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Dedicated to Ratip BERKER on the occasion of his seventy - fifth birthday

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ORD. PROF. DR. RATİP BERKER
and
MECHANICS

The colleagues, former students and the editors of this journal
take honour
in dedicating this special issue of the
BULLETIN OF THE TECHNICAL UNIVERSITY OF ISTANBUL
to
RATİP BERKER
Eminent Scientist, Teacher and Administrator

ON THE OCCASION OF
HIS SEVENTY - FIFTH BIRTHDAY

Honorary Editor
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CONSERVATION LAWS FOR ONE - DIMENSIONAL FINITE ELASTODYNAMICS

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Abstract : The field equations of one - dimensional nonlinear elastodynamics are expressed in terms of appropriate exterior differential forms. By explicitly contructing a closed ideal of the ring generated by these differential forms it is shown that the conservation law structure of the field equations is determined by two scalar functions satisfying a number of linear partial differential equations. The solutions of this set of equations are found and an exhaustive list of the conservation laws are given. The linear case is also studied.

1. INTRODUCTION

The identification of the conservation law structure admitted by a given set of field equations, that is the entire set of equations in the form of conservation law which are satisfied identically whenever the original field equations are satisfied becomes quite important, particularly due to the advent of some very effective methods to construct exact solutions of some nonlinear differential equations. It has been shown that in certain nonlinear systems the existence of infinitely many conservation laws usually implies that the system is amenable to the use of some methods such as inverse spectral or Bäcklund transformations. Apart from its theoretical interest nontrivial conservation laws (those which are not direct consequences of natural balance laws) in continuum mechanics provide quite practical tools for asymptotic analyses of the behaviour of the field in the neighbourhood of singularities such as cracks. One of the earliest specimen in this context is the J - integral in linearly elastic bodies [1].

A general systematic approach to deduce the conservation laws by employing Cartan's exterior differential forms was proposed by Wahlquist and Estabrook [2]. Although this problem can also be attacked (and has already been attacked by many authors) by the theory of Lie groups the inherent simplicity and elegancy of the calculus of exterior forms makes it a far better tool for this purpose, at least in this author's opinion.

The present work aims to extend the method of Wahlquist and Estabrook to the one - dimensional finite elastodynamics. The field equations are written as 2 - forms in an eight dimensional manifold. Then the conservation laws are derived from the properties of the ring of ideals and the well - known Frobenius' theorem. It is observed that conservation laws constitute a finite set which is slightly enlarged in case of linear elastodynamics. The static version of this problem has been treated by Knowles and Sternherg [3] and Olver [4] by using Lie formalism and Noether's theorem. We should also mentioned that conservation laws in linear elastodynamics have been derived by Fletcher [5] by use of Lie's theory.

2. FIELD EQUATIONS

We consider one - dimensional motion of a homogeneous hyperelastic solid which is free from initial stresses and body forces. Then the field equations in material coordinates can be written as [6][1]

$$A_{kl}(\mathrm{p})\,\frac{\partial^2 u_l}{\partial X^2}=\frac{\partial^2 u_k}{\partial t^2}\,,\qquad (k,l=1,2,3) \tag{2.1}$$

where $X_1=X$ is the material coordinate along the motion, $\mathbf{u}=\mathbf{u}(X,t)$ is the displacement vector and

$$\mathrm{p}=\frac{\partial \mathrm{u}}{\partial X} \tag{2.2}$$

is the displacement gradient. The symmetric matrix $\mathbf{A}$ is defined by

$$A_{kl}(\mathrm{p})=\frac{1}{\rho_0}\,\frac{\partial^2\Sigma}{\partial p_k\partial p_l} \tag{2.3}$$

where $\Sigma=\Sigma(\mathrm{p})$ is the stress potential and ρ_0 is the density of the medium in reference configuration. The first Piola - Kirchhoff stress tensor is

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[1] Throughout the text we adopt the summation convention for repeated indices.

$$T_k = T_{1k} = \frac{\partial \Sigma}{\partial p_k} \tag{2.4}$$

By introducing the velocity vector

$$\mathbf{v} = \frac{\partial \mathbf{u}}{\partial t} \tag{2.5}$$

we can replace the second order system (2.1) by the following first order system

$$A_{kl}(\mathbf{p}) \frac{\partial p_l}{\partial X} - \frac{\partial v_k}{\partial t} = 0, \qquad \frac{\partial v_k}{\partial X} - \frac{\partial p_k}{\partial t} = 0 \tag{2.6}$$

