

**LOW-ORDER CONTROLLER DESIGN FOR TIME-DELAY
SYSTEMS**

**M.Sc. Thesis by
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**ZAMAN GECİKMELİ SİSTEMLER İÇİN DÜŞÜK MERTEBELİ
KONTROLÖR TASARIMI**

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FOREWORD

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Barış Samim NESİMİOĞLU

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LOW-ORDER CONTROLLER DESIGN FOR TIME-DELAY SYSTEMS

SUMMARY

In this thesis, the main purpose is to develop solutions on designing low-order (P, PI and PID type) controllers for time-delay systems. Based on this objective, new ideas are proposed in this work.

The first one of these ideas is the derivation of all stabilizing proportional controllers for first order time-delay systems in a simpler manner than the existing way in the literature. Since in industry, most of the processes are modelled as first or second order systems and most of them are controlled with P or PI or PID type controllers, providing such a derivation is significant. The achieved results are identical to existing results in the literature and offer more comprehensiveness and facility to the reader.

Since this superiority comes from that of employed stability analysis over the Hermite-Biehler Theorem for Quasipolynomials (which is utilized in the derivation existing in the literature) for the systems possessing single time delay, these two methods are presented firstly, in order that the reader comprehend this fact and follow this work with ease. To facilitate matters and to evidence this superiority same examples are given for both of the stability methods.

The most important achievement that comes into being in this thesis is to find all proportional controllers providing time delay independent stability. In other words, these values of proportional controllers dispel the effects of time delay on stability. This amazing fact is based on calculating the proportional controllers for which a root crossing from the imaginary axis cannot take place. To achieve this, the required conditions are derived by making use of the stability analysis that is mentioned above, and then, some root locus concepts are utilized. As the last, the detailed examples are given to enlighten the concepts.

ZAMAN GECİKMELİ SİSTEMLER İÇİN DÜŞÜK MERTEBELİ KONTROLÖR TASARIMI

ÖZET

Bu tezde güdülen temel amaç, zaman gecikmeli sistemler için düşük mertebeli (P, PI ve PID tip kontrolörler) kontrolör tasarımı konusunda yeni açılımlar getirmektir. Bu amaçtan yola çıkılarak, bu çalışmada yeni fikirler önerilmiştir.

Önerilen fikirlerden ilki, zaman gecikmesine sahip birinci mertebeden sistemleri kararlı yapacak tüm P tipi kontrolörlerin kümesini, literatürde var olan yöntemden daha kolay bir şekilde hesaplamaktır. Endüstride var olan süreçlerin çoğu birinci ve ikinci mertebeden modellenebildiği, ve bu süreçlerin de büyük bir çoğunluğu P, PI veya PID tip kontrolörlerle kontrol edildiği için, böyle bir çalışmanın yapılması önemlidir. Nitekim, varılan sonuçlar, literatürde olanlarla aynıdır; ancak kolaylık ve anlaşılabilirlik açısından okuyucu için avantaj sunmaktadır.

Bahsedilen bu avantaj, aslında kullanılan kararlılık analizinin tek zaman gecikmesine sahip sistemler için, literatürde var olan yöntemde kullanılmış olan “kuasipolinomlara genişletilmiş Hemite-Biehler teoremi”ne olan üstünlüğünden ileri geldiğinden, okuyucunun bu üstünlüğü anlaması ve çalışmanın geri kalanını daha rahat bir şekilde takip edebilmesi açısından, öncelikli olarak bu iki yöntem tanıtılmıştır. Ayrıca, bu üstünlüğün belgelenmesini kolaylaştırmak adına, her iki yöntem için de aynı örneklere yer verilmiştir.

Bu tezde ortaya çıkan en önemli başarı ise, zaman gecikmesinden bağımsız bir kararlılık (veya kararsızlık) sunan tüm P tipi kontrolörlerin bulunmasıdır. Bir başka deyişle, bulunan bu kontrolör değerleri, zaman gecikmesinin kararlılık üzerindeki etkisini yok eder. Bu ilginç ve şaşırtıcı durum, imajiner eksenden kök geçişlerinin olamayacağı kontrolör değerlerinin bulunmasında gizlidir. Bunu başarmak için ise, gerekli koşullar, yukarıda belirtilen kararlılık analizinden türetilmiş ve daha sonra bazı temel kök eğrisi kuralları kullanılmıştır. En son olarak ise anlatılan yöntemin daha rahat anlaşılabilmesi açısından, detaylı örneklere yer verilmiştir.

1. INTRODUCTION

As known, control systems can be divided into two main groups in terms of including time delay: delay-free systems and time-delay systems. The first one, when linear time-invariant systems are considered, has been examined by the researchers for many decades, hence there exist several methods not only on analysis, but also on design concepts. However, for time-delay systems, most cases in analysis, even determining stability which is the first and foremost important requirement of the design, are not so straightforward as the delay-free case. First of all, the effect of time-delay on stability cannot be predicted at first sight. In most of the cases, the time-delay has an instabilizing effect (emergence of time delay leads the system to instability) whereas in some case an unstable delay-free system could be stable with the emergence of time delay. Thus, with the consideration of other concepts on control theory, analyzing and designing of time-delay systems are much more complicated than those of for delay-free systems.

When the industrial applications is considered, time delay, in other words “dead time” may appear in many processes, and most of them are likely to be caused by the following phenomena [1] :

- a) The requirement of time while transporting quantities such as energy, mass or information.
- b) If the entire system consists of a considerable number of low-order systems to be connected in series, the sum of the time lags between each system.
- c) The required time for controllers especially if they are implementing a possible complex control algorithm and the required time for sensors when analyzing data (possibly results in a feedback delay).

In general, from the process point of view, controlling processes possessing considerable amount of time delay are rather difficult than the ones without time delay via employing standard feedback controllers due to following reasons [1]:

- a) The effects of predictable and unpredictable disturbances to the output or the controlled variable may not have been observed, until a considerable amount of time has passed.
- b) The effects of the control action at a moment are felt in the controlled variable after a considerable time has elapsed.
- c) The present control action may have been generated to amend the actual error which is originated before a considerable time.

On the other hand, from the mathematical point of view, for time-delay systems, the closed-loop characteristic expressions constitute the ones named “quasipolynomials”, which possess infinite number of roots, resulting in the establishment of the stability quite difficult than that of delay-free systems.

Therefore, from both point of views, designing controllers, especially the low order ones such as P, PI and PID to obtain a desired closed-loop response is a quite difficult task and have to be handled with a great care.

1.1 Purpose of the Thesis

When designing a controller for a system, the knowledge of all stabilizing controller parameters, especially for simple controllers such as P, PI and PID type, becomes an important task. This concept differs from tuning of controller, and possesses considerable advantages compared to the tuning concept. Although the knowledge of all stabilizing parameters facilitates the controller design, there do not exist several methods for obtaining all stabilizing parameters for simple controllers such as P, PI and PID when the time-delay case is considered. The main objective of this thesis is to develop solutions for simple controller design for time-delay systems.

1.2 Theoretical Background

As it might be expected, analyzing the stability of a system has to be well understood before designing a controller especially for time-delay systems, since the analyzing methods of stability of time-delay systems are not straightforward. However, most of the methods in the literature are not suitable for arbitrary linear time-invariant systems, most of them work with restricted transfer functions. In this thesis, two of the most general and convenient stability methods are presented in order to meet the stated objectives.

1.3 Achievements

In this thesis, a simpler analytical determination (compared to the way followed in [2]) of all stabilizing proportional controllers for first order time-delay systems is presented. Moreover, and as the most essential, all proportional controllers providing a time delay independent stability for an arbitrary linear time-invariant system is considered.

2. TIME DELAY AND STABILITY

As mentioned in the last section, stability is the first and foremost important requirement of the design in most of the cases. However, determining stability of time-delay systems are much more complicated than that of the delay-free systems. From the mathematical point of view, the closed-loop characteristic equation of a time-delay system becomes a quasipolynomial which has infinite number of roots. This fact is a sequel to impossibility of determining the stability of a time-delay system using the well-known stability tools which are used for delay-free systems such as Routh-Hurwitz criterion, Hermite-Biehler theorem etc. The same things can be considered for the design case: the well-known design tools for delay-free systems such as root locus technique, the generalized version of Hermite-Biehler theorem which is presented in [3], etc. are unusable for time-delay systems since these tools are based on the root distribution of the closed-loop system.

In the literature, there exist many methods for determining the stability [4-12]. However, some of them are restricted to different classes of transfer functions [7-8], and some of them are restricted to the magnitude of the time delay [5]. Moreover, most of them do not lead to simple criteria.

In this part of the thesis the probable types of time-delay in the control loop is shown and the characteristic equations of time delay systems (quasipolynomials) are formed firstly. After that, two important stability analysis methods are introduced. As the first, the extension of Hermite-Biehler Theorem to quasipolynomials [2] are described briefly in order to compare the ways of determining all stabilizing proportional controllers for first-order time-delay systems achieved in [2] with achieved in this thesis. And as the last, the stability method presented in [9] is described briefly, which forms the basis of the works in this thesis.

