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Delikli sonsuz bir ortamda iki boyutlu bir termoelastisite problemi

A two dimensional thermoelastic problem for an infinite medium with cavity

by

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Bu çalışma düzlem şekil değiştirme halinde, izole edilmiş bir deliğe sahip ve sonsuzda verilen bir doğrultuda düzgün bir ısı akısına maruz bir elastik ortam için kompleks potansiyellerin belirlenmesi ile ilgilidir. Birim daire üzerine özel tipte bir tasvir fonksiyonu için potansiyellerin açık ifadeleri verilmiştir.

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The paper is concerned with the determination of complex potentials for an infinite elastic medium in plane strain with an insulated cavity subjected to a uniform heat flux in a given direction at infinity. For a special type of mapping function into a unit circle explicit expressions for potentials are provided.

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1. Introduction

This paper deals with a rather simple application of Kolossof-Muskhelishvili solution to two-dimensional thermoelastic problems. The solution of the problem is reduced to the determination of two complex potentials. The fundamental equations become entirely similar to those in the ordinary theory of elasticity with a fictitious body force density due to the temperature distribution. The fundamental equations are summarized in Section 2. The restrictions on complex potentials and their basic structure are briefly discussed in Section 3. The thermoelastic mixed boundary value problem involving only thermal agents is formulated in Section 4 and a series solution is constructed. The general theory is applied in Section 5 to the study of an infinite elastic medium with a cavity subjected to a uniform heat flux at infinity. The shape of the cavity is given by a special mapping function into a unit circle which generates many interesting curves such as curvilinear polygons which can be thought as approximation to rectilinear ones. The complex potentials are determined explicitly although the passage to stress components requires rather complicated but trivial algebra. Finally in Section 6 potentials are given for a special form of the cavity.

2. Fundamental Equations

We consider two-dimensional thermoelastic problems for a cylindrical region of infinite extent in x_3 - direction. It is assumed that the body is free from body forces but subjected to a given two-dimensional temperature distribution which may be time dependent. However, all the inertial effects will be neglected throughout the analysis. $x_1\,x_2$ - coordinate plane is any plane perpendicular to x_3 - axis. The solution of such problems depends on the determination of two functions $\phi(x_1, x_2)$ and $U(x_1, x_2)$ which satisfy the biharmonic and Poisson's equations [1]

$$\begin{aligned} \nabla^4 \phi &= 0 \\ \nabla^2 U &= cT \end{aligned} \tag{2.1}$$

where T is the function characterizing the temperature distribution and

$$c = \frac{2\mu(3\lambda+2\mu)\alpha}{\lambda+2\mu} = \frac{E\alpha}{1-\nu} \tag{2.2}$$

In (2.2), λ and μ are the conventional Lamé's constants of the elastic solid, E and ν are Young's modulus and Poisson's ratio and α is the coefficient of thermal expansion. We emphasize again the fact that the time variable may enter into the arguments of the functions ϕ and U as merely a parameter if the thermal state is not a steady one since, then, the temperature distribution is given as a solution of the classical diffusion equation.

The solution of the biharmonic equation may be written as [2]

$$\phi = z\overline{\Omega}(\bar{z}) + \bar{z}\,\Omega(z) + \omega(z) + \overline{\omega}(\bar{z}) \tag{2.3}$$

where $\Omega(z)$ and $\omega(z)$ are arbitrary complex potentials and $z = x_1 + ix_2$

With the knowledge of ϕ and U the stress tensor $t_{\alpha\beta}$, the displacement u_α [greek indices = 1, 2] and the rotation r, are expressed as follows

$$
\begin{aligned}
\Theta &= t_{11}+t_{22}=4\,[\Omega'(z)+\bar{\Omega}'(\bar{z})]-cT \\
\Phi &= t_{11}-t_{22}+2it_{12}=-4\left[z\,\bar{\Omega}''(\bar{z})+\bar{\omega}'(\bar{z})-\frac{\partial^2 U}{\partial \bar{z}^2}\right] \\
\mu D &= \mu\,(u_1+iu_2)=\varkappa\,\Omega(z)-z\,\bar{\Omega}'(\bar{z})-\bar{\omega}'(\bar{z})+\frac{\partial U}{\partial \bar{z}} \\
2\mu\, r_3 &= i\,(\varkappa+1)\,[\Omega'(z)-\bar{\Omega}'(\bar{z})]
\end{aligned}
\tag{2.4}
$$

where

$$\varkappa=\frac{\lambda+2\mu}{\lambda+\mu} \tag{2.5}$$

Primes denote differentiations with respect to arguments and $U(x_1, x_2)=U(z, \bar{z})$.

The resultant force and the moment, with respect to the origin, of the tractions acting on the portion of the boundary between a fixed and a variable point are given by

$$F=X_1+iX_2=\int_{s_0}^{s}(t_1+it_2)\,ds=-2i\left[z\bar{\Omega}'(\bar{z})+\Omega(z)+\bar{\omega}(\bar{z})-\frac{\partial U}{\partial \bar{z}}\right]$$

$$
\begin{aligned}
M=\int_{s_0}^{s}(t_2\,x_1-t_1\,x_2)\,ds &= -z\bar{z}\,[\Omega'(z)+\bar{\Omega}'(\bar{z})]-z\,\omega'(z) \\
&\quad -\bar{z}\,\bar{\omega}'(\bar{z})+\omega(z)+\bar{\omega}(\bar{z})+z\,\frac{\partial U}{\partial z}+\bar{z}\,\frac{\partial U}{\partial \bar{z}}-U
\end{aligned}
\tag{2.6}
$$

apart from arbitrary constants which may be absorbed into the potentials since they do not effect stresses. In (2.6), s denotes arc length on the boundary, **n** is unit exterior normal of the boundary curve and $t_\alpha=t_{\alpha\beta}n_\beta$.