Let us now define in the eight dimensional manifold M whose coordinates are $\{\mathbf{p}, \mathbf{v}, X, t\}$ the following differential 2 - forms

$$\omega_k = dv_k \wedge dX + A_{kl}(\mathbf{p})\, dp_l \wedge dt = dv_k \wedge dX + \frac{1}{\rho_0} d\left(\frac{\partial \Sigma}{\partial p_k}\right) \wedge dt$$
$$\gamma_k = dv_k \wedge dt + dp_k \wedge dX \tag{2.7}$$

where $\wedge$ denotes the exterior product. Any differentiable solution of (2.6) will annual the forms (2.7). For $\mathbf{p} = \mathbf{p}(X, t)$, $\mathbf{v} = \mathbf{v}(X, t)$ we have indeed

$$\omega_k = \left(A_{kl} \frac{\partial p_l}{\partial X} - \frac{\partial v_k}{\partial t}\right) dXdt, \qquad \gamma_k = \left(\frac{\partial v_k}{\partial X} - \frac{\partial p_k}{\partial t}\right) dX \wedge dt$$

Hence the solution set $(\mathbf{p}, \mathbf{v})$ is the annihilator of the forms ω_k and γ_k at the point (X, t) of the physical space. Therefore to find a solution to (2.6) is reduced to determine a two - dimensional submanifold H of M on which $\omega_k|_H = \gamma_k|_H = 0$. Obviously the exterior derivatives of ω_k and γ_k vanish, i.e., $d\omega_k = 0$, $d\gamma_k = 0$. Hence these forms constitutes a *closed ideal* of the ring of differential forms since any local two - dimensional (in X, t) surface element (tangent subspace of M at a certain point) which annuls (ω_k, γ_k) also annuls their exterior derivatives. Then Cartan's theorem [7] insures that these elementary tangent subspaces can be patched together to form the global solution manifold.

The existence of conservation laws associated with equations (2.6) is strictly related to the existence of *exact* 2 - forms belonging to the ring generated by (ω_k, γ_k). We thus search for such forms in the following fashion

$$\Omega=\varphi_k\omega_k+\psi_k\gamma_k \tag{2.8}$$

where the coefficients φ_k and γ_k are function of $(\mathbf{p}, \mathbf{v}, X, t)$. Obviously on the solution submanifold H the form Ω also vanishes. If we can determine now φ_k and γ_k in such a way that Ω is closed, that is, $d\Omega=0$ then also $d\Omega|_H=0$. Therefore the global solution manifold is an annihilator of these forms Ω. On the other hand, since $d\Omega=0$ we know that Ω is *locally exact* [8], namely, there exist a 1 - form σ such locally

$$\Omega=d\sigma \tag{2.9}$$

On the solution manifold 1 - form σ is expressible as

$$\sigma=\Phi dt-\Psi dX \tag{2.10}$$

where Φ and Ψ are functions of $(\mathbf{p}, \mathbf{v}, X, t)$ and if $\mathbf{p}=\mathbf{p}(X, t)$, $\mathbf{v}=\mathbf{v}(X, t)$ we then have

$$\Omega=\left(\frac{\partial\Phi}{\partial X}+\frac{\partial\Psi}{\partial t}\right)dX\wedge dt$$

Since the solution submanifold annuls Ω, if $\mathbf{p}$ and $\mathbf{v}$ is a solution of (2.6) then the relation

$$\frac{\partial\Phi}{\partial X}+\frac{\partial\Psi}{\partial t}=0 \tag{2.11}$$

is satisfied. This is none other than the desired conservation law. Therefore the construction of conservation laws depends the determination of the functions φ_k and ψ_k such that $d\Omega=0$ and their numbers are determined by the number of such independent functions.