2.1 Characteristic Equations for Time-Delay Systems

At this part, the characteristic equations of time delay systems will be derived as it is achieved in [13].

In time domain, time delay can be represented simply by the following block diagram (Figure 2.1). This figure represents a time delay with a magnitude of T .

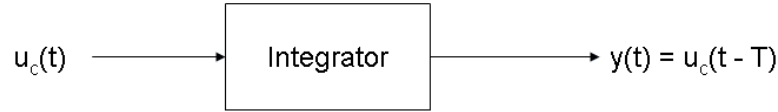


Figure 2.1 Representation of time delay

To simplify matters, when a first-order feedback system is considered, delay could present in the feedback loop (generally caused by sensor delays). The block diagram for such a system can be shown in Figure 2.2.

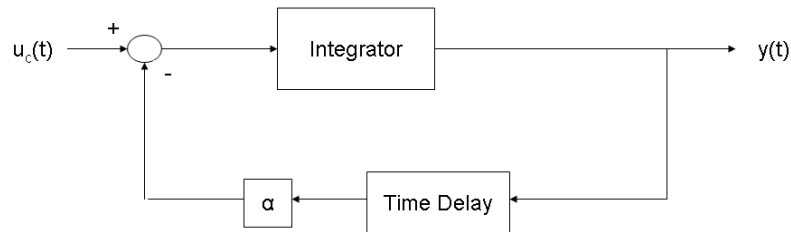


Figure 2.2 A first-order system with a delay in the feedback loop

The input-output equation of this first-order system can be given by the following expression:

$$u_c(t) = \dot{y}(t) + \alpha y(t - T) \quad (2.1)$$

Or alternatively, the delay could appear within the loop. A simple feedback first-order system possesses a delay within the loop is shown in the figure 2.3

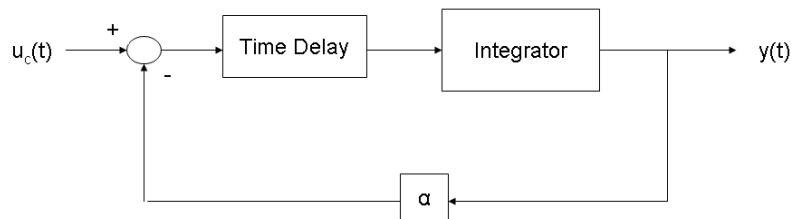


Figure 2.3 A first-order system with a delay within the loop

The input-output equation of such a system can be expressed as

$$u_c(t-T) = \dot{y}(t) + \alpha y(t-T) \quad (2.2)$$

Moreover, the delay could exist in the input. Such a simple feedback first-order system is shown in the figure 2.4.

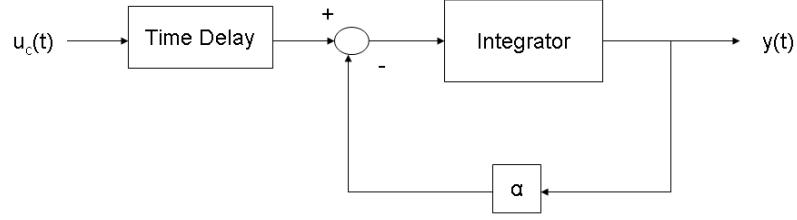


Figure 2.4 A first-order system with a delay in the input

The input-output equation of such a feedback first-order system with a delay in the input can be expressed as

$$u_c(t-T) = \dot{y}(t) + \alpha y(t) \quad (2.3)$$

In order to extend the concept, consider an n^{th} order system with n feedback and possesses n different time delays in the feedback loops. Such a system can be depicted in the figure 2.5.

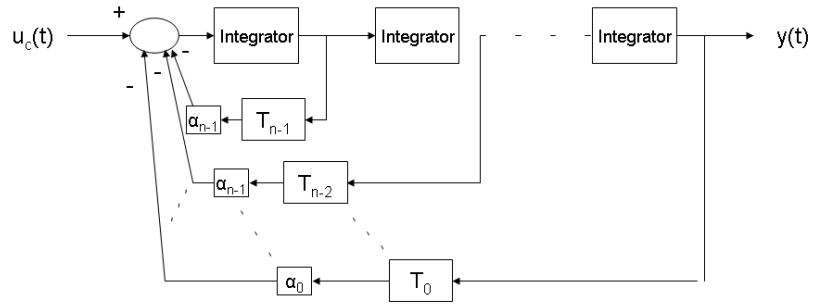


Figure 2.5 An n^{th} order integrator with n different delays in the feedback loop

This system can be stated in state space form with the following equation:

$$\dot{X}(t) = A_n X(t) + \sum_{i=0}^{n-1} A_i X(t-T_i) + Bu(t) \quad (2.4)$$

It is worth noting that $X(t)$, $\dot{X}(t)$, $X(t-T_i)$ and B are vectors having a dimension of $(n \times 1)$ and A_i s are matrices with a dimension of $(n \times n)$. If the characteristic equation is formed in s-domain by using (2.4), the following can be obtained:

$$T(s) = \det \left| sI - A_n - \sum_{i=0}^{n-1} e^{-sT_i} A_i \right| \quad (2.5)$$

$$T(s) = P_0(s) + \sum_{j=1}^p e^{-sL_j} P_j(s) \quad (2.6)$$

It should be noted that L_j s are appropriate sums of T_i s in the expression (2.5).

In the expression (2.6), as one might expect, there are p different delays. If these delays are integer multiples of a common positive number, say β , then these delays are stated to be commensurate and characteristic equation of such a system can be expressed by the following equation:

$$T(s) = P_0(s) + P_1(s)e^{-s\beta} + P_2(s)e^{-2s\beta} + \dots + P_p(s)e^{-ps\beta} \quad (2.7)$$

It is worth noting that throughout this thesis; only the systems possessing a single time delay in the feedback loop or within the loop are going to be considered. Then for such a system, the expression (2.7) becomes

$$P_0(s) + P_1(s)e^{-sL} \quad (2.8)$$

where $P_0(s)$ denotes the denominator of the open-loop transfer function whereas $P_1(s)$ denotes the numerator. To illustrate, such a system possessing a single time delay within the loop can be seen in the Figure 2.6.

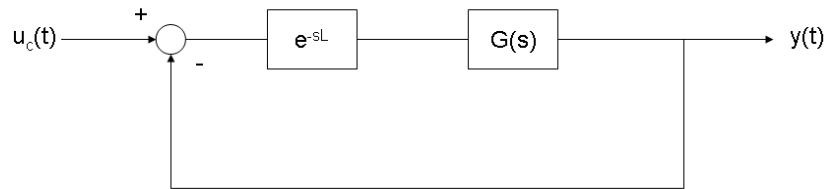


Figure 2.6 A system with a single delay

In this case generally the delay is caused by mass transportation. As one might expect, the open-loop transfer function that is depicted in the Figure 2.6 can be expressed as

$$G_0(s) = G(s)e^{-sL} = \frac{N(s)}{D(s)}e^{-sL} \quad (2.9)$$

and the closed-loop characteristic expression becomes

$$T(s) = D(s) + N(s)e^{-sL} \quad (2.10)$$

From the mathematical point of view, the expression (2.10) is a quasipolynomial, and has infinite number of roots. Thus, analyzing stability of a quasipolynomial is much more complicated than that of a polynomial. In terms of possible root distributions, when control systems are considered, quasipolynomials can be divided into two main groups.

Retarded type time-delay systems: In this case, the highest power of $T(s)$ does not include the term of e^{-sL} , i.e. $\deg N(s) < \deg D(s)$. For this type of systems infinite numbers of roots are in the left half s-plane, thus there can only be finite number of right half s-plane roots. In other words, for this type of time-delay systems, the requirement of the stability is equal to proving the absence of the right half s-plane roots.

Neutral type time-delay systems: In this case, the highest power of $T(s)$ include the term of e^{-sL} , i.e. $\deg N(s) = \deg D(s)$. For all of neutral systems, proving the absence of the right half s-plane roots is not sufficient, since the imaginary axis could be approached by a certain root chain and this could lead the system to instability. For this type of time-delay systems, the stability can be guaranteed by proving that no roots whose real parts are greater than a negative number, say α , exists, i.e. all the roots lie to the left of the line $\text{Re}[s] = \alpha$ in the s-plane.

As a consequence, if the system is strictly proper, then the closed-loop characteristic equation results in a retarded type quasipolynomial, whereas if the system is biproper, then the closed-loop characteristic equation becomes a neutral one.

2.2 Determination of the Stability of Time-Delay Systems

As determining the stability of quasipolynomials are much more complicated than that of polynomials, some researches developed and have still been developing for this treatment. However, most of the methods for this case are not directly applicable such as Routh-Hurwitz criterion in delay-free case, they include more complex mathematical calculations or longer procedures [4,6,9,11,12]. On the other hand, some of the methods are relative simple and easy to employ, yet they are not derived for general class of quasipolynomials, they can be used for only quasipolynomials with different restrictions [4,6]. In this part of the thesis, two of the most general ones are going to be described. In the first subsection, the extension of Hermite-Biehler theorem to quasipolynomials are going to be introduced in order that the reader could compare the way of analytical determination of all stabilizing controllers for first order time-delay systems presented in [2] with the way achieved in this thesis. In the second subsection, the stability method presented in [9] which forms the basis the achievements in this thesis is going to be described briefly. The first one is for systems with p commensurate delays and determines the stability independently of the delay term, whereas the latter one is for systems possessing single time delay and determines the stability of the system with respect to the delay term, L .

