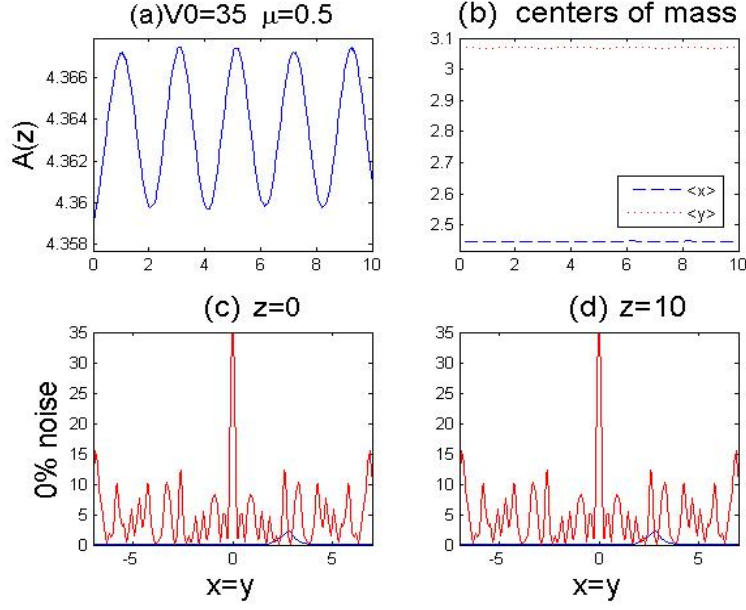


**Figure 5.20:** Same as Fig.(5.18) with  $\mu = 0.5$  and  $V_0 = 12.5$

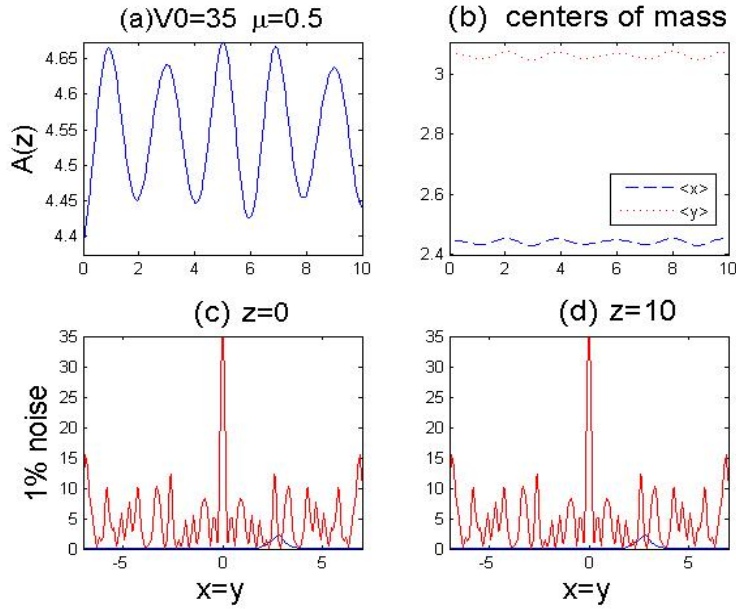
Now, we increase the depth ( $V_0$ ) of the lattice and repeat the same procedure with the same propagation constant ( $\mu = 0.5$ ) for dislocated lattice soliton. The nonlinear stability properties of Penrose-7 solitons with  $V_0 = 35$  and  $\mu = 0.5$  are investigated in Fig.(5.21) and Fig.(5.22). In Fig.(5.21), fundamental soliton (which obtained by SR method) taken as initial condition without noise and, in Fig.(5.22) fundamental soliton is perturbed with 1% random noise in the amplitude and phase.

As can be seen from Fig.(5.21) and Fig.(5.22), peak amplitudes of the fundamental solitons ( $A(z) = \max_{x,y} |\tilde{u}(x, y, z)|$ ) oscillate with the propagation distance  $z$  and the center of mass in the  $x$  and  $y$  axis nearly stay at the same place. This suggests that the fundamental Penrose-7 solitons are nonlinearly stable in this parameter regime.

As a result, deeper potential (increased values of  $V_0$ ) does improve the nonlinear stability of the fundamental solitons in Penrose-7 lattice similar to what has been observed for the dislocated and periodic lattices.