3. Restrictions on Complex Potentials

In most of the problems one requires stresses and displacements to be single-valued. If we employ the notation $[\;]_P^P$ to denote the jump in the enclosed function at a point P when a closed contour, which lies entirely

* It should be recalled that $t_{33}=\nu(t_{11}+t_{22})$, $t_{13}=t_{23}=0$.

in the body and which starts and ends at P, is traced. Therefore one should write

$$[\Theta]_P^P=[\phi]_P^P=[D]_P^P=0 \tag{3.1}$$

or since T is single-valued

$$\begin{aligned} &[\Omega'(z)+\bar{\Omega}'(\bar{z})]_P^P=0\\ &\left[z\,\bar{\Omega}''(\bar{z})+\bar{\omega}''(\bar{z})-\frac{\partial^2 U}{\partial \bar{z}^2}\right]_P^P=0\\ &\left[\varkappa\,\Omega(z)-z\,\bar{\Omega}'(\bar{z})-\bar{\omega}'(\bar{z})+\frac{\partial U}{\partial \bar{z}}\right]_P^P=0 \end{aligned} \tag{3.2}$$

If we differentiate the last of (3.2) with respect to z and $\bar{z}$ and consider [cf. $(2.1)_1$]

$$4\,\frac{\partial^2 U}{\partial z\,\partial \bar{z}}=cT \tag{3.3}$$

thereby conclude that $\partial^2 U/\partial z\partial\bar{z}$ is single - valued, that is $[\partial^2 U/\partial z\,\partial\bar{z}]_P^P=0$ we find

$$\begin{aligned} &[\varkappa\,\Omega'(z)-\bar{\Omega}'(\bar{z})]_P^P=0\\ &\left[z\bar{\Omega}''(\bar{z})-\bar{\omega}''(\bar{z})-\frac{\partial^2 U}{\partial \bar{z}^2}\right]_P^P=0 \end{aligned} \tag{3.4}$$

The combination of (3.2) and (3.4) results in the following conditions

$$\begin{aligned} &[\Omega'(z)]_P^P=0\\ &[\bar{\omega}'(\bar{z})]_P^P=\left[\varkappa\,\Omega(z)+\frac{\partial U}{\partial \bar{z}}\right]_P^P \end{aligned} \tag{3.5}$$

If the region occupied by the body is finite and simply-connected then the regularity of the stress field requires the functions $\Omega(z)$ and $\omega(z)$ to be holomorphic in that region. Thus the condition (3.5) are satisfied automatically and stresses and displacements become single-valued. However, if the region is multiply-connected the complex potentials may contain nonholomorphic parts. Let us consider $n+1$ non-intersecting closed curves C_0, C_1, C_2, . . . ,C_n of which C_0 encloses all the remaining curves (Fig. 1) and may recede to infinity. The region R occupies the two-dimensional space between these curves. If we require stresses and displacements to be single-valued in R we assume that

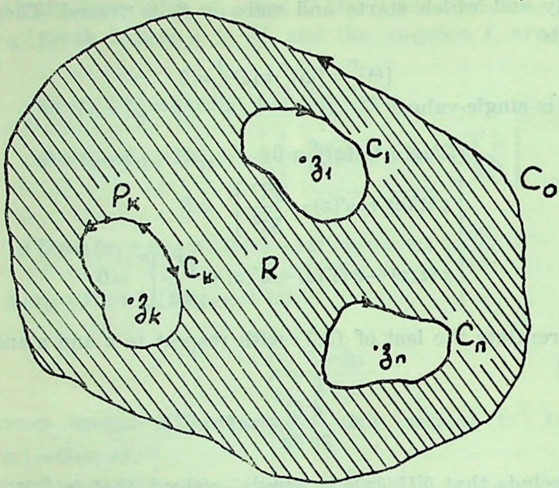

Fig. 1

$$\Omega'(z) = \sum_{k=1}^{n} \frac{A_k}{z - z_k} + \Omega_0'(z)$$

(3.6)

$$\omega'(z) = \sum_{k=1}^{n} B_k \log(z - z_k) + \omega_0'(z)$$

or

$$\Omega(z) = \sum_{k=1}^{n} A_k \log(z - z_k) + \bar{\Omega}_0(z)$$

(3.7)

$$\omega(z) = \sum_{k=1}^{n} B_k (z - z_k) \log(z - z_k) + \bar{\omega}_0(z)$$

where z_k is an arbitrary interior point to the area enclosed by C_k, A_k and B_k are complex constant and $\bar{\Omega}_0(z)$ and $\bar{\omega}_0(z)$ are holomorphic in R except possibly at infinity. Since $\Omega(z)$ and $\omega'(z)$ increase by $2\pi i A_k$ and $2\pi i B_k$ upon one revolution on C_k in the positive sense, (3.5) requires that

$$2\pi i \varkappa A_k + 2\pi i H_k = -2\pi i \bar{B}_k$$

where $2\pi i H_k$ is the increment in $\partial U/\partial \bar{z}$ on the same contour per one revolution z_k in the positive sense (Fig. 1), *i. e.*,

$$H_k = \frac{1}{2\pi i}\left[\frac{\partial U}{\partial \bar{z}}\right]_{P_k}^{P_k} \tag{3.8}$$