2.2.1 The Hermite-Biehler theorem for quasipolynomials

In this section, the results derived in [2] are briefly presented. It is worth to note that this method could determine the stability of quasipolynomials with p commensurate delays, however the systems in the examples are going to possess single time delays because throughout this thesis, only the systems with a single time delay are of interest.

Consider a system with p different delays which are arranged in an ascending order, i.e.

$$L_i < L_{i+1} \quad (2.11)$$

has the characteristic quasipolynomial as follows:

$$T(s) = d(s) + e^{-sL_1}n_1(s) + e^{-sL_2}n_2(s) + \dots e^{-sL_p}n_p(s) \quad (2.12)$$

For this quasipolynomial, the following assumptions are made:

A₁: $\deg d(s) = n$ and $\deg n_i(s) \leq n$ for $i = 1, 2, \dots, p$

A₂: $L_i = \beta L_1$, $i = 2, 3, \dots, p$, i.e. the system has commensurate delays.

Since multiplying $T(s)$ with e^{sL_p} does not change the result in terms of root locations (note that it has no finite zeros) multiplying $T(s)$ with e^{sL_p} gives

$$T^*(s) = (e^s)^{L_p} d(s) + (e^s)^{(L_p - L_1)} n_1(s) + (e^s)^{(L_p - L_2)} n_2(s) + \dots n_p(s) \quad (2.13)$$

In $T^*(s)$, the term containing the highest powers of s and e^s is defined as the principal term. Due to Pontryagin, if a function of the form $f(s, e^s)$ (note that this is the form of $T^*(s)$) has not a principal term, then it has infinite number of zeros with arbitrary large positive real parts. Indeed this result is identical to the assumption A_1 that is stated above.

Then the following theorem can be stated which determines if a particular quasipolynomial of the form $T^*(s)$ under the assumptions of A_1 and A_2 [2]:

Theorem 2.1: Denote the real and imaginary parts of $T^*(j\omega)$ as $T_r(\omega)$ and $T_i(\omega)$ respectively such that

$$T^*(j\omega) = T_r(\omega) + jT_i(\omega) \quad (2.14)$$

Then $T^*(s)$ is stable if and only if the following conditions hold

1) $T_r(\omega)$ and $T_i(\omega)$ have only simple roots and these roots interlace.

2) $\frac{d \arctan \frac{T_i(\omega)}{T_r(\omega)}}{d\omega} > 0$ for some $\omega \in (-\infty, \infty)$ (monotonic phase increase property).

As one might expect, if the derivative in the second condition of the theorem is calculated, the following expression is obtained:

$$\frac{\frac{dT_i(\omega)}{d\omega} T_r(\omega) - \frac{dT_r(\omega)}{d\omega} T_i(\omega)}{T_r^2(\omega) + T_i^2(\omega)} > 0 \quad (2.15)$$

Thus, the condition 2 in the theorem can be reduced to a simpler condition as follows:

$$\frac{dT_i(\omega)}{d\omega}T_r(\omega) - \frac{dT_r(\omega)}{d\omega}T_i(\omega) > 0 \quad (2.16)$$

Even if checking the second condition in the above theorem is a straightforward one, checking the realness of the roots of $T_r(\omega)$ and $T_i(\omega)$ in the first condition is not a straightforward one. Hence, the following lemma can be given in order to determine the realness of the roots of these functions due to Pontryagin [2]:

Lemma 2.1: Let n and p denote the highest powers of s and e^s respectively in $T^*(s)$. For a constant number, say β , for which the coefficients of terms of highest degree do not vanish when $w = \beta$ in the expressions $T_r(\omega)$ and $T_i(\omega)$. Then, in order that $T_r(\omega)$ and $T_i(\omega)$ have only real roots, the necessary and sufficient condition is that in each of the intervals

$$-2\ell\pi + \beta \leq \omega \leq 2\ell\pi + \beta, \quad \ell = \ell_0, \ell_0 + 1, \ell_0 + 2 \dots \quad (2.17)$$

$T_r(\omega)$ or $T_i(\omega)$ have exactly $4\ell p + n$ roots for a sufficiently large ℓ_0 .

Then for these concepts to be more comprehensive, the following examples are given.

2.2.1.1 Example 2.1

Determine the stability of the unity feedback system with the open-loop transfer function with a delay of 0.5 seconds:

$$G(s) = \frac{5}{s+3} e^{-0.5s} \quad (2.18)$$

As one might expect, the closed loop characteristic expression becomes:

$$T(s) = s + 3 + 5e^{-0.5s} \quad (2.19)$$

If $T^*(s)$ is formed by multiplying $T(s)$ with $e^{0.5s}$, the following is obtained:

$$T^*(s) = (s+3)e^{0.5s} + 5 \quad (2.20)$$

By letting $s = j\omega$ in (2.19), if the $T(j\omega)$ is decomposed into real and imaginary parts as mentioned in (2.14), the following is obtained:

$$T_r(j\omega) = 5 + 3\cos(0.5\omega) - \omega\sin(0.5\omega) \quad (2.21)$$

and

$$T_i(j\omega) = w \cos(0.5\omega) + 3 \sin(0.5\omega) \quad (2.22)$$

If the first condition (monotonic phase increase) of the theorem is checked for $\omega_0 = 0$, the following can be written:

$$T_i'(0)T_r(0) = (2.5)(8) > 0 \quad (2.23)$$

As (2.22) states, the first condition is satisfied. In order to check the second condition of the theorem that if $T_r(\omega)$ and $T_i(\omega)$ have the interlacing property and have only real roots, the roots of $T_r(\omega)$ and $T_i(\omega)$ have to be determined. However, both of them infinite number of roots. Considering the fact that they have the interlacing property at high frequencies since the phasor of $\frac{5}{s+3} \Big|_{s=j\omega}$ tends to zero, as ω tends to

infinity, which ensures the monotonic phase increase property, the interlacing condition is required to be checked only up to a finite frequency. In figure 2.7, it can be verified that the interlacing property of $T_r(\omega)$ and $T_i(\omega)$ is satisfied. However, as mentioned in theorem, unless proving the realness of the roots of $T_r(\omega)$ or $T_i(\omega)$ is not achieved, the stability is not guaranteed even if the interlacing property holds.

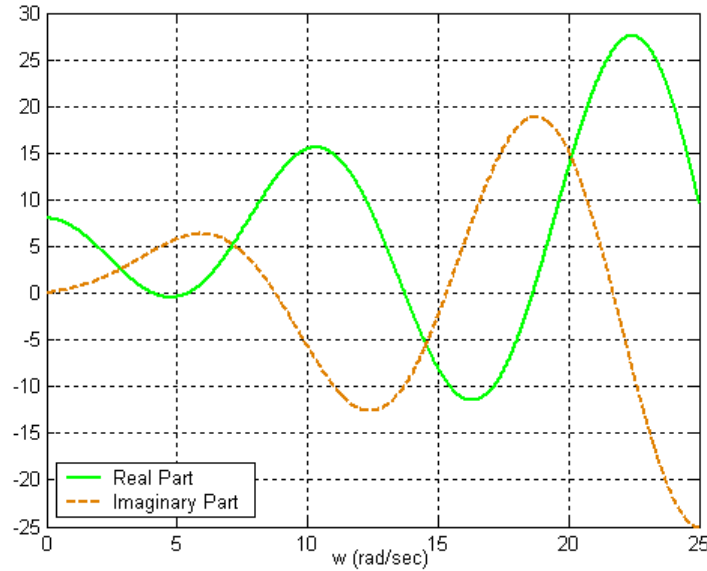


Figure 2.7 The real and imaginary parts of $T^*(j\omega)$ in Example 2.1

Yet, checking this condition is not a straightforward one. To facilitate the matters, Lemma 2.1 can be used. In this example, the realness of the roots of $T_i(\omega)$ is going to be checked. To achieve this, by letting $A = 0.5s$, the expression (2.19) is rewritten as:

$$\tan A = \frac{-2}{3} A \quad (2.24)$$

As one might expect, the roots of (2.19) are the same as the solution of (2.24). However, (2.24) is not an algebraic equation and hence, finding an analytical solution is difficult to find. In order to have an idea of the nature of roots, the graphical illustration of (2.24) can be seen in figure 2.8.

As mentioned in Lemma 2.1, (2.24) must have $4\ell p + n$ roots (where $p=1$ and $n=1$ in this example) for a sufficiently large ℓ_0 in the interval of: $-2\ell\pi + \beta \leq \omega \leq 2\ell\pi + \beta$, $\ell = \ell_0, \ell_0 + 1, \ell_0 + 2 \dots$ (2.25)

β is an appropriate constant for which the term coming from the principal term in $T^*(s)$ does not vanish at $\omega = \beta$ (in this example this term is $\omega \cos(0.5\omega)$). Due to this restriction, if β is chosen $\frac{\pi}{4}$, then $T_i(j\omega)$ has to have 5, 9, 13... roots for $\ell_0 = 1$, $\ell_0 = 2$, $\ell_0 = 3 \dots$ respectively.

