__İSTANBUL TECHNICAL UNIVERSITY ★ INSTITUTE OF SCIENCE AND TECHNOLOGY__

GEODESICS IN CYLINDRICAL SYMMETRIC SPACETIMES

MSc. Thesis by

Fuat İlkehan VARDARLI

Department : **Physics Engineering**

Programme: **Physics Engineering**

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(509041107)

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Supervisor (Chairman): **Prof. Dr. Neşe ÖZDEMİR**

Members of the Examining Committee **Prof.Dr. Ayşe Hümeyra BİLGE (İ.T.Ü.)**

Prof.Dr. Mahmut HORTAÇSU (İ.T.Ü.)

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BAZI SİLİNDİRİK SİMETRİK UZAYLARDA JEODEZİKLER

YÜKSEK LİSANS TEZİ

Fuat İlkehan VARDARLI
(509041107)

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Tez Danışmanı : Prof.Dr. Neşe ÖZDEMİR
Diğer Jüri Üyeleri Prof.Dr. Ayşe Hümeyra Bilge (İ.T.Ü.)
Prof.Dr. Mahmut HORTAÇSU (İ.T.Ü.)

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CONTENTS

ABBREVATIONS

GR	:	General Relativity
GUT	:	Grand Unified Theories
CMB	:	Cosmic Microwave Background
EEF	:	Einstein Field Equations

LIST OF SYMBOLS

$\mu, \nu, \rho, \kappa, \lambda, ...$	:	Spacetime indices
$i, j, k, l, m...$	:	Spatial indices
$G^{\mu\nu}$	:	Einstein Field Tensor
$T^{\mu\nu}$	:	Energy-Momentum Tensor
u^{μ}	:	Four velocity
P^{μ}	:	Four momentum
$F^{\mu\nu}$	:	Electromagnetic Field Tensor
$R^{\nu}_{\kappa\lambda\mu}$	:	Riemanian Curvature Tensor
$R_{\mu\nu}$	:	Ricci Tensor
R	:	Ricci Scalar
$\Gamma^{\lambda}_{\nu\mu}$	:	Christoffel Symbols

GEODESICS IN CYLINDRICAL SYMMETRIC SPACETIMES

SUMMARY

In general relativity a physical object or an observer follows timelike curve in the space-time which is called the worldline of the particle. This curve runs from past to the future in general, but in some space-times these curves intersect themselves and form closed timelike curves. It means that the worldline of the particle is not connected to earlier times and the causality principle is violated. According to the causality principle, each event in spacetime is a consequence of its cause and related to it directly in every rest frame. This contradiction can be disregarded in some cases by transferring a great amount of energy to the system via an external force allowing the existence of the closed timelike curves. If the trajectories of the particles are geodesics that is, if the timelike curves are geodesics then the external force to be applied will be null and one cannot get closed timelike curves without violating causality principle.

In this work we study the geodesics of some cylindrical spacetimes and examine the conditions for the geodesics to be timelike and null. After energy and the angular momentum for the timelike and null geodesics are calculated in Gödel-cosmic string spacetime, conditions required for the existence of the closed timelike curves are obtained. Furthermore, in the rotating cylindrical black hole spacetime with cosmological constant the geodesic structure is obtained. In this spacetime conditions necessary for the existence of the closed timelike curves are also obtained.

BAZI SİLİNDİRİK SİMETRİK UZAYLARDA JEODEZİKLER

ÖZET

Genel görelilikte fiziksel bir nesne veya bir gözlemci parçacığın "dünyaçizgisi" olarak anılan, uzayzamandaki zamansal eğriyi izler. Genel olarak eğri, geçmişten geleceğe uzanırken bazı uzayzamanlarda bu eğriler kesişir ve kapalı zamansal eğriler oluşturur. Bunun anlamı, parçacığın dünyaçizgisinin önceki zamanlara bağlı olmamasıdır ve nedensellik ihlal edilir. Nedensellik ilkesine göre, uzayzamandaki her olay bir sebebin sonucudur ve her durağan çerçevede sebebe doğrudan bağlıdır. Bu çelişki, bazı durumlarda, kapalı zamansal eğrilerin varlığını sağlayan çok büyük miktarda enerjinin bir dış kuvvet tarafından sisteme uygulanmasıyla dikkate alınmayabilir. Parçacıkların yörüngelerinin jeodezikler olduğu durumda, yani zamansal eğrilerin jeodezik olması halinde uygulanması gereken dış kuvvet sıfırdır ve nedensellik ilkesi ihlal edilmeden kapalı zamansal eğriler bulunamaz.

Bu çalışmada, bazı silindirik uzayzamanlarda jeodezikler ve bu jeodeziklerin zamansal veya ışıksal eğriler olması durumları incelenmiştir. Gödel-Kozmik Sicim uzayzamanında zamansal ve sıfır jeodezikler için enerji ve açısal momentum hesaplanmış, kapalı zamansal eğrilerin oluşumu için gerekli şartlar elde edilmiştir. İkinci olarak, kozmolojik sabitli, dönen silindirik kara delik uzayzamanında jeodezik yapısı çalışılmış, bu uzayzaman için kapalı zamansal eğrilerin bulunma koşulları elde edilmiştir.

1. INTRODUCTION

The gravitational field of the matter (any type) will affect the total geometry of the spacetime and this effect can be given by EEF which relate the metric field and curvature to the total matter field (energy) distribution. The nonlinearity of the field equations, does not allow to present precise motion of the matter in the background spacetime. In the absence of the gravitational field, i.e. in the flat spacetime, particles and light rays follow straight lines. But, in the presence of the gravitational field, the path of the particles and the radiation can not be determined precisely and to be determined as a principle feature of the general relativity.

After getting a solution of the Einstein field equations corresponding matter (energy) distribution, it remains to determine the motion of the free-falling particles in this field (background spacetime). In general relativity, it is a hypothesis that small freely moving bodies follow geodesic trajectories. That is, the free-falling or inertial motion occurs along the timelike and null geodesics of the spacetime parameterized by proper time τ as

$$ds^2 = -m^2 d\tau^2 \,. \tag{1.1}$$

In the case of a manifold with torsion-free and metric-compatible connection a geodesic curve is also an autoparallel curve. That is geodesics are such curves that, their tangent vectors are parallel transported.

In a coordinate system the tangent vector u to a curve γ in a coordinate frame x^λ can be given as

$$u^\lambda = \frac{dx^\lambda}{d\tau} \,. \tag{1.2}$$

Parallel transport equation of the tangent vector becomes

$$\nabla_u u = \frac{D}{d\tau}\frac{dx^\lambda}{d\tau} = 0 \tag{1.3}$$

where $D/d\tau = (dx^\lambda/d\tau)\nabla_\lambda$, which is the covariant derivative along the curve x^λ.

Then, with the definition of covariant derivative, one can obtain

$$\frac{d^2x^\lambda}{d\tau^2}+\Gamma^\lambda_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}=0 \tag{1.4}$$

and it is called geodesic equation. The norm of the tangent vector determine the character of geodesic. If $u^\lambda u_\lambda = -1$ then the geodesic is timelike and if $u^\lambda u_\lambda = 1$ the geodesic is spacelike . If $u^\lambda u_\lambda = 0$ the geodesic is called null geodesic. It is important to note that $u^\lambda u_\lambda$ as a constant of motion. In the meaning that, once we obtain the value of $u^\lambda u_\lambda$ at a point on the geodesic it will be constant along the geodesic curve. For example, a spacelike geodesic at one point cannot be timelike at another point [1].

The idea of having closed timelike curves in a spacetime violates the cosmic censorship conjecture. But in some cases the external forces act on the system such that closed timelike curves may form. Extremely great amount of energy is necessary for this external force. The best well known example of the spacetimes having closed timelike curves are Gödel type spaces. The existence and their stability are examined widely in the literature[2]. Therefore, it will be interesting to study other spacetimes having closed timelike curves on them.

In this thesis we study the timelike and/or lightlike geodesics of some cylindrical spacetimes first and examine the existence of the closed timelike curves. In the following chapter we give the information about geodesic mechanism. Section 3 is devoted to the geodesics of Gödel-cosmic string spacetime [3] and in chapter 4 we examine the trajectory of the timelike and null particles in the background spacetime of rotating cylindrical black hole [4].

2. GEODESIC EQUATIONS : METHODS

Freely-moving particles or light rays in given spacetime follow trajectories given by a set of second order differential equations; geodesic equation (1.4). This equation can be calculated by different methods. In a standard metric space, a geodesic on a Riemannian manifold M is defined as a curve $\gamma(\tau)$ minimizes the length of the curve. Explicitly, we can write the length of any curve as

$$s = \int_{\gamma} \sqrt{\pm g(\gamma, \dot{\gamma})}\, d\tau \tag{2.1}$$

where $\dot{\gamma}$ represents the derivative with respect to τ, and is a vector. $-(+)$ sign stands for timelike (spacelike) particles and it is equal to zero for radiation. Let S be action than

$$\delta S = -mc\delta \int ds = -mc \int \delta ds = 0 \tag{2.2}$$

$$\delta ds^2 = \delta\left(g_{\mu\nu} dx^{\mu} dx^{\nu}\right) = 2ds\delta ds = dx^{\mu} dx^{\nu} \frac{\partial g_{\mu\nu}}{\partial x^{\lambda}} \delta x^{\lambda} + 2 g_{\mu\nu} dx^{\mu} d\delta x^{\nu} \tag{2.3}$$

$$\Rightarrow \delta S = -mc \int \left(\frac{1}{2} \frac{dx^{\mu}}{ds} \frac{dx^{\nu}}{ds} \frac{\partial g_{\mu\nu}}{\partial x^{\lambda}} \delta x^{\lambda} + g_{\mu\lambda} \frac{dx^{\mu}}{ds} \frac{d\delta x^{\lambda}}{ds} \right) ds = 0 \tag{2.4}$$

$$\Rightarrow \delta S = -mc \int \left(\frac{1}{2} \frac{dx^{\mu}}{ds} \frac{dx^{\nu}}{ds} \frac{\partial g_{\mu\nu}}{\partial x^{\lambda}} \delta x^{\lambda} - \frac{d}{ds}\left(g_{\mu\lambda} \frac{dx^{\mu}}{ds} \right) \delta x^{\lambda} \right) ds = 0 \tag{2.5}$$

$$\Rightarrow \int \left(\frac{1}{2} \frac{dx^{\mu}}{ds} \frac{dx^{\nu}}{ds} \frac{\partial g_{\mu\nu}}{\partial x^{\lambda}} - \frac{d}{ds}\left(g_{\mu\lambda} \frac{dx^{\mu}}{ds} \right) \right) \delta x^{\lambda} ds = 0 \tag{2.6}$$

$$\Rightarrow \frac{1}{2} u^{\mu} u^{\nu} \frac{\partial g_{\mu\nu}}{\partial x^{\lambda}} - \frac{d}{ds}\left(g_{\mu\lambda} u^{\mu} \right) = 0 \tag{2.7}$$

$$\Rightarrow u^{\mu} u^{\nu} \frac{\partial g_{\mu\nu}}{\partial x^{\lambda}} - 2 g_{\mu\lambda} \frac{du^{\mu}}{ds} - 2u^{\mu} \frac{dg_{\mu\lambda}}{ds} = 0 \tag{2.8}$$

$$\Rightarrow u^{\mu} u^{\nu} \frac{\partial g_{\mu\nu}}{\partial x^{\lambda}} - 2 g_{\mu\lambda} \frac{du^{\mu}}{ds} - 2u^{\mu} \frac{dx^{\sigma}}{ds} \frac{\partial g_{\mu\lambda}}{\partial x^{\sigma}} = 0 \tag{2.9}$$

$$\Rightarrow u^{\mu} u^{\nu} \frac{\partial g_{\mu\nu}}{\partial x^{\lambda}} - 2 g_{\mu\lambda} \frac{du^{\mu}}{ds} - 2u^{\mu} u^{\sigma} \frac{\partial g_{\mu\lambda}}{\partial x^{\sigma}} = 0 \tag{2.10}$$

$$\Rightarrow u^{\mu} u^{\nu} \frac{\partial g_{\mu\nu}}{\partial x^{\lambda}} - 2 g_{\mu\lambda} \frac{du^{\mu}}{ds} - u^{\mu} u^{\sigma} \left(\frac{\partial g_{\mu\lambda}}{\partial x^{\sigma}} + \frac{\partial g_{\sigma\lambda}}{\partial x^{\mu}} \right) = 0 \tag{2.11}$$

$$\Rightarrow g_{\mu\lambda} \frac{du^{\mu}}{ds} + u^{\mu} u^{\sigma} \frac{1}{2} \left(\frac{\partial g_{\mu\lambda}}{\partial x^{\sigma}} + \frac{\partial g_{\sigma\lambda}}{\partial x^{\mu}} - \frac{\partial g_{\mu\sigma}}{\partial x^{\lambda}} \right) = 0 \tag{2.12}$$