Hence

$$B_k = -(\varkappa \bar{A}_k + \bar{H}_k) \tag{3.9}$$

Therefore complex potentials take the following forms

$$\Omega(z) = \sum_{k=1}^{n} A_k \log(z - z_k) + \Omega_0(z)$$

$$\omega'(z) = -\sum_{k=1}^{n} (\varkappa \bar{A}_k + \bar{H}_k)\log(z - z_k) + \omega_0'(z) \tag{3.10}$$

Now let F_k be the resultant force acting on C_k. It follows from $(2.6)_1$, (3.5) that

$$\begin{aligned} F_k &= -2i\left[z\bar{\Omega}'(\bar{z}) + \Omega(z) + \bar{\omega}'(\bar{z}) - \frac{\partial U}{\partial \bar{z}}\right]_{P_k}^{P_k} \\ &= -2i(1+\varkappa)[\Omega(z)]_{P_k}^{P_k} = -4\pi(1+\varkappa)A_k \end{aligned} \tag{3.11}$$

where we should note that the positive sense in evaluating F_k is clockwise.

Henceforth we shall deal only with stress and displacement fields due to thermal sources. Therefore we have $A_k = 0$ for all cases and (3.10) will be reduced to the forms

$$\Omega(z) = \Omega_0(z)$$

$$\omega'(z) = -\sum_{k=1}^{n} \bar{H}_k \log(z - z_k) + \omega_0'(z) \tag{3.12}$$

which are fundamental in the subsequent analysis.

When the region R extends to infinity then we can write the asymptotic forms for large $|z|$ as

$$\Omega(z) \to \Omega_\infty(z); \quad \Omega_0(z) \to \Omega_\infty(z), \quad |z| \to \infty$$

$$\omega'(z) \to -\bar{H} \log z + \omega'_\infty(z); \quad \omega_0'(z) \to \omega'_\infty(z), \ |z| \to \infty$$

where

$$H = \sum_{k=1}^{n} H_k \tag{3.13}$$

The behaviour of complex potentials as $|z|$ tends to infinity are determined by setting stresses equal to zero at infinity. For large $|z|$ let us assume that the following asymptotic expansion for temperature is valid

$$T = \frac{1}{2}\left[T_\infty(z) + \bar{T}_\infty(\bar{z}) + \frac{A_\infty}{z} + \frac{A_\infty}{\bar{z}} + 0\left(\frac{1}{z^2}, \frac{1}{\bar{z}^2}\right)\right]$$

where A_∞ is constant and we considered the fact that T is a real function. Thus in conjunction with (2.1) this gives

$$\frac{\partial^2 U}{\partial \bar{z}^2} = \frac{cz}{8}\left[\bar{T}'_\infty(\bar{z}) - \frac{A_\infty}{\bar{z}^2} + 0\left(\frac{1}{z^2}, \frac{1}{\bar{z}^2}\right)\right]$$

Hence as $|z| \to \infty$

$$T \to \frac{1}{2}[T_\infty(z) + \bar{T}_\infty(\bar{z})], \quad \frac{\partial^2 U}{\partial \bar{z}^2} \to \frac{cz}{8} \bar{T}_\infty(\bar{z}) \tag{3.14}$$

If we introduce (3.14) into $(2.4)_{1,2}$ we obtain

$$\Omega'_\infty(z) = \frac{c}{8} T_\infty(z) + iC_\infty + 0\left(\frac{1}{z^2}\right)$$

$$\omega''_\infty(z) = 0\left(\frac{1}{z^2}\right)$$

where C_∞ is a real constant. One can easily see that

$$r_3 = (1+\varkappa)\frac{C_\infty}{z^2}, \quad |z| \to \infty$$

Therefore for vanishing rotation an infinity we have $C_\infty = 0$ and (3.12) reduces to

$$\Omega'(z)=\frac{c}{8}\,T_{\infty}(z)+0\left(\frac{1}{z^2}\right)$$

$$|z|\to\infty \qquad (3.15)$$

$$\omega''(z)=-\frac{\overline{H}}{z}+0\left(\frac{1}{z^2}\right)$$

In general $T_{\infty}(z)$ will be of the form

$$T_{\infty}(z)=\sum_{k=0}^{m} q_k\, z^k \qquad (3.16)$$

Therefore one writes

$$\Omega(z)=\frac{c}{8}\sum_{k=0}^{m}\frac{q_k}{k+1}\,z^{k+1}+0\left(\frac{1}{z}\right)$$

$$|z|\to\infty$$

$$\omega'(z)=-\overline{H}\log z+0\left(\frac{1}{z}\right) \qquad (3.17)$$

4. Thermoelastic Boundary Value Problems

Let $C=C_0\cup C_1\cup\ldots\cup C_n$. Since we consider here only the behaviour of elastic material under pure thermal agents, boundary value problems can then be formulated as follows: Tractions are zero along a portion C_s of the boundary contour C and displacements are zero along the remaining part $C_u=C-C_s$. It follows from $(2.4)_3$ and $(2.6)_1$ that the boundary conditions may be expressed collectively as

$$\alpha\Omega(t)+t\overline{\Omega}'(\bar t)+\overline{\omega}'(\bar t)=\frac{\partial U(t,\bar t)}{\partial t} \quad \text{on } C \qquad (4.1)$$

where $t\in C$ and $\alpha=1$ on C_s and $\alpha=-\varkappa$ on C_u. $U(z,\bar z)$ is the particular solution of $(2.1)_2$.