3. CONSTRUCTION OF CONSERVATION LAWS

Since $d\omega_k$ and $d\gamma_k$ vanish identically the exterior derivative of (2.8) is

$$d\Omega=d\varphi_k\wedge\omega_k+d\psi_k\wedge\gamma_k \tag{3.1}$$

φ_k, γ_k are 0 - forms. Hence their exterior derivatives are simply ordinary differentials and we obtain

$$d\varphi_k = \frac{\partial \varphi_k}{\partial p_l} dp_l + \frac{\partial \varphi_k}{\partial v_l} dv_l + \frac{\partial \varphi_k}{\partial X} dX + \frac{\partial \varphi_k}{\partial t} dt$$

and a similar expression for $d\psi_k$. Therefore employing (2.7) in (3.1) and collecting similar terms we obtain

$$\begin{aligned} d\Omega = & \left(\frac{\partial \varphi_k}{\partial p_l} - \frac{\partial v_l}{\partial \varphi_k}\right) dp_l \wedge dv_k \wedge dX + \frac{\partial \varphi_k}{\partial v_l} dv_l \wedge dv_k \wedge dX \\ & + \left(\frac{\partial \varphi_k}{\partial t} - \frac{\partial \psi_k}{\partial X}\right) dv_k \wedge dX \wedge dt + A_{kl} \frac{\partial \varphi_k}{\partial p_m} dp_m \wedge dp_l \wedge dt \\ & + (A_{kl} \frac{\partial \varphi_k}{\partial v_m} - \frac{\partial \psi_m}{\partial p_l}) dv_m \wedge dp_l \wedge dt - (A_{kl} \frac{\partial \varphi_k}{\partial X} - \frac{\partial \psi_l}{\partial t}) dp_l \wedge dX \wedge dt \\ & + \frac{\partial \psi_k}{\partial v_l} dv_l \wedge dv_k \wedge dt + \frac{\partial \psi_k}{\partial p_l} dp_l \wedge dp_k \wedge dX = 0 \end{aligned}$$

In order to satisfy this identity the coeffitients of the independent 3 - forms should be set equal to zero by taking into account the antisymmetry property of the exterior product. Hence we obtain

$$\begin{aligned} & \frac{\partial \varphi_k}{\partial p_l} - \frac{\partial \psi_l}{\partial v_k} = 0 \\ & \frac{\partial \varphi_k}{\partial v_l} - \frac{\partial \varphi_l}{\partial v_k} = 0 \\ & \frac{\partial \varphi_k}{\partial t} - \frac{\partial \psi_k}{\partial X} = 0 \\ & A_{kl} \frac{\partial \varphi_k}{\partial p_m} - A_{km} \frac{\partial \varphi_k}{\partial p_l} = 0 \\ & A_{kl} \frac{\partial \varphi_k}{\partial v_m} - \frac{\partial \psi_m}{\partial p_l} = 0 \\ & A_{kl} \frac{\partial \varphi_k}{\partial X} - \frac{\partial \psi_l}{\partial t} = 0 \\ & \frac{\partial \psi_k}{\partial v_l} - \frac{\partial \psi_l}{\partial v_k} = 0 \\ & \frac{\partial \psi_k}{\partial p_l} - \frac{\partial \psi_l}{\partial p_k} = 0 \end{aligned} \tag{3.2}$$

On the other hand the exterior derivative of (2.10) is simply

$$d\sigma=\frac{\partial\Phi}{\partial p_k}\,dp_k\wedge dt+\frac{\partial\Phi}{\partial v_k}\,dv_k\wedge dt+\frac{\partial\Phi}{\partial X}\,dX\wedge dt-\frac{\partial\Psi}{\partial p_k}\,dp_k\wedge dX$$
$$-\frac{\partial\Psi}{\partial v_k}dv_k\wedge dX-\frac{\partial\Psi}{\partial t}\,dt\wedge dX \qquad (3.3)$$

and in view of (2.7), Ω is given by

$$\Omega=\varphi_k dv_k\wedge dX+A_{kl}\varphi_k dp_l\wedge dt+\psi_k dv_k\wedge dt+\psi_k dp_k\wedge dX \qquad (3.4)$$

Equating (3.3) and (3.4) and comparing the similar terms we obtain

$$\varphi_k=-\frac{\partial\Psi}{\partial v_k}\,,\qquad A_{kl}\varphi_k=\frac{\partial\Phi}{\partial p_l}\,,$$
$$\qquad (3.5)$$
$$\psi_k=\frac{\partial\Phi}{\partial v_k}=-\frac{\partial\Psi}{\partial p_k}\,,\quad \frac{\partial\Phi}{\partial X}+\frac{\partial\Psi}{\partial t}=0$$

Eliminating φ_k and ψ_k in Eqs. (3.5) we derive the following overdetermined system of linear partial differential equations to specify the functions Φ and Ψ

$$\frac{\partial\Phi}{\partial v_k}+\frac{\partial\Psi}{\partial p_k}=0$$
$$\frac{\partial\Phi}{\partial p_k}+A_{kl}\frac{\partial\Psi}{\partial v_l}=0 \qquad (3.6)$$
$$\frac{\partial\Phi}{\partial X}+\frac{\partial\Psi}{\partial t}=0$$

It is straightforward to see that Eqs. (3.2) become just identities if we introduce the relations (3.5) with functions Φ and Ψ satisfying (3.6).