$\sigma \to \nu$

$$\Rightarrow g^{\rho\lambda} g_{\mu\lambda} \frac{du^{\mu}}{ds} + u^{\mu} u^{\nu} \frac{1}{2} g^{\rho\lambda} \left(\frac{\partial g_{\mu\lambda}}{\partial x^{\nu}} + \frac{\partial g_{\nu\lambda}}{\partial x^{\mu}} - \frac{\partial g_{\mu\nu}}{\partial x^{\lambda}} \right) = 0 \tag{2.13}$$

$$\Rightarrow \frac{du^{\rho}}{ds} + \Gamma^{\rho}_{\mu\nu} u^{\mu} u^{\nu} = 0 \tag{2.14}$$

Having derived the geodesic equation, we should say some words about the parametrization of a geodesic path. From the form of (2.13) , it is clear that a transformation

$$s \to \tau = as + b,$$

for some constants a and b, leaves the equation (2.1) invariant. Any parameter related to the proper time in this way, is called an affine parameter, and is as good as the proper time for parameterizing a geodesic.

The primary usefulness of geodesics in GR is that they are the paths followed by unaccelerated particles. In fact, the geodesic equation can be thought of as the generalization of Newton's law $\mathbf{f} = m\mathbf{a}$ for the case $\mathbf{f} = 0$. It is also possible to introduce forces by adding terms to the right hand side of (2.1). As an example, we can write the equation of motion for a particle of mass m and charge q in GR as below [1]

$$\frac{d^2x^\lambda}{d\tau^2}+\Gamma^\lambda_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}=\frac{q}{m}F^\lambda_{\ \rho}\frac{dx^\rho}{d\tau} \tag{2.15}$$

where $F^\lambda_{\ \rho}$ is the electromagnetic tensor which is a function of coordinates. Indeed, one can show that, this equation is equivalent to equation (2.1). To prove this equivalence, we can rewrite the equation (2.15) as

$$\frac{d^2x^\lambda}{d\tau^2}+\Gamma^\lambda_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}=\alpha\frac{dx^\lambda}{d\tau} \tag{2.16}$$

where $\alpha=\alpha\left(x^\mu\right)$. By making a change of parameter $\sigma=\sigma\left(\tau\right)$, we obtain $d/d\tau=(d\sigma/d\tau)(d/d\sigma)$. Then we can transform equation (2.16) as below

$$\frac{d\sigma}{d\tau}\frac{d}{d\sigma}\left(\frac{d\sigma}{d\tau}\frac{dx^\lambda}{d\sigma}\right)+\Gamma^\lambda_{\mu\nu}\left(\frac{d\sigma}{d\tau}\right)^2\frac{dx^\mu}{d\sigma}\frac{dx^\nu}{d\sigma}=\alpha\frac{d\sigma}{d\tau}\frac{dx^\rho}{d\sigma}. \tag{2.17}$$

That is

$$\frac{d^2\sigma}{d^2\tau}\frac{dx^\lambda}{d\sigma}+\left(\frac{d\sigma}{d\tau}\right)^2\left(\frac{d^2x^\lambda}{d\sigma^2}+\Gamma^\lambda_{\mu\nu}\frac{dx^\mu}{d\sigma}\frac{dx^\nu}{d\sigma}\right)=\alpha\frac{d\sigma}{d\tau}\frac{dx^\rho}{d\sigma}. \tag{2.18}$$

Therefore we choose $\sigma\left(\lambda\right)$ such that

$$\frac{d^2\sigma}{d^2\tau}=\alpha\left(x^\lambda\left(\tau\right)\right)\frac{d\sigma}{d\tau}, \tag{2.19}$$

then

$$\frac{d^2x^\lambda}{d\sigma^2}+\Gamma^\lambda_{\mu\nu}\frac{dx^\mu}{d\sigma}\frac{dx^\nu}{d\sigma}=0 \tag{2.20}$$

Another relatively easy method which does not require to calculate Christoffell symbols is as follows : Consider a curve parameterised by τ

$$\frac{ds}{d\tau} = \sqrt{\left| g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} \right|} = \sqrt{\left| g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu \right|} \tag{2.21}$$

The total length from a to b along this curve can be calculated as below

$$s = \int_a^b ds = \int_a^b \frac{ds}{d\tau} d\tau = \int_a^b \sqrt{\left| g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu \right|} d\tau \tag{2.22}$$

$\sqrt{\left| g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu \right|}$ can be thought as the *lagrangian* of the system, and s as the *action*. To simplify the calculations, one can choose the affine parametrization $ds/d\tau = 1$, so we have $\pounds(x^\mu, \dot{x}^\mu) = g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu$. Then, the "action" becomes

$$s = \int_a^b \pounds(x^\mu, \dot{x}^\mu) d\tau \tag{2.23}$$

Minimizing this action, leads us to Euler-Lagrange equations as expected :

$$\delta s = \delta \int_a^b \pounds(x^\mu, \dot{x}^\mu) d\tau = 0 \Rightarrow \frac{d}{d\tau} \left(\frac{\partial \pounds}{\partial \dot{x}^\mu} \right) - \frac{\partial \pounds}{\partial x^\mu} = 0 \tag{2.24}$$

So, in the second method, we obtain the lagrangian from the metric and using Euler- Lagrange equations, we find the equations of motion and the orbits. Since we have $ds^2 = 0$ for light, the lagrangian becomes

$$\pounds = -m^2 = g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu \quad \text{(for timelike geodesics)} \tag{2.25}$$

$$\pounds = 0 = g_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu} \quad \text{(for null geodesics)} \tag{2.26}$$

3. GÖDEL-COSMIC STRING SOLUTION

In 1949, Gödel discovered a solution of Einstein's field equations which, unlike the Friedmann-Lemaitre solutions of relativistic cosmology, is incompatible with Weyl's postulate and does not allow the possibility of defining a universal cosmic time. Here, we consider Gödel-cosmic string solution which is Gödel universe including a cosmic string and calculate timelike geodesics of the spacetime [3].

Cosmic strings are one-dimensional topological defects that may have been formed during phase transitions in the early universe. The spacetime metric associated with a cosmic string is

$$ds^2 = -c^2dt^2 + dr^2 + r^2d\theta^2 + b^2r^2\sin^2\theta d\phi^2, \tag{3.1}$$

where b is a dimensionless constant parameter characteristic of the cosmic string such that

$$b = 1 - 4\mu. \tag{3.2}$$

Here μ, $0 \le \mu \le \frac{1}{4}$, is connected with the linear mass density ρ along the cosmic string via $\mu = G\rho/c^2$.The idea that GUT-scale cosmic strings with dimensionless parameter $\mu \sim 10^{-6}$ can be the source of the primordial perturbations needed for galaxy formation is excluded by recent measurements of the CMB anisotropy. However, a mixture of topological defects and inflation is consistent with the current CMB data. It is, therefore, interesting to investigate the nonlinear superposition of cosmic strings with other gravitational fields in order to determine the modifications in the standard physical consequences of general relativity due to the presence of cosmic strings and other cosmological defects.

In this section, the geodesics in Gödel-cosmic string solution is examined using the Euler-Lagrange equations of motion.

3.1. The Spacetime

The Gödel spacetime corresponds to a rigidly rotating infinitely long cylinder of pressure-free matter (dust) with constant density $\rho = \Omega^2/4\pi$ and a cosmological constant $\Lambda = -\Omega^2$. Einstein's equations can be written as below

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi T_{\mu\nu} \tag{3.3}$$

where $T_{\mu\nu} = \rho u_\mu u_\nu$. The corresponding metric is

$$ds^2 = -dx_0^2 + dx_1^2 - e^{\sqrt{2}\Omega x_1} dx_2^2 - dx_3^2 - 2\sqrt{2} e^{\sqrt{2}\Omega x_1} dx_0 dx_2 \tag{3.4}$$

With an appropriate choice of coordinate system, this metric can be written as below

$$\begin{aligned} ds^2 = -dt^2 + dr^2 + dz^2 - \frac{2}{\Omega^2} \sinh^2\left(\frac{\Omega r}{\sqrt{2}}\right) \left(\sinh^2\left(\frac{\Omega r}{\sqrt{2}}\right) - 1\right) d\varphi^2 \\ + \frac{4}{\Omega} \sinh^2\left(\frac{\Omega r}{\sqrt{2}}\right) d\varphi dt \end{aligned} \tag{3.5}$$

where the following transformations are done

$$\begin{gathered} e^{\sqrt{2}\Omega x_1} = \cosh\left(\sqrt{2}\Omega r\right) + \cos\varphi \sinh\left(\sqrt{2}\Omega r\right), \\ \sqrt{2}\Omega x_2 = \sin\varphi \sinh\left(\sqrt{2}\Omega r\right), \\ \tan\left(\frac{1}{2}\left[\Omega x_0 + t + \varphi\right]\right) = e^{\sqrt{2}\Omega r} \tan\frac{\varphi}{2}. \end{gathered}$$

It is interesting to note that, in the above transformation, as $\Omega \to 0$, we recover $x_1 = r\cos\varphi$, $x_2 = r\sin\varphi$, as expected. We expect that the Gödel-cosmic string solution of Einstein's field equations should have the metric

$$ds^2 = -dt^2 + dr^2 + dz^2 - \frac{2b^2}{\Omega^2} s^2 \left(s^2 - 1\right) d\varphi^2 + \frac{4b}{\Omega} s^2 d\varphi dt \tag{3.6}$$

where $s = \sinh\left(\Omega r/\sqrt{2}\right)$. This solution is obtained from (3.3) by the transformation $\varphi \to b\varphi$. One can verify explicitly that (3.4) indeed satisfies the Einstein field equations for a fluid with $u^\mu = (1,0,0,0)$ in cylindrical coordinates. The vorticity vector of the fluid is given by

$$\omega^\mu = \frac{1}{2\sqrt{-g}} \varepsilon^{\mu\nu\rho\sigma} u_\nu u_{\rho;\sigma}, \tag{3.7}$$

where $\varepsilon^{\mu\nu\rho\sigma}$ is the totally anti-symmetric tensor. The Gödel-cosmic string spacetime has constant vorticity only in the z-direction

$$\omega^z = \Omega \tag{3.8}$$

which indicates that as before, Ω is the rotation frequency of the Gödel-cosmic string universe. Moreover, for $\Omega \to 0$, we recover the metric including a cosmic string in cylindrical coordinates; thus, we can conclude that the metric (3.4) includes a cosmic string with string parameter b.