**Figure 5.21:** Evolution of soliton situated at the Penrose-7 potential with  $V_0 = 35$  and  $\mu = 0.5$ . (a) Peak amplitude  $A(z)$  as a function of the propagation distance. The initial condition is taken as the fundamental soliton, (b) Center of mass in the  $x$  and  $y$  coordinates, (c) Cross section along the diagonal axis of a fundamental soliton at the minimum superimposed on the Penrose-7 potential at  $z = 0$ , (d) Cross section along the diagonal axis of the fundamental soliton at the minimum superimposed on the Penrose-7 potential after the propagation ( $z = 10$ ).



**Figure 5.22:** Same as Fig.(5.21), added 1% noise to fundamental soliton in amplitude and phase.

## 6. CONCLUSION AND RECOMMENDATIONS

The purpose of this study was to investigate band gap structures and stability properties of localized solitons in various lattices (i.e., periodic, nonperiodic or defected lattices). Firstly, we give solution of the model (i.e., NLS equation with an external potential which is taken to be the periodic lattice) that governs nonlinear wave propagation, by perturbation and spectral methods and we compare results of these methods. We observed that the perturbation method is in good agreement with the spectral methods and, the obtained results are supported by previous studies. After that we solved the model by the numerical algorithm (named SR method) and, we obtained the first nonlinear band gap structure of the periodic lattice by SR method. By comparing the results obtained by the use of the numerical and analytical methods, it has been shown that the SR method is in good agreement with the perturbation and the spectral methods.

After obtaining fundamental solitons by SR method, we investigate stability properties of the solitons in periodic lattices. We investigate the nonlinear stability properties of fundamental solitons according to Vakhitov-Kolokolov (VK) stability criterion and by a direct computational method. It is observed that the numerical simulations are in agreement with VK stability criterion.

Dislocated lattice solitons are also investigated. We obtained that the first nonlinear band gap for dislocated lattice solitons is narrower than that of lattice solitons without dislocation and, there is a marked difference between soliton powers of dislocated lattice solitons and lattice solitons without dislocation. We investigated the nonlinear stability properties of dislocated lattice solitons and, we showed that numerical simulations are in agreement with VK stability criterion.

In the last part of the study, Penrose type lattice solitons are investigated. Same numerical algorithm (SR Method) has been used to investigate the Penrose type

lattice solitons namely for Penrose-5 and Penrose-7 solitons. By plotting on axis mode profiles of Penrose-5 and Penrose-7 solitons, we observed that Penrose-7 lattice solitons have similar properties (having dimples etc.) with Penrose-5 lattice solitons. Finally, we investigated the nonlinear stability properties of Penrose-7 solitons and, we showed that numerical simulations are in good agreement with VK stability criterion.

In this study, it is shown that, adding a defect to a periodic lattice decreases the band size and also effects the nonlinear stability properties in a negative way. Such a comparison can also be made between crystal and quasicrystal lattices. Having irregularities or quasicrystal properties impoverish both the band size and nonlinear stability of the fundamental solitons. Both in the dislocated lattice solitons and Penrose-7 solitons, it has been observed that, deeper lattices (increased potential depth) help to arrest collapse which usually occurs in the existence of irregularities such as defects or quasicrystal structures. As a result, using deeper lattices is shown to be an effective method as the collapse arrest mechanism.

As a future project, investigating the linear stability properties of dislocated lattice solitons and Penrose-7 solitons may be considered. Also, investigating double and triple solitons and vortex structures (solitons carrying angular momentum) in dislocated and Penrose-7 lattices can be a very interesting research topic. Since collapse is one of the leading research interests appearing in the current literature, applying other collapse arrest mechanisms such as adding a nonlinear saturation to the physical model can be considered as a future study. On the other hand, second band solitons have been observed experimentally but currently, the numerical algorithm(SR) is not able to calculate such solitons. Therefore, modifying SR to generate the second band solitons may be one of the most important research priorities.

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## **CURRICULUM VITAE**

Mahmut Bağcı was born in 1983 in Adıyaman. After graduated Adıyaman high school, he began his undergraduate education at İstanbul Technical University Urban and Regional Planning Department in 2001. He passed Physics Engineering Department in 2003 and graduated from this department in 2007. He began his graduate education at İstanbul Technical University Mathematics Programme in 2007 and he continues this education.