From now on we shall consider infinite regions with a hole of arbitrary shape. Let the function $z=z(\sigma)$ map conformally the region R in z-plane to the interior of a unit circle in σ-plane. The boundary conditions then become

$$\alpha f(\tau)+\frac{z(\tau)}{\overline{z}'(1/\tau)}\,\overline{f}'(1/\tau)+\overline{h}(1/\tau)=V(\tau) \qquad (4.2)$$

in σ-plane and on the unit circle. In (4.2) we define

$$\begin{aligned}
&f(\sigma)=\Omega[z(\sigma)]\\
&h(\sigma)=\omega'[z(\sigma)]\\
&V(\tau)=\left.\frac{\partial U(t,\bar{t})}{\partial \bar{t}}\right|_{t\in C\,,\;t=t(\tau)}
\end{aligned}\tag{4.3}$$

and note that on the unit circle we can write

$$\tau=e^{i\theta}\,,\quad \bar{\tau}=e^{-i\theta}=\frac{1}{\tau}\tag{4.4}$$

We consider a class of regions for which the mapping function is represented by [1].

$$z=\frac{\gamma}{\sigma}+\sum_{k=0}^{\infty}\gamma_k\,\sigma^k\,,\qquad |\sigma|<1\tag{4.5}$$

Thus

$$f(\sigma)=\Omega(z)=\frac{c}{8}\sum_{k=0}^{m}\frac{q_k\gamma^{k+1}}{k+1}\,\frac{1}{\sigma^{k+1}}+f_0(\sigma)$$

$$h(\sigma)=\omega'(z)=\bar{H}\log\sigma+h_0(\sigma)\tag{4.6}$$

where

$$f_0(\sigma)=\sum_{k=1}^{\infty}a_k\,\sigma^k$$

$$\tag{4.7}$$

$$h_0(\sigma)=\sum_{k=0}^{\infty}b_k\,\sigma^k$$

with unknown coefficients a_k and b_k. Therefore the boundary condition is now

$$\alpha f_0(\tau)+\frac{z(\tau)}{\bar{z}'(1/\tau)}\bar{f}'_0(1/\tau)+\bar{h}_0(1/\tau)=Q(\tau)\tag{4.8}$$

where

$$Q(\tau)=V(\tau)-\frac{\alpha c}{8}\sum_{k=0}^{m}\frac{q_k\gamma^{k+1}}{k+1}\,\frac{1}{\tau^{k+1}}+$$

$$+\frac{z(\tau)}{\bar{z}'(1/\tau)}\ \frac{c}{8}\sum_{k=0}^{m}\bar{q}_k\ \bar{\gamma}^{k+1}\ \tau^{k+2}-H\log\tau \qquad |\tau|=1 \tag{4.9}$$

It view of (3.9) it is clear that H is so chosen that $Q(\tau)$ will be a single-valued function. If we assume that $z(\tau)/\bar{z}'(1/\tau)$ and $Q(\tau)$ are expandable into complex Fourier series on the circle $|\tau|=1$ we have

$$\frac{z(\tau)}{\bar{z}'(1/\tau)}=\sum_{k=-\infty}^{\infty}c_k\,\tau^k$$

$$Q(\tau)=\sum_{k=-\infty}^{\infty}C_k\,\tau^k \tag{4.10}$$

where

$$c_k=\frac{1}{2\pi}\int_0^{2\pi}\frac{z(\theta)}{\bar{z}'(\theta)}e^{-ik\theta}d\theta\ ,\ C_k=\frac{1}{2\pi}\int_0^{2\pi}Q(\theta)e^{-ik\theta}d\theta\ ,\ \tau=e^{i\theta}$$

If we introduce (4.10) into (4.8) and arrange the resulting expression and equate the like powers of τ we obtain

$$\alpha a_k+\sum_{m=1}^{\infty}m\bar{a}_m\,c_{m+k-1}=C_k \qquad (k=1,2,\cdots)$$

$$\bar{b}_k+\sum_{m=1}^{\infty}m\bar{a}_m\,c_{m-k-1}=C_k \qquad (k=0,1,2\cdots) \tag{4.11}$$

This set of equations determine a_k and b_k and consequently $f_0(\sigma)$ and $h_0(\sigma)$ by which we can construct the complex potentials $\Omega(z)$ and $\omega'(z)$.

5. Insulated Curvilinear Polygonal Cavity Subjected to a Uniform Heat Flux at Infinity.

As an example to the above analysis we consider an infinite elastic medium with a cavity of curvilinear polygonal shape. The body is subjected to a given uniform heat flux Q_∞ in a given direction (Fig. 2). The mapping function [3]

$$z = \frac{na}{\sigma} + b\sigma^n \tag{5.1}$$

provides the desired tool. Putting $\sigma = \tau = e^{i\theta}$ ($\theta \in [0, 2\pi]$ we see that the point z determined by (5.1) traces various curves in z-plane by taking diverse values for n, a, b. Here we assume that a and b are real and $a > b$. For instance if we take $n = 1$, $a = (a' + b')/2$, $b = (a' - b')/2$ the cavity in z-plane becomes an ellipse with major and minor semi-axes a' and b' respectively, and major axis coinciding with x_1 - axis.