3.2. Euler-Lagrange Equations

Following the second method, lagrangian for Gödel-cosmic string metric can be written as below:

$$\pounds = -m^2 = -\dot{t}^2 + \dot{r}^2 + \dot{z}^2 - \frac{2b^2}{\Omega^2} s^2 \left(s^2 - 1\right) \dot{\varphi}^2 + \frac{4b}{\Omega} s^2 \dot{\varphi}\dot{t} \tag{3.9}$$

where $\dot{x}^\mu = \partial x^\mu / \partial \tau$. Then, Euler-Lagrange equations becomes

$$for\ t : \frac{d}{d\tau}\left(-2\dot{t} + \frac{4b}{\Omega} s^2 \dot{\varphi}\right) = 0 \tag{3.10}$$

$$\Rightarrow \dot{t} - \frac{2b}{\Omega} s^2 \dot{\varphi} = E \tag{3.11}$$

$$for\ \varphi : \frac{d}{d\tau}\left(-\frac{4b^2}{\Omega^2} s^2 \left(s^2-1\right)\dot{\varphi} + \frac{4b}{\Omega} s^2 \dot{t}\right) = 0 \tag{3.12}$$

$$\frac{b^2}{\Omega^2} s^2 \left(s^2-1\right)\dot{\varphi}^2 - \frac{b}{\Omega} s^2 \dot{t} = L \tag{3.13}$$

Solving equations (3.6) and (3.8) for $\dot{t}$ and $\dot{\varphi}$, we obtain

$$\frac{b^2}{\Omega^2} s^2 \left(s^2-1\right)\dot{\varphi} - \frac{b}{\Omega} s^2 \left(E + \frac{2b}{\Omega} s^2 \dot{\varphi}\right) = L \tag{3.14}$$

$$\Rightarrow \frac{4b^2}{\Omega^2} s^4 \dot{\varphi} - \frac{4b^2}{\Omega^2} s^2 \dot{\varphi} - \frac{4b}{\Omega} s^2 E - \frac{2b^2}{\Omega^2} s^4 \dot{\varphi} = L \tag{3.15}$$

$$\left(\frac{2b^2}{\Omega^2} s^4 - \frac{4b^2}{\Omega^2} s^2\right)\dot{\varphi} = L + \frac{4b}{\Omega} s^2 E \tag{3.16}$$

$$\dot{\varphi} = \frac{L\Omega^2 + 4b\Omega s^2 E}{2bs^2\left(s^2-1\right)} \tag{3.17}$$

$$\Rightarrow \dot{t} = E + \frac{2b}{\Omega} s^2 \left(\frac{L\Omega^2 + 4b\Omega s^2 E}{2bs^2\left(s^2-1\right)}\right) = E + \left(\frac{L\Omega + 4bs^2 E}{\left(s^2-1\right)}\right)$$
$$\frac{\left(s^2-1\right)E + L\Omega + 4bs^2 E}{\left(s^2-1\right)} = \frac{\left(\left(1+4b\right)s^2-1\right)E + L\Omega}{\left(s^2-1\right)} \tag{3.18}$$

$$for\ z : \frac{d}{d\tau}\left(2\dot{z}\right) = 0 \Rightarrow \dot{z} = \Phi \tag{3.19}$$

$$z = C_1\tau + C_2 \tag{3.20}$$

We can use the lagrangian to calculate $\dot{r}^2$

$$
\begin{aligned}
\dot{r}^2 = \dot{t}^2 - \Phi^2 - m^2 + \frac{2b^2}{\Omega^2}s^2\left(s^2-1\right)\dot{\varphi}^2 - \frac{4b}{\Omega}s^2\dot{\varphi}\dot{t}& \\
= \left(\frac{\left((1+4b)s^2-1\right)E + L\Omega}{\left(s^2-1\right)}\right)^2& \\
+ \frac{2b^2}{\Omega^2}s^2\left(s^2-1\right)\left(\frac{L\Omega^2+4b\Omega s^2E}{2bs^2\left(s^2-1\right)}\right)^2& \\
-\frac{4b}{\Omega}s^2\left(\frac{L\Omega^2+4b\Omega s^2E}{2bs^2\left(s^2-1\right)}\right)\left(\frac{\left((1+4b)s^2-1\right)E+L\Omega}{\left(s^2-1\right)}\right) - \Phi^2 - m^2&
\end{aligned}
\tag{3.21}
$$

Collecting in the same denominator $s^2(s^2-1)^2$, and simplifying the numerator of the first three expressions (A) on the right hand side of this equation

$$
\begin{aligned}
A = \left((1+4b)s^2-1\right)^2 s^2E^2 + \Omega^2s^2L^2 + 2\Omega\left((1+4b)s^2-1\right)s^2EL& \\
+\frac{\Omega^4\left(s^2-1\right)L^2 + 16b^2\Omega^2s^4\left(s^2-1\right)E^2 + 8\Omega^3bs^2\left(s^2-1\right)EL}{2\Omega^2}& \\
-8bs^4\left((1+4b)s^2-1\right)E^2 - 2s^2\Omega^2L^2 - \left(8bs^4\Omega + 2s^2\Omega\left((1+4b)s^2-1\right)\right)EL&
\end{aligned}
\tag{3.22}
$$

$$
\begin{aligned}
\Rightarrow A = \left[2\Omega^2s^2\left((1+4b)s^2-1\right)\left(s^2+4bs^2-1-8bs^2\right) + 16b^2\Omega^2s^4\left(s^2-1\right)\right]E^2& \\
+\left[\left(2s^2+s^2-1-4s^2\right)\Omega^4\right]L^2& \\
+\left[\left((1+4b)s^2-1\right)\left(2s^2+4s^2\right)\Omega^3 + 8\Omega^3bs^2\left(s^2-1\right)\right]EL&
\end{aligned}
\tag{3.23}
$$

$$
\begin{aligned}
\Rightarrow A = \left[2\Omega^2s^2\left(s^2+4bs^2-1\right)\left(s^2-4bs^2-1\right) + 16b^2\Omega^2s^6 - 16b^2\Omega^2s^4\right]E^2& \\
+\left[-\left(s^2+1\right)\Omega^4\right]L^2& \\
+\left[\left(s^2+4bs^2-1\right)6s^2\Omega^3 + 8\Omega^3bs^4 - 8\Omega^3bs^2\right]EL&
\end{aligned}
\tag{3.24}
$$

$$\Rightarrow A=\left[2\Omega^2s^6-4\Omega^2s^6-32\Omega^2b^2s^6+2\Omega^2s^4+16b^2\Omega^2s^6-16b^2\Omega^2s^4\right]E^2-c^2\Omega^4L^2$$
$$+\left[6s^4+24s^4b-6s^2+8bs^4-8bs^2\right]\Omega^3EL \quad (3.25)$$

$$\Rightarrow A=\left[s^4\left(1-s^2\right)-8b^2s^4\left(s^2+1\right)\right]2\Omega^2E^2-c^2\Omega^4L^2$$
$$+\left[6s^4+24s^4b-6s^2+8bs^4-8bs^2\right]\Omega^3EL \quad (3.26)$$

$$\Rightarrow A=\left[\left(1-s^2\right)-8b^2c^2\right]2s^4\Omega^2E^2-c^2\Omega^4L^2+\left[6s^2\left(s^2-1\right)+8bs^2\left(4s^2-1\right)\right]\Omega^3EL \quad (3.27)$$

where $c=\cosh\left(\Omega r/\sqrt{2}\right)$. Then finally

$$\dot{r}^2=\frac{\left[\left(1-s^2\right)-8b^2c^2\right]2s^4\Omega^2E^2-c^2\Omega^4L^2+\left[6s^2\left(s^2-1\right)+8bs^2\left(4s^2-1\right)\right]\Omega^3EL}{s^2\left(s^2-1\right)^2}$$
$$-\Phi^2-m^2 \quad (3.28)$$

3.3. Closed Timelike Curves

The existence of the closed timelike curves can be expressed mathematically as follows [2] : Let us denote γ, a closed curve given in its parametric form by,

$$t=t_0;\quad r=r_0;\quad z=z_0;\quad \varphi=[0,2\pi] \quad (3.29)$$

where t_0, r_0 and z_0 are constants. When γ is parametrized with an arbitrary parameter σ, we have a timelike curve if

$$g_{\mu\nu}\frac{dx^{\mu}}{d\tau}\frac{dx^{\nu}}{d\tau} < 0 \tag{3.30}$$

with the metric signature $(-,+,+,+)$. In the Gödel-cosmic string spacetime [3], this condition reduce to $g_{\varphi\varphi} < 0$ i.e.,

$$-\frac{2b^2}{\Omega^2}s^2\left(s^2-1\right) < 0 \tag{3.31}$$

4. CYLINDRICAL ROTATING BLACK HOLE

There are many solutions which contains cylindrical symmetry, like the solutions of Levi- Civita, Chazy-Curzon and the stationary solution of Lewis. In this section, we will analyze the geodesics around a cylindrical symmetric, rotating blackhole which is found by Lemos in 1994 [4].

4.1. The Spacetime

Einstein Hilbert action in four dimension is given by

$$S = \frac{1}{16\pi G} \int d^4x \sqrt{-g}\,(R - 2\Lambda) \tag{4.1}$$

where g is the determinant of the metric and R the Ricci scalar. We assume that the spacetime is cylindrically symmetric and time-independent, i.e, there are three Killing vectors: $\partial/\partial z$ which corresponds to the translational symmetry along the z axis, $\partial/\partial\varphi$ which has closed periodic trajectories around the z axis and $\partial/\partial t$ corresponding to the invariance under time translations. The solution which satisfies the equations of motion (4.1), can be written as below

$$\begin{aligned} ds^2 = -\left[\left(\lambda^2 - \frac{\omega^2}{\alpha^2}\right)\alpha^2 r^2 - \frac{b\lambda^2}{\alpha r}\right] dt^2 - 2\frac{\omega b}{\alpha^3 r} d\varphi dt + \frac{dr^2}{\alpha^2 r^2 - \frac{b}{\alpha r}} \\ + \left[\left(\lambda^2 - \frac{\omega^2}{\alpha^2}\right) r^2 + \frac{b\omega^2}{\alpha^5 r}\right] d\varphi^2 + \alpha^2 r^2 dz^2 \end{aligned} \tag{4.2}$$

$$-\infty < t < \infty, \quad 0 \le r < \infty, \quad 0 \le \varphi < 2\pi \quad -\infty < z < \infty.$$

Here r is the radial circumferential coordinate, $\alpha^2 = -\Lambda/3 > 0$, ω and λ are constant parameters corresponding to the angular momentum of the spacetime and b is a constant which is related to the mass

$$b = \frac{1}{4}\left(-M + 3M\sqrt{1 - \frac{8J^2\alpha^2}{9M^2}}\right) \tag{4.3}$$

where M and J are the ADM mass and angular momentum respectively. Equation (4.2) represents a rotating blackhole with an event horizon at $\alpha r = b^{1/3}$.Since the Kretschmann scalar is given by $R_{abcd}R^{abcd} = 24\alpha^4\left(1 + b^2/2\alpha^6 r^6\right)$, there is a scalar polynomial singularity at $\alpha r = 0$.

