İSTANBUL TECHNICAL UNIVERSITY ★ INSTITUTE OF SCIENCE AND TECHNOLOGY

(3+3+2) WARPED-LIKE PRODUCT MANIFOLDS WITH $SPIN(7)$ HOLONOMY

Ph.D. Thesis by

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Department : MATHEMATICS ENGINEERING

Programme : MATHEMATICS ENGINEERING

FEBRUARY 2008

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Date of Submission	:	**4 February 2008**
Date of Examination	:	**21 November 2008**

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FEBRUARY 2008

İSTANBUL TEKNİK ÜNİVERSİTESİ ★ FEN BİLİMLERİ ENSTİTÜSÜ

$SPIN(7)$ HOLONOMİSİNE SAHİP (3+3+2) WARPED-BENZERİ ÇARPIM MANİFOLDLARI

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Tezin Enstitüye Verildiği Tarih : 4 Şubat 2008
Tezin Savunulduğu Tarih : 21 Kasım 2008

Tez Danışmanı : Prof. Dr. Ayşe H. BİLGE

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ŞUBAT 2008

ACKNOWLEDGEMENTS

I would like to thank my supervisor Prof. Dr. Ayşe H. Bilge for supports, advices during the preparation of this thesis and and giving me guidance throughout the process of this work with remarkable patience and conscientiousness. I should also like to thank Prof. Dr. Tekin Dereli, Prof. Dr. Vahap Erdoğdu and Prof. Dr. Zerrin Şentürk for many comments and invaluable support during this work.

Additionally I would also like to thank Prof. Dr. Şahin Koçak for guidance and continual encouragement throughout thesis. He taught me many fundamental concepts and their relations with his excellent and genuine manner of presentation.

I would like to thank Prof. Dr. Thomas Friedrich, Prof. Dr. Ilka Agricola, Dr. Simon Chiossi, Dr. Richard Cleyton and Dr. Takayoshi Ootsuka for many fruitful discussions and comments.

I also wish to give my very special thanks my wife and my little son for living with the thesis as well as giving me love and support all through the years.

This thesis was supported by TUBITAK under Project No: 106T558 and TUBITAK-BIDEB Research Programme 2214.

February 2008 Selman UĞUZ

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LIST OF SYMBOLS

$\mathbf{M}$	:	Differentiable manifold
$\mathbf{V}$	:	Vector space
$\mathbf{V}^*$	:	Dual vector space of V
$(\mathbf{r},\mathbf{s})$	:	Tensor of r-covariant, s-contravariant
$\mathbf{T^r_s}(\mathbf{V})$	:	Set of tensors of r-covariant, s-contravariant
$\Sigma^r(V)$	:	Set of symmetric tensors of r-covariant
$\wedge^{\mathbf{r}}(\mathbf{V})$	:	Set of alternating tensors of r-covariant
$\otimes$	:	Tensorial product
$\wedge$	:	Wedge product
$\mathbf{d}$	:	Exterior derivative
$\mathbf{v_p}$	:	Tangent vector at p
$\mathbf{T_pM}$	:	Tangent vector space at p
$\mathbf{T^*_pM}$	:	Cotangent vector space at p
$\mathbf{TM}$	:	Tangent bundle
$\mathbf{T^*M}$	:	Cotangent bundle
$\chi(\mathbf{M})$	:	Vector fields on M
$\mathbf{g}$	:	Metric tensor
∇	:	Connection
Γ^k_{ij}	:	Second type Christoffel symbol
$\mathbf{R^l_{ijk}}$	:	Curvature tensor
$\mathbf{T}$	:	Torsion tensor
$\mathbf{K}(\mathbf{M})$	:	Sectional curvature of M
$\mathbf{w}$	:	Connection one-form matrix
$\mathscr{R}$	:	Curvature 2-form matrix
$\mathbb{R}$	:	Real numbers
$\mathbb{C}$	:	Complex numbers
$\mathbb{H}$	:	Quaternions
$\mathbb{O}$	:	Octonions
$\mathbb{R}^{\mathbf{n}}$	:	n-dimensional Real vector space
$\mathbf{SO}(\mathbf{n})$	:	Special orthogonal group
$\mathbf{SU}(\mathbf{n})$	:	Special unitary group
$\mathbf{Sp}(\mathbf{n})$	:	Symplectic group
$\mathbf{\Omega}$	:	Bonan form
$\|,\|$	:	Norm
$\langle,\rangle$	:	Inner product
$*$	:	Hodge dual

(3+3+2) WARPED-LIKE PRODUCT MANIFOLDS WITH $SPIN(7)$ HOLONOMY

SUMMARY

In the theory of Riemannian holonomy groups there are two exceptional cases, the holonomy group G_2 in 7-dimensional and the holonomy group $Spin(7)$ in 8-dimensional manifolds. In the present thesis, we investigate the structure of Riemannian manifolds whose holonomy group is a subgroup of $Spin(7)$, for a special case.

Manifolds with $Spin(7)$ holonomy are characterized by the existence of a 4-form, called the ***Bonan form*** (Cayley form or Fundamental form), which is self-dual in the Hodge sense, $Spin(7)$ invariant and closed. In Chapter 2, we review two methods for the construction of the Bonan form, based on the octonionic multiplication and the triple vector cross products on octonions.

In Chapter 3, we survey a metric with $Spin(7)$ holonomy on $S^3 \times S^3 \times \mathbb{R}^2$ given by Yasui and Ootsuka. By using a specific tensor formula called the 2-vector condition given there, we obtain conditions on the commutators of orthonormal vector fields for the existence of a metric with $Spin(7)$ holonomy on an arbitrary 8-manifold.

In Chapter 4, we define ***"(3+3+2) warped-like product manifolds"*** as a generalization of multiply warped product manifolds, by allowing the fiber metric to be non block diagonal, on a manifold $M = F \times B$, where the base B is a two dimensional Riemannian manifold, the fibre F is a 6-manifold of the form $F = F_1 \times F_2$ where F_i's $(i = 1,2)$ are complete, connected and simply connected Riemannian 3-manifolds. In the Yasui-Ootsuka solution, the underlying manifold is of this type and the fibers are assumed to be S^3. In this thesis we prove that if the specific Bonan form given in Yasui-Ootsuka is closed, then the fibre spaces F_i's are isometric to S^3. This implies that the Yasui-Ootsuka solution is unique in the class of $(3+3+2)$ warped-like product metrics admitting the $Spin(7)$ structure determined by the Bonan form given in Yasui-Ootsuka.

Finally we briefly discuss the conclusions of the study and the directions for future research.

$SPIN(7)$ HOLONOMİSİNE SAHİP (3+3+2) WARPED-BENZERİ ÇARPIM MANİFOLDLARI

ÖZET

Riemann holonomi teorisinde ayrıcalıklı iki holonomi grubu vardır. Bunlar 7-boyutlu manifoldlar üzerinde olan G_2 ve 8-boyutlu manifoldlar üzerinde olan $Spin(7)$ holonomi gruplarıdır. Bu tez çalışmasında, holonomi grubu, $Spin(7)$'nin bir alt grubu olan Riemann manifoldlarının yapısı özel bir durum için incelenmiştir.

2. Bölümde, $Spin(7)$ holonomi grubuna sahip manifoldlar, *Bonan formu* (Cayley formu veya esas form) olarak adlandırılan, Hodge anlamında kendine eş, $Spin(7)$ invaryant ve kapalı bir 4-formun varlığı ile karakterize edilir. Oktonion çarpımı ve oktonionlar üzerinde tanımlı üçlü vektör çarpımı kullanılarak Bonan formunun elde edilme yolları tartışılmıştır.

3. Bölümde, Yasui-Ootsuka tarafından $S^3 \times S^3 \times \mathbb{R}^2$ manifoldu üzerinde verilen metrik incelenmiştir. Bu makalede verilen 2-vektör şartı adlı bir tensör denklemi kullanılarak, herhangi bir manifold üzerindeki metriğin $Spin(7)$ holonomiye sahip olması için, ortonormal vektör alanlarının komütatörlerinin sağlaması gereken koşullar elde edilmiştir.

4. Bölümde, çoklu warped çarpım manifoldlarının, lif metriği diagonal olmayan bir genellemesi olan *"(3+3+2) warped-benzeri çarpım manifoldları"* tanımlanmıştır. Bu manifoldlar $M = F \times B$ şeklinde olup, B iki boyutlu bir Riemann manifoldu, F_i, $(i = 1,2)$ bağlantılı, basit bağlantılı, tam Riemann 3-manifoldlar ve $F = F_1 \times F_2$ şeklinde 6-boyutlu bir Riemann manifolddur. Yasui-Ootsuka çözümünde, lif uzayları 3-küreler olarak alınan bu tip manifold örneği çalışılmıştır. Bu tez çalışmasında, Yasui-Ootsuka tarafından verilen Bonan formu, yukarıdaki koşullar altında kapalı olduğunda, liflerin F_i $(i = 1,2)$ S^3'e isometrik olduğu ispatlanmıştır. Bu sonuç ise, Yasui-Ootsuka çalışmasındaki Bonan formu tarafından belirlenen $Spin(7)$ yapısına sahip, $(3+3+2)$ warped-benzeri çarpım sınıfları içerisinde, Yasui-Ootsuka çözümünün tekliğini göstermektedir.

Son bölümde çalışmamızın sonuçlarını irdelenip, ileride çalışılabilecek araştırma konuları tartışılmıştır.

1. INTRODUCTION

In this thesis we will study Riemannian manifolds whose holonomy group is contained in $Spin(7)$. These manifolds are characterized by the existence of a 4-form, called the Bonan form (or the fundamental form, Cayley form) their geometry is very rich and in particular they are Ricci-flat [9].

In Riemannian geometry, there is a unique torsion free metric connection ∇, called the Levi-Civita connection which defines parallel the transport of vectors parallelly along curves. When we transport vectors around a closed curve, their final position can be different from their initial position. This change is expressed as a *holonomy transformation*. The set of all such changes constitutes a group of transformations which is called the *holonomy group* (see Section 2.4.2).

The holonomy group of a Riemannian manifold was defined by Élie Cartan in 1923 and proved to be an efficient tool in the study of Riemannian manifolds (see [16, 17, 34, 42] for further details). Cartan gave a classification of holonomy groups for irreducible, simply-connected, Riemannian symmetric manifolds by using the theory of Lie groups. The list of possible holonomy groups of irreducible, simply-connected, non-symmetric Riemannian manifolds which is called *Berger's list* (see Table 2.1) was given by Marcel Berger in 1955 [4].

Berger's list includes the group $SO(n)$ as the generic case, $U(n)$, $SU(n)$ in $2n$-dimensions, $Sp(n)$, $Sp(n)Sp(1)$ in $4n$-dimensions and two special cases, G_2 holonomy in 7-dimensions and $Spin(7)$ holonomy in 8-dimensions. Manifolds with holonomy groups $U(n)$, $SU(n)$, $Sp(n)$, $Sp(n)Sp(1)$ are denoted as manifolds with *special holonomy* and the two special cases are described as manifolds with *exceptional holonomy*. When Berger's list was presented, the existence of manifolds with special holonomy was an open problem (see Section 2.4.3.1).

The existence of manifolds with exceptional holonomy was first demonstrated by R.Bryant in 1987 [10], then complete examples were given by R. Bryant and S.

Salamon in 1989 [11] and the first compact examples were found by D. Joyce in 1996 [30]. The study of manifolds with exceptional holonomy and the construction of explicit examples is still an active research area in mathematics and physics.

The thesis is organized as follows.

In Chapter 2, we first overview certain basic concepts from Riemannian geometry and present the definitions to set up our notational conventions. Then warped product and multiply warped product manifolds are also reviewed. Finally the classification of Riemannian manifolds with special holonomy is given in detail.

In Chapter 3, we concentrate on manifolds with $Spin(7)$ holonomy and review the structure of a certain 4-form called the *Bonan form* Ω [9]. An explicit construction of the Bonan form is presented in two different ways, using the structure constants of octonionic algebra in Section 3.1.1 and the vector cross products on octonions in Section 3.1.2. We note that there are many different multiplication tables of octonions in the literature [2]. In order to obtain the same Bonan form (see the equation (3.11)) in these two different constructions, we use two different multiplication rules of octonions respectively given in Table 3.1 and Table A.1 (see Appendix A.1).

In Section 3.2.1, we give an overview of the method given by Yasui-Ootsuka [45] and obtain the explicit form of the equations for the existence of a metric with $Spin(7)$ holonomy in terms of vector fields for the general case (see the equation (3.31)). Then in Section 3.2.2, we present the $Spin(7)$ metric (equation (3.57)) obtained by Yasui-Ootsuka [45] on $S^3 \times S^3 \times \mathbb{R}^3$, that we call the *"Yasui-Ootsuka solution."*

In Chapter 4, we start to work with the explicit $Spin(7)$ metric on $S^3 \times S^3 \times \mathbb{R}^2$ given in the equation (3.57) and look whether one could obtain other solutions by relaxing some of their assumptions, in particular without requiring the three dimensional submanifolds to be S^3. With this purpose, we define ***warped-like product metrics*** as a general framework for our metric ansatz, by allowing the fiber metric to be non block diagonal as presented in Section 4.2.

In Section 4.3, we work with a specific $(3+3+2)$ warped-like product manifold

$$M = F_1 \times F_2 \times B \tag{1.1}$$

and a specific $Spin(7)$ structure. In Section 4.4, we prove that, when the base B is two dimensional, the fibre F is a 6-manifold of the form $F = F_1 \times F_2$ such that F_i's $(i = 1,2)$ are complete, connected and simply connected 3-manifolds and the metric is given by the equation (4.28), then the connection of the fibers is completely determined by the requirement that the Bonan 4-form given in the equation (3.11) be closed.

With the global assumptions given above, it is concluded that the fibers $(F_i, i = 1,2)$ are isometric to 3-spheres S^3. It follows that the Yasui-Ootsuka solution is unique in the class of $(3+3+2)$ warped-like product metrics defined by the equation (4.28) admitting the $Spin(7)$ structure determined by the Bonan form given in the equation (3.11).

In Chapter 5, conclusions of the study will also be discussed briefly and some of the relevant concepts of our work will be presented for future studies.

2. PRELIMINARIES

2.1 Basic Definitions

In this section we will briefly recall certain basic facts from differential geometry that are used in the thesis. These will also be helpful to set up our notational conventions.

Definition 2.1. An n-dimensional *manifold* M is a topological space such that each point has a neighborhood homeomorphic to an open subset of the Euclidean space $\mathbb{R}^n$.

In addition, we assume that M is also Hausdorff and second countable to ensure that the manifold embeds in some finite-dimensional Euclidean space [34]. We also note that the Hausdorff condition is an essential part of the definition, because there are locally Euclidean spaces which are non-Hausdorff.

Let M be a manifold, a pair (U,ϕ) is called a coordinate chart for M if $U \subset M$ is an open set and ϕ is a homeomorphism of U to an open subset $\phi(U) \subset \mathbb{R}^n$. Two charts (U_1,ϕ_1) and (U_2,ϕ_2) are called C^∞-compatible if whenever $U_1 \cap U_2$ is non-empty, the mapping

$$\phi_1 \circ \phi_2^{-1} : \phi_2(U_1 \cap U_2) \to \phi_1(U_1 \cap U_2) \tag{2.1}$$

is a diffeomorphism.

An atlas is a family of charts (U_α, ϕ_α) where any two are C^∞-compatible and $M = \bigcup_{\alpha \in I} U_\alpha$, where I is an index set. The manifold M with a smooth differentiable structure is called a differentiable manifold.

Definition 2.2. Let M be a differentiable manifold and $\alpha : I \longrightarrow M$ be a differentiable curve in M. Let $C^\infty(M)$ be the set of functions on M that are differentiable at p. Suppose that $\alpha(0) = p \in M$. A *tangent vector* to the curve α at $t = 0$ is a function $\alpha_p' : C^\infty(M) \longrightarrow \mathbb{R}$ given by

$$\alpha_p'(f) = \frac{d(f \circ \alpha)}{dt}|_{t=0} \tag{2.2}$$

where $f \in C^\infty(M)$.

A tangent vector denoted by v_p is the tangent vector at $t=0$ of some curve $\alpha : I \longrightarrow M$ with $\alpha(0) = p$. The tangent vectors at p satisfy the following properties [34],

$$v_p : C^\infty(M) \longrightarrow \mathbb{R} \tag{2.3}$$

$$\begin{aligned} &i)\ v_p(af+bg) = av_p(f) + bv_p(g) \qquad \text{(linearity)} \\ &ii)\ v_p(fg) = v_p(f)g + v_p(g)f \qquad \text{(Leibniz rule)} \end{aligned}$$

Definition 2.3. Let M be a manifold, for $p \in M$, the set of tangent vectors at p is called the *tangent space* of M at p and denoted by T_pM.

$$T_pM = \{v_p \mid v_p : C^\infty(M) \longrightarrow \mathbb{R}\}. \tag{2.4}$$

The tangent space at p is a vector space. Let $v_p, w_p \in T_pM$, $\lambda \in \mathbb{R}$ then

$$\begin{aligned} (v+w)_p &= v_p + w_p, \\ (\lambda v)_p &= \lambda v_p. \end{aligned} \tag{2.5}$$

All tangent spaces of a manifold is called the *tangent bundle*.

$$TM = \cup_{p \in M} T_pM = \{(p, v_p) \mid p \in M,\ v_p \in T_pM\}. \tag{2.6}$$

If M is an n-dimensional manifold, then the tangent bundle is a $2n$-dimensional manifold [34].

Definition 2.4. A *vector field* v on a differentiable manifold M is a correspondence that associates a vector $v_p \in T_pM$ to each point $p \in M$

$$\begin{aligned} v: \quad & M \longrightarrow TM \\ & p \rightarrow v(p) = v_p \in T_pM. \end{aligned} \tag{2.7}$$

The set of smooth vector fields is a vector space and it is denoted by $\chi(M)$.

Definition 2.5. A real *tensor* ϕ on a vector space V is a multi-linear map

$$\phi : \underbrace{V \times \ldots \times V}_{r} \times \underbrace{V^* \times \ldots \times V^*}_{s} \longrightarrow \mathbb{R} \tag{2.8}$$

where V^* is the dual space of V.

The integers r and s are called respectively covariant and contravariant orders of ϕ. The set of all tensors on V of covariant order r and contravariant order s is denoted by $T_s^r(V)$. We will say ϕ is (r,s)-type tensor field. If $s=0$, $\phi \in T_0^r(V)$ is called r-covariant tensor space.

Let ϕ be a r-covariant tensor in $T_0^r(V)$ and $S(r)$ denote the permutation group of the set of natural numbers $\{1,2,...,r\}$. Note that $\sigma \in S(r) \Longrightarrow \sigma : (1,2,...,r) \longrightarrow (\sigma(1),\sigma(2),...,\sigma(r))$ and

$$sgn\sigma = \begin{cases} 1, & \sigma \quad is \quad even, \\ -1, & \sigma \quad is \quad odd. \end{cases} \tag{2.9}$$

ϕ is *alternating*(or *anti-symmetric*) if

$$\phi(X_1,.....,X_r) = sgn\sigma\phi(X_{\sigma(1)},.....X_{\sigma(r)}), \tag{2.10}$$

ϕ is *symmetric* if

$$\phi(X_1,.....,X_r) = \phi(X_{\sigma(1)},.....X_{\sigma(r)}), \tag{2.11}$$

for every $\sigma \in S(r)$.

The set of all alternating r-covariant tensors and all symmetric r-covariant tensors in $T^r(T_pM)$ is denoted by $\Lambda^r T_pM$ and $\Sigma^r T_pM$ respectively. An alternating r-covariant tensor field of order r on a manifold M is called an *exterior differential form* or *r-form*.

Definition 2.6. Let M be an n-dimensional differentiable manifold and $\Lambda(TM)$ be the direct sum of all the spaces $\Lambda^p(TM)$. There exists a unique multi-linear map

$$d : \Lambda(TM) \to \Lambda(TM)$$

which satisfies following conditions

$i)$ $\quad f \in \Lambda^0(TM) = C^\infty(M)$ then, $\; df\;$ is the total derivative of $\; f$,

$ii)$ $\quad \phi \in \Lambda^r(TM)\;$ and $\;\varphi \in \Lambda^s(TM)\;$ then,

$$d(\phi \wedge \varphi) = d\phi \wedge \varphi + (-1)^r \phi \wedge d\varphi,$$

$iii)$ $\quad d^2 = 0.$

The map d is called *exterior differentiation or exterior derivative.*

2.2 Riemannian Geometry

In this section, we present certain fundamental concepts in Riemannian geometry.

2.2.1 Metric Tensor

Let M be a differentiable manifold and $p \in M$. A Riemannian metric g is a type $(2,0)$ tensor field on M

$$g : TM \times TM \longrightarrow \mathbb{R} \tag{2.12}$$

which satisfies the following conditions at each point $p \in M$ and for each $u, v \in TM$,

$$i)\ g_p(u,v) = g_p(v,u)$$
$$ii)\ g_p(u,u) \geq 0, \qquad \text{(equality holds iff u} = 0)$$

where $g_p = g\,|_p$. In a local coordinate basis $\{x^i\}$, we can write the metric tensor g as

$$g = \sum_{i,j=1}^{n} g_{ij} dx^i dx^j. \tag{2.13}$$

2.2.2 Hodge Duality

Let V be an n-dimensional oriented real inner product space. Then there is a linear transformation called the Hodge star operator [43]

$$* : \Lambda(V) \rightarrow \Lambda(V) \tag{2.14}$$

which is given by the requirement that for any orthonormal basis $\{e_1, e_2, ..., e_n\}$ of V

$$\begin{aligned} *(e_1 \wedge ... \wedge e_n) &= \pm 1, \\ *(e_1 \wedge ... \wedge e_p) &= \pm e_{p+1} \wedge ... \wedge e_n. \end{aligned} \tag{2.15}$$

The Hodge star operator has the following property on $\Lambda^p(V)$

$$** = (-1)^{p(n-p)}. \tag{2.16}$$

2.2.3 Connections

Let M be a C^∞ manifold, a connection on M is a map

$$\nabla : \chi(M) \times \chi(M) \to \chi(M), \tag{2.17}$$

which satisfies the following three conditions

$$\begin{aligned} &i) \quad \nabla_X(aY+bZ) = a\nabla_X Y + b\nabla_X Z, \\ &ii) \quad \nabla_{(fX+gY)}Z = f\nabla_X Z + g\nabla_Y Z, \\ &iii) \ \nabla_X(fY) = (Xf)Y + f\nabla_X Y, \end{aligned}$$

where $X,Y,Z \in \chi(M)$, $a,b \in R$ and $f,g \in C^\infty(M)$.

The torsion tensor $\mathbf{T}$ of a connection ∇ is a $(2,1)$ type tensor defined by

$$\begin{aligned} \mathbf{T}: \quad & TM \times TM \longrightarrow TM \\ & \mathbf{T}(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y] \end{aligned} \tag{2.18}$$

where $X,Y \in TM$. A connection ∇ with $\mathbf{T}=0$ is said to be *torsion-free* or *symmetric connection*. Note that ∇ and $\mathbf{T}$ are defined without a metric. If there is a metric g on M, then we state the compatibility g and ∇ as

$$\nabla_X(g(u,v)) = (\nabla_X g)(u,v) + g(\nabla_X u, v) + g(u, \nabla_X v), \tag{2.19}$$

where X is a tangent vector. The fundamental theorem of Riemannian geometry is as follows.

Theorem 2.7. *Given a Riemannian manifold M, there exists a unique torsion-free connection ∇ compatible with g called the Levi-Civita connection.*

In local coordinates, the Levi-Civita connection can be given in terms of the Christoffel symbols of the second kind Γ^k_{ij}. Let ∇ be a connection on M and $\partial_i = \frac{\partial}{\partial x^i}$, the Christoffel symbols Γ^k_{ij} is written as

$$\nabla_{\partial_i}\partial_j = \sum_k \Gamma^k_{ij}\partial_k. \tag{2.20}$$

As the connection is torsion-free,

$$\nabla_{\partial_i}\partial_j - \nabla_{\partial_j}\partial_i = [\partial_i, \partial_j]. \tag{2.21}$$

Since

$$[\partial_i, \partial_j] = 0, \tag{2.22}$$

we have $\nabla_{\partial_i}\partial_j = \nabla_{\partial_j}\partial_i$. This expression implies that

$$\Gamma^k_{ij} = \Gamma^k_{ji}. \tag{2.23}$$

If we use compatibility of g with ∇, we obtain

$$\partial_i\langle\partial_j, \partial_\sigma\rangle + \partial_j\langle\partial_i, \partial_\sigma\rangle - \partial_\sigma\langle\partial_i, \partial_j\rangle = 2\langle\partial_\sigma, \nabla_{\partial_i}\partial_j\rangle. \tag{2.24}$$

From this, we have

$$\partial_i(g_{j\sigma}) + \partial_j(g_{i\sigma}) - \partial_\sigma(g_{ij}) = 2\sum_\sigma \Gamma^k_{ij} g_{\sigma k}. \tag{2.25}$$

If we multiply both sides with the inverse of $g_{\sigma k}$, we obtain the classical expression of Christoffel symbols Γ^k_{ij} in terms of metric components in a local coordinate basis as follows [15]

$$\Gamma^k_{ij} = \frac{1}{2}\sum_{\sigma=1}^{n} g^{k\sigma}\left(\frac{\partial g_{\sigma j}}{\partial x^i} + \frac{\partial g_{\sigma i}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^\sigma}\right). \tag{2.26}$$

2.2.4 Parallel Translation and Geodesics

Let M be a Riemannian manifold and α be a smooth curve in M defined on the interval I. Let us choose a point $p \in M$ on the curve and denote by T its tangent vector at this point

$$\alpha : I \to M, \quad p = \alpha(t_0), \quad T = \alpha_p^{'}(t). \tag{2.27}$$

Let $\varphi = (x^1, x^2, ...x^n)$ be a local coordinate system, (U, φ) be a chart and Y be a vector field on U.

Definition 2.8. A vector field Y on $\alpha : [a,b] \to M$ is *parallel* on α if $\nabla_T Y = 0$.

In local coordinates, T and Y can be written as

$$T = \alpha_p^{'}(t) = \sum_{i=1}^{n} \frac{d\alpha^i}{dt}\partial_i = \sum_{i=1}^{n} T^i \partial_i, \tag{2.28}$$

$$Y(t) = \sum_{j=1}^{n} Y^j \partial_j, \tag{2.29}$$

where $\alpha(t) = \left(\alpha^1(t), \alpha^2(t), ..., \alpha^n(t)\right)$ and $\partial_i = \frac{\partial}{\partial x^i}$. By using the definition, we obtain the equations of parallel displacement in local coordinates as

$$\nabla_T Y = \sum_{k=1}^{n} \left[\frac{dY^k}{dt} + \sum_{i,j=1}^{n} T^i Y^j \Gamma_{ij}^k \right] \partial_k \tag{2.30}$$

$$= \frac{dY^k}{dt} + \sum_{i,j=1}^{n} T^i Y^j \Gamma_{ij}^k = 0 \tag{2.31}$$

where $k = 1, 2, 3, ...n$.

Definition 2.9. A curve α is a *geodesic* if $\nabla_T T = 0$.

If we take $Y = T$ and $Y^k = \frac{d\alpha^k(t)}{dt}$, the equation of the geodesics on a manifold M in local coordinates is given by

$$\frac{d^2\alpha^k}{dt^2} + \sum_{i,j=1}^{n} \frac{d\alpha^i}{dt} \frac{d\alpha^j}{dt} \Gamma_{ij}^k = 0, \tag{2.32}$$

where $k = 1, 2, 3, ...n$.

2.2.5 Curvature and the Ricci Tensor

We present the definition of Riemannian curvature tensor and the Ricci tensor as follows.