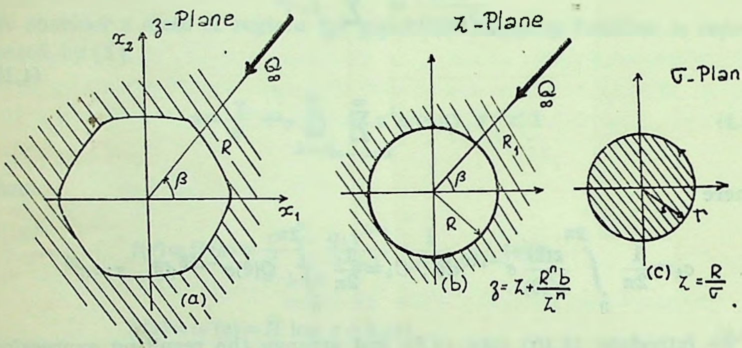

Fig. 2

For regular polygonal shapes of $n+1$ sides we have $a = nb$ and one of the vertices lies on x_1 - axis.

If we write, n being a positive integer,

$$\zeta = \frac{R}{\sigma}, \quad R = na \tag{5.2}$$

(5.1) is transformed to

$$z = \zeta + \frac{R^n b}{\zeta^n}$$

which in turn maps the region R into the exterior of a circle with radius R in ζ - plane. Since the real axis in z - plane is mapped to the real axis of ζ - plane the direction of heat flux in the latter plane remains unchanged (Fig. 2). The solution of the heat conduction problem in ζ - plane requires to find the temperature T satisfying

$$
\begin{aligned}
&\nabla^2 T=0 \quad \text{in } R_1 \\
&-k\,\nabla T=-Q_\infty \cos(\varphi-\beta)\,, \quad \rho\to\infty \\
&\mathbf{n}\,.\,\nabla T=\frac{\partial T}{\partial r}=0 \qquad \text{on} \quad \rho=R
\end{aligned}
\tag{5.3}
$$

where we write $\zeta=\rho\, e^{i\varphi}$ and k is the conductivity. The solution is

$$T=q_\infty\left(\rho+\frac{R^2}{\rho}\right)\cos(\varphi-\beta) \tag{5.4}$$

where

$$q_\infty=\frac{Q_\infty}{k} \tag{5.5}$$

We can easily transform (5.4) to

$$T=\frac{q_\infty}{2}\left[\zeta e^{-i\beta}+\bar{\zeta}e^{i\beta}+\frac{R^2\, e^{i\beta}}{\zeta}+\frac{R^2\, e^{-i\beta}}{\bar{\zeta}}\right] \tag{5.6}$$

Recalling that $R=na$ we can find by (5.2)

$$T(\sigma\,,\,\bar{\sigma})=\frac{q_\infty na}{2}\left[\sigma\, e^{i\beta}+\bar{\sigma}\, e^{-i\beta}+\frac{e^{-i\beta}}{\sigma}+\frac{e^{i\beta}}{\bar{\sigma}}\right] \tag{5.7}$$

As $|z|\to\infty$ we see from (5.1) that $|\sigma|\to 0$. Hence

$$T(\sigma\,,\bar{\sigma})\to\frac{1}{2}\,[T_0(\sigma)+\bar{T}_0(\bar{\sigma})]\,,\quad |\sigma|\to 0$$

where

$$T_0(\sigma)=\frac{q_\infty na\, e^{-i\beta}}{\sigma} \tag{5.8}$$

or when $|\sigma|\to 0$, $z\to na/\sigma$ and we find from (5.8) that

$$T_\infty(z)=q_\infty\, e^{-i\beta} z$$

Therefore the coefficients in (3.16) become $q_k=0$, $k\neq 1$ and

$$q_1=q_\infty\, e^{-i\beta} \tag{5.9}$$

Similarly in (4.5) we write

$$\gamma=na\,;\quad \gamma_k=0\,,\quad k\neq n\,,\quad \gamma_n=b \tag{5.10}$$

Since

$$z'(\sigma)=-\frac{na}{\sigma^2}+nb^{n-1}$$

we have from $(2.1)_1$ and (5.7)

$$\frac{\partial U}{\partial \bar{z}}=\frac{c}{4}\int T(\sigma,\bar{\sigma})\,z'(\sigma)\,d\sigma=\frac{q_\infty nac}{8}\Big[\frac{nbe^{i\beta}}{n+1}\sigma^{n+1}$$

$$+be^{-i\beta}\bar{\sigma}\sigma^n+\frac{nbe^{-i\beta}}{n-1}\sigma^{n-1}+b\,e^{i\beta}\frac{\sigma^n}{\bar{\sigma}}-ane^{i\beta}\log\sigma$$

$$+ane^{-i\beta}\frac{\bar{\sigma}}{\sigma}+\frac{ane^{-i\beta}}{2\sigma^2}+\frac{ane^{i\beta}}{\sigma\bar{\sigma}}\Big],\quad n\neq 1 \tag{5.11}$$

$$=\frac{q_\infty ac}{8}\Big[(be^{-i\beta}-ae^{i\beta})\log\sigma+\frac{b\,e^{i\beta}}{2}\sigma^2+be^{-i\beta}\sigma\bar{\sigma}$$

$$+be^{i\beta}\frac{\sigma}{\bar{\sigma}}+ae^{-i\beta}\frac{\bar{\sigma}}{\sigma}+\frac{ae^{-i\beta}}{2\sigma^2}+\frac{ae^{i\beta}}{\sigma\bar{\sigma}}\Big],\quad n=1$$

Therefore on the unit circle $|\sigma|=1$, $\sigma=\tau=e^{i\theta}$ we have from (5.11)

$$V(\tau)=\frac{q_\infty nac}{8}\Big[\frac{2n+1}{n+1}be^{i\beta}\tau^{n+1}+\frac{2n-1}{n-1}be^{-i\beta}\tau^{n-1}$$

$$-ane^{i\beta}\log\tau+\frac{3ane^{-i\beta}}{2\tau^2}+ane^{i\beta}\Big]\quad n\neq 1 \tag{5.12}$$

$$=\frac{q_\infty ac}{8}\Big[(be^{-i\beta}-ae^{i\beta})\log\tau+\frac{3be^{i\beta}}{2}\tau^2$$

$$+\frac{3ae^{-i\beta}}{2\tau^2}+ae^{i\beta}-be^{-i\beta}\Big],\quad n=1$$

We can determine H from (4.9) by eliminating the logarithmic term in $Q(\tau)$. We thus obtain

$$H=-\frac{q_\infty n^2a^2ce^{i\beta}}{8},\quad n\neq 1$$

$$=\frac{q_\infty ac}{8}(be^{-i\beta}-ae^{i\beta}),\quad n=1 \tag{5.13}$$

Henceforth we shall proceed in the analysis by assuming that $n>1$. The case $n=1$ can be handled just as easily as the one under consideration.

Hence from (4.9) we have

$$Q(\tau)=\frac{q_\infty nac}{8}\left[\frac{2n+1}{n+1}\,be^{i\beta}\,\tau^{n+1}+\frac{2n-1}{n-1}\,be^{-i\beta}\,\tau^{n-1}\right.$$

$$\left.+ane^{i\beta}+\frac{ane^{-i\beta}}{\tau^2}+ane^{i\beta}\,\frac{na\,\tau^{n+1}+b\,\tau^{2n+2}}{-na\,\tau^{n+1}+bn}\right] \tag{5.14}$$

If we recall that $b/a<1$, $|\tau|=1$ we can expand the last term in (5.14) as

$$\frac{na\,\tau^{n+1}+b\,\tau^{2n+2}}{-na\,\tau^{n+1}+bn}=-\left[\frac{b}{an}\,\tau^{n+1}+\left(1+\frac{b^2}{na^2}\right)\sum_{m=0}^{\infty}\left(\frac{b}{a\,\tau^{n+1}}\right)^m\right] \tag{5.15}$$

Hence we obtain

$$Q(\tau)=\frac{q_\infty nac}{8}\left[\frac{nbe^{i\beta}}{n+1}\,\tau^{n+1}+\frac{(2n-1)be^{-i\beta}}{n-1}\,\tau^{n-1}\right.$$

$$\left.+\frac{ane^{-i\beta}}{\tau^2}+ane^{i\beta}-ae^{i\beta}\left(n+\frac{b^2}{a^2}\right)\sum_{m=0}^{\infty}\left(\frac{b}{a\,\tau^{n+1}}\right)^m\right] \tag{5.16}$$

Therefore

$$C_{n-1}=\frac{(2n-1)\,nq_\infty abce^{-i\beta}}{8(n-1)}$$

$$C_{n+1}=\frac{n^2\,q_\infty abce^{i\beta}}{8(n+1)} \tag{5.17}$$

$$C_k=0\,,\quad k>0\,,\quad k\neq n-1\,,\quad n+1$$

On the other hand we see from (5.15) that

$$\frac{z(\tau)}{\bar{z}'(1/\tau)}=\frac{na\,\tau^{n-2}+b\,\tau^{2n-1}}{-na\,\tau^{n+1}+bn}=-\frac{b}{an}\,\tau^{n-2}-\left(1+\frac{b^2}{na^2}\right)\frac{1}{\tau^3}\sum_{m=0}^{\infty}\left(\frac{b}{a\,\tau^{n+1}}\right)^m \tag{5.18}$$

Hence

$$c_{n-2}=-\frac{b}{an}\,,\quad n\geq 2$$

$$c_k=0\,,\quad k>0\,,\quad k\neq n-2 \tag{5.19}$$

Thus in (4.11), $m+k-1=n-2$ whence $m=n-k-1\geq 1$ and $k\leq -2$ and we obtain the following set of equations for $k\leq n-2$

$$\alpha a_1+(n-2)\,\bar{a}_{n-2}\,c_{n-2}=C_1$$
$$\alpha a_2+(n-3)\,\bar{a}_{n-3}\,c_{n-2}=C_2$$
$$\vdots$$
$$\alpha a_{n-2}+\bar{a}_1\,c_{n-2}\qquad=C_{n-2}$$

from which we have

$$a_k=\frac{\alpha C_k-(n-k-1)c_{n-2}\bar{C}_{n-k-1}}{\alpha^2-k(n-k-1)\,c_{n-2}\,\bar{c}_{n-2}}\,,\quad k\leq n-2 \tag{5.20}$$

One can easily see that

$$a_k=\frac{C_k}{\alpha}\,,\quad k>n-2 \tag{5.21}$$

We assume that the cavity wall is free from stresses so that α should be taken equal to 1. In the present case (5.20) and (5.21) reduce to

$$\begin{aligned} a_{n-1}&=C_{n-1}\\ a_{n+1}&=C_{n+1}\\ \text{other } a_k&=0 \end{aligned} \tag{5.22}$$

Hence the complex potential $f_0(\sigma)$ becomes [cf. (4.7)]

$$f_0(\sigma)=\frac{n(2n-1)\,q_\infty abce^{-i\beta}}{8(n-1)}\,\sigma^{n-1}+\frac{n^2q_\infty abce^{i\beta}}{8(n+1)}\,\sigma^{n+1} \tag{5.23}$$

Although we determine the function $h_0(\sigma)$ through (4.11) it is more convenient now to employ a direct method utilizing the Cauchy's teorem. It follows from (4.8) that

$$h_0(\tau)=\bar{Q}(1/\tau)-\frac{\bar{z}(1/\tau)}{z'(\tau)}\,f_0'(\tau)-\bar{f}_0(1/\tau) \tag{5.24}$$

Since $h_0(\sigma)$ is an holomorphic function we can write

$$h_0(\sigma)=\frac{1}{2\pi i}\int_\Gamma\frac{h_0(\tau)}{\tau-\sigma}\,d\tau$$

where Γ denotes the unit circle in σ - plane. Thus

$$h_0(\sigma)=\frac{1}{2\pi i}\int_\Gamma\frac{\bar{Q}(1/\tau)}{\tau-\sigma}\,d\tau-\frac{1}{2\pi i}\int_\Gamma\frac{\bar{z}(1/\tau)}{z'(\tau)}\,\frac{f_0'(\tau)}{\tau-\sigma}\,d\tau$$

It is straightforward to see that the integral of the last term in (5.24) vanishes. On the other hand

$$\frac{\bar{z}(1/\tau)}{z'(\tau)}=-\frac{na\tau+b/\tau^n}{nb\tau^{n-1}-na/\tau^2}=-\left(\frac{b}{na\tau^{n-2}}+\tau^3\right)\left(1+\frac{b}{a}\tau^{n+1}+\frac{b^2}{a^2}\tau^{2n+2}+\dots\right)$$

$$=-\frac{b}{na}\tau^{-(n-2)}-\left(1+\frac{b^2}{na^2}\right)\tau^3-\dots$$

$$\bar{Q}(1/\tau)=\frac{nq_\infty ac}{8}\left[\frac{nbe^{-i\beta}}{n+1}\frac{1}{\tau^{n+1}}+\frac{(2n-1)be^{i\beta}}{n-1}\frac{1}{\tau^{n-1}}+ane^{-i\beta}\right.$$

$$\left.+ane^{i\beta}\tau^2-ae^{-i\beta}\left(n+\frac{b^2}{a^2}\right)\sum_{m=0}^{\infty}\left(\frac{b}{a}\tau^{n+1}\right)^m\right]$$

When $k>0$

$$\frac{1}{2\pi i}\int_\Gamma\frac{d\tau}{\tau^k(\tau-\sigma)}=\left[\frac{1}{\sigma^k}+\frac{(-1)^{k-1}(k-1)!}{(k-1)!(-\sigma)^k}\right]=0$$

Therefore (5.24) is evaluated as follows

$$h_0(\sigma)=\frac{nq_\infty ac}{8}\left[ane^{-i\beta}+ane^{i\beta}\sigma^2-e^{-i\beta}\left(n+\frac{b^2}{a^2}\right)\sum_{m=0}^{\infty}\left(\frac{b}{a}\sigma^{n+1}\right)^m\right]$$

$$-\frac{\bar{z}(1/\sigma)}{z'(\sigma)}f_0'(\sigma)$$

Since $\left|\frac{b}{a}\sigma^{n+1}\right|<1$ we have

$$\sum_{m=0}^{\infty}\left(\frac{b}{a}\sigma^{n+1}\right)^m=\frac{1}{1-\frac{b}{a}\sigma^{n+1}}$$

and from (5.23) and (5.1)

$$f_0'(\sigma)=\frac{nq_\infty ac}{8}\left[(2n-1)be^{-i\beta}\sigma^{n-2}+nbe^{i\beta}\sigma^n\right]$$

$$\frac{\bar{z}(1/\sigma)}{z'(\sigma)}=\frac{1}{\sigma^{n-2}}\frac{na\sigma^{n+1}+b}{nb\sigma^{n+1}-na}$$

and we find

$$h_0(\sigma)=\frac{nq_\infty ac}{8}\,\frac{n\left(an+\dfrac{b^2}{a}\right)e^{i\beta}\sigma^2+n(n-1)\,be^{-i\beta}\sigma^{n+1}+(n-1)\dfrac{b^2}{a}e^{-i\beta}}{n\left(1-\dfrac{b}{a}\sigma^{n+1}\right)} \tag{5.25}$$

Finally we obtain the complex potentials in σ - plane from (4.6) and (5.13):

$$f(\sigma)=\frac{nq_\infty ac}{8}\left[\frac{nbe^{i\beta}}{n+1}\sigma^{n+1}+\frac{(2n-1)be^{-i\beta}}{n-1}\sigma^{n-1}+\frac{nae^{-i\beta}}{2\sigma^2}\right]$$

$$h(\sigma)=\frac{nq_\infty ac}{8}\left[\frac{n(n-1)be^{-i\beta}\sigma^{n+1}+na[n+(b^2/a^2)e^{i\beta}\sigma^2]}{n\left(1-\dfrac{b}{a}\sigma^{n+1}\right)}\right.$$

$$\left.+\frac{(n-1)(b^2/a)e^{-i\beta}}{n\left(1-\dfrac{b}{a}\sigma^{n+1}\right)}+nae^{-i\beta}\log\sigma\right] \tag{5.26}$$

6. Special Cases

If we approximate the shape of the cavity to a regular polygon of $n+1$ sides we have to take $a=nb$. It is straightforward to see that the radius of the circle in which the regular polygon is inscribed is given by

$$\rho=b(n^2+1)$$

so that by means of this parameter we have

$$b=\frac{\rho}{n^2+1},\qquad a=\frac{n\rho}{n^2+1} \tag{6.1}$$

Hence the complex potentials become expressible in the following forms:

$$f(\sigma)=\frac{n^2q_\infty c\rho^2}{8(n^2+1)^2}\left[\frac{ne^{i\beta}}{n+1}\sigma^{n+1}+\frac{(2n-1)\,e^{-i\beta}}{n-1}\sigma^{n-1}+\frac{n^2\,e^{-i\beta}}{2\sigma^2}\right]$$

$$h(\sigma)=\frac{n^2q_\infty c\rho^2}{8(n+1)^2}\left[n(n\log\sigma-n+1)^{-i\beta}+\frac{n^3+1}{n}\,\frac{ne^{i\beta}\sigma^2+(n-1)\,e^{-i\beta}}{n-\sigma^{n+1}}\right] \tag{6.2}$$

The potentials in z - plane are obtainable through the inverse of the mapping (5.1).

i. Circular Hole. If we let $n\to\infty$, $b\to 0$ but $\lim bn^2=\rho$ we obtain the case of a circular hole (Fig. $2a$):

$$f(\sigma)=\frac{q_\infty c\rho^2 e^{-i\beta}}{16\sigma^2}$$

$$h(\sigma)=\frac{q_\infty c\rho^2}{8}\left(e^{i\beta}\sigma^2+e^{-i\beta}\log\sigma\right) \tag{6.3}$$

with

$$z=\frac{\rho}{\sigma} \tag{6.4}$$

Hence

$$\Omega(z)=\frac{q_\infty c e^{-i\beta}}{16}z^2$$

$$\omega'(z)=\frac{q_\infty c\rho^2}{8}\left(\frac{\rho^2 c^{i\beta}}{z^2}+e^{-i\beta}\log\frac{\rho}{z}\right) \tag{6.5}$$

These results coincide with those obtained previously [4].

ii. Equilateral Triangular Hole. Let $n=2$ (Fig. 3)

$$f(\sigma)=\frac{q_\infty c\rho^2}{50}\left[\frac{2}{3}e^{i\beta}\sigma^3+3e^{-i\beta}\sigma+\frac{2e^{-i\beta}}{\sigma^2}\right]$$

$$h(\sigma)=\frac{q_\infty c\rho^2}{50}\left[2(2\log\sigma-1)e^{-i\beta}+\frac{9}{2}\,\frac{2e^{i\beta}\sigma^2+e^{-i\beta}}{2-\sigma^3}\right] \tag{6.6}$$

with

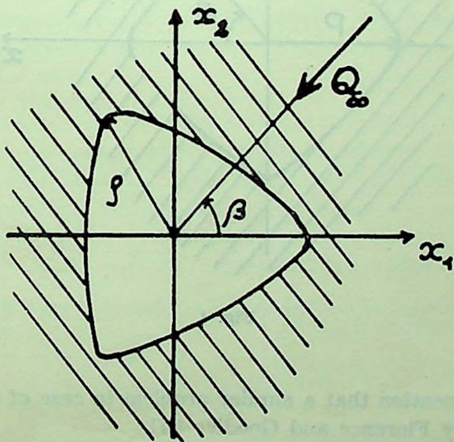

Fig. 3

$$z=\frac{\rho}{5}\left(\frac{4}{\sigma}+\sigma^{?}\right) \tag{6.7}$$

iii. Square Hole. Let $n=3$ (Fig.4)

$$f(\sigma)=\frac{9q_{\infty}c\rho^{2}}{800}\left[\frac{3}{4}\;e^{i\beta}\sigma^{4}+\frac{5}{2}\;e^{-i\beta}\sigma^{2}+\frac{9e^{-i\beta}}{2\sigma^{2}}\right]$$

$$h(\sigma)=\frac{9q_{\infty}c\rho^{2}}{800}\left[3(3\log\sigma-2)+\frac{28}{3}\;\frac{3e^{i\beta}\sigma^{2}+2e^{-i\beta}}{3-\sigma^{4}}\right] \tag{6.8}$$

with

$$z=\frac{\rho}{10}\left(\frac{9}{\sigma}+\sigma^{3}\right) \tag{6.9}$$

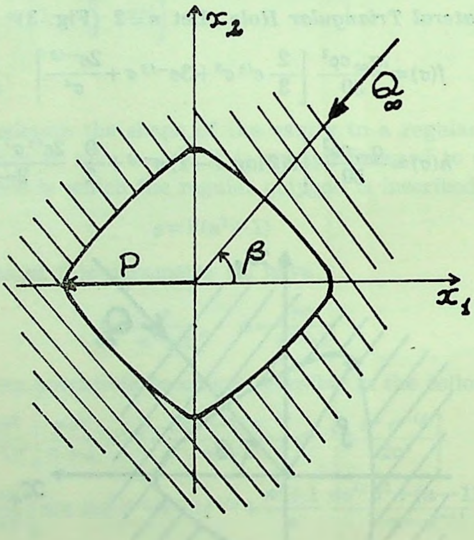

Fig. 4

It is worth to mention that a similar problem in case of an ovaloid hole was attacked by Florence and Goodier [5].

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