It follows from $(3.6)_{1,2}$ that

$$\frac{\partial^2\Phi}{\partial v_k\partial p_l}=-\frac{\partial^2\Psi}{\partial p_k\partial p_l}=-A_{lm}\frac{\partial^2\Psi}{\partial v_m\partial v_k}$$

This implies that the relation

$$A_{km}\frac{\partial^2\Psi}{\partial v_l \partial v_m}=A_{lm}\frac{\partial^2\Psi}{\partial v_k \partial v_m} \tag{3.7}$$

should be valid for an arbitrary symmetric tensor A_{kl}. It is straightforward to see that (3.7) is satisfied if and only if

$$\frac{\partial^2\Psi}{\partial v_k \partial v_l}=\rho_0\psi(\mathbf{p}, \mathbf{v}, X, t)\,\delta_{kl} \tag{3.8}$$

that is, if

$$\frac{\partial^2\Psi}{\partial v_1^2}=\frac{\partial^2\Psi}{\partial v_2^2}=\frac{\partial^2\Psi}{\partial v_3^2}=\rho_0\psi, \qquad \frac{\partial^2\Psi}{\partial v_k \partial v_l}=0, \qquad (k\neq l)$$

where ψ is an arbitrary function. The last relation results in

$$\Psi(\mathbf{p}, \mathbf{v}, X, t)=\Psi_1(\mathbf{p}, v_1, X, t)+\Psi_2(\mathbf{p}, v_2, X, t)+\Psi_3(\mathbf{p}, v_3, X, t)$$

so that we have

$$\frac{\partial^2\Psi_1(.\,, v_1, .\,.)}{\partial v_1^2}=\frac{\partial^2\Psi_2(.\,, v_2, .\,, .)}{\partial v_2^2}=\frac{\partial^2\Psi_3(.\,, v_3, .\,, .)}{\partial v_3^2}$$
$$=\rho_0\Psi(\mathbf{p}, v_1, v_2, v_3, X, t)$$

These equations can only be satisfied if ψ independent of $\mathbf{v}$, i.e., if

$$\psi=\psi(\mathbf{p}, X, t) \tag{3.9}$$

Then the integration gives the following result

$$\Psi=\frac{1}{2}\rho_0\psi(\mathbf{p}, X, t)v_kv_k+\lambda_k(\mathbf{p}, X, t)v_k+\mu(\mathbf{p}, X, t) \tag{3.10}$$

where λ_k and μ are arbitrary functions of $\mathbf{p}$, X, t. If we introduce (3.10) into $(3.6)_{1,2}$ we obtain

$$\begin{aligned}\frac{\partial\Phi}{\partial v_k}&=-\frac{1}{2}\rho_0\frac{\partial\psi}{\partial p_k}v_mv_m-\frac{\partial\lambda_m}{\partial p_k}v_m-\frac{\partial\mu}{\partial p_k}=A_k\\ \frac{\partial\Phi}{\partial p_k}&=-\rho_0A_{kl}(\mathbf{p})\psi v_l-A_{kl}(\mathbf{p})\lambda_l\end{aligned} \tag{3.11}$$

The integrability condition for $(3.11)_1$ is

$$e_{lkm}\frac{\partial A_k}{\partial v_m}=0$$

or

$$e_{lkm}\left(\rho_0\frac{\partial\psi}{\partial p_k}v_m+\frac{\partial\lambda_m}{\partial p_k}\right)=0$$

Since ψ and λ_m are independent of $\mathbf{v}$ the above expression can only be satisfied if

$$e_{lkm}\frac{\partial\psi}{\partial p_k}=0,\qquad e_{lkm}\frac{\partial\lambda_m}{\partial p_k}=0$$

These relations imply that

$$\psi=\psi(X,t),\qquad \lambda_m=\frac{\partial\Lambda}{\partial p_m}\tag{3.12}$$

where $\Lambda=\Lambda(\mathbf{p},X,t)$ is an arbitrary function. It follows from $(3.6)_2$, (3.8), (3.12) and $(3.11)_2$ that

$$\frac{\partial^2\Phi}{\partial v_k\partial p_l}=-\rho_0\psi A_{kl}$$

$$=-\psi\frac{\partial^2\Sigma}{\partial p_k\partial p_l}=-\frac{\partial^3\Lambda}{\partial p_k\partial p_m\partial p_l}v_m-\frac{\partial^2\mu}{\partial p_k\partial p_l}$$

whence we deduce that

$$\frac{\partial^3\Lambda}{\partial p_k\partial p_l\partial p_m}=0,\quad \frac{\partial^2\mu}{\partial p_k\partial p_l}=\psi\frac{\partial^2\Sigma}{\partial p_k\partial p_l}$$

We thus have

$$\frac{\partial^2\Lambda}{\partial p_k\partial p_l}=\rho_0\Lambda_{kl}(X,t),\qquad \frac{\partial^2}{\partial p_k\partial p_l}(\mu-\psi\Sigma)=0\tag{3.13}$$

where Λ_{kl} is an arbitrary *symmetric* tensor. The integration of Eqs. (3.13) yields immediately

$$\Lambda = \frac{1}{2} \rho_0 \Lambda_{kl} p_k p_l + \rho_0 \Lambda_k (X, t) p_k$$

$$\mu = \psi \Sigma + \mu_k (X, t) p_k + \mu_0 (X, t) \tag{3.14}$$

$$\lambda_k = \rho_0 \Lambda_{kl} p_l + \rho_0 \Lambda_k$$

where Λ_k, μ_k and μ_0 are arbitrary functions of X and t. An arbitrary function of X, t in the expression for Λ is neglected because it is immaterial in calculating λ_k. Then the function Ψ takes the following form

$$\Psi = \psi (X, t) (\Sigma + \frac{1}{2} \rho_0 v_k v_k) + \rho_0 \Lambda_{kl} (X, t) p_l v_k$$

$$+ \rho_0 \Lambda_k (X, t) v_k + \mu_k (X, t) p_k + \mu_0 (X, t) \tag{3.15}$$

Substituting (3.15) into $(3.6)_1$ we obtain

$$\frac{\partial \Phi}{\partial v_k} = -\psi \frac{\partial \Sigma}{\partial p_k} - \rho_0 \Lambda_{kl} v_l - \mu_k$$

the integration of which gives

$$\Phi = -\psi \frac{\partial \Sigma}{\partial p_k} v_k - \frac{1}{2} \rho_0 \Lambda_{kl} v_k v_l - \mu_k v_k + \pi (\mathrm{p}, X, t) \tag{3.16}$$

where π is an arbitrary function of its arguments. Let us introduce now (3.15) and (3.16) into $(3.6)_2$. We have

$$-\psi \frac{\partial^2 \Sigma}{\partial p_k \partial p_l} v_l + \frac{\partial \pi}{\partial p_k} + \psi \frac{\partial^2 \Sigma}{\partial p_k \partial p_l} v_l + \frac{\partial^2 \Sigma}{\partial p_k \partial p_l} \Lambda_{ml} p_m + \frac{\partial^2 \Sigma}{\partial p_k \partial p_l} \Lambda_l = 0$$

from which we obtain

$$\frac{\partial}{\partial p_k} (\pi + \frac{\partial \Sigma}{\partial p_l} \Lambda_l) = - \frac{\partial^2 \Sigma}{\partial p_k \partial p_n} p_m \Lambda_{mn} \tag{3.17}$$

The integrability condition of (3.17) is

$$A_{km} \Lambda_{ml} = \Lambda_{km} A_{ml}$$

from which we again deduce that

$$\Lambda_{kl}=\Lambda_0(X,t)\delta_{kl} \tag{3.18}$$

Introducing (3.18) into (3.17) we obtain

$$\frac{\partial}{\partial p_k}\left(\pi+\frac{\partial\Sigma}{\partial p_l}\Lambda_l\right)=-\Lambda_0\frac{\partial}{\partial p_k}\left(\frac{\partial\Sigma}{\partial p_l}p_l-\Sigma\right)$$

Therefore we have

$$\pi=\Lambda_0\left(\Sigma-\frac{\partial\Sigma}{\partial p_k}p_k\right)-\Lambda_k\frac{\partial\Sigma}{\partial p_k}+\pi_0(X,t) \tag{3.19}$$

where $\pi_0(X,t)$ is an arbitrary function.