Definition 2.10. The *Riemannian curvature tensor* R is a $(3,1)$ type tensor field defined by

$$\begin{aligned} R: \quad TM \times TM \times TM \to \quad & TM \\ (X,Y,Z) \mapsto \quad & R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z. \end{aligned} \tag{2.33}$$

We prove the tensoriality of R as

$$\begin{aligned} R(fX, gY)hZ &= f\nabla_X\{g\nabla_Y(hZ)\} - g\nabla_Y\{f\nabla_X(hZ)\} - fX(g)\nabla_Y(hZ) \\ &+gY(g)\nabla_X(hZ) - fg\nabla_{[X,Y]}(hZ) \\ &= fg\nabla_X\{Y(h)Z + h\nabla_Y Z\} - gf\nabla_Y\{X(h)Z + h\nabla_X Z\} \\ &- fg[Z,Y](h)Z - fgh\nabla_{[X,Y]}(hZ), \\ &= fgh\{\nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z\} \\ &= fghR(X,Y)Z, \end{aligned} \tag{2.34}$$

where $f,g,h \in C^\infty(M)$. If we use local coordinates, then

$$\begin{aligned} R(\partial_i,\partial_j)\partial_k &= \nabla_{\partial_i}\nabla_{\partial_j}\partial_k - \nabla_{\partial_j}\nabla_{\partial_i}\partial_k - \nabla_{[\partial_i,\partial_j]}\partial_k \\ &= \nabla_{\partial_i}\nabla_{\partial_j}\partial_k - \nabla_{\partial_j}\nabla_{\partial_i}\partial_k. \end{aligned} \tag{2.35}$$

By the properties of covariant differentiation, we have

$$\begin{aligned} R(\partial_i,\partial_j)\partial_k &= \nabla_{\partial_i}(\Gamma^\alpha_{jk})\partial_\alpha - \nabla_{\partial_j}(\Gamma^\alpha_{ik})\partial_\alpha \\ &= \partial_i(\Gamma^\alpha_{jk})\partial_\alpha + \sum_\alpha \Gamma^\alpha_{jk}\nabla_{\partial_i}\partial_\alpha - \partial_j(\Gamma^\alpha_{ik})\partial_\alpha - \sum_\alpha \Gamma^\alpha_{ik}\nabla_{\partial_j}\partial_\alpha \\ &= \partial_i(\Gamma^\alpha_{jk})\partial_\alpha + \sum_{\alpha,\sigma} \Gamma^\alpha_{jk}\Gamma^\sigma_{i\alpha}\partial_\sigma - \partial_j(\Gamma^\alpha_{ik})\partial_\alpha - \sum_{\alpha,\sigma} \Gamma^\alpha_{ik}\Gamma^\sigma_{j\alpha}\partial_\sigma \\ &= [\partial_i(\Gamma^\alpha_{jk}) - \partial_j(\Gamma^\alpha_{ik})]\partial_\alpha + \sum_{\alpha,\sigma}(\Gamma^\alpha_{jk}\Gamma^\sigma_{i\alpha} - \Gamma^\alpha_{ik}\Gamma^\sigma_{j\alpha})\partial_\sigma. \end{aligned} \tag{2.36}$$

If we replace the index α with σ in the first summation, we obtain

$$\begin{aligned} R(\partial_i,\partial_j)\partial_k &= \left[\sum_\sigma \partial_i(\Gamma^\sigma_{jk}) - \partial_j(\Gamma^\sigma_{ik})\right]\partial_\sigma + \sum_\alpha\sum_\sigma(\Gamma^\alpha_{jk}\Gamma^\sigma_{i\alpha} - \Gamma^\alpha_{ik}\Gamma^\sigma_{j\alpha})\partial_\sigma \\ &= \sum_\sigma\left[\partial_i(\Gamma^\sigma_{jk}) - \partial_j(\Gamma^\sigma_{ik}) + \sum_\alpha(\Gamma^\alpha_{jk}\Gamma^\sigma_{i\alpha} - \Gamma^\alpha_{ik}\Gamma^\sigma_{j\alpha})\right]\partial_\sigma \\ &= \sum_\sigma R^\sigma_{kij}\partial_\sigma, \end{aligned} \tag{2.37}$$

where

$$R^\sigma_{kij} = \frac{\partial}{\partial x^i}\Gamma^\sigma_{jk} - \frac{\partial}{\partial x^j}\Gamma^\sigma_{ik} + \sum_\alpha(\Gamma^\alpha_{jk}\Gamma^\sigma_{i\alpha} - \Gamma^\alpha_{ik}\Gamma^\sigma_{j\alpha}). \tag{2.38}$$

The R^σ_{kij} are the components of the curvature tensor [15]. By lowering the σ, we obtain the fourth rank tensor $R_{\sigma kij} = g_{\sigma\alpha}R^\alpha_{kij}$ as

$$R_{\sigma kij} = \frac{1}{2}\left(\frac{\partial^2 g_{\sigma j}}{\partial x^k \partial x^i} + \frac{\partial^2 g_{ki}}{\partial x^\sigma \partial x^j} - \frac{\partial^2 g_{\sigma i}}{\partial x^k \partial x^j} - \frac{\partial^2 g_{kj}}{\partial x^\sigma \partial x^i}\right) + g_{\alpha\beta}(\Gamma^\beta_{\sigma j}\Gamma^\alpha_{ki} - \Gamma^\beta_{\sigma i}\Gamma^\alpha_{kj}). \tag{2.39}$$

The fourth rank tensor $R_{\sigma kij}$ satisfies the following identities.

$$\begin{aligned} &R_{\sigma kij} = -R_{k\sigma ij}, \quad R_{\sigma kij} = -R_{\sigma kji}, \\ &R_{\sigma kij} = R_{ij\sigma k}, \quad R_{\sigma iij} = R_{\sigma kjj} = 0, \\ &R_{\sigma kij} + R_{\sigma ijk} + R_{\sigma jki} = 0. \end{aligned} \tag{2.40}$$

Definition 2.11. The Ricci tensor denoted by R_{ij} is defined by

$$R_{ij} = \sum_k R^k_{ikj}. \tag{2.41}$$

In terms of local coordinates, the components of the Ricci tensor can be computed from the following formula [15]

$$R_{ij} = \sum_{\sigma,\rho} \left(\frac{\partial}{\partial x^j} \Gamma^{\sigma}_{i\sigma} - \frac{\partial}{\partial x^{\sigma}} \Gamma^{\sigma}_{ij} + \Gamma^{\sigma}_{\rho j} \Gamma^{\rho}_{i\sigma} - \Gamma^{\sigma}_{\rho\sigma} \Gamma^{\rho}_{ij} \right). \tag{2.42}$$

Ricci-flat manifolds are Riemannian manifolds whose Ricci tensor vanishes. We give two more definitions related to the Ricci tensor.

Definition 2.12. The Ricci scalar is denoted by R and is defined by

$$R = \sum_{i,j} g^{ij} R_{ij}. \tag{2.43}$$

Definition 2.13. A Riemannian manifold (M, g) is said to be an Einstein manifold if

$$R_{ij} = k g_{ij}, \tag{2.44}$$

where k is a constant.

2.3 Warped and Multiply Warped Products

We present the definitions of warped and multiply warped product manifolds. Then using these definitions, a generalization of multiply warped product manifolds that we call "*warped-like product manifolds*" will be given in Chapter 4. The idea of warped product manifolds is a decomposition of the manifolds into a product of fiber and base spaces $M = F \times B$. More details are given in [39].

Definition 2.14. **[39]** Let (F, g_F), (B, g_B) be Riemannian manifolds and $f > 0$ be smooth function on B. A *warped product manifold* is a product manifold $M = F \times B$ equipped with the metric

$$g = \pi_B^* g_B + (f \circ \pi_B)\, \pi_1^* g_F, \tag{2.45}$$

where $\pi_1 : F \times B \longrightarrow F$ and $\pi_B : F \times B \longrightarrow B$ are the natural projections.

In local coordinates the first block that depends on the coordinates of the first group of coordinates is multiplied by a function of the second group of coordinates. Then if $f = 1$, then $F \times B$ reduces to a Riemannian product manifold.

If the definition holds an open subset of M, then M is called *locally warped product* manifold. A generalization of the notion of warped product metrics is the "*multiply warped products*", defined as follows [23].

Definition 2.15. Let (F_i, g_{F_i}), $i = 1,2,...,n$ and (B, g_B) be Riemannian manifolds. Let $f_i > 0$ be smooth functions on B. A *multiply warped product manifold* is the product manifold

$$F_1 \times F_2 \times ... \times F_n \times B \tag{2.46}$$

with base (B, g_B), fibers (F_i, g_{F_i}), $i = 1,2,...,n$, warping functions $f_i > 0$ and equipped with the metric

$$g = \pi_B^* g_B + \sum_{i=1}^{n} (f_i \circ \pi_B) \pi_i^* g_{F_i}, \tag{2.47}$$

where $\pi_B : F_1 \times F_2 \times ... \times F_n \times B \longrightarrow B$ and $\pi_i : F_1 \times F_2 \times ... \times F_n \times B \longrightarrow F_i$ are the natural projections on B and F_i respectively.

In this scheme, the metric is block diagonal, with the metrics of the F_i's are multiplied by a conformal factor depending on the coordinates of the base.

2.4 Riemannian Holonomy

2.4.1 Preliminaries

In the literature, Riemannian manifolds (M, g) are given special names coming from the holonomy group classification list presented by M. Berger. Berger's list, presented in Table 2.1 gives the possible holonomy groups of irreducible, simply-connected and non-symmetric Riemannian manifolds (see Theorem 2.18).

Table 2.1: Berger's list

Cases	Holonomy groups	Real dimension
i	$SO(n)$	$n \geq 2$
ii	$U(m)$	$n = 2m,\ m \geq 2$
iii	$SU(m)$	$n = 2m,\ m \geq 2$
iv	$Sp(m)$	$n = 4m,\ m \geq 1$
v	$Sp(m)Sp(1)$	$n = 4m,\ m \geq 1$
vi	G_2	7
vii	$Spin(7)$	8

Manifolds with holonomy $SO(n)$ constitute the generic case, while all others are denoted as manifolds with special holonomy and the last two cases are described as manifolds with exceptional holonomy.

The original list included $Spin(9)$, but D. Aleksevskii [1] and Gray-Brown [26] modified the original statement of the Berger's list, i.e. they excluded the $Spin(9)$ holonomy in 16-dimensions by showing manifolds with this holonomy group are symmetric.

After Berger introduced his classification list, whether all subgroups given in the list could occur as the holonomy group of a Riemannian manifold (M,g) were open problem. The existence problem is solved by the following authors case by case. Here we present the historical summary of these structures as follows (see for detailed history in [31]).

The existence of manifolds with $Hol(g) \subseteq SU(n)$ and $Hol(g) \subseteq Sp(n)$ was shown by E. Calabi which gave the first local construction of explicit metrics with $SU(n)$ and $Sp(n)$ holonomy [13].

S.T. Yau proved the existence compact manifolds with $SU(n)$ holonomy by using his solution of the Calabi conjecture [46]. Yau's solution implies that any compact Kahler manifold with vanishing first Chern class admits a unique metric with $SU(n)$ holonomy. In the literature, manifolds with holonomy group $SU(n)$ are called *Calabi-Yau manifolds* [31].

Explicit examples of complete metrics with $Hol(g) \subseteq Sp(n)$ on compact manifolds were given by Fujuki [24].

The existence of manifolds with G_2 and $Spin(7)$ holonomy was first constructed by R.Bryant [10], who gave some examples of explicit, incomplete manifolds. Then R. Bryant and S. Salamon found explicit, complete metrics with exceptional holonomy on noncompact manifolds [11]. The first examples of metrics with G_2 and $Spin(7)$ holonomy on compact manifolds were constructed by D. Joyce [30] as mentioned in the introduction.

The research on construction of explicit metric examples on manifolds with exceptional holonomy group continues [20, 32, 33, 35].

2.4.2 The Holonomy Group

In this section we present the definition of the holonomy group of a Riemannian manifold and discuss some of its important properties. For further details of the holonomy group of a Riemannian manifold, we refer to the books by Berger [5], Besse [6] and Salamon [41].

Let M be a manifold and g be a Riemannian metric on M. The Levi-Civita connection ∇ defines the parallel transport of vectors along a curve.

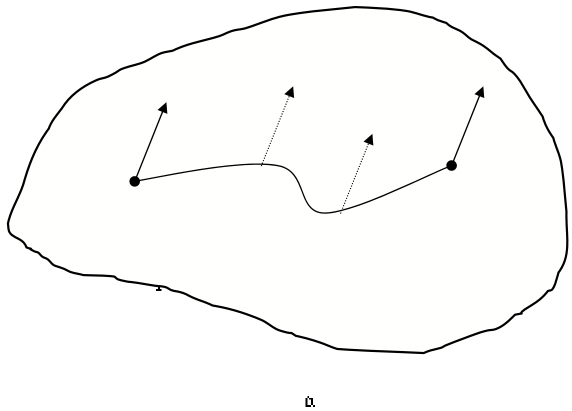

Figure 2.1: Parallel translation of vector field.

Let $\alpha : [0,1] \longrightarrow M$ be a smooth curve with $\alpha(0) = p$ and $\alpha(1) = q$. By using parallel translation, ∇ defines a linear map

$$P_\alpha : T_pM \longrightarrow T_qM \tag{2.48}$$

which preserves vector addition and scalar multiplication, which is an isometry as the metric is covariantly constant $\nabla g = 0$. If we choose a smooth closed curve passing from $p \in M$, then parallel transport defines a self-isometry of T_pM.

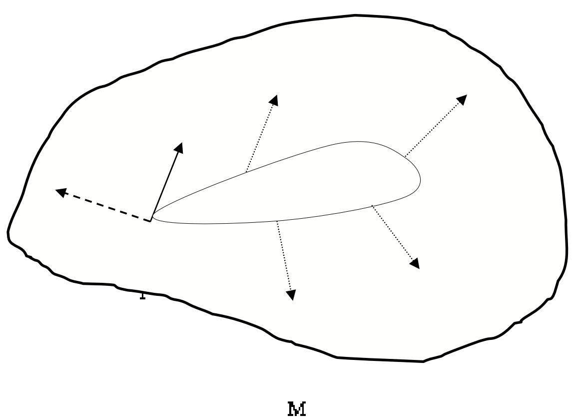

Figure 2.2: Holonomy group at the point p.

The set of all loops based at p gives rise to a group of isometries of T_pM which is called the *holonomy group* based at p and it is denoted by $Hol_p(M)$.

$Hol_p(M)$ is a subgroup of the group of all isometries at the point p, i.e. it is isomorphic to a subgroup of the orthogonal group $O(n)$. If the manifold is connected, then the holonomy groups based at the different points are conjugate subgroups of $O(n)$,

$$P_\alpha Hol_p(M) P_\alpha^{-1} = Hol_q(M) \tag{2.49}$$

where α is any smooth curve from p to q in M.

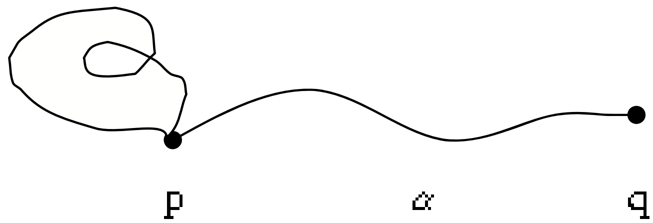

Figure 2.3: Holonomy group for connected manifolds.

This implies that we can drop the base point of M and just define the holonomy group $Hol(M)$ as a subgroup of $O(n)$ up to conjugation. A Riemannian metric on an orientable manifold has holonomy group $SO(n)$, but for some metrics it can be a subgroup, in which case the manifold is said to have special holonomy as mentioned before.

If we choose contractible closed curves (i.e. closed curves that can be contracted to a point) at p in M, then we get a new subgroup of the holonomy group at p which is called *restricted (reduced)* holonomy group and denoted by $Hol_0(M)$. It is an important property that the restricted holonomy group of M $(Hol_0(M))$ is a normal subgroup of $Hol(M)$. It follows the definition, if the manifold M is simply-connected, then

$$Hol_0(M) = Hol(M). \tag{2.50}$$

We state below some of the basic properties of the holonomy groups and restricted holonomy groups without proofs.

Proposition 2.16. **[34]** *Let (M,g) be a n-dimensional connected manifold. Then*

- $i)$ $Hol_0(M)$ *is closed, connected Lie subgroup of* $SO(n)$,
- $ii)$ $Hol_0(M)$ *is the identity component of* $Hol(M)$,
- $iii)$ *There is a surjective group homomorphism* $\phi : \pi_1(M) \longrightarrow Hol(M)/Hol_0(M)$,
- $iv)$ $Hol_0(M)$ *is trivial if and only if g is flat.*

2.4.3 The Classification of Riemannian Holonomy Groups

Let (M,g) be a connected Riemannian manifold. From the definition, we know that if M is a simply connected manifold, then the holonomy and restricted (reduced) holonomy groups coincide. Thus we study the simply connected manifolds to avoid fundamental groups and global topology.

We look at the holonomy group of Riemannian product manifolds. The following proposition gives the holonomy classification of reducible manifolds.

Proposition 2.17. **[31]** *Let (M_1,g_1) and (M_2,g_2) be Riemannian manifolds. Then the product metric $g_1 \times g_2$ has holonomy*

$$Hol(g_1 \times g_2) = Hol(g_1) \times Hol(g_2). \tag{2.51}$$

In classifying of the holonomy groups, we restrict us to the irreducible case as the holonomy group of a reducible manifold is the product of the holonomy groups of the components.

Riemannian symmetric spaces that are generic types of manifolds which can be written as the quotient of two Lie groups $M = G/H$ [34]. E. Cartan proved that the holonomy group of $M = G/H$ is just H. Thus Cartan obtained the classification list of the holonomy groups of all irreducible, simply connected, Riemannian symmetric manifolds [17].

The research on holonomy is focused on the determination of the restricted holonomy groups of irreducible non-symmetric Riemannian manifolds. The main result which is called the Berger's holonomy classification theorem is presented in the following section.

2.4.3.1 Holonomy Groups Classification: Berger's List

M. Berger proved the following result usually referred as *Berger's Theorem* and gave the list of possible holonomy groups called Berger's list in the literature. More detailed treatment can be found in the book by Berger [5].

Theorem 2.18. **[Berger]** *Let (M,g) be an n-dimensional simply-connected,irreducible and non-symmetric manifold. Then the holonomy group $Hol(g)$ is given in Table 2.1.*

Special holonomy groups given in the Berger's list are important in the study of Riemannian manifolds. We state some definitions and geometric properties for each case in the Berger's list to understand the importance of these holonomy groups.

(i) $Hol(g) = SO(n)$ is the holonomy group of generic situation.

(ii) Metrics g with $Hol(g) \subseteq U(n)$ defines Kahler metrics and Kahler geometry. These are the natural class of the complex manifolds [34].

(iii) If $Hol(g) \subseteq SU(n)$, then g is said to be ***Calabi-Yau metric*** and M is called ***Calabi-Yau manifold*** [31]. Since $SU(n)$ is a subset of $U(m)$, the Calabi-Yau metrics are automatically Kahler. If g is Kahler, then $Hol_0(g) \subseteq SU(n)$ if and only if the metric is Ricci-flat. Hence Calabi-Yau metrics are Ricci-flat Kahler metrics.

The first explicit examples of complete Calabi-Yau metrics were given by Calabi [14]. The existence of compact manifolds with $SU(n)$ holonomy was shown by Yau and it was obtained from the Yau's solution of the Calabi conjecture [46]. The well-known example is the $K3$ (complex) surface which has a set of metrics with holonomy $SU(2)$ [31].

(iv) Metrics g with holonomy group contained in $Sp(n)$ are called ***hyperkahler metrics***. Since $Sp(n) \subset SU(2n) \subset U(2n)$, hyperkahler metrics are Ricci-flat and Kahler. The explicit examples of complete metrics with $Sp(n)$ holonomy were obtained by Calabi [12]. The metrics on compact manifolds with $Sp(n)$ holonomy can be also obtained from Yau's solution of the Calabi conjecture [31]. The first compact examples were given by Fujuki [24] with $Sp(2)$ holonomy and Beauville [3] with $Sp(n)$ holonomy.

(v) If $Hol(g) \subseteq Sp(n)Sp(1)$ for the dimension $m \geq 2$, then g is said to be ***quaternionic Kahler metric***. These metrics are Einstein, but not Kahler and not Ricci-flat [31]. Detailed work on quaternionic Kahler manifolds was presented by Salamon [40].

(vi)-(vii) Metrics g with holonomy group contained in G_2 and $Spin(7)$ are called ***exceptional holonomy metrics*** and manifolds with G_2 and $Spin(7)$ holonomy are called ***exceptional manifolds***. Sometimes these are called G_2 and $Spin(7)$ manifolds with respect to their holonomy groups in the literature.

After two years later of Joyce's work [30], the existence of manifolds with G_2 and $Spin(7)$ holonomy on compact manifolds was obtained by Kovalev [36] in a different way.

The following Figure 2.4 is presented as a nice summary the geometry of the Berger's list and taken from the book by Salamon [41].

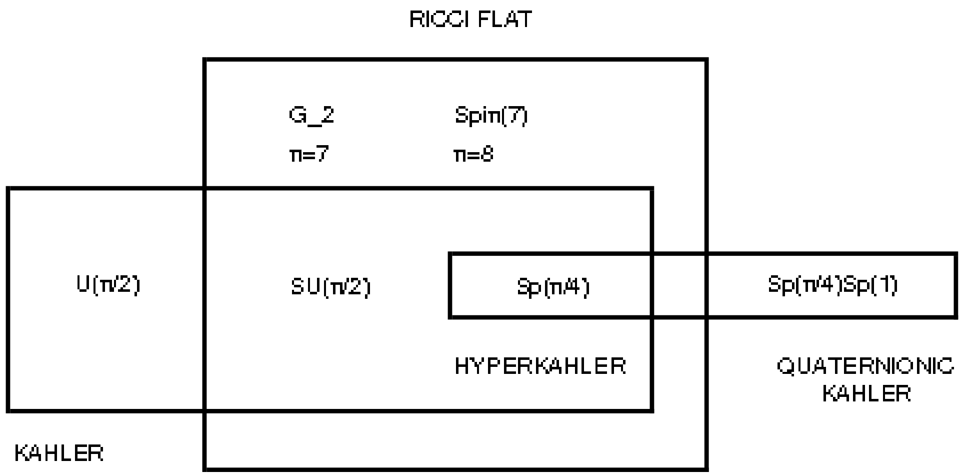

Figure 2.4: Salamon's illustration of the holonomy groups of the Berger's list.

2.4.3.2 Berger's List and Normed Algebras over $\mathbb{R}$

An alternative approach to the Berger's list is given by using four *division algebras* or *skew-fields*: the real numbers $\mathbb{R}$, the complex numbers $\mathbb{C}$, the quaternions $\mathbb{H}$ and the octonions $\mathbb{O}$ [41].

There is a relation between the Berger's list and these four division algebras. The groups in the Berger list are the group of automorphisms (or subgroups) of $\mathbb{R}^n$,$\mathbb{C}^n$,$\mathbb{H}^n$ and $\mathbb{O}$. In other words, $SO(n)$ is the group of automorphisms of $\mathbb{R}^n$, $U(n)$ and $SU(n)$ are the group of automorphisms of $\mathbb{C}^n$, $Sp(n)$and $Sp(n)Sp(1)$ are the group of automorphisms of $\mathbb{H}^n$, G_2 is the group of automorphisms of $Im\ \mathbb{O} \cong \mathbb{R}^7$, $Spin(7)$ is the group of automorphisms of $\mathbb{O} \cong \mathbb{R}^8$. We summarize Berger's list and normed algebras relation in Table 2.2.

2.4.3.3 Classification Table

In the light of the Riemannian holonomy studies up to now, we present the following classification Figure 2.5 to understand the history of the Riemannian holonomy theory. It is important to note that there is no complete classification

Table 2.2: Berger's list via division algebras

Cases	Berger's list	Vector space	Real dimension
i	$SO(n)$	$\mathbb{R}^n$	$n \geq 2$
ii	$U(m)$	$\mathbb{C}^m$	$n = 2m,\ m \geq 2$
iii	$SU(m)$	$\mathbb{C}^m$	$n = 2m,\ m \geq 2$
iv	$Sp(m)$	$\mathbb{H}^m$	$n = 4m,\ m \geq 1$
v	$Sp(m)Sp(1)$	$\mathbb{H}^m$	$n = 4m,\ m \geq 1$
vi	G_2	$Im\ \mathbb{O}$	7
vii	$Spin(7)$	$\mathbb{O}$	8

for non simply-connected manifolds yet, as indicated in the figure (See [38,44] for further details).

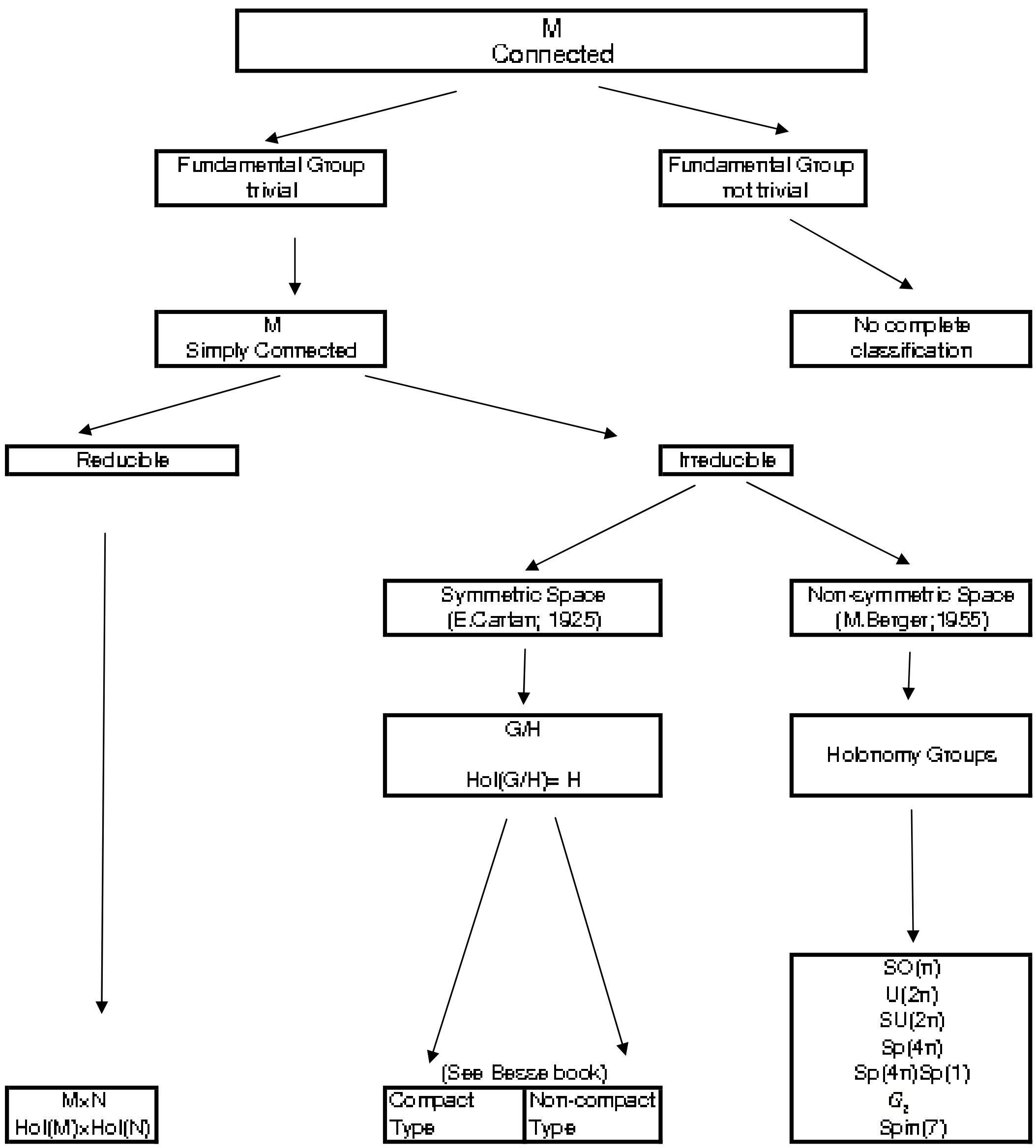

Figure 2.5: Classification of Riemannian holonomy.

2.4.4 Explicit Examples

In this section we examine for the computation of the holonomy groups and restricted holonomy groups for simple examples.

Example 2.19. Let M be the Euclidean space $M = \mathbb{R}^n$ with its usual inner product. The Euclidean space $\mathbb{R}^n$ is a simply-connected (trivial) manifold, hence the holonomy and restricted holonomy groups coincide. As $\mathbb{R}^n$ is also a flat manifold, all parallel translation maps are identity $P_\alpha = I_d$ for all closed curves α. Hence, $Hol(\mathbb{R}^n) = Hol_0(\mathbb{R}^n) = \{I_d\}$.

Example 2.20. Let M be the cylinder $M = S^1 \times \mathbb{R}$. As it is a flat manifold, the restricted holonomy group is $Hol_0(M) = \{I_d\}$. Although the cylinder is not a simply-connected manifold, the holonomy and restricted holonomy groups are the same. Hence $Hol_0(M) = Hol(M) = \{I_d\}$.

Example 2.21. Let M be the cone with its tip removed. As it is flat, the restricted holonomy group is $Hol_0(M) = \{I_d\}$. Since the cone is not a simply-connected manifold, there are closed curves which are not contractible on the cone. If α is a non-contractible closed curve, then vectors are rotated with respect to vertex angle of the cone. If the first rotation angle between the vectors is denoted by θ, then the parallel translation map is $e^{i\theta}$. If we rotate the vectors n-times, we get the parallel translation maps are $e^{in\theta}$, where $n \in \mathbb{Z}$. Hence we obtain the holonomy group of the cone at p as follows

$$Hol(p) = \{e^{in\theta} \mid n \in \mathbb{Z}\} \cong \mathbb{Z}. \tag{2.52}$$

Example 2.22. Finally, let M be the 2-sphere $M = S^2$. Since the sphere S^n is a simply-connected manifold for $n \geq 2$ [15], every closed curve on the sphere S^2 is a contractible curve. As the sphere S^2 is not a flat manifold, the holonomy group of S^2 is nontrivial. Note that on great circles of the sphere, the parallel translation map is identity as in the Example 2.20. For any other circle of the sphere, the parallel translation is a rotation by a fixed angle related to the vertex angle of a cone tangent to the circle.