4.2. Geodesics, Euler-Lagrange Equations

The Lagrangian for this metric can be written as below

$$\surd ounds = -1 = -\left[\left(\lambda^2 - \frac{\omega^2}{\alpha^2}\right)\alpha^2 r^2 - \frac{b\lambda^2}{\alpha r}\right]\dot{t}^2 - 2\frac{\omega b}{\alpha^3 r}\dot{\varphi}\dot{t} + \frac{\dot{r}^2}{\alpha^2 r^2 - \frac{b}{\alpha r}} + \left[\left(\lambda^2 - \frac{\omega^2}{\alpha^2}\right)r^2 + \frac{b\omega^2}{\alpha^5 r}\right]\dot{\varphi}^2 + \alpha^2 r^2 \dot{z}^2 \tag{4.4}$$

To simplify the calculations, we define two new radial functions :

$$A(r) = \left(\lambda^2 - \frac{\omega^2}{\alpha^2}\right)\alpha^2 r^2 - \frac{b\lambda^2}{\alpha r} \tag{4.5}$$

$$B(r) = \left(\lambda^2 - \frac{\omega^2}{\alpha^2}\right)r^2 + \frac{b\omega^2}{\alpha^5 r} \tag{4.6}$$

With these new functions, lagrangian becomes

$$\pounds = -1 = -A\dot{t}^2 - 2\frac{\omega b}{\alpha^3 r}\dot{\varphi}\dot{t} + \frac{\dot{r}^2}{\alpha^2 r^2 - \frac{b}{\alpha r}} + B\dot{\varphi}^2 + \alpha^2 r^2 \dot{z}^2 \tag{4.7}$$

Euler-Lagrange equations become

$$\frac{d}{ds}\left(-2A\dot{t}-\frac{2\omega b}{\alpha^3 r}\dot{\varphi}\right)=0 \tag{4.8}$$

$$\frac{d}{ds}\left(-\frac{2\omega b}{\alpha^3 r}\dot{t}+2B\dot{\varphi}\right)=0 \tag{4.9}$$

$$\frac{d}{ds}\left(2\alpha^2 r^2 \dot{z}\right)=0 \Rightarrow \dot{z}=\frac{\Omega}{\alpha^2 r^2} \tag{4.10}$$

Then, we have

$$A\dot{t}+\frac{\omega b}{\alpha^3 r}\dot{\varphi}=E \tag{4.11}$$

$$-\frac{\omega b}{\alpha^3 r}\dot{t}+B\dot{\varphi}=L \tag{4.12}$$

Solving equations (4.10) and (4.11) for $\dot{t}$ and $\dot{\varphi}$, we obtain

$$A\frac{\omega b}{\alpha^3 r}\dot{t}+\frac{\omega^2 b^2}{\alpha^6 r^2}\dot{\varphi}=E\frac{\omega b}{\alpha^3 r} \tag{4.13}$$

$$-A\frac{\omega b}{\alpha^3 r}\dot{t}+AB\dot{\varphi}=LA \tag{4.14}$$

$$\Rightarrow \dot{\varphi}=\frac{\frac{\omega b E}{\alpha^3 r}+AL}{AB+\frac{\omega^2 b^2}{\alpha^6 r^2}} \quad ; \quad \dot{t}=\frac{EB-\frac{\omega b L}{\alpha^3 r}}{AB+\frac{\omega^2 b^2}{\alpha^6 r^2}} \tag{4.15}$$

Then, the radial geodesic becomes

$$\frac{\dot{r}^2}{\alpha^2 r^2 - \frac{b}{\alpha r}}$$

$$= A\frac{\left(EB - \frac{\omega b L}{\alpha^3 r}\right)^2}{\left(AB + \frac{\omega^2 b^2}{\alpha^6 r^2}\right)^2} + 2\frac{\omega b}{\alpha^3 r}\frac{\left(EB - \frac{\omega b L}{\alpha^3 r}\right)\left(\frac{\omega b E}{\alpha^3 r} + AL\right)}{\left(AB + \frac{\omega^2 b^2}{\alpha^6 r^2}\right)^2} - B\frac{\left(\frac{\omega b E}{\alpha^3 r} + AL\right)^2}{\left(AB + \frac{\omega^2 b^2}{\alpha^6 r^2}\right)^2} - \left(\frac{\Omega^2}{\alpha^2 r^2} + m^2\right) \tag{4.16}$$

Then, $\dot{r}^2$ becomes

$$\frac{\dot{r}^2}{\left(\alpha^2 r^2 - \frac{b}{\alpha r}\right)} = \frac{\left((BE^2 - AL^2) - \frac{2\omega b E L}{\alpha^3 r}\right)}{\left(\frac{\omega^2 b^2}{\alpha^6 r^2} + AB\right)} - \left(\frac{\Omega^2}{\alpha^2 r^2} + m^2\right) \tag{4.17}$$

In order to calculate null geodesics $(m = 0)$ lying on the $z = const.$ plane, we write $\dot{r}$ as below

$$\Rightarrow \dot{r} = \sqrt{\frac{\left(\alpha^2 r^2 - \frac{b}{\alpha r}\right)\left((BE^2 - AL^2) - \frac{2\omega b E L}{\alpha^3 r}\right)}{\left(\frac{\omega^2 b^2}{\alpha^6 r^2} + AB\right)}} \tag{4.18}$$

Integrating this expression will give us the relation between r and the proper time of the particle τ; $\tau = \tau(r)$:

$$\tau = \int \sqrt{\frac{\left(\frac{\omega^2 b^2}{\alpha^6 r^2} + AB\right)}{\left(\alpha^2 r^2 - \frac{b}{\alpha r}\right)\left((BE^2 - AL^2) - \frac{2\omega b E L}{\alpha^3 r}\right)}}\, dr \tag{4.19}$$

Simplifying this expression is a messy work. As a first step we calculate the product AB

$$A = \alpha^2\left(\lambda^2 - \frac{\omega^2}{\alpha^2}\right) r^2 - \frac{b\lambda^2}{\alpha r} = P_1 r^2 - \frac{P_2}{r} \tag{4.20}$$

$$B = \left(\lambda^2 - \frac{\omega^2}{\alpha^2}\right) r^2 + \frac{b\omega^2}{\alpha^5 r} = Q_1 r^2 + \frac{Q_2}{r} \tag{4.21}$$

$$
\begin{aligned}
AB = \left(P_1 r^2 - \frac{P_2}{r}\right)\left(Q_1 r^2 + \frac{Q_2}{r}\right) \\
= P_1 Q_1 r^4 + \frac{P_1 Q_2}{r} - \frac{P_2 Q_1}{r} - \frac{P_2 Q_2}{r^2} \\
= \alpha^2\left(\lambda^2 - \frac{\omega^2}{\alpha^2}\right)^2 r^4 + \alpha^2\left(\lambda^2 - \frac{\omega^2}{\alpha^2}\right)\frac{b\omega^2}{\alpha^5} r - \left(\lambda^2 - \frac{\omega^2}{\alpha^2}\right)\frac{b\lambda^2}{\alpha} r - \frac{b^2\omega^2\lambda^2}{\alpha^6 r^2} \\
= \left(\lambda^2\alpha^2 - \omega^2\right)^2 r^4 + r\left(\lambda^2\alpha^2 - \omega^2\right)\left(\frac{b\omega^2}{\alpha^5} - \frac{b\lambda^2}{\alpha^3}\right) - \frac{b^2\omega^2\lambda^2}{\alpha^6 r^2} \\
= \frac{1}{\alpha^2}\left(\lambda^2\alpha^2 - \omega^2\right)^2 r^4 + \frac{b}{\alpha^5}\left(\lambda^2\alpha^2 - \omega^2\right)\left(\omega^2 - \lambda^2\alpha^2\right) r - \frac{b^2\omega^2\lambda^2}{\alpha^6 r^2} \\
= \frac{\left(\lambda^2\alpha^2 - \omega^2\right)^2 \alpha^4 r^6 + -b\left(\lambda^2\alpha^2 - \omega^2\right)^2 \alpha r^3 - b^2\omega^2\lambda^2}{\alpha^6 r^2} \qquad (4.22)
\end{aligned}
$$

Then, the numerator becomes

$$
\begin{aligned}
\frac{\omega^2 b^2}{\alpha^6 r^2} + AB = \frac{\left(\lambda^2\alpha^2 - \omega^2\right)^2 \alpha^4 r^6 - b\left(\lambda^2\alpha^2 - \omega^2\right)^2 \alpha r^3 + \omega^2 b^2\left(1 - \lambda^2\right)}{\alpha^6 r^2} \\
= \frac{n_1 r^6 + n_2 r^3 + n_3}{\alpha^6 r^2} \qquad (4.23)
\end{aligned}
$$

where

$$
\begin{aligned}
n_1 &= \alpha^8\left(\lambda^2 - \frac{\omega^2}{\alpha^2}\right)^2 \\
n_2 &= -\frac{n_1 b}{\alpha^3} \\
n_3 &= \omega^2 b^2\left(1 - \lambda^2\right).
\end{aligned}
$$

Then the integral becomes

$$
\tau = \int \sqrt{\frac{n_1 r^6 + n_2 r^3 + n_3}{\alpha^6 r^2\left[\left(\alpha^2 r^2 - \frac{b}{\alpha r}\right)\left(\left(BE^2 - AL^2\right) - \frac{2\omega bEL}{\alpha^3 r}\right)\right]}} dr \qquad (4.24)
$$

The denominator can be simplified as below

$$
\begin{aligned}
&\alpha^6 r^2\left[\left(\alpha^2 r^2-\frac{b}{\alpha r}\right)\left(\left(BE^2-AL^2\right)-\frac{2\omega bEL}{\alpha^3 r}\right)\right]\\
&\qquad=\frac{\alpha^6 r^2}{\alpha^4 r^2}\left[\left(\alpha^3 r^3-b\right)\left(\left(BE^2-AL^2\right)\alpha^3 r-2\omega bEL\right)\right]\\
&=\alpha^2\left[\left(\alpha^3 r^3-b\right)\left(\left(Q_1E^2r^2+\frac{Q_2}{r}E^2-P_1L^2r^2+\frac{P_2}{r}L^2\right)\alpha^3 r-2\omega bEL\right)\right]\\
&=\left(\alpha^5 r^3-b\alpha^2\right)\left(Q_1E^2\alpha^3r^3+Q_2\alpha^3E^2-P_1L^2r^3\alpha^3+P_2L^2\alpha^3\right)\\
&\qquad-2\omega bEL\alpha^5r^3+2\omega bELb\alpha^2\\
&=Q_1E^2\alpha^8r^6+Q_2\alpha^8E^2r^3-P_1L^2\alpha^8r^6+P_2L^2\alpha^8r^3-Q_1E^2\alpha^5br^3-Q_2b\alpha^5E^2\\
&\qquad+P_1L^2b\alpha^3r^3+P_2L^2b\alpha^5-2\omega bEL\alpha^5r^3+2\omega bELb\alpha^2\\
&=\left(Q_1E^2\alpha^8-P_1L^2\alpha^8\right)r^6\\
&+\left(Q_2\alpha^8E^2+P_2L^2\alpha^8-Q_1E^2\alpha^5b+P_1L^2b\alpha^3-2\omega bEL\alpha^5\right)r^3\\
&\qquad+P_2L^2b\alpha^5-Q_2b\alpha^5E^2+2\omega bELb\alpha^2\\
&=k_1r^6+k_2r^3+k_3
\end{aligned}
\tag{4.25}
$$

where

$$
\begin{aligned}
k_1 &= \alpha^8\frac{\left(\lambda^2\alpha^2-\omega^2\right)}{\alpha^2}\left(E^2-\alpha^2L^2\right)=\alpha^6\left(\lambda^2\alpha^2-\omega^2\right)\left(E^2-\alpha^2L^2\right)\\
k_2 &= b\alpha^3L^2\left(\omega^2\left(\alpha^2+\alpha^8-1\right)+\lambda^2\left(1-\alpha^8+\alpha^4\right)\right)\\
k_3 &= b^2\left(\lambda^2\alpha^4L^2-\omega^2E^2+2\omega\alpha^2EL\right)
\end{aligned}
$$

Finally we can write the integral as below

$$
\tau=\int\sqrt{\frac{n_1r^6+n_2r^3+n_3}{k_1r^6+k_2r^3+k_3}}dr \tag{4.26}
$$

It is not possible to evaluate this integral in this form. However, by choosing $k_1=k_2=0$, we find AppellF1 hypergeometric series as the result of the integral. Let us take $E=\alpha L$,then, $k_1=0$.And $k_2=0$ gives us the condition:

$$\frac{\omega^2}{\lambda^2} = \frac{\left(\alpha^8 - \alpha^4 - 1\right)}{\left(\alpha^2 + \alpha^8 - 1\right)} \tag{4.27}$$

Under the condition $E = \alpha L$, k_3 becomes:

$$k_3 = b^2 L^2 \left(\lambda^2 \alpha^4 - \omega^2 \alpha^2 + 2\omega\alpha^3\right)$$

Now, in order to find $\varphi = \varphi\left(r\right)$, we will calculate $\dot{\varphi}^2/\dot{r}^2$. One can find $\dot{\varphi}^2$ as below

$$\dot{\varphi} = \frac{\frac{\omega bE}{\alpha^3 r} + AL}{AB + \frac{\omega^2 b^2}{\alpha^6 r^2}} = \frac{\frac{\omega bE}{\alpha^3 r} + AL}{\frac{n_1 r^6 + n_2 r^3 + n_3}{\alpha^6 r^2}} = \frac{\left(\frac{\omega bE}{\alpha^3 r} + AL\right)\alpha^6 r^2}{n_1 r^6 + n_2 r^3 + n_3} \tag{4.28}$$

$$\frac{\omega bE}{\alpha^3 r} + AL = \frac{\omega bE}{\alpha^3 r} + P_1 L r^2 - \frac{P_2 L}{r} = \frac{1}{\alpha^3 r}\left(\omega bE + P_1 L\alpha^3 r^3 - P_2 L\alpha^3\right) \tag{4.29}$$

$$\begin{aligned}\left(\frac{\omega bE}{\alpha^3 r} + AL\right)\alpha^6 r^2 &= \left(\omega bE + LP_1\alpha^3 r^3 - P_2 L\alpha^3\right)\alpha^3 r \\ &= LP_1\alpha^3 r^4 + \left(\omega b\alpha^3 E - P_2 L\alpha^6\right) r \\ &= L\alpha^3\left(\lambda^2\alpha^2 - \omega^2\right) r^4 + \left(\omega b\alpha^3 E - b\lambda^2 L\alpha^5\right) r = Fr^4 + Gr \end{aligned} \tag{4.30}$$

where

$$\begin{aligned} F &= L\alpha^3\left(\lambda^2\alpha^2 - \omega^2\right) \\ G &= \left(\omega b\alpha^3 E - b\lambda^2 L\alpha^5\right) \end{aligned}$$

$$\Rightarrow \dot{\varphi} = \frac{Fr^4 + Gr}{n_1 r^6 + n_2 r^3 + n_3} \tag{4.31}$$

$$\dot{\varphi}^2 = \frac{\left(Fr^4 + Gr\right)^2}{\left(n_1 r^6 + n_2 r^3 + n_3\right)^2} \tag{4.32}$$

$$\begin{aligned}
\Rightarrow \left(\frac{d\varphi}{dr}\right)^2 = \frac{\dot{\varphi}^2}{\dot{r}^2} &= \frac{\left(Fr^4 + Gr\right)^2}{\left(n_1 r^6 + n_2 r^3 + n_3\right)^2} \frac{n_1 r^6 + n_2 r^3 + n_3}{k_1 r^6 + k_2 r^3 + k_3} \\
&= \frac{r^2 \left(m_1 r^6 + m_2 r^3 + m_3\right)}{\left(n_1 r^6 + n_2 r^3 + n_3\right)\left(k_1 r^6 + k_2 r^3 + k_3\right)}
\end{aligned} \tag{4.33}$$

$$\varphi = \int \sqrt{\frac{\left(Fr^4 + Gr\right)^2}{\left(n_1 r^6 + n_2 r^3 + n_3\right)\left(k_1 r^6 + k_2 r^3 + k_3\right)}} dr \tag{4.34}$$

Again, choosing $k_1 = k_2 = 0$, one can obtain AppellF1 hypergeometric series.

4.3. Closed Circular Orbits

If we consider $z = const.$ $(\Omega = 0)$ only, we get orbits of the particle in (r, θ) plane. Let us assume that $\dot{r}^2$ as effective potential $\left(V_{eff}\right)$. To get circular orbits, we should set $V_{eff} = 0$ and $V'_{eff} = \frac{\partial V_{eff}}{\partial r} = 0$ simultaneously. If we want the orbits's stableness, V''_{eff} should be less than zero simultaneously.

$$\frac{V_{eff}}{\left(\alpha^2 r^2 - \frac{b}{\alpha r}\right)} = \frac{\left(\left(BE^2 - AL^2\right) - \frac{2\omega bEL}{\alpha^3 r}\right)}{\left(\frac{\omega^2 b^2}{\alpha^6 r^2} + AB\right)} - m^2 = 0 \tag{4.35}$$

or

$$\left(\left(BE^2 - AL^2\right) - \frac{2\omega bEL}{\alpha^3 r}\right) = \left(\frac{\omega^2 b^2}{\alpha^6 r^2} + AB\right) \tag{4.36}$$

$V'_{eff} = 0$ gives

$$m^2 \left(A_r B + AB_r - \frac{2\omega^2 b^2}{\alpha^6 r^3}\right) = \left(B_r E^2 - A_r L^2 + \frac{2\omega b}{\alpha^3 r^2} EL\right) \tag{4.37}$$

and V''_{eff} gives

$$\begin{aligned} m^2 \left(A_{rr} B + 2A_r B_r + AB_{rr} + \frac{6\omega^2 b^2}{\alpha^6 r^4}\right) &= B_{rr} E^2 - A_{rr} L^2 - \frac{4\omega bEL}{\alpha^3 r} \\ &= \frac{2}{r^2}\left(BE^2 - AL^2 - \frac{2\omega b}{\alpha^3 r}\right) \end{aligned} \tag{4.38}$$

where $A_r = \frac{\partial A}{\partial r}$ and $A_{rr} = \frac{\partial^2 A}{\partial r^2}$.One can solve these three equations simultaneously for E, L and r. For

$$r_0 = \frac{b^{1/3}\left(-\alpha^2\lambda^2 + \omega^2 + \sqrt{\alpha^4\lambda^4 + 2\alpha^2\left(-20 + 19\lambda^2\right)\omega^2 + \omega^4}\right)^{1/3}}{10^{1/3}\left(\alpha^5\lambda^2 - \alpha^3\omega^2\right)^{1/3}} \tag{4.39}$$

we found the values of E and L .(see appendix)

4.4. Closed Timelike Curves/Geodesics

Under the coordinate transformation $r \to -r$, the line element is also a solution of Einstein's field equations:

$$ds^2 = -\left[\left(\lambda^2 - \frac{\omega^2}{\alpha^2}\right)\alpha^2 r^2 + \frac{b\lambda^2}{\alpha r}\right] dt^2 + 2\frac{\omega b}{\alpha^3 r} d\varphi dt + \frac{dr^2}{\alpha^2 r^2 + \frac{b}{\alpha r}}$$
$$+ \left[\left(\lambda^2 - \frac{\omega^2}{\alpha^2}\right) r^2 - \frac{b\omega^2}{\alpha^5 r}\right] d\varphi^2 + \alpha^2 r^2 dz^2 \quad (4.40)$$

The existence of the closed timelike curves is examined as in the case of Gödel-cosmic string spacetime : Let us denote γ, a closed curve given in its parametric form by,

$$t = t_0; \quad r = r_0; \quad z = z_0; \quad \varphi = [0, 2\pi] \tag{4.41}$$

where t_0, r_0 and z_0 are constants. When γ is parametrized with an arbitrary parameter τ, we have a timelike curve if

$$g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} < 0 \tag{4.42}$$

with the metric signature $(-,+,+,+)$. In the rotating black hole spacetime [4], this condition reduce to $g_{\varphi\varphi} < 0$ i.e.,

$$\left(\lambda^2 - \frac{\omega^2}{\alpha^2}\right) r_0^2 - \frac{b\omega^2}{\alpha^5 r_0} < 0 \tag{4.43}$$

$$\Rightarrow r_0 < \left(\frac{b\omega^2}{(\alpha^2\lambda^2-\omega^2)\,\alpha^3}\right)^{1/3} \tag{4.44}$$

In order to find closed timelike geodesics, we must set $a^r = 0$, the four-acceleration vector in the radial direction, in other words,

$$\Gamma^r_{\varphi\varphi} = 0 \tag{4.45}$$

$$-\frac{\left(b+\alpha^3 r_0^3\right)\left(2\alpha^5\lambda^2 r_0^3 + b\omega^2 - 2\alpha^3 r_0^3\omega^2\right)}{2\alpha^6 r_0^3} = 0 \tag{4.46}$$

We solve ω from this equation and substitute in the (4.43) we obtain

$$\omega = \frac{\sqrt{2}\alpha^{5/2}\lambda r_0^{3/2}}{\sqrt{-b+2\alpha^3 r_0^3}} \tag{4.47}$$

$$\frac{3b\lambda^2 r_0^2}{b-2\alpha^3 r_0^3} < 0 \tag{4.48}$$

5. CONCLUSION

In this work we study the geodesics of some cylindrical spacetimes and examine the conditions for the geodesics to be timelike and null. After energy and the angular momentum for the timelike and null geodesics are calculated in Gödel-cosmic string spacetime, conditions required for the existence of the closed timelike curves are obtained. Furthermore, in the rotating cylindrical black hole spacetime with cosmological constant the geodesic structure is obtained. In this spacetime conditions necessary for the existence of the closed timelike curves are also obtained.

```
a1 = ((λ^2 - ω^2 / α^2) α^2 r^2 - b λ^2 / (α r))
a2 = ((λ^2 - ω^2 / α^2) r^2 + b ω^2 / (α^5 r))
a1r = D[a1, r]
a2r = D[a2, r]
a1rr = D[a1, r, r]
a2r = D[a2, r, r]
```

$$-\frac{b\lambda^2}{r\alpha} + r^2\alpha^2\left(\lambda^2 - \frac{\omega^2}{\alpha^2}\right)$$

$$\frac{b\omega^2}{r\alpha^5} + r^2\left(\lambda^2 - \frac{\omega^2}{\alpha^2}\right)$$

$$\frac{b\lambda^2}{r^2\alpha} + 2r\alpha^2\left(\lambda^2 - \frac{\omega^2}{\alpha^2}\right)$$

$$-\frac{b\omega^2}{r^2\alpha^5} + 2r\left(\lambda^2 - \frac{\omega^2}{\alpha^2}\right)$$

$$-\frac{2b\lambda^2}{r^3\alpha} + 2\alpha^2\left(\lambda^2 - \frac{\omega^2}{\alpha^2}\right)$$

$$\frac{2b\omega^2}{r^3\alpha^5} + 2\left(\lambda^2 - \frac{\omega^2}{\alpha^2}\right)$$

```
veff = FullSimplify[
  a2 * ee^2 - a1 * le^2 - 2 * w * b * ee * le / (α^3 * r) - m^2 * (a1 * a2 + w^2 b^2 / α^6 / r^2)]
veff1 = FullSimplify[m^2 * (a1r * a2 + a1 * a2r - 2 * w^2 * b^2 / α^6 * r^3) -
   a2r * ee^2 + a1r * le^2 - 2 * w * b * ee * le / α^3 * r^2]
```

$$-a1\,le^2 + a2\left(ee^2 - a1\,m^2\right) - \frac{b\,w\left(b\,m^2\,w + 2\,ee\,le\,r\,\alpha^3\right)}{r^2\alpha^6}$$

$$a2r\left(-ee^2 + a1\,m^2\right) + a1r\left(le^2 + a2\,m^2\right) - \frac{2\,b\,r^2\,w\left(b\,m^2\,r\,w + ee\,le\,\alpha^3\right)}{\alpha^6}$$

```
Solve[veff == 0, ee]
```

$$ee = \frac{b\,le\,r\,w\,\alpha^3 - \sqrt{b^2\,le^2\,r^2\,w^2\,\alpha^6 + a2\,b^2\,m^2\,r^2\,w^2\,\alpha^6 + a1\,a2\,le^2\,r^4\,\alpha^{12} + a1\,a2^2\,m^2\,r^4\,\alpha^{12}}}{a2\,r^2\,\alpha^6}$$

$$\frac{b\,le\,r\,w\,\alpha^3 - \sqrt{b^2\,le^2\,r^2\,w^2\,\alpha^6 + a2\,b^2\,m^2\,r^2\,w^2\,\alpha^6 + a1\,a2\,le^2\,r^4\,\alpha^{12} + a1\,a2^2\,m^2\,r^4\,\alpha^{12}}}{a2\,r^2\,\alpha^6}$$

Simplify[Solve[veff1 == 0, le]]

$$\Big\{\Big\{le \to -\sqrt{\left(-\left(2\,a2r\,b^4 m^2 r^3 w^4 - 4\,a2r\,b^4 m^2 r^5 w^4 + 2\,a2\,b^4 m^2 r^6 w^4 - 4\,a2\,b^4 m^2 r^8 w^4 + 3\,a1r\,a2\,a2r\,b^2 m^2 r^2 w^2 \alpha^6 + a1\,a2r^2 b^2 m^2 r^2 w^2 \alpha^6 + 2\,a1r\,a2^2 b^2 m^2 r^5 w^2 \alpha^6 + 4\,a1\,a2\,a2r\,b^2 m^2 r^5 w^2 \alpha^6 + 2\,a1r\,a2^2 b^2 m^2 r^7 w^2 \alpha^6 - 2\,a1\,a2\,a2r\,b^2 m^2 r^7 w^2 \alpha^6 + 2\,a1\,a2^2 b^2 m^2 r^8 w^2 \alpha^6 - a1r^2 a2^3 m^2 r^4 \alpha^{12} + a1\,a1r\,a2^2 a2r\,m^2 r^4 \alpha^{12} - 2\sqrt{\left(b^2 m^4 r^2 \left(a2r + a2\,r^3\right)^2 w^2 \left(b^2 w^2 + a1\,a2\,r^2 \alpha^6\right) \left(b^4 r^4 \left(1 - 2\,r^2\right)^2 w^4 + b^2 \left(a1\,r^3 \left(2\,a2r - 2\,a2r\,r^2 + a2\,r^3\right) + a1r \left(a2r + 2\,a2\,r^3 - 2\,a2\,r^5\right)\right) w^2 \alpha^6 + a1\,a1r\,a2\,a2r\,r^2 \alpha^{12}\right)\right)}\right) \Big/ \left(r^2 \alpha^6 \left(-a1r^2 a2^2 r^2 \alpha^6 + a1\,r^2 \left(4\,a2r\,b^2 r\,w^2 + 4\,a2\,b^2 r^4 w^2 - a1\,a2r^2 \alpha^6\right) + 2\,a1r \left(2\,a2r\,b^2 w^2 + 2\,a2\,b^2 r^3 w^2 + a1\,a2\,a2r\,r^2 \alpha^6\right)\right)\right)}\Big\},$$

$$\Big\{le \to \sqrt{\left(-\left(2\,a2r\,b^4 m^2 r^3 w^4 - 4\,a2r\,b^4 m^2 r^5 w^4 + 2\,a2\,b^4 m^2 r^6 w^4 - 4\,a2\,b^4 m^2 r^8 w^4 + 3\,a1r\,a2\,a2r\,b^2 m^2 r^2 w^2 \alpha^6 + a1\,a2r^2 b^2 m^2 r^2 w^2 \alpha^6 + 2\,a1r\,a2^2 b^2 m^2 r^5 w^2 \alpha^6 + 4\,a1\,a2\,a2r\,b^2 m^2 r^5 w^2 \alpha^6 + 2\,a1r\,a2^2 b^2 m^2 r^7 w^2 \alpha^6 - 2\,a1\,a2\,a2r\,b^2 m^2 r^7 w^2 \alpha^6 + 2\,a1\,a2^2 b^2 m^2 r^8 w^2 \alpha^6 - a1r^2 a2^3 m^2 r^4 \alpha^{12} + a1\,a1r\,a2^2 a2r\,m^2 r^4 \alpha^{12} - 2\sqrt{\left(b^2 m^4 r^2 \left(a2r + a2\,r^3\right)^2 w^2 \left(b^2 w^2 + a1\,a2\,r^2 \alpha^6\right) \left(b^4 r^4 \left(1 - 2\,r^2\right)^2 w^4 + b^2 \left(a1\,r^3 \left(2\,a2r - 2\,a2r\,r^2 + a2\,r^3\right) + a1r \left(a2r + 2\,a2\,r^3 - 2\,a2\,r^5\right)\right) w^2 \alpha^6 + a1\,a1r\,a2\,a2r\,r^2 \alpha^{12}\right)\right)}\right) \Big/ \left(r^2 \alpha^6 \left(-a1r^2 a2^2 r^2 \alpha^6 + a1\,r^2 \left(4\,a2r\,b^2 r\,w^2 + 4\,a2\,b^2 r^4 w^2 - a1\,a2r^2 \alpha^6\right) + 2\,a1r \left(2\,a2r\,b^2 w^2 + 2\,a2\,b^2 r^3 w^2 + a1\,a2\,a2r\,r^2 \alpha^6\right)\right)\right)}\Big\},$$

$$\Big\{le \to -\sqrt{\left(-\left(2\,a2r\,b^4 m^2 r^3 w^4 - 4\,a2r\,b^4 m^2 r^5 w^4 + 2\,a2\,b^4 m^2 r^6 w^4 - 4\,a2\,b^4 m^2 r^8 w^4 + 3\,a1r\,a2\,a2r\,b^2 m^2 r^2 w^2 \alpha^6 + a1\,a2r^2 b^2 m^2 r^2 w^2 \alpha^6 + 2\,a1r\,a2^2 b^2 m^2 r^5 w^2 \alpha^6 + 4\,a1\,a2\,a2r\,b^2 m^2 r^5 w^2 \alpha^6 + 2\,a1r\,a2^2 b^2 m^2 r^7 w^2 \alpha^6 - 2\,a1\,a2\,a2r\,b^2 m^2 r^7 w^2 \alpha^6 + 2\,a1\,a2^2 b^2 m^2 r^8 w^2 \alpha^6 - a1r^2 a2^3 m^2 r^4 \alpha^{12} + a1\,a1r\,a2^2 a2r\,m^2 r^4 \alpha^{12} + 2\sqrt{\left(b^2 m^4 r^2 \left(a2r + a2\,r^3\right)^2 w^2 \left(b^2 w^2 + a1\,a2\,r^2 \alpha^6\right) \left(b^4 r^4 \left(1 - 2\,r^2\right)^2 w^4 + b^2 \left(a1\,r^3 \left(2\,a2r - 2\,a2r\,r^2 + a2\,r^3\right) + a1r \left(a2r + 2\,a2\,r^3 - 2\,a2\,r^5\right)\right) w^2 \alpha^6 + a1\,a1r\,a2\,a2r\,r^2 \alpha^{12}\right)\right)}\right) \Big/ \left(r^2 \alpha^6 \left(-a1r^2 a2^2 r^2 \alpha^6 + a1\,r^2 \left(4\,a2r\,b^2 r\,w^2 + 4\,a2\,b^2 r^4 w^2 - a1\,a2r^2 \alpha^6\right) + 2\,a1r \left(2\,a2r\,b^2 w^2 + 2\,a2\,b^2 r^3 w^2 + a1\,a2\,a2r\,r^2 \alpha^6\right)\right)\right)}\Big\},$$

$$\Big\{le \to \sqrt{\left(-\left(2\,a2r\,b^4 m^2 r^3 w^4 - 4\,a2r\,b^4 m^2 r^5 w^4 + 2\,a2\,b^4 m^2 r^6 w^4 - 4\,a2\,b^4 m^2 r^8 w^4 + 3\,a1r\,a2\,a2r\,b^2 m^2 r^2 w^2 \alpha^6 + a1\,a2r^2 b^2 m^2 r^2 w^2 \alpha^6 + 2\,a1r\,a2^2 b^2 m^2 r^5 w^2 \alpha^6 + 4\,a1\,a2\,a2r\,b^2 m^2 r^5 w^2 \alpha^6 + 2\,a1r\,a2^2 b^2 m^2 r^7 w^2 \alpha^6 - 2\,a1\,a2\,a2r\,b^2 m^2 r^7 w^2 \alpha^6 + 2\,a1\,a2^2 b^2 m^2 r^8 w^2 \alpha^6 - a1r^2 a2^3 m^2 r^4 \alpha^{12} + a1\,a1r\,a2^2 a2r\,m^2 r^4 \alpha^{12} + 2\sqrt{\left(b^2 m^4 r^2 \left(a2r + a2\,r^3\right)^2 w^2 \left(b^2 w^2 + a1\,a2\,r^2 \alpha^6\right) \left(b^4 r^4 \left(1 - 2\,r^2\right)^2 w^4 + b^2 \left(a1\,r^3 \left(2\,a2r - 2\,a2r\,r^2 + a2\,r^3\right) + a1r \left(a2r + 2\,a2\,r^3 - 2\,a2\,r^5\right)\right) w^2 \alpha^6 + a1\,a1r\,a2\,a2r\,r^2 \alpha^{12}\right)\right)}\right) \Big/ \left(r^2 \alpha^6 \left(-a1r^2 a2^2 r^2 \alpha^6 + a1\,r^2 \left(4\,a2r\,b^2 r\,w^2 + 4\,a2\,b^2 r^4 w^2 - a1\,a2r^2 \alpha^6\right) + 2\,a1r \left(2\,a2r\,b^2 w^2 + 2\,a2\,b^2 r^3 w^2 + a1\,a2\,a2r\,r^2 \alpha^6\right)\right)\right)}\Big\}\Big\}$$

le1

```
le = -√(-(2 a2r b^4 m^2 r^3 w^4 - 4 a2r b^4 m^2 r^5 w^4 + 2 a2 b^4 m^2 r^6 w^4 - 4 a2 b^4 m^2 r^8 w^4 +
        3 a1r a2 a2r b^2 m^2 r^2 w^2 α^6 + a1 a2r^2 b^2 m^2 r^2 w^2 α^6 + 2 a1r a2^2 b^2 m^2 r^5 w^2 α^6 +
        4 a1 a2 a2r b^2 m^2 r^5 w^2 α^6 + 2 a1r a2^2 b^2 m^2 r^7 w^2 α^6 - 2 a1 a2 a2r b^2 m^2 r^7 w^2 α^6 +
        2 a1 a2^2 b^2 m^2 r^8 w^2 α^6 - a1r^2 a2^3 m^2 r^4 α^12 + a1 a1r a2^2 a2r m^2 r^4 α^12 -
        2 √(b^2 m^4 r^2 (a2r + a2 r^3)^2 w^2 (b^2 w^2 + a1 a2 r^2 α^6) (b^4 r^4 (1 - 2 r^2)^2 w^4 + b^2 (a1 r^3 (2 a2r -
                  2 a2r r^2 + a2 r^3) + a1r (a2r + 2 a2 r^3 - 2 a2 r^5)) w^2 α^6 + a1 a1r a2 a2r r^2 α^12))) /
      (r^2 α^6 (-a1r^2 a2^2 r^2 α^6 + a1 r^2 (4 a2r b^2 r w^2 + 4 a2 b^2 r^4 w^2 - a1 a2r^2 α^6) +
         2 a1r (2 a2r b^2 w^2 + 2 a2 b^2 r^3 w^2 + a1 a2 a2r r^2 α^6))));

Simplify[ee]

-  1
  ----------
  a2 r^2 α^6
  (b r w α^3 √(-(2 a2r b^4 m^2 r^3 w^4 - 4 a2r b^4 m^2 r^5 w^4 + 2 a2 b^4 m^2 r^6 w^4 - 4 a2 b^4 m^2 r^8 w^4 + 3 a1r a2 a2r b^2 m^2
           r^2 w^2 α^6 + a1 a2r^2 b^2 m^2 r^2 w^2 α^6 + 2 a1r a2^2 b^2 m^2 r^5 w^2 α^6 + 4 a1 a2 a2r b^2 m^2 r^5 w^2 α^6 +
          2 a1r a2^2 b^2 m^2 r^7 w^2 α^6 - 2 a1 a2 a2r b^2 m^2 r^7 w^2 α^6 + 2 a1 a2^2 b^2 m^2 r^8 w^2 α^6 -
          a1r^2 a2^3 m^2 r^4 α^12 + a1 a1r a2^2 a2r m^2 r^4 α^12 - 2 √(b^2 m^4 r^2 (a2r + a2 r^3)^2 w^2
              (b^2 w^2 + a1 a2 r^2 α^6) (b^4 r^4 (1 - 2 r^2)^2 w^4 + b^2 (a1 r^3 (2 a2r - 2 a2r r^2 + a2 r^3) +
                   a1r (a2r + 2 a2 r^3 - 2 a2 r^5)) w^2 α^6 + a1 a1r a2 a2r r^2 α^12))) /
        (r^2 α^6 (-a1r^2 a2^2 r^2 α^6 + a1 r^2 (4 a2r b^2 r w^2 + 4 a2 b^2 r^4 w^2 - a1 a2r^2 α^6) +
            2 a1r (2 a2r b^2 w^2 + 2 a2 b^2 r^3 w^2 + a1 a2 a2r r^2 α^6)))) +
    √(((b^2 w^2 + a1 a2 r^2 α^6) (-2 a2r b^4 m^2 r^3 w^4 + 4 a2r b^4 m^2 r^5 w^4 - 2 a2 b^4 m^2 r^6 w^4 +
          4 a2 b^4 m^2 r^8 w^4 + a1r a2 a2r b^2 m^2 r^2 w^2 α^6 - a1 a2r^2 b^2 m^2 r^2 w^2 α^6 + 2 a1r a2^2 b^2 m^2 r^5 w^2 α^6 -
          2 a1r a2^2 b^2 m^2 r^7 w^2 α^6 + 2 a1 a2 a2r b^2 m^2 r^7 w^2 α^6 + 2 a1 a2^2 b^2 m^2 r^8 w^2 α^6 +
          a1 a1r a2^2 a2r m^2 r^4 α^12 - a1^2 a2 a2r^2 m^2 r^4 α^12 + 2 √(b^2 m^4 r^2 (a2r + a2 r^3)^2 w^2
              (b^2 w^2 + a1 a2 r^2 α^6) (b^4 r^4 (1 - 2 r^2)^2 w^4 + b^2 (a1 r^3 (2 a2r - 2 a2r r^2 + a2 r^3) +
                   a1r (a2r + 2 a2 r^3 - 2 a2 r^5)) w^2 α^6 + a1 a1r a2 a2r r^2 α^12)))) /
       (-a1r^2 a2^2 r^2 α^6 + a1 r^2 (4 a2r b^2 r w^2 + 4 a2 b^2 r^4 w^2 - a1 a2r^2 α^6) +
         2 a1r (2 a2r b^2 w^2 + 2 a2 b^2 r^3 w^2 + a1 a2 a2r r^2 α^6))))

le2

le2 =
  √(-(2 a2r b^4 m^2 r^3 w^4 - 4 a2r b^4 m^2 r^5 w^4 + 2 a2 b^4 m^2 r^6 w^4 - 4 a2 b^4 m^2 r^8 w^4 + 3 a1r a2 a2r b^2 m^2 r^2 w^2 α^6 +
        a1 a2r^2 b^2 m^2 r^2 w^2 α^6 + 2 a1r a2^2 b^2 m^2 r^5 w^2 α^6 + 4 a1 a2 a2r b^2 m^2 r^5 w^2 α^6 +
        2 a1r a2^2 b^2 m^2 r^7 w^2 α^6 - 2 a1 a2 a2r b^2 m^2 r^7 w^2 α^6 +
        2 a1 a2^2 b^2 m^2 r^8 w^2 α^6 - a1r^2 a2^3 m^2 r^4 α^12 + a1 a1r a2^2 a2r m^2 r^4 α^12 -
        2 √(b^2 m^4 r^2 (a2r + a2 r^3)^2 w^2 (b^2 w^2 + a1 a2 r^2 α^6) (b^4 r^4 (1 - 2 r^2)^2 w^4 + b^2 (a1 r^3 (2 a2r - 2 a2r
                  r^2 + a2 r^3) + a1r (a2r + 2 a2 r^3 - 2 a2 r^5)) w^2 α^6 + a1 a1r a2 a2r r^2 α^12))) /
     (r^2 α^6 (-a1r^2 a2^2 r^2 α^6 + a1 r^2 (4 a2r b^2 r w^2 + 4 a2 b^2 r^4 w^2 - a1 a2r^2 α^6) +
        2 a1r (2 a2r b^2 w^2 + 2 a2 b^2 r^3 w^2 + a1 a2 a2r r^2 α^6))));
```

```
Simplify[ee]
```

```
-      1
  ─────────
  a2 r^2 α^6
 (b r w α^3 √((-(2 a2r b^4 m^2 r^3 w^4 - 4 a2r b^4 m^2 r^5 w^4 + 2 a2 b^4 m^2 r^6 w^4 - 4 a2 b^4 m^2 r^8 w^4 + 3 a1r a2 a2r b^2 m^2
        r^2 w^2 α^6 + a1 a2r^2 b^2 m^2 r^2 w^2 α^6 + 2 a1r a2^2 b^2 m^2 r^5 w^2 α^6 + 4 a1 a2 a2r b^2 m^2 r^5 w^2 α^6 +
       2 a1r a2^2 b^2 m^2 r^7 w^2 α^6 - 2 a1 a2 a2r b^2 m^2 r^7 w^2 α^6 + 2 a1 a2^2 b^2 m^2 r^8 w^2 α^6 -
       a1r^2 a2^3 m^2 r^4 α^12 + a1 a1r a2^2 a2r m^2 r^4 α^12 - 2 √(b^2 m^4 r^2 (a2r + a2 r^3)^2 w^2
            (b^2 w^2 + a1 a2 r^2 α^6) (b^4 r^4 (1 - 2 r^2)^2 w^4 + b^2 (a1 r^3 (2 a2r - 2 a2r r^2 + a2 r^3) +
                  a1r (a2r + 2 a2 r^3 - 2 a2 r^5)) w^2 α^6 + a1 a1r a2 a2r r^2 α^12))) /
      (r^2 α^6 (-a1r^2 a2^2 r^2 α^6 + a1 r^2 (4 a2r b^2 r w^2 + 4 a2 b^2 r^4 w^2 - a1 a2r^2 α^6) +
          2 a1r (2 a2r b^2 w^2 + 2 a2 b^2 r^3 w^2 + a1 a2 a2r r^2 α^6)))) +
   √(((b^2 w^2 + a1 a2 r^2 α^6) (-2 a2r b^4 m^2 r^3 w^4 + 4 a2r b^4 m^2 r^5 w^4 - 2 a2 b^4 m^2 r^6 w^4 +
          4 a2 b^4 m^2 r^8 w^4 + a1r a2 a2r b^2 m^2 r^2 w^2 α^6 - a1 a2r^2 b^2 m^2 r^2 w^2 α^6 + 2 a1r a2^2 b^2 m^2 r^5 w^2 α^6 -
          2 a1r a2^2 b^2 m^2 r^7 w^2 α^6 + 2 a1 a2 a2r b^2 m^2 r^7 w^2 α^6 + 2 a1 a2^2 b^2 m^2 r^8 w^2 α^6 +
          a1 a1r a2^2 a2r m^2 r^4 α^12 - a1^2 a2 a2r^2 m^2 r^4 α^12 + 2 √(b^2 m^4 r^2 (a2r + a2 r^3)^2 w^2
              (b^2 w^2 + a1 a2 r^2 α^6) (b^4 r^4 (1 - 2 r^2)^2 w^4 + b^2 (a1 r^3 (2 a2r - 2 a2r r^2 + a2 r^3) +
                    a1r (a2r + 2 a2 r^3 - 2 a2 r^5)) w^2 α^6 + a1 a1r a2 a2r r^2 α^12)))) /
      (-a1r^2 a2^2 r^2 α^6 + a1 r^2 (4 a2r b^2 r w^2 + 4 a2 b^2 r^4 w^2 - a1 a2r^2 α^6) +
        2 a1r (2 a2r b^2 w^2 + 2 a2 b^2 r^3 w^2 + a1 a2 a2r r^2 α^6))))
```

```
le3

le = -√((-(2 a2r b^4 m^2 r^3 w^4 - 4 a2r b^4 m^2 r^5 w^4 + 2 a2 b^4 m^2 r^6 w^4 - 4 a2 b^4 m^2 r^8 w^4 +
          3 a1r a2 a2r b^2 m^2 r^2 w^2 α^6 + a1 a2r^2 b^2 m^2 r^2 w^2 α^6 + 2 a1r a2^2 b^2 m^2 r^5 w^2 α^6 +
          4 a1 a2 a2r b^2 m^2 r^5 w^2 α^6 + 2 a1r a2^2 b^2 m^2 r^7 w^2 α^6 - 2 a1 a2 a2r b^2 m^2 r^7 w^2 α^6 +
          2 a1 a2^2 b^2 m^2 r^8 w^2 α^6 - a1r^2 a2^3 m^2 r^4 α^12 + a1 a1r a2^2 a2r m^2 r^4 α^12 +
          2 √(b^2 m^4 r^2 (a2r + a2 r^3)^2 w^2 (b^2 w^2 + a1 a2 r^2 α^6) (b^4 r^4 (1 - 2 r^2)^2 w^4 + b^2 (a1 r^3 (2 a2r -
                    2 a2r r^2 + a2 r^3) + a1r (a2r + 2 a2 r^3 - 2 a2 r^5)) w^2 α^6 + a1 a1r a2 a2r r^2 α^12))) /
     (r^2 α^6 (-a1r^2 a2^2 r^2 α^6 + a1 r^2 (4 a2r b^2 r w^2 + 4 a2 b^2 r^4 w^2 - a1 a2r^2 α^6) +
          2 a1r (2 a2r b^2 w^2 + 2 a2 b^2 r^3 w^2 + a1 a2 a2r r^2 α^6))));
```

Simplify[**ee**]

$$-\frac{1}{a2\, r^2\, \alpha^6}\Bigg(b\, r\, w\, \alpha^3 \sqrt{\Big(-\Big(2\, a2r\, b^4\, m^2\, r^3\, w^4 - 4\, a2r\, b^4\, m^2\, r^5\, w^4 + 2\, a2\, b^4\, m^2\, r^6\, w^4 - 4\, a2\, b^4\, m^2\, r^8\, w^4 + 3\, a1r\, a2\, a2r\, b^2\, m^2\, r^2\, w^2\, \alpha^6 + a1\, a2r^2\, b^2\, m^2\, r^2\, w^2\, \alpha^6 + 2\, a1r\, a2^2\, b^2\, m^2\, r^5\, w^2\, \alpha^6 + 4\, a1\, a2\, a2r\, b^2\, m^2\, r^5\, w^2\, \alpha^6 + 2\, a1r\, a2^2\, b^2\, m^2\, r^7\, w^2\, \alpha^6 - 2\, a1\, a2\, a2r\, b^2\, m^2\, r^7\, w^2\, \alpha^6 + 2\, a1\, a2^2\, b^2\, m^2\, r^8\, w^2\, \alpha^6 - a1r^2\, a2^3\, m^2\, r^4\, \alpha^{12} + a1\, a1r\, a2^2\, a2r\, m^2\, r^4\, \alpha^{12} + 2\sqrt{\Big(b^2\, m^4\, r^2\, (a2r + a2\, r^3)^2\, w^2\, (b^2\, w^2 + a1\, a2\, r^2\, \alpha^6)\, \big(b^4\, r^4\, (1 - 2\, r^2)^2\, w^4 + b^2\, \big(a1\, r^3\, (2\, a2r - 2\, a2r\, r^2 + a2\, r^3) + a1r\, (a2r + 2\, a2\, r^3 - 2\, a2\, r^5)\big)\, w^2\, \alpha^6 + a1\, a1r\, a2\, a2r\, r^2\, \alpha^{12}\big)\Big)}\Big) \Big/ \Big(r^2\, \alpha^6\, \big(-a1r^2\, a2^2\, r^2\, \alpha^6 + a1\, r^2\, (4\, a2r\, b^2\, r\, w^2 + 4\, a2\, b^2\, r^4\, w^2 - a1\, a2r^2\, \alpha^6) + 2\, a1r\, (2\, a2r\, b^2\, w^2 + 2\, a2\, b^2\, r^3\, w^2 + a1\, a2\, a2r\, r^2\, \alpha^6)\big)\Big)\Big)} + \sqrt{\Big(\Big(\Big((b^2\, w^2 + a1\, a2\, r^2\, \alpha^6)\, \Big(2\, a2r\, b^4\, m^2\, r^3\, w^4 - 4\, a2r\, b^4\, m^2\, r^5\, w^4 + 2\, a2\, b^4\, m^2\, r^6\, w^4 - 4\, a2\, b^4\, m^2\, r^8\, w^4 - a1r\, a2\, a2r\, b^2\, m^2\, r^2\, w^2\, \alpha^6 + a1\, a2r^2\, b^2\, m^2\, r^2\, w^2\, \alpha^6 - 2\, a1r\, a2^2\, b^2\, m^2\, r^5\, w^2\, \alpha^6 + 2\, a1r\, a2^2\, b^2\, m^2\, r^7\, w^2\, \alpha^6 - 2\, a1\, a2\, a2r\, b^2\, m^2\, r^7\, w^2\, \alpha^6 - 2\, a1\, a2^2\, b^2\, m^2\, r^8\, w^2\, \alpha^6 - a1\, a1r\, a2^2\, a2r\, m^2\, r^4\, \alpha^{12} + a1^2\, a2\, a2r^2\, m^2\, r^4\, \alpha^{12} + 2\sqrt{\Big(b^2\, m^4\, r^2\, (a2r + a2\, r^3)^2\, w^2\, (b^2\, w^2 + a1\, a2\, r^2\, \alpha^6)\, \big(b^4\, r^4\, (1 - 2\, r^2)^2\, w^4 + b^2\, \big(a1\, r^3\, (2\, a2r - 2\, a2r\, r^2 + a2\, r^3) + a1r\, (a2r + 2\, a2\, r^3 - 2\, a2\, r^5)\big)\, w^2\, \alpha^6 + a1\, a1r\, a2\, a2r\, r^2\, \alpha^{12}\big)\Big)}\Big)\Big) \Big/ \Big(a1r^2\, a2^2\, r^2\, \alpha^6 + a1\, r^2\, (-4\, a2r\, b^2\, r\, w^2 - 4\, a2\, b^2\, r^4\, w^2 + a1\, a2r^2\, \alpha^6) - 2\, a1r\, (2\, a2r\, b^2\, w^2 + 2\, a2\, b^2\, r^3\, w^2 + a1\, a2\, a2r\, r^2\, \alpha^6)\Big)\Big)}\Bigg)$$

le4

le =

$$\mathbf{\sqrt{\Big(-\Big(2\, a2r\, b^4\, m^2\, r^3\, w^4 - 4\, a2r\, b^4\, m^2\, r^5\, w^4 + 2\, a2\, b^4\, m^2\, r^6\, w^4 - 4\, a2\, b^4\, m^2\, r^8\, w^4 + 3\, a1r\, a2\, a2r\, b^2\, m^2\, r^2\, w^2\, \alpha^6 + a1\, a2r^2\, b^2\, m^2\, r^2\, w^2\, \alpha^6 + 2\, a1r\, a2^2\, b^2\, m^2\, r^5\, w^2\, \alpha^6 + 4\, a1\, a2\, a2r\, b^2\, m^2\, r^5\, w^2\, \alpha^6 + 2\, a1r\, a2^2\, b^2\, m^2\, r^7\, w^2\, \alpha^6 - 2\, a1\, a2\, a2r\, b^2\, m^2\, r^7\, w^2\, \alpha^6 + 2\, a1\, a2^2\, b^2\, m^2\, r^8\, w^2\, \alpha^6 - a1r^2\, a2^3\, m^2\, r^4\, \alpha^{12} + a1\, a1r\, a2^2\, a2r\, m^2\, r^4\, \alpha^{12} + 2\sqrt{\Big(b^2\, m^4\, r^2\, (a2r + a2\, r^3)^2\, w^2\, (b^2\, w^2 + a1\, a2\, r^2\, \alpha^6)\, \big(b^4\, r^4\, (1 - 2\, r^2)^2\, w^4 + b^2\, \big(a1\, r^3\, (2\, a2r - 2\, a2r\, r^2 + a2\, r^3) + a1r\, (a2r + 2\, a2\, r^3 - 2\, a2\, r^5)\big)\, w^2\, \alpha^6 + a1\, a1r\, a2\, a2r\, r^2\, \alpha^{12}\big)\Big)}\Big) \Big/ \Big(r^2\, \alpha^6\, \big(-a1r^2\, a2^2\, r^2\, \alpha^6 + a1\, r^2\, (4\, a2r\, b^2\, r\, w^2 + 4\, a2\, b^2\, r^4\, w^2 - a1\, a2r^2\, \alpha^6) + 2\, a1r\, (2\, a2r\, b^2\, w^2 + 2\, a2\, b^2\, r^3\, w^2 + a1\, a2\, a2r\, r^2\, \alpha^6)\big)\Big)\Big)};}$$

Simplify[**ee**]

$$
\begin{aligned}
&\frac{1}{\mathrm{a2}\, r^2\, \alpha^6}\Bigg(b\, r\, w\, \alpha^3 \sqrt{\Big(-\Big(2\,\mathrm{a2r}\, b^4 m^2 r^3 w^4 - 4\,\mathrm{a2r}\, b^4 m^2 r^5 w^4 + 2\,\mathrm{a2}\, b^4 m^2 r^6 w^4 - 4\,\mathrm{a2}\, b^4 m^2 r^8 w^4 + 3\,\mathrm{a1r}\,\mathrm{a2}\,\mathrm{a2r}\, b^2} \\
&\quad m^2 r^2 w^2 \alpha^6 + \mathrm{a1}\,\mathrm{a2r}^2 b^2 m^2 r^2 w^2 \alpha^6 + 2\,\mathrm{a1r}\,\mathrm{a2}^2 b^2 m^2 r^5 w^2 \alpha^6 + \\
&\quad 4\,\mathrm{a1}\,\mathrm{a2}\,\mathrm{a2r}\, b^2 m^2 r^5 w^2 \alpha^6 + 2\,\mathrm{a1r}\,\mathrm{a2}^2 b^2 m^2 r^7 w^2 \alpha^6 - 2\,\mathrm{a1}\,\mathrm{a2}\,\mathrm{a2r}\, b^2 m^2 r^7 w^2 \alpha^6 + \\
&\quad 2\,\mathrm{a1}\,\mathrm{a2}^2 b^2 m^2 r^8 w^2 \alpha^6 - \mathrm{a1r}^2 \mathrm{a2}^3 m^2 r^4 \alpha^{12} + \mathrm{a1}\,\mathrm{a1r}\,\mathrm{a2}^2\,\mathrm{a2r}\, m^2 r^4 \alpha^{12} + \\
&\quad 2\sqrt{\Big(b^2 m^4 r^2 \left(\mathrm{a2r} + \mathrm{a2}\, r^3\right)^2 w^2 \left(b^2 w^2 + \mathrm{a1}\,\mathrm{a2}\, r^2 \alpha^6\right)\Big(b^4 r^4 \left(1 - 2 r^2\right)^2 w^4 + b^2 \Big(\mathrm{a1}\, r^3 \Big(2\,\mathrm{a2r} -} \\
&\qquad 2\,\mathrm{a2r}\, r^2 + \mathrm{a2}\, r^3\Big) + \mathrm{a1r}\left(\mathrm{a2r} + 2\,\mathrm{a2}\, r^3 - 2\,\mathrm{a2}\, r^5\right)\Big) w^2 \alpha^6 + \mathrm{a1}\,\mathrm{a1r}\,\mathrm{a2}\,\mathrm{a2r}\, r^2 \alpha^{12}\Big)\Big)\Big) \Big/ \\
&\quad \Big(r^2 \alpha^6 \Big(-\mathrm{a1r}^2 \mathrm{a2}^2 r^2 \alpha^6 + \mathrm{a1}\, r^2 \left(4\,\mathrm{a2r}\, b^2 r\, w^2 + 4\,\mathrm{a2}\, b^2 r^4 w^2 - \mathrm{a1}\,\mathrm{a2r}^2 \alpha^6\right) + \\
&\qquad 2\,\mathrm{a1r}\left(2\,\mathrm{a2r}\, b^2 w^2 + 2\,\mathrm{a2}\, b^2 r^3 w^2 + \mathrm{a1}\,\mathrm{a2}\,\mathrm{a2r}\, r^2 \alpha^6\right)\Big)\Big)\Big) - \\
&\quad \sqrt{\Big(\Big(\left(b^2 w^2 + \mathrm{a1}\,\mathrm{a2}\, r^2 \alpha^6\right)\Big(2\,\mathrm{a2r}\, b^4 m^2 r^3 w^4 - 4\,\mathrm{a2r}\, b^4 m^2 r^5 w^4 + 2\,\mathrm{a2}\, b^4 m^2 r^6 w^4 - 4\,\mathrm{a2}\, b^4 m^2 r^8 w^4 -} \\
&\quad \mathrm{a1r}\,\mathrm{a2}\,\mathrm{a2r}\, b^2 m^2 r^2 w^2 \alpha^6 + \mathrm{a1}\,\mathrm{a2r}^2 b^2 m^2 r^2 w^2 \alpha^6 - 2\,\mathrm{a1r}\,\mathrm{a2}^2 b^2 m^2 r^5 w^2 \alpha^6 + \\
&\quad 2\,\mathrm{a1r}\,\mathrm{a2}^2 b^2 m^2 r^7 w^2 \alpha^6 - 2\,\mathrm{a1}\,\mathrm{a2}\,\mathrm{a2r}\, b^2 m^2 r^7 w^2 \alpha^6 - 2\,\mathrm{a1}\,\mathrm{a2}^2 b^2 m^2 r^8 w^2 \alpha^6 - \\
&\quad \mathrm{a1}\,\mathrm{a1r}\,\mathrm{a2}^2\,\mathrm{a2r}\, m^2 r^4 \alpha^{12} + \mathrm{a1}^2\,\mathrm{a2}\,\mathrm{a2r}^2 m^2 r^4 \alpha^{12} + 2\sqrt{\Big(b^2 m^4 r^2 \left(\mathrm{a2r} + \mathrm{a2}\, r^3\right)^2 w^2} \\
&\qquad \left(b^2 w^2 + \mathrm{a1}\,\mathrm{a2}\, r^2 \alpha^6\right)\Big(b^4 r^4 \left(1 - 2 r^2\right)^2 w^4 + b^2 \Big(\mathrm{a1}\, r^3 \left(2\,\mathrm{a2r} - 2\,\mathrm{a2r}\, r^2 + \mathrm{a2}\, r^3\right) + \\
&\qquad\quad \mathrm{a1r}\left(\mathrm{a2r} + 2\,\mathrm{a2}\, r^3 - 2\,\mathrm{a2}\, r^5\right)\Big) w^2 \alpha^6 + \mathrm{a1}\,\mathrm{a1r}\,\mathrm{a2}\,\mathrm{a2r}\, r^2 \alpha^{12}\Big)\Big)\Big)\Big) \Big/ \\
&\quad \Big(\mathrm{a1r}^2 \mathrm{a2}^2 r^2 \alpha^6 + \mathrm{a1}\, r^2 \left(-4\,\mathrm{a2r}\, b^2 r\, w^2 - 4\,\mathrm{a2}\, b^2 r^4 w^2 + \mathrm{a1}\,\mathrm{a2r}^2 \alpha^6\right) - \\
&\quad 2\,\mathrm{a1r}\left(2\,\mathrm{a2r}\, b^2 w^2 + 2\,\mathrm{a2}\, b^2 r^3 w^2 + \mathrm{a1}\,\mathrm{a2}\,\mathrm{a2r}\, r^2 \alpha^6\right)\Big)\Big)\Bigg)
\end{aligned}
$$

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BIOGRAPHY

Fuat İlkehan VARDARLI was born in İstanbul, Turkey in 1978. He graduated from Saint Benoit French College in 1998. He obtained his BSc. degree in 2004 from Yıldız Technical University, Department of Physics . He started MSc education at İstanbul Technical University in 2004. He is working currently at ITU Physics Department as a research assistant.