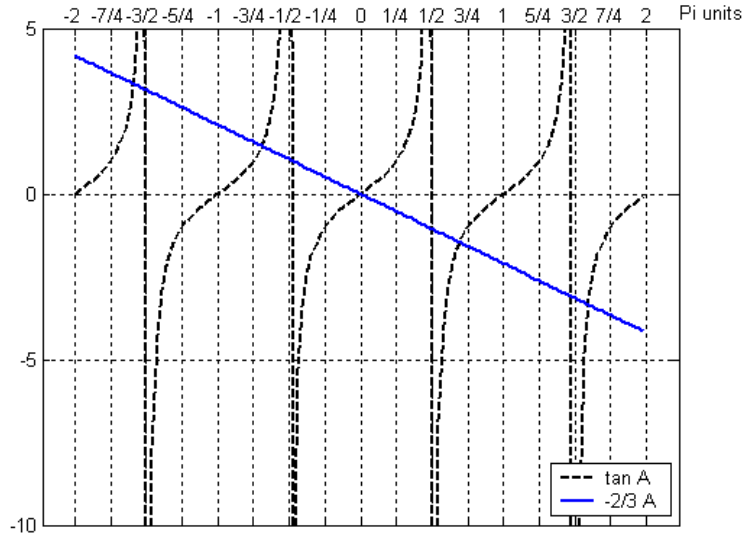


Figure 2.8 The graphical illustration of (2.24)

As can be observed from the figure 2.8, the roots of $T_i(\omega)$ lie in the interval of $\left(\frac{(2k+1)}{2}\pi, \frac{(4k+3)}{4}\pi\right)$, for $k=1,2,\dots$ (it is worth noting that, the roots are getting closer to $\frac{(2k+1)}{2}\pi$ as k increases.). Remembering that $T_i(\omega)$ is an odd function, then it can be stated that for $\ell_0=1$ and $\beta=\frac{\pi}{4}$, $T_i(\omega)$ has 2 roots in the interval of $A \in \left(0, \frac{9\pi}{4}\right)$ and so does for $A \in \left(-\frac{7\pi}{4}, 0\right)$. More commonly, it can be observed that for each of the intervals

$$A \in \left((2\ell_0\pi + \frac{\pi}{4}), \left((2(\ell_0+1)\pi + \frac{\pi}{4})\right)\right), \ell_0=1,2,\dots \quad (2.26)$$

and

$$A \in \left(\left(-2(\ell_0+1)\pi + \frac{\pi}{4}\right), \left(-2\ell_0\pi + \frac{\pi}{4}\right)\right) \ell_0=1,2,\dots \quad (2.27)$$

$T_i(\omega)$ has 2 more zeros. Moreover for $A=0$, there is one more root, and it can be concluded that, condition (2.25) is satisfied for $\ell_0=1,2,\dots$, and hence it can be stated that $T_i(\omega)$ has only real roots. Then it can finally be stated that the time-delay system is stable, when the delay is 0.5 seconds.

2.2.1.2 Example 2.2

Consider the same system under the delay of 3 seconds and determine the stability.

If the procedure stated in theorem 1 is applied to this system, $T_r(\omega)$ and $T_i(\omega)$ can be written as follows:

$$T_r(j\omega) = 5 + 3\cos(3\omega) - \omega\sin(3\omega) \quad (2.28)$$

and

$$T_i(j\omega) = \omega\cos(3\omega) + 3\sin(3\omega) \quad (2.29)$$

First, it should be determined whether $T^*(j\omega)$ has monotonic phase increase property. If this is checked for $\omega_0=0$, the following is obtained:

$$T_i'(0)T_r(0) = (10)(8) > 0 \quad (2.30)$$

The second condition is determining whether $T_r(\omega)$ and $T_i(\omega)$ have the interlacing property and if they have, checking that $T_r(\omega)$ or $T_i(\omega)$ have only real roots. As mentioned in the last example, verifying the interlacing property, only up to a finite frequency is sufficient. In figure (2.9), the graphical illustration of $T_r(\omega)$ and $T_i(\omega)$ is shown.

As can be observed from the figure (2.9), it can be stated that, the interlacing property does not hold for $T_r(\omega)$ and $T_i(\omega)$. Then it can be calculated that the system given in (2.18) becomes unstable when the delay increases from 0.5 second to 3 seconds.

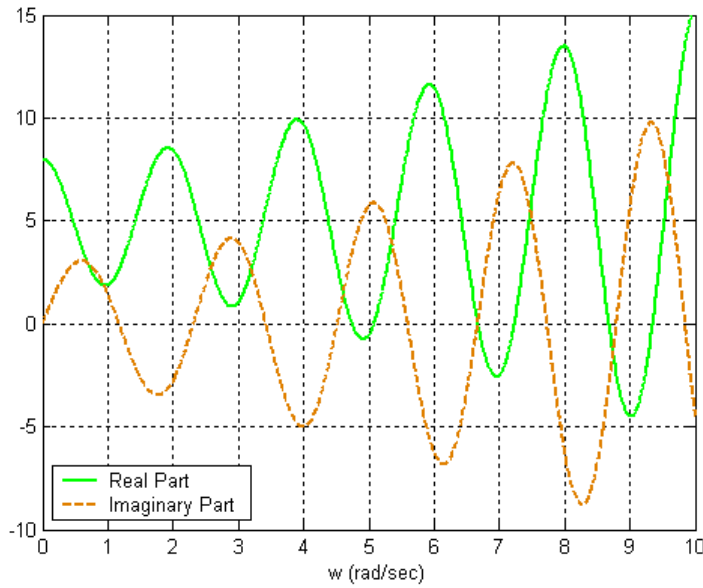


Figure 2.9 The real and imaginary parts of $T^*(j\omega)$ in Example 2.2

2.2.1.3 Example 2.3

Determine the stability of a second order unity feedback system with the open loop transfer function possessing a time delay of 1 second:

$$G(s) = \frac{\sqrt{221}}{s^2 + 5s + 15} e^{-s} \quad (2.31)$$

As described in theorem 1 and applied in the previous examples, if $T^*(s)$ is formed, the following can be written:

$$T^*(s) = T(s)e^s = (s^2 + 5s + 15)e^s + \sqrt{221} \quad (2.32)$$

If $T^*(s)$ is decomposed into its real and imaginary parts by letting $s = j\omega$ in (2.32), the following is obtained:

$$T_r(\omega) = \sqrt{221} + 15 \cos \omega - 5\omega \sin \omega - \omega^2 \cos \omega \quad (2.33)$$

$$T_i(\omega) = 15 \sin \omega + 5\omega \cos \omega - \omega^2 \sin \omega \quad (2.34)$$

Firstly, the monotonic phase increase property is checked. If $\omega_0 = 0$ is selected to be checked to facilitate matters:

$$T_i'(0)T_r(0) = (20)(15 + \sqrt{221}) > 0 \quad (2.35)$$

From (2.34), it can be seen that the monotonic phase increase property holds for $T^*(j\omega)$. The second condition requiring to be checked is the interlacing property of $T_r(\omega)$ and $T_i(\omega)$. Moreover, it is worth noting that, as mentioned in the previous examples, since the same arguments are valid for this example, the interlacing property is verified only up to a finite frequency. In figure 2.10, the graphical illustration of $T_r(\omega)$ and $T_i(\omega)$ are revealed.

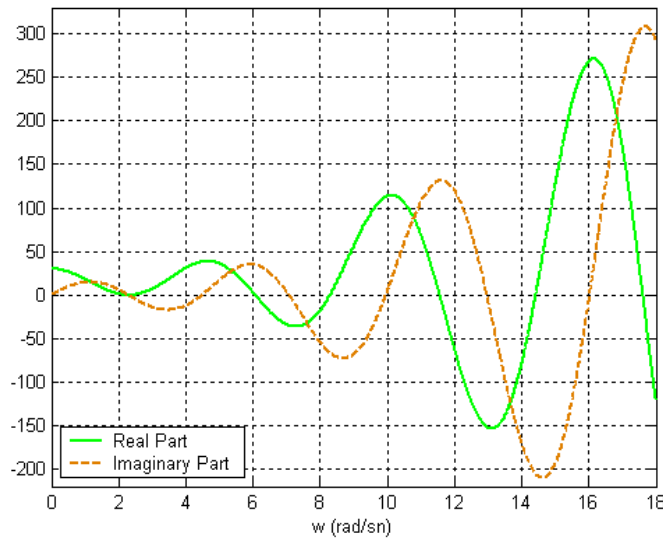


Figure 2.10 The real and imaginary parts of $T^*(j\omega)$ in Example 2.3

From figure 2.10, it is clear that, the interlacing property holds for $T_r(\omega)$ and $T_i(\omega)$. Then the next step is determining whether all the roots of $T_r(\omega)$ or $T_i(\omega)$ are real. To achieve this, if equation (2.34) is rewritten as follows:

$$\tan \omega = \frac{5\omega}{15 - \omega^2} \quad (2.36)$$

Note that, the roots of (2.34) are identical to the solutions of (2.36). However, (2.36) is not an algebraic equation, and hence it is difficult to find an analytical solution for it. In figure 2.12, the graphical illustration of (2.36) is shown to observe the nature of the solutions.