The circle S^1 can be written as follows,

$$S^1 = \{(x,y) \mid x^2 + y^2 = 1\} = \{e^{i\theta} \mid sin^2\theta + cos^2\theta = 1, \theta \in [0,2\pi]\}. \tag{2.53}$$

S^1 has a group structure with summation operation, and group elements are θ which in $[0, 2\pi]$. Now we prove that there is a group isomorphism between S^1 and $SO(2)$. Let h be a map from S^1 to $SO(2)$,

$$\begin{aligned} h: S^1 &\longrightarrow SO(2) \\ e^{i\theta} &\longmapsto \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}. \end{aligned} \tag{2.54}$$

It is easily verified that this map is one to one, onto and a group isomorphism. This isomorphism implies that set of translation maps are $SO(2)$. Hence the holonomy group of the sphere is given by

$$Hol_0(S^2) = Hol(S^2) = SO(2). \tag{2.55}$$

3. MANIFOLDS WITH $SPIN(7)$ HOLONOMY

In this chapter we start with a review of the geometry of manifolds with $Spin(7)$ holonomy. A manifold (M,g) with $Spin(7)$ holonomy is a real orientable 8-dimensional Riemannian manifold whose holonomy group $Hol(g)$ is contained in $Spin(7)$. The key of the construction of these manifolds is a globally defined 4-form Ω which is called the *Bonan form* (*Cayley form* or *fundamental form*) in the literature [9].

In Section 3.1, we present two methods for the construction the Bonan 4-form based on the structure constants of octonionic algebra [28], and on vector cross products of octonions [7,27].

In Section 3.2, we study the construction of explicit examples of manifolds with $Spin(7)$ holonomy. In Section 3.2.1, we give an overview of the method given by Yasui-Ootsuka [45] and obtain the explicit form of the equations for the existence of a metric with $Spin(7)$ in terms of vector fields for the general case. Then in Section 3.2.2, we present the $Spin(7)$ metric structure obtained by Yasui-Ootsuka [45] on $S^3 \times S^3 \times \mathbb{R}^3$.

The existence of the globally defined 4-form has remarkable properties; closedness, self-duality in the Hodge sense and $Spin(7)$ invariance [10]. Conversely, if the Bonan form is closed, then the manifold has $Spin(7)$ holonomy, as given by R. Bryant (see Proposition 3.1) in [11]. These features of the Bonan form are the main tools for the construction of a $Spin(7)$ holonomy manifold.

3.1 The Bonan Form on $\mathbb{R}^8$

In the literature, there are different ways to construct the Bonan form Ω. In Section 3.1.1, we present a method of the construction Ω on $\mathbb{R}^8$ via octonionic algebra. In Section 3.1.2, we use triple vector cross products on octonions given in [27] to obtain an alternative definition.

A 4-form Ω on $\mathbb{R}^8 = \{(x^1, x^2, ..., x^8)\}$ can be written as

$$\Omega = \frac{1}{4!} \sum_{\alpha\beta\gamma\delta} \Omega_{\alpha\beta\gamma\delta}\; dx^\alpha \wedge dx^\beta \wedge dx^\gamma \wedge dx^\delta, \tag{3.1}$$

where $\Omega_{\alpha\beta\gamma\delta}$ are functions on $\mathbb{R}^8$.

For notational convenience, we identify dx^i with e^i, local orthonormal basis for the cotangent bundle of $\mathbb{R}^8$ and we write it as

$$\Omega = \frac{1}{4!} \sum_{\alpha\beta\gamma\delta} \Omega_{\alpha\beta\gamma\delta}\; e^\alpha \wedge e^\beta \wedge e^\gamma \wedge e^\delta. \tag{3.2}$$

In the following we shall omit the wedge symbol in exterior products of forms, i.e. we write

$$e^{ab} = e^a \wedge e^b,\; e^{abc} = e^a \wedge e^b \wedge e^c,\; e^{abcd} = e^a \wedge e^b \wedge e^c \wedge e^d. \tag{3.3}$$

In Sections 3.1.1 and 3.1.2, we shall present two different methods for the computation of the components $\Omega_{\alpha\beta\gamma\delta}$ of Ω using octonions.

3.1.1 Obtaining the Bonan Form via Octonionic Algebra

The method of the construction discussed below is related to the structure constants of octonionic algebra $\mathbb{O}$ [2, 28]. The octonions are the largest of the four normed division algebras over the real numbers $\mathbb{R}$. The octonions form an 8-dimensional non-associative algebra generated by the eight elements

$$\mathbb{O} = span\ \{1, \mathbf{e}_a \ \ | \ \ a = 1, 2, .., 7\}, \tag{3.4}$$

satisfying the multiplication rule given in Table 3.1. Note that the multiplication is not unique and there are many ways to construct such a table [2].

By using Table 3.1, the structure constants of octonions denoted by φ_{abc} can be written as

$$\mathbf{e}_a \mathbf{e}_b = \varphi_{abc} \mathbf{e}_c - \delta_{ab}, \tag{3.5}$$

where φ_{abc} are totally antisymmetric with

$$\varphi_{abc} = 1, \tag{3.6}$$

for the following set of indices

$$(abc) = (123), (516), (624), (435), (471), (673), (572), \tag{3.7}$$

Table 3.1: The multiplication of octonions

	1	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$	$\mathbf{e}_4$	$\mathbf{e}_5$	$\mathbf{e}_6$	$\mathbf{e}_7$
1	1	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$	$\mathbf{e}_4$	$\mathbf{e}_5$	$\mathbf{e}_6$	$\mathbf{e}_7$
$\mathbf{e}_1$	$\mathbf{e}_1$	-1	$\mathbf{e}_3$	-$\mathbf{e}_2$	$\mathbf{e}_7$	-$\mathbf{e}_6$	$\mathbf{e}_5$	$-\mathbf{e}_4$
$\mathbf{e}_2$	$\mathbf{e}_2$	-$\mathbf{e}_3$	-1	$\mathbf{e}_1$	$\mathbf{e}_6$	$\mathbf{e}_7$	-$\mathbf{e}_4$	-$\mathbf{e}_5$
$\mathbf{e}_3$	$\mathbf{e}_3$	$\mathbf{e}_2$	-$\mathbf{e}_1$	-1	-$\mathbf{e}_5$	$\mathbf{e}_4$	$\mathbf{e}_7$	-$\mathbf{e}_6$
$\mathbf{e}_4$	$\mathbf{e}_4$	-$\mathbf{e}_7$	-$\mathbf{e}_6$	$\mathbf{e}_5$	-1	-$\mathbf{e}_3$	$\mathbf{e}_2$	$\mathbf{e}_1$
$\mathbf{e}_5$	$\mathbf{e}_5$	$\mathbf{e}_6$	-$\mathbf{e}_7$	-$\mathbf{e}_4$	$\mathbf{e}_3$	-1	-$\mathbf{e}_1$	$\mathbf{e}_2$
$\mathbf{e}_6$	$\mathbf{e}_6$	-$\mathbf{e}_5$	$\mathbf{e}_4$	-$\mathbf{e}_7$	-$\mathbf{e}_2$	$\mathbf{e}_1$	-1	$\mathbf{e}_3$
$\mathbf{e}_7$	$\mathbf{e}_7$	$\mathbf{e}_4$	$\mathbf{e}_5$	$\mathbf{e}_6$	-$\mathbf{e}_1$	-$\mathbf{e}_2$	-$\mathbf{e}_3$	-1

and zero otherwise.

Then Ω_{abcd} is given by the following formula [28],

$$\begin{aligned} \Omega_{abc8} &= \varphi_{abc}, \\ \Omega_{abcd} &= \frac{1}{3!} \sum_{efg} \varepsilon_{abcdefg} \varphi_{efg} \end{aligned} \tag{3.8}$$

where $\varepsilon_{abcdefg}$ is totally antisymmetric constants. Thus the components Ω_{abcd} of Ω:

$$\Omega_{abcd} = 1, \tag{3.9}$$

for the following set of elements

$$\begin{aligned} (abcd) \;\; = \;\; & \{(1238),(5168),(6248),(4358),(4718),(6738),(5728), \\ & (4567),(7423),(3751),(6172),(2635),(5214),(1346)\} \end{aligned} \tag{3.10}$$

and zero otherwise. The explicit expression of the Bonan form is given by

$$\begin{aligned} \Omega \quad & = e^{1238} + e^{5168} + e^{6248} + e^{4358} + e^{4718} + e^{6738} + e^{5728} \\ & + e^{4567} + e^{7423} + e^{3751} + e^{6172} + e^{2635} + e^{5214} + e^{1346}. \end{aligned} \tag{3.11}$$

The 4-form Ω is self dual in the Hodge sense, that is,

$$\Omega = *\Omega, \tag{3.12}$$

hence the 8-form $\Omega \wedge \Omega$ coincides with the volume form of $\mathbb{R}^8$. In addition, it is invariant under the action of $Spin(7)$ [37].

The second method of the construction Ω is given by using vector cross products on octonions in the following sections.

3.1.2 Obtaining the Bonan Form via Vector Cross Products on Octonions

The Bonan form $\mathbf{\Omega}$ is given in terms of triple vector cross products [27] as

$$\Omega(x,y,z,w) = \langle x, y\times z\times w\rangle, \tag{3.13}$$

where $x,y,z,w \in \mathbb{O}$.

Now we compute the explicit expression of $\mathbf{\Omega}$ via triple vector cross products. Note that we use a new octonionic multiplication table (see Appendix A) chosen differently from Table 3.1 to obtain the same Bonan form as the one given in the equation (3.11).

The non-zero set of $\mathbf{\Omega}(\mathbf{e}_\alpha,\mathbf{e}_\beta,\mathbf{e}_\gamma,\mathbf{e}_\delta)$ can be obtained by the following formula given in [7] as

$$\Omega(\mathbf{e}_\alpha,\mathbf{e}_\beta,\mathbf{e}_\gamma,\mathbf{e}_\delta) = \langle \mathbf{e}_\alpha, \mathbf{e}_\beta(\overline{\mathbf{e}_\gamma}\mathbf{e}_\delta)\rangle, \tag{3.14}$$

where $\mathbf{e}_\alpha$'s are the basis of the octonions and $\overline{\mathbf{e}_\gamma}$ denotes the conjugate of the element $\mathbf{e}_\gamma$. We find the non-zero index set of the Bonan form as follows;

$$\Omega(\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3,\mathbf{e}_4) = \langle \mathbf{e}_1,\mathbf{e}_2(\overline{\mathbf{e}_3}\mathbf{e}_4)\rangle = \langle \mathbf{e}_1,\mathbf{e}_2(-\mathbf{e}_3\mathbf{e}_4)\rangle = \langle \mathbf{e}_1,\mathbf{e}_2(-\mathbf{e}_5)\rangle = \langle \mathbf{e}_1,\mathbf{e}_7\rangle = 0,$$

$$\Omega(\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3,\mathbf{e}_5) = \langle \mathbf{e}_1,\mathbf{e}_2(\overline{\mathbf{e}_3}\mathbf{e}_5)\rangle = \langle \mathbf{e}_1,\mathbf{e}_2(-\mathbf{e}_3\mathbf{e}_5)\rangle = \langle \mathbf{e}_1,\mathbf{e}_2(-\mathbf{e}_4)\rangle = \langle \mathbf{e}_1,\mathbf{e}_6\rangle = 0,$$

$$\dots$$

$$\Omega(\mathbf{e}_3,\mathbf{e}_6,\mathbf{e}_7,\mathbf{e}_8) = \langle \mathbf{e}_3,\mathbf{e}_6(\overline{\mathbf{e}_7}\mathbf{e}_8)\rangle = \langle \mathbf{e}_3,\mathbf{e}_6(-\mathbf{e}_7\mathbf{e}_8)\rangle = \langle \mathbf{e}_3,\mathbf{e}_6(-\mathbf{e}_7)\rangle = \langle \mathbf{e}_3,\mathbf{e}_3\rangle = 1,$$

$$\Omega(\mathbf{e}_4,\mathbf{e}_5,\mathbf{e}_6,\mathbf{e}_7) = \langle \mathbf{e}_4,\mathbf{e}_5(\overline{\mathbf{e}_6}\mathbf{e}_7)\rangle = \langle \mathbf{e}_4,\mathbf{e}_5(-\mathbf{e}_6\mathbf{e}_7)\rangle = \langle \mathbf{e}_4,\mathbf{e}_5\mathbf{e}_3\rangle = \langle \mathbf{e}_4,\mathbf{e}_4\rangle = 1,$$

$$\dots$$

$$\Omega(\mathbf{e}_4,\mathbf{e}_6,\mathbf{e}_7,\mathbf{e}_8) = \langle \mathbf{e}_4,\mathbf{e}_6(\overline{\mathbf{e}_7}\mathbf{e}_8)\rangle = \langle \mathbf{e}_4,\mathbf{e}_6(-\mathbf{e}_7\mathbf{e}_8)\rangle = \langle \mathbf{e}_4,\mathbf{e}_6(-\mathbf{e}_7)\rangle = \langle \mathbf{e}_4,\mathbf{e}_3\rangle = 0,$$

$$\Omega(\mathbf{e}_5,\mathbf{e}_6,\mathbf{e}_7,\mathbf{e}_8) = \langle \mathbf{e}_5,\mathbf{e}_6(\overline{\mathbf{e}_7}\mathbf{e}_8)\rangle = \langle \mathbf{e}_5,\mathbf{e}_6(-\mathbf{e}_7\mathbf{e}_8)\rangle = \langle \mathbf{e}_5,\mathbf{e}_6(-\mathbf{e}_7)\rangle = \langle \mathbf{e}_5,\mathbf{e}_3\rangle = 0.$$

Then the full set of non-zero indices of $\mathbf{\Omega}(\mathbf{e}_\alpha,\mathbf{e}_\beta,\mathbf{e}_\gamma,\mathbf{e}_\delta)$ is found in Appendix A as follows

$$\begin{aligned}&\{(1238),(1245),(2167),(1346),(1357),(1478),(5168),\\ &(3247),(2356),(2468),(2578),(4358),(3678),(4567)\}.\end{aligned} \tag{3.15}$$

Hence using triple vector cross products on octonions, we obtain the same explicit expression of the Bonan form given in the equation (3.11) as

$$\begin{aligned}\Omega \quad &= e^{1238}+e^{1245}-e^{1267}+e^{1346}+e^{1357}+e^{1478}-e^{1568} \\ &\quad - e^{2347}+e^{2356}+e^{2468}+e^{2578}-e^{3458}+e^{3678}+e^{4567}.\end{aligned}$$

3.2 Construction of a Manifold with $Spin(7)$ Holonomy

The following proposition reveals the importance of the Bonan form Ω in the construction of a manifold with $Spin(7)$ holonomy and is given by R. Bryant in [10].

Proposition 3.1. [Bryant] *The holonomy group of Riemannian metric defined by the Bonan form Ω is contained in $Spin(7)$ if and only if $d\Omega = 0$.*

3.2.1 A Vector Field Method for the Construction of Manifolds with $Spin(7)$ Holonomy

In this section we present the method given by Y.Yasui and T. Ootsuka in 2001 [45] for the construction of a manifold with $Spin(7)$ holonomy. In their approach, the condition $d\Omega = 0$ is converted to an expression in terms of vector fields and the specific solution discussed in Section 3.2.2 is obtained.

Note that as Ω is a 4-form in eight dimensions, the $d\Omega = 0$ gives 56 equations involving exterior derivatives of the basis 1-forms. Applying the method given in [45], we obtain equivalently, 56 equations involving the commutators of tangent vector fields (see Appendix C.3). These equations given in Appendix C in explicit form are new. Since in the derivation of our main result, we have used directly the condition $d\Omega = 0$, we did not make use of the equations in Appendix (C.3), but we note that they are general expressions valid for any background and provide ready to use expressions for metrical ansatz in terms of vector fields.

We now present the vector field method for the construction of a manifold with $Spin(7)$ holonomy.

Proposition 3.2. [Y. Yasui - T. Ootsuka] *Let M be a simply connected eight dimensional manifold, and $dvol$ be the volume form on M. Let V_α, $(\alpha = 1,2,...,8)$ be linearly independent vector fields on M and W^α be the one-forms dual to V_α. Suppose that the vector fields V_α satisfy the following two conditions:*

i. *the volume-preservation condition:* $L_{V_\alpha} dvol = 0$.

ii. *the 2-vector condition:* $\sum_{\alpha\beta\gamma\delta} \Omega_{\alpha\beta\gamma\delta}[V_\alpha \wedge V_\beta, V_\gamma \wedge V_\delta]_{SN} = 0$,

where $\Omega_{\alpha\beta\gamma\delta}$ are given in the equations (3.9)-(3.10) *and* $[\ ,\ ]_{SN}$ *is the Schouten-Nijenhuis bracket, i.e.*

$$\begin{aligned}[V_\alpha \wedge V_\beta, V_\gamma \wedge V_\delta]_{SN} \quad &= [V_\alpha, V_\gamma] \wedge V_\beta \wedge V_\delta - [V_\alpha, V_\delta] \wedge V_\beta \wedge V_\gamma \\ &-[V_\beta, V_\gamma] \wedge V_\alpha \wedge V_\delta + [V_\beta, V_\delta] \wedge V_\alpha \wedge V_\gamma.\end{aligned}$$

Then the metric with $Spin(7)$ holonomy is

$$g = \phi \sum_\alpha W^\alpha \otimes W^\alpha \tag{3.16}$$

where $\phi^2 = dvol(V_1, V_2, ..., V_8)$ and the corresponding Bonan 4-form is given by

$$\Omega = \frac{1}{4!}\phi^2 \sum_{\alpha\beta\gamma\delta} \Omega_{\alpha\beta\gamma\delta}\, e^{\alpha\beta\gamma\delta}, \tag{3.17}$$

where $e^\alpha = \sqrt{\phi} W^\alpha$.

Since the proof of Proposition 3.2 is just briefly outlined in the work by Yasui-Ootsuka, we present the proof in details and use in the next sections.

Proof. We prove that Ω satisfies the three fundamental properties given in the previous sections, i.e. it is self-dual in the Hodge sense, $Spin(7)$ invariant and closed.

Self-duality and $Spin(7)$ invariance: If we replace the 1-form

$$e^\alpha = \sqrt{\phi} W^\alpha, \tag{3.18}$$

in the equations (3.16) and (3.16) respectively, then

$$\begin{aligned}\Omega &= \frac{1}{4!} \sum_{\alpha\beta\gamma\delta} \Omega_{\alpha\beta\gamma\delta}\, e^{\alpha\beta\gamma\delta}, \\ g &= \sum_\alpha e^\alpha \otimes e^\alpha.\end{aligned} \tag{3.19}$$

Hence we get the same expression as given in the equation (3.11) on $\mathbb{R}^8$. It follows that Ω is a $Spin(7)$ invariant and self-dual 4-form in the Hodge sense as mentioned in Section 3.1.1.

Closedness: We now show that Ω is closed form, that is $d\Omega = 0$. For this we rewrite Ω in the form

$$\Omega = \frac{1}{4!}\sum_{\alpha\beta\gamma\delta}\Omega_{\alpha\beta\gamma\delta} i_{V_\alpha} i_{V_\beta} i_{V_\gamma} i_{V_\delta} dvol, \tag{3.20}$$

where i_{V_α} denotes the inner derivation with respect to V_α. We calculate $d\Omega$ by using the following formula successively

$$L_{V_\alpha} i_{V_\beta} - i_{V_\beta} L_{V_\alpha} = i_{[V_\alpha, V_\beta]}, \tag{3.21}$$

$$L_{V_\alpha} = d i_{V_\alpha} + i_{V_\alpha} d. \tag{3.22}$$

Then

$$\begin{aligned}
d\Omega \quad &= \frac{1}{4!}\sum_{\alpha\beta\gamma\delta}\Omega_{\alpha\beta\gamma\delta} d i_{V_\alpha} i_{V_\beta} i_{V_\gamma} i_{V_\delta} dvol = \frac{1}{4!}\sum_{\alpha\beta\gamma\delta}\Omega_{\alpha\beta\gamma\delta}\ (L_{V_\alpha} - i_{V_\alpha} d i_{V_\beta} i_{V_\gamma} i_{V_\delta}) dvol \\
&= \frac{1}{4!}\sum_{\alpha\beta\gamma\delta}\Omega_{\alpha\beta\gamma\delta}\ (L_{V_\alpha} i_{V_\beta} i_{V_\gamma} i_{V_\delta} - i_{V_\alpha} d i_{V_\beta} i_{V_\gamma} i_{V_\delta}) dvol \\
&= \frac{1}{4!}\sum_{\alpha\beta\gamma\delta}\Omega_{\alpha\beta\gamma\delta}\ (L_{V_\alpha} i_{V_\beta} i_{V_\gamma} i_{V_\delta} - i_{V_\alpha} L_{V_\beta} i_{V_\gamma} i_{V_\delta} + i_{V_\alpha} i_{V_\beta} i_{V_\beta} d i_{V_\gamma} i_{V_\delta}) dvol \\
&= \frac{1}{4!}\sum_{\alpha\beta\gamma\delta}\Omega_{\alpha\beta\gamma\delta}\ (L_{V_\alpha} i_{V_\beta} i_{V_\gamma} i_{V_\delta} - i_{V_\alpha} L_{V_\beta} i_{V_\gamma} i_{V_\delta} + i_{V_\alpha} i_{V_\beta} L_{V_\gamma} i_{V_\delta} - i_{V_\alpha} i_{V_\beta} i_{V_\gamma} d i_{V_\delta}) dvol \\
&= \frac{1}{4!}\sum_{\alpha\beta\gamma\delta}\Omega_{\alpha\beta\gamma\delta}\ (L_{V_\alpha} i_{V_\beta} i_{V_\gamma} i_{V_\delta} - i_{V_\alpha} L_{V_\beta} i_{V_\gamma} i_{V_\delta} + i_{V_\alpha} i_{V_\beta} L_{V_\gamma} i_{V_\delta} - i_{V_\alpha} i_{V_\beta} i_{V_\gamma} L_{V_\delta} \\
&\quad + i_{V_\alpha} i_{V_\beta} i_{V_\gamma} i_{V_\delta} d) dvol \\
&= \frac{1}{4!}\sum_{\alpha\beta\gamma\delta}\Omega_{\alpha\beta\gamma\delta}\ (i_{[V_\alpha, V_\beta]} i_{V_\gamma} i_{V_\delta} + i_{V_\beta} i_{[V_\alpha, V_\gamma]} i_{V_\delta} + i_{V_\beta} i_{V_\gamma} i_{[V_\alpha, V_\delta]} + i_{V_\beta} i_{V_\gamma} i_{V_\delta} L_{V_\alpha} \\
&\quad - i_{V_\alpha} i_{[V_\beta, V_\gamma]} i_{V_\delta} - i_{V_\alpha} i_{V_\gamma} i_{[V_\delta, V_\delta]} - i_{V_\alpha} i_{V_\gamma} i_{V_\delta} L_{V_\beta} + i_{V_\alpha} i_{V_\beta} i_{[V_\gamma, V_\delta]} \\
&\quad + i_{V_\alpha} i_{V_\beta} i_{V_\delta} L_{V_\gamma} - i_{V_\alpha} i_{V_\beta} i_{V_\gamma} L_{V_\delta} + i_{V_\alpha} i_{V_\beta} i_{V_\gamma} i_{V_\delta} d) dvol.
\end{aligned}$$

Furthermore using the antisymmetry properties of $\Omega_{\alpha\beta\gamma\delta}$ and the closedness of top form $dvol$, i.e. $d(dvol) = 0$, we get the following equation

$$d\Omega = \frac{1}{4!}\sum_{\alpha\beta\gamma\delta}\Omega_{\alpha\beta\gamma\delta}\ \left(6 i_{[V_\alpha, V_\beta]} i_{V_\gamma} i_{V_\delta} - 4 i_{V_\alpha} i_{V_\beta} i_{V_\gamma} L_{V_\delta}\right) dvol. \tag{3.23}$$

If we put the volume preservation condition $L_{V_\alpha} dvol = 0$ and the 2-vector condition, then we obtain

$$d\Omega = \frac{6}{4!} \sum_{\alpha\beta\gamma\delta} \Omega_{\alpha\beta\gamma\delta} i_{[V_\alpha,V_\beta]} i_{V_\gamma} i_{V_\delta} dvol \tag{3.24}$$

and

$$\sum_{\alpha\beta\gamma\delta} \Omega_{\alpha\beta\gamma\delta} [V_\alpha \wedge V_\beta, V_\gamma \wedge V_\delta]_{SN} = -4 \sum_{\alpha\beta\gamma\delta} \Omega_{\alpha\beta\gamma\delta} [V_\alpha, V_\beta] \wedge V_\gamma \wedge V_\delta = 0. \tag{3.25}$$

If we use the following property

$$\sum_{\alpha\beta\gamma\delta} i_{[V_\alpha,V_\beta]} i_{V_\gamma} i_{V_\delta} dvol = i_{\sum_{\alpha\beta\gamma\delta}[V_\alpha,V_\beta]\wedge V_\gamma \wedge V_\delta} dvol, \tag{3.26}$$

then the equation (3.24) is written as

$$d\Omega = \frac{6}{4!} i_{\sum_{\alpha\beta\gamma\delta} \Omega_{\alpha\beta\gamma\delta}[V_\alpha,V_\beta]\wedge V_\gamma \wedge V_\delta} dvol. \tag{3.27}$$

By using the equation (3.25), this gives $d\Omega = 0$. This completes the proof that Ω is the Bonan 4-form and g is the metric with $Spin(7)$ holonomy on M. □

We shall now obtain the conditions implied by the 2-vector condition of Proposition 3.2. If we expand the Schouten-Nijenhuis bracket by using the antisymmetry properties of $\Omega_{\alpha\beta\gamma\delta}$, then we can write

$$\sum_{\alpha\beta\gamma\delta} \Omega_{\alpha\beta\gamma\delta} [V_\alpha \wedge V_\beta, V_\gamma \wedge V_\delta]_{SN} = 4 \sum_{\alpha\beta\gamma\delta} \Omega_{\alpha\beta\gamma\delta} [V_\alpha, V_\gamma] \wedge V_\beta \wedge V_\delta. \tag{3.28}$$

The explicit expression of $\sum_{\alpha\beta\gamma\delta} \Omega_{\alpha\beta\gamma\delta} [V_\alpha, V_\gamma] \wedge V_\beta \wedge V_\delta$ is given in Appendix B.1. If we rearrange the expression given in Appendix B.1, then we obtain

$$\begin{aligned}
\sum_{\alpha\beta\gamma\delta} \Omega_{\alpha\beta\gamma\delta} [V_\alpha, V_\gamma] \wedge V_\beta \wedge V_\delta \quad &= \\
([V_6,V_7]-[V_4,V_5]-[V_3,V_8])\wedge V_1 \wedge V_2 \quad &+ \quad (-[V_5,V_7]+[V_2,V_8]-[V_4,V_6])\wedge V_1 \wedge V_3 + \\
([V_3,V_6]-[V_7,V_8]+[V_2,V_5])\wedge V_1 \wedge V_4 \quad &+ \quad ([V_3,V_7]-[V_2,V_4]+[V_6,V_8])\wedge V_1 \wedge V_5 + \\
(-[V_5,V_8]-[V_3,V_4]-[V_2,V_7])\wedge V_1 \wedge V_6 \quad &+ \quad ([V_2,V_6]+[V_4,V_8]-[V_3,V_5])\wedge V_1 \wedge V_7 + \\
([V_5,V_6]-[V_2,V_3]-[V_4,V_7])\wedge V_1 \wedge V_8 \quad &+ \quad (-[V_1,V_8]-[V_5,V_6]+[V_4,V_7])\wedge V_2 \wedge V_3 + \\
(-[V_1,V_5]-[V_3,V_7]-[V_6,V_8])\wedge V_2 \wedge V_4 \quad &+ \quad ([V_3,V_6]-[V_7,V_8]+[V_1,V_4])\wedge V_2 \wedge V_5 + \\
([V_4,V_8]+[V_1,V_7]-[V_3,V_5])\wedge V_2 \wedge V_6 \quad &+ \quad ([V_5,V_8]+[V_3,V_4]-[V_1,V_6])\wedge V_2 \wedge V_7 + \\
([V_1,V_3]-[V_5,V_7]-[V_4,V_6])\wedge V_2 \wedge V_8 \quad &+ \quad ([V_2,V_7]-[V_1,V_6]+[V_5,V_8])\wedge V_3 \wedge V_4 +
\end{aligned}$$

$$
\begin{aligned}
&(-[V_4,V_8]-[V_2,V_6]-[V_1,V_7])\wedge V_3\wedge V_5 &+& \quad ([V_1,V_4]+[V_2,V_5]-[V_7,V_8])\wedge V_3\wedge V_6+\\
&([V_6,V_8]-[V_2,V_4]+[V_1,V_5])\wedge V_3\wedge V_7 &+& \quad ([V_4,V_5]-[V_1,V_2]-[V_6,V_7])\wedge V_3\wedge V_8+\\
&(-[V_1,V_2]+[V_3,V_8]-[V_6,V_7])\wedge V_4\wedge V_5 &+& \quad ([V_5,V_7]-[V_1,V_3]-[V_2,V_8])\wedge V_4\wedge V_6+\\
&(-[V_5,V_6]+[V_2,V_3]-[V_1,V_8])\wedge V_4\wedge V_7 &+& \quad (-[V_3,V_5]+[V_1,V_7]+[V_2,V_6])\wedge V_4\wedge V_8+\\
&(-[V_2,V_3]-[V_4,V_7]+[V_1,V_8])\wedge V_5\wedge V_6 &+& \quad ([V_4,V_6]-[V_1,V_3]-[V_2,V_8])\wedge V_5\wedge V_7+\\
&([V_2,V_7]-[V_1,V_6]+[V_3,V_4])\wedge V_5\wedge V_8 &+& \quad ([V_1,V_2]-[V_4,V_5]-[V_3,V_8])\wedge V_6\wedge V_7+\\
&(-[V_2,V_4]+[V_3,V_7]+[V_1,V_5])\wedge V_6\wedge V_8 &+& \quad (-[V_1,V_4]-[V_3,V_6]-[V_2,V_5])\wedge V_7\wedge V_8.
\end{aligned}
\tag{3.29}
$$

If we write the commutator as

$$\left[V_\alpha,V_\beta\right]=\sum_\gamma c_{\alpha\beta\gamma}V_\gamma, \tag{3.30}$$

then the rearranged equation (3.29) can be written as

$$\sum_{\alpha\beta\gamma\delta}\Omega_{\alpha\beta\gamma\delta}[V_\alpha,V_\gamma]\wedge V_\beta\wedge V_\delta=\sum_{\alpha\beta\gamma}C_{\alpha\beta\gamma}V_\alpha\wedge V_\beta\wedge V_\gamma, \tag{3.31}$$

where $C_{\alpha\beta\gamma}$ are linear combinations of the coefficients of $c_{\alpha\beta\gamma}$ $(\alpha,\beta,\gamma=1,2,...,8)$. Hence we obtain a new set of 56 linear equations which should be zero in order to satisfy the 2-vector condition given in Appendix C.3.