Employing (3.18) and (3.19) in (3.15) and (3.16) we finally arrive at the following expressions

$$\Phi=-\psi(X,t)\frac{\partial\Sigma}{\partial p_k}v_k+\Lambda_0(X,t)\left(\Sigma-\frac{\partial\Sigma}{\partial p_k}p_k-\frac{1}{2}\rho_0v_kv_k\right)-\Lambda_k(X,t)\frac{\partial\Sigma}{\partial p_k}$$
$$-\mu_k(X,t)v_k+\pi_0(X,t) \tag{3.20}$$

$$\Psi=\psi(X,t)\left(\Sigma+\frac{1}{2}\rho_0v_kv_k\right)+\rho_0\Lambda_0(X,t)p_kv_k+\rho_0\Lambda_k(X,t)v_k+\mu_k(X,t)p_k$$
$$+\mu_0(X,t)$$

We introduce now (3.20) into $(3.6)_3$ to obtain

$$\left(\frac{\partial\psi}{\partial t}-\frac{\partial\Lambda_0}{\partial X}\right)\frac{1}{2}\rho_0v_kv_k+\left(-\frac{\partial\psi}{\partial X}\frac{\partial\Sigma}{\partial p_k}+\rho_0\frac{\partial\Lambda_0}{\partial t}p_k-\frac{\partial\mu_k}{\partial X}+\rho_0\frac{\partial\Lambda_k}{\partial t}\right)v_k$$
$$+\left(\frac{\partial\psi}{\partial t}+\frac{\partial\Lambda_0}{\partial X}\right)\Sigma-\left(\frac{\partial\Lambda_k}{\partial X}+\frac{\partial\Lambda_0}{\partial X}p_k\right)\frac{\partial\Sigma}{\partial p_k}+\frac{\partial\mu_k}{\partial t}p_k+\frac{\partial\pi_0}{\partial X}+\frac{\partial\mu_0}{\partial t}=0 \tag{3.21}$$

To satisfy this equation we should have

$$\frac{\partial\psi}{\partial t}-\frac{\partial\Lambda_0}{\partial X}=0$$

$$-\frac{\partial\psi}{\partial X}\frac{\partial\Sigma}{\partial p_k}+\rho_0\frac{\partial\Lambda_0}{\partial t}p_k=0$$

$$-\frac{\partial\mu_k}{\partial X}+\rho_0\frac{\partial\Lambda_k}{\partial t}=0 \tag{3.22}$$

$$\left(\frac{\partial\psi}{\partial t}+\frac{\partial\Lambda_0}{\partial X}\right)\Sigma-\left(\frac{\partial\Lambda_k}{\partial X}+\frac{\partial\Lambda_0}{\partial X}p_k\right)\frac{\partial\Sigma}{\partial p_k}+\frac{\partial\mu_k}{\partial t}p_k=0$$

$$\frac{\partial\pi_0}{\partial X}+\frac{\partial\mu_0}{\partial t}=0$$

Eq. $(3.22)_2$ leeds to the result

$$\frac{\partial\psi}{\partial X}=0,\quad \frac{\partial\Lambda_0}{\partial t}=0$$

It is easy to see that this should be valid even in the linear elasticity. Hence we have

$$\psi=\psi(t),\qquad \Lambda_0=\Lambda_0(X)$$

Then $(3.22)_1$ yields

$$\frac{d\psi}{dt}-\frac{d\Lambda_0}{dX}=0$$

or

$$\psi=\alpha t+\beta,\qquad \Lambda_0=\alpha X+\gamma \tag{3.23}$$

where α, β, γ are constants. Thus $(3.22)_4$ is reduced to

$$\alpha\left(2\Sigma-\frac{\partial\Sigma}{\partial p_k}p_k\right)-\frac{\partial\Lambda_k}{\partial X}\frac{\partial\Sigma}{\partial p_k}+\frac{\partial\mu_k}{\partial t}p_k=0 \tag{3.24}$$

In *nonlinear materials* (3.24) can only be satisfied if we take

$$\frac{\partial\Lambda_k}{\partial X}=0,\quad \frac{\partial\mu_k}{\partial t}=0$$

and

$$\alpha=0 \tag{3.25}$$

since $2\Sigma=(\partial\Sigma/\partial p_k)$ implies that Σ is a homogeneous quadratic function and this clearly corresponds to a linear material. Therefore we have

$$\Lambda_k=\Lambda_k(t),\qquad \mu_k=\mu_k(X)$$

whence we deduce in view of $(3.22)_3$ that

$$-\frac{d\mu_k}{dX}+\rho_0\frac{d\Lambda_k}{dt}=0$$

and

$$\Lambda_k=\alpha_k t+\beta_k\,,\qquad \mu_k=\rho_0\alpha_k X+\gamma_k \tag{3.26}$$

where α_k, β_k and γ_k are constants.

It is quite obvious that we can assume that $\pi_0=\mu_0=0$ without any loss of generality since they do not involve field variables. Therefore, for nonlinear materials, expression (3.20) become

$$\begin{aligned}\Phi&=-\beta\frac{\partial\Sigma}{\partial p_k}v_k+\gamma(\Sigma-\frac{\partial\Sigma}{\partial p_k}p_k-\frac{1}{2}\rho_0 v_k v_k)-\alpha_k(t\frac{\partial\Sigma}{\partial p_k}+\rho_0 X v_k)\\&\quad-\beta_k\frac{\partial\Sigma}{\partial p_k}-\gamma_k v_k\\&=-\beta T_k v_k+\gamma(L-T_k p_k)-\alpha_k(tT_k+\rho_0 X v_k)-\beta_k T_k-\gamma_k v_k\\\Psi&=\beta(\Sigma+\frac{1}{2}\rho_0 v_k v_k)+\gamma\rho_0 v_k p_k+\alpha_k(\rho_0 t v_k-\rho_0 X p_k)\\&\quad+\beta_k\rho_0 v_k+\gamma_k p_k\end{aligned}\tag{3.27}$$

where we define the *Lagrangian density* by

$$L=\Sigma-\frac{1}{2}\rho_0|\mathbf{v}|^2 \tag{3.28}$$

Thus the corresponding conservation laws can be listed as follows

$$\begin{aligned}&(\beta) && \frac{\partial}{\partial t}(\Sigma-\frac{1}{2}\rho_0|\mathbf{v}|^2)-\frac{\partial}{\partial X}(T_k v_k)=0\\&(\beta_k) && \frac{\partial}{\partial t}(\rho_0 v_k)-\frac{\partial T_k}{\partial X}=0\\&(\gamma_k) && \frac{\partial p_k}{\partial t}-\frac{\partial v_k}{\partial X}=0\\&(\alpha_k) && \frac{\partial}{\partial t}[\rho_0(tv_k+Xp_k)]-\frac{\partial}{\partial X}(tT_k+\rho_0 X v_k)=0\\&(\gamma) && \frac{\partial}{\partial t}(\rho_0 v_k p_k)+\frac{\partial}{\partial X}(L-T_k p_k)=0\end{aligned}\tag{3.29}$$

Obviously $(3.29)_1$ corresponds to the conservation of energy, $(3.29)_2$ to the balance of momentum and $(3.29)_3$ is the compatibility relation $(2.6)_2$. The last four laws are imposed by the mathematical structure of the field equations. It is easy to check that they vanish identically for a solution set of the field equations.

In *linear materials* one still obtains (3.23). However the interpretation of Eq. (3.24) should be modified drastically. In such materials the stress potential is expressible as

$$\Sigma=\frac{1}{2}\, a_{kl}p_k p_l \tag{3.30}$$

where a_{kl} is a symmetric, positive definite tensor. In this case we have

$$2\Sigma=\frac{\partial\Sigma}{\partial p_k}\, p_k$$

Therefore the constant α is not necessarily zero. Eqs. $(3.22)_3$ and (3.24) lead to

$$\begin{aligned} \frac{\partial\Lambda_k}{\partial t}&=\frac{1}{\rho_0}\,\frac{\partial\mu_k}{\partial X} \\ \frac{\partial\mu_k}{\partial t}&=a_{kl}\,\frac{\partial\Lambda_l}{\partial X} \end{aligned} \tag{3.31}$$

In addition to the particular solution (3.26) these equations admit the following general solution obtainable by using standard techniques

$$\begin{aligned} \Lambda_k&=\sum_{\alpha=1}^{3}\,[U_\alpha\;(X+\lambda_\alpha\, t)+V_\alpha\;(X-\lambda_\alpha\, t)]n_k{}^{(\alpha)} \\ \mu_k&=\rho_0\sum_{\alpha=1}^{3}\,\lambda_\alpha\,[U_\alpha\;(X+\lambda_\alpha\, t)-V_\alpha\;(X-\lambda_\alpha\, t)]n_k{}^{(\alpha)} \end{aligned} \tag{3.32}$$

where $\rho_0\lambda^2_\alpha$ $(\alpha=1,2,3)$ are the eigenvalues of the tensor a_{kl}, i.e., they are the roots of the equation

$$\det(a_{kl}-\rho_0\lambda^2\delta_{kl})=0$$

$\mathbf{n}(\alpha)$ are the mutually orthogonal eigenvectors of a_{kl}. U_α and V_α are arbitrary functions of their arguments. If we introduce (3.32) into (3.20) the conservation law yields

$$\frac{\partial\Phi}{\partial X}+\frac{\partial\Psi}{\partial t}=\ldots-\sum_{\alpha=1}^{3}(U'_\alpha+V'_\alpha)T(\alpha)-\rho_0\sum_{\alpha=1}^{3}\lambda_\alpha(U'_\alpha-V'_\alpha)v(\alpha)$$
$$+\rho_0\sum_{\alpha=1}^{3}\lambda_\alpha(U'_\alpha-V'_\alpha)v(\alpha)+\rho_0\sum_{\alpha=1}^{3}\lambda^2_\alpha(U'_\alpha+V'_\alpha)p(\alpha)=0 \tag{3.33}$$

where primes denote differentiation with respect to the argument and we define

$$T(\alpha)=T_k n_k(\alpha), \qquad v(\alpha)=v_k n_k(\alpha), \qquad p(\alpha)=p_k n_k(\alpha)$$

Obviously (3.33) reduces to

$$\frac{\partial\Phi}{\partial X}+\frac{\partial\Psi}{\partial t}=\ldots+\sum_{\alpha=1}(U'_\alpha+V'_\alpha)(\rho_0\lambda^2_\alpha p(\alpha)-T(\alpha))=0$$

This reveals no important property since the expressions

$$T(\alpha)=\rho_0\lambda^2_\alpha\ p(\alpha) \qquad (\alpha=1,2,3)$$

are none other than the constitutive relations in a coordinate system taken along the eigenvectors. In fact if we multiply the constitutive relations

$$T_k=a_{kl}p_l$$

by an eigenvector we have

$$T(\alpha)=n_k(\alpha)\ T_k=n_k(\alpha)\ a_{kl}p_l=\rho_0\lambda^2_\alpha\ n_l(\alpha)\ p_l=\rho_0\lambda^2_\alpha\ p(\alpha)$$

Therefore the only meaningful additional conservation law in lineer theory arises from the terms corresponding to the parameter α in (3.23), namely,

$$\Phi = \ldots + \alpha[-tT_k v_k + X(L - T_k p_k)]$$

$$\Psi = \ldots + \alpha[t(\Sigma + \frac{1}{2}\rho_0 |\mathbf{v}|^2) + X\rho_0 p_k v_k]$$

which is

$$(\alpha) \quad \frac{\partial}{\partial t}[t(\Sigma + \frac{1}{2}\rho_0 |\mathbf{v}|^2) + X\rho_0 p_k v_k] + \frac{\partial}{\partial X}[-tT_k v_k + X(L - T_k p_k)] = 0 \tag{3.34}$$

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BİR BOYUTLU SONLU ELASTODİNAMİKTE KORUNUM YASALARI

Özet : Bir boyutlu lineer olmayan elastodinamiğin alan denklemleri uygun dış diferansiyel formlar cinsinden ifade edilmiştir. Bu diferansiyel formların ürettigi halkanın kapalı ideali inşa edilerek alan denklemlerinin korunum yasaları yapısının bazı lineer kısmî diferansiyel denklemleri sağlayan iki skaler fonksiyon yardımıyla belirlendiği gösterilmiştir. Bu denklem takımının çözümü bulunmuş ve korunum yasalarının tüketici bir listesi verilmiştir. Lineer durum da incelenmiştir.