ISTANBUL TECHNICAL UNIVERSITY ★ GRADUATE SCHOOL OF SCIENCE, ENGINEERING AND TECHNOLOGY

CHAMELEON GRAVITY

M.Sc. THESIS

Sibel BORAN

Department of Physics Engineering

Physics Engineering Programme

JUNE 2012

ISTANBUL TECHNICAL UNIVERSITY ★ GRADUATE SCHOOL OF SCIENCE, ENGINEERING AND TECHNOLOGY

CHAMELEON GRAVITY

M.Sc. THESIS

Sibel BORAN
(509081112)

Department of Physics Engineering

Physics Engineering Programme

Thesis Advisor: Assoc. Prof. Dr. A. Savaş ARAPOĞLU

JUNE 2012

İSTANBUL TEKNİK ÜNİVERSİTESİ ★ FEN BİLİMLERİ ENSTİTÜSÜ

BUKALEMUN KÜTLE ÇEKİMİ

YÜKSEK LİSANS TEZİ

Sibel BORAN
(509081112)

Fizik Mühendisliği Anabilim Dalı

Fizik Mühendisliği Programı

Tez Danışmanı: Doç. Dr. Savaş ARAPOĞLU

HAZİRAN 2012

Sibel BORAN, a **M.Sc.** student of ITU **Graduate School of Science, Engineering and Technology** 509081112 successfully defended the thesis entitled **"CHAMELEON GRAVITY"**, which she prepared after fulfilling the requirements specified in the associated legislations, before the jury whose signatures are below.

Thesis Advisor :	**Assoc. Prof. Dr. A. Savaş ARAPOĞLU** Istanbul Technical University	....
Jury Members :	**Prof. Dr. Neşe Özdemir** Istanbul Technical University	....
	Assoc. Prof. Dr. Aybike Özer Istanbul Technical University	....

Date of Submission : 03 May 2012
Date of Defense : 04 June 2012

To the memory of dear my friend Orkun DURMUŞ (1986-2012)..

FOREWORD

I want to thank to my advisor Assoc. Prof. Dr. A. Savaş ARAPOĞLU who enlightens my way with his great help and guidance throughout this thesis. I was able to complete this thesis with his support and endless patience by overcoming the difficulties. For giving me the chance to work with him, I am grateful to him. Working with him has been a great luck for me.

June 2012 Sibel BORAN
(Physicist)

TABLE OF CONTENTS

Page

ABBREVIATIONS

BAO	**:** Baryon Acoustic Oscillations
BD	**:** Brans-Dicke
CDM	**:** Cold Dark Matter
CI	**:** Conformal Invariance
CMBR	**:** Cosmic Microwave Background Radiation
CT	**:** Conformal Transformation
EF	**:** Einstein Frame
EP	**:** Equivalence Principle
EEP	**:** Einstein' s Equivalence Principle
FRW	**:** Friedmann-Robertson-Walker
GG	**:** Galileo Galilei Satellite Experiment
GR	**:** General Relativity
JF	**:** Jordan Frame
KK	**:** Kaluze-Klein
LHC	**:** Large Hadron Collider
SCP	**:** Supernova Cosmology Project
SEE	**:** Satellite Energy Exchange Project
SEP	**:** Strong Equivalence Principle
ST	**:** Scalar-Tensor
STEP	**:** The Satellite Test of the Equivalence Principle
WEP	**:** Weak Equivalence Principle
WMAP	**:** Wilkinson Microwave Anisotropy Probe

LIST OF TABLES

LIST OF FIGURES

Page

LIST OF SYMBOLS

$(-,+,+,+)$	**:** The signature of the metric
$\mathbf{c} = \mathbf{\hbar} = \mathbf{1}$	**:** The speed of the light and Planck' s constant
$\dot{}$	**:** Dots denote differentiation with respect to time.
$'$	**:** Primes correspond to time independent derivative.
∂_μ or subscript $,\mu$	**:** Ordinary or Partial Derivative
∇_μ or subscript $;\mu$	**:** Covariant Derivative
$\Box \equiv \mathbf{g}^{\mu\nu}\nabla_\mu\nabla_\nu$	**:** d' Alembertian Operator
$\mathbf{G}$	**:** The Newtonian Gravitational Constant
$\kappa \equiv \frac{\mathbf{8\pi G}}{\mathbf{c^4}}$	**:** Constant
$\eta_{\mu\nu}$	**:** Minkowskian Metric Tensor
$\mathbf{g}_{\mu\nu}$	**:** Lorentzian or pseudo-Riemannian Metric Tensor
$\mathbf{g}$	**:** Determinant of the $g_{\mu\nu}$
$\Gamma^\rho{}_{\mu\nu}$	**:** The Affine connection or Schwarzs-Christoffel Symbol
$\mathbf{R}^\rho{}_{\mu\lambda\nu}$	**:** Riemann Tensor with respect to $g_{\mu\nu}$
$\mathbf{R}^\lambda{}_{\mu\lambda\nu} \equiv \mathbf{R}_{\mu\nu}$	**:** Ricci Tensor with respect to $g_{\mu\nu}$
$\mathbf{R}$	**:** Ricci Curvature Scalar with respect to $g_{\mu\nu}$
$\mathbf{T}^{\mu\nu}$	**:** Energy-momentum tensor
φ, ϕ	**:** Scalar Fields
ψ	**:** Matter Field
Λ	**:** Cosmological Constant

CHAMELEON GRAVITY

SUMMARY

At the end of the last millennium, the cosmological observations, such as SN Ia, CMB radiation, allowed us in order to witness a revolutionary discovery of the accelerated expansion universe model.

Adding a scalar field into the currently accepted theory of the large scale structure of the Universe, Einstein's general theory of relativity, under the title of quintessence models is given to load more clear meaning in terms of theoretically as an alternative way for this important discovery. One may follow such a route only on the observational ground; but, if one would like to relate these scalar fields to the theories which are candidate of being the theory of everything, one may easily go into the trouble of violating some other important observations, such as Solar System observations.

In general terms, in cosmology, known the most basic problem is still cosmological constant problem and the scalar field models can not be defined as a definite address for the solution to this problem. On the other hand; in particular, the scalar field models arising from the mysterious vacuum energy, value of which estimates as almost seventy two percent of the Universe for present, help us to able to make the explanation about the accelerated expansion universe model, with the help of the fundamental feature of the scalar field known as simplicity.

The philosophy of the model is that cosmological scalar fields, such as quintessence, have not yet been detected in local tests of the equivalence principle because we happen to live in a dense environment. The cosmological implications of such a scenario are investigated in detail and many of the important results present in the literature are re-arranged and are re-derived.

Which is the subject of this thesis, the chameleon gravity is a novel scenario where a scalar field acquires a mass which depends on the local matter density. According this scenario, while the field is massive on Earth, where the density is high, it is essentially free in the Solar System, where the density is low. All existing tests of gravity are satisfied. For this scenario, the near future experiment results are awaited with the curiosity and eagerly.

The inconsistencies between the measurements in the laboratory and the expectations can be ignored by the new and surprising outcomes arising from differently behaviors of the scalar field in the regions of high density than in the regions of low density. Because they are able to hide so well from our observations and experiments, this scalar fields are therefore called as a chameleon field. Their physical properties, such as their mass, depend on the environment. Eventually, for the scalar fields, the environment

density is significant in order to able to detect the existence of these fields and so these fields can also disappear due to environment density.

BUKALEMUN KÜTLE ÇEKİMİ

ÖZET

Geride bıraktığımız yakın geçmişte, SNIa ve CMB ışıması gibi kosmolojik gözlemler ivmelenerek genişleyen (şuan ki) evren modelinin devrim niteliğindeki keşfine tanıklık etmemiz için bize fırsat tanıdı.

Bu önemli keşfe teorik açıdan daha da netlik kazandırmak için alternatif bir yol olarak, evrenin büyük ölçekteki yapısının teorisi olarak halen kabul gören Einstein' ın genel görelilik teorisine, quintessence modelleri başlığı altında eklenen skaler alan modelleri gösterilebilir. Bu alanların aynı zamanda, alternatif kütle çekimi teorisinden olan Skaler-Tensor teoriler kaynaklı olduğu da bilinir.

Yalnızca gözlemlere dayalı olarak elde edilen verileri inceleyip, bunlara fiziksel olarak anlam yükleyebilmek için bu yolu izlemek kolaylık sağlarken, herşeyin teorisi olma yolunda aday olan teorilerle bu skaler alanlar ilişkilendirilmek istenildiğinde, Güneş Sistemi gözlemleri gibi diğer bazı önemli gözlemlerde izlenilen bu yolun bozunmaya uğrayacağını görmek mümkündür.

Genel açıdan bakıldığında, kosmoloji de halen yanıt aranan temel bir sorun olan, kosmolojik sabit sorununa kesin çözüm için skaler alan modelleri adres olarak gösterilemez. Çünkü sahip oldukları basitlik özelliği nedeniyle bu modellerin potensiyel değerleri neredeyse sıfır olacak kadar çok küçük değer olarak kabul edilir. Evrenin ivmelenme fikrinin ortaya atılmasını sağlayan modelin sahip olduğu dinamikler (serbestlik dereceleri) açısındansa, potansiyelin sfr olarak kabul grebileceği gerçek bir vakum durumuna ulaşmanın henüz yetersiz kalacağı savunulur.

Öte yandan; özel olarak, kosmolojik veri ve gözlemlerden yardımla şuan ki değeri evrenin yaklaşık yüzde yetmiş ikisini oluşturduğu öngörülen gizemli vakum enerjisi kaynaklı quintessence skaler alan modelleri sahip olduğu temel özelliği olan basitliği sayesinde ivmelenerek genişleyen evren modeli hakkında bilgi sahibi olmamız için bize yardımcı olur.

İvmelenerek genişleyen evren modelinin fikrini oluşturabilmek için, potensiyelin giderek yavaşça azaldığı kabul edilerek, skaler alanın kütlesinin de buna bağlı olarak etkin kütle şeklindeki eldesine gidilir. Buradaki önemli nokta, bu etkin kütlenin fiziğin önemli bir diğer kolu olan parçacık fiziğindeki elde edilen parçacık kütlesi ölçümleri ile karşlaştırıldığında çok çok küçük olması gerekliliğidir ($m_\phi \equiv \sqrt{V''(\phi)} \leq 3H_o \leq 10^{-42}$ Gev). Quintessence skaler alanları bu tür özellikteki potensiyele sahip oldukları için bir anlamda bu tezin temelini oluşturan kaynak skaler alanlar olarak da kabul edilip, alglanabilinir.

Modelin felsefesini oluşturan quintessence gibi skaler alanlar, yoğun bir ortamda bulunulduğu için eşdeğerlik ilkesinin yerel testlerinde henüz tespit edilememiş olup;

kosmolojik etkileri detayla incelenilen ve önemli sonuçları literatürde geniş yer tutan pekçok senaryo için yeniden düzenlemeye, türetmeye gidebilmek için faydalıdır.

Bu tezin konusu olan ve gelecekteki yapılması planlanan deneylerinin de gözlemsel sonuçları merak ve heyecanla beklenen bukalemun kütle çekimi yerel madde yoğunluğuna bağlı olarak skaler alanın kütle kazanmasını anlatan yeni bir senaryodur. Bu senaryoya göre, skaler alan madde yoğunluğunun oldukça fazla olduğu yeryüzünde kütlelenirken, yoğunluğun oldukça düşük olduğu Güneş Sistemi' nde aslında serbest olup, varolan tüm kütle çekimi testlerinde sağlanır.

Bu senaryo ile ilgili skaler alan kaynağını ortaya çıkarmak için pekçok çalışma mevcuttur. Bunların bilinen birkaç örneği olarak, teoriksel kuantum skaler alanların üyesi olan, sicim teorisi kaynaklı dilaton, radion alanları verilebilir. Fakat, yukarıda da belirtildiği gibi, biz bu tez boyunca bu senaryoyu daha iyi kavrayabilmek için , teoriksel kuantum olmayan, vakum enerjisi temelli quintessence skaler alanları ile ilgili olacağız.

Laboratuvar deneylerinin ölçümleri ile beklentiler arasındaki tutarsızlıklar, skaler alanların düşük yoğunluklu bölgelerden yüksek yoğunluklu bölgelerdeki farklı davranışları sebebiden kaynaklanan yeni ve şaşırtıcı sonuçlarla gözardı edilebilir. Çünkü bu alanlar kendilerini gözlem ve deneylerden ustalıkla gizleyebilirler; bu yüzdendir ki, bu tür özelliğe sahip alanlar bukalemun alanlar olarak isimlendirilirler. Böylelikle, bu alanların fiziksel özelliği olan kütlelerinin de bulunduğu ortama bağlılığı doğrulanır; yani, bu alanların varlığının fark edilmesinde ya da alanların deney ve gözlemlerde gözden kaybolmasında, alanların bulunduğu ortam yoğunluğunun önemi büyüktür.

Örneğin, yeryüzü (Dünya) gibi yüksek yoğunluğa sahip bölgelerde, skaler alan büyük bir kütleye sahip olup, eşdeğerlik ilkesinin bozunmasının gizlenmesi üssel olarak gerçekleşir. Düşük yoğunluğa sahip yılıdızlar, gök cisimleri arası bölgelerde ise alanların kütleleri yaklaşık olarak bugünkü Hubble parametresi boyutundadır. Çok daha düşük yoğunluklu bölgeler olarak bilinen Güneş Sistemi deneylerinin gerçekleştiği bölgelerin yerel madde yoğunluğu Dünya' nın sahip olduğu yoğunluktan çok daha düşüktür. Güneş Sistemi gözlemlerinde skaler alanların hareketlerinin gözlenmesi ise ince kabuk mekanizması olarak tanımlanan yeni bir mekanizma tarafından engellenir.

Maddeye herhangi bir skaler alan bağlanmasının etkisinin gözlemlerce fark edilmesinin önlenmesinin sebebi olarak, bu özel mekanizmanın varlığının geliştirilmesi gösterilir. Öyle ki, yeteri kadar küçük nesneler bu mekanizmadan etkilenmezler, böylece bu küçük nesnelerin sahip oldukları kütle yoğunları tamamiyle dış ortamlarına eklenir. Fakat, Dünya ile Güneş gibi büyük nesneler arasında kuvvetle aracılanan skaler alanlar, kütle çekimi deneylerinin Güneş Sistemi' nde gerçekleşmesini sağlayan ince kabuk mekanizmasının varlığı sayesinde gözden kaybolabilirler.

Kısaca skaler alanlar için, değişim aralığı ile bağlanmalar arası ilişki şöyle özetlenebilinir:

Küçük bağlanmalarda büyük değişimler gözlenmez; çünkü ince kabuk mekanizmasının etkisi küçük nesneler üzerinde yoktur ve maddeyle skaler alanların etkileşim aralıkları kısadır. Büyük nesneler için büyük değişimler söz konusudur. Çünkü etkileşim aralıkları uzun olup, maddeye bağlanmaları daha güçlü olduğundan skaler alanlar için ortama daha hızlı uyum sağlama ve deneylerden çok daha iyi gizlenme imkanı gelişir.

SEE projesi, STEP, Galileo Galilei ve MICROSCOPE gibi uydu deneylerinden, yakın gelecekte alınması umulan sonuçlar bukalemun alanlarının var olabileceğinin ispatını olumlu yönde destekler nitelikte olacaktır. Şayet quintessence ve kütle çekimi arasında bağdaştırıcı ve deneylerle de pekiştirilen somut sonuçlara ulaşılabilinirse, quintessence alanlarının bukalemun alanlarının kaynağı olabileceği fikri de gerçeklik kazanmş olacaktır.

1. INTRODUCTION

The current accelerated expansion of the universe has been confirmed by many independent observations. The supporting evidence comes from the supernovae Ia data [1–4], cosmic microwave background radiation [1, 5–7], and the large scale structure of the universe [1, 8, 9]. Although the cosmological constant is arguably the simplest explanation and the best fit to all observational data, its theoretical value predicted by quantum field theory is many orders of magnitude greater than the value to explain the current acceleration of the universe. This problematic nature of cosmological constant has motivated an intense research for alternative explanations. A scalar field component, in the framework of Einstein's general theory of relativity, with a negative pressure, for example, can give results consistent with the observations.

The attempts to explain the accelerated expansion of the universe with scalar fields are named as Quintessence Models. The scalar fields are also used in the so-called inflation models, which people believe that the first accelerated expansion stage of the early universe. Obviously, the scalar fields are the first candidates for explaining the observational data unless the current theories and the observed matter forms are able to explain them. The main reason for this is, possibly, the simple nature of scalar fields in comparison to spinor or vector fields.

Another scalar field which is the target of a huge research and many state-of-the-art accelerators is the Higgs field of Standard Model of Particle Physics. For example, the most important motivation behind the LHC experiment in CERN is to detect the Higgs particle actually.

There are also theoretical reasons to consider scalar fields as fundamental constituents of nature: For example, they arise naturally in the Kaluza-Klein theories, superstring theories, and M-theory; or they are proposed to cure some conceptual problems of currently accepted theories. But, as of the beginning of 21st century, fundamental scalar fields exist only as hypothetical structures in physics, and we can summarize the places and their classical or quantum character as follows:

- Hypothetical non-quantum scalar fields:
  - Scalar fields in the so-called scalar-tensor theories of gravity,
  - Inflations.
- Hypothetical quantum scalar fields:
  - Higgs particle, giving mass by interactions with massless particles,
  - Dilatons and moduli fields etc., quantum fields appearing in superstring theory and M-theory.

The scalar fields that we have mentioned above are not observed as of 2012; and the scalar fields arising naturally in the string theory are generally coupled to matter with gravitational strength, and therefore lead to unacceptably large violations of the Equivalence Principle. Thus one expects that there must be a mechanism suppress the Equivalence-Principle-violating contributions of such scalar fields. Khoury and Weltman [10] have suggested a coupling which gives the scalar field a mass depending on the local density of matter. The idea is that the mass of the scalar field is not constant in space and time, but rather depends on the environment, in particular, on the local matter density: In regions of high density, such as on Earth, the mass of the field can be sufficiently large to satisfy constraints on EP violations and fifth force; meanwhile, on cosmological scales where the matter density is 10^{30} times smaller, the mass of the field can be of order H_0, the present day value of the Hubble constant, thus allowing the field to evolve cosmologically today. The philosophy, therefore, is that cosmological scalar fields, such as quintessence, have not yet been detected in local tests of the EP because we happen to live in a dense environment. Since their physical characteristics depend sensitively on their environment, such scalar fields are dubbed as chameleons.

This thesis is the summary of chameleon gravity written by a beginner level theoretical physicist. The plan of the thesis is as follow: in chapter 2 and 3, Einstein's general theory of relativity and the current state of the cosmology in the framework of general relativity are summarized. In chapter 4, the scalar-tensor theories are explained and, finally, in chapter 5 the rudiments of the chameleon gravity is presented.

2. EINSTEIN'S GENERAL THEORY OF RELATIVITY

2.1 Prelude

Einstein's General Theory of Relativity (GR) is the currently accepted relativistic theory of gravitation. The key idea of the theory is gravity is the result of the curvature of spacetime. This makes gravity different from the other fundamental interactions since the other forces are represented by some fundamental fields living on spacetime but gravity stems from the spacetime itself which means, in the context of general relativity, the dynamical field giving rise to gravitation is the metric tensor describing the curvature of spacetime, instead of an additional field propagating through spacetime. In this sense, gravity is really a special interaction and the principle leading to this specialness is the Principle of Equivalence.

The principle of equivalence is formalized in different forms: Weak EP, EEP, and SEP. The WEP states that "if an uncharged test body is placed at an initial event in spacetime and given an initial velocity there, then its subsequent trajectory will be independent of its internal structure and composition". The EEP states that "in small enough regions of spacetime, the laws of physics reduce to those of special relativity; it is impossible to detect the existence of a gravitational field by means of local experiments". It is the EEP that implies/suggests that there should at least one second rank tensor field which reduces in the local freely falling frame, to a metric conformal with the Minkowski one and therefore we should attribute the action of gravity to the curvature of spacetime described by the metric. On the other hand, the WEP implies that spacetime is endowed with a family of preferred trajectories which are the world lines of freely falling test bodies; but the existence of metric is not suggested by the WEP; but this universality is the origin of the claim that gravity is not actually a force, but a feature of spacetime. Therefore the first step to understand gravity is to understand the curvature of spacetime, how this curvature is described mathematically, and how it is relevant to gravity. However, it is important to emphasize the fact that we can not prove that gravity should be thought of as the curvature of spacetime; but, instead we can propose the idea, derive its consequences, and see if the result is a reasonable fit to our experience of the world.

The appropriate mathematical structure used to describe curvature is that of a differentiable manifold, and a manifold is essentially a set that looks locally like R^n, i.e., flat space but globally different. We intuitively expect that the notion of curvature depends exclusively on the metric, but it is not immediately clear how curvature is related to any given metric. However, a more careful treatment shows that the curvature depends on a quantity called connection, and connections may or may not depend on the metric. The connection provides a way to relate vectors in the tangent spaces of nearby points on a manifold. There is a unique connection derived from the metric and used in Einstein's General Theory of Relativity, called the Christoffel symbol given by the equation (**2.1**)

$$\Gamma^{\rho}{}_{\mu\nu} = \frac{1}{2} g^{\rho\lambda} (g_{\nu\lambda,\mu} + g_{\lambda\mu,\nu} - g_{\mu\nu,\lambda}) \qquad \textbf{(2.1)}$$

This object is not a tensor, although it seems so; and the fundamental use of a connection is to take a covariant derivative, a kind of derivative which transforms like a tensor (**2.2**):

$$\nabla_{\sigma} T^{\mu_1\mu_2..\mu_k}{}_{\nu_1\nu_2..\nu_l} = \partial_{\sigma} T^{\mu_1\mu_2..\mu_k}{}_{\nu_1\nu_2..\nu_l} + \Gamma^{\mu_1}_{\sigma\lambda} T^{\lambda\mu_2..\mu_k}{}_{\nu_1\nu_2..\nu_l} + \Gamma^{\mu_2}_{\sigma\lambda} T^{\mu_1\lambda..\mu_k}{}_{\nu_1\nu_2..\nu_l} + ..$$

$$- (\Gamma^{\lambda}{}_{\sigma\nu_1} T^{\mu_1\mu_2..\mu_k}{}_{\lambda\nu_2..\nu_l} + \Gamma^{\lambda}{}_{\sigma\nu_2} T^{\mu_1\mu_2..\mu_k}{}_{\nu_1\lambda..\nu_l} + ..) \qquad \textbf{(2.2)}$$

The mathematical structure used to describe the curvature of a manifold is the Riemann tensor which is a $(1,3)$ tensor obtained from the connection by (**2.3**)

$$R^{\rho}{}_{\mu\lambda\nu} = \Gamma^{\rho}{}_{\nu\mu,\lambda} - \Gamma^{\rho}{}_{\lambda\mu,\nu} + \Gamma^{\rho}{}_{\lambda\sigma}\Gamma^{\sigma}{}_{\nu\mu} - \Gamma^{\rho}{}_{\nu\sigma}\Gamma^{\sigma}{}_{\lambda\mu} \qquad \textbf{(2.3)}$$

2.2 Gravity as Geometry

In order to examine the physics of gravitation we should be able to answer the following questions: "How does the gravitation influence the matter?" and "how does the matter determine the gravitational field?". The hard part is to find the equation which governs the response of spacetime curvature to the presence of matter and/or energy. In the previous section we have considered the necessary quantities to define the curvature of spacetime. But it is not enough to conclude all from these considerations that the gravity is the result of geometry. We have to quantify this proposal in the light of experiments. We expect that some, or all, components of the Riemann tensor or some other tensors derived from it, must be related to the energy-momentum of the particles which is collectively described by the energy-momentum tensor, $T^{\mu\nu}$.

Eventually Einstein derived the equation (**2.4**):

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = G_{\mu\nu}{}^{M} \equiv \kappa T_{\mu\nu}{}^{M} \quad \textbf{(2.4)}$$

This equation is known as the Einstein's equation (**A.7**). The tensor $R_{\mu\nu}$ is the Ricci tensor and it is obtained from the Riemann tensor through the following contraction ():

$$R^{\rho}{}_{\mu\lambda\nu} \underset{\rho\to\lambda}{\to} R^{\lambda}{}_{\mu\lambda\nu} \equiv R_{\mu\nu} = \Gamma^{\lambda}{}_{\nu\mu,\lambda} - \Gamma^{\lambda}{}_{\lambda\mu,\nu} + \Gamma^{\lambda}{}_{\lambda\sigma}\Gamma^{\sigma}{}_{\nu\mu} - \Gamma^{\lambda}{}_{\nu\sigma}\Gamma^{\sigma}{}_{\lambda\mu} \quad \textbf{(2.5)}$$

The left-hand side of the Einstein's equation is the measure of curvature of spacetime and the right-hand side is the measure of the energy and momentum of the matter in the spacetime. Although this equation seems simple in this form (thanks to the power of tensor notation), it contains 10 coupled differential equations for the 10 components of the metric in 4-dimensional spacetime. Actually the aim is to find the components of the metric in the presence of a specified type of matter.

The second physical ingredient of the Einstein's General Theory of Relativity is about the response of matter to spacetime curvature. We expect that the free particles follow the shortest path in spacetime, and in the context of GR since we consider gravity as the manifestation of curvature and not as an interaction their parametrized paths $x^{\mu}(\lambda)$ obey the geodesic equation(**2.6**):

$$\frac{d^2 x^{\rho}(\lambda)}{d\lambda^2} + \Gamma^{\rho}{}_{\mu\nu} \frac{dx^{\mu}(\lambda)}{d\lambda} \frac{dx^{\nu}(\lambda)}{d\lambda} = 0 \quad \textbf{(2.6)}$$

2.3 Assumptions of General Relativity

The first experimental verification of General Relativity was the Eddington's measurement of light deflection in 1919, four years after the appearance of the theory.

But until 1960s, General Relativity was not the object of systematic experimental tests. In 1960, Pound and Rebka [11, 12] measured the gravitational redshift of light proposed by Einstein in 1907. This experiment is considered as one of the three classical tests of GR together with the perihelion shift of Mercury and light deflection measured by Eddington. On the other hand, it was later realized that some of the gravitational experiments [12–14] do not test the validity of specific field equations but test the validity of principles; for instance, the gravitational redshift experiments test the validity of the principle of equivalence. Then, in the light of this fact, a number of alternative theories of gravitation had been proposed, many of which were therefore indistinguishable as far as some of the tests were concerned. Therefore, because of the fact that experiments tests principles and not specific theories/field equations, it is important to highlight the specific assumptions of GR to see its difference from other alternative gravitation theories.

First of all, GR is a metric theory: There exists a metric, $g_{\mu\nu}$, and $\nabla_\mu T^{\mu\nu} = 0$ where ∇_μ is the covariant derivative defined with the Christoffel connection of this metric and $T^{\mu\nu}$ is the energy-momentum tensor of the matter fields. Actually the geodesic motion can be derived from this second condition of the metric postulate. Theories satisfying these postulates are called as metric theories. However, even if the metric postulates are adopted, GR is not the only theory that satisfies them and there are extra restrictions that should be imposed in order to be led uniquely to this theory.

The assumptions made to reach GR are

- $\Gamma^\rho{}_{\mu\nu} = \Gamma^\rho{}_{\nu\mu}$, that is the connection is symmetric (or spacetime is torsionless),
- $\nabla_\lambda g_{\mu\nu} = 0$, that is, the connection is a metric one,
- No fields other than the metric mediate the gravitational interaction,
- The field equations should be second order partial differential equations,
- The field equations should be covariant (or the action should be diffeomorphism invariant).

2.4 Einstein's Equation from an Action

GR is a classical theory and therefore no reference to an action is really physically required; one could just stay with the field equations. However the Lagrangian formulation of a theory has some advantages. Other than its elegance it has two important reasons to develop a Lagrangian formulation for GR: One is that at the quantum level the action is indeed acquires a physical meaning and one expects that a more fundamental theory of gravity will give an affective low energy gravitational action at a suitable limit; the second one is that it is much easier to compare alternative gravity theories through their actions rather than by their field equations. For these reasons we will follow the Lagrangian formulation.

Since one of the assumptions of GR is "no fields other than the metric mediate the gravitational interaction", we expect that the general structure of the action should include a Lagrangian for gravity which depends only on the metric and a Lagrangian for the matter which depends on the matter fields. Furthermore, for the matter Lagrangian we have one basic requirement: Its variation with respect to the metric must give the energy-momentum tensor, since it is this quantity on the right-hand side of the Einstein's equation. Therefore, we define the equation (**2.7**) as

$$T_{\mu\nu}{}^{M} \equiv -\frac{2}{\sqrt{-g}}\frac{\delta S_M}{\delta g^{\mu\nu}} \tag{2.7}$$

where

$$S_M = \int d^4x\sqrt{-g}\mathscr{L}_M(g_{\mu\nu},\psi) \tag{2.8}$$

is the matter action, ψ collectively denoting the matter fields.

The gravitational action is

$$S_{EH} = \frac{1}{2\kappa}\int d^4x\sqrt{-g}R \tag{2.9}$$

where the subscript EH stands for Einstein-Hilbert and this action has the name Einstein-Hilbert action. The factor, R in this equation is the Ricci scalar and the constant factor in front of the integral is

$$\kappa = \frac{8\pi G}{c^4} \tag{2.10}$$

and it is chosen with some anticipation that the field equations at some appropriate limit will give us the equations of Newtonian gravity.

Then, finally, the variation of the action $S = S_{EH} + S_M$ with respect to the (inverse) metric $g^{\mu\nu}$ gives the Einstein' s Equation (**A.7**).

3. COSMOLOGY

3.1 The Basics of Cosmology

The GR is the most convenient theory to study the evolution of the Universe at large/cosmological scales; that is, the Einstein equation, (**2.4**), is the equation that we use to study the evolution of the Universe. But we have two more ingredients to explicitly study the large scale structure of the Universe: The metric tensor that describes the Universe contributing to the left-hand side of the Einstein equation and the matter content of the universe that contributes to the right-hand side. To determine the metric of the universe at cosmological scales we make a very important assumption which simplifies many of the complexities which we encounter at smaller scales (for example, at the scale of our solar system): The main assumption of cosmology is that the Universe is homogeneous and isotropic on large scales (at scales > 100Mpc). With this assumption the metric takes the form (**3.1**):

$$ds^2 = -dt^2 + a^2(t)[\frac{dr^2}{1-kr^2} + r^2 d\theta^2 + r^2 sin^2\theta d\phi^2] \tag{3.1}$$

known as the Friedmann-Robertson-Walker (FRW) metric. The constant k is related to the spatial curvature of the spacetime and can take values $1, 0, -1$ depending on whether the Universe is spatially closed, flat or open, respectively. The time dependent function $a(t)$ is called the scale factor; it depends only on time because of the assumptions of homogeneity and isotropy. To describe the energy-matter components of the Universe, we make another assumption about the matter content of the universe by taking the energy-momentum tensor in the form of perfect fluid as (**3.2**):

$$T^{\mu\nu} = (\rho + p)u^{\mu}u^{\nu} + pg^{\mu\nu} \tag{3.2}$$

where $u^{\mu} = (1, 0, 0, 0)$ denotes the 4-velocity of the observer comoving with the fluid and ρ and p are the proper energy density and pressure of the fluid. Actually the assumption of the homogeneity and isotropy of the Universe require the components of the energy-momentum tensor everywhere to take the form $T^{00} = \rho(t)$, $T^{0i} = 0$, and $T^{ij} \sim a^{-2}(t)p(t)$, ($i, j$ denote the spatial coordinate components.).

Furthermore, in description of the matter content of the Universe we also assume that matter/energy content satisfies an equation of state of the form $p = \omega\rho$, where ω is called as the equation of state parameter. This type of parametrization is also useful in classifying the matter type, beyond its simple form; for example, the cold matter/dust is characterized by $\omega = 0$, the hot matter/radiation by $\omega = 1/3$, and the vacuum energy by $\omega = -1$. The space components of the conservation law of energy-momentum, $\bigtriangledown_\mu T^{\mu i} = 0$, give us a useful relation (**3.3**):

$$\dot{\rho} + 3\frac{\dot{a}}{a}(p+\rho) = 0 \qquad (\textbf{3.3})$$

this equation combined with the equation of state gives the equation (**3.4**):

$$\rho \propto a^{-3-3\omega} \qquad (\textbf{3.4})$$

Substituting the FRW metric and energy-momentum tensor into the Einstein equation, we get the Friedmann equations (**3.5**) and (**3.6**):

$$(\frac{\dot{a}}{a})^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2} \qquad (\textbf{3.5})$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p) \qquad (\textbf{3.6})$$

Note that the Friedmann equations, together with the equation (**3.4**), are equations for the scale factor $a(t)$. The equation (**3.5**) is an equation for $\dot{a}$, telling us about the velocity of the expansion or contraction of the Universe.

3.2 Acceleration

The equation (**3.6**) involves $\ddot{a}$, telling us about the acceleration of the expansion or the contraction.

Notice that k does not appear in this equation, i.e., the acceleration does not depend on the characteristics of the spatial curvature.

Why is the acceleration of the Universe is interesting? The equation (**3.6**) implies by simple intuition that gravity is always an attractive force if caused by ordinary matter/energy; because $\rho + 3p \geqslant 0$ for ordinary matter/energy and this in turn implies that $\ddot{a} \leqslant 0$, that is, the expansion will always be slowed by gravity. But what we observe today is not consistent with this expectation [3]. If the expansion of the Universe is accelerating today, instead of the naive expectation of the deceleration, it implies that, in the framework of the GR, there must be an exotic matter/energy with equation of state parameter $\omega < -1/3$ and the contribution of this new type of matter/energy must be greater than the ordinary matter/energy. To be more precise, the accelerated expansion is a phase in which (**3.7**)

$$\ddot{a} \geqslant 0 \tag{3.7}$$

It does not seem possible for any kind of baryonic matter to satisfy the equation (**3.7**), which directly implies that a period of accelerated expansion in the Universe evolution can only be achieved within the framework of GR if some new form of matter field with special characteristics is introduced.

Actually, we believe that there was another accelerated phase of the Universe, called inflation [12, 15, 16], which is necessary especially to cure some of the conceptual shortcomings of the Standard Big Bang Model of the Universe. These problems are the so-called horizon problem, flatness problem, and the monopole problem. All these problems are solved in a natural way by a rapid expansion of the universe in a short period of time, approximately 10^{-35}s after the Big Bang. There are some indirect observations supporting such a scenario.

The acceleration that we are mostly interested in this thesis is not the inflation happened early in the history of the Universe but the current accelerated phase of the Universe which is indeed unexpected in the framework of GR and flat, matter-dominated Universe model. First observational data which indicate an accelerating universe were published in 1998 by "High-z Supernova Team" [1,3] an international group of astronomers and in 1999 by "Supernova Cosmology Project" (SCP) [1,2] at Lawrence Berkeley National Laboratory. Using type Ia supernovae as the standard candles, they basically plotted observed brightness of different SNe Ia against their redshifts. What they observed basically was that the distant supernovae were fainter than expected!

3.3 The Current Status of the Universe

The supernovae data is not enough alone to conclude that the Universe is expanding in an accelerated manner. The current values of critical cosmological parameters are obtained through complicated, mostly indirect, and intermingling observations. Therefore, to state conclusively about the current acceleration of the Universe, we need some other parameters provided by some other observations.

In order to mention properly about these observations and the parameters obtained through them, it is better to make some definitions. One of the most important of all these parameters is the Hubble parameter (**3.8**) defined through the relation

$$H(t) = \frac{\dot{a}}{a} \tag{3.8}$$

The value of it at any instant is called the Hubble Constant; it is being called as a constant because its value at any instant is the same everywhere in the Universe. Another important parameter is the critical density of the Universe (**3.9**):

$$\rho_C = \frac{3H^2}{8\pi G} \tag{3.9}$$

which is important in characterizing the spatial curvature of the Universe. The parameter which measures the contribution of each matter/energy component of the Universe to the total density of the Universe is the density parameter defined as the equation (**3.10**):

$$\Omega = \frac{\rho}{\rho_C} \tag{3.10}$$

which is considered for each matter/energy species. The density parameter for the curvature is also defined as (**3.11**)

$$\Omega_k = -\frac{k}{a^2H^2} \tag{3.11}$$

The first Friedmann equation, (**3.5**), implies that

$$\Omega + \Omega_k = 1 \tag{3.12}$$

The experiments in order to infer whether a period of accelerated expansion has occurred are the followings: The most recent dataset is that of the Wilkinson

Microwave Anisotropy Probe (WMAP) observations [35] combined with data from supernovae and galaxy surveys in many cases. The WMAP data indicates that

$$\Omega_k = -0.015 \pm ... \qquad (3.13)$$

that is, Ω is very close to unity and the Universe appears to be spatially flat: The most important observation supporting the supernovae data is the spatial flatness of the Universe, obtained from the observations of Cosmic Microwave Background Radiation (CMBR).

However, observational data hold more surprises. Even though Ω is measured to be very close to unity, the contribution of matter to it, Ω_M, is only order of 24%. Therefore there seems to be some unknown form of energy density of the universe, called dark energy. What is more, observations indicate that, if one tries to model dark energy as a perfect fluid with an equation of state of the form $p = \omega\rho$ then $\omega_{DE} = -1,06...$. Since it is the dominant energy condition today, this implies that the universe should be undergoing an accelerated expansion currently as well. This is also what was found earlier using supernovae surveys [4, 12]. This was the first evidence for cosmic acceleration, and have arguably provided the most effective restrictions on the dark energy equation of state parameter.

The observations do not seem to stop here: As mentioned in the previous paragraph, Ω_M accounts for approximately 24% of the energy density of the Universe. However, one also has to ask how much of this amount is actually ordinary baryonic matter. Observations indicate that the contribution of baryons, Ω_B, is approximately 4%, leaving 20% of the total energy content of the universe and some 83% of the matter content to be accounted for by some unknown unobserved form of matter, called dark matter. Differently from dark energy, dark matter has the gravitational characteristics of ordinary matter (hence the name). However, it is not directly observed since it appears to interact very weakly if at all. Currently the dark matter is mostly treated as being cold and not baryonic, since these characteristics appear to be in good accordance with the data.

The current phenomenological model of the universe explaining all the data mentioned above is called as the ΛCDM (Λ Cold Dark Matter) model, which is sometimes also called the Concordance model. Λ in this model is the cosmological constant included to explain the current accelerated expansion of the universe. The parameters of this model are as follows:

Table 3.1: Cosmological Parameters [33, 34].

Parameter	WMAP7 alone	WMAP7 + BAO + H_0
Hubble parameter: h	0.704 ± 0.025	0.702 ± 0.014
Cold dark matter density: $\Omega_{CDM} = \frac{\rho_{CDM}}{\rho_C}$		
Value: $\Omega_{CDM}h^2$	0.112 ± 0.006	0.113 ± 0.004
Baryon density: $\Omega_B = \frac{\rho_B}{\rho_C}$		
Value: $\Omega_B h^2$	0.0225 ± 0.0006	0.0226 ± 0.0005
Cosmological constant: Ω_Λ		
Value: Ω_Λ	0.73 ± 0.03	0.725 ± 0.016
Radiation density: Ω_R		
Value: $\Omega_R h^2$	0.134 ± 0.006	0.135 ± 0.004
Dark energy		
equation of state parameter: ω		$\sharp -0.98 \pm 0.05$

$\sharp$ Extended model parameter [34].

Table 3.2: Astrophysical Constants [34].

Quantity	Symbol, equation	Value
Speed of light	c	$299792458 ms^{-1}$
Newtonian		
gravitational constant	G_N	$6.6738(8)x10^{-11} m^3 kg^{-1} s^{-2}$
Planck mass	$\sqrt{\hbar c/G_N}$	$1.22093(7)x10^{19} GeV/c^2$
		$= 2.17651(13)x10^{-8} kg$
Planck length	$\sqrt{\hbar G_N/c^3}$	$1.61620(10)x10^{-35} m$
Standard		
gravitational acceleration	g_N	$9.80665 ms^{-2} \approx \pi^2$
Critical		
density of the Universe	$\rho_C = 3H_0^2/8\pi G_N$	$2.77536627x10^{11} h^2 M_\odot Mpc^{-3}$
Currently CMB temperature	T_0	$2.7255(6) K$

4. SCALAR-TENSOR THEORY

4.1 Reasons of Renewed Interest on Scalar-Tensor Theories

The scalar field is the nature's the simplest field and before the Einstein's GR, G. Nordström proposed a scalar theory of gravity in 1912. Today, the reason for still considering scalar fields in gravity theories is not this simplestness but some more technical reasons: Considering the gravitational coupling constant as time dependent, the gravity sector of low energy effective action of string theories, and, currently, the accelerated expansion of the universe. A gravitational scalar field is an essential feature of supergravity, superstring, and M-theories.

In 1961, Brans and Dicke suggested a new theory alternative to GR [17]. Their theory consists of a scalar field and this scalar field together with the metric tensor describe the gravity. The main motivation behind this new theory is to incorporate the Mach' s principle which was reformulated by Dicke in a clearer form as – the gravitational constant should be a function of the mass distribution of the Universe– and they thought that a time dependent gravitational constant would satisfy the principle. The scalar field that they introduced would play the role of the inverse gravitational coupling.

Currently, the astonishing discovery of the accelerated expansion of the universe stimulated a renewed interest on theories including scalar fields. These scalar fields can be considered as either an exotic form of matter/energy in Einstein's GR or a component governing the gravitational effects in the theory in addition to the metric tensor.

4.1.1 Brans-Dicke theory

Brans-Dicke theory is the prototype for gravitational theories alternative to GR. The original motivation for Brans-Dicke theory was to formulate a new theory which includes the Mach's principle which is not (explicitly) included in GR. The action of the theory is given by (**4.1**):

$$S_{BD} = \int d^4x\sqrt{-g}\{\phi_{BD}R - \frac{\omega}{\phi_{BD}}g^{\mu\nu}\nabla_\mu\phi_{BD}\nabla_\nu\phi_{BD} - V(\phi_{BD})\} + S_M \qquad \textbf{(4.1)}$$

where

$$S_M = \int d^4x\sqrt{-g}\mathscr{L}_M \qquad \textbf{(4.2)}$$

is the action for all matter fields in the theory and ω is a dimensionless parameter and $\phi_{BD} \equiv \frac{1}{16\pi}\phi$. One should note that matter is not directly coupled to the scalar field ϕ, that is, $\mathscr{L}_M$ is independent of ϕ. This is pharesed technically as the "minimal coupling to matter"; but the scalar field is non-minimally coupled to the gravity. The effects of gravity in Brans-Dicke theory are described by the metric tensor and the scalar field.

The field equations obtained by varying the action with respect to the metric tensor and the scalar field respectively are the equations (**4.3**) and (**4.4**):

$$G_{\mu\nu}{}^{BD} = \frac{8\pi}{\phi}T_{\mu\nu}{}^{M}$$

$$+\frac{\omega}{\phi^2}(\nabla_\mu\phi\nabla_\nu\phi - \frac{1}{2}g_{\mu\nu}\nabla^\alpha\phi\nabla_\alpha\phi) + \frac{1}{\phi}(\nabla_\mu\nabla_\nu\phi - g_{\mu\nu}\Box\phi) - \frac{V(\phi)}{2\phi}g_{\mu\nu} \qquad \textbf{(4.3)}$$

$$\Box\phi = \frac{1}{2+3\omega}(8\pi T^M + \phi V(\phi)' - 2V(\phi)) \qquad \textbf{(4.4)}$$

Taking the trace of the equation following from the scalar field equation of motion is

$$R^{BD} = -\frac{8\pi}{\phi}T^M + \frac{\omega}{\phi^2}\nabla^\alpha\phi\nabla_\alpha\phi + \frac{3\Box\phi}{\phi} + \frac{2V(\phi)}{\phi} \qquad \textbf{(4.5)}$$

The form of the action (**4.1**) or of the field equation (**4.3**) suggest that the Brans-Dicke field, ϕ, plays the role of the inverse gravitational coupling

$$G_{eff} = \frac{1}{\phi} \qquad \textbf{(4.6)}$$

The parameter ω is the only free parameter of the theory. From a theoretical point of view, a value of ω of order unity would be natural, and it does appear in the low energy limit of string theories. However, values of ω of this order are excluded by the available tests of gravitational theories in the weak field limit, for a massless and for a light scalar field ϕ. A light scalar is one that has a range larger than the size of the Solar System or of the laboratory used to test gravity. Solar system experiments constrain the value of this free parameter as [18]

$$\omega > 40000 \tag{4.7}$$

4.2 Scalar-Tensor Theories

Scalar-Tensor theory of gravity is the generalization of Brans-Dicke theory. General action form is given as (**4.8**):

$$S_{ST} = \int d^4x\sqrt{-g}\{f(\phi)R - \frac{1}{2}h(\phi)g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi - U(\phi)\} + S_M \tag{4.8}$$

where

$$S_M = \int d^4x\sqrt{-g}\mathscr{L}_M(g_{\mu\nu},\Psi) \tag{4.9}$$

corresponds to matter action of the scalar-tensor theories. $\phi = \phi(x,t)$ is scalar field, $f(\phi)$, $h(\phi)$ and $U(\phi)$ are the functions to specify the form of the scalar-tensor theory.

For the ST theory, the field equations are (**4.10**) and (**4.11**):

$$G_{\mu\nu}{}^{ST} = \frac{1}{f(\phi)}(\frac{1}{2}T_{\mu\nu}{}^{\phi} + \frac{1}{2}T_{\mu\nu}{}^{M} + \nabla_\mu\nabla_\nu f(\phi) - g_{\mu\nu}\Box f(\phi)) \tag{4.10}$$

where $T_{\mu\nu}{}^{\phi} = -g_{\mu\nu}(\frac{1}{2}h(\phi)(\nabla\phi)^2 - U(\phi)) + h(\phi)\nabla_\mu\phi\nabla_\nu\phi$.

and

$$h(\phi)\Box\phi + \frac{1}{2}h(\phi)^{'}g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi + U(\phi)^{'} + f(\phi)^{'}R = 0 \tag{4.11}$$

Eventually, it is mentioned about the relation between ST theory and BD theory. By choosing these functions as $f(\phi) = \frac{\phi}{16\pi}$, $h(\phi) = \frac{\omega}{8\pi\phi}$ and $U(\phi) = V(\phi)$, where ω is a coupling constant, the ST action turns out to be the generalized BD action form.

4.3 Conformal Transformation

In 1919, the conformal transformation (CT) was firstly asserted by Weyl, who aimed to unify gravitation and electromagnetism formulation. In 1973, Weyl' s theory was reformulated and was used by Dirac.

Many high energy physics theories and many classical theories are now formulated by using a CT mapping the Jordan Conformal Frame (JF) to the Einstein Conformal

Frame[1] (EF). The CT techniques are used to generate the solution techniques if solution is known in one conformal frame, but not in another. In the literature, the use of CT techniques has become widespread

- on gravitational physics;
  - on alternative gravitational theories to GR: BD theory, generalized ST theories, nonlinear theory of gravity, Kaluze-Klein theories,
  - on studies of the scalar fields nonminimally coupled to gravity,
  - on the unified theories in multidimensional spaces.
- on cosmology.

By performing a CT the spacetime metric and by redefining the scalar field, new dynamical variables, $\tilde{g}_{\mu\nu}$ and $\tilde{\varphi}$, are obtained. The point dependent rescaled new set of dynamical variables, $(\tilde{g}_{\mu\nu}, \tilde{\varphi})$, is obtained in the Einstein Conformal Frame, as opposed to $(g_{\mu\nu}, \varphi)$ which constitutes the Jordan Conformal Frame, with given a spacetime $(M, g_{\mu\nu})$, where M is a smooth manifold of the dimension $n \geq 2$ and $g_{\mu\nu}$ is a Lorentzian or Riemannian metric.

For the $g_{\mu\nu}$, g, $\Gamma^{\rho}{}_{\mu\nu}$, $R^{\rho}{}_{\mu\lambda\nu}$, and R the CTs from the JF to the EF are shown as the equations (**4.12**), (**4.13**), (**4.14**), (**4.15**), (**4.16**) and (**4.17**):

$$g_{\mu\nu} \underset{CT}{\rightarrow} \tilde{g}_{\mu\nu} = \Omega^2 g_{\mu\nu} \tag{4.12}$$

$$g \underset{CT}{\rightarrow} \tilde{g} = \Omega^{2n} g \tag{4.13}$$

$$\Gamma^{\rho}{}_{\mu\nu} \underset{CT}{\rightarrow} \tilde{\Gamma}^{\rho}{}_{\mu\nu} = \Gamma^{\rho}{}_{\mu\nu} + \Omega^{-1}(\delta^{\rho}_{\mu}\nabla_{\nu}\Omega + \delta^{\rho}_{\nu}\nabla_{\mu}\Omega - g_{\mu\nu}\nabla^{\rho}\Omega) \tag{4.14}$$

$$\begin{aligned} R^{\rho}{}_{\mu\lambda\nu} \underset{CT}{\rightarrow} \tilde{R}^{\rho}{}_{\mu\lambda\nu} &= R^{\rho}{}_{\mu\lambda\nu} + 2g_{\mu[\lambda}\nabla_{\nu]}\nabla^{\rho}(\ln\Omega) \\ &-2\delta^{\rho}_{[\lambda}\nabla_{\nu]}\nabla_{\mu}(\ln\Omega) - 2g_{\mu[\lambda}\nabla_{\nu]}(\ln\Omega)\nabla^{\rho}(\ln\Omega) \\ &+2\delta^{\rho}_{[\lambda}\nabla_{\nu]}(\ln\Omega)\nabla_{\mu}(\ln\Omega) - 2g_{\mu[\nu}\delta^{\rho}_{\lambda]}\nabla_{\alpha}(\ln\Omega)\nabla^{\alpha}(\ln\Omega) \end{aligned} \tag{4.15}$$

$$\begin{aligned} R^{\rho}{}_{\mu\lambda\nu} \underset{CT}{\rightarrow} \tilde{R}^{\rho}{}_{\mu\lambda\nu} \underset{\lambda\rightarrow\rho}{\rightarrow} \tilde{R}_{\mu\nu} &= R_{\mu\nu} + (2-n)(\nabla_{\mu}\nabla_{\nu}(\ln\Omega) - \nabla_{\mu}(\ln\Omega)\nabla_{\nu}(\ln\Omega) \\ &+ g_{\mu\nu}(\nabla(\ln\Omega))^2) - g_{\mu\nu}\nabla^2(\ln\Omega) \end{aligned} \tag{4.16}$$

[1] as the terminological, frame denotes a set of dynamical variables of the theory.

$$\tilde{R} \equiv \tilde{g}^{\mu\nu}\tilde{R}_{\mu\nu} = \Omega^{-2}[R + (1-n)(2\nabla^2(\ln\Omega) - (2-n)(\nabla(\ln\Omega))^2)] \qquad \textbf{(4.17)}$$

for $n \geq 2$. Especially, for $n = 4$ spacetime dimensions, the transformation of the Ricci scalar is written as the equation (**4.18**)

$$\tilde{R} = \Omega^{-2}[R - \frac{6\Box\Omega}{\Omega}],$$

$$\tilde{R} = \Omega^{-2}[R - \frac{12\Box(\sqrt{\Omega})}{(\sqrt{\Omega})} + \frac{3g^{\mu\nu}\nabla_\mu\Omega\nabla_\nu\Omega}{\Omega^2}] \qquad \textbf{(4.18)}$$

Here $\Omega = \Omega(x)$ is called the conformal or Weyl factor, which is known as a nowhere vanishing, regular function.

The general CTs are not diffeomorphism of the manifold, M, and rescaled metric is not simply the metric and is different from $g_{\mu\nu}$. It describes the different gravitational fields and different physics. Nevertheless, in the conformal isometry, which originates from diffeomorphism, the metric is left unchanged even if metric coordinate representation of the metric varies.

Since only when a physical frame is uniquely determined in the theory and its observable predictions are meaningful, to make a physically comparison between the conformally transformed frames to each other is important. The determination of which conformal frame is the physical[2] one is still a problem. In order to provide the physical equivalence between two conformal frames, the advocated idea is that the units of length, mass, and time(they must scale with appropriate powers of the BD scalar ϕ) are varying in the EF [17].

To study the mathematically more convenient, when the theories were formulated for two conformal frames, firstly, mathematical equivalence had been established between these frames. Because the space solutions of the theory in one frame are isomorphic to the space solutions in the conformally related frame [19]. The mathematical equivalence between the two frames a prior implies nothing about their physical equivalence.

[2]as the term physical theory denotes one that is theoretically consistent and predicts the values of some observables that can, at least in principle, be measured in experiments performed in four macroscopic spacetime dimensions.

For the alternative theories of gravity, the JF formulation is unphysical. Since the Einstein gravity is essentially the only viable classical theory of gravity, the EF formulation is the only possible one for a classical theory. This statement is strictly correct only if the purely gravitational part of the action (without matter) is considered: In fact, when matter is included into the action, in general it exhibits an anomalous coupling to the scalar field which does not occur in GR. The EF is the physical one (and the JF and all the other conformal frames are unphysical) for the following classes of theories:

- (generalized) ST theories of gravity are described by the general form action as

$$S = \int d^4x\sqrt{-g}\left\{f(\phi)R - \frac{\omega(\phi)}{\phi}g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi + \Lambda(\phi)\right\} + S_M$$

  which includes Brans-Dicke theory as a special case.

- classical Kaluza-Klein theories,

- nonlinear theories of gravity whose gravitational part is described by the Lagrangian density, $\mathscr{L}_G = \sqrt{-g}f(R)$.

Furthermore, for the ST theories, the Einstein-Hilbert and the Palatini actions are equivalent in the EF, but not in the JF, the formulation, in the EF, is therefore accepted as the base of relation between its action and GR theory by some authors. Some others find difficulties in quantizing the scalar field fluctuations in the linear approximation in the JF, but not in the EF (quantization and the conformal transformation do not commute.). In terms of the compactification of the extra dimensions in higher dimensional theories, others claim that the EF is forced upon us [19].

In addition to writing mathematical and physical equivalence between the frames, to write the conformal invariance (CI) theories is significant, quantum field theory in curved spaces (Birrell and Davis, 1982), statical mechanics and string theories (Dita and Georgescu, 1989) can be given as the examples [19]. Since the laws of physics must be invariant under a transformation of units, conformal invariance must also be followed [17].

4.3.1 The conformal transformation of Brans-Dicke theory

To obtain conformally transformed BD action to EF, the dynamics of the theory should be rescaled. By determining new scaling factor as $\Omega^2 \equiv G\phi$ and by taking BD scalar field redefinition as $\tilde{\phi}(\phi) \equiv \sqrt{\frac{2\omega+3}{16\pi G}} \ln(\frac{\phi}{\phi_0})$, where $\phi \neq 0$, $\omega > \frac{-3}{2}$ and $\phi_0^{-1} = G$, in the EF, the transformed BD action is finally obtained as (**4.19**):

$$\tilde{S}_{BD} = \int d^4x\sqrt{-\tilde{g}}\{\frac{\tilde{R}}{16\pi G} - \frac{1}{2}\tilde{g}^{\mu\nu}\tilde{\nabla}_\mu\tilde{\phi}\tilde{\nabla}_\nu\tilde{\phi}\} + \tilde{S}_M \tag{4.19}$$

where $\tilde{\nabla}_\mu$ is the covariant derivative operator of the rescaled metric $\tilde{g}_{\mu\nu}$. In the EF, matter part of action becomes as (**4.20**):

$$\tilde{S}_M = \int d^4x\sqrt{-\tilde{g}}e^{\{-8\sqrt{\frac{\pi G}{2\omega+3}}\tilde{\phi}\}}\mathscr{L}_M(\tilde{g}) \tag{4.20}$$

The gravitational part of the action contains only Einstein gravity, but a free scalar field acting as a source of gravitational always appears. In the JF, The gravitational field is described by both the metric tensor $g_{\mu\nu}$ and the BD scalar, ϕ. Nonetheless, in the EF, gravitational field is only described by the metric tensor, $\tilde{g}_{\mu\nu}$; but the scalar field, $\tilde{\phi}$, which is now a form of matter, remind its fundamental role in the old frame. Also, the rest of the matter part the Lagrangian in th EF is multiplied by an exponential factor, an anomalous coupling to the scalar field, $\tilde{\phi}$, is thus displayed.

In the EF, field equation and wave equation for conformally transformed BD theory are given by the equations (**4.21**) and (**4.22**):

$$\tilde{G}_{\mu\nu}{}^{BD} = \tilde{T}_{\mu\nu}{}^{\phi} + \tilde{T}_{\mu\nu}{}^{M} \tag{4.21}$$

where $\tilde{T}_{\mu\nu}{}^{\phi} = \tilde{\nabla}_\mu\phi\tilde{\nabla}_\nu\phi - \frac{1}{2}\tilde{g}_{\mu\nu}(\tilde{\nabla}\phi)^2$.

$$\Box\phi = \zeta^2\tilde{T}^M \tag{4.22}$$

where $\zeta^2 = M_{Pl}{}^{-2}\frac{1}{6+\varepsilon\xi}$; $\varepsilon\xi = 4\omega$ and $M_{Pl}{}^{-2} = 8\pi G$; then, $\zeta^2 = \frac{4}{3+2\omega}$.

With the help of BD theory and by writing geodesic equation in two frames, the sharp difference between the EF and the JF can be deduced. While the geodesics equation is generated in the EF, the expression of fifth force occurs.

By using the conformally transformed to the EF metric expression, $\tilde{g}_{\mu\nu}$, and by using the BD conformal factor, $\Omega = \sqrt{G\phi}$. In the EF, the conservation of $\tilde{T}^{\mu\nu}{}_M$ is found as (**4.23**):

$$\tilde{\nabla}_\mu \tilde{T}^{\mu\nu} = -\sqrt{\frac{4\pi G}{2\omega+3}}\tilde{T}\tilde{\nabla}^\nu \tilde{\phi} \tag{4.23}$$

By considering a dust fluid with $p = 0$, the $\tilde{T}_{\mu\nu} = \tilde{\rho}\tilde{u}_\mu\tilde{u}_\nu$ and $\tilde{T} = -\tilde{\rho}$ are obtained and an affine parameter λ along the fluid worldlines with tangent $\tilde{u}^\mu$ is introduced. By substituting these into the equation (**4.23**), the following equation is found as

$$\tilde{u}_\mu(\frac{d\tilde{\rho}}{d\lambda} + \tilde{\rho}\tilde{\nabla}^\alpha \tilde{u}_\alpha) + \tilde{\rho}(\frac{d\tilde{u}_\mu}{d\lambda} - \sqrt{\frac{4\pi G}{2\omega+3}}\tilde{\rho}\tilde{\nabla}_\mu\tilde{\phi}) = 0$$

From above equation, the following expressions are obtained as $\frac{d\tilde{\rho}}{d\lambda} + \tilde{\rho}\tilde{\nabla}^\alpha \tilde{u}_\alpha = 0$ and $\frac{d\tilde{u}_\mu}{d\lambda} - \sqrt{\frac{4\pi G}{2\omega+3}}\tilde{\rho}\tilde{\nabla}_\mu\tilde{\phi} = 0$.

The geodesics equation is then modified according to (**4.24**):

$$\frac{d^2x^\mu}{d\lambda^2} + \tilde{\Gamma}^\mu{}_{\nu\alpha}\frac{dx^\nu}{d\lambda}\frac{dx^\alpha}{d\lambda} = \sqrt{\frac{4\pi G}{2\omega+3}}\tilde{\nabla}_\mu\tilde{\phi} \tag{4.24}$$

in the EF. There is a fifth force proportional to the gradient of $\tilde{\phi}$, the couples in the same way to any massive test particle. Due to this coupling, scalar-tensor theories in the EF are nonmetric theories. The universality of free fall[3] is violated by the fifth force correction to the geodesic equation because of the spacetime dependence of $\tilde{\nabla}_\mu\tilde{\phi}$. The interaction range of the fifth force becomes short in the region of high density, which allows the possibility that the models are compatible with local gravity density.

The study of CI property of BD theory helps to solve the problems arising in the $\omega \to \infty$ limit of BD theory, this limit is supposed to give back GR, but it fails to do so when $T = 0$.

[3] all bodies fall with the same acceleration in the gravitational field, independently of their mass and composition.

5. CHAMELEON GRAVITY

5.1 Chameleon

In our universe, the evidence of the existence of the nearly massless scalar field can be detected with the help of the cosmic acceleration model. Dark (vacuum) energy model is accepted as the base and the most general form of the cosmological models. In the literature, the quintessence scalar field model known as the candidate of hypothetical non-quantum scalar fields is the source of the vacuum energy scalar field model. This scalar field slowly rolls and its flat potential slowly rolls down by evolving on the cosmological time scales today and its mass must be order the present Hubble parameter, $H_0 \sim 10^{-42}$ GeV. That is, the existence of the scalar field with a mass of order the present Hubble parameter, H_0, can be given as the evidence for the accelerated expansion of the Universe.

If the scalar field exists and couples to matter, it is assumed that its coupling to matter must be tuned to unnaturally small values in order to satisfy the tests of the EP and this scalar field mediates fifth force which is suppressed in the laboratory and in the interaction between large bodies like planets, but which may be detectable between small test masses in space. For example, if the scalar field couples strongly to matter, it should have been detected by now as a fifth force. Because the coupling between the nearly massless or light scalar field and matter is relevant to fifth force. Not only the EP violation and also fifth force depend on the environment of the scalar field.

To evade the EP and fifth force constraints, the main constraint is given as the mass of the scalar field be sufficiently large on the Earth. This constraint was put forward by J. Khoury and A. Weltman as cosmological evolution of the scalar field [20]. According to their idea, the scalar field can cosmologically evolve while having couplings to matter of order unity, i.e., $\beta_i \sim O(1)$ and the scalar field acquires a mass whose a magnitude depends on the local matter density. Because the cosmological effects of the scalar field do not arise from large violation of the EP in the laboratory, due to very dense environment.

Furthermore, in the high density region (like the Earth), the field has a large mass and the interaction range is typically of order 1 mm on the Earth, the resulting violation of the EP are exponentially suppressed. In the low density region (like interstellar space), the mass of the field can be of the order of the present Hubble parameter, thereby making the fields potential candidates for causing the acceleration of the Universe and also in the terrestrial experiments, the large mass of the field suppresses its interaction with matter.

As final, in much lower density region, local matter density is much smaller than the density of the Earth and in the solar system experiments, the interaction range is of order $10 - 10^4$ AU and in observations of the solar system the action of the field is suppressed by a new mechanism known as a thin shell mechanism. This mechanism is used in order to suppress the detectable effect of any coupling to matter. While sufficiently small objects do not suffer from thin shell suppression, and thus their entire mass contributes to the exterior field, the scalar field mediated force between large objects, like between the Earth and the Sun, is suppressed by thin sell effect, which thereby ensures that solar system experiments of gravity are satisfied.

J. Khoury and A. Weltman have predicted that the near future experiments, which will test gravity in space, should observe corrections of order unity to Newton' s constant compared to its measured value on the Earth, due to fifth force contributions which are crucial in space but exponentially suppressed on the Earth [20].

If there are the inconsistencies between the measurements in the laboratory and the expectations, these can be ignored by the new and surprising outcomes arising from differently behaviors of the scalar field in the regions of high density than in the regions of low density. The field is able to hide so well from our observations and experiments. This scalar field is thus called as a chameleon field. Its physical properties, such as its mass, depend on the environment. Moreover, in region of high density, the chameleon blends with its environment and becomes essentially invisible to search for EP violation and fifth force.

5.2 Chameleon Gravity: Action and Field Equations

The chameleon mechanism is a way to give an effective mass to a light scalar field via field self-interaction and interaction with matter [21]. The chameleon scalar field, ϕ, is conformally coupled to matter. That is, matter experiences a metric which is a conformal transformation of the Einstein metric. Usually, these fields and their potentials stem from the scalar-tensor theories.

In the EF, the action for the chameleon field is given by (**5.1**):

$$S = \int d^4x\sqrt{-g}\frac{R}{2\kappa} + S_\phi + S_M \qquad \textbf{(5.1)}$$

where S_ϕ is the pure scalar field action part of and S_M is the matter action part of the total action, they are given by in the following forms, respectively

$$S_\phi = \int d^4x\sqrt{-g}\{-\frac{1}{2}(\nabla\phi)^2 - V(\phi)\} \qquad \textbf{(5.2)}$$

$$S_M = -\int d^4x\mathscr{L}_M(g_{\mu\nu}, \psi_M) \qquad \textbf{(5.3)}$$

If $(2\kappa)^{-1}$ is rewritten in terms of the reduced Planck mass as $\frac{M_{Pl}^2}{2}$, then the action becomes (**5.4**)

$$S = \int d^4x\sqrt{-g}\{\frac{M_{Pl}^2}{2}R - \frac{1}{2}(\nabla\phi)^2 - V(\phi) - \frac{1}{\sqrt{-g}}\mathscr{L}_M(g_{\mu\nu}, \psi_M)\} \qquad \textbf{(5.4)}$$

where $V(\phi)$ is the potential term, $\mathscr{L}_M(g_{\mu\nu}, \psi_M)$ is the Lagrangian density of the matter.

By taking the small variation of the chameleon field action with respect to ϕ, the field/the wave equation is obtained as

$$\nabla^2\phi = V_{,\phi} + \sum_i \frac{2\beta_i}{M_{Pl}}\frac{1}{\sqrt{-g}}\frac{\delta\mathscr{L}_M}{\delta g_{\mu\nu}}g_{\mu\nu} \qquad \textbf{(5.5)}$$

in the EF.

In order to write the action of the chameleon field in the EF, taken as $\tilde{g}_{\mu\nu}$ metric tensor in the JF is transformed to the EF as (**5.6**):

$$\tilde{g}_{\mu\nu}{}^i \equiv \tilde{g}_{\mu\nu} \to g_{\mu\nu} = \Omega^{-2}\tilde{g}_{\mu\nu} \qquad \textbf{(5.6)}$$

where the conformal factor becomes $\Omega = \Omega(x) = e^{\beta_i \phi / M_{Pl}}$, β_i' s are dimensionless coupling constants, in principle one for each matter species(in the EF and JF, the metric tensor and transformed one will be taken as ${g_{\mu\nu}}^i \equiv g_{\mu\nu}$ and ${\tilde{g}_{\mu\nu}}^i \equiv \tilde{g}_{\mu\nu}$, for simplicity). $c, \hbar = 1$ are taken and the metric signature is accepted as $(-,+,+,+)$. The reduced Planck mass is taken as $M_{Pl} \equiv (8\pi G)^{-1/2} \approx 2.44 \cdot 10^{18}$GeV.

Then, the transformation from the JF to the EF for the determinant of the $g_{\mu\nu}$ is given by the equation (**5.7**):

$$\tilde{g} \rightarrow g = (e^{\beta_i \phi / M_{Pl}})^{-2n} \tilde{g} \tag{5.7}$$

where n is the dimension of the spacetime.

In the EF, the chameleon equation of motion is obtained by defining the trace of the standard form $T^{\mu\nu}$.

$$T \equiv T^{\mu\nu} g_{\mu\nu} = -\rho + 3p = -(1 - 3\omega_i) p \tag{5.8}$$

By using the metric (**3.1**), with the transformed scale factor, a, is obtained as (**5.9**):

$$\tilde{a} \rightarrow a \equiv \Omega^{-1} \tilde{a} \rightarrow a \equiv e^{-\beta_i \phi / M_{Pl}} \tilde{a} \tag{5.9}$$

for each species, i.

In the equation (**5.6**) given metric tensor, $g^{\mu\nu}$, and its components and time-time, and space-space components of the ${\Gamma^\rho}_{\mu\nu}$ are found in terms of the transformed scale factor, a, the equation (**5.9**). By using the conserved energy momentum tensor, the equation of motion is found as (**5.10**):

$$\nabla^2 \phi = V_{,\phi} + \sum_i {\rho}^i e^{(1-3\omega_i)\beta_i \phi / M_{Pl}} (1 - 3\omega_i) \frac{\beta_i}{M_{Pl}} \tag{5.10}$$

Also, the chameleon equation of the motion in the same frame becomes simply the equation (**5.11**):

$$\nabla^2 \phi = V_{eff,\phi} \tag{5.11}$$

In the EF, a single effective potential can be expressed by the equation (**5.12**):

$$V_{eff}(\phi) = V(\phi) + \sum_i {\rho}^i e^{(1-3\omega_i)\beta_i \phi / M_{Pl}} \tag{5.12}$$

where the $V_{eff}(\phi)$ consists of the itself interaction term, $V(\phi)$, plus an exponential term, $\sum_i \rho^i e^{(1-3\omega_i)\beta_i\phi/M_{Pl}}$, due to the conformal coupling. It is realized that really under certain conditions on the self interaction term and the coupling, this effective potential term has a minimum which depends on the local matter density. As a result, "the scalar field acquires a mass which increases with local matter density". For

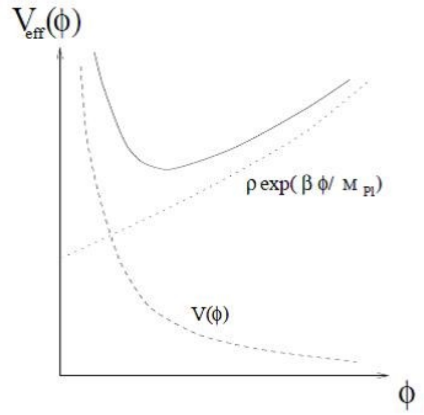

Figure 5.1: The chameleon effective potential, V_{eff}.
V_{eff}(solid curve) is the sum of two contributions: First one the actual potential, $V(\phi)$ (dashed curve), and the other one its coupling to matter density,ρ (dotted curve).

cosmic acceleration, a bare potential $V(\phi)$ is chosen via slow roll mechanism in the quintessence model. As in this model, also in here $V(\phi)$ should be monotonically decreasing function of ϕ. To come up with a mechanism, which makes field acts as a cosmological constant only today, would be a cumbersome, so without loss of generality; namely, the positive direction, the potential $V(\phi)$ is assumed that it has always been rolling down a potential slope in. To obtain the behavior of the chameleon field, it is assumed that $V(\phi)$ is of the runaway form, in the following sense:

- $lim_{\phi\to 0}V(\phi) = \infty$,
- $V(\phi)$ is C^∞, bounded below, and decreasing,
- $V_{,\phi}$ is negative and increasing,
- $V_{,\phi\phi}$ is positive and decreasing.

As will be seen, a chameleon field has mass as $m_\phi = \sqrt{V_{eff,\phi\phi}}$, it is not constant but changes with local matter density. The effective potential,(**5.12**), can be rewritten as the equation (**5.13**):

$$V_{eff}(\phi) = V(\phi) + \rho_M A(\phi) \tag{5.13}$$

where $A(\phi) = \sum_i \rho^i e^{(1-3\omega_i)\beta_i\phi/M_{Pl}}$

If the field, which depends on ρ^i, is taken as a finite value, $\phi = \phi_{min}$, the last obtained equation of motion, (**5.10**), becomes the equation (**5.14**):

$$\nabla^2\phi \equiv V_{eff,\phi}(\phi_{min}) = V_{,\phi}(\phi_{min}) + \sum_i \rho^i e^{(1-3\omega_i)\beta_i\phi_{min}/M_{Pl}}(1-3\omega_i)\frac{\beta_i}{M_{Pl}} \tag{5.14}$$

This equation shows that if any ρ^i increases, ϕ_{min} decreases, since $V_{,\phi}$ and $e^{(1-3\omega_i)\beta_i\phi\ M_{Pl}}$ are an increasing functions of the ϕ.

The associated mass with the scalar field, ϕ, is given by the equation (**5.15**):

$$m^2 \equiv V_{eff,\phi\phi}(\phi) = V_{,\phi\phi} + \sum_i \rho^i e^{(1-3\omega_i)\beta_i\phi/M_{Pl}}(1-3\omega_i)^2\frac{\beta_i^2}{M_{Pl}^2} \tag{5.15}$$

if $\phi = \phi_{min}$, then $m^2 \equiv m^2_{min}$.

The last equation shows $\phi_{min} \ll M_{Pl}$, since $V_{,\phi\phi}$ is a decreasing function of the ϕ, m_{min} is expected to increase.

Also, in the case where all the $(1-3\omega_i)\beta_i$ are equal, it is also easy to show that m_{min} increase with ρ^i, regardless of the magnitude of ϕ_{min}. For instance, if let $(1-3\omega_i)\beta_i = B$ for each i, where $B > 0$ is constant. Then, equation (**5.15**) becomes the following equation (**5.16**):

$$m^2_{min} = V_{,\phi\phi}(\phi_{min}) + \frac{B}{M_{Pl}}\sum_i \rho^i e^{(1-3\omega_i)\beta_i\phi_{min}/M_{Pl}}(1-3\omega_i)^2\frac{\beta_i}{M_{Pl}} \tag{5.16}$$

where

$$m^2_{min} = V_{,\phi\phi}(\phi_{min}) - \frac{\beta_i}{M_{Pl}}V_{,\phi}(\phi_{min}) \tag{5.17}$$

The given conditions below of the $V(\phi)$ is really verified.

- $V_{,\phi\phi}$ is a decreasing function of ϕ and $V_{,\phi}$ is an increasing function of ϕ.
- ϕ_{min} is a decreasing function of each ρ^i and then m_{min} is an increasing function of each ρ^i.

for the quintessence model, two principal potential types are generally used. The first one is the exponential potential allows chameleon behavior while causing cosmic acceleration; but, the other one, which is the inverse power law potential, can not cause acceleration without violating, by several orders of magnitude, existing experimental bounds for laboratory detection. In the limit of $V(\phi)$, the crucial difference between these potential functions is realized.

Ratra-Peebles or the inverse power law potential and the exponential potential, are respectively given as (**5.18**) and (**5.19**):

$$V(\phi) = \frac{M^{4+n}}{\phi^n} \tag{5.18}$$

$$V(\phi) = M^4 exp \frac{M^n}{\phi^n} \tag{5.19}$$

and their behavior of in the limit of $V(\phi)$ are given below

$$\lim_{\phi \to \infty} \frac{M^{4+n}}{\phi^n} = 0 \tag{5.20}$$

$$\lim_{\phi \to \infty} M^4 exp \frac{M^n}{\phi^n} = M^4 \tag{5.21}$$

Present day observations of the the accelerated expansion universe model estimates the value of the constant which differ by several orders of magnitude between the two potential.

5.2.1 The static spherically symmetric solution

In a static chameleon field, the equation for the chameleon force acting on a test mass and an approximate solution will be given. The interaction of the chameleon field with the matter is encapsulated by the conformal coupling of the equations (**5.6**) and (**5.7**), so is the interaction of the spacetime geometry with the matter. Since the matter fields ${\psi_M}^i$ couple to ${g_{\mu\nu}}^i$ instead of to $g_{\mu\nu}$ (in the EF), worldlines of free test particles of species i are the geodesics of ${g_{\mu\nu}}^i$ rather than of $g_{\mu\nu}$ (in the EF).

In the JF, the geodesic equation of the worldline x^μ of the test mass of species i is

$$\ddot{x}^\rho + \tilde{\Gamma}^\rho{}_{\mu\nu} \dot{x}^\mu \dot{x}^\nu = 0 \tag{5.22}$$

where a dot denotes differentiation with respect to proper time, $\tilde{\tau}$.

In the EF, the geodesic equation of the worldline x^{μ} of the test mass of species i becomes

$$\ddot{x}^{\rho} + \Gamma^{\rho}{}_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu} + \frac{\beta_i}{M_{Pl}}(2\phi_{,\mu}\dot{x}^{\mu}\dot{x}^{\nu} + g^{\rho\sigma}\phi_{,\sigma}) = 0 \qquad \textbf{(5.23)}$$

In the left-hand side of this equation the second term is the gravitational term, and the third term is denoted to be chameleon force term. With the help of the equation (**5.6**), $\tilde{g}_{\mu\nu,\sigma} = (\frac{2\beta_i}{M_{Pl}}\phi_{,\sigma}g_{\mu\nu} + g_{\mu\nu,\sigma})e^{2\beta_i\phi/M_{Pl}}$ is used above.

In the nonrelativistic limit, a test mass m of species i in a static chameleon field, ϕ, experiences a force $\vec{F}_{\phi}$ given by $\frac{\vec{F}_{\phi}}{m} = -\frac{\beta_i}{M_{Pl}}\vec{\nabla}\phi$. Thus, ϕ is the potential for the chameleon force. On the body with thin shell, the definition for this force (or fifth force) is given as the same equation. Since all field variations are confined to small region near body surface, this region is called as a thin shell and also the fifth force is produced in a body with thin shell becomes very small $\frac{\Delta R}{R} \ll 1$, where R denotes the radius of the body and ΔR denotes the thickness of shell. The relations between

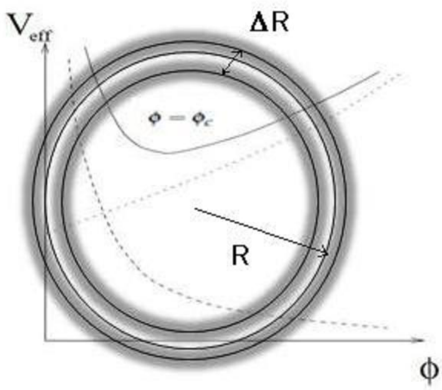

Figure 5.2: Thin shell for large ρ.

the couplings and the variation ranges for the scalar field can be summarized in the following

- For small coupling (long range, small mass) no thin shell effect, no major changes.

- For large coupling (short range, large mass) thin shell effect, major changes; because, the stronger the matter coupling provides the faster adaption and the better hiding from experiments, for scalar fields.

A chameleon field has mass as $m_\phi = \sqrt{V_{eff,\phi\phi}}$, it is not constant but changes with local matter density as was found the equation (**5.15**).

Outside the body, the fifth force is small correction to Newton' s law. Because the chameleon field is basically constant inside a big, spherical body and field varies only inside a thin shell close to outer edge of the body. Therefore, thanks to the existence of a thin shell regime, the local tests can be evaded.

Now, the our aim is to find time independent solutions of the scalar field, $\phi(\vec{x})$, (A) for static spherically symmetric matter distributions $\tilde{\rho}(r)$ of a single pressure free matter species in the weak field limit. We shall work in the weak gravity regime, assuming that the new potential is small everywhere and the backreaction of the energy density in the field, ϕ, is negligible; therefore, the metric tensor can be approximated to the one of the Minkowski spacetime, i.e., Assuming $g_{\mu\nu} \approx \eta_{\mu\nu}$ [10,20,22,23]. Then, the equation $\vec{\nabla}^2\phi = V_{eff,\phi} = V_{,\phi} + \frac{\beta}{M_{Pl}}\rho e^{\beta\phi/M_{Pl}}$ becomes

$$\frac{d^2\phi(r)}{dr^2} + \frac{2}{r}\frac{d\phi(r)}{dr} = V_{,\phi}(\phi(r)) + \frac{\beta}{M_{Pl}}\tilde{\rho}(r)e^{4\beta\phi(r)/M_{Pl}} \quad \textbf{(5.24)}$$

The spherically symmetric solutions to equation (**5.24**) are briefly discussed in the following:

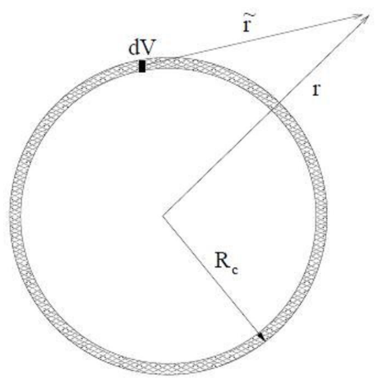

Figure 5.3: Sphere figure.
For spherically symmetric approximation solution.

$$\phi = \begin{cases} \phi \to \phi_\infty & \text{if r > R outside the sphere. Yukawa potential is recognized.} \\ \phi \to \phi_C & \text{if r < R inside the sphere. To recognize the } \phi\text{, approximation methods are used.} \end{cases}$$

i) Outside the sphere; the harmonic oscillator approximation is used.

ii) Inside the sphere;

- ⋆ $\phi \gg \phi_C$; linear approximation and

- ⋆ $\phi \sim \phi_C = \begin{cases} \text{the low contrast solution} & \text{if } R_C = R \\ \text{the thick shell solution} & \text{if } R_C = 0 \\ \text{the thin shell solution} & \text{if } 0 < R_C < R \end{cases}$

are used.

here, in order to divide the interval, $[0,R]$, on which $\phi \sim \phi_C$ as $[0,R_C]$, and on which $\phi \gg \phi_C$ as $[R_C,R]$ R_C is defined. To remain as a undivided interval $[0,R]$, $R_C = 0$ or $R_C = R$ is taken.

5.3 Chameleon Cosmology

For this cosmology, the assumption of a spatial flatness is given. Initially, the first Friedmann cosmological equation for the chameleon field in the EF is derived. By assuming that a flat, homogeneous, isotropic universe, FRW universe, with the metric $g_{\mu\nu} = e^{2\beta_i\phi/M_{Pl}}\mathrm{diag}(-1,a^2,a^2,a^2)$ and ϕ to be spatially homogeneous scalar field, the most general form of the wave equation becomes (**5.25**):

$$\ddot{\phi} + 3H\dot{\phi} = -V_{eff,\phi}(\phi) \qquad \textbf{(5.25)}$$

The equation (**5.11**) gives the right-hand side of the last equation and the wave equation for the FRW universe, $\nabla^2\phi = -(\ddot{\phi} + 3H\dot{\phi})$, equals to the left-hand side of the the last equation.

Suppose that the Universe is composed of ϕ, pressure free matter with density ρ_M coupled to ϕ by a coupling constant β, and radiation with density ρ_R. For the matter of species i conformally coupled to ϕ.

By using the equations (**2.7**), (**5.6**) and (**5.7**), for $n = 4$ (four-dimensions), $T^{\mu\nu} = \tilde{T}^{\mu\nu}e^{6\beta_i\phi/M_{Pl}}$ and $\tilde{T} \equiv \tilde{T}^{\mu\nu}\tilde{g}_{\mu\nu} = \tilde{\rho} + 3\tilde{p}$ are obtained. By using the time-time component of the $\tilde{T}^{\mu\nu}$ and $\tilde{T}$ is given as $\tilde{T}^{00}\tilde{g}_{00} = -\tilde{\rho}$. By substituting these into the time-time component of the T^{00} , (A), in the EF, becomes

$$T^{00} = \rho e^{(1-3\omega_i)\beta_i\phi/M_{Pl}} \qquad \textbf{(5.26)}$$

Then, the first Friedmann Equation becomes the equation (**5.27**):

$$3H^2M_{Pl}^2 = \frac{1}{2}\dot{\phi}^2 + V(\phi) + \rho_M e^{\beta\phi/M_{Pl}} + \rho_R \qquad \textbf{(5.27)}$$

where $\Omega_M \equiv e^{\beta\phi_{min}/M_{Pl}}\frac{\rho_M}{\rho_C}$ or this equation can be rewritten as

$$\rho_c \equiv \frac{1}{2}\dot{\phi}^2 + V(\phi) + \rho_M e^{\beta\phi/M_{Pl}} + \rho_R \qquad \textbf{(5.28)}$$

In addition to the calculated ρ_c, on the cosmological scales, as an example for the large ϕ, the behavior of two potentials can be examined.

Firstly, the Ratra-Peebles potential, (**5.18**), the matter is described by a single species with $\tilde{\rho} = \rho_M$ and $\omega_i \approx 0$.

By using the definition of energy density in the EF,

$$\rho \equiv e^{3(1+\omega_i)\beta_i\phi/M_{Pl}}\tilde{\rho} \qquad \textbf{(5.29)}$$

Putting this into the equation (**5.14**)

$$V_{,\phi}(\phi_{min}) + \sum_i \rho^i e^{(1-3\omega_i)\beta_i\phi_{min}/M_{Pl}}(1-3\omega_i)\frac{\beta_i}{M_{Pl}} = 0 \qquad \textbf{(5.30)}$$

is found. Then,

$$V_{,\phi}(\phi_{min}) + \sum_i (\tilde{\rho}e^{3(1+3\omega_i)\beta_i\phi/M_{Pl}})e^{(1-3\omega_i)\beta_i\phi_{min}/M_{Pl}}(1-3\omega_i)\frac{\beta_i}{M_{Pl}} = 0$$

Assuming that $\phi = \phi_\infty$ today and taking the (**5.18**) potential

$$V_{,\phi}(\phi_\infty) + \sum_i \tilde{\rho}e^{3\beta_i\phi_\infty/M_{Pl}}e^{\beta_i\phi_{min}/M_{Pl}}\frac{\beta_i}{M_{Pl}} = 0$$

$$\Rightarrow -n\frac{M^{4+n}}{\phi_\infty^{n+1}} + \frac{\beta}{M_{Pl}}\tilde{\rho}e^{4\beta\phi_\infty/M_{Pl}} = 0$$

$$\Rightarrow n\frac{M^{4+n}}{\phi_\infty^n}\frac{1}{\phi} = \frac{\beta}{M_{Pl}}\rho_M e^{4\beta\phi_\infty/M_{Pl}} \qquad \textbf{(5.31)}$$

where the definition of the density for the dark energy, ρ_{DE}, $\rho_{DE} = V(\phi_\infty) = \frac{M^{4+n}}{\phi_\infty^n}$, and $\rho\phi_\infty^n = \Omega_M$; then,

$$\Rightarrow n\frac{M^{4+n}}{\phi_\infty^n}\frac{1}{\phi_\infty} = \frac{\beta}{M_{Pl}}\rho_M e^{4\beta\phi_\infty/M_{Pl}}$$

$$\Rightarrow n\rho_{DE}\frac{\phi_\infty^n}{\phi_\infty^{n+1}} = \frac{\beta}{M_{Pl}}\rho_M e^{4\beta\phi_\infty/M_{Pl}} \tag{5.32}$$

by multiplying 4 both of sides this equation and using $\rho\phi_\infty^n \cong \Omega_{DE}$ and ,

$$4n\frac{\Omega_{DE}}{\Omega_M}\frac{M_{Pl}}{4\beta e^{4\beta\phi_{(\infty)}/M_{Pl}}} = \phi_\infty \tag{5.33}$$

In here, the Lambert function (or product-log) can be defined as

$$W(4n(\frac{1-\Omega_M}{\Omega_M})) = 4n\frac{\Omega_{DE}}{\Omega_M}e^{-4\beta\phi_\infty/M_{Pl}} \tag{5.34}$$

Finally,

$$\Rightarrow \phi_\infty = \frac{M_{Pl}}{4\beta}W(\frac{1-\Omega_M}{\Omega_M}) \tag{5.35}$$

is written. By supposing $n = \beta = 1$ and $\Omega_M \approx 0.237$

Second, for the exponential potential, (**5.19**), this potential can be analyzed: If $\phi \gg M$ today, for $\rho_{DE} \approx V(\phi) \approx M^4$, and so $M \approx (\rho_{DE})^{\frac{1}{4}} \approx 2.40 \cdot 10^{-3}$eV.

Finally, it can easily be seen that there is an important difference between two potential:

For large ϕ, $M^4 e^{\frac{M^n}{\phi^n}} \approx M^4 + \frac{M^{4+n}}{\phi^n}$. That is, for $\phi \ll M$, these potentials differ by the constant M^4 from each other. Because this constant comes to dominate in the case of the exponential potential.

5.3.1 The cosmological evolution of the chameleon field

From the chameleon field/wave equation, (**5.11**), the chameleon seeks out the minimum of the V_{eff}. But, now ϕ_{min} changes over time, as ρ_M is diluted by the expansion of the universe.

The most general form of chameleon wave equation, (**5.11**), imply that the characteristic time for the evolution of the $\phi_{min}(t)$ is roughly that of the evolution of

ρ_M, i.e., $\left|\frac{\rho_M}{\dot{\rho}_M}\right| \sim H^{-1}$; where H^{-1} is the Hubble time. ϕ keeps up with an attractor solution ϕ_{min} from the early universe until today. The field $\phi(t)$ can be found by making the damped. For this, the harmonic oscillator approximation (A) is used. The last expression of the wave equation, (**5.25**), is rewritten as

$$\ddot{\phi} + 3H\dot{\phi} = \omega(\phi - \phi_{min}) \tag{5.36}$$

where the ω is given by $\omega^2 = \frac{dV_{eff,\phi}}{d\phi} = V_{eff,\phi\phi}(\phi_{min}) = m^2_{min}$, for $\phi = \phi_{min}$. The characteristic frequency of the oscillator is found as (**5.37**):

$$\omega^2 = m_{min} \tag{5.37}$$

- If the oscillation is underdamped, then its characteristic response time is the m^{-1}_{min}. The condition for this: $2m_{min} > 3H$.
- In order for $\phi(t)$ keep up with $\phi_{min}(t)$, it must have to be $m^{-1}_{min} \ll H^{-1}$. That is, $m_{min} \gg H$, which is consistent with the oscillator being underdamped.

By assuming that β is of order unity and $n \gtrsim \frac{1}{2}$,

i) If $\phi(t)$ is slowly rolling, then $m_{min} \gg H$ from the earliest time, i.e. the end of inflation, until today (with two cases as $\phi_{min} \lesssim M$ and $\phi_{min} \gg M$.) (*Proof 1.*).

ii) If $m_{min} \gg H$, then $\phi(t)$ is slowly rolling. But $m_{min} \gg H$ also gives $\phi(t) \approx \phi_{min}(t)$, and therefore $\phi(t)$ is slowly rolling too. If $\phi(t) \approx \phi_{min}(t)$ at some initial time, then $\phi(t) \approx \phi_{min}(t)$ from that initial time until today. However, in the future, dilution of ρ_M and ρ_R will allow $V(\phi) \approx M^4$ to dominate the energy budget. At this stage, the universe will be expanding exponentially as a de Sitter spacetime, so it will become constant (*Proof 2.*).

In the early universe, the for the chameleon field, assuming that the universe undergoes an initial period of inflation driven by an inflation field with $\omega \approx -1$ which is coupled to ϕ in the same manner as matter. Then, the equation (**5.10**) with $\phi = \phi_{min}$ can be rewritten as

$$V_{eff}(\phi_{min}) = V(\phi_{min}) + \sum_i \rho e^{(1-3\omega_i)\beta_i\phi_{min}/M_{Pl}} \tag{5.38}$$

The energy density of the inflation field is taken as $\rho = \rho_{vac}$, by using $\omega \approx -1$ and $V(\phi_{min}) \approx M^4 e^{M^n/\phi^n}$, $V_{eff}(\phi)$ is rewritten as the following equation.

$$V_{eff}(\phi) \approx M^4 e^{\frac{M^n}{\phi^n}} + \rho_{vac} e^{4\beta_i \phi / M_{Pl}} \quad \textbf{(5.39)}$$

Since ρ_{vac} is roughly constant during inflation, the potential function $V_{eff}(\phi)$ is time dependent, so the equation

$$\ddot{\phi} + 3H\dot{\phi} = -V_{eff,\phi}(\phi) \quad \textbf{(5.40)}$$

The last equation is really the equation of a damped oscillator. The harmonic oscillator approximation is valid if it is assumed that initial conditions must be such $m \gtrsim H$.

At the end of inflation, the inflation decays into mostly radiation plus some matter and small excitations of the field ϕ. Because $1 - 3\omega = 0$ for radiation, it does not couple to ϕ and so ϕ_{min} increases dramatically during reheating.

As the universe expands and cools during the radiation era, matter species decouple from the heat bath one by one. When a species decouples, the perfect fluid approximation $T^{\mu\nu} g_{\mu\nu} \approx -\rho$ for matter becomes invalid for a time because the decoupling particles have relativistic velocities. This provides kicks to the equation (**5.40**), which drive ϕ back towards ϕ_{min}.

When the radiation era ends and the matter era begins, the driving term in the equation (**5.40**) comes to dominate the friction term, so ϕ converges to ϕ_{min} and then follows it.

5.4 Detecting the Chameleon Field

Chameleon scalar fields become a candidate for dark energy if quintessence and gravity experiments can be reconciled. But more concrete expectations about the existence of the chameleon scalar fields in the Nature will be given by outcomes of the near future experiments, which will test gravity in space, including searching for a fifth force between two test body.

The near future experiments are known as the satellite experiments; for examples, the Satellite Energy Exchange (SEE) project [24], STEP [25], Galileo Galilei(GG) [26] and MICROSCOPE [27]. According to scenario of J. Khoury and A. Weltman, SEE project experiment should observe corrections of order unity to Newton' s constant compared to its measured value on the Earth, due to fifth force contributions which are important in space but exponentially suppressed on the Earth [20]. It is expected to test the parameter, η, of the Eöt-Wash experiment, which is briefly known as universalities of free fall in orbit, to very high accuracy with the help of the other satellite experiments; STEP [25], Galileo Galilei (GG) [26] and MICROSCOPE [27] (expected measurements accuracy of 10^{-18}, 10^{-17} and 10^{-15}, respectively.). J. Khoury and A. Weltman predict that violation of the strong EP signals can be observed by these experiments. According to their scenario, the signal will be larger than the ground based Eöt-Wash bound of 10^{-13}, for a wide range of parameters [20].

If an effective Newton' s constant can measure different from that on the Earth by the SEE project, or if an EP violating signals larger than from expected Eöt-Wash experiment are observed by STEP, this will strongly indicate that a mechanism, known as a chameleon, of the form proposed here is realized in the Nature: Otherwise, to make explanation about the differences between measurements in the laboratory and those in orbit is very difficult. These new and surprising outcomes are directly caused from consequence of the different behavior of scalar field in the regions, which have the different densities.

REFERENCES

[1] **Arapoğlu, S., Deliduman, C., and Ekşi, K.Y.** (2011). Constraints on perturbative f(R) gravity via neutron stars, JCAP 1107 (2011) 020 [arXiv:gr-qc/1003.3179v3].

[2] **Perlmutter, S.** et al (1999). [Supernova Cosmology Project Collaboration], Measurements of Omega and Lambda from 42 high redshift supernovae, Astrophys. J. Vol. 517, pp. 565 [arXiv:astro-ph/9812133].

[3] **Riess, A.G.** et al (1998). [Supernova Search Team Collaboration], Observational evidence from supernovae for an accelerating universe and a cosmological constant, Astron. J. Vol. 116, pp. 1009-1038 [arXiv:astro-ph/9805201].

[4] **Riess, A.G.** et al (2004). [Supernova Search Team Collaboration], Type Ia supernova discoveries at z 1 from the Hubble Space Telescope: Evidence for past deceleration and constraints on dark energy evolution, Astrophys. J. Vol. 607, pp. 665 [arXiv:astro-ph/0402512].

[5] **Komatsu, E.** et al (2009). [WMAP Collaboration], Five-Year Wilkinson Microwave Anisotropy Probe (WMAP) Observations: Cosmological Interpretation, Astrophys. J. Suppl. Vol. 180, pp. 330 [arXiv:astro-ph/0803.0547].

[6] **Spergel, D.N.** et al (2003). [WMAP Collaboration], First year Wilkinson Microwave Anisotropy Probe (WMAP) observations: Determination of cosmological parameters, Astrophys. J. Suppl. Vol. 148, pp. 175 [arXiv:astro-ph/0302209].

[7] **Spergel, D.N.** et al (2007). [WMAP Collaboration], Wilkinson Microwave Anisotropy Probe (WMAP) three year results: implications for cosmology, Astrophys. J. Suppl. Vol. 170, pp. 377 [arXiv:astro-ph/0603449].

[8] **Tegmark, M.** et al (2004). [SDSS Collaboration], Cosmological parameters from SDSS and WMAP, Phys. Rev. D 69, 103501 [arXiv:astro-ph/0310723].

[9] **Tegmark, M.** et al (2006). [SDSS Collaboration], Cosmological Constraints from the SDSS Luminous Red Galaxies, Phys. Rev. D 74, 123507 [arXiv:astro-ph/0608632].

[10] **Khoury, J. and Weltman, A.** (2004). Chameleon Fields: Awaiting Surprises for Tests of Gravity in Space, Phys. Rev. Lett. 93, 171104 [arXiv:astro-ph/0309300v3].

[11] **Pound, R.V. and Rebka G.A.** (1960). Phys. Rev. Lett. 4, 337.

[12] **Sotiriou, T.P.** (2007). *Modified Actions for Gravity: Theory and Phenomenology* (PhD. Thesis), SISSA [arXiv:gr-qc/0710.4438].

[13] **Dicke, R.H.** (1960). Am. J. Phys. 28, 344.

[14] **Schiff, L.I.** (1960). Am. J. Phys. 28, 340.

[15] **Guth, A.H.** (1981). Inflationary universe: A possible solution to the horizon and flatness problems, Phys. Rev. D 23, 347.

[16] **Starobinsky, A.A.** (1980). Phys. Lett. B 91, 99.

[17] **Faraoni, V.** (2004). Cosmology In Scalar-Tensor Gravity, Kluwer Academic Publishers, First Edition.

[18] **Fujii, Y. and Maeda, K.** (2004). The Scalar-Tensor Theory Of Gravitation, Cambridge University Press, First Edition.

[19] **Faraoni, V., Gunzig, E., and Nardone, P.** (1998). Conformal Transformations In Classical Gravitational Theories And In Cosmology [arXiv:gr-qc/9811047v1].

[20] **Khoury, J. and Weltman, A.** (2003). Chameleon Cosmology, Phys. Rev. D 69, 044026 [arXiv:astro-ph/0309411v2].

[21] **Brax, Ph., van de Bruck, C., and Davis, A.C.** (2004). Is the Radion a Chameleon?, JCAP 0411 (2004) 004 [arXiv:astro-ph/0408464v1].

[22] **Sivestri, A.** (2011). Scalar radiation from Chameleon-shielded regions, Phys. Rev. Lett. 106 (2011) 251101 [arXiv:astro-ph 1103.4013v2].

[23] **Waterhouse, T.P.** (2006). An Introduction to Chameleon Gravity [arXiv:astro-ph/0611816v1].

[24] **Sanders, A.J.** et al (1999). Meas. Sci. Technol. 10, 514.

[25] **Mester, J.** et al (2001). Class. Quant. Grav. 18, 2475. `http://einstein.stanford.edu/STEP/`, date: 03.05.2012.

[26] **Nobili, A.M.** et al (2000). Class. Quant. Grav. 17, 2347. `http://eotvos.dm.unipi.it/nobili/`, date: 03.05.2012.

[27] **Touboul, P.** et al (2002). Acta Astronaut 50, 443. `http://www.onera.fr/dmph/accelerometre/index.html` or `http://smsc.cnes.fr/MICROSCOPE/`, date: 03.05.2012

[28] **Brax, Ph. and Martin, J.** (2002). Quintessence And The Accelerating Universe [arXiv:astro-ph/0210533v1].

[29] **Carroll, S.** (2004). SPACETIME AND GEOMETRY An Introduction to General Relativity, Addison-Wesley New York, Second Edition.

[30] **Cliftona, T., Ferreiraa, P.G., Padillab, A., and Skordisb, C.** (2011). Modified Gravity and Cosmology, Phys. Rept. 513 (2012) 1-189 [arXiv:astro-ph/1106.2476].

[31] **de Felice, A. and Tsujikawa, S.** (2010). f(R) Theories Living Rev. Rel. Vol. 13 No. 3, pp. 1-161 [arXiv:gr-qc/1002.4928v2].

[32] **Mizuno, S. and Maeda, K.** (2001). Quintessence in a Brane World [arXiv:hep-ph/0108012v1].

[33] **Url-1** `http://pdg.lbl.gov/2011/reviews/rpp2011-rev-cosmological-parameters.pdf`, date: 03.05.2012.

[34] **Url-2** `http://pdg.lbl.gov/2011/reviews/rpp2011-rev-astrophysical-constants.pdf`, date: 03.05.2012.

[35] **Url-3** `http://www.nu.to.infn.it/exp/all/wmap/`, date: 03.05.2012.

APPENDICES

APPENDIX A.1 : Derivation of Einstein' s field equations
APPENDIX A.2 : Derivation of the field equation for the chameleon field
APPENDIX A.3 : Derivation of the solutions to cosmological chameleon field equations

APPENDIX A.1

Derivation of Einstein' s field equations

In order to obtain Einstein field equation, with the help of the Euler-Lagrange equation of motion with respect to $g^{\mu\nu}$[1] and by using variation principle $\frac{\delta S_{EH}}{\delta g_{\mu\nu}}$ as

$$\frac{\delta S_{EH}}{\delta g^{\mu\nu}} = -\partial_\mu\left(\frac{\delta \mathscr{L}_{EH}}{\delta(\partial_\mu g^{\mu\nu})}\right) + \frac{\delta \mathscr{L}_{EH}}{\delta g^{\mu\nu}} = 0 \tag{A.1}$$

where The Euler-Lagrange Equations for a Field Theory in a flat spacetime can be summarized as

$$L = \int d^3x \mathscr{L}(\phi^i, \partial_\mu \phi^i) \tag{A.2}$$

So the action is $S = \int dt L = \int d^4x \mathscr{L}(\phi^i, \partial_\mu \phi^i)$,

The Lagrangian density, $\mathscr{L}$, is a Lorentz scalar and all of the EsoM can be derived from the Lagrangian density. The action should be the unchanged under small variations of the fields, ϕ^i,

$$\phi^i \to \phi^i + \delta\phi^i$$

$$\partial_\mu(\phi^i) \to \partial_\mu \phi^i + \delta(\partial_\mu \phi^i) = \partial_\mu \phi^i + \partial_\mu(\delta\phi^i)$$

Then,

$$\mathscr{L}(\phi^i, \partial_\mu \phi^i) \to \mathscr{L}(\phi^i + \delta\phi^i, \partial_\mu \phi^i + \partial_\mu \delta\phi^i) \tag{A.3}$$

$$= \mathscr{L}(\phi^i, \partial_\mu \phi^i) + \frac{\partial \mathscr{L}}{\partial \phi^i}\delta\phi^i + \frac{\partial \mathscr{L}}{\partial(\partial_\mu \phi^i)}\partial_\mu(\delta\phi^i). \tag{A.4}$$

Corresponding action expression becomes $S \to S + \delta S$, where explicitly,

$$\delta S = \int d^4x\left(\frac{\partial \mathscr{L}}{\partial \phi^i}\delta\phi^i + \frac{\partial \mathscr{L}}{\partial(\partial_\mu \phi^i)}\partial_\mu(\delta\phi^i)\right)$$

in here, to find the required summation integrand term, $\delta\phi^i$, by integrating the second term,

$$\int d^4x \frac{\partial \mathscr{L}}{\partial(\partial_\mu \phi^i)}\partial_\mu(\delta\phi^i) = \int d^4x \partial_\mu\left(\frac{\partial \mathscr{L}}{\partial(\partial_\mu \phi^i)}\delta\phi^i\right) - \int d^4x \partial_\mu\left(\frac{\partial \mathscr{L}}{\partial(\partial_\mu \phi^i)}\right)\delta\phi^i$$

[1] for simplicity, variation of the action is taken with respect to inverse of $g_{\mu\nu}$

finally,

$$\delta S = \int d^4x[\frac{\partial \mathscr{L}}{\partial \phi^i} - \partial_\mu \frac{\partial \mathscr{L}}{\partial(\partial_\mu \phi^i)}](\delta \phi^i)$$

This satisfies $\delta S = \int d^4x \frac{\delta S}{\delta \phi^i} \delta \phi^i$, the final EsoM for the field theory are thus:

$$\frac{\delta S}{\delta \phi^i} = \frac{\partial \mathscr{L}}{\partial \phi^i} - \partial_\mu (\frac{\partial \mathscr{L}}{\partial(\partial_\mu \phi^i)}) = 0 \quad \textbf{(A.5)}$$

The Einstein-Hilbert action is given by

$$S_{EH} = \int d^4x \sqrt{-g} \frac{R}{2\kappa} + S_M \quad \textbf{(A.6)}$$

Firstly, S_{EH} can be rewritten by separating with respect to the gravitational and the matter action parts.

$S_{EH} = S_G + S_M,$

$S_{EH} = \int d^4x \mathscr{L}_G(g,R) + \int d^4x \mathscr{L}_M(g_{\mu\nu}, \psi),$

$S_G = \int d^4x \sqrt{-g} \frac{R}{16\pi G},$

$S_G = \frac{1}{16\pi G} \int d^4x \sqrt{-g} g^{\mu\nu} R_{\mu\nu}$, in here, $R = g^{\mu\nu} R_{\mu\nu}$ was used.

$S_M = \int d^4x \mathscr{L}_M(g_{\mu\nu}, \psi).$

Their variations are taken with respect to $g^{\mu\nu}$,

$\delta S_{EH} = \delta S_G + \delta S_M;$

$\delta S_G = \frac{1}{16\pi G} \int d^4x \delta(\sqrt{-g} g^{\mu\nu} R_{\mu\nu}),$

$\delta S_G = \frac{1}{16\pi G} \int d^4x \left\{ g^{\mu\nu} R_{\mu\nu} \delta \sqrt{-g} + \sqrt{-g} R_{\mu\nu} \delta g^{\mu\nu} + \sqrt{-g} g^{\mu\nu} \delta R_{\mu\nu} \right\}.$

$\delta S_M = \int d^4x \delta(\mathscr{L}_M(g_{\mu\nu}, \psi)).$

where $\delta g^{\mu\nu} = \delta g^{\mu\nu}$,

$\delta \sqrt{-g} = -\frac{1}{2} \sqrt{-g} g_{\mu\nu} \delta g^{\mu\nu},$

$\delta R^{\rho}{}_{\mu\lambda\nu} = \nabla_\lambda (\delta \Gamma^{\rho}{}_{\nu\mu}) - \nabla_\nu (\delta \Gamma^{\rho}{}_{\lambda\mu}).$

- Firstly, to calculate the $\delta \sqrt{-g}$,

$\left| g_{\mu\nu} \right| = g \equiv |-g| = -\sqrt{-g}\sqrt{-g}$ this special case, because of metric signature of metric is accepted to be negative in four dimensions.

$\delta\sqrt{-g} = -\frac{1}{2}\frac{1}{\sqrt{-g}}\delta g,$

Let S square matrix $S = e^{\ln S}$, the variation of S matrix is given as $\frac{1}{|S|}\delta|S| = Tr(S^{-1}\delta S)$ where $\ln|S| = Tr(\ln S)$.

Now let $S = g_{\mu\nu}$, the variation of g is taken as

$\frac{1}{|g_{\mu\nu}|}\delta|g_{\mu\nu}| = Tr(g^{\mu\nu}\delta g_{\mu\nu}),$

$\delta g = gg^{\mu\nu}\delta g_{\mu\nu},$

or using $g_{\mu\nu}\delta g^{\mu\nu}g^{\mu\nu}\delta g_{\mu\nu} = 1$

$\delta g = -gg_{\mu\nu}\delta g^{\mu\nu}$

$\delta\sqrt{-g} = -\frac{1}{2}\frac{1}{\sqrt{-g}}(-gg_{\mu\nu}\delta g^{\mu\nu})$; where $g \equiv |-g| = -\sqrt{-g}\sqrt{-g},$

$\delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g}g_{\mu\nu}\delta g^{\mu\nu}$ is found.

- To find the $\delta R_{\mu\nu}$,

By starting the variation of the Christoffel Symbols:

$\Gamma^{\rho}{}_{\mu\nu} = \frac{1}{2}g^{\rho\lambda}(g_{\nu\lambda,\mu} + g_{\lambda\mu,\nu} - g_{\mu\nu,\lambda}),$

By using $g_{\mu\nu} \to g_{\mu\nu} + \delta g_{\mu\nu}$; $\delta g^{\mu\nu} = -g^{\mu\alpha}g^{\nu\beta}\delta g_{\alpha\beta}$ and $\Gamma^{\rho}{}_{\mu\nu} \to \Gamma^{\rho}{}_{\mu\nu} + \delta\Gamma^{\rho}{}_{\mu\nu}$.

$\delta\Gamma^{\rho}{}_{\mu\nu} = -\frac{1}{2}[g_{\nu\lambda}\nabla_{\mu}(\delta g^{\rho\lambda}) + g_{\lambda\mu}\nabla_{\nu}(\delta g^{\rho\lambda}) - g_{\nu\alpha}g_{\mu\beta}\nabla^{\rho}(\delta g^{\alpha\beta})]$

in terms of $\delta g^{\rho\lambda}$.

Then, for the variation of the Riemann tensor:

$R^{\rho}{}_{\mu\lambda\nu} = \Gamma^{\rho}{}_{\nu\mu,\lambda} - \Gamma^{\rho}{}_{\lambda\mu,\nu} + \Gamma^{\rho}{}_{\lambda\sigma}\Gamma^{\sigma}{}_{\nu\mu} - \Gamma^{\rho}{}_{\nu\sigma}\Gamma^{\sigma}{}_{\lambda\mu}.$

By using $\Gamma^{\rho}{}_{\mu\nu} \to \Gamma^{\rho}{}_{\mu\nu} + \delta\Gamma^{\rho}{}_{\mu\nu}$ and $R^{\rho}{}_{\mu\lambda\nu} \to R^{\rho}{}_{\mu\lambda\nu} + \delta R^{\rho}{}_{\mu\lambda\nu}$.

$\delta R^{\rho}{}_{\mu\lambda\nu} = \nabla_{\lambda}(\delta\Gamma^{\rho}{}_{\nu\mu}) - \nabla_{\nu}(\delta\Gamma^{\rho}{}_{\lambda\mu})$ is found.

Then, $\lambda \to \rho$, the variation for the Ricci Tensor can be obtained as

$\delta R_{\mu\nu} = \nabla_{\rho}(\delta\Gamma^{\rho}{}_{\nu\mu}) - \nabla_{\nu}(\delta\Gamma^{\rho}{}_{\rho\mu}).$

Finally,

$\delta S_G = \frac{1}{16\pi G}\int d^4x\left\{R\delta\sqrt{-g} + \sqrt{-g}R_{\mu\nu}\delta g^{\mu\nu} + \sqrt{-g}g^{\mu\nu}\delta R_{\mu\nu}\right\},$

$\delta S_G = \frac{1}{16\pi G}\int d^4x\sqrt{-g}\left\{(-\frac{1}{2}Rg_{\mu\nu} + R_{\mu\nu})\delta g^{\mu\nu} + g^{\mu\nu}[\nabla_{\rho}(\delta\Gamma^{\rho}{}_{\nu\mu}) - \nabla_{\nu}(\delta\Gamma^{\rho}{}_{\rho\mu})]\right\},$

$\delta S_G = \frac{1}{16\pi G}\int d^4x\sqrt{-g}\left\{(R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu})\delta g^{\mu\nu} + \nabla_\sigma[(g^{\mu\nu}\delta\Gamma^{\rho}{}_{\nu\mu}) - (g^{\mu\sigma}\delta\Gamma^{\rho}{}_{\rho\mu})]\right\}$,

the last term equals to zero, from the total derivative. $\frac{\delta S_M}{\delta g^{\mu\nu}} = \frac{1}{\sqrt{-g}}\frac{\delta \mathscr{L}_M}{\delta g^{\mu\nu}} \equiv -\frac{1}{2}T_{\mu\nu}{}^{M}$.

Now, by using the obtained variation calculations, the variation of the Einstein-Hilbert action is taken with respect to $\delta g^{\mu\nu}$.

$\frac{\delta S_{EH}}{\delta g^{\mu\nu}} = \frac{\delta S_G}{\delta g^{\mu\nu}} + \frac{\delta S_M}{\delta g^{\mu\nu}} = 0,$

$\frac{\delta S_{EH}}{\delta g^{\mu\nu}} = \frac{1}{\sqrt{-g}}\frac{\delta \mathscr{L}_{EH}}{\delta g^{\mu\nu}} = \frac{1}{\sqrt{-g}}(\frac{\delta \mathscr{L}_G}{\delta g^{\mu\nu}} + \frac{\delta \mathscr{L}_M}{\delta g^{\mu\nu}}) = 0,$

$\frac{\delta S_{EH}}{\delta g^{\mu\nu}} = \frac{1}{16\pi G}(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R - \frac{1}{2}T_{\mu\nu}{}^{M}) = 0$

$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi G T_{\mu\nu}{}^{M}$

is found or can also be written as

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = G_{\mu\nu}{}^{M} \qquad \textbf{(A.7)}$$

where $G_{\mu\nu}{}^{M} \equiv \kappa T_{\mu\nu}{}^{M}$; $\kappa \equiv \frac{8\pi G}{c^4}$.

Especially, for $T_{\mu\nu}{}^{M} = 0$, the vacuum solution of the EFE' s can be obtained as

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 0 \qquad \textbf{(A.8)}$$

APPENDIX A.2

Derivation of the field equation for the chameleon field

To find the field equation for the chameleon field in the EF, the variation of the following given form action is taken as

$$S = \int d^4x\sqrt{-g}\{\frac{M_{Pl}^2}{2}R - \frac{1}{2}(\nabla\phi)^2 - V(\phi) - \frac{1}{\sqrt{-g}}\mathscr{L}_M(g_{\mu\nu}, \psi_M)\} \qquad \textbf{(A.9)}$$

where

$S_\phi = \int d^4x\sqrt{-g}\{-\frac{1}{2}(\nabla\phi)^2 - V(\phi)\}$ is the pure scalar field action part with the potential, $V(\phi)$,

$S_M = -\int d^4x\mathscr{L}_M(g_{\mu\nu}, \psi_M)$ is the matter action part of the total action.

$(2\kappa)^{-1}$ is rewritten in terms of the reduced Planck mass as $\frac{M_{Pl}^2}{2}$.

$\mathscr{L}_{(m)}(g_{\mu\nu}, \psi_M)$ is the Lagrangian density of the matter.

By taking the small variation of this action with respect to ϕ,

- $\delta S_\phi = -\int d^4x\delta(\sqrt{-g}\frac{1}{2}(\nabla\phi)^2 + V(\phi))$,

$-\delta(\sqrt{-g}\frac{1}{2}(\nabla\phi)^2 + V(\phi)) = -\frac{\sqrt{-g}}{2}(\partial_\mu(\phi+\delta\phi)\partial^\mu(\phi+\delta\phi) + 2V(\phi+\delta\phi))$,

where $\phi \to \phi + \delta\phi$,

$\partial_\mu(\phi+\delta\phi)\partial^\mu(\phi+\delta\phi = \partial_\mu\phi\partial^\mu\phi + \partial_\mu\phi\partial^\mu\delta\phi + \partial_\mu\delta\phi\partial^\mu\phi + O(\delta\phi)^2$,

$V(\phi+\delta\phi) = V(\phi) + V(\phi)'\delta\phi \Rightarrow V(\phi)' = \frac{\delta V(\phi)}{\delta\phi}$.

Then, by substituting these into the variation of the action,

$-\delta(\sqrt{-g}\frac{1}{2}(\nabla\phi)^2 + V(\phi)) = \frac{\sqrt{-g}}{2}(-g_{\mu\nu}(\partial_\mu\phi\partial_\nu\delta\phi + \partial_\mu\delta\phi\partial_\nu\phi) - 2V(\phi)'\delta\phi)$,

where

$-g_{\mu\nu}(\partial_\mu\phi\partial_\nu\delta\phi + \partial_\mu\delta\phi\partial_\nu\phi) = -2g^{\mu\nu}\partial_\mu\phi\partial_\nu\delta\phi$,

$-g_{\mu\nu}(\partial_\mu\phi\partial_\nu\delta\phi + \partial_\mu\delta\phi\partial_\nu\phi) = -2\partial_\nu((g^{\mu\nu}\partial_\mu\phi)\delta\phi) + 2(\partial_\nu g^{\mu\nu})\partial_\mu\phi\delta\phi + 2g^{\mu\nu}(\partial_\nu\partial_\mu\phi)\delta\phi$,

where $-2\partial_\nu((g^{\mu\nu}\partial_\mu\phi)\delta\phi) = 0$ from the total derivative and $\partial_\nu g^{\mu\nu} = 0$.

$\Rightarrow -2g^{\mu\nu}\partial_\mu\phi\partial_\nu\delta\phi = 2g^{\mu\nu}(\partial_\nu\partial_\mu\phi)\delta\phi$ or

$\Rightarrow -2g^{\mu\nu}\partial_\mu\phi\partial_\nu\delta\phi = 2\Box\phi\delta\phi$,

$-\delta(\sqrt{-g}\frac{1}{2}(\nabla\phi)^2+V(\phi))=\frac{\sqrt{-g}}{2}(2\Box\phi-2V(\phi)^{'}(\delta\phi)$ or

$\Box\phi=\nabla_\mu\nabla^\mu\phi=\nabla^2\phi$.

Finally,

$\delta S_\phi=\int d^4x\sqrt{-g}(\nabla^2\phi-V(\phi)^{'})(\delta\phi)$ is found.

- $\delta S_M=-\int d^4x\delta(\frac{1}{\sqrt{-g}}\mathscr{L}_M(g_{\mu\nu},\psi_M))$,

due to the Lagrangian density of the matter part doesnot depend on the scalar field and the scalar field, ϕ, the chameleon should conformally be coupled to matter, while the variation of the this action part, by using the chain rule and taking the transformed metric, $g_{\mu\nu}$, in the EF, as (**5.6**), $\tilde{g}_{\mu\nu}{}^{i}\equiv\tilde{g}_{\mu\nu}\rightarrow g_{\mu\nu}=\Omega^{-2}\tilde{g}_{\mu\nu}$ with $\Omega=e^{\beta_i\phi/M_{Pl}}$.

Hence,

$\frac{\delta S_M}{\delta\phi}=-\frac{1}{\sqrt{-g}}\frac{\delta\mathscr{L}_M}{\delta\phi}=-\frac{1}{\sqrt{-g}}\frac{\delta\mathscr{L}_M}{\delta g_{\mu\nu}}\frac{\delta g_{\mu\nu}}{\delta\phi}$,

$\Rightarrow-\frac{1}{\sqrt{-g}}\frac{\delta\mathscr{L}_M}{\delta g_{\mu\nu}}\frac{\delta(e^{2\beta_i\phi/M_{Pl}}\tilde{g}_{\mu\nu})}{\delta g_{\mu\nu}}=-\frac{1}{\sqrt{-g}}\frac{\delta\mathscr{L}_M}{\delta g_{\mu\nu}}\frac{2\beta_i}{M_{Pl}}e^{2\beta_i\phi/M_{Pl}}\tilde{g}_{\mu\nu}\delta\phi$, where equation (**5.6**) was used.

$\Rightarrow-\frac{1}{\sqrt{-g}}\frac{\delta\mathscr{L}_M}{\delta g_{\mu\nu}}\frac{(e^{2\beta_i\phi/M_{Pl}}\tilde{g}_{\mu\nu})}{\delta g_{\mu\nu}}=-\frac{1}{\sqrt{-g}}\frac{\delta\mathscr{L}_M}{\delta g_{\mu\nu}}\frac{2\beta_i}{M_{Pl}}g_{\mu\nu}\delta\phi.$

$\Rightarrow\delta S_M=-\sum_i\frac{2\beta_i}{M_{Pl}}\frac{1}{\sqrt{-g}}\frac{\delta\mathscr{L}_M}{\delta g_{\mu\nu}}g_{\mu\nu}(\delta\phi)$ is obtained.

Eventually, the variation of the total action for the ϕ,

$\delta S=\delta S_\phi+\delta S_M$,

$\delta S=\int d^4x\sqrt{-g}\{\nabla^2\phi-V_{,\phi}-\sum_i\frac{2\beta_i}{M_{Pl}}\frac{1}{\sqrt{-g}}\frac{\delta\mathscr{L}_M}{\delta g_{\mu\nu}}g_{\mu\nu}\}(\delta\phi)$,

By using the variational principle, $\frac{\delta S}{\delta\phi}=0$, then,

$\nabla^2\phi-V_{,\phi}-\sum_i\frac{2\beta_i}{M_{Pl}}\frac{1}{\sqrt{-g}}\frac{\delta\mathscr{L}_M}{\delta g_{\mu\nu}}g_{\mu\nu}=0$,

Finally,

$$\nabla^2\phi=V_{,\phi}+\sum_i\frac{2\beta_i}{M_{Pl}}\frac{1}{\sqrt{-g}}\frac{\delta\mathscr{L}_M}{\delta g_{\mu\nu}}g_{\mu\nu} \qquad \textbf{(A.10)}$$

is obtained in the EF.

In the EF, the chameleon field equation is rewritten in terms of the energy density of the matter, for each species i, ρ can obtain.

For this: By using the (**3.1**) metric, the transformed scale factor (**5.9**), a, and the transformed metric tensor (**5.6**),$g_{\mu\nu}$ to the EF,

$\tilde{g}_{\mu\nu} \rightarrow g_{\mu\nu} \equiv \Omega^{-2}\tilde{g}_{\mu\nu}$

$g_{\mu\nu} \equiv e^{-2\beta_i\phi/M_{Pl}}\tilde{g}_{\mu\nu}$

diag $(1, a^2, a^2, a^2) = e^{-2\beta_i\phi/M_{Pl}}$ diag$(-1, \tilde{a}^2, \tilde{a}^2, \tilde{a}^2)$

then, the transformed scale factor is obtained as

$a \equiv e^{-\beta_i\phi/M_{Pl}}\tilde{a}$

$g_{\mu\nu} = \text{diag}(-e^{2\beta_i\phi/M_{Pl}}, a^2, a^2, a^2),$

$g^{\mu\nu} = \text{diag}(-e^{-2\beta_i\phi/M_{Pl}}, a^{-2}, a^{-2}, a^{-2}).$

By using these the components of the Christoffel symbols are computed as

Recalling $\Gamma^{\rho}{}_{\mu\nu} \equiv \frac{1}{2}g^{\rho\lambda}(g_{\nu\lambda,\mu} + g_{\lambda\mu,\nu} - g_{\mu\nu,\lambda})$.

- $\Gamma^{i}{}_{0i} = \frac{1}{2}g^{i\lambda}(g_{i\lambda,0} + g_{\lambda 0,i} - g_{0i,\lambda})$; $(i = 1,2,3)$ and $(\lambda = 0,1,2,3)$.

  Then, $\Gamma^{1}{}_{01} = \Gamma^{2}{}_{02} = \Gamma^{3}{}_{03} = a^{-1}a_{,0}$

- $\Gamma^{0}{}_{ii} = \frac{1}{2}g^{i\lambda}(g_{i\rho,i} + g_{\lambda i,i} - g_{ii,\lambda})$; $(i = 1,2,3)$ and $(\lambda = 0,1,2,3)$.

  Then, $\Gamma^{0}{}_{11} = \Gamma^{0}{}_{22} = \Gamma^{0}{}_{33} = e^{-2\beta_i\phi/M_{Pl}}aa_{,0}$

are found. With the help of these, to expand the conservation equation,

$T^{\mu\nu}{}_{;\rho} = T^{\mu\nu}{}_{;\rho} + \Gamma^{\mu}{}_{\rho\lambda}T^{\lambda\nu} + \Gamma^{\nu}{}_{\rho\lambda}T^{\lambda\mu}.$

Using the conservation equation, $0 = T^{0\nu}{}_{;\nu}$.

For the index $\mu = 0$, its components can be found as

- $0 = T^{0\nu}{}_{;\nu} + \Gamma^{0}{}_{\nu\lambda}T^{\lambda\nu} + \Gamma^{\nu}{}_{\nu\lambda}T^{\lambda 0},$
- $0 = T^{00}{}_{;0} + \Gamma^{0}{}_{\nu\lambda}T^{\lambda\nu} + \Gamma^{\nu}{}_{\nu 0}T^{00},$

where, $T^{00}{}_{;0} = (-\rho g^{00})_{,0}$ with $g^{00} = -e^{-2\beta_i\phi/M_{Pl}}$,

$\Gamma^{0}{}_{\nu\lambda}T^{\lambda\nu} = (3)(e^{-2\beta_i\phi/M_{Pl}}aa_{,0})(pg^{\lambda\nu})$, where $(\lambda, \nu = i = 1,2,3)$,

$\Gamma^{0}{}_{\nu\lambda}T^{\lambda\nu} = 3e^{-2\beta_i\phi/M_{Pl}}aa_{,0}pg^{ii}$ with $g^{ii} = a^{-2}$,

$\Gamma^{0}{}_{\nu\lambda}T^{\lambda\nu} = 3e^{-2\beta_i\phi/M_{Pl}}a^{-1}a_{,0}p$

and

$\Gamma^{\nu}{}_{\nu 0}T^{00} = 3a^{-1}a_{,0}((-e^{-2\beta_i\phi/M_{Pl}})(-\rho)) \Rightarrow \Gamma^{\nu}{}_{\nu 0}T^{00} = 3e^{-2\beta_i\phi/M_{Pl}}\rho a^{-1}a_{,0}$

are found. Now, by using $-\rho+3p=-(1-3\omega_i)\rho$ and $p=-\frac{\rho}{3}(1-3\omega_i-1)$ from equation of state $p=\omega_i\rho$,

$0=e^{-2\beta_i\phi/M_{(Pl)}}(\rho_{,0}+3a^{-1}a_{,0}(1+\omega_i)\rho)$ is obtained.

In here, each of the both sides of the last expression, multiplying by $e^{2\beta_i\phi/M_{Pl}}a^{3(1+\omega_i)}$,

$0=a^{3(1+\omega_i)}\rho_{,0}+3(1+\omega_i)a^{3(1+\omega_i)-1}a_{,0}\rho,$

$0=(a^{3(1+\omega_i)}\rho)_{,0},$

$0=((\tilde{a}e^{-\beta_i\phi/M_{Pl}})^{3(1+\omega_i)}\rho)_{,0}.$

It is realized that in the EF the quantity of the energy density transforms as

$$\rho\equiv e^{3(1+\omega_i)\beta_i\phi/M_{Pl}}\tilde{\rho}. \qquad \textbf{(A.11)}$$

This obeys the continuity equation (**3.4**), $\tilde{\rho}\propto\tilde{a}^{-3(1+\omega_i)}$(in the JF) if $\phi\ll M_{Pl}$; then $\tilde{\rho}\approx\rho$.

Finally, for each species i, $\tilde{\rho}$ can be found in the EF:

For the JF, $\tilde{T}\equiv\tilde{T}^{\mu\nu}\tilde{g}_{\mu\nu}=-\tilde{\rho}+3\tilde{p}=-(1-3\omega_i)\tilde{\rho}$ from this $\tilde{\rho}=-\frac{1}{(1-3\omega_i)}\tilde{T}^{\mu\nu}\tilde{g}_{\mu\nu}\Rightarrow$ $\tilde{\rho}=(-\frac{1}{(1-3\omega_i)})(-\frac{2}{\sqrt{-\tilde{g}}}\frac{\partial\mathscr{L}_M}{\partial\tilde{g}_{\mu\nu}}\tilde{g}_{\mu\nu}),$

is obtained. Substituting the found $\tilde{\rho}$ into (**A.11**) as

$\rho\equiv e^{3(1+\omega_i)\beta_i\phi/M_{Pl}}(-\frac{1}{(1-3\omega_i)})(-\frac{2}{\sqrt{-\tilde{g}}}\frac{\partial\mathscr{L}_M}{\partial\tilde{g}_{\mu\nu}}\tilde{g}_{\mu\nu})$

here, the equation (**5.7**) is used, for $n=4$

$\rho=e^{3(1+\omega_i)\beta_i\phi/M_{Pl}}\frac{1}{(1-3\omega_i)}\frac{2}{\sqrt{-e^{8\beta_i\phi/M_{Pl}}g}}\frac{\partial\mathscr{L}_M}{\partial\tilde{g}_{\mu\nu}}\tilde{g}_{\mu\nu}$

By putting $\tilde{\rho}^i\equiv\rho$ in the EF,

$\tilde{\rho}^i=e^{-(1-3\omega_i)\beta_i\phi/M_{Pl}}\frac{1}{(1-3\omega_i)}\frac{2}{\sqrt{-g}}\frac{\partial\mathscr{L}_M}{\partial\tilde{g}_{\mu\nu}}\tilde{g}_{\mu\nu},$

Furthermore, this equation can be written as

$$\tilde{\rho}^i e^{(1-3\omega_i)\beta_i\phi/M_{Pl}}(1-3\omega_i)=\frac{2}{\sqrt{-g}}\frac{\partial\mathscr{L}_M}{\partial\tilde{g}_{\mu\nu}}\tilde{g}_{\mu\nu}. \qquad \textbf{(A.12)}$$

By substituting the (**A.12**) into the chameleon equation of motion the ϕ dependence, (**A.10**), is obtained as

$\nabla^2\phi=V_{,\phi}(\phi)+\sum_i\tilde{\rho}^i e^{(1-3\omega_i)\beta_i\phi/M_{Pl}}(1-3\omega_i)\frac{\beta_i}{M_{Pl}}$

in the EF.

By integrating, the dynamics for ϕ in terms of a single effective potential can be expressed by

$$V_{eff}(\phi) \equiv V(\phi) + \sum_i \tilde{\rho}^i e^{(1-3\omega_i)\beta_i\phi/M_{Pl}} \tag{A.13}$$

Also, the chameleon equation of the motion in the EF becomes simply

$$\nabla^2\phi = V_{eff,\phi}(\phi) \tag{A.14}$$

The static spherically symmetric solution

Our aim is to find time independent solutions $\phi(\vec{x})$, for spherically symmetric matter distributions $\tilde{\rho}(r)$ of a single pressure free matter species in the weak field limit.

Assuming that $\tilde{g}_{\mu\nu} = \tilde{\eta}_{\mu\nu}$, and the equation (A) can be rewritten as

$$\frac{d^2\phi}{dr^2} + \frac{2}{r}\frac{d\phi}{dr} = V_{,\phi}(\phi(r)) + \frac{\beta}{M_{Pl}}\tilde{\rho}(r)e^{4\beta\phi(r)/M_{Pl}} \qquad (\textbf{A.15})$$

where R is homogeneous spherical mass of radius, ρ is density. Sitting in a background matter distribution with density ρ_∞, with $\rho_c > \rho_\infty$. Then,

$$\begin{cases} \tilde{\rho}(r) = \rho_c & \text{if } r < R \\ \tilde{\rho}(r) = \rho_\infty & \text{if } r > R \end{cases}$$

$$\begin{cases} \phi_c \equiv \phi_{min}(\rho = \rho_c) & \text{if } m_c \equiv m(\phi_c) \\ \phi_\infty \equiv \phi_{min}(\rho = \rho_{min}) & \text{if } m_{min} \equiv m(\phi_{min}) \end{cases}$$

can be defined, where $\phi_c < \phi_\infty$.

The spherically symmetric solutions for the equation (**A.15**) will be discussed below:

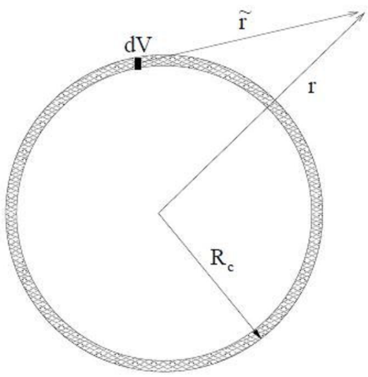

Figure A.1: Sphere figure.

$$\phi = \begin{cases} \phi \to \phi_\infty & \text{if r > R outside the sphere. Yukawa potential is recognized.} \\ \phi \to \phi_c & \text{if r < R inside the sphere. To recognize the } \phi \text{, approximation methods are used.} \end{cases}$$

where

i) Outside the sphere; the harmonic oscillator approximation is used

ii) Inside the sphere;

- $\star$ $\phi \gg \phi_c$; linear approximation and

$$\star \ \phi \sim \phi_c = \begin{cases} \text{the low contrast solution} & \text{if } R_c = R \\ \text{the thick shell solution} & \text{if } R_c = 0 \\ \text{the thin shell solution} & \text{if } 0 < R_c < R \end{cases}$$

are used.

here, in order to divide the interval, $[0,R]$, as $[0,R_c]$, on which $\phi \sim \phi_c$, and $[R_c,R]$, on which $\phi \gg \phi_c$ R_c is defined. To remain as a undivided interval $[0,R]$, $R_c = 0$ or $R_c = R$ is taken.

i) Outside the sphere, $r > R$, the solutions: ϕ drives towards ϕ_∞. The harmonic oscillator approximation is used. The (**5.25**) is rewritten as

The equation (**A.15**) with the help of the equation (**5.36**) becomes

$\frac{d^2\phi}{dr^2} + \frac{2}{r}\frac{d\phi}{dr} = m_\infty^2(\phi - \phi_\infty)$.

The general solution to this differential equation is found as

$$\phi(r) = A\frac{e^{-m_\infty(r-R)}}{r} + B\frac{e^{m_\infty(r-R)}}{r} + \phi_\infty$$

for dimensionless constants A and B.

Imposing the condition that $\phi \to \phi_\infty$ as $r \to \infty$ gives $B = 0$

$$\phi(r) = A\frac{e^{-m_\infty(r-R)}}{r} + \phi_\infty$$

where $e^{-m_{(\infty)}(r-R)}$ is known as an exponential suppression factor.

ii) Inside the sphere, $r < R$, the solutions: ϕ drives towards ϕ_c. According to the divided intervals, there as three approximations are used.

Let define R_c to divide the interval $[0,R]$ into two interval as $[0,R_c]$ on which $\phi \sim \phi_c$ and $[R_c,R]$ on which $\phi \gg \phi_c$. The other case is accepted as $R_c = 0$ or $R_c = R$, so that the interval $[0,R]$ remains undivided.

Linear approximation is used if at $[R_c,R]$, $\phi \gg \phi_c$. In this case the harmonic oscillator approximation to V_{eff} is not valid. But, for $\phi > \phi_{min}$, the bare potential V decays quickly and the term $\rho e^{\beta\phi/M_{Pl}}$ comes to dominate. In particular,

$$V_{eff,\phi}(\phi) \approx \frac{\beta}{M_{Pl}}\rho_c e^{4\beta\phi/M_{Pl}} \approx \frac{\beta}{M_{Pl}}\rho_c \qquad \textbf{(A.16)}$$

Since, today, in all cases; $\phi \ll M_{Pl}$!

Now, the equation (**A.15**) takes the form as $\frac{d^2\phi}{dr^2} + \frac{2}{r}\frac{d\phi}{dr} \approx \frac{\beta}{M_{Pl}}\rho_c$

with the general solution,

$$\phi(r) = \frac{\beta}{6M_{Pl}}\rho_c r^2 + \frac{C}{r} + D\phi_c$$

for dimensionless constants C and D.

Harmonic oscillator approximation is used if at $[0, R_c]$,

$$\phi \sim \phi_c = \begin{cases} \text{the low contrast solution} & \text{if } R_c = R \\ \text{the thick shell solution} & \text{if } R_c = 0 \\ \text{the thin shell solution} & \text{if } 0 < R_c < R \end{cases}$$

By using the harmonic oscillator approximation,

$$V_{eff,\phi}(\phi) \approx m_c^2(\phi - \phi_c) \quad \textbf{(A.17)}$$

then, the solution is given by

$$\phi(r) = A\frac{e^{-m_\infty(r-R)}}{r} + B\frac{e^{m_\infty(r-R)}}{r} + \phi_\infty$$

for this, the solution becomes

$$\phi(r) = E\frac{e^{-m_c r}}{r} + F\frac{e^{m_c(r-R)}}{r} + \phi_c$$

for dimensionless constants E and F. By smoothly gluing together these solutions and imposing $\frac{d\phi}{dr} \to 0$ and $r \to 0$ to ensure continuity of the three dimensional solutions at the origin.

By recalling the boundary conditions: $\begin{cases} \textbf{i.} \lim_{r\to R^-}\phi(r) = \lim_{r\to R^+}\phi(r) \\ \textbf{ii.} \lim_{r\to R^-}\frac{d\phi(r)}{dr} = \lim_{r\to R^+}\frac{d\phi(r)}{dr} \end{cases}$

There are three cases:

Case 1 . Low-contrast solution, if $R_c = R$. Solutions used are

$$\begin{cases} \phi(r) = F\frac{e^{m_c(r-R)} - e^{-m_c(r+R)}}{r} + \phi_c; r < R \\ \phi(r) = A\frac{e^{-m_\infty(r-R)}}{r} + \phi_\infty; r > R \end{cases}$$

The gradients of these solutions are

$$\begin{cases} \frac{d\phi(r)}{dr} = F\frac{(m_c e^{m_c(r-R)} + m_c e^{-m_c(r+R)})r + e^{m_c(r-R)} - e^{-m_c(r+R)}}{r^2}; r < R \\ \frac{d\phi(r)}{dr} = A\frac{-m_\infty e^{-m_\infty(r-R)}r + e^{-m_\infty(r-R)}}{r^2}; r > R \end{cases}$$

We can show their calculations following forms, $\phi(r) = F\frac{e^{m_c(r-R)}-e^{-m_c(r+R)}}{r} + \phi_c;\ r < R$

(coming from inside the sphere solution equation $\phi(r) = E\frac{e^{-m_c r}}{r} + F\frac{e^{m_c(r-R)}}{r} + \phi_c$ ($\lim_{r\to 0} \to \frac{d\phi}{dr} \to 0; E = -Fe^{-m_c R}$).)

and

$\phi(r) = A\frac{e^{-m_\infty(r-R)}}{r} + \phi_\infty;\ r > R.$

(coming from outside the sphere solution equation $\phi(r) = A\frac{e^{m_\infty(r-R)}}{r} + \phi_\infty$.)

Their gradients are

$\frac{d\phi(r)}{dr} = F\frac{(m_c e^{m_c(r-R)}+m_c e^{-m_c(r+R)})r+e^{m_c(r-R)}-e^{-m_c(r+R)}}{r^2};\ r < R.$

$\frac{d\phi(r)}{dr} = A\frac{-m_\infty e^{-m_\infty(r-R)}r+e^{-m_\infty(r-R)}}{r^2};\ r > R.$

By using the boundary conditions, the constans are obtained below,

From condition **i.**:

$F\frac{e^{m_c(R-R)}-e^{-m_c(R+R)}}{R^2} + \phi_c = A\frac{e^{-m_\infty(R-R)}}{R} + \phi_\infty,$

$\Rightarrow F\frac{1-e^{m_c 2R}}{R} + \phi_c = A\frac{1}{R} + \phi_\infty$ is obtained, let (I).

From condition **ii.**:

$F\frac{(m_c Re^{m_c(R-R)}-e^{m_c(R-R)})+m_c Re^{-m_c(R+R)}+e^{-m_c(R+R)}}{R^2} = A\frac{-m_\infty Re^{-m_\infty(R-R)}-e^{-m_\infty(R-R)}}{R^2},$

$\Rightarrow F\frac{(m_c R-1)+(m_c R+1)e^{-m_c 2R}}{R^2} = A\frac{-(m_\infty R+1)}{R^2}$ is obtained, let(II).

the solution to the obtained equations (I) and (II):

$\phi_\infty - \phi_c = F(\frac{1-e^{-m_c 2R}}{R}) - A\frac{1}{R},$

(here, by typing the equal value of equation From(II))

$\phi_\infty - \phi_c = A((\frac{-(m_\infty R+1)}{(m_c R-1)+(m_c R+1)e^{-m_c 2R}})(\frac{1-e^{-m_c 2R}}{R}) - \frac{1}{R}),$

$\frac{(\phi_\infty-\phi_c)((m_c R-1)-(m_c R+1)e^{-m_c 2R})R}{-(m_\infty R+1)(1-e^{-m_c 2R})+((m_c R-1)+(m_c R+1)e^{-m_c 2R})} = A,$

$$\therefore A = \frac{(\phi_\infty - \phi_c)}{m_\infty + m_c + (m_c - m_\infty)e^{-2m_c R}}((1 - m_c R) - (1 + m_c R)e^{-2m_c R})$$

is found.

For the obtaining F,

From(II), $F((m_cR-1)+(m_cR+1)e^{-2m_cR}) = A(-(m_\infty R+1))$,

$\Rightarrow F = A\frac{(m_\infty R+1)}{(1-m_cR)-(1+m_cR)e^{-2m_cR}}$,

(by substituting the found A value in here)

$$\therefore F = \frac{(\phi_\infty - \phi_c)}{m_\infty + m_c + (m_c - m_\infty)e^{-2m_cR}}(m_\infty R+1)$$

is found.

Case 2 . The thick-shell solution, if $R_c = 0$. Solutions are given as

$$\begin{cases} \phi(r) = \frac{\beta}{6M_{Pl}}\rho_c r^2 + D\phi_c; r < R \\ \phi(r) = A\frac{e^{-m_\infty(r-R)}}{r} + \phi_\infty; r > R \end{cases}$$

Let' s show

$\phi(r) = \frac{\beta}{6M_{Pl}}\rho_c r^2 + D\phi_c;\ r < R.$

(coming from inside the sphere solution equation $\phi(r) = \frac{\beta}{6M_{Pl}}\rho_c r^2 + \frac{C}{r} + D\phi_c$, where the boundary cond. at the origin requires simply $C = 0$.)

and

$\phi(r) = A\frac{e^{-m_\infty(r-R)}}{r} + \phi_\infty;\ r > R.$

(coming from outside the sphere solution equation $\phi(r) = A\frac{e^{-m_\infty(r-R)}}{r} + B\frac{e^{m_\infty(r-R)}}{r} + \phi_\infty$.)

Their gradients are

$$\begin{cases} \frac{d\phi(r)}{dr} = \frac{\beta}{6M_{Pl}}2r\rho_c; r < R \\ \frac{d\phi(r)}{dr} = A(\frac{-m_\infty e^{-m_\infty(r-R)}r - e^{-m_\infty(r-R)}}{r^2}); r > R \end{cases}$$

By using the boundary conditions, the constants are found as

From condition **i.**:

$\frac{\beta}{6M_{Pl}}\rho_c R^2 + D\phi_c = A\frac{1}{R} + \phi_\infty$ is obtained, let (III).

From condition **ii.**:

$\frac{\beta}{3M_{Pl}}\rho_c R = A\frac{-m_\infty R-1}{R^2}$ is found, let (IV).

$$\therefore A = -\frac{\beta}{3M_{Pl}}\rho_c R^3 \frac{1}{1+m_\infty R}$$

is found by using (IV).

From (III), $D = \frac{1}{\phi_c}(A\frac{1}{R} + \phi_\infty - \frac{\beta}{6M_{Pl}}\rho_c R^2)$,

(by putting the found A value into here)

$$\therefore D = \frac{\phi_\infty}{\phi_{(c)}} - (\frac{1}{1+Rm_\infty} + \frac{1}{2})\frac{\beta\rho_c R^2}{3M_{Pl}}$$

Case 3 . The thin-shell solution, if $0 < R_{(c)} < R$. Solutions are

$$\begin{cases} \phi(r) = F\frac{e^{m_c(r-R_c)} - e^{-m_c(r+R_c)}}{r} + \phi_c; r \in (0, R_c) \\ \phi(r) = \frac{\beta}{6M_{Pl}}\rho_c r^2 + \frac{C}{r} + D\phi_c; r \in (R_c, R) \\ \phi(r) = A\frac{e^{-m_\infty(r-R)}}{r} + \phi_\infty; r \in (R, \infty) \end{cases}$$

Now, the solution is divided between three regions.

The gradients of solutions are

$$\begin{cases} \frac{d\phi(r)}{dr} = F\frac{(m_c e^{m_c(r-R_c)} + m_c e^{-m_c(r+R_c)})r - e^{m_c(r-R_c)} + e^{-m_c(r+R_c)}}{r^2}; r \in (0, R_c) \\ \frac{d\phi(r)}{dr} = \frac{\beta}{3M_{Pl}} r\rho_c - \frac{C}{r}; r \in (R_c, R) \\ \frac{d\phi(r)}{dr} = A\frac{-m_\infty e^{-m_\infty(r-R)}r - e^{-m_\infty(r-R)}}{r^2}; r \in (R, \infty) \end{cases}$$

in this case, four continuity equations are obtained by using the boundary conditions:

For two at R_c:

From condition **i.**:

$F(\frac{1-e^{-m_c 2R_c}}{R_c}) + \phi_c = \frac{\beta}{6M_{Pl}}\rho_c R_c^2 + \frac{C}{R_c} + D\phi_c$,

From condition **ii.**:

$F(\frac{(m_c e^{m_c(r-R_c)} + m_c e^{-m_c(r+R_c)})r - e^{m_c(r-R_c)} + e^{-m_c(r+R_c)}}{r^2}) = \frac{\beta}{6M_{Pl}}\rho_c 2R_c - \frac{C}{R_c^2}$, at $r = R_c$.

$\Rightarrow F(\frac{m_c R_c - 1 + m_c R_c e^{-m_c 2R_c} + e^{-m_c 2R_c}}{R_c^2}) = \frac{\beta}{6M_{Pl}}\rho_c 2R_c - \frac{C}{R_c^2}$.

For two at R:

From condition **i.**:

$\frac{\beta}{6M_{Pl}}\rho_c R^2 + \frac{C}{R} + D\phi_c = A\frac{1}{R} + \phi_\infty$,

From condition **ii.**:

$\frac{\beta}{3M_{Pl}}\rho_c R - \frac{C}{R^2} = A(-m_\infty R - 1)\frac{1}{R^2}$.

For this system, we still need one more equation to tell us what R_c is. In order to determine $R_c \in [0, R]$, the two approximation equations (**A.16**) and (**A.17**) have been used. While for $r \in (0, R_c)$ the harmonic approximation, (**A.17**), is better, for $r \in (R_c, R)$ the linear approximation, (**A.16**), is better. How to be made a better description for the R_c?

As ϕ increases from ϕ_c, the harmonic approximation $m_c^2(\phi - \phi_c)$ increases without bound from 0. At some point, $m_c^2(\phi - \phi_c) = \frac{\beta}{M_{Pl}}\rho_c$. But, it is known that $V_{eff,\phi}(\phi) \approx V_{,\phi}(\phi) + \frac{\beta}{M_{Pl}}\rho_c < \frac{\beta}{M_{Pl}}\rho_c$, because of $V_{,\phi} < 0$.

Hence, while the harmonic(Taylor) approximation is the better approximation for the $m_c^2(\phi - \phi_c) < \frac{\beta}{M_{Pl}}\rho_c$.

Once $m_c^2(\phi - \phi_c) > \frac{\beta}{M_{Pl}}\rho_c$, it should be switched to the linear approximation, therefore R_c may be defined by the following procedures:

- Try $R_c = R$ and compute the low-contrast solution $\phi(r)$.

If $m_c^2(\phi(R) - \phi_c) < \frac{\beta}{M_{Pl}}\rho_c$, then the solution is valid and $R_c = R$.

- Try $R_c = 0$ and compute the thick-shell solution $\phi(r)$.

If $m_c^2(\phi(0) - \phi_c) > \frac{\beta}{M_{Pl}}\rho_c$, then the solution is valid and $R_c = 0$.

- Otherwise, R_c is defined by the equation $m_c^2(\phi(R_c) - \phi_c) = \frac{\beta}{M_{Pl}}\rho_c$ and the computed solution $\phi(r)$ is given by the thin-shell solution.

By substituting the $r \to R_c^-$ limit of the thin-shell solution, this becomes $m_c^2(\phi(R_c) - \phi_c) = \frac{\beta}{M_{Pl}}\rho_c$, $m_c^2 F(\frac{1-e^{2m_cR_c}}{R_c^2}) = \frac{\beta}{M_{Pl}}\rho_c$,

$$\therefore F = \frac{\beta\rho_c R_c}{m_c^2 M_{Pl}(1 - e^{-2m_cR_c})}$$

The thin shell suppression factor; we need to analyze the behavior of the field ϕ outside the sphere, for it is there that experiments to measure the chameleon force must be take place. By recalling that in the limit $m_\infty R \ll 1$ and (in the thin-shell case) $F = 0$, $\frac{(R-R_c)}{R} \ll 1$, the thin and thick shell solutions. But, low-contrast solution is simply the $R_c \to R$ limit of the thin-shell solution. In all cases, the exterior approximation solution is

$$\phi(r) = A\frac{e^{-m_\infty(r-R)}}{r} + B\frac{e^{m_\infty(r-R)}}{r} + \phi_\infty$$

where the dimensionless constant A tells us about the magnitude of ϕ and thus of the chameleon force.

In the thick-shell case, assuming $m_\infty R \ll 1$; $A = -\frac{\beta}{3M_{Pl}}\rho_c\frac{R^3}{1+m_\infty R}$,

$A \approx -\frac{\beta}{M_{(Pl)}}\frac{1}{4\pi}\frac{4\pi}{3}R^3\rho_{(c)}$ let $(\star)$.

In the thin-shell case, assuming that $\phi = \phi_c$ for $r < R_c$, which translates for us to $F = 0$. Then, the two constant equations are found as

$F(\frac{1-e^{-m_c 2R_c}}{R_c}) + \phi_c = \frac{\beta}{6M_{Pl}}\rho_c R_c^2 + \frac{C}{R_c} + D\phi_c$

and

$F(\frac{m_c R_c - 1 + m_c R_c e^{-m_c 2R_c} + e^{-m_c 2R_c}}{R_c^2}) = \frac{\beta}{3M_{Pl}}\rho_c R_c - \frac{C}{R_c^2}$

These will give in the following results

$$\therefore C = \frac{\beta}{3M_{Pl}}\rho_c R_c^3$$

and

$$\therefore D = 1 - \frac{\beta\rho_c R_c^2}{2M_{Pl}}\frac{1}{\phi_c}$$

Substituting the constants into the equation

$\frac{\beta}{3M_{Pl}}\rho_c R - \frac{C}{R^2} = A(-m_\infty R - 1)\frac{1}{R^2}$, at R;

$\Rightarrow \frac{\beta}{3M_{Pl}}\rho_c R - \frac{\beta}{3M_{Pl}}\rho_c R_c^3\frac{1}{R^2} = A(-m_\infty R - 1)\frac{1}{R^2}$,

$\Rightarrow \frac{\beta}{3M_{Pl}}\rho_c(R^3 - R_c^3)\frac{1}{R^2} \approx -A\frac{1}{R^2}$,

$\Rightarrow A \approx -\frac{\beta}{3M_{Pl}}\rho_c(R^3 - R_c^3)$.

Now, substituting the found values of A, C, and D into the equation

$\frac{\beta}{6M_{Pl}}\rho_c R^2 + \frac{C}{R} + D\phi_c = A\frac{1}{R} + \phi_\infty$, at R;

$\frac{\beta}{6M_{Pl}}\rho_c R - \frac{\beta}{3M_{Pl}}\rho_c R_c^3\frac{1}{R^2} + (1 - \frac{\beta\rho_c R_c^2}{2M_{Pl}}\frac{1}{\phi_c})\phi_c = -\frac{\beta}{3M_{Pl}}\rho_c(R^3 - R_{(c)}^3)\frac{1}{R} + \phi_\infty$.

$\Rightarrow \frac{1}{2}\frac{\beta}{M_{Pl}}\rho_c R^2 - \frac{1}{2}\frac{\beta}{M_{Pl}}\rho_c R_c^2 + \phi_c = \phi_\infty$,

$\Rightarrow R^2 - R_c^2 = (\phi_\infty - \phi_c)\frac{2M_{Pl}}{\beta\rho_c}$ is obtained.

Finally, Taylor-expand equation; $A \approx -\frac{\beta}{M_{Pl}}\frac{1}{4\pi}\frac{4\pi}{3}R^3\rho_c$, in R_c about R to get

$A \approx -\frac{\beta}{3M_{Pl}}\rho_c(R^3 - R_c^3)$,

$\Rightarrow A \approx -\frac{\beta}{3M_{Pl}}\rho_c \frac{3}{2}R(R^2 - R_c^2),$

$\Rightarrow A \approx -\frac{\beta}{3M_{Pl}}\rho_c \frac{3}{2}R(\phi_\infty - \phi_c)\frac{2M_{Pl}}{\beta\rho_c},$

here, by using thick-shell case $A \approx -\frac{\beta}{M_{Pl}}\frac{1}{4\pi}\frac{4\pi}{3}R^3\rho_c$.

$A = -\frac{\beta}{4\pi}(\frac{4\pi}{3}R^3\rho_c)\frac{3M_{Pl}R(\phi_\infty-\phi_c)}{\beta\rho_c R^2}$ is obtained, let ($\star\star$).

is obtained.

The external solution approximation can also be written in the following form:

$\phi_{thick}(r) \approx -\frac{\beta}{4\pi M_{Pl}}(\frac{4\pi}{3}R^3\rho_c)\frac{e^{-m_\infty(r-R)}}{r} + \phi_\infty,$

(coming from A ($\star$) the term $-\frac{\beta}{4\pi M_{Pl}}(\frac{4\pi}{3}R^3\rho_c)$)

$\phi_{thin}(r) \approx -\frac{\beta}{4\pi M_{Pl}}(\frac{4\pi}{3}R^3\rho_c)(3\frac{M_{Pl}(\phi_\infty-\phi_c)}{\beta\rho_c R^2})\frac{e^{-m_\infty(r-R)}}{r} + \phi_\infty,$

(coming from A ($\star\star$) the term $-\frac{\beta}{4\pi M_{Pl}}(\frac{4\pi}{3}R^3\rho_c)(3\frac{M_{Pl}(\phi_\infty-\phi_c)}{\beta\rho_c R^2})$)

The difference between the thin-shell solution and the thick-shell external solution is the the thin shell suppression factor, $3\frac{\Delta R}{R}$,

where $\frac{\Delta R}{R} \equiv \frac{M_{Pl}}{\beta}(\phi_\infty - \phi_c)\frac{1}{\rho_c R^2}$.

This factor is used to show the difference between the thick-shell case ($\Delta R \gtrsim R$) and thin-shell case ($\Delta R \ll R$) by the Weltmann-Khoury.

In here, the criterion, for the thick-shell condition:

$m_c^2(\phi(0) - \phi_c) > \frac{\beta}{M_{Pl}}\rho_c$ with $R_c = 0$, can be translated in terms of $\frac{\Delta R}{R}$ as

$m_c^2(\phi_\infty - \phi_c - \frac{\beta\rho_c R^2}{2M_{Pl}}) > \frac{\beta}{M_{Pl}}\rho_c,$

$m_c^2(\phi_\infty - \phi_c) - \beta\rho_c R^2\frac{1}{2} > \beta\rho_c\frac{1}{m_c^2},$

$M_{Pl}(\phi_{(\infty)} - \phi_c)\frac{1}{\beta\rho_c R^2} - \frac{1}{2} > \frac{1}{m_c^2 R^2},$

where $M_{Pl}(\phi_\infty - \phi_c) \equiv \frac{\Delta R}{R},$

$\frac{\Delta R}{R} > \frac{1}{m_c^2 R^2} + \frac{1}{2}.$

Also, in the general form, a chameleon suppression factor is given by

$W \equiv -A[\frac{\beta}{4\pi M_{Pl}}(\frac{4}{3}\pi R^3\rho_c]^{-1},$

$\Rightarrow W = -A\frac{3M_{Pl}}{\beta R^3 \rho_c}$.

Clearly, in the thick-shell case, the chameleon suppression factor is taken as $W \approx 1$, and in the thin-shell and in the low-contrast cases it is taken as $0 < W < 1$.

The chameleon suppression factor gives us a means of quantifying the thickness of the shell of a object in a given background density.

APPENDIX A.3

Derivation of the solutions to cosmological chameleon field equations

To obtain the cosmological equations for the chameleon filed, a flat, homogeneous, isotropic universe is taken with the metric (**3.1**) and ϕ is accepted to be homogeneous.

$\nabla^2\phi = g^{\mu\nu}\nabla_\mu\nabla_\nu\phi$ with $\nabla_\mu \to \partial_\mu$,

$\nabla^2\phi = g^{\mu\nu}(\partial_\mu\partial_\nu\phi - \Gamma^{\rho}{}_{\nu\mu}\phi_{,\rho})$,

$\nabla^2\phi = g^{00}\partial_0\partial_0\phi - g^{ii}\Gamma^{0}{}_{ii}\phi_{,0}$ with $(i = 1,2,3)$,

$\nabla^2\phi = (-1)\phi_{,00} - (3a^{-2})\Gamma^{0}{}_{11}\phi_{,0}$,

$\nabla^2\phi = -\ddot{\phi} - 3a^{-2}(a\dot{a})\dot{\phi}$

$$\nabla^2\phi = -(\ddot{\phi} + 3H\dot{\phi}) \qquad \textbf{(A.18)}$$

In the EF, by using the equations (**A.14**) and (**A.18**):

$\ddot{\phi} + 3H\dot{\phi} = -V_{eff,\phi}(\phi)$

is obtained as the usual result for a spatially homogeneous scalar field. With the help of these, the Friedmann equations can be derived. Suppose that the universe is composed of ϕ, pressure free matter with density ρ_M coupled to ϕ by a coupling constant β, and radiation with density ρ_R. For the matter of species i conformally coupled to ϕ.

To obtain the first Friedmann Equation in the EF, $\tilde{T}^{\mu\nu}$ is used. The conformal transformation from the JF to the EF of the $\tilde{T}^{\mu\nu}$ is given by $T^{\mu\nu} = \tilde{T}^{\mu\nu}e^{6\beta_i\phi/M_{Pl}}$.

In the JF, $\tilde{T}$ is $\tilde{T}^{\mu\nu}\tilde{g}_{\mu\nu} = \tilde{\rho} + 3\tilde{p}$ for $n = 4$. By using the time-time component of $\tilde{T}^{\mu\nu}$, $\tilde{T}$ is found as $\tilde{T}^{00}\tilde{g}_{00} = -\tilde{\rho}$.

In the EF, T^{00} is found as $T^{00} = \tilde{T}^{00}e^{6\beta_i\phi/M_{Pl}} = -\frac{\tilde{\rho}}{\tilde{g}_{00}}e^{6\beta_i\phi/M_{Pl}} = -\tilde{\rho}e^{2\beta_i\phi/M_{Pl}}$,

where $\tilde{g}_{00} = g_{00}e^{2\beta_i\phi/M_{Pl}}$ and $g_{00} = -e^{2\beta_i\phi/M_{Pl}}$,

$$T^{00} = \tilde{\rho}e^{(1-3\omega_i)\beta_i\phi/M_{Pl}}, \qquad \textbf{(A.19)}$$

By using (**A.11**), the Friedmann Equation becomes

$$3H^2M_{Pl}^2 = \frac{1}{2}\dot{\phi}^2 + V(\phi) + \rho_M e^{\beta\phi/M_{Pl}} + \rho_R \qquad \textbf{(A.20)}$$

or

$$\rho_c \equiv \frac{1}{2}\dot{\phi}^2 + V(\phi) + \rho_M e^{\beta\phi/M_{Pl}} + \rho_R \qquad \textbf{(A.21)}$$

with $\Omega_M \equiv \frac{\rho_M e^{\beta\phi_{min}/M_{Pl}}}{\rho_c}$.

CURRICULUM VITAE

Name Surname: SİBEL BORAN

Place and Date of Birth: İstanbul / 31.05.1983

E-Mail: sibel.boran@hotmail.com

B.Sc.: IKU Physics Department

M.Sc.: ITU Physics Engineering Department