3.2.2 An Example of Manifold with $Spin(7)$ Holonomy: $S^3\times S^3\times\mathbb{R}^2$

We illustrate the method by applying to the eight dimensional manifold

$$M=S^3\times S^3\times\mathbb{R}^2, \tag{3.32}$$

as given in [45]. Let (x,y) be the coordinates on $\mathbb{R}^2$ and

$$\theta^i,\ \theta^{\hat{i}}\quad i=1,2,3\ \text{ and }\ \hat{i}=i+3, \tag{3.33}$$

be the left invariant 1-forms on the 3-spheres satisfying the following relations

$$
\begin{aligned}
&d\theta^1=-\theta^{23},\quad d\theta^2=\theta^{13},\quad d\theta^3=-\theta^{12},\\
&d\theta^{\hat{1}}=-\theta^{\hat{2}\hat{3}},\quad d\theta^{\hat{2}}=\theta^{\hat{1}\hat{3}},\quad d\theta^{\hat{3}}=-\theta^{\hat{1}\hat{2}},
\end{aligned}
\tag{3.34}
$$

Thus

$$\{dx,dy,\theta^1,\theta^2,\theta^3,\theta^{\hat{1}},\theta^{\hat{2}},\theta^{\hat{3}}\} \tag{3.35}$$

are global sections of the cotangent bundle T^*M and the duals of these global sections are respectively

$$\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \theta_1, \theta_2, \theta_3, \theta_{\hat{1}}, \theta_{\hat{2}}, \theta_{\hat{3}}\}. \tag{3.36}$$

The commutator of the vector fields θ_i and $\theta_{\hat{i}}$ $(i=1,2,3)$ on S^3 are given by

$$\begin{aligned}
&[\theta_i, \theta_{\hat{i}}] = 0, && i = 1,2,3 \\
&[\theta_1, \theta_2] = \theta_3, && [\theta_1, \theta_3] = -\theta_2, && [\theta_2, \theta_3] = \theta_1, \\
&[\theta_{\hat{1}}, \theta_{\hat{2}}] = \theta_{\hat{3}}, && [\theta_{\hat{1}}, \theta_{\hat{3}}] = -\theta_{\hat{2}}, && [\theta_{\hat{2}}, \theta_{\hat{3}}] = \theta_{\hat{1}}.
\end{aligned} \tag{3.37}$$

We start with the ansatz for vector fields $V_\alpha \quad (\alpha = 1,2,...,6)$

$$\begin{aligned}
V_i &= a(x)\left[\alpha_{11}(y)\theta_i + \alpha_{12}(y)\theta_{\hat{i}}\right], && (i = 1,2,3), \\
V_{\hat{i}} &= b(x)\left[\alpha_{21}(y)\theta_i + \alpha_{22}(y)\theta_{\hat{i}}\right], && (i = 1,2,3),
\end{aligned} \tag{3.38}$$

and without loss of generality, we take V_7 and V_8 as below,

$$\begin{aligned}
V_7 &= \beta(y)\frac{\partial}{\partial x}, \\
V_8 &= c(x)\frac{\partial}{\partial y}.
\end{aligned} \tag{3.39}$$

The volume form $dvol$ is chosen as

$$dvol = dx\ dy\ \theta^1\theta^2\theta^3\theta^{\hat{1}}\theta^{\hat{2}}\theta^{\hat{3}}. \tag{3.40}$$

To construct the explicit expression of the metric with $Spin(7)$ holonomy on $M = S^3 \times S^3 \times \mathbb{R}^2$, we check that these vector fields satisfy the two conditions given in Proposition 3.2.

1- The volume-preservation condition: $L_{V_\alpha} dvol = 0$.

We use the formula $L_{V_\alpha} = di_{V_\alpha} + i_{V_\alpha} d$ in the computation. Since V_α for $\alpha = (1,2,3,...,6)$ has no components along dx and dy,

$$L_{V_\alpha} dx = i_{V_\alpha} d(dx) + d(i_{V_\alpha} dx) = 0$$

and

$$L_{V_\alpha} dy = i_{V_\alpha} d(dy) + d(i_{V_\alpha} dy) = 0.$$

The formula $L_{V_\alpha} dvol$ is written as

$$\begin{aligned} L_{V_\alpha} dvol &= d i_{V_\alpha} dvol + i_{V_\alpha} d(dvol) \\ &= d i_{V_\alpha} dvol. \end{aligned} \tag{3.41}$$

For ease of computation, we write the equation (3.38) as

$$\begin{aligned} V_i &= a_i(x,y)\theta_i + \hat{a}_i(x,y)\theta_{\hat{i}}, \\ V_{\hat{i}} &= b_i(x,y)\theta_i + \hat{b}_i(x,y)\theta_{\hat{i}}, \end{aligned} \tag{3.42}$$

where $i = 1,2,3$. By linearity of i,

$$\begin{aligned} L_{V_i} dvol &= d\left[a_i(x,y) i_{\theta_i} dvol + \hat{a}_i(x,y) i_{\theta_{\hat{i}}} dvol\right] \\ &= da_i i_{\theta_i} dvol + d\hat{a}_i i_{\theta_{\hat{i}}} dvol + a_i d(i_{\theta_i} dvol) + \hat{a}_i d(i_{\theta_{\hat{i}}} dvol). \end{aligned} \tag{3.43}$$

But $i_{\theta_i} dvol$ and $i_{\theta_{\hat{i}}} dvol$ has components along dx and dy, hence their exterior product with da_i and $d\hat{a}_i$ gives zero. Thus

$$L_{V_i} dvol = a_i d(i_{\theta_i} dvol) + \hat{a}_i d(i_{\theta_{\hat{i}}} dvol), \tag{3.44}$$

for $i = 1,2,3$. Let $i = 1$, then

$$\begin{aligned} L_{V_1} dvol &= a_1 d(i_{\theta_1} dvol) + \hat{a}_1 d(i_{\theta_{\hat{1}}} dvol) \\ &= a_1 d(dx\, dy\ \theta^{23}\theta^{\hat{1}\hat{2}\hat{3}}) + \hat{a}_1 d(dx\, dy\ \theta^{123}\theta^{\hat{2}\hat{3}}) \\ &= 0. \end{aligned} \tag{3.45}$$

Similar procedures apply for V_i $i = 2,3$ and $V_{\hat{i}}$ $i = 1,2,3$. We omit the details. We now check that $L_{V_8} dvol = 0$. The computation for V_7 is similar and it is omitted.

$$\begin{aligned} L_{V_8} dvol &= d i_{V_8} dvol \\ &= d i_{c(x)\frac{\partial}{\partial y}} \left(dx\, dy\ \theta^{123}\theta^{\hat{1}\hat{2}\hat{3}}\right) \\ &= d\left(-c(x) dx \theta^{123}\theta^{\hat{1}\hat{2}\hat{3}}\right). \end{aligned} \tag{3.46}$$

Using $d(cdx) = 0$, we obtain

$$\begin{aligned} L_{V_8} dvol &= -c(x) dx\, d(\theta^{123}\theta^{\hat{1}\hat{2}\hat{3}}) \\ &= -c(x) dx \left(d(\theta^{123})\theta^{\hat{1}\hat{2}\hat{3}} - \theta^{123} d(\theta^{\hat{1}\hat{2}\hat{3}})\right). \end{aligned}$$

By using the equation (3.34), it is seen that $d(\theta^{123}) = d(\theta^{\hat{1}\hat{2}\hat{3}}) = 0$. Then we complete the volume preservation condition of the V_8, i.e, $L_{V_8} dvol = 0$.

Hence we conclude that the vector fields V_α given in the equation (3.38) satisfy the volume-preservation condition $L_{V_\alpha} dvol = 0 \quad$ for $(\alpha = 1,2,3,...,8)$.

2- The 2-vector condition; $\sum_{\alpha\beta\gamma\delta} \Omega_{\alpha\beta\gamma\delta} [V_\alpha \wedge V_\beta, V_\gamma \wedge V_\delta]_{SN} = 0$

We need to the commutator of the vector fields to use of the 56 linear equations given in Appendix C.3. Thus we start to compute the commutator of the vector fields V_α by using commutator relations of θ_i and $\theta_{\hat{i}}$ given in the equations (3.37). We use the notation dot $\dot{}$ and prime $'$ to denote the differentiations with respect to x and y.

If we choose $\alpha_{11} = 2$ and $\alpha_{21} = 0$ in the equation (3.38) as given by [45], then the vector fields are

$$
\begin{aligned}
V_1 &= a(x)\left(2\theta_1 + \alpha_{12}(y)\theta_{\hat{1}}\right), \quad V_2 = a(x)\left(2\theta_2 + \alpha_{12}(y)\theta_{\hat{2}}\right), \\
V_3 &= a(x)\left(2\theta_3 + \alpha_{12}(y)\theta_{\hat{3}}\right), \quad V_4 = b(x)\alpha_{22}(y)\theta_{\hat{1}}, \\
V_5 &= b(x)\alpha_{22}(y)\theta_{\hat{2}}, \qquad\qquad V_6 = b(x)\alpha_{22}(y)\theta_{\hat{3}}, \\
V_7 &= \beta(y)\frac{\partial}{\partial x}, \qquad\qquad\qquad V_8 = c(x)\frac{\partial}{\partial y}.
\end{aligned}
\tag{3.47}
$$

We can write the $\theta_i, \theta_{\hat{i}}, \frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ from the equation (3.47) as

$$
\begin{aligned}
\theta_{\hat{1}} &= \frac{V_4}{b\alpha_{22}}, & \theta_{\hat{2}} &= \frac{V_5}{b\alpha_{22}}, \\
\theta_{\hat{3}} &= \frac{V_6}{b\alpha_{22}}, & \theta_1 &= \frac{V_1}{2a} - \frac{\alpha_{12}V_4}{2b\alpha_{22}}, \\
\theta_2 &= \frac{V_2}{2a} - \frac{\alpha_{12}V_5}{2b\alpha_{22}}, & \theta_3 &= \frac{V_3}{2a} - \frac{\alpha_{12}V_6}{2b\alpha_{22}} \\
\frac{\partial}{\partial x} &= \frac{V_7}{\beta}, & \frac{\partial}{\partial y} &= \frac{V_8}{c}.
\end{aligned}
\tag{3.48}
$$

If we put the equation (3.48) in the commutators of the vector fields, then we obtain the followings

$$
\begin{aligned}
&[V_1, V_2] = 2aV_3 + \left(\frac{a^2\alpha_{12}^2 - 2a^2\alpha_{12}}{b\alpha_{22}}\right)V_6, \\
&[V_1, V_3] = -2aV_2 + \left(\frac{a^2\alpha_{12}^2 - 2a^2\alpha_{12}}{b\alpha_{22}}\right)V_5, \\
&[V_1, V_4] = 0, \quad [V_1, V_5] = a\alpha_{12}V_6, \\
&[V_1, V_6] = -a\alpha_{12}V_5, \quad [V_1, V_7] = -\frac{\beta\dot{a}}{a}V_1,
\end{aligned}
$$

$$
\begin{aligned}
&[V_1, V_8] = -\frac{ac\alpha_{12}'}{b\alpha_{22}} V_4, \\
&[V_2, V_3] = 2aV_1 + \left(\frac{a^2\alpha_{12}^2 - 2a^2\alpha_{12}}{b\alpha_{22}} \right) V_4, \\
&[V_2, V_4] = -a\alpha_{12} V_6, \quad [V_2, V_5] = 0, \quad [V_2, V_6] = a\alpha_{12} V_4, \\
&[V_2, V_7] = -\frac{\beta \dot{a}}{a} V_2, \quad [V_2, V_8] = -\frac{ac\alpha_{12}'}{b\alpha_{22}} V_5, \\
&[V_3, V_4] = a\alpha_{12} V_5, \quad [V_3, V_5] = -a\alpha_{12} V_4, \quad [V_3, V_6] = 0, \\
&[V_3, V_7] = -\frac{\beta \dot{a}}{a} V_3, \quad [V_3, V_8] = -\frac{ac\alpha_{12}'}{b\alpha_{22}} V_6, \\
&[V_4, V_5] = b\alpha_{22} V_6, \quad [V_4, V_6] = -b\alpha_{22} V_5, \\
&[V_4, V_7] = -\frac{\beta \dot{b}}{b} V_4, \quad [V_4, V_8] = -\frac{c\alpha_{22}'}{\alpha_{22}} V_4, \\
&[V_5, V_6] = b\alpha_{22} V_4, \quad [V_5, V_7] = -\frac{\beta \dot{b}}{b} V_5, \\
&[V_5, V_8] = -\frac{c\alpha_{22}'}{\alpha_{22}} V_5, \\
&[V_6, V_7] = -\frac{\beta \dot{b}}{b} V_6, \quad [V_6, V_8] = -\frac{c\alpha_{22}'}{\alpha_{22}} V_6, \\
&[V_7, V_8] = \frac{\beta \dot{c}}{c} V_8 - \frac{c\beta'}{\beta} V_7.
\end{aligned} \tag{3.49}
$$

Hence we obtain the following c_{ijk} (see the equation (3.30)) which are different from zero

$$
\begin{aligned}
&c_{123} = -c_{132} = c_{231} = 2a, \\
&c_{171} = c_{272} = c_{373} = -\frac{\beta \dot{a}}{a}, \\
&c_{456} = -c_{465} = c_{564} = b\alpha_{22}, \\
&c_{474} = c_{575} = c_{676} = -\frac{\beta \dot{b}}{b}, \\
&c_{184} = c_{285} = c_{386} = -\frac{ac\alpha_{12}'}{b\alpha_{22}}, \\
&c_{484} = c_{585} = c_{686} = -\frac{c\alpha_{22}'}{\alpha_{22}}, \\
&c_{126} = -c_{135} = c_{234} = \frac{a^2\alpha_{12}^2 - 2a^2\alpha_{12}}{b\alpha_{22}}, \\
&c_{787} = -\frac{c\beta'}{\beta}, \quad c_{788} = \frac{\beta \dot{c}}{c}, \\
&c_{156} = -c_{165} = -c_{246} = c_{264} = c_{345} = -c_{354} = a\alpha_{12}.
\end{aligned} \tag{3.50}
$$

If we put these values in the equations of Appendix C.3, then we get the following set of five different first order non-linear differential equations.

$$
\begin{aligned}
&\frac{ac\alpha_{12}'}{b\alpha_{22}} - b\alpha_{22} - 2\frac{\beta \dot{a}}{a} - \frac{\beta \dot{b}}{b} = 0, \\
&a - a\alpha_{12} + \frac{c\alpha_{22}'}{\alpha_{22}} = 0,
\end{aligned}
$$

$$
\begin{aligned}
&\frac{a^2\alpha_{12}^2-2a^2\alpha_{12}}{b\alpha_{22}}+\frac{ac\alpha_{12}'}{b\alpha_{22}}-\frac{\beta\dot{b}}{b}=0,\\
&2a-2a\alpha_{12}+\frac{c\alpha_{22}'}{\alpha_{22}}+\frac{c\beta'}{\beta}=0,\\
&\frac{a^2\alpha_{12}^2-2a^2\alpha_{12}}{b\alpha_{22}}-b\alpha_{22}-\frac{\beta\dot{a}}{a}-\frac{\beta\dot{b}}{b}-\frac{\beta\dot{c}}{c}=0.
\end{aligned}
\tag{3.51}
$$

The solution of the equations given in the (3.51) is

$$
\begin{aligned}
&\alpha_{12}=1-tanh(y),\\
&\alpha_{22}=\beta=sech(y),\\
&a=c,\\
&\dot{a}=\frac{1}{2}(\frac{a^3}{b}-ab),\\
&\dot{b}=-2a^2.
\end{aligned}
\tag{3.52}
$$

From the solution, it is seen that the y dependent functions $\alpha_{12},\alpha_{22},\beta$ are solved but x dependent functions a,b are given by differential equations. Then the vector fields are

$$
\begin{aligned}
&V_1=a(x)\left(2\theta_1+(1-tanh(y))\theta_{\hat{1}}\right),\\
&V_2=a(x)\left(2\theta_2+(1-tanh(y))\theta_{\hat{2}}\right),\\
&V_3=a(x)\left(2\theta_3+(1-tanh(y))\theta_{\hat{3}}\right),\\
&V_4=b(x)sech(y)\theta_{\hat{1}},\\
&V_5=b(x)sech(y)\theta_{\hat{2}},\\
&V_6=b(x)sech(y)\theta_{\hat{3}},\\
&V_7=sech(y)\frac{\partial}{\partial x},\\
&V_8=a(x)\frac{\partial}{\partial y}.
\end{aligned}
\tag{3.53}
$$

To obtain explicit metric expression, we will compute the dual of vector fields and the function ϕ (see Proposition 3.2) defined by Yasui and Ootsuka. We obtain the following dual of vector fields as follows

$$
\begin{aligned}
&W^1=\frac{1}{2a}\theta^1,\qquad W^2=\frac{1}{2a}\theta^2,\\
&W^3=\frac{1}{2a}\theta^3,\\
&W^4=\frac{1}{bsech(y)}\left(\theta^{\hat{1}}+\frac{(1-tanh(y))}{2}\theta^1\right),
\end{aligned}
$$

$$
\begin{aligned}
W^5 &= \frac{1}{bsech(y)}\left(\theta^{\hat{2}} + \frac{(1-tanh(y))}{2}\theta^2\right),\\
W^6 &= \frac{1}{bsech(y)}\left(\theta^{\hat{3}} + \frac{(1-tanh(y))}{2}\theta^3\right),\\
W^7 &= \frac{1}{sech(y)}dx,\\
W^8 &= \frac{1}{a}dy. \qquad (3.54)
\end{aligned}
$$

Then ϕ is obtained as

$$
\begin{aligned}
\phi^2 &= dvol(V_1, V_2, ..., V_8)\\
&= dx\wedge dy\wedge\theta^1\wedge\theta^2\wedge\theta^3\wedge\theta^{\hat{1}}\wedge\theta^{\hat{2}}\wedge\theta^{\hat{3}}(V_1, V_2, ..., V_8)\\
&= \begin{vmatrix}
\langle dx, V_1\rangle & \langle dx, V_2\rangle & ... & \langle dx, V_8\rangle\\
\langle dy, V_1\rangle & \langle dy, V_2\rangle & ... & \langle dy, V_8\rangle\\
\langle \theta^1, V_1\rangle & \langle \theta^1, V_2\rangle & ... & \langle \theta^1, V_8\rangle\\
\langle \theta^2, V_1\rangle & \langle \theta^2, V_2\rangle & ... & \langle \theta^2, V_8\rangle\\
\langle \theta^3, V_1\rangle & \langle \theta^3, V_2\rangle & ... & \langle \theta^3, V_8\rangle\\
\langle \theta^{\hat{1}}, V_1\rangle & \langle \theta^{\hat{1}}, V_2\rangle & ... & \langle \theta^{\hat{1}}, V_8\rangle\\
\langle \theta^{\hat{2}}, V_1\rangle & \langle \theta^{\hat{2}}, V_2\rangle & ... & \langle \theta^{\hat{2}}, V_8\rangle\\
\langle \theta^{\hat{3}}, V_1\rangle & \langle \theta^{\hat{3}}, V_2\rangle & ... & \langle \theta^{\hat{3}}, V_8\rangle
\end{vmatrix}\\
&= a^4 b^3 sech^4(y). \qquad (3.55)
\end{aligned}
$$

The orthonormal frame $e^\alpha = \sqrt{\phi}W^\alpha$ can be written as follows (see the equation (3.18))

$$
\begin{aligned}
e^i &= \frac{1}{2}b^{\frac{3}{4}}sech(y)\theta^i, & i = 1,2,3\\
e^{\hat{i}} &= \frac{1}{2}ab^{-\frac{1}{4}}(1-tanh(y))\theta^i + ab^{-\frac{1}{4}}\theta^{\hat{i}}, & i = 1,2,3\\
e^7 &= ab^{\frac{3}{4}}dx,\\
e^8 &= b^{\frac{3}{4}}sech(y)dy, & (3.56)
\end{aligned}
$$

where the functions $a(x)$, $b(x)$ and their relations are given in the equation (3.52). From the orthonormal basis given in the equation (3.56), we write the following $Spin(7)$ holonomy metric on $S^3\times S^3\times\mathbb{R}^2$ given in [45] as

$$
\begin{aligned}
g &= \sum_{\alpha=1}^{8} e^\alpha \otimes e^\alpha\\
g &= \sqrt{a^4b^3}dx^2 + \sqrt{b^3}sech^2(y)\left(dy^2 + \frac{1}{4}(\theta^i)^2\right) + \sqrt{\frac{a^4}{b}}\left(\theta^{\hat{i}} + \frac{(1-tanh(y))}{2}\theta^i\right)^2 \qquad (3.57)
\end{aligned}
$$

where a and b depend on x and satisfy the differential equations

$$\begin{aligned} \dot{a} &= \frac{1}{2}\left(\frac{a^3}{b} - ab\right), \\ \dot{b} &= -2a^2. \end{aligned} \tag{3.58}$$

After then we call the explicit Spin(7) holonomy metric given in the equation (3.57) a *"Yasui-Ootsuka solution"* on $S^3 \times S^3 \times \mathbb{R}^2$.

4. (3+3+2) WARPED-LIKE PRODUCT MANIFOLDS WITH $SPIN(7)$ HOLONOMY

The aim of this chapter is to prove the uniqueness of the metric with $Spin(7)$ holonomy on $S^3 \times S^3 \times \mathbb{R}^2$ [45], (i.e. *the Yasui-Ootsuka solution* as given in the equation (3.57)), in the class of $(3+3+2)$ warped-like product metrics defined in the equation (4.28) which admit the $Spin(7)$ structure determined by the Bonan form Ω given by the equation (3.11).

In Section 4.2, we note that the Yasui-Ootsuka metric given by the equation (3.38) is a generalization of warped product metrics and we define *" (3+3+2) warped-like product metrics"* as a general framework for our metrical ansatz, by allowing the fiber metric to be non block diagonal in a multiply warped product [23].

In Section 4.3 and Section 4.4, we work with a $(3+3+2)$ warped-like manifold $M = F_1 \times F_2 \times B$ defined in the equation (4.28), and we prove that, when the base B is two dimensional, the fibre F is a 6-manifold of the form $F = F_1 \times F_2$ such that the fibers F_i's $(i = 1,2)$ are complete, connected and simply connected 3-manifolds and the metric is given by the equation (4.28), then the connection of the fibers is completely determined by the requirement that the Bonan form given in the equation (3.11) be closed. With the global assumptions given on the fibers above, we conclude that the fibers F_i's $(i = 1,2)$ are isometric to 3-spheres S^3 (see Theorem 4.5).

In Section 4.5, we prove that the Yasui-Ootsuka solution is unique in the class of metrics given by the equation (4.28) admitting the $Spin(7)$ structure defined by the equation (3.11), by giving the gauge transformations to this metric explicitly (see Corollary 4.10).

4.1 Preliminaries

Assuming from the existence of a globally defined Bonan form, the problem of proving that M has holonomy in $Spin(7)$ is reduced to the local problem of checking that Ω is a closed form (see Proposition 3.1). We shall do this for the metric ansatz given by the equation (4.28) and the Bonan form given in the equation (3.11).

Our notation is as follows. e_i and e^i $(i = 1, \ldots, n)$ denote respectively local orthonormal frames for the tangent and the cotangent bundles. This gives rise to local bases for k-forms denoted by

$$\begin{aligned} e^{ij} &= e^i \wedge e^j, \\ e^{ijk} &= e^i \wedge e^j \wedge e^k, \\ e^{ijkl} &= e^i \wedge e^j \wedge e^k \wedge e^l, \quad \ldots \end{aligned} \tag{4.1}$$

If we have a local orthonormal frame e^i $(i = 1, ..., n)$ on n-dimensional manifold M, then the metric g is written as

$$g = \sum_{i=1}^{n} e^i \otimes e^i. \tag{4.2}$$

We shall now rewrite the Bonan form Ω given in the equation (3.11) in a suitable form for our purposes. Starting with the equation (3.11), the Bonan form is given by

$$\begin{aligned} \Omega \quad &= e^{1238} - e^{1568} + e^{2468} - e^{3458} + e^{1478} + e^{3678} + e^{2578} \\ &+ e^{4567} - e^{2347} + e^{1357} - e^{1267} + e^{2356} + e^{1245} + e^{1346}. \end{aligned}$$

We rearrange it as

$$\begin{aligned} \Omega &= (e^{14} + e^{25} + e^{36})e^{78} + e^{1245} + e^{1346} + e^{2356} \\ &+ (e^{123} - e^{156} + e^{246} - e^{345})e^8 + (e^{456} - e^{234} + e^{135} - e^{126})e^7. \end{aligned} \tag{4.3}$$

Then relabeling the indices

$$\hat{1} = 4, \quad \hat{2} = 5 \qquad \hat{3} = 6, \tag{4.4}$$

that is,

$$\begin{aligned} e^4 &\longrightarrow e^{\hat{1}}, \\ e^5 &\longrightarrow e^{\hat{2}}, \\ e^6 &\longrightarrow e^{\hat{3}}, \end{aligned} \tag{4.5}$$

we obtain the following form

$$\begin{aligned} \Omega &= (e^{1\hat{1}}+e^{2\hat{2}}+e^{3\hat{3}})e^{78}-(e^{1\hat{1}2\hat{2}}+e^{1\hat{1}3\hat{3}}+e^{2\hat{2}3\hat{3}}) \\ &\quad +(e^{123}-e^{1\hat{2}\hat{3}}-e^{\hat{1}2\hat{3}}-e^{\hat{1}\hat{2}3})e^8+(e^{\hat{1}\hat{2}\hat{3}}-e^{\hat{1}23}-e^{1\hat{2}3}-e^{12\hat{3}})e^7. \end{aligned} \tag{4.6}$$

When we introduce new variables β, μ and ν as

$$\begin{aligned} \beta &= e^{1\hat{1}}+e^{2\hat{2}}+e^{3\hat{3}}, \\ \mu &= e^{123}-e^{1\hat{2}\hat{3}}-e^{\hat{1}2\hat{3}}-e^{\hat{1}\hat{2}3}, \\ \nu &= e^{\hat{1}\hat{2}\hat{3}}-e^{\hat{1}23}-e^{1\hat{2}3}-e^{12\hat{3}}, \end{aligned} \tag{4.7}$$

Ω is written as

$$\Omega = \beta e^{78}+\mu e^8+\nu e^7-\frac{1}{2}\beta^2 \tag{4.8}$$

and its exterior derivative is

$$d\Omega = d\beta e^{78}+\beta d(e^{78})+d\mu e^8-\mu de^8+d\nu e^7-\nu de^7-\beta d\beta. \tag{4.9}$$

Since Ω is a 4-form on an eight dimensional manifold, the $d\Omega = 0$ gives 56 equations for the partial derivatives of the coefficient functions in the basis 1-forms. These equations would be analogues of the 56 equations involving the commutators of tangent vector fields (see Appendix C.3). We shall not give them in the general case but investigate only a special case in the next sections.