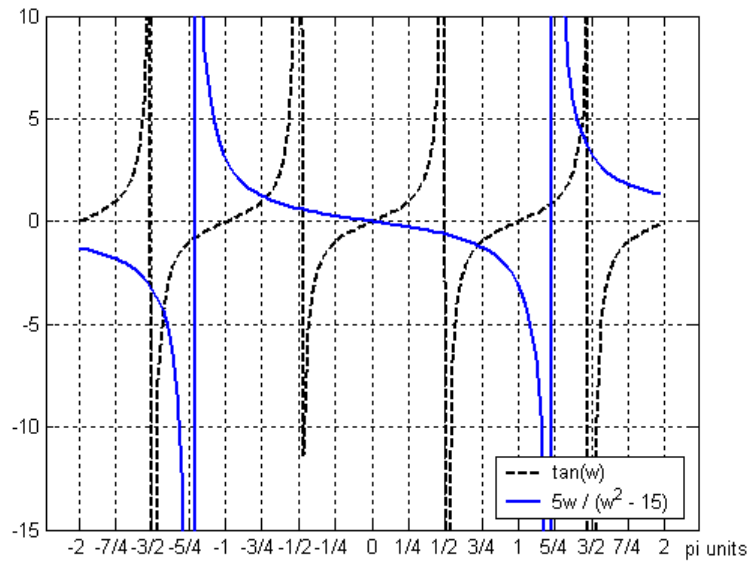


Figure 2.11 The graphical illustration of (2.36)

As can be observed from the figure 2.11, there is one more root of $T_i(\omega)$ for each value of k , in the interval of $\left(\frac{4k+5}{4}\pi, \frac{4(k+1)+5}{4}\pi\right)$, $k = 0, 1, 2, \dots$, for each value of k , and there exist two more roots at the interval of $\left[0, \frac{3\pi}{4}\right]$. Remembering that $T_i(\omega)$ is an odd function of ω , then it can be stated that $T_i(\omega)$ has one more root for each value of k , in the interval of $\left(\frac{-4k-5}{4}\pi, \frac{-4(k+1)-5}{4}\pi\right)$, $k = 0, 1, 2, \dots$, and one

extra root in the interval of $\left(\frac{-3}{4}\pi, 0\right)$. The numbers of roots of $T_i(\omega)$ have to be 6, 10, 14... for $\ell_0 = 1, \ell_0 = 2, \ell_0 = 3 \dots$ respectively in order that $T_i(\omega)$ has only real roots as mentioned in Lemma 1 and checked in Example 2.1. If β is chosen such that $\beta = \frac{\pi}{4}$ for which the principal term comes from $T^*(s)$ does not vanish $((-\omega^2 + 15)\sin \omega \neq 0)$ as done in the example 2.1, it can be stated that, $T_i(\omega)$ has 6, 10, 14... roots for $\ell_0 = 1, 2, 3 \dots$ respectively. Then, it can be concluded that, all roots of $T_i(\omega)$ are real, and hence the system given in (2.30) is stable under a time delay of 1 second.

2.2.1.4 Example 2.4

Consider the same system given in (2.30) with a delay of 2 seconds and determine whether the system is stable or not.

If the procedure stated in theorem 1 is applied to this system, $T_r(\omega)$ and $T_i(\omega)$ can be written as follows:

$$T_r(\omega) = -\omega^2 \cos 2\omega + 15 \cos 2\omega - 5\omega \sin 2\omega + \sqrt{221} \quad (2.37)$$

and

$$T_i(\omega) = -\omega^2 \sin 2\omega + 5\omega \cos 2\omega + 15 \sin 2\omega \quad (2.38)$$

If the monotonic phase increase property is checked:

$$T_i'(0)T_r(0) = (35)(15 + \sqrt{221}) > 0 \quad (2.39)$$

From (2.39), it is clear that the monotonic phase increase property holds for $T^*(j\omega)$. Then the next step is checking the interlacing property for $T_r(\omega)$ and $T_i(\omega)$ (it should be noted that this is verified only up to a finite frequency). The plots of $T_r(\omega)$ and $T_i(\omega)$ is shown in Figure 2.12. From the figure, it is obvious that the roots of $T_r(\omega)$ and $T_i(\omega)$ do not have the interlacing property. Thus, it can be concluded that the system given in (2.30) is unstable, under a time delay of 2 seconds.

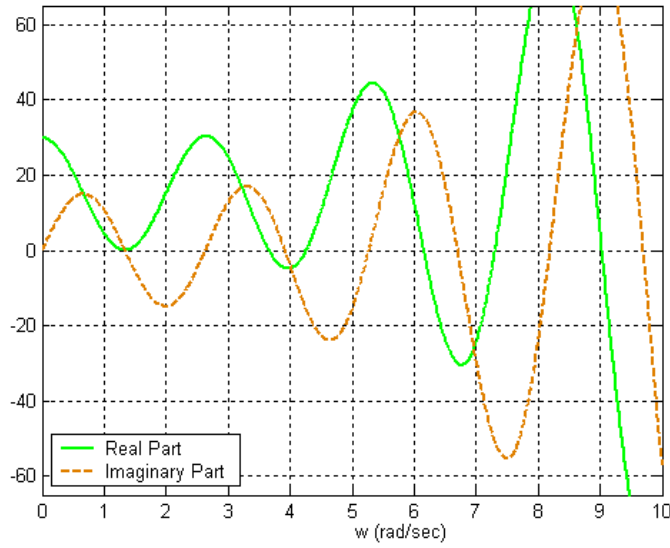


Figure 2.12 The plots of $T_r(\omega)$ and $T_i(\omega)$ in Example 2.4.

2.2.2 Walton-Marshall method

In this section the stability method presented in [9] is going to be described briefly. Compared to the Hermite Biehler theorem, this method is suitable for the systems possess a single time delay, and determines the stability in terms of the delay term, L .

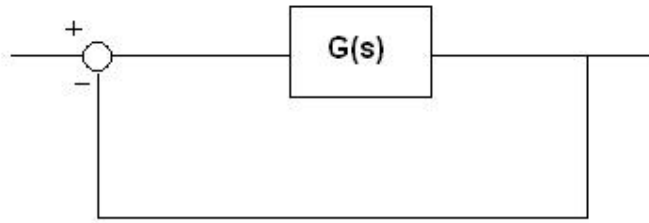


Figure 2.13 A unity feedback control system possessing a single time delay

For a given control system as depicted in Figure 1 let the transfer function be of the form:

$$G(s) = \frac{N(s)}{D(s)} e^{-Ls} = \frac{n_m s^m + n_{m-1} s^{m-1} + \dots + n_1 s + n_0}{d_n s^n + d_{n-1} s^{n-1} + \dots + d_1 s + d_0} e^{-Ls}, L > 0 \quad (2.40)$$

and assume that $D(s)$ and $N(s)$ have no common roots, i.e. they are coprime polynomials. Then, the method can be described in three steps:

Step 1: First, determine the closed loop system's stability under the delay-free case i.e. whether all the poles of the closed loop system are in the left half-plane or not.

Step 2: For an infinitesimally small L , infinite number of new roots arise in addition to the delay-free case and these roots appear at infinity. At this step, the half-plane where these roots arise is determined. To summarize, the results can be stated as follows and the reader is referred to [9] for a detailed explanation.

i) if $\deg N(s) > \deg D(s)$: For this case, infinite number of new roots arise at the right half s -plane and the system is unstable for $\forall L > 0$.

ii) if $\deg N(s) = \deg D(s)$: For this case, the closed-loop quasipolynomial is of neutral type and in order to determine where the new roots arise should be analysed further. After this detailed analysis in [9], if the condition

$$\frac{|d_n|}{|n_n|} > 1 \quad (2.41)$$

holds, all the new roots arise in the left half plane, i.e. the system could be stable for some L , otherwise, the system is unstable for $\forall L > 0$.

iii) if $\deg N(s) < \deg D(s)$: For this case, the quasipolynomial is of retarded type and infinite number of new roots arise at the right half plane. Therefore, the system could be stable for some L .