Now we discuss generalizations of warped product manifolds and define *(3+3+2) warped-like product* manifolds that we shall use.

4.2 (3+3+2) Warped-Like Product Manifolds

We recall that the Yasui-Ootsuka solution on

$$M = S^3 \times S^3 \times \mathbb{R}^2 \tag{4.10}$$

is given by the following (global) orthonormal frame

$$\begin{aligned}
e^i &= \frac{1}{2} b^{\frac{3}{4}} sech(y)\theta^i, & i = 1,2,3\\
e^{\hat{i}} &= \frac{1}{2} ab^{-\frac{1}{4}}(1 - tanh(y))\theta^i + ab^{-\frac{1}{4}}\theta^{\hat{i}}, & i = 1,2,3\\
e^7 &= ab^{\frac{3}{4}} dx,\\
e^8 &= b^{\frac{3}{4}} sech(y)dy,
\end{aligned} \tag{4.11}$$

where the local sections of the cotangent bundle of each S^3 respectively by $\theta^i, \theta^{\hat{i}}$ and the functions $a(x)$, $b(x)$ are given in the equation (3.52). Thus the metric is

$$\begin{aligned}
g &= \underbrace{\left[\sqrt{a^4 b^3}dx^2 + \sqrt{b^3} sech^2(y)dy^2\right]}_{\pi_B^* g_B}\\
&+ \frac{1}{4}\left[\sqrt{b^3} sech^2(y) + \sqrt{\frac{a^4}{b}}(1 - tanh(y))^2\right] \underbrace{\sum_{i=1}^{3}(\theta^i)^2}_{\pi_1^* g_{S^3}} + \sqrt{\frac{a^4}{b}} \underbrace{\sum_{i=1}^{3}(\theta^{\hat{i}})^2}_{\pi_2^* g_{S^3}}\\
&+ \sqrt{\frac{a^4}{b}}[1 - tanh(y)] \sum_{i=1}^{3} \theta^i \theta^{\hat{i}},
\end{aligned} \tag{4.12}$$

where $\pi_B : S^3 \times S^3 \times \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ and $\pi_i : S^3 \times S^3 \times \mathbb{R}^2 \longrightarrow S^3$ are the natural projections on $\mathbb{R}^2$ and S^3 respectively.

Defining the functions

$$\begin{aligned}
&f_1 = \frac{1}{4}\left[\sqrt{b^3} sech^2(y) + \sqrt{\frac{a^4}{b}}(1 - tanh(y))^2\right], \quad f_2 = \sqrt{\frac{a^4}{b}},\\
&h = \sqrt{\frac{a^4}{b}}[1 - tanh(y)]
\end{aligned} \tag{4.13}$$

and the 2-form ω

$$\omega = \sum_{i=1}^{3} \theta^i \theta^{\hat{i}}, \tag{4.14}$$

we can write g as

$$g = \pi_B^* g_B + \sum_{i=1}^{2}(f_i \circ \pi_B)\pi_i^* g_{F_i} + h\omega, \tag{4.15}$$

where $\pi_B : S^3 \times S^3 \times \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ and $\pi_i : S^3 \times S^3 \times \mathbb{R}^2 \longrightarrow S^3$ are the natural projections on $\mathbb{R}^2$ and S^3 respectively. The matrix of g with respect to the (global) frame

$$\{\theta^1, \theta^2, \theta^3, \theta^{\hat{1}}, \theta^{\hat{2}}, \theta^{\hat{3}}, e^7, e^8\} \tag{4.16}$$

is

$$g = \begin{pmatrix} f_1 I_3 & \frac{h}{2} I_3 & 0 \\ \frac{h}{2} I_3 & f_2 I_3 & 0 \\ 0 & 0 & I_2 \end{pmatrix} \tag{4.17}$$

where I_3 and I_2 are identity matrices of size 3 and 2, and zeroes denote zero matrices of appropriate sizes. Note that if h were zero, the metric given in the equation (4.15) would be a multiply warped product with a block diagonal matrix with respect to an appropriate frame [23].

We shall discuss below possible generalizations of this structure. Let M be the topologically product manifold

$$M = F_1 \times F_2 \times ... \times F_k \times B, \tag{4.18}$$

where $dim\ F_a = n_a$, $(a = 1, ..., k)$, $dim\ B = n$. Assume that these manifolds are equipped with Riemannian metrics g_{F_a} and g_B respectively. Let $U_a \subset F_a$ and $V \subset B$ be coordinate neighborhoods on F_a and B respectively, and let

$$U_1 \times U_2 \times ... \times U_k \times V. \tag{4.19}$$

Denote the local sections of the cotangent bundle of each F_a respectively by $\{\theta_a^i\}_{i=1}^{n_a}$, the local coordinates of each F_a by $\{y_a^i\}_{i=1}^{n_a}$, and the local coordinates on B by $x^1, x^2, ..., x^n$.

We can define a metric on M by choosing linearly independent local sections of the cotangent bundle T^*M and declaring these to be orthonormal. The most general local orthonormal frame corresponding to a fiber-base decomposition is given by

$$e_a^i = \sum_{b=1}^{k} \sum_{j=1}^{n_b} A_{aj}^{bi} \theta_b^j, \qquad i = 1, ..., n_a,\ a = 1, .., k \tag{4.20}$$

$$e_B^i = \sum_{j=1}^{n} a_{Bj}^i\ dx^j, \qquad i = 1, ..., n \tag{4.21}$$

where

$$A^{bi}_{aj} = A^{bi}_{aj}(x^1, x^2, ..., x^n), \quad a^i_{Bj} = a^i_{Bj}(x^1, x^2, ..., x^n). \tag{4.22}$$

First note that

$$\sum_{i=1}^{n} e^i_B \otimes e^i_B = \pi^*_B g_B, \tag{4.23}$$

where $\pi_B : F_1 \times F_2 \times ... \times F_k \times B \longrightarrow B$ is the natural projection on B. Then the metric g is written by

$$\begin{aligned} g &= \pi^*_B g_B + \sum_{a=1}^{k} \sum_{i=1}^{n_a} e^i_a \otimes e^i_a \\ &= \pi^*_B g_B + \sum_{a=1}^{k} \sum_{i=1}^{n_a} \left[\left(\sum_{b=1}^{k} \sum_{j=1}^{n_b} A^{bi}_{aj} \theta^j_b \right) \otimes \left(\sum_{c=1}^{k} \sum_{l=1}^{n_c} A^{ci}_{al} \theta^l_c \right) \right] \\ &= \pi^*_B g_B + \sum_{b,c=1}^{k} \sum_{j=1}^{n_b} \sum_{l=1}^{n_c} \left(\sum_{a=1}^{k} \sum_{i=1}^{n_a} A^{bi}_{aj} A^{ci}_{al} \right) \ \theta^j_b \otimes \theta^l_c. \end{aligned} \tag{4.24}$$

If we split the summation for c into the cases $b = c$ and $c \neq b$, we obtain

$$\begin{aligned} g &= \pi^*_B g_B + \sum_{b=1}^{k} \sum_{j,l=1}^{n_b} \left(\sum_{a=1}^{k} \sum_{i=1}^{n_a} A^{bi}_{aj} A^{bi}_{al} \right) \ \theta^j_b \otimes \theta^l_b \\ &+ \sum_{b=1}^{k} \sum_{c=b+1}^{k} \sum_{j=1}^{n_b} \sum_{l=1}^{n_c} \left(\sum_{a=1}^{k} \sum_{i=1}^{n_a} A^{bi}_{aj} A^{ci}_{al} \right) \ (\theta^j_b \otimes \theta^l_c + \theta^l_b \otimes \theta^j_c). \end{aligned} \tag{4.25}$$

If $\{\tilde{\theta}^j_b\}^{n_b}_{j=1}$ is any other local orthonormal frame for F_b, then

$$\theta^j_b = \sum_{r=1}^{n_b} P^j_{br} \ \tilde{\theta}^r_b. \tag{4.26}$$

Under these frame rotations g transforms as

$$\begin{aligned} g &= \pi^*_B g_B + \sum_{b=1}^{k} \sum_{c=1}^{k} \sum_{j=1}^{n_b} \sum_{l=1}^{n_c} \left(\sum_{a=1}^{k} \sum_{i=1}^{n_a} A^{bi}_{aj} A^{ci}_{al} \right) \left[\left(\sum_{r=1}^{n_b} P^j_{br} \tilde{\theta}^r_b \right) \otimes \left(\sum_{s=1}^{n_s} P^l_{cs} \tilde{\theta}^s_c \right) \right] \\ &= \pi^*_B g_B + \sum_{b=1}^{k} \sum_{r,s=1}^{n_b} \left(\sum_{a=1}^{k} \sum_{i=1}^{n_a} \sum_{j,l=1}^{n_b} A^{bi}_{aj} A^{bi}_{al} P^j_{br} P^l_{bs} \right) \tilde{\theta}^r_b \otimes \tilde{\theta}^s_b \\ &+ \sum_{b=1}^{k} \sum_{c=b+1}^{k} \sum_{r=1}^{n_b} \sum_{s=1}^{n_c} \left(\sum_{a=1}^{k} \sum_{i=1}^{n_a} \sum_{j=1}^{n_b} \sum_{l=1}^{n_c} A^{bi}_{aj} A^{ci}_{al} P^j_{br} P^l_{cs} \right) \tilde{\theta}^r_b \otimes \tilde{\theta}^s_c. \end{aligned} \tag{4.27}$$

Hence the coefficients of $\tilde{\theta}^s_b \otimes \tilde{\theta}^s_c$ given in the equation (4.27) is in general a function depending on the coordinates of the base B and the fibers F_a and F_b.

The problem we are dealing is modeled on an 8-dimensional warped-like product manifold $M = F_1 \times F_2 \times B$ where F_1, F_2 are 3-dimensional and B is a 2-dimensional manifold.

Definition 4.1. Let $M = F_1 \times F_2 \times B$ be an 8-dimensional topologically product manifold where F_1, F_2 are 3-manifolds and B is a 2-manifold, each equipped with Riemannian metrics. Let $\theta^i, \theta^{\hat{i}}$ be orthonormal sections of the cotangent bundles of F_1 and F_2 respectively and x, y be local coordinates on B. If the metric on M is defined by the following orthonormal frame

$$\begin{aligned} e^i &= A(x,y)\theta^i + B(x,y)\theta^{\hat{i}}, \quad i = 1,2,3 \\ e^{\hat{i}} &= \hat{A}(x,y)\theta^i + \hat{B}(x,y)\theta^{\hat{i}}, \quad i = 1,2,3 \\ e^{i+6} &= a_{i1}(x,y)dx + a_{i2}(x,y)dy, \quad i = 1,2 \end{aligned} \tag{4.28}$$

where $A, B, \hat{A}, \hat{B}$ and a_{ij} $(i, j = 1,2)$ are functions on base manifold B, then we call (M, e^i) $(i = 1,2,...,8)$ a *“(3+3+2) warped-like product"* manifold.

The metric is then given by (4.28) is written as

$$\begin{aligned} g &= \sum_{i=1}^{3}\left(e^i \otimes e^i + e^{\hat{i}} \otimes e^{\hat{i}}\right) + \sum_{i=1}^{2} e^{i+6} \otimes e^{i+6} \\ &= \underbrace{(a_{11}^2 + a_{21}^2)dx \otimes dx + (a_{12}^2 + a_{22}^2)dy \otimes dy + (a_{11}a_{12} + a_{22}a_{21})dx \otimes dy}_{\pi_B^* g_B} \\ &\quad + (A^2 + \hat{A}^2)\underbrace{\sum_{i=1}^{3}\theta^i \otimes \theta^i}_{\pi_1^* g_{F_1}} + (B^2 + \hat{B}^2)\underbrace{\sum_{i=1}^{3}\theta^{\hat{i}} \otimes \theta^{\hat{i}}}_{\pi_2^* g_{F_2}} + 2(AB + \hat{A}\hat{B})\underbrace{\sum_{i=1}^{3}\theta^i \otimes \theta^{\hat{i}}}_{\pi_{12}^* \omega} \\ &= \pi_B^* g_B + (f_1 \circ \pi_B)\pi_1^* g_{F_1} + (f_2 \circ \pi_B)\pi_2^* g_{F_2} + (h_{12} \circ \pi_B)\pi_{12}^* \omega, \end{aligned} \tag{4.29}$$

where

$$f_1 = A^2 + \hat{A}^2, \quad f_2 = B^2 + \hat{B}^2, \quad h_{12} = 2(AB + \hat{A}\hat{B}), \tag{4.30}$$

$\pi_B : F_1 \times F_2 \times \times B \longrightarrow B$, $\pi_b : F_1 \times F_2 \times B \longrightarrow F_b$ and $\pi_{12} : F_1 \times F_2 \times B \longrightarrow F_1 \times F_2$ are the natural projections on B, F_b and $F_1 \times F_2$ respectively, and $\omega = \sum_{i=1}^{3} \theta^i \theta^{\hat{i}}$ on $F_1 \times F_2$.

Remark 4.2. Since all 3-manifolds are paralellizable [29], (i.e. F_i's $(i = 1,2)$ are paralellizable), then the fiber F is also a paralellizable 6-manifold. Let $\theta^i, \theta^{\hat{i}}$ $(i = 1,2,3)$ be (global) orthonormal sections of the cotangent bundles of F_1 and F_2 respectively. Then the set of 3-forms in the fiber F, i.e. $\Lambda^3(F)$, includes two closed 3-forms. These are the volume forms of the F_i's $(i = 1,2)$ given by

$$vol_{F_1} = \theta^{123} \quad \text{and} \quad vol_{F_2} = \theta^{\hat{1}\hat{2}\hat{3}}. \tag{4.31}$$

After fixing orthonormal basis for F_1 and F_2, we define an almost complex structure J by $\theta_i \to \theta_{\hat{i}}$, $\theta^{\hat{i}} \to -\theta_i$ where θ_i and $\theta_{\hat{i}}$ are the duals of the θ^i and $\theta^{\hat{i}}$ respectively. The complex volume form Ψ and the Kähler form ω on F are defined by

$$\begin{aligned} \Psi &= \Psi^+ + i\Psi^- = (\theta^1 + iJ\theta^1)(\theta^2 + iJ\theta^2)(\theta^3 + iJ\theta^3), \\ \omega &= \sum_{i=1}^{3} \theta^i \wedge J\theta^i. \end{aligned} \tag{4.32}$$

Using $J\theta^i = \theta^{\hat{i}}$ for $i = 1,2,3$, we obtain

$$\begin{aligned} \Psi^+ &= \theta^{123} - \theta^{1\hat{2}\hat{3}} - \theta^{\hat{1}2\hat{3}} - \theta^{\hat{1}\hat{2}3}, \\ \Psi^- &= \theta^{\hat{1}23} + \theta^{1\hat{2}3} + \theta^{12\hat{3}} - \theta^{\hat{1}\hat{2}\hat{3}}, \\ \omega &= \theta^1\theta^{\hat{1}} + \theta^2\theta^{\hat{2}} + \theta^3\theta^{\hat{3}}. \end{aligned} \tag{4.33}$$

Hence the fiber space F admits a special almost Hermitian 6-dimensional manifold structure, i.e. a six-dimensional smooth manifold endowed with an $SU(3)$-structure. Such a manifold is characterized by its complex volume form $\Psi = \Psi^+ + i\Psi^-$ and its Kähler form ω [34].

4.3 Bonan Form and $(3+3+2)$ Warped-Like Product Structure

In this section, we present the Bonan form Ω in terms of the $(3+3+2)$ warped-like product structure on $M = F \times \mathbb{R}^2$.

Proposition 4.3. *Let F be a 6-dimensional Riemannian manifold of the form $F = F_1 \times F_2$ and F_i $(i = 1,2)$ be 3-manifolds. Let $\theta^i, \theta^{\hat{i}}$ $(i = 1,2,3)$ be orthonormal sections of the cotangent bundles of F_1 and F_2 respectively. Let $(M = F \times \mathbb{R}^2,\ e^i)$ be an 8-dimensional $(3+3+2)$ warped-like product manifold given in Definition 4.1. Then the form given in the equation (3.11) is written as*

$$\Omega = -\tfrac{1}{2} f^2\omega^2 + \phi_1^+ m_1 + \phi_2^+ m_2 + \phi_1^- n_1 + \phi_2^- n_2 + f\omega e^{78},$$

where

$$\begin{aligned} &\omega = \theta^{1\hat{1}} + \theta^{2\hat{2}} + \theta^{3\hat{3}},\ \ \phi_1^+ = \theta^{123},\ \ \phi_1^- = \theta^{\hat{1}\hat{2}\hat{3}}, \\ &\phi_2^+ = \theta^{1\hat{2}\hat{3}} + \theta^{\hat{1}2\hat{3}} + \theta^{\hat{1}\hat{2}3},\ \ \phi_2^- = \theta^{\hat{1}23} + \theta^{1\hat{2}3} + \theta^{12\hat{3}}, \end{aligned} \tag{4.34}$$

and f, m_i, n_i, $(i=1,2)$ *are*

$$\begin{aligned} f &= A\hat{B}-B\hat{A}, \\ m_1 &= [A^3-3A\hat{A}^2]e^8+[\hat{A}^3-3A^2\hat{A}]e^7, \\ m_2 &= [AB^2-2B\hat{A}\hat{B}-A\hat{B}^2]e^8+[\hat{A}\hat{B}^2-2AB\hat{B}-B^2\hat{A}]e^7, \\ n_1 &= [B^3-3B\hat{B}^2]e^8+[\hat{B}^3-3B^2\hat{B}]e^7, \\ n_2 &= [A^2B-2A\hat{A}\hat{B}-B\hat{A}^2]e^8+[\hat{A}^2\hat{B}-2AB\hat{A}-A^2\hat{B}]e^7. \end{aligned} \tag{4.35}$$

Proof. When we substitute the $(3+3+2)$ warped-like product structure given by the equation (4.28) into the expressions of the Bonan form given in the equation (3.11), then we obtain

$$\beta = f\omega \tag{4.36}$$

where

$$f = A\hat{B}-B\hat{A}, \quad \omega = \theta^{1\hat{1}}+\theta^{2\hat{2}}+\theta^{3\hat{3}}. \tag{4.37}$$

And we get μ and ν as

$$\begin{aligned} \mu &= [A^3-3A\hat{A}^2]\theta^{123}+[AB^2-2B\hat{A}\hat{B}-A\hat{B}^2](\theta^{1\hat{2}\hat{3}}+\theta^{\hat{1}2\hat{3}}+\theta^{\hat{1}\hat{2}3}) \\ &\quad +[B^3-3B\hat{B}^2]\theta^{\hat{1}\hat{2}\hat{3}}+[A^2B-2A\hat{A}\hat{B}-B\hat{A}^2](\theta^{1\hat{2}3}+\theta^{\hat{1}23}+\theta^{12\hat{3}}). \end{aligned}$$

$$\begin{aligned} \nu &= [\hat{A}^3-3A^2\hat{A}]\theta^{123}+[\hat{A}\hat{B}^2-2AB\hat{B}-B^2\hat{A}](\theta^{1\hat{2}\hat{3}}+\theta^{\hat{1}2\hat{3}}+\theta^{\hat{1}\hat{2}3}) \\ &\quad +[\hat{B}^3-3B^2\hat{B}]\theta^{\hat{1}\hat{2}\hat{3}}+[\hat{A}^2\hat{B}-2AB\hat{A}-A^2\hat{B}](\theta^{1\hat{2}3}+\theta^{\hat{1}23}+\theta^{12\hat{3}}). \end{aligned} \tag{4.38}$$

We introduce new variables to simplify the notation $\phi_i^{\pm}$ $(i=1,2)$ as

$$\begin{aligned} \phi_1^+ &= \theta^{123}, \quad \phi_2^+ = \theta^{1\hat{2}\hat{3}}+\theta^{\hat{1}2\hat{3}}+\theta^{\hat{1}\hat{2}3}, \\ \phi_1^- &= \theta^{\hat{1}\hat{2}\hat{3}}, \quad \phi_2^- = \theta^{\hat{1}23}+\theta^{1\hat{2}3}+\theta^{12\hat{3}}. \end{aligned} \tag{4.39}$$

Then we can write

$$\mu e^8+\nu e^7 = \phi_1^+ m_1+\phi_2^+ m_2+\phi_1^- n_1+\phi_2^- n_2, \tag{4.40}$$

where the coefficient 1-forms m_i and n_i $(i=1,2)$ are given in the equation (4.35). Hence we write the Bonan form Ω on M as follows

$$\Omega = \frac{1}{2}f^2\omega^2+\phi_1^+ m_1+\phi_2^+ m_2+\phi_1^- n_1+\phi_2^- n_2+f\omega e^{78}. \tag{4.41}$$

This completes the proof. □

Hence Proposition 4.3 implies that the Bonan form Ω is written as a linear combination of the forms in the exterior algebra of the fiber with coefficients in the exterior algebra of the base space.

Remark 4.4. If the metric on $F \times B$ is a Riemannian product where B is diffeomorphic to $\mathbb{R}^2$, then the Bonan form Ω on $F \times \mathbb{R}^2$ is given in [31]

$$\Omega = \omega e^{78} + \Psi^+ e^8 + e^7 \Psi^- - \frac{1}{2}\omega^2, \tag{4.42}$$

where $\Psi = \Psi^+ + i\Psi^-$ is complex volume form, ω is Kähler form and e^7, e^8 are orthonormal frame of the cotangent bundle of $\mathbb{R}^2$. We see that in the case of a $(3+3+2)$ warped-like product, this linear combination will involve

$$\{\omega, \omega^2, \phi_1^+, \phi_1^- \ \phi_2^+, \phi_2^-\}, \tag{4.43}$$

where $\phi_1^+ = \theta^{123}$ and $\phi_1^- = \theta^{\hat{1}\hat{2}\hat{3}}$ are the volume forms of the fibers F_i's $(i = 1,2)$ respectively.

Corresponding to the decomposition of the manifold as “fiber" and “base", the exterior algebra has the following direct sum decomposition,

$$\Lambda^p(M) = \bigoplus_{a+k=p} \Lambda^{(a,k)}(M), \tag{4.44}$$

where $a = 1, \ldots, 6$ and $k = 1,2$, i.e. in our case the fiber is 6-dimensional and the base is 2-dimensional, this fiber structure gives a decomposition of the exterior algebra as follows.

$$\begin{aligned}
\Lambda^1(M) &= \Lambda^{1,0} \oplus \Lambda^{0,1}, \\
\Lambda^2(M) &= \Lambda^{2,0} \oplus \Lambda^{1,1} \oplus \Lambda^{0,2}, \\
\Lambda^3(M) &= \Lambda^{3,0} \oplus \Lambda^{2,1} \oplus \Lambda^{1,2}, \\
\Lambda^4(M) &= \Lambda^{4,0} \oplus \Lambda^{3,1} \oplus \Lambda^{2,2}, \\
\Lambda^5(M) &= \Lambda^{5,0} \oplus \Lambda^{4,1} \oplus \Lambda^{3,2}, \\
\Lambda^6(M) &= \Lambda^{6,0} \oplus \Lambda^{5,1} \oplus \Lambda^{4,2}, \\
\Lambda^7(M) &= \Lambda^{6,1} \oplus \Lambda^{5,2}, \\
\Lambda^8(M) &= \Lambda^{6,2}.
\end{aligned} \tag{4.45}$$

Under the exterior derivative these summands are mapped as

$$d: \quad \Lambda^{(a,k)}(M) \longrightarrow \Lambda^{(a+1,k)} \oplus \Lambda^{(a,k+1)}. \tag{4.46}$$

We can refine this decomposition by splitting the components for each fiber as

$$\Lambda^p(M) = \bigoplus_{a+b+k=p} \Lambda^{(a,b,k)}(M), \tag{4.47}$$

where a and b range from 1 to 3 and $k = 1,2$ as before, i.e.,

$$\begin{aligned}
\Lambda^1(M) &= \Lambda^{1,0,0} \oplus \Lambda^{0,1,0} \oplus \Lambda^{0,0,1}, \\
\Lambda^2(M) &= \Lambda^{2,0,0} \oplus \Lambda^{1,1,0} \oplus \Lambda^{0,2,0} \oplus \Lambda^{1,0,1} \oplus \Lambda^{0,1,1} \oplus \Lambda^{0,0,2}, \\
\Lambda^3(M) &= \Lambda^{3,0,0} \oplus \Lambda^{2,1,0} \oplus \Lambda^{1,2,0} \oplus \Lambda^{0,3,0} \oplus \Lambda^{2,0,1} \oplus \Lambda^{1,1,1} \oplus \Lambda^{0,2,1} \oplus \Lambda^{1,0,2} \oplus \Lambda^{0,1,2}, \\
\Lambda^4(M) &= \Lambda^{3,1,0} \oplus \Lambda^{2,2,0} \oplus \Lambda^{1,3,0} \oplus \Lambda^{3,0,1} \oplus \Lambda^{2,1,1} \oplus \Lambda^{1,2,1} \oplus \Lambda^{0,3,1} \oplus \Lambda^{2,0,2} \oplus \Lambda^{1,1,2} \\
&\quad \oplus \Lambda^{0,2,2}, \\
\Lambda^5(M) &= \Lambda^{3,2,0} \oplus \Lambda^{2,3,0} \oplus \Lambda^{3,1,1} \oplus \Lambda^{2,2,1} \oplus \Lambda^{1,3,1} \oplus \Lambda^{3,0,2} \oplus \Lambda^{2,1,2} \oplus \Lambda^{1,2,2} \oplus \Lambda^{0,3,2}, \\
\Lambda^6(M) &= \Lambda^{3,3,0} \oplus \Lambda^{3,2,1} \oplus \Lambda^{2,3,1} \oplus \Lambda^{3,1,2} \oplus \Lambda^{2,2,2} \oplus \Lambda^{1,3,2}, \\
\Lambda^7(M) &= \Lambda^{3,3,1} \oplus \Lambda^{3,2,2} \oplus \Lambda^{2,3,2}, \\
\Lambda^8(M) &= \Lambda^{3,3,2}.
\end{aligned} \tag{4.48}$$

The effect of the exterior derivative is given by

$$d: \quad \Lambda^{(a,b,k)}(M) \longrightarrow \Lambda^{(a+1,b,k)} \oplus \Lambda^{(a,b+1,k)} \oplus \Lambda^{(a,b,k+1)}. \tag{4.49}$$

After defining the structure of the $(3+3+2)$ warped-like product manifolds, we study the $Spin(7)$ holonomy metrics on these type of manifolds and prove a main theorem related to the $(3+3+2)$ warped-like product manifolds with $Spin(7)$ holonomy in the following section.

4.4 (3+3+2) Warped-Like Product Manifolds with $Spin(7)$ Holonomy

We consider the case where the eight dimensional manifold has a $3+3+2$ decomposition and for simplicity we assume that the base is $\mathbb{R}^2$. We will prove that under suitable global assumptions the fibers are isometric to S^3.

Theorem 4.5. *Let M be diffeomorphic to $F \times B$, where the base B is a two dimensional Riemannian manifold diffeomorphic to $\mathbb{R}^2$, the fibre F is a 6-manifold of the form*

$F = F_1 \times F_2$, and F_i's $(i = 1,2)$ are complete, connected and simply connected 3-manifolds. Let the metric on M be a $(3+3+2)$ warped-like product with the following orthonormal frame

$$\begin{aligned} e^i &= A(x,y)\theta^i + B(x,y)\theta^{\hat{i}}, \quad i = 1,2,3 \\ e^{\hat{i}} &= \hat{A}(x,y)\theta^i + \hat{B}(x,y)\theta^{\hat{i}}, \quad i = 1,2,3 \\ e^{i+6} &= a_{i1}(x,y)dx + a_{i2}(x,y)dy, \quad i = 1,2 \end{aligned}$$

and Ω be the Bonan form on M given by

$$\Omega = -\tfrac{1}{2} f^2\omega^2 + \phi_1^+ m_1 + \phi_2^+ m_2 + \phi_1^- n_1 + \phi_2^- n_2 + f\omega e^{78},$$

where

$$\begin{aligned} &\omega = \theta^{1\hat{1}} + \theta^{2\hat{2}} + \theta^{3\hat{3}}, \ \ f = A\hat{B} - B\hat{A}, \ \ \phi_1^+ = \theta^{123}, \ \ \phi_1^- = \theta^{\hat{1}\hat{2}\hat{3}}, \\ &\phi_2^+ = \theta^{1\hat{2}\hat{3}} + \theta^{\hat{1}2\hat{3}} + \theta^{\hat{1}\hat{2}3}, \ \ \phi_2^- = \theta^{\hat{1}23} + \theta^{1\hat{2}3} + \theta^{12\hat{3}}, \end{aligned}$$

and m_i, n_i, $i = 1,2$ are 1-forms written as

$$\begin{aligned} m_1 &= [A^3 - 3A\hat{A}^2]e^8 + [\hat{A}^3 - 3A^2\hat{A}]e^7, \\ m_2 &= [AB^2 - 2B\hat{A}\hat{B} - A\hat{B}^2]e^8 + [\hat{A}\hat{B}^2 - 2AB\hat{B} - B^2\hat{A}]e^7, \\ n_1 &= [B^3 - 3B\hat{B}^2]e^8 + [\hat{B}^3 - 3B^2\hat{B}]e^7, \\ n_2 &= [A^2B - 2A\hat{A}\hat{B} - B\hat{A}^2]e^8 + [\hat{A}^2\hat{B} - 2AB\hat{A} - A^2\hat{B}]e^7. \end{aligned}$$

If $d\Omega = 0$, then F_1 and F_2 are isometric to S^3.

The crucial step in the proof of this theorem is to find projections of the 5-form $d\Omega$ into subspaces of $\Lambda^5(M)$ determined by the $(3+3+2)$ warped-like product structure. We prove the main theorem a slightly different way in [8].

Proposition 4.6. *Let (M, e^i) be an 8-dimensional warped-like product manifold as in Theorem 4.5 If $d\Omega = 0$, then the following three conditions hold.*

$$\begin{aligned} &i) \quad \omega d\omega = 0, \\ &ii) \quad f df \omega^2 = d\phi_2^+ m_2 + d\phi_2^- n_2, \\ &iii) \quad f d\omega e^{78} = \phi_1^+ dm_1 + \phi_2^+ dm_2 + \phi_1^- dn_1 + \phi_2^- dn_2, \end{aligned}$$

where $\omega, \phi_i^\pm$, m_i, n_i, $(i = 1,2)$ and f are given in the equation (4.34) and (4.35).

Proof. For ease of our computation, we write the Bonan form Ω given in the equation (4.41) as

$$\Omega = \left[-\tfrac{1}{2}f^2\omega^2\right] + \left[\phi_1^+ m_1 + \phi_2^+ m_2 + \phi_1^- n_1 + \phi_2^- n_2\right] + \left[f\omega e^{78}\right]. \tag{4.50}$$

The terms in the brackets belong to subspaces

$$\Lambda^{4,0}, \quad \Lambda^{3,1} \quad \text{and} \quad \Lambda^{2,2}$$

respectively. Note that

$$df\ e^{78} = d(e^{78}) = 0, \tag{4.51}$$

since the base of the multi-warped product is two dimensional. Similarly, as each F_i is three dimensional, their volume forms are closed, i.e,

$$d\phi_1^+ = d\phi_1^- = 0. \tag{4.52}$$

Then $d\Omega = 0$ reduces to

$$\begin{aligned} d\Omega \quad &= \left[-f^2\omega d\omega\right] + \left[-fdf\omega^2 + d\phi_2^+ m_2 + d\phi_2^- n_2\right] \\ &+ \left[fd\omega e^{78} - \phi_1^+ dm_1 - \phi_2^+ dm_2 - \phi_1^- dn_1 - \phi_2^- dn_2\right], \end{aligned} \tag{4.53}$$

where the terms in the brackets belong respectively to

$$\Lambda^{5,0}(M), \quad \Lambda^{4,1}(M) \quad \text{and} \quad \Lambda^{3,2}(M).$$

The closedness of Ω gives us the three equations of Proposition 4.6. This completes the proof. $\square$

Now we prove that the third condition of the Proposition 4.6 fixes the exterior derivatives of the θ^i's and $\theta^{\hat{i}}$'s completely for the manifold M in Theorem 4.5.

Lemma 4.7. *Let (M, e^i) be an 8-dimensional warped-like product manifold as in Theorem 4.5. If*

$$fd\omega e^{78} - \phi_1^+ dm_1 - \phi_2^+ dm_2 - \phi_1^- dn_1 - \phi_2^- dn_2 = 0,$$

then

$$\begin{aligned} d\theta^1 &= c_1\theta^{23}, & d\theta^{\hat{1}} &= c_2\theta^{\hat{2}\hat{3}}, \\ d\theta^2 &= -c_1\theta^{13}, & d\theta^{\hat{2}} &= -c_2\theta^{\hat{1}\hat{3}}, \\ d\theta^3 &= c_1\theta^{12}, & d\theta^{\hat{3}} &= c_2\theta^{\hat{1}\hat{2}}, \end{aligned} \tag{4.54}$$

where c_1 and c_2 are arbitrary constants.

Proof. Write the exterior derivative m_1, m_2, n_1, n_2 are of the following form

$$dm_1 = u_1 e^{78}, \quad dm_2 = u_2 e^{78},$$
$$dn_1 = v_1 e^{78}, \quad dn_2 = v_2 e^{78}, \tag{4.55}$$

where u_1, u_2, v_1, v_2 are functions on the base manifold B. Then we can factorize e^{78} in the condition and obtain

$$[f d\omega] - \left[\phi_1^+ u_1\right] - \left[\phi_2^+ u_2\right] - \left[\phi_1^- v_1\right] - \left[\phi_2^- v_2\right] = 0. \tag{4.56}$$

In the equation (4.56), the terms in the brackets belong respectively to subspaces

$$\Lambda^{(2,1,0)} \oplus \Lambda^{(1,2,0)}, \quad \Lambda^{(3,0,0)}, \quad \Lambda^{(1,2,0)}, \quad \Lambda^{(0,3,0)} \quad \text{and} \quad \Lambda^{(2,1,0)}.$$

This implies that $u_1 = v_1 = 0$, that is,

$$dm_1 = dn_1 = 0. \tag{4.57}$$

Thus we obtain

$$f d\omega = \phi_2^+ u_2 + \phi_2^- v_2. \tag{4.58}$$

If we write explicitly ω, ϕ_2^+ and ϕ_2^-, then

$$fd(\theta^{1\hat{1}} + \theta^{2\hat{2}} + \theta^{3\hat{3}}) = (\theta^{1\hat{2}\hat{3}} + \theta^{\hat{1}2\hat{3}} + \theta^{\hat{1}\hat{2}3})u_2 + (\theta^{\hat{1}23} + \theta^{1\hat{2}3} + \theta^{12\hat{3}})v_2.$$

When we rearrange the equality,

$$\begin{aligned}
&(fd\theta^1 - v_2\theta^{23})\theta^{\hat{1}} - (fd\theta^{\hat{1}} + u_2\theta^{\hat{2}\hat{3}})\theta^1 \\
&+(fd\theta^2 + v_2\theta^{13})\theta^{\hat{2}} - (fd\theta^{\hat{2}} - u_2\theta^{\hat{1}\hat{3}})\theta^2 \\
&+(fd\theta^3 - v_2\theta^{12})\theta^{\hat{3}} - (fd\theta^{\hat{3}} + u_2\theta^{\hat{1}\hat{2}})\theta^3 = 0,
\end{aligned}$$

we obtain

$$d\theta^1 = \tfrac{v_2}{f}\theta^{23}, \quad d\theta^2 = -\tfrac{v_2}{f}\theta^{13}, \quad d\theta^3 = \frac{v_2}{f}\theta^{12}, \tag{4.59}$$
$$d\theta^{\hat{1}} = -\tfrac{u_2}{f}\theta^{\hat{2}\hat{3}}, \quad d\theta^{\hat{2}} = \tfrac{u_2}{f}\theta^{\hat{1}\hat{3}}, \quad d\theta^{\hat{3}} = -\frac{u_2}{f}\theta^{\hat{1}\hat{2}}. \tag{4.60}$$

If we take the exterior derivative of $d\theta^1 = \tfrac{v_2}{f}\theta^{23}$, we get

$$d\left(\frac{v_2}{f}\right)\theta^{23} + \frac{v_2}{f}d\theta^2\theta^3 - \frac{v_2}{f}\theta^2 d\theta^3 = 0. \tag{4.61}$$

Using the equation (4.59), it is seen that

$$d\left(\frac{v_2}{f}\right)=0, \tag{4.62}$$

and in similar way,

$$d\left(\frac{u_2}{f}\right)=0. \tag{4.63}$$

It implies that, $\frac{v_2}{f}, \frac{u_2}{f}$ are constants. This proves the Lemma 4.7, if the constants are chosen as c_1 and c_2. □

The connection forms on any 3-dimensional Riemannian manifolds F_1 and F_2 can be given as follows;

$$\begin{aligned}
d\theta^1 &= w_{12}\theta^2+w_{13}\theta^3, & d\theta^{\hat{1}} &= \eta_{12}\theta^{\hat{2}}+\eta_{13}\theta^{\hat{3}},\\
d\theta^2 &= -w_{12}\theta^1+w_{23}\theta^3, & d\theta^{\hat{2}} &= -\eta_{12}\theta^{\hat{1}}+\eta_{23}\theta^{\hat{3}},\\
d\theta^3 &= -w_{13}\theta^1-w_{23}\theta^2, & d\theta^{\hat{3}} &= -\eta_{13}\theta^{\hat{1}}-\eta_{23}\theta^{\hat{2}},
\end{aligned} \tag{4.64}$$

where w_{ij} and η_{ij} are connection one-forms. If we expand these one forms, we get

$$\begin{aligned}
d\theta^1 &= (w_{12}^1\theta^1+w_{12}^3\theta^3)\theta^2+(w_{13}^1\theta^1+w_{13}^2\theta^2)\theta^3,\\
d\theta^2 &= (-w_{12}^2\theta^2-w_{12}^3\theta^3)\theta^1+(w_{23}^1\theta^1+w_{23}^2\theta^2)\theta^3,\\
d\theta^3 &= (-w_{13}^2\theta^2-w_{12}^3\theta^3)\theta^1+(-w_{23}^1\theta^1-w_{23}^3\theta^3)\theta^2,
\end{aligned} \tag{4.65}$$

$$\begin{aligned}
d\theta^{\hat{1}} &= (\eta_{12}^1\theta^{\hat{1}}+\eta_{12}^3\theta^{\hat{3}})\theta^{\hat{2}}+(\eta_{13}^1\theta^{\hat{1}}+\eta_{13}^2\theta^{\hat{2}})\theta^{\hat{3}},\\
d\theta^{\hat{2}} &= (-\eta_{12}^2\theta^{\hat{2}}-\eta_{12}^3\theta^{\hat{3}})\theta^{\hat{1}}+(\eta_{23}^1\theta^{\hat{1}}+\eta_{23}^2\theta^{\hat{2}})\theta^{\hat{3}},\\
d\theta^{\hat{3}} &= (-\eta_{13}^2\theta^{\hat{2}}-\eta_{12}^3\theta^{\hat{3}})\theta^{\hat{1}}+(-\eta_{23}^1\theta^{\hat{1}}-\eta_{23}^3\theta^{\hat{3}})\theta^{\hat{2}}.
\end{aligned} \tag{4.66}$$

Then $d\theta^i$'s are

$$\begin{aligned}
d\theta^1 &= w_{12}^1\theta^{12}+w_{13}^1\theta^{13}+(w_{13}^2-w_{12}^3)\theta^{23},\\
d\theta^2 &= w_{12}^2\theta^{12}+(w_{12}^3+w_{23}^1)\theta^{13}-w_{13}^1\theta^{23},\\
d\theta^3 &= (w_{13}^2-w_{23}^1)\theta^{12}-w_{12}^2\theta^{13}+w_{12}^1\theta^{23}.
\end{aligned} \tag{4.67}$$

And similarly $d\theta^{\hat{i}}$'s are

$$\begin{aligned}
d\theta^{\hat{1}} &= \eta_{12}^1\theta^{\hat{1}\hat{2}}+\eta_{13}^1\theta^{\hat{1}\hat{3}}+(\eta_{13}^2-\eta_{12}^3)\theta^{\hat{2}\hat{3}},\\
d\theta^{\hat{2}} &= \eta_{12}^2\theta^{\hat{1}\hat{2}}+(\eta_{12}^3+\eta_{23}^1)\theta^{\hat{1}\hat{3}}-\eta_{13}^1\theta^{\hat{2}\hat{3}},\\
d\theta^{\hat{3}} &= (\eta_{13}^2-\eta_{23}^1)\theta^{\hat{1}\hat{2}}-\eta_{12}^2\theta^{\hat{1}\hat{3}}+\eta_{12}^1\theta^{\hat{2}\hat{3}}.
\end{aligned} \tag{4.68}$$

Here we compute the connection one-form matrix w for 3-manifold F_1. A similar procedure applies for F_2. If we choose the constant c as

$$(w_{13}^2 - w_{23}^1) = (w_{13}^2 - w_{12}^3) = -(w_{12}^3 + w_{23}^1) = c, \tag{4.69}$$

then

$$\begin{aligned} d\theta^1 &= (w_{13}^2 - w_{12}^3)\theta^{23} = c\theta^{23}, \\ d\theta^2 &= (w_{12}^3 + w_{23}^1)\theta^{13} = -c\theta^{13}, \\ d\theta^3 &= (w_{13}^2 - w_{23}^1)\theta^{12} = c\theta^{12}. \end{aligned} \tag{4.70}$$

Hence the connection one-form matrix is obtained as follows,

$$w = \begin{pmatrix} 0 & -\frac{c}{2}\theta^3 & \frac{c}{2}\theta^2 \\ \frac{c}{2}\theta^3 & 0 & -\frac{c}{2}\theta^1 \\ -\frac{c}{2}\theta^2 & \frac{c}{2}\theta^1 & 0 \end{pmatrix} = -\frac{c}{2}\begin{pmatrix} 0 & \theta^3 & -\theta^2 \\ -\theta^3 & 0 & \theta^1 \\ \theta^2 & -\theta^1 & 0 \end{pmatrix}. \tag{4.71}$$

The curvature 2-form matrix $\mathscr{R}$ can be computed as

$$\begin{aligned} \mathscr{R} \quad &= dw - ww \\ &= -\frac{c^2}{2}\begin{pmatrix} 0 & \theta^{12} & \theta^{13} \\ -\theta^{12} & 0 & \theta^{23} \\ -\theta^{13} & -\theta^{23} & 0 \end{pmatrix} + \frac{c^2}{4}\begin{pmatrix} 0 & \theta^{12} & \theta^{13} \\ -\theta^{12} & 0 & \theta^{23} \\ -\theta^{13} & -\theta^{23} & 0 \end{pmatrix} \\ &= -\frac{c^2}{4}\begin{pmatrix} 0 & \theta^{12} & \theta^{13} \\ -\theta^{12} & 0 & \theta^{23} \\ -\theta^{13} & -\theta^{23} & 0 \end{pmatrix}. \end{aligned} \tag{4.72}$$

The components of the curvature two-form matrix are obtained as below

$$\begin{aligned} \mathscr{R}_2^1 &= -\frac{c^2}{4}\theta^1\theta^2, \\ \mathscr{R}_3^1 &= -\frac{c^2}{4}\theta^1\theta^3, \\ \mathscr{R}_3^2 &= -\frac{c^2}{4}\theta^2\theta^3. \end{aligned} \tag{4.73}$$

The the curvature two-form elements $\mathscr{R}_i^j$ are related to the Riemannian curvature elements as

$$\mathscr{R}_i^j = dw_i^k - w_k^j w_i^k = R_{ikl}^j w^k w^l. \tag{4.74}$$

Then we obtain

$$R_{1212} = R_{1313} = R_{2323} = -\frac{c^2}{4}. \tag{4.75}$$

Riemannian curvature elements describe the sectional curvature $K_{ij} = R_{ijji}$ which implies that the sectional curvature of F_1 is positive, i.e.

$$K(F_1) = \frac{c^2}{4} > 0. \tag{4.76}$$

Similar result is obtained for F_2. We will complete the proof of Theorem 4.5 by using the following classical result given in [34].

Theorem 4.8. **[Kobayashi-Nomizu]** *Any two connected, simply connected complete Riemannian manifolds of constant curvature k are isometric to each other.*

Proof of Theorem 4.5. Since the equation (4.54) describes the Lie algebra $su(2)$, it follows that if the fibers are connected and simply connected, then they are diffeomorphic to S^3 [43]. Alternatively, using the equation (4.54), we obtained that the sectional curvatures of F_1 and F_2 were positive, i.e, $K(F_i) = \frac{c^2}{4} > 0. \quad \text{for} \quad i = 1,2$. Then by the Theorem 4.8, it follows that F_1 and F_2 are isometric to S^3. This completes the proof of Theorem 4.5. $\square$

Remark 4.9. For the existence of the solution, we have to find $A, B, \hat{A}, \hat{B}$ and a_{ij} $(i, j = 1,2)$ such that the conditions $i)$ and $ii)$ of Proposition 4.6 are satisfied. From the exterior derivatives of the basis 1-forms θ^i and $\theta^{\hat{i}}$, it is seen that the condition $i)$ of Proposition 4.6 holds identically. The condition $ii)$ is to be solved, but instead of this computation, we will use the Yasui-Ootsuka solution in the following section.

4.5 Comparison with the Yasui-Ootsuka ansatz

In this section we first give the explicit gauge transformations from the equation (4.28) to the Yasui-Ootsuka metric (see the equations (3.38)-(3.39)), then we show that the Yasui-Ootsuka solution satisfies the conditions of Proposition 4.8, hence it is the unique solution for a $(3+3+2)$ warped-like manifold of the form given in the equation (4.28) admitting the Spin(7) structure defined by the Bonan form in the equation (3.11).

Since every two-dimensional Riemannian metric can be diagonalized [15], in the equation (4.28) we can take

$$\begin{aligned} e^7 &= \tilde{a}_{11} dx, \\ e^8 &= \tilde{a}_{22} dy, \end{aligned} \tag{4.77}$$

as in [45]. We will show that we can also set $B = 0$ in the equation (4.28) by a frame transformation and obtain exactly the Yasui-Ootsuka metrical ansatz given in the equation (4.11).

An orthogonal transformation of the cotangent frame $\{e^i, e^{\hat{i}}\}$ is given by

$$\begin{pmatrix} \tilde{e^i} \\ \tilde{e}^{\hat{i}} \\ \tilde{e}^{i+6} \end{pmatrix} = \begin{pmatrix} P & Q & 0 \\ \hat{P} & \hat{Q} & 0 \\ 0 & 0 & I \end{pmatrix} \begin{pmatrix} e^i \\ e^{\hat{i}} \\ e^{i+6} \end{pmatrix}. \tag{4.78}$$

Thus we obtain the cotangent frame $\{e^i, e^{\hat{i}}\}$ as

$$\begin{aligned} \tilde{e}^i &= \sum_j \left(P^i_j e^j + Q^i_j e^{\hat{j}} \right), \quad i = 1,2,3 \\ \tilde{e}^{\hat{i}} &= \sum_j \left(\hat{P}^i_j e^j + \hat{Q}^i_j e^{\hat{j}} \right), \quad i = 1,2,3 \end{aligned} \tag{4.79}$$

where $P, Q, \hat{P}, \hat{Q}$ satisfy

$$\begin{aligned} PP^t + QQ^t &= I, \\ P\hat{P}^t + Q\hat{Q}^t &= 0, \\ \hat{P}\hat{P}^t + \hat{Q}\hat{Q}^t &= I. \end{aligned} \tag{4.80}$$

The new basis elements $\tilde{e}^i, \tilde{e}^{\hat{i}}$ can be written now as

$$\begin{aligned} \tilde{e}^i &= \tilde{A}\theta^i + \tilde{B}\theta^{\hat{i}}, \\ \tilde{e}^{\hat{i}} &= \tilde{\hat{A}}\theta^i + \tilde{\hat{B}}\theta^{\hat{i}}, \end{aligned} \tag{4.81}$$

where

$$\begin{aligned} \tilde{A} &= AP + \hat{A}Q, \quad \tilde{B} = BP + \hat{B}Q, \\ \tilde{\hat{A}} &= A\hat{P} + \hat{A}\hat{Q}, \quad \tilde{\hat{B}} = B\hat{P} + \hat{B}\hat{Q}. \end{aligned} \tag{4.82}$$

If we require $\tilde{B} = 0$, then

$$P = -\frac{\hat{B}}{B} Q. \tag{4.83}$$

Substituting this in the equation (4.82) with $\tilde{A}$, we see that

$$\tilde{A} I = \left(\hat{A} - \frac{A\hat{B}}{B} \right) Q, \tag{4.84}$$

hence

$$Q = Q_0(x, y) I, \tag{4.85}$$

that is, Q is the proportional to identity. Then from the first condition given in the equation (4.80), we can determine Q_0 and obtain P and Q as

$$Q = \pm \frac{B}{\sqrt{B^2 + \hat{B}^2}} I, \tag{4.86}$$

$$P = \mp \frac{\hat{B}}{\sqrt{B^2 + \hat{B}^2}} I. \tag{4.87}$$

Then we determine $\hat{P}$ from the equation $P\hat{P}^t + Q\hat{Q}^t = 0$, and substituting in $\hat{A}$, we see that $\hat{Q}$ is also proportional to identity and determine $\hat{P}$ and $\hat{Q}$ as

$$\hat{Q} = \varepsilon \frac{\hat{B}}{\sqrt{B^2 + \hat{B}^2}} I \tag{4.88}$$

$$\hat{P} = \varepsilon \frac{B}{\sqrt{B^2 + \hat{B}^2}} I, \tag{4.89}$$

where $\varepsilon^2 = 1$. The transformation matrix

$$\begin{pmatrix} P & Q \\ \hat{P} & \hat{Q} \end{pmatrix} = \frac{1}{\sqrt{B^2 + \hat{B}^2}} \begin{pmatrix} \mp\hat{B} & \pm B \\ \varepsilon B & \varepsilon\hat{B} \end{pmatrix} \tag{4.90}$$

is clearly orthogonal and the coefficients of the new frame are

$$\begin{aligned}
\tilde{A} &= \mp \frac{f}{\sqrt{B^2 + \hat{B}^2}}, \\
\tilde{B} &= 0, \\
\tilde{\hat{A}} &= \varepsilon \frac{AB + \hat{A}\hat{B}}{\sqrt{B^2 + \hat{B}^2}}, \\
\tilde{\hat{B}} &= \varepsilon \sqrt{B^2 + \hat{B}^2},
\end{aligned} \tag{4.91}$$

hence we obtain exactly the metric ansatz given in the equation (4.11). Comparing with the equation (4.28), we can see that

$$\begin{aligned}
A &= \frac{1}{2} b^{\frac{3}{4}} sech(y), \\
B &= 0, \\
\hat{A} &= \frac{1}{2} a b^{-\frac{1}{4}} (1 - tanh(y)), \\
\hat{B} &= a b^{-\frac{1}{4}}, \\
a_{11} &= a b^{\frac{3}{4}}, \quad a_{12} = 0, \\
a_{21} &= 0, \qquad a_{22} = b^{\frac{3}{4}} sech(y).
\end{aligned} \tag{4.92}$$

By a straight forward computation using (3.58), it can be seen that the three conditions given in Proposition 4.6 are satisfied, hence we obtain a direct proof that the Yasui-Ootsuka solution given in the equation (3.57) is a $Spin(7)$ holonomy metric. Thus we have the following corollary.

Corollary 4.10. *Let M be the $(3+3+2)$ warped-like product manifold defined in the equation* (4.28) *which admits the Spin(7) structure defined by the Bonan form given in the equation* (3.11). *Then the metric given by* (3.57) *is unique up to gauge transformations.*

5. CONCLUSION AND DISCUSSION

In this thesis we have focused on a special case of Riemannian manifolds whose holonomy group is contained in $Spin(7)$.

As we have seen, a manifold with $Spin(7)$ holonomy is an 8-dimensional real orientable manifold as $Spin(7)$ is a subgroup of $SO(8)$. A remarkable feature of this manifold is the existence of a nowhere vanishing 4-form which is called Bonan form denoted by Ω and which can be written locally as in the equation (3.11). The Bonan form has a key role in the construction of a manifold with $Spin(7)$ holonomy. In the literature, a few explicit examples of $Spin(7)$ holonomy manifolds are constructed locally case by case, by using the fundamental properties of the Bonan form, i.e. self-duality, $Spin(7)$ invariance and closedness [11, 25].

We have presented explicit construction methods of the Bonan form Ω on $\mathbb{R}^8$ by using octonionic multiplication rule [2, 45] and the triple vector cross product of octonions [7, 22].

We have surveyed an explicit metric on a manifold with $Spin(7)$ holonomy and worked the metric given by Yasui and Ootsuka on the manifold $S^3 \times S^3 \times \mathbb{R}^2$. In their method, they use the vector fields ansatz (see the equation (3.38)) to satisfy the volume-preserving vector fields condition and a specific tensor formula called the 2-vector condition. Then a $Spin(7)$ metric structure on $S^3 \times S^3 \times \mathbb{R}^2$ is obtained from the solutions of the first order non-linear differential equations (see the equation (3.51)) as given in the Section 3.2.2. We called this $Spin(7)$ metric solution as the *"Yasui-Ootsuka solution"* on $S^3 \times S^3 \times \mathbb{R}^2$.

Applying the method given in [45] we obtain equivalently, 56 equations involving the commutators of tangent vector fields (see Appendix C.3). The explicit expression form of these linear equations is new one and can be used in future work.

In our work, instead of using the method given by Yasui-Ootsuka (see Section 4.2), it is used a differential form ansatz as a $(3+3+2)$ *warped-like product metric* which is a generalization of multiply-warped product metric [8].

In the thesis we prove that, when the base B is two dimensional, the fibre F is a 6-manifold of the form $F = F_1 \times F_2$ such that F_i's $(i = 1,2)$ are complete, connected and simply connected 3-manifolds and the metric is given by the equation (4.28), and M has the $Spin(7)$ structure determined by the Bonan form given in the equation (3.11), then the connection of the fibers is completely determined by the requirement that the Bonan form be closed. With the global assumptions given above, it is concluded that the fibers F_i's are isometric to S^3.

This implies that the Yasui-Ootsuka solution given in the equation (3.57) on $S^3 \times S^3 \times \mathbb{R}^2$ is unique in the class of the $(3+3+2)$ warped-like product metrics admitting the $Spin(7)$ structure determined by the Bonan form given in the equation (3.11).

We recall that as the Bonan form Ω is a 4-form, then $d\Omega = 0$ gives 56 equations involving exterior derivatives of the basis 1-forms. The connection on any 8-dimensional manifolds is determined by the connection 1-forms, i.e. 8x7/2=28, that is, 28x8=224 parameters. If the manifold is of type $SO(8)$ or $Spin(7)$, then there are respectively 28 and 21 free parameters. Since there are 56 equations, this shows that the solution is not unique under the above conditions.

In the case of the $3+3+2$ warped-like product metric, there are 9 parameters on each 3-manifolds F_i $(i = 1,2)$. Then there are totally 18 parameters and 56 equations as mentioned above. Under some special conditions, it is not surprising to obtain a unique solution as given in Corollary 4.10.

Finally we will present some problems related to the warped-like product manifolds with exceptional holonomy groups. These can be given as research topics for future studies.

Alternative splitting of 8-dimensional manifolds

We have defined the warped-like product manifold in general case and studied a special case $(3+3+2)$ warped-like product manifolds with $Spin(7)$ holonomy. Any alternative splitting of 8-dimensional warped-like product manifolds with $Spin(7)$ holonomy can also be studied.

G_2 case

We have seen that the exceptional holonomy groups of Riemannian manifolds are G_2 and $Spin(7)$ which occur in 7 and 8-dimensional manifolds respectively. An obvious question is that what can we do for warped-like product manifolds with G_2 holonomy [18, 19]. And it can be also studied any alternative splitting of 7-dimensional warped-like product manifolds with G_2 holonomy.

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A. MULTIPLICATION TABLE OF OCTONIONS

There are many ways to construct the multiplication table of octonions. Among these are Cayley-Dickson process, Fano plane, etc. [2,28]. We choose the following multiplication table of octonions to obtain the Bonan form via vector cross products on octonions in Section 3.1.2.

Let us choose the following octonion basis elements as follows

$$\mathbb{O} = span\ \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4, \mathbf{e}_5, \mathbf{e}_6, \mathbf{e}_7, \mathbf{e}_8 = 1\}. \tag{A.1}$$

Then the multiplication of $\mathbf{e}_i$'s is chosen as follows.

Table A.1: Octonion multiplication table

	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$	$\mathbf{e}_4$	$\mathbf{e}_5$	$\mathbf{e}_6$	$\mathbf{e}_7$	$\mathbf{e}_8$
$\mathbf{e}_1$	-1	$-\mathbf{e}_3$	$\mathbf{e}_2$	$-\mathbf{e}_7$	$\mathbf{e}_6$	$-\mathbf{e}_5$	$\mathbf{e}_4$	$\mathbf{e}_1$
$\mathbf{e}_2$	$\mathbf{e}_3$	-1	$-\mathbf{e}_1$	$-\mathbf{e}_6$	$-\mathbf{e}_7$	$\mathbf{e}_4$	$\mathbf{e}_5$	$\mathbf{e}_2$
$\mathbf{e}_3$	$-\mathbf{e}_2$	$\mathbf{e}_1$	-1	$\mathbf{e}_5$	$-\mathbf{e}_4$	$-\mathbf{e}_7$	$\mathbf{e}_6$	$\mathbf{e}_3$
$\mathbf{e}_4$	$\mathbf{e}_7$	$\mathbf{e}_6$	$-\mathbf{e}_5$	-1	$\mathbf{e}_3$	$-\mathbf{e}_2$	$-\mathbf{e}_1$	$\mathbf{e}_4$
$\mathbf{e}_5$	$-\mathbf{e}_6$	$\mathbf{e}_7$	$\mathbf{e}_4$	$-\mathbf{e}_3$	-1	$\mathbf{e}_1$	$-\mathbf{e}_2$	$\mathbf{e}_5$
$\mathbf{e}_6$	$\mathbf{e}_5$	$-\mathbf{e}_4$	$\mathbf{e}_7$	$\mathbf{e}_2$	$-\mathbf{e}_1$	-1	$-\mathbf{e}_3$	$\mathbf{e}_6$
$\mathbf{e}_7$	$-\mathbf{e}_4$	$-\mathbf{e}_5$	$-\mathbf{e}_6$	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$	-1	$\mathbf{e}_7$
$\mathbf{e}_8$	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$	$\mathbf{e}_4$	$\mathbf{e}_5$	$\mathbf{e}_6$	$\mathbf{e}_7$	1

The Non-Zero Set of $\Omega(\mathbf{e}_\alpha, \mathbf{e}_\beta, \mathbf{e}_\gamma, \mathbf{e}_\delta)$ in Section 3.1.2

The Bonan form on R^8 is written as

$$\Omega = \frac{1}{4!} \sum_{\alpha\beta\gamma\delta} \Omega_{\alpha\beta\gamma\delta}\ e^\alpha \wedge e^\beta \wedge e^\gamma \wedge e^\delta,$$

where e^α's belong to an local orthonormal basis for the cotangent bundle of $\mathbb{R}^8$. Then $\mathbf{\Omega}_{\alpha\beta\gamma\delta}$ is defined by the following formula.

$$\Omega_{\alpha\beta\gamma\delta} = \Omega(\mathbf{e}_\alpha, \mathbf{e}_\beta, \mathbf{e}_\gamma, \mathbf{e}_\delta) = \langle \mathbf{e}_\alpha, \mathbf{e}_\beta(\overline{\mathbf{e}_\gamma}\mathbf{e}_\delta)\rangle, \tag{A.2}$$

where $\overline{\mathbf{e}_\alpha}$ is conjugation of $\mathbf{e}_\alpha$ and $\mathbf{e}_\alpha$'s $(\alpha = 1, 2..., 8)$ are given in the equation (A.1). The non-zero elements are computed respectively as

$$\Omega(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4) = \langle \mathbf{e}_1, \mathbf{e}_2(\overline{\mathbf{e}_3}\mathbf{e}_4)\rangle = \langle \mathbf{e}_1, \mathbf{e}_2(-\mathbf{e}_3\mathbf{e}_4)\rangle = \langle \mathbf{e}_1, \mathbf{e}_2(-\mathbf{e}_5)\rangle = \langle \mathbf{e}_1, \mathbf{e}_7\rangle = 0,$$
$$\Omega(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_5) = \langle \mathbf{e}_1, \mathbf{e}_2(\overline{\mathbf{e}_3}\mathbf{e}_5)\rangle = \langle \mathbf{e}_1, \mathbf{e}_2(-\mathbf{e}_3\mathbf{e}_5)\rangle = \langle \mathbf{e}_1, \mathbf{e}_2(-\mathbf{e}_4)\rangle = \langle \mathbf{e}_1, \mathbf{e}_6\rangle = 0,$$
$$\Omega(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_6) = \langle \mathbf{e}_1, \mathbf{e}_2(\overline{\mathbf{e}_3}\mathbf{e}_6)\rangle = \langle \mathbf{e}_1, \mathbf{e}_2(-\mathbf{e}_3\mathbf{e}_6)\rangle = \langle \mathbf{e}_1, \mathbf{e}_2(\mathbf{e}_7)\rangle = \langle \mathbf{e}_1, \mathbf{e}_5\rangle = 0,$$

$$\Omega(\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3,\mathbf{e}_7)=\langle\mathbf{e}_1,\mathbf{e}_2(\overline{\mathbf{e}_3}\mathbf{e}_7)\rangle=\langle\mathbf{e}_1,\mathbf{e}_2(-\mathbf{e}_3\mathbf{e}_7)\rangle=\langle\mathbf{e}_1,\mathbf{e}_2(-\mathbf{e}_6)\rangle=\langle\mathbf{e}_1,-\mathbf{e}_4\rangle=0,$$
$$\Omega(\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3,\mathbf{e}_8)=\langle\mathbf{e}_1,\mathbf{e}_2(\overline{\mathbf{e}_3}\mathbf{e}_8)\rangle=\langle\mathbf{e}_1,\mathbf{e}_2(-\mathbf{e}_3\mathbf{e}_8)\rangle=\langle\mathbf{e}_1,\mathbf{e}_2(-\mathbf{e}_3)\rangle=\langle\mathbf{e}_1,\mathbf{e}_1\rangle=1,$$
$$\Omega(\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_4,\mathbf{e}_5)=\langle\mathbf{e}_1,\mathbf{e}_2(\overline{\mathbf{e}_4}\mathbf{e}_5)\rangle=\langle\mathbf{e}_1,\mathbf{e}_2(-\mathbf{e}_4\mathbf{e}_5)\rangle=\langle\mathbf{e}_1,\mathbf{e}_2(-\mathbf{e}_3)\rangle=\langle\mathbf{e}_1,\mathbf{e}_1\rangle=1,$$
$$\Omega(\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_4,\mathbf{e}_6)=\langle\mathbf{e}_1,\mathbf{e}_2(\overline{\mathbf{e}_4}\mathbf{e}_6)\rangle=\langle\mathbf{e}_1,\mathbf{e}_2(-\mathbf{e}_4\mathbf{e}_6)\rangle=\langle\mathbf{e}_1,\mathbf{e}_2\mathbf{e}_2\rangle=\langle\mathbf{e}_1,-1\rangle=0,$$
$$\Omega(\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_4,\mathbf{e}_7)=\langle\mathbf{e}_1,\mathbf{e}_2(\overline{\mathbf{e}_4}\mathbf{e}_7)\rangle=\langle\mathbf{e}_1,\mathbf{e}_2(-\mathbf{e}_4\mathbf{e}_7)\rangle=\langle\mathbf{e}_1,\mathbf{e}_2\mathbf{e}_1\rangle=\langle\mathbf{e}_1,\mathbf{e}_3\rangle=0,$$
$$\Omega(\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_4,\mathbf{e}_8)=\langle\mathbf{e}_1,\mathbf{e}_2(\overline{\mathbf{e}_4}\mathbf{e}_8)\rangle=\langle\mathbf{e}_1,\mathbf{e}_2(-\mathbf{e}_4\mathbf{e}_8)\rangle=\langle\mathbf{e}_1,\mathbf{e}_2(-\mathbf{e}_4)\rangle=\langle\mathbf{e}_1,\mathbf{e}_6\rangle=0,$$
$$\Omega(\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_5,\mathbf{e}_6)=\langle\mathbf{e}_1,\mathbf{e}_2(\overline{\mathbf{e}_5}\mathbf{e}_6)\rangle=\langle\mathbf{e}_1,\mathbf{e}_2(-\mathbf{e}_5\mathbf{e}_6)\rangle=\langle\mathbf{e}_1,\mathbf{e}_2(-\mathbf{e}_1)\rangle=\langle\mathbf{e}_1,-\mathbf{e}_3\rangle=0,$$
$$\Omega(\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_5,\mathbf{e}_7)=\langle\mathbf{e}_1,\mathbf{e}_2(\overline{\mathbf{e}_5}\mathbf{e}_7)\rangle=\langle\mathbf{e}_1,\mathbf{e}_2(-\mathbf{e}_5\mathbf{e}_7)\rangle=\langle\mathbf{e}_1,\mathbf{e}_2\mathbf{e}_2)\rangle=\langle\mathbf{e}_1,-1\rangle=0,$$
$$\Omega(\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_5,\mathbf{e}_8)=\langle\mathbf{e}_1,\mathbf{e}_2(\overline{\mathbf{e}_5}\mathbf{e}_8)\rangle=\langle\mathbf{e}_1,\mathbf{e}_2(-\mathbf{e}_5\mathbf{e}_8)\rangle=\langle\mathbf{e}_1,\mathbf{e}_2(-\mathbf{e}_5)\rangle=\langle\mathbf{e}_1,\mathbf{e}_7\rangle=0,$$
$$\Omega(\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_6,\mathbf{e}_7)=\langle\mathbf{e}_1,\mathbf{e}_2(\overline{\mathbf{e}_6}\mathbf{e}_7)\rangle=\langle\mathbf{e}_1,\mathbf{e}_2(-\mathbf{e}_6\mathbf{e}_7)\rangle=\langle\mathbf{e}_1,\mathbf{e}_2\mathbf{e}_3\rangle=\langle\mathbf{e}_1,-\mathbf{e}_1\rangle=-1,$$
$$\Omega(\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_6,\mathbf{e}_8)=\langle\mathbf{e}_1,\mathbf{e}_2(\overline{\mathbf{e}_6}\mathbf{e}_8)\rangle=\langle\mathbf{e}_1,\mathbf{e}_2(-\mathbf{e}_6\mathbf{e}_8)\rangle=\langle\mathbf{e}_1,\mathbf{e}_2(-\mathbf{e}_6)\rangle=\langle\mathbf{e}_1,-\mathbf{e}_4\rangle=0,$$
$$\Omega(\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_7,\mathbf{e}_8)=\langle\mathbf{e}_1,\mathbf{e}_2(\overline{\mathbf{e}_7}\mathbf{e}_8)\rangle=\langle\mathbf{e}_1,\mathbf{e}_2(-\mathbf{e}_7\mathbf{e}_8)\rangle=\langle\mathbf{e}_1,\mathbf{e}_2(-\mathbf{e}_7)\rangle=\langle\mathbf{e}_1,-\mathbf{e}_5\rangle=0,$$
$$\Omega(\mathbf{e}_1,\mathbf{e}_3,\mathbf{e}_4,\mathbf{e}_5)=\langle\mathbf{e}_1,\mathbf{e}_3(\overline{\mathbf{e}_4}\mathbf{e}_5)\rangle=\langle\mathbf{e}_1,\mathbf{e}_3(-\mathbf{e}_4\mathbf{e}_5)\rangle=\langle\mathbf{e}_1,\mathbf{e}_3(-\mathbf{e}_3)\rangle=\langle\mathbf{e}_1,1\rangle=0,$$
$$\Omega(\mathbf{e}_1,\mathbf{e}_3,\mathbf{e}_4,\mathbf{e}_6)=\langle\mathbf{e}_1,\mathbf{e}_3(\overline{\mathbf{e}_4}\mathbf{e}_6)\rangle=\langle\mathbf{e}_1,\mathbf{e}_3(-\mathbf{e}_4\mathbf{e}_6)\rangle=\langle\mathbf{e}_1,\mathbf{e}_3\mathbf{e}_2\rangle=\langle\mathbf{e}_1,\mathbf{e}_1\rangle=1,$$
$$\Omega(\mathbf{e}_1,\mathbf{e}_3,\mathbf{e}_4,\mathbf{e}_7)=\langle\mathbf{e}_1,\mathbf{e}_3(\overline{\mathbf{e}_4}\mathbf{e}_7)\rangle=\langle\mathbf{e}_1,\mathbf{e}_3(-\mathbf{e}_4\mathbf{e}_7)\rangle=\langle\mathbf{e}_1,\mathbf{e}_3\mathbf{e}_1\rangle=\langle\mathbf{e}_1,-\mathbf{e}_2\rangle=0,$$
$$\Omega(\mathbf{e}_1,\mathbf{e}_3,\mathbf{e}_4,\mathbf{e}_8)=\langle\mathbf{e}_1,\mathbf{e}_3(\overline{\mathbf{e}_4}\mathbf{e}_8)\rangle=\langle\mathbf{e}_1,\mathbf{e}_3(-\mathbf{e}_4\mathbf{e}_8)\rangle=\langle\mathbf{e}_1,\mathbf{e}_3(-\mathbf{e}_4)\rangle=\langle\mathbf{e}_1,-\mathbf{e}_5\rangle=0,$$
$$\Omega(\mathbf{e}_1,\mathbf{e}_3,\mathbf{e}_5,\mathbf{e}_6)=\langle\mathbf{e}_1,\mathbf{e}_3(\overline{\mathbf{e}_5}\mathbf{e}_6)\rangle=\langle\mathbf{e}_1,\mathbf{e}_3(-\mathbf{e}_5\mathbf{e}_6)\rangle=\langle\mathbf{e}_1,\mathbf{e}_3(-\mathbf{e}_1)\rangle=\langle\mathbf{e}_1,\mathbf{e}_2\rangle=0,$$
$$\Omega(\mathbf{e}_1,\mathbf{e}_3,\mathbf{e}_5,\mathbf{e}_7)=\langle\mathbf{e}_1,\mathbf{e}_3(\overline{\mathbf{e}_5}\mathbf{e}_7)\rangle=\langle\mathbf{e}_1,\mathbf{e}_3(-\mathbf{e}_5\mathbf{e}_7)\rangle=\langle\mathbf{e}_1,\mathbf{e}_3\mathbf{e}_2\rangle=\langle\mathbf{e}_1,\mathbf{e}_1\rangle=1,$$
$$\Omega(\mathbf{e}_1,\mathbf{e}_3,\mathbf{e}_5,\mathbf{e}_8)=\langle\mathbf{e}_1,\mathbf{e}_3(\overline{\mathbf{e}_5}\mathbf{e}_8)\rangle=\langle\mathbf{e}_1,\mathbf{e}_3(-\mathbf{e}_5\mathbf{e}_8)\rangle=\langle\mathbf{e}_1,\mathbf{e}_3(-\mathbf{e}_5)\rangle=\langle\mathbf{e}_1,\mathbf{e}_4\rangle=0,$$
$$\Omega(\mathbf{e}_1,\mathbf{e}_3,\mathbf{e}_6,\mathbf{e}_7)=\langle\mathbf{e}_1,\mathbf{e}_3(\overline{\mathbf{e}_6}\mathbf{e}_7)\rangle=\langle\mathbf{e}_1,\mathbf{e}_3(-\mathbf{e}_6\mathbf{e}_7)\rangle=\langle\mathbf{e}_1,\mathbf{e}_3\mathbf{e}_3\rangle=\langle\mathbf{e}_1,-1\rangle=0,$$
$$\Omega(\mathbf{e}_1,\mathbf{e}_3,\mathbf{e}_6,\mathbf{e}_8)=\langle\mathbf{e}_1,\mathbf{e}_3(\overline{\mathbf{e}_6}\mathbf{e}_8)\rangle=\langle\mathbf{e}_1,\mathbf{e}_3(-\mathbf{e}_6\mathbf{e}_8)\rangle=\langle\mathbf{e}_1,\mathbf{e}_3(-\mathbf{e}_6)\rangle=\langle\mathbf{e}_1,\mathbf{e}_7\rangle=0,$$
$$\Omega(\mathbf{e}_1,\mathbf{e}_3,\mathbf{e}_7,\mathbf{e}_8)=\langle\mathbf{e}_1,\mathbf{e}_3(\overline{\mathbf{e}_7}\mathbf{e}_8)\rangle=\langle\mathbf{e}_1,\mathbf{e}_3(-\mathbf{e}_7\mathbf{e}_8)\rangle=\langle\mathbf{e}_1,\mathbf{e}_3(-\mathbf{e}_7)\rangle=\langle\mathbf{e}_1,-\mathbf{e}_6\rangle=0,$$
$$\Omega(\mathbf{e}_1,\mathbf{e}_4,\mathbf{e}_5,\mathbf{e}_6)=\langle\mathbf{e}_1,\mathbf{e}_4(\overline{\mathbf{e}_5}\mathbf{e}_6)\rangle=\langle\mathbf{e}_1,\mathbf{e}_4(-\mathbf{e}_5\mathbf{e}_6)\rangle=\langle\mathbf{e}_1,\mathbf{e}_4(-\mathbf{e}_1)\rangle=\langle\mathbf{e}_1,-\mathbf{e}_7\rangle=0,$$
$$\Omega(\mathbf{e}_1,\mathbf{e}_4,\mathbf{e}_5,\mathbf{e}_7)=\langle\mathbf{e}_1,\mathbf{e}_4(\overline{\mathbf{e}_5}\mathbf{e}_7)\rangle=\langle\mathbf{e}_1,\mathbf{e}_4(-\mathbf{e}_5\mathbf{e}_7)\rangle=\langle\mathbf{e}_1,\mathbf{e}_4\mathbf{e}_2\rangle=\langle\mathbf{e}_1,\mathbf{e}_6\rangle=0,$$
$$\Omega(\mathbf{e}_1,\mathbf{e}_4,\mathbf{e}_5,\mathbf{e}_8)=\langle\mathbf{e}_1,\mathbf{e}_4(\overline{\mathbf{e}_5}\mathbf{e}_8)\rangle=\langle\mathbf{e}_1,\mathbf{e}_4(-\mathbf{e}_5\mathbf{e}_8)\rangle=\langle\mathbf{e}_1,\mathbf{e}_4(-\mathbf{e}_5)\rangle=\langle\mathbf{e}_1,-\mathbf{e}_3\rangle=0,$$
$$\Omega(\mathbf{e}_1,\mathbf{e}_4,\mathbf{e}_6,\mathbf{e}_7)=\langle\mathbf{e}_1,\mathbf{e}_4(\overline{\mathbf{e}_6}\mathbf{e}_7)\rangle=\langle\mathbf{e}_1,\mathbf{e}_4(-\mathbf{e}_6\mathbf{e}_7)\rangle=\langle\mathbf{e}_1,\mathbf{e}_4\mathbf{e}_3\rangle=\langle\mathbf{e}_1,-\mathbf{e}_5\rangle=0,$$
$$\Omega(\mathbf{e}_1,\mathbf{e}_4,\mathbf{e}_6,\mathbf{e}_8)=\langle\mathbf{e}_1,\mathbf{e}_4(\overline{\mathbf{e}_6}\mathbf{e}_8)\rangle=\langle\mathbf{e}_1,\mathbf{e}_4(-\mathbf{e}_6\mathbf{e}_8)\rangle=\langle\mathbf{e}_1,\mathbf{e}_4(-\mathbf{e}_6)\rangle=\langle\mathbf{e}_1,\mathbf{e}_2\rangle=0,$$
$$\Omega(\mathbf{e}_1,\mathbf{e}_4,\mathbf{e}_7,\mathbf{e}_8)=\langle\mathbf{e}_1,\mathbf{e}_4(\overline{\mathbf{e}_7}\mathbf{e}_8)\rangle=\langle\mathbf{e}_1,\mathbf{e}_4(-\mathbf{e}_7\mathbf{e}_8)\rangle=\langle\mathbf{e}_1,\mathbf{e}_4(-\mathbf{e}_7)\rangle=\langle\mathbf{e}_1,\mathbf{e}_1\rangle=1,$$
$$\Omega(\mathbf{e}_1,\mathbf{e}_5,\mathbf{e}_6,\mathbf{e}_7)=\langle\mathbf{e}_1,\mathbf{e}_5(\overline{\mathbf{e}_6}\mathbf{e}_7)\rangle=\langle\mathbf{e}_1,\mathbf{e}_5(-\mathbf{e}_6\mathbf{e}_7)\rangle=\langle\mathbf{e}_1,\mathbf{e}_5\mathbf{e}_3\rangle=\langle\mathbf{e}_1,\mathbf{e}_4\rangle=0,$$
$$\Omega(\mathbf{e}_1,\mathbf{e}_5,\mathbf{e}_6,\mathbf{e}_8)=\langle\mathbf{e}_1,\mathbf{e}_5(\overline{\mathbf{e}_6}\mathbf{e}_8)\rangle=\langle\mathbf{e}_1,\mathbf{e}_5(-\mathbf{e}_6\mathbf{e}_8)\rangle=\langle\mathbf{e}_1,\mathbf{e}_5(-\mathbf{e}_6)\rangle=\langle\mathbf{e}_1,-\mathbf{e}_1\rangle=-1,$$
$$\Omega(\mathbf{e}_1,\mathbf{e}_5,\mathbf{e}_7,\mathbf{e}_8)=\langle\mathbf{e}_1,\mathbf{e}_5(\overline{\mathbf{e}_7}\mathbf{e}_8)\rangle=\langle\mathbf{e}_1,\mathbf{e}_5(-\mathbf{e}_7\mathbf{e}_8)\rangle=\langle\mathbf{e}_1,\mathbf{e}_5(-\mathbf{e}_7)\rangle=\langle\mathbf{e}_1,\mathbf{e}_2\rangle=0,$$
$$\Omega(\mathbf{e}_1,\mathbf{e}_6,\mathbf{e}_7,\mathbf{e}_8)=\langle\mathbf{e}_1,\mathbf{e}_6(\overline{\mathbf{e}_7}\mathbf{e}_8)\rangle=\langle\mathbf{e}_1,\mathbf{e}_6(-\mathbf{e}_7\mathbf{e}_8)\rangle=\langle\mathbf{e}_1,\mathbf{e}_6(-\mathbf{e}_7)\rangle=\langle\mathbf{e}_1,\mathbf{e}_3\rangle=0,$$
$$\Omega(\mathbf{e}_2,\mathbf{e}_3,\mathbf{e}_4,\mathbf{e}_5)=\langle\mathbf{e}_2,\mathbf{e}_3(\overline{\mathbf{e}_4}\mathbf{e}_5)\rangle=\langle\mathbf{e}_2,\mathbf{e}_3(-\mathbf{e}_4\mathbf{e}_5)\rangle=\langle\mathbf{e}_2,\mathbf{e}_3(-\mathbf{e}_3)\rangle=\langle\mathbf{e}_2,1\rangle=0,$$
$$\Omega(\mathbf{e}_2,\mathbf{e}_3,\mathbf{e}_4,\mathbf{e}_6)=\langle\mathbf{e}_2,\mathbf{e}_3(\overline{\mathbf{e}_4}\mathbf{e}_6)\rangle=\langle\mathbf{e}_2,\mathbf{e}_3(-\mathbf{e}_4\mathbf{e}_6)\rangle=\langle\mathbf{e}_2,\mathbf{e}_3\mathbf{e}_2\rangle=\langle\mathbf{e}_2,\mathbf{e}_1\rangle=0,$$
$$\Omega(\mathbf{e}_2,\mathbf{e}_3,\mathbf{e}_4,\mathbf{e}_7)=\langle\mathbf{e}_2,\mathbf{e}_3(\overline{\mathbf{e}_4}\mathbf{e}_7)\rangle=\langle\mathbf{e}_2,\mathbf{e}_3(-\mathbf{e}_4\mathbf{e}_7)\rangle=\langle\mathbf{e}_2,\mathbf{e}_3\mathbf{e}_1\rangle=\langle\mathbf{e}_2,-\mathbf{e}_2\rangle=-1,$$
$$\Omega(\mathbf{e}_2,\mathbf{e}_3,\mathbf{e}_4,\mathbf{e}_8)=\langle\mathbf{e}_2,\mathbf{e}_3(\overline{\mathbf{e}_4}\mathbf{e}_8)\rangle=\langle\mathbf{e}_2,\mathbf{e}_3(-\mathbf{e}_4\mathbf{e}_8)\rangle=\langle\mathbf{e}_2,\mathbf{e}_3(-\mathbf{e}_4)\rangle=\langle\mathbf{e}_2,-\mathbf{e}_5\rangle=0,$$
$$\Omega(\mathbf{e}_2,\mathbf{e}_3,\mathbf{e}_5,\mathbf{e}_6)=\langle\mathbf{e}_2,\mathbf{e}_3(\overline{\mathbf{e}_5}\mathbf{e}_6)\rangle=\langle\mathbf{e}_2,\mathbf{e}_3(-\mathbf{e}_5\mathbf{e}_6)\rangle=\langle\mathbf{e}_2,\mathbf{e}_3(-\mathbf{e}_1)\rangle=\langle\mathbf{e}_2,\mathbf{e}_2\rangle=1,$$
$$\Omega(\mathbf{e}_2,\mathbf{e}_3,\mathbf{e}_5,\mathbf{e}_7)=\langle\mathbf{e}_2,\mathbf{e}_3(\overline{\mathbf{e}_5}\mathbf{e}_7)\rangle=\langle\mathbf{e}_2,\mathbf{e}_3(-\mathbf{e}_5\mathbf{e}_7)\rangle=\langle\mathbf{e}_2,\mathbf{e}_3\mathbf{e}_2\rangle=\langle\mathbf{e}_2,\mathbf{e}_1\rangle=0,$$
$$\Omega(\mathbf{e}_2,\mathbf{e}_3,\mathbf{e}_5,\mathbf{e}_8)=\langle\mathbf{e}_2,\mathbf{e}_3(\overline{\mathbf{e}_5}\mathbf{e}_8)\rangle=\langle\mathbf{e}_2,\mathbf{e}_3(-\mathbf{e}_5\mathbf{e}_8)\rangle=\langle\mathbf{e}_2,\mathbf{e}_3(-\mathbf{e}_5)\rangle=\langle\mathbf{e}_2,\mathbf{e}_4\rangle=0,$$
$$\Omega(\mathbf{e}_2,\mathbf{e}_3,\mathbf{e}_6,\mathbf{e}_7)=\langle\mathbf{e}_2,\mathbf{e}_3(\overline{\mathbf{e}_6}\mathbf{e}_7)\rangle=\langle\mathbf{e}_2,\mathbf{e}_3(-\mathbf{e}_6\mathbf{e}_7)\rangle=\langle\mathbf{e}_2,\mathbf{e}_3\mathbf{e}_3\rangle=\langle\mathbf{e}_2,-1\rangle=0,$$
$$\Omega(\mathbf{e}_2,\mathbf{e}_3,\mathbf{e}_6,\mathbf{e}_8)=\langle\mathbf{e}_2,\mathbf{e}_3(\overline{\mathbf{e}_6}\mathbf{e}_8)\rangle=\langle\mathbf{e}_2,\mathbf{e}_3(-\mathbf{e}_6\mathbf{e}_8)\rangle=\langle\mathbf{e}_2,\mathbf{e}_3(-\mathbf{e}_6)\rangle=\langle\mathbf{e}_2,\mathbf{e}_7\rangle=0,$$
$$\Omega(\mathbf{e}_2,\mathbf{e}_3,\mathbf{e}_7,\mathbf{e}_8)=\langle\mathbf{e}_2,\mathbf{e}_3(\overline{\mathbf{e}_7}\mathbf{e}_8)\rangle=\langle\mathbf{e}_2,\mathbf{e}_3(-\mathbf{e}_7\mathbf{e}_8)\rangle=\langle\mathbf{e}_2,\mathbf{e}_3(-\mathbf{e}_7)\rangle=\langle\mathbf{e}_2,-\mathbf{e}_6\rangle=0,$$
$$\Omega(\mathbf{e}_2,\mathbf{e}_4,\mathbf{e}_5,\mathbf{e}_6)=\langle\mathbf{e}_2,\mathbf{e}_4(\overline{\mathbf{e}_5}\mathbf{e}_6)\rangle=\langle\mathbf{e}_2,\mathbf{e}_4(-\mathbf{e}_5\mathbf{e}_6)\rangle=\langle\mathbf{e}_2,\mathbf{e}_4(-\mathbf{e}_1)\rangle=\langle\mathbf{e}_2,-\mathbf{e}_7\rangle=0,$$
$$\Omega(\mathbf{e}_2,\mathbf{e}_4,\mathbf{e}_5,\mathbf{e}_7)=\langle\mathbf{e}_2,\mathbf{e}_4(\overline{\mathbf{e}_5}\mathbf{e}_7)\rangle=\langle\mathbf{e}_2,\mathbf{e}_4(-\mathbf{e}_5\mathbf{e}_7)\rangle=\langle\mathbf{e}_2,\mathbf{e}_4\mathbf{e}_2\rangle=\langle\mathbf{e}_2,\mathbf{e}_6\rangle=0,$$
$$\Omega(\mathbf{e}_2,\mathbf{e}_4,\mathbf{e}_5,\mathbf{e}_8)=\langle\mathbf{e}_2,\mathbf{e}_4(\overline{\mathbf{e}_5}\mathbf{e}_8)\rangle=\langle\mathbf{e}_2,\mathbf{e}_4(-\mathbf{e}_5\mathbf{e}_8)\rangle=\langle\mathbf{e}_2,\mathbf{e}_4(-\mathbf{e}_5)\rangle=\langle\mathbf{e}_2,-\mathbf{e}_3\rangle=0,$$
$$\Omega(\mathbf{e}_2,\mathbf{e}_4,\mathbf{e}_6,\mathbf{e}_7)=\langle\mathbf{e}_2,\mathbf{e}_4(\overline{\mathbf{e}_6}\mathbf{e}_7)\rangle=\langle\mathbf{e}_2,\mathbf{e}_4(-\mathbf{e}_6\mathbf{e}_7)\rangle=\langle\mathbf{e}_2,\mathbf{e}_4\mathbf{e}_3\rangle=\langle\mathbf{e}_2,-\mathbf{e}_5\rangle=0,$$
$$\Omega(\mathbf{e}_2,\mathbf{e}_4,\mathbf{e}_6,\mathbf{e}_8)=\langle\mathbf{e}_2,\mathbf{e}_4(\overline{\mathbf{e}_6}\mathbf{e}_8)\rangle=\langle\mathbf{e}_2,\mathbf{e}_4(-\mathbf{e}_6\mathbf{e}_8)\rangle=\langle\mathbf{e}_2,\mathbf{e}_4(-\mathbf{e}_6)\rangle=\langle\mathbf{e}_2,\mathbf{e}_2\rangle=1,$$
$$\Omega(\mathbf{e}_2,\mathbf{e}_4,\mathbf{e}_7,\mathbf{e}_8)=\langle\mathbf{e}_2,\mathbf{e}_4(\overline{\mathbf{e}_7}\mathbf{e}_8)\rangle=\langle\mathbf{e}_2,\mathbf{e}_4(-\mathbf{e}_7\mathbf{e}_8)\rangle=\langle\mathbf{e}_2,\mathbf{e}_4(-\mathbf{e}_7)\rangle=\langle\mathbf{e}_2,\mathbf{e}_1\rangle=0,$$

$$
\begin{aligned}
\Omega(\mathbf{e}_2,\mathbf{e}_5,\mathbf{e}_6,\mathbf{e}_7) &= \langle \mathbf{e}_2, \mathbf{e}_5(\overline{\mathbf{e}_6}\mathbf{e}_7)\rangle = \langle \mathbf{e}_2, \mathbf{e}_5(-\mathbf{e}_6\mathbf{e}_7)\rangle = \langle \mathbf{e}_2, \mathbf{e}_5\mathbf{e}_3\rangle = \langle \mathbf{e}_2, \mathbf{e}_4\rangle = 0,\\
\Omega(\mathbf{e}_2,\mathbf{e}_5,\mathbf{e}_6,\mathbf{e}_8) &= \langle \mathbf{e}_2, \mathbf{e}_5(\overline{\mathbf{e}_6}\mathbf{e}_8)\rangle = \langle \mathbf{e}_2, \mathbf{e}_5(-\mathbf{e}_6\mathbf{e}_8)\rangle = \langle \mathbf{e}_2, \mathbf{e}_5(-\mathbf{e}_6)\rangle = \langle \mathbf{e}_2, -\mathbf{e}_1\rangle = 0,\\
\Omega(\mathbf{e}_2,\mathbf{e}_5,\mathbf{e}_7,\mathbf{e}_8) &= \langle \mathbf{e}_2, \mathbf{e}_5(\overline{\mathbf{e}_7}\mathbf{e}_8)\rangle = \langle \mathbf{e}_2, \mathbf{e}_5(-\mathbf{e}_7\mathbf{e}_8)\rangle = \langle \mathbf{e}_2, \mathbf{e}_5(-\mathbf{e}_7)\rangle = \langle \mathbf{e}_2, \mathbf{e}_2\rangle = 1,\\
\Omega(\mathbf{e}_2,\mathbf{e}_6,\mathbf{e}_7,\mathbf{e}_8) &= \langle \mathbf{e}_2, \mathbf{e}_6(\overline{\mathbf{e}_7}\mathbf{e}_8)\rangle = \langle \mathbf{e}_2, \mathbf{e}_6(-\mathbf{e}_7\mathbf{e}_8)\rangle = \langle \mathbf{e}_2, \mathbf{e}_6(-\mathbf{e}_7)\rangle = \langle \mathbf{e}_2, \mathbf{e}_3\rangle = 0,\\
\Omega(\mathbf{e}_3,\mathbf{e}_4,\mathbf{e}_5,\mathbf{e}_6) &= \langle \mathbf{e}_3, \mathbf{e}_4(\overline{\mathbf{e}_5}\mathbf{e}_6)\rangle = \langle \mathbf{e}_3, \mathbf{e}_4(-\mathbf{e}_5\mathbf{e}_6)\rangle = \langle \mathbf{e}_3, \mathbf{e}_4(-\mathbf{e}_1)\rangle = \langle \mathbf{e}_3, -\mathbf{e}_7\rangle = 0,\\
\Omega(\mathbf{e}_3,\mathbf{e}_4,\mathbf{e}_5,\mathbf{e}_7) &= \langle \mathbf{e}_3, \mathbf{e}_4(\overline{\mathbf{e}_5}\mathbf{e}_7)\rangle = \langle \mathbf{e}_3, \mathbf{e}_4(-\mathbf{e}_5\mathbf{e}_7)\rangle = \langle \mathbf{e}_3, \mathbf{e}_4\mathbf{e}_2\rangle = \langle \mathbf{e}_3, \mathbf{e}_6\rangle = 0,\\
\Omega(\mathbf{e}_3,\mathbf{e}_4,\mathbf{e}_5,\mathbf{e}_8) &= \langle \mathbf{e}_3, \mathbf{e}_4(\overline{\mathbf{e}_5}\mathbf{e}_8)\rangle = \langle \mathbf{e}_3, \mathbf{e}_4(-\mathbf{e}_5\mathbf{e}_8)\rangle = \langle \mathbf{e}_3, \mathbf{e}_4(-\mathbf{e}_5)\rangle = \langle \mathbf{e}_3, -\mathbf{e}_3\rangle = -1,\\
\Omega(\mathbf{e}_3,\mathbf{e}_4,\mathbf{e}_6,\mathbf{e}_7) &= \langle \mathbf{e}_3, \mathbf{e}_4(\overline{\mathbf{e}_6}\mathbf{e}_7)\rangle = \langle \mathbf{e}_3, \mathbf{e}_4(-\mathbf{e}_6\mathbf{e}_7)\rangle = \langle \mathbf{e}_3, \mathbf{e}_4\mathbf{e}_3\rangle = \langle \mathbf{e}_3, -\mathbf{e}_5\rangle = 0,\\
\Omega(\mathbf{e}_3,\mathbf{e}_4,\mathbf{e}_6,\mathbf{e}_8) &= \langle \mathbf{e}_3, \mathbf{e}_4(\overline{\mathbf{e}_6}\mathbf{e}_8)\rangle = \langle \mathbf{e}_3, \mathbf{e}_4(-\mathbf{e}_6\mathbf{e}_8)\rangle = \langle \mathbf{e}_3, \mathbf{e}_4(-\mathbf{e}_6)\rangle = \langle \mathbf{e}_3, \mathbf{e}_2\rangle = 0,\\
\Omega(\mathbf{e}_3,\mathbf{e}_4,\mathbf{e}_7,\mathbf{e}_8) &= \langle \mathbf{e}_3, \mathbf{e}_4(\overline{\mathbf{e}_7}\mathbf{e}_8)\rangle = \langle \mathbf{e}_3, \mathbf{e}_4(-\mathbf{e}_7\mathbf{e}_8)\rangle = \langle \mathbf{e}_3, \mathbf{e}_4(-\mathbf{e}_7)\rangle = \langle \mathbf{e}_3, \mathbf{e}_1\rangle = 0,\\
\Omega(\mathbf{e}_3,\mathbf{e}_5,\mathbf{e}_6,\mathbf{e}_7) &= \langle \mathbf{e}_3, \mathbf{e}_5(\overline{\mathbf{e}_6}\mathbf{e}_7)\rangle = \langle \mathbf{e}_3, \mathbf{e}_5(-\mathbf{e}_6\mathbf{e}_7)\rangle = \langle \mathbf{e}_3, \mathbf{e}_5\mathbf{e}_3\rangle = \langle \mathbf{e}_3, \mathbf{e}_4\rangle = 0,\\
\Omega(\mathbf{e}_3,\mathbf{e}_5,\mathbf{e}_6,\mathbf{e}_8) &= \langle \mathbf{e}_3, \mathbf{e}_5(\overline{\mathbf{e}_6}\mathbf{e}_8)\rangle = \langle \mathbf{e}_3, \mathbf{e}_5(-\mathbf{e}_6\mathbf{e}_8)\rangle = \langle \mathbf{e}_3, \mathbf{e}_5(-\mathbf{e}_6)\rangle = \langle \mathbf{e}_3, -\mathbf{e}_1\rangle = 0,\\
\Omega(\mathbf{e}_3,\mathbf{e}_5,\mathbf{e}_7,\mathbf{e}_8) &= \langle \mathbf{e}_3, \mathbf{e}_5(\overline{\mathbf{e}_7}\mathbf{e}_8)\rangle = \langle \mathbf{e}_3, \mathbf{e}_5(-\mathbf{e}_7\mathbf{e}_8)\rangle = \langle \mathbf{e}_3, \mathbf{e}_5(-\mathbf{e}_7)\rangle = \langle \mathbf{e}_3, \mathbf{e}_2\rangle = 0,\\
\Omega(\mathbf{e}_3,\mathbf{e}_6,\mathbf{e}_7,\mathbf{e}_8) &= \langle \mathbf{e}_3, \mathbf{e}_6(\overline{\mathbf{e}_7}\mathbf{e}_8)\rangle = \langle \mathbf{e}_3, \mathbf{e}_6(-\mathbf{e}_7\mathbf{e}_8)\rangle = \langle \mathbf{e}_3, \mathbf{e}_6(-\mathbf{e}_7)\rangle = \langle \mathbf{e}_3, \mathbf{e}_3\rangle = 1,\\
\Omega(\mathbf{e}_4,\mathbf{e}_5,\mathbf{e}_6,\mathbf{e}_7) &= \langle \mathbf{e}_4, \mathbf{e}_5(\overline{\mathbf{e}_6}\mathbf{e}_7)\rangle = \langle \mathbf{e}_4, \mathbf{e}_5(-\mathbf{e}_6\mathbf{e}_7)\rangle = \langle \mathbf{e}_4, \mathbf{e}_5\mathbf{e}_3\rangle = \langle \mathbf{e}_4, \mathbf{e}_4\rangle = 1,\\
\Omega(\mathbf{e}_4,\mathbf{e}_5,\mathbf{e}_6,\mathbf{e}_8) &= \langle \mathbf{e}_4, \mathbf{e}_5(\overline{\mathbf{e}_6}\mathbf{e}_8)\rangle = \langle \mathbf{e}_4, \mathbf{e}_5(-\mathbf{e}_6\mathbf{e}_8)\rangle = \langle \mathbf{e}_4, \mathbf{e}_5(-\mathbf{e}_6)\rangle = \langle \mathbf{e}_4, -\mathbf{e}_1\rangle = 0,\\
\Omega(\mathbf{e}_4,\mathbf{e}_5,\mathbf{e}_7,\mathbf{e}_8) &= \langle \mathbf{e}_4, \mathbf{e}_5(\overline{\mathbf{e}_7}\mathbf{e}_8)\rangle = \langle \mathbf{e}_4, \mathbf{e}_5(-\mathbf{e}_7\mathbf{e}_8)\rangle = \langle \mathbf{e}_4, \mathbf{e}_5(-\mathbf{e}_7)\rangle = \langle \mathbf{e}_4, \mathbf{e}_2\rangle = 0,\\
\Omega(\mathbf{e}_4,\mathbf{e}_6,\mathbf{e}_7,\mathbf{e}_8) &= \langle \mathbf{e}_4, \mathbf{e}_6(\overline{\mathbf{e}_7}\mathbf{e}_8)\rangle = \langle \mathbf{e}_4, \mathbf{e}_6(-\mathbf{e}_7\mathbf{e}_8)\rangle = \langle \mathbf{e}_4, \mathbf{e}_6(-\mathbf{e}_7)\rangle = \langle \mathbf{e}_4, \mathbf{e}_3\rangle = 0,\\
\Omega(\mathbf{e}_5,\mathbf{e}_6,\mathbf{e}_7,\mathbf{e}_8) &= \langle \mathbf{e}_5, \mathbf{e}_6(\overline{\mathbf{e}_7}\mathbf{e}_8)\rangle = \langle \mathbf{e}_5, \mathbf{e}_6(-\mathbf{e}_7\mathbf{e}_8)\rangle = \langle \mathbf{e}_5, \mathbf{e}_6(-\mathbf{e}_7)\rangle = \langle \mathbf{e}_5, \mathbf{e}_3\rangle = 0.
\end{aligned}
$$

Hence $\mathbf{\Omega}_{\alpha\beta\gamma\delta}$ is non-zero for the following set of indices

$$
\begin{aligned}
&\{(1238),(1245),(2167),(1346),(1357),(1478),(5168),\\
&(3247),(2356),(2468),(2578),(4358),(3678),(4567)\}. \qquad \text{(A.3)}
\end{aligned}
$$

B. THE SCHOUTEN-NIJENHUIS BRACKET

In the work by [45], the Schouten-Nijenhuis bracket is used in the construction of manifolds with $Spin(7)$ holonomy. This expression is used to obtain the 56 linear equations given in Appendix C.

$$
\begin{aligned}
&\sum_{\alpha\beta\gamma\delta} \Omega_{\alpha\beta\gamma\delta}[V_\alpha, V_\gamma] \wedge V_\beta \wedge V_\delta = \\
&\quad [V_1,V_3]\wedge V_2\wedge V_8 - [V_1,V_8]\wedge V_2\wedge V_3 - [V_2,V_3]\wedge V_1\wedge V_8 + [V_2,V_8]\wedge V_1\wedge V_3 \\
&- [V_1,V_2]\wedge V_3\wedge V_8 - [V_3,V_8]\wedge V_1\wedge V_2 + [V_5,V_6]\wedge V_1\wedge V_8 - [V_5,V_8]\wedge V_1\wedge V_6 \\
&- [V_1,V_6]\wedge V_5\wedge V_8 + [V_1,V_8]\wedge V_5\wedge V_6 + [V_1,V_5]\wedge V_6\wedge V_8 - [V_6,V_8]\wedge V_5\wedge V_1 \\
&- [V_4,V_6]\wedge V_2\wedge V_8 - [V_6,V_8]\wedge V_2\wedge V_4 - [V_2,V_4]\wedge V_6\wedge V_8 + [V_2,V_8]\wedge V_6\wedge V_4 \\
&+ [V_2,V_6]\wedge V_4\wedge V_8 - [V_4,V_8]\wedge V_6\wedge V_2 + [V_4,V_5]\wedge V_3\wedge V_8 - [V_4,V_8]\wedge V_3\wedge V_5 \\
&- [V_3,V_5]\wedge V_4\wedge V_8 + [V_3,V_8]\wedge V_4\wedge V_5 + [V_3,V_4]\wedge V_5\wedge V_8 - [V_5,V_8]\wedge V_4\wedge V_3 \\
&- [V_1,V_4]\wedge V_7\wedge V_8 - [V_4,V_8]\wedge V_7\wedge V_1 + [V_1,V_7]\wedge V_4\wedge V_8 + [V_7,V_8]\wedge V_4\wedge V_1 \\
&- [V_4,V_7]\wedge V_1\wedge V_8 - [V_1,V_8]\wedge V_4\wedge V_7 - [V_3,V_6]\wedge V_7\wedge V_8 - [V_6,V_8]\wedge V_7\wedge V_3 \\
&+ [V_3,V_7]\wedge V_6\wedge V_8 + [V_7,V_8]\wedge V_6\wedge V_3 - [V_6,V_7]\wedge V_3\wedge V_8 - [V_3,V_8]\wedge V_6\wedge V_7 \\
&- [V_2,V_5]\wedge V_7\wedge V_8 - [V_5,V_8]\wedge V_7\wedge V_2 + [V_2,V_7]\wedge V_5\wedge V_8 + [V_7,V_8]\wedge V_5\wedge V_2 \\
&- [V_5,V_7]\wedge V_2\wedge V_8 - [V_2,V_8]\wedge V_5\wedge V_7 + [V_4,V_6]\wedge V_5\wedge V_7 - [V_4,V_7]\wedge V_5\wedge V_6 \\
&- [V_5,V_6]\wedge V_4\wedge V_7 + [V_5,V_7]\wedge V_4\wedge V_6 - [V_4,V_5]\wedge V_6\wedge V_7 - [V_6,V_7]\wedge V_4\wedge V_5 \\
&- [V_2,V_7]\wedge V_4\wedge V_3 + [V_3,V_7]\wedge V_4\wedge V_2 + [V_2,V_4]\wedge V_7\wedge V_3 - [V_3,V_4]\wedge V_7\wedge V_2 \\
&+ [V_4,V_7]\wedge V_2\wedge V_3 - [V_2,V_3]\wedge V_7\wedge V_4 - [V_3,V_7]\wedge V_5\wedge V_1 - [V_1,V_3]\wedge V_5\wedge V_7 \\
&+ [V_5,V_7]\wedge V_3\wedge V_1 + [V_1,V_5]\wedge V_3\wedge V_7 + [V_3,V_5]\wedge V_7\wedge V_1 - [V_1,V_7]\wedge V_3\wedge V_5 \\
&+ [V_6,V_7]\wedge V_1\wedge V_2 + [V_2,V_6]\wedge V_1\wedge V_7 - [V_1,V_7]\wedge V_6\wedge V_2 + [V_1,V_2]\wedge V_6\wedge V_7 \\
&+ [V_1,V_6]\wedge V_7\wedge V_2 + [V_2,V_7]\wedge V_6\wedge V_1 + [V_2,V_3]\wedge V_6\wedge V_5 - [V_2,V_5]\wedge V_6\wedge V_3 \\
&+ [V_3,V_6]\wedge V_2\wedge V_5 - [V_5,V_6]\wedge V_2\wedge V_3 - [V_2,V_6]\wedge V_3\wedge V_5 - [V_3,V_5]\wedge V_2\wedge V_6 \\
&- [V_1,V_5]\wedge V_2\wedge V_4 + [V_4,V_5]\wedge V_2\wedge V_1 + [V_1,V_2]\wedge V_5\wedge V_4 + [V_2,V_4]\wedge V_5\wedge V_1 \\
&+ [V_2,V_5]\wedge V_1\wedge V_4 - [V_2,V_4]\wedge V_5\wedge V_1 + [V_1,V_4]\wedge V_3\wedge V_6 - [V_1,V_6]\wedge V_3\wedge V_4 \\
&- [V_3,V_4]\wedge V_1\wedge V_6 + [V_3,V_6]\wedge V_1\wedge V_4 - [V_1,V_3]\wedge V_4\wedge V_6 - [V_4,V_6]\wedge V_1\wedge V_3.
\end{aligned}
\tag{B.1}
$$

C. THE SET OF 56 LINEAR EQUATIONS

When we write the commutator V_α and V_β given in the equation (3.30) as

$$\left[V_\alpha, V_\beta\right] = \sum_\gamma c_{\alpha\beta\gamma} V_\gamma, \tag{C.1}$$

then the Schouten-Nijenhuis bracket can be written as

$$\sum_{\alpha\beta\gamma\delta} \Omega_{\alpha\beta\gamma\delta}[V_\alpha, V_\gamma] \wedge V_\beta \wedge V_\delta = \sum_{\alpha\beta\gamma} C_{\alpha\beta\gamma} V_\alpha \wedge V_\beta \wedge V_\gamma, \tag{C.2}$$

where $C_{\alpha\beta\gamma}$ are collection of the coefficients of $c_{\alpha\beta\gamma}$ $(\alpha, \beta, \gamma = 1, 2, ..., 8)$. Thus we obtain the set of 56 linear equations which should be zero in order to satisfy the 2-vector condition.

$$\begin{aligned}
C_{123} &= -c_{181} - c_{282} - c_{383} - c_{453} + c_{462} + c_{471} - c_{561} + c_{572} + c_{673} = 0,\\
C_{124} &= -c_{151} - c_{252} - c_{362} - c_{371} - c_{384} - c_{454} + c_{674} - c_{681} + c_{782} = 0,\\
C_{125} &= c_{141} + c_{242} + c_{361} - c_{372} - c_{385} - c_{455} + c_{675} - c_{682} - c_{781} = 0,\\
C_{126} &= c_{171} + c_{272} + c_{342} - c_{351} - c_{386} - c_{456} + c_{481} + c_{582} + c_{676} = 0,\\
C_{127} &= -c_{161} - c_{262} + c_{341} + c_{352} - c_{387} - c_{457} - c_{482} + c_{581} + c_{677} = 0,\\
C_{128} &= c_{131} + c_{232} - c_{388} - c_{458} - c_{461} + c_{472} - c_{562} - c_{571} + c_{678} = 0,\\
C_{134} &= -c_{161} - c_{253} + c_{271} + c_{284} - c_{363} - c_{464} - c_{574} + c_{581} + c_{783} = 0,\\
C_{135} &= -c_{171} + c_{243} - c_{261} + c_{285} - c_{373} - c_{465} - c_{481} - c_{575} - c_{683} = 0,\\
C_{136} &= c_{141} + c_{251} + c_{273} + c_{286} + c_{343} - c_{466} - c_{576} + c_{583} - c_{781} = 0,\\
C_{137} &= c_{151} - c_{241} - c_{263} + c_{287} + c_{353} - c_{467} - c_{483} - c_{577} + c_{681} = 0,\\
C_{138} &= -c_{121} + c_{233} + c_{288} + c_{451} - c_{468} + c_{473} - c_{563} - c_{578} - c_{671} = 0,\\
C_{145} &= -c_{121} + c_{244} + c_{255} + c_{365} - c_{374} + c_{381} - c_{671} - c_{684} - c_{785} = 0,\\
C_{146} &= -c_{131} + c_{256} + c_{274} - c_{281} + c_{344} + c_{366} + c_{571} + c_{584} - c_{786} = 0,\\
C_{147} &= -c_{181} + c_{231} + c_{257} - c_{264} + c_{354} + c_{367} - c_{484} - c_{561} - c_{787} = 0,\\
C_{148} &= c_{171} + c_{234} + c_{258} + c_{261} - c_{351} + c_{368} + c_{474} - c_{564} - c_{788} = 0,\\
C_{156} &= c_{181} - c_{231} - c_{246} + c_{275} + c_{345} + c_{376} - c_{471} + c_{585} + c_{686} = 0,\\
C_{157} &= -c_{131} - c_{247} - c_{265} - c_{281} + c_{355} + c_{377} + c_{461} - c_{485} + c_{687} = 0,\\
C_{158} &= -c_{161} + c_{235} - c_{248} + c_{271} + c_{341} + c_{378} + c_{475} - c_{565} + c_{688} = 0,\\
C_{167} &= c_{121} - c_{266} - c_{277} - c_{347} + c_{356} - c_{381} - c_{451} - c_{486} - c_{587} = 0,\\
C_{168} &= c_{151} + c_{236} - c_{241} - c_{278} - c_{348} + c_{371} + c_{476} - c_{566} - c_{588} = 0,\\
C_{178} &= -c_{141} + c_{237} - c_{251} + c_{268} - c_{358} - c_{361} + c_{477} + c_{488} - c_{567} = 0,\\
C_{234} &= c_{153} - c_{162} - c_{184} + c_{272} + c_{373} + c_{474} - c_{564} + c_{582} + c_{683} = 0,\\
C_{235} &= -c_{143} - c_{172} - c_{185} - c_{262} - c_{363} + c_{475} - c_{482} - c_{565} + c_{783} = 0,\\
C_{236} &= c_{142} - c_{173} - c_{186} + c_{252} + c_{353} + c_{476} - c_{483} - c_{566} - c_{782} = 0,\\
C_{237} &= c_{152} + c_{163} - c_{187} - c_{242} - c_{343} + c_{477} - c_{567} - c_{583} + c_{682} = 0,\\
C_{238} &= -c_{122} - c_{133} - c_{188} + c_{452} + c_{463} + c_{478} - c_{568} + c_{573} - c_{672} = 0,
\end{aligned}$$

$$
\begin{aligned}
C_{245} &= -c_{122}-c_{144}-c_{155}-c_{364}-c_{375}+c_{382}-c_{672}-c_{685}+c_{784}=0,\\
C_{246} &= -c_{132}-c_{156}-c_{174}-c_{282}+c_{354}-c_{376}-c_{484}+c_{572}-c_{686}=0,\\
C_{247} &= -c_{157}+c_{164}-c_{182}+c_{232}-c_{344}-c_{377}-c_{562}-c_{584}-c_{687}=0,\\
C_{248} &= -c_{134}-c_{158}+c_{172}+c_{262}-c_{352}-c_{378}+c_{464}+c_{574}-c_{688}=0,\\
C_{256} &= c_{146}-c_{175}+c_{182}-c_{232}+c_{355}+c_{366}-c_{472}-c_{485}-c_{786}=0,\\
C_{257} &= -c_{132}+c_{147}+c_{165}-c_{282}-c_{345}+c_{367}+c_{462}-c_{585}-c_{787}=0,\\
C_{258} &= -c_{135}+c_{148}-c_{162}+c_{272}+c_{342}+c_{368}+c_{465}+c_{575}-c_{788}=0,\\
C_{267} &= c_{122}+c_{166}+c_{177}-c_{346}-c_{357}-c_{382}-c_{452}+c_{487}-c_{586}=0,\\
C_{268} &= -c_{136}+c_{152}+c_{178}-c_{242}-c_{358}+c_{372}+c_{466}+c_{488}+c_{576}=0,\\
C_{278} &= -c_{137}-c_{142}-c_{168}-c_{252}+c_{348}-c_{362}+c_{467}+c_{577}+c_{588}=0,\\
C_{345} &= -c_{123}-c_{165}+c_{174}+c_{264}+c_{275}+c_{383}+c_{484}+c_{585}-c_{673}=0,\\
C_{346} &= -c_{133}-c_{144}-c_{166}-c_{254}+c_{276}-c_{283}+c_{573}+c_{586}+c_{784}=0,\\
C_{347} &= -c_{154}-c_{167}-c_{183}+c_{233}+c_{244}+c_{277}-c_{563}+c_{587}-c_{684}=0,\\
C_{348} &= c_{124}-c_{168}+c_{173}+c_{263}+c_{278}-c_{353}-c_{454}+c_{588}+c_{674}=0,\\
C_{356} &= -c_{145}-c_{176}+c_{183}-c_{233}-c_{255}-c_{266}-c_{473}-c_{486}+c_{785}=0,\\
C_{357} &= -c_{133}-c_{155}-c_{177}+c_{245}-c_{267}-c_{283}+c_{463}-c_{487}-c_{685}=0,\\
C_{358} &= c_{125}-c_{163}-c_{178}-c_{268}+c_{273}+c_{343}-c_{455}-c_{488}+c_{675}=0,\\
C_{367} &= c_{123}+c_{147}-c_{156}+c_{246}+c_{257}-c_{383}-c_{453}-c_{686}-c_{787}=0,\\
C_{368} &= c_{126}+c_{148}+c_{153}-c_{243}+c_{258}+c_{373}-c_{456}+c_{676}-c_{788}=0,\\
C_{378} &= c_{127}-c_{143}+c_{158}-c_{248}-c_{253}-c_{363}-c_{457}+c_{677}+c_{688}=0,\\
C_{456} &= -c_{126}+c_{135}+c_{184}-c_{234}+c_{285}+c_{386}-c_{474}-c_{575}-c_{676}=0,\\
C_{457} &= -c_{127}-c_{134}+c_{185}-c_{235}-c_{284}+c_{387}+c_{464}+c_{565}-c_{677}=0,\\
C_{458} &= -c_{128}-c_{164}-c_{175}-c_{265}+c_{274}+c_{344}+c_{355}+c_{388}-c_{678}=0,\\
C_{467} &= c_{124}-c_{137}+c_{186}-c_{236}-c_{287}-c_{384}-c_{454}+c_{566}+c_{577}=0,\\
C_{468} &= -c_{138}+c_{154}-c_{176}-c_{244}-c_{266}-c_{288}+c_{356}+c_{374}+c_{578}=0,\\
C_{478} &= -c_{144}-c_{177}-c_{188}+c_{238}-c_{254}-c_{267}+c_{357}-c_{364}-c_{568}=0,\\
C_{567} &= c_{125}+c_{136}+c_{187}-c_{237}+c_{286}-c_{385}-c_{455}-c_{466}-c_{477}=0,\\
C_{568} &= c_{155}+c_{166}+c_{188}-c_{238}-c_{245}-c_{276}-c_{346}+c_{375}-c_{478}=0,\\
C_{578} &= -c_{138}-c_{145}+c_{167}-c_{255}-c_{277}-c_{288}-c_{347}-c_{365}+c_{468}=0,\\
C_{678} &= c_{128}-c_{146}-c_{157}+c_{247}-c_{256}-c_{366}-c_{377}-c_{388}-c_{458}=0.
\end{aligned}
\tag{C.3}
$$

CIRCULUM VITAE

Selman UĞUZ was born in Polatlı in 1978. He graduated from the Polatlı High School in 1994. He obtained his BSc. degree in 1998 from Dokuz Eylül University, Department of Mathematics Education. He won YLE examination and came for his graduate education to Istanbul Technical University. He obtained M.Sc. degree in 2002 from Istanbul Technical University, Department of Mathematics Engineering. He started PhD education at the same department in 2002. He has been working in Istanbul Technical University, Institute of Science and Technology as a research assistant since 2000.