Step 3: For $L > 0$, if the system has a root pair on the imaginary axis, the following equation is obtained:

$$e^{-j\omega L} = \frac{-D(j\omega)}{N(j\omega)} \quad (2.42)$$

This equation includes two conditions to be satisfied:

Condition 1:

$$\left| \frac{-D(j\omega)}{N(j\omega)} \right| = 1 \quad (2.43)$$

then it can be written:

$$\sqrt{(-D(j\omega))(-D(-j\omega))} = \sqrt{(N(j\omega))(N(-j\omega))} \quad (2.44)$$

if the square of both sides is taken:

$$D(j\omega)D(-j\omega) = N(j\omega)N(-j\omega) \quad (2.45)$$

the following is obtained:

$$D(j\omega)D(-j\omega) - N(j\omega)N(-j\omega) = 0 \quad (2.46)$$

Let the above polynomial be denoted by $W(\omega^2)$ since it is a polynomial of ω^2 . Then it can be stated that only for a zero of $W(\omega^2)$, say $\pm \omega_i$, the system could have an imaginary axis root pair which is placed at $\pm j\omega_i$. However, the value of L corresponding to a root crossing at the particular ω_i must be calculated. This can be achieved by satisfying the following condition:

Condition 2:

$$\cos(\omega_i L) = \operatorname{Re} \left\{ \frac{-D(j\omega_i)}{N(j\omega_i)} \right\} \quad (2.47)$$

and

$$\sin(\omega_i L) = \operatorname{Im} \left\{ \frac{D(j\omega)}{N(j\omega)} \right\} \quad (2.48)$$

This means that if the system has a pair of imaginary axis root, these can only be located at the zeros of (2.46). For a particular ω_i obtained from (2.46), the system has a pair of roots which are placed at $\mp j\omega_i$, for the value of L which is obtained from the equations (2.47) and (2.48). Here it should be noted that for $-\omega_i$, however at the first glance, (2.47) and (2.48) seems different, after some algebraic effort, the equations derived for $-\omega_i$ give the same with (2.47) and (2.48). Therefore, for a zero of $W(\omega^2)$, $(\pm \omega_i), -\omega_i$ is not going to be taken into account during the rest of the thesis.

Another point worth to note is that, from the above equations (2.47) and (2.48), L is periodic with a period of $\frac{2\pi}{\omega_i}$. It means that, once the smallest $L = L_{0_i}$ is found from (2.47) and (2.48), a root pair crosses the imaginary axis at the same ω_i for the delay term L_{k_i} 's satisfying:

$$L_{k_i} = L_{0_i} + \frac{2k\pi}{\omega_i}, k = 1, 2, 3... \quad (2.49)$$

To summarize, let the above procedures be briefly stated. Step 1 is straightforward: the roots of the delay-free case is calculated. If the system is strictly proper, or proper satisfying the condition stated in the step 2, for an infinitesimally small L, infinite number of new roots arise in the left half plane. For increasing L, some roots are placed at the imaginary axis for a particular value of ω and L. These values can be calculated from the equation (2.46) and (2.47)-(2.48) respectively at step3. Here it should be noted that if $W(\omega^2)$ has no positive real zeros, then for any positive value of L, no roots cross the imaginary axis and the stability of time delay system is the same as the delay-free case which is calculated at step 1.

After finding ω_i 's from $W(\omega^2)$ and the corresponding values of L_{0_i} 's, for each ω_i 's the direction of crossings should be found since the roots could cross the imaginary axis from left to right (destabilizing) or vice versa (stabilizing). This can be achieved by taking the derivative of s with respect to L in the closed loop characteristic equation (by calculating the derivative of $\frac{ds}{dL}$ from the characteristic equation of the closed loop system, which is an implicit function of s and L, i.e. $T(s, L) = 0$). After that, letting $s = j\omega_i$ in this expression and determining the sign of $\text{Re}\left(\frac{ds}{dL}\right)$ give the result. If

i) $\text{Sgn Re}\left(\frac{ds}{dL}\right)_{s=j\omega_i} > 0$, then the crossing root pair is destabilizing, i.e. it crosses

from left half s-plane to right half s-plane.

ii) $\text{Sgn Re}\left(\frac{ds}{dL}\right)_{s=j\omega_i} < 0$, then the crossing root pair is stabilizing, i.e. it crosses from

right half s-plane to left half s-plane.

After some algebraic effort, it can be shown that $\text{Sgn}\left(\text{Re}\left[\frac{ds}{dL}\right]_{s=j\omega_i}\right) =$

$\text{Sgn}\left([W'(\omega^2)]_{\omega=\omega_i^2}\right)$. (For details of this effort, see [9].)

The most important point that should have been noted that the crossing direction is independent of the delay term, and for a particular ω_i obtained from (2.46), all crossings are at the same direction for all L_{k_i} 's.

2.2.2.1 Example 2.5

Consider the same system given in example 2.1, where the open loop transfer function is

$$G(s) = \frac{5}{s+3} e^{-sL} \quad (2.50)$$

It is worth noting that the value of time delay is not specified as done in example 2.1, as this method determines the stability of the time-delay systems in terms of time delay.

The closed loop characteristic equation of the system can be represented as:

$$T(s, L) = s + 3 + 5e^{-sL} = 0 \quad (2.51)$$

Then if the procedure explained in this section is applied:

- 1) For $L = 0$ (delay – free case) the system is stable.
- 2) Since $\deg(D(s)) > \deg(N(s))$, the system could be stable for some L.
- 3) $W(\omega^2) = \omega^2 - 16$ is found and $W'(\omega^2) = 1$ which is constant and positive.

Then it can be stated that there is only one possible ω^2 , which is the only one root of $W(\omega^2)$, for which a pair of root can move towards the other half plane for a suitable

value of L and since $\text{Sgn} \left[\text{Re} \left(\frac{ds}{dL} \right)_{s=j4} \right] = \text{Sgn} [W'(\omega^2)]_{\omega^2=16} = 1 > 0$, then the

crossing pair of roots (which crosses the imaginary axis at the frequency of $\omega = \pm 4$) is a destabilizing pair.

In order to determine corresponding L_0 for $\omega = 4$, the following equations are obtained:

$$\cos(\omega L) = \text{Re} \left[\frac{-j\omega - 3}{5} \right] \quad (2.52)$$

and

$$\sin(\omega L) = \text{Im}\left[\frac{j\omega + 3}{5}\right] \quad (2.53)$$

for $\omega = 4$ which are reduced to

$$\cos(4L) = \frac{-3}{5} \quad (2.54)$$

and

$$\sin(4L) = \frac{4}{5} \quad (2.55)$$

respectively. If the equations (2.54) and (2.55) are solved, L_0 is found such that:

$$L_0 = 0,176\pi \quad (2.56)$$

Then the corresponding L_k for which a pair of roots cross the imaginary axis can be written as:

$$L_k = L_0 + \frac{2k\pi}{\omega} = 0,176\pi + k\frac{\pi}{2}, \quad k = 1, 2, 3, \dots \quad (2.57)$$

Then it can be concluded that when $L \in (0, 0.176\pi)$, the systems stability is the same as the delay-free case (in this case, for $L = 0$, the system is stable). For a delay term $L \in (0.176\pi, 0.676\pi)$, a pair of roots of the system is in the right half plane. In general, for $L \in \left(0.176\pi + \frac{(k-1)\pi}{2}, 0.176\pi + \frac{k\pi}{2}\right)$, $k = 1, 2, 3, \dots$, k pair of roots will be in the right half plane.

As can be noticed, the results found in example 2.5 by using the stability method presented in [8] are identical to those of example 2.1 and 2.2 in which the stabilities are analyzed via Hermite-Biehler theorem for time delay values 0.5 seconds and 2 seconds, respectively.

2.2.2.2 Example 2.6

Consider the same system with a unity feedback in example 2.3 in which the open loop transfer function is given as:

$$G(s) = \frac{\sqrt{221}}{s^2 + 5s + 15} e^{-sL} \quad (2.58)$$

Here, it is noted that, L is not specified as this stability method determines the stability in terms of time delay.

If the closed loop characteristic equation is formed, the following can be written:

$$T(s, L) = s^2 + 5s + 15 + \sqrt{221}e^{-sL} = 0 \quad (2.59)$$

Then, if the procedure explained in this section is applied as it is done in the previous example:

- 1) For $L = 0$ (delay – free case), the system is stable.
- 2) As $\deg(D(s)) > \deg(N(s))$, the system could be stable for some L .
- 3) $W(\omega^2) = \omega^4 - 5\omega^2 + 4$, which has roots at $\omega^2 = 1$ and $\omega^2 = 4$.

If the behaviour of $W'(\omega^2)$ at these roots is analyzed, then the following results are obtained:

- i) For $\omega^2 = 1$, $W'(\omega^2) = 2\omega^2 - 5 < 0$, then this root is stabilizing.
- ii) For $\omega^2 = 4$, $W'(\omega^2) = 2\omega^2 - 5 > 0$, then this root is destabilizing.

So as to determine the corresponding L_{0_i} values for ω_i 's ($\omega_1 = \pm 1$ and $\omega_2 = \pm 2$), the following equations are obtained:

- i) For $\omega_1 = 1$;

$$\cos(\omega L) = \operatorname{Re} \left[\frac{\omega^2 - 5j\omega - 15}{\sqrt{221}} \right] \quad (2.60)$$

and

$$\sin(\omega L) = \operatorname{Im} \left[\frac{-\omega^2 + 5j\omega + 15}{\sqrt{221}} \right] \quad (2.61)$$

for $\omega_1 = 1$, which are reduced to:

$$\cos(L) = \frac{-14\sqrt{221}}{221} \quad (2.62)$$

and

$$\sin(L) = \frac{5\sqrt{221}}{221} \quad (2.63)$$

If the equations (2.62) and (2.63) are solved, the corresponding value of L_0 is found such that:

$$L_{0_1} = 0,891\pi \quad (2.64)$$

Then the corresponding L_k for which a pair of roots cross the imaginary axis from left half plane to the right half plane can be written as:

$$L_{k_1} = 0.891\pi + \frac{2k\pi}{\omega_1} = 0.891\pi + 2k\pi, \quad k = 1, 2, 3 \dots \quad (2.65)$$

ii) For $\omega_2 = 2$;

$$\cos(\omega L) = \operatorname{Re} \left[\frac{\omega^2 - 5j\omega - 15}{\sqrt{221}} \right] \quad (2.66)$$

and

$$\sin(\omega L) = \operatorname{Im} \left[\frac{-\omega^2 + 5j\omega + 15}{\sqrt{221}} \right] \quad (2.67)$$

for $\omega_2 = 2$ which are reduced to:

$$\cos(L) = \frac{-11\sqrt{221}}{221} \quad (2.68)$$

and

$$\sin(L) = \frac{10\sqrt{221}}{221} \quad (2.69)$$

Then the corresponding value of L_{0_2} is found such that:

$$L_{0_2} = 0.383\pi \quad (2.70)$$

After that, the following can be written the corresponding values of L_{k_2} :

$$L_{k_2} = 0.383\pi + \frac{2k\pi}{\omega_2} = 0.383\pi + k\pi, \quad k = 1, 2, 3 \dots \quad (2.71)$$

Then, it can be concluded that, for $L \in (0, 0.383\pi)$, the system's stability is the same as the delay-free case (in this case, the system is stable for $L = 0$). For $L \in (0.383\pi, 0.891\pi)$, a pair of roots cross to the right half s-plane (by crossing the imaginary axis at $\omega = \pm 2$). For $L \in (0.891\pi, 1.383\pi)$, these roots move backwards to the s-right half plane (by crossing the imaginary axis at $\omega = \pm 1$), so the system is stable again. After that interval, due to the periodicity of L_{0_2} is shorter than that of L_{0_1} , permanent instability occurs when $L \geq 1.383\pi$. Then, it is finally stated that for the values of the time delay lie in the following intervals:

$$L \in (0, 0.383\pi) \cup (0.891\pi, 1.383\pi) \quad (2.72)$$

As can be observed, the results found in example 2.6 are identical to those of in the examples 2.3 and 2.4 in which the stability of the same system is analyzed under the time delay values of 1 second and 2 seconds, respectively. Based on the presented examples, it can be argued that the stability method presented in [9] is much more convenient and straightforward than the Hermite-Biehler theorem for quasipolynomials, especially for the systems possessing a single time delay.

3. ANALYTICAL DETERMINATION OF ALL STABILIZING PROPORTIONAL CONTROLLERS FOR 1ST ORDER TIME-DELAY SYSTEMS

As mentioned in section 2, in contrast to delay-free systems, determination of the stability is a difficult task which results in designing controllers. This can also be evidenced by the achievements of the researchers in the last two decade for delay-free systems: In [3], determination of all stabilizing controllers is achieved for proportional controllers by generalizing the well-known Hermite-Biehler theorem for polynomials which have zeros on the imaginary axis and/or in the right half plane. In [14], the technique is extended to PID controllers. However, these methods require a search over a set of signums, which increases with the degree of the system in an exponential manner. . In [15], the searching algorithm is improved by proving that different sets of signums could not correspond to the same gain interval. In [16], a faster way of calculating all stabilizing gains is presented, and this is extended to PID controllers. In [17], all stabilizing values of PID controllers are calculated via stability boundary locus. However, determining an equation for the stability boundary locus (in most of the cases it has to be drawn point by point) is the main disadvantage of this method. In [18], by using the Kronecker summation in the state space model of the system, an equation for the stability boundary locus is derived.

On the other hand, the achievements are much more limited for time-delay systems compared to delay-free case: In [2], all stabilizing P and PI controllers are calculated for first-order time-delay systems in terms of open loop gain, time delay and time constant of the plant, by extending the Hermite-Biehler theorem for quasi-polynomials as expressed in detail in the last section. In [19], the results are extended to the PID controllers, and in [20], all stabilizing values of proportional controllers are calculated for second-order time-delay systems. In [21], all stabilizing values of PID controllers are calculated for second-order time-delay systems. However, in all of these works, the reader has to deal with a trigonometric equation, which is difficult to solve analytically. All stabilizing proportional controllers are calculated

for single time-delay systems that have no imaginary axis roots except for one at the origin are found in [22], by the help of the interlacing property of the real and imaginary part of quasi-polynomials at high frequencies. This idea is extended to PID controllers in [23]. However, these methods need a stabilizing set of parameters to be selected at the beginning, which is difficult to determine, especially for PID controllers. In [24], which is the one of the most general one, all stabilizing P, PI and PID controllers are calculated for an arbitrary LTI system with time-delay. This method finds the intersection of all stabilizing low-order controllers for all values of time delays lie in a range $[0, L]$.

Nevertheless, in terms of industrial processes, the system that has to be controlled is usually modelled a first or second order one, and in most of the cases, the employed controller is P, PI or PID type. Thus, an analytical characterization of all stabilizing controllers is quite important for first and second order systems. Indeed, especially for P type controllers, this treatment can be solved via Nyquist theorem, however deriving an analytical solution gives the explicit relationships between the stability and the other parameters such as time delay (L), time constant (τ), and the open loop gain (k_s) of the plant.

In this part of the thesis, a method of determining all stabilizing proportional controllers for first order time-delay systems is proposed. This method possesses advantages over the one presented in [2] in terms of simplicity and comprehensiveness. The derived method is based on the stability criterion presented in the section 2.2.2 whereas the one presented in [2] is based on the Hermite-Biehler theorem for quasipolynomials.

In order that the reader can compare the ways in ease, the method presented in [2] is briefly explained. After that explanation, the way proposed in this is expressed in detail. In both of these sections, the considered system is a characteristic type first order system, which has a transfer function of

$$G(s) = \frac{k_s}{1 + \tau s} e^{-sL} \quad (3.1)$$

where k_s is the open loop gain τ is the time constant of the plant and L denotes the time delay. Assume that this system is controlled with a proportional controller, i.e. $C(s) = k_p$. Such a system with a unity feedback can be shown in the figure 3.1. It is

worth noting that, in the rest of this part of the thesis, it is assumed that $k_s > 0$ and $L > 0$.

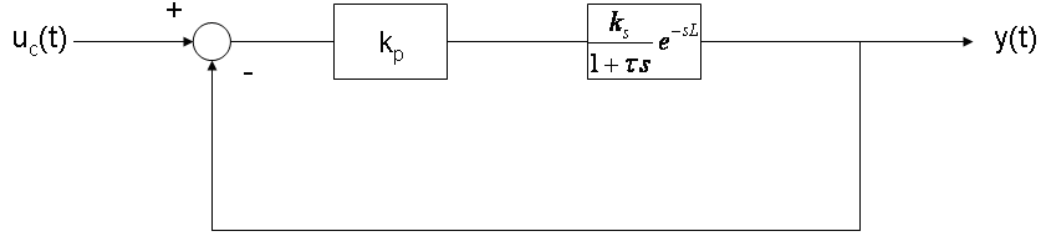


Figure 3.1 A first order time-delay system with a unity feedback

As can be derived with ease, the closed-loop characteristic equation of such a system given in (3.1) is

$$\tau s + 1 + k_p k_s e^{-sL} = 0 \quad (3.2)$$

In both of the sections, the cases are considered separately in terms of open-loop stability. As one might expect, for an open loop stable system, the time constant of the plant is greater than zero, i.e.

$$\tau > 0 \quad (3.3)$$

whereas the time constant of the plant is smaller than zero for an open-loop unstable system, i.e.

$$\tau < 0 \quad (3.4)$$

3.1 Determination Via Hermite-Biehler Theorem

As mentioned in section 2.2.1, the expressions used in this section will be of the form, with considering (3.2)

$$T^*(s) = T(s)e^{sL} = (\tau s + 1)e^{sL} + k_s k_p \quad (3.5)$$

If (3.5) is evaluated at $s = j\omega$ and decomposed into its real and imaginary parts as in the section 2.2.1 and the examples therein:

$$T_r(\omega) = \cos(L\omega) - \tau\omega \sin(L\omega) + k_p k_s \quad (3.6)$$

$$T_i(\omega) = \sin(L\omega) + \tau\omega \cos(L\omega) \quad (3.7)$$

After this brief introduction, two different cases are going to be dealt in terms of open-loop stability:

3.1.1 Open-loop stable plant

Firstly, in order that the system given in (3.2) (and inherently in (3.5)) is stable, as mentioned in the section 2.2.1, the monotonic phase increase property must hold. If ω_0 is chosen such that $\omega_0 = 0$, the following is obtained:

$$T_i'(0)T_r(0) = (\tau + L)(1 + k_p k_s) > 0 \quad (3.8)$$

In order that (3.8) is satisfied, by remembering the fact that $\tau > 0$ and $L > 0$, the following condition can be written:

$$k_p > \frac{-1}{k_s} \quad (3.9)$$

The second step is checking the interlacing property and the realness of the roots of $T_i(\omega)$ or $T_r(\omega)$. In contrast to the way followed in the examples in the section 2.2.1, since in this case the system is not specified (the behaviour of system varies as T , L and k_s changes), the realness of the roots of $T_i(\omega)$ is checked firstly. If (3.7) is rewritten as with the change of the variables $A \triangleq L\omega$,

$$T_i(A) = \sin A + \frac{\tau}{L} A \cos A \quad (3.10)$$

$$T_r(A) = \cos A - \frac{\tau}{L} A \sin A + k_p k_s \quad (3.11)$$

and for the roots of $T_i(\omega)$, (3.10) becomes

$$\tan(A) = \frac{-\tau}{L} A \quad (3.12)$$

It is clear that the roots of $T_i(\omega)$ are identical to the solutions of (3.12). However, (3.12) is not an algebraic equation and finding an analytical solution is quite difficult. In Figure 3.2, the graphical illustration of the equation (3.12) is shown.

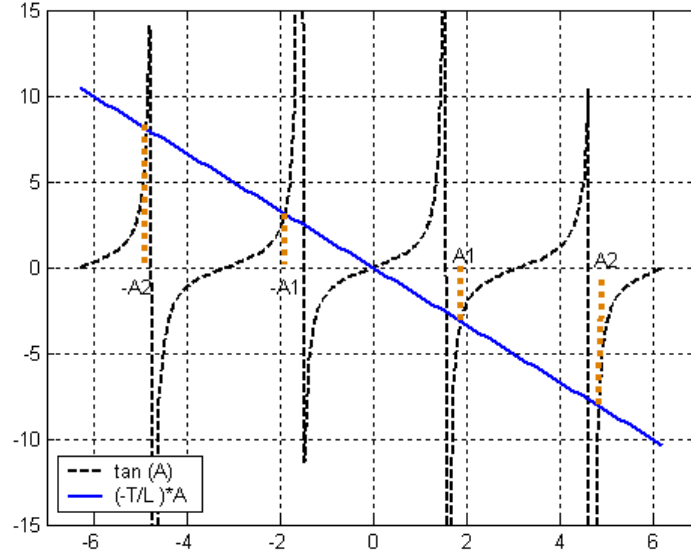


Figure 3.2 The plot of (3.12)

It can be observed from figure 3.2 that $T_i(A)$ has two more roots in each of the interval of $\left(\frac{4k+1}{2}\pi, 2(k+1)\pi\right)$, $k=0,1,2,\dots$, (say $A_b, A_{b+1}, \dots$, $b=1,3,5,\dots$) (it is evident that the roots get closer to $\frac{4k+1}{2}\pi$ as k increases. Remembering that $T_i(A)$ is an odd function, it follows that $T_i(A)$ has two more roots in each of the interval $\left(-2(k+1)\pi, -\frac{4k+1}{2}\pi\right)$, $k=0,1,2,\dots$. By keeping in mind that $T_i(A)$ has a root at the origin (say A_0), then it can be possible to show that for an appropriate constant, say β , for which the term in $T_i(A)$ and $T_r(A)$ coming from the principal term does not vanish, $T_i(A)$ has $4\ell p + n$ roots (in this case $p=1$ and $n=1$) in the interval of $-2\ell\pi + \beta \leq \omega \leq 2\ell\pi + \beta$, $\ell = \ell_0, \ell_0 + 1, \ell_0 + 2, \dots$ (3.13)

for sufficiently large ℓ_0 .

After proving the realness of roots of $T_i(A)$, the interlacing property should be checked. Considering (3.10) and (3.11), for the roots of $T_i(A)$, ($A_0, A_1, A_2, \dots$ as mentioned above) where

$$A_0 = 0, A_1 \in \left(\frac{\pi}{2}, \pi \right), A_2 \in \left(\frac{3\pi}{2}, 2\pi \right) \dots \quad (3.14)$$

For these roots, $T_r(A)$ must satisfy the following:

$$T_r(A_0) > 0, T_r(A_1) < 0, T_r(A_2) > 0, T_r(A_3) < 0 \dots \quad (3.15)$$

for $A_0, A_1, A_2, A_3 \dots$, it can be written respectively (by using (3.11) and (3.12)):

$$T_r(A_0) = 1 + k_p k_s > 0 \Rightarrow k_p > \frac{-1}{k_s} \triangleq M_0 \quad (3.16)$$

$$T_r(A_1) = \frac{-\tau}{L} \sqrt{A_1^2 + \frac{L^2}{\tau^2}} + k_p k_s < 0 \Rightarrow k_p < \frac{\tau}{k_s L} \sqrt{A_1^2 + \frac{L^2}{\tau^2}} \triangleq M_1 \quad (3.17)$$

$$T_r(A_2) = \frac{\tau}{L} \sqrt{A_2^2 + \frac{L^2}{\tau^2}} + k_p k_s > 0 \Rightarrow k_p > \frac{-\tau}{k_s L} \sqrt{A_2^2 + \frac{L^2}{\tau^2}} \triangleq M_2 \quad (3.18)$$

$$T_r(A_3) = \frac{-\tau}{L} \sqrt{A_3^2 + \frac{L^2}{\tau^2}} + k_p k_s < 0 \Rightarrow k_p < \frac{\tau}{k_s L} \sqrt{A_3^2 + \frac{L^2}{\tau^2}} \triangleq M_3 \quad (3.19)$$

and so on. As can be observed from (3.16-3.19), $M_0, M_2, M_4 \dots$ defines a lower bound on k_p whereas $M_1, M_3, M_5 \dots$ defines an upper bound on it. As one might expect, $M_0 > M_2 > M_4 > \dots$ and $M_1 < M_3 < M_5 < \dots$, thus it can be stated that the intersection of all these inequalities stated in (3.16-3.19) give the following inequality:

$$M_0 < k_p < M_1 \Rightarrow \frac{-1}{k_s} < k_p < \frac{\tau}{k_s L} \sqrt{A_1^2 + \frac{L^2}{\tau^2}} \quad (3.20)$$

where A_1 is the solution of equation (3.12) in the interval of $\left(\frac{\pi}{2}, \pi \right)$.

As one might expect, since the upper bound of k_p varies with L, then the relation that how the range of k_p changes with respect to L can be determined. In order to achieve this, denoting the upper bound of k_p as k_{pu} , if the derivative of the upper bound of k_p with respect to L is taken, the following is obtained:

$$\frac{dk_{p_u}}{dL} = \frac{\tau^2 A_1}{k_s L^3 \sqrt{\frac{\tau^2}{L^2} A_1^2 + 1}} \left(-A_1 + L \frac{dA_1}{dL} \right) \quad (3.21)$$

As mentioned before and during the section, since $L > 0$, $k_s > 0$, $\tau > 0$ and $A_1 > 0$, it can be written:

$$\frac{\tau^2 A_1}{k_s L^3 \sqrt{\frac{\tau^2}{L^2} A_1^2 + 1}} > 0 \quad (3.22)$$

Then, the sign of the term $\left(-A_1 + L \frac{dA_1}{dL} \right)$ is required so as to determine the sign of

$\frac{dk_{p_u}}{dL}$. Remembering A_1 is the solution of equation (3.12), if this expression is

differentiated with respect to L :

$$\sec^2(A_1) \frac{dA_1}{dL} = \frac{\tau}{L^2} A_1 - \frac{\tau}{L} d \frac{A_1}{dL} \quad (3.23)$$

considering (3.12) and after some algebraic manipulation, the following can be had:

$$\frac{dA_1}{dL} = \frac{\tau A_1}{L^2 + \tau^2 A_1^2 + \tau L} \quad (3.24)$$

If this expression is considered in $\left(-A_1 + L \frac{dA_1}{dL} \right)$, the following can be written:

$$-A_1 + L \frac{dA_1}{dL} = - \left(1 - \frac{1}{\frac{L}{\tau} + \frac{\tau}{L} A_1^2 + 1} \right) A_1 \quad (3.25)$$

Then it can be straightforward to prove that

$$\frac{1}{\frac{L}{\tau} + \frac{\tau}{L} A_1^2 + 1} < 1 \quad (3.26)$$

and it can be stated that the expression $\left(-A_1 + L \frac{dA_1}{dL} \right)$ is negative and hence it can be

concluded that

$$\frac{dk_{p_u}}{dL} < 0 \quad (3.27)$$

Thus, it is deduced that a stabilizing controller set of an open-loop stable first order system will always stabilize the same system possessing smaller time delay.

3.1.2 Open-loop unstable plant

As applied in the last section, in order that the system given in (3.2) (and inherently in (3.5)) is stable, the monotonic phase increase property must hold. If ω_0 is chosen such that $\omega_0 = 0$, the following is obtained:

$$T_i'(0)T_r(0) = (\tau + L)(1 + k_p k_s) > 0 \quad (3.28)$$

In order that (3.28) is satisfied, considering the fact that $(1 + k_p k_s) < 0$ for closed-loop stability for delay-free system, then the following condition has to be hold:

$$(\tau + L) < 0 \Rightarrow \frac{-\tau}{L} > 1 \quad (3.29)$$

The second step is checking the interlacing property and the realness of the roots of $T_i(\omega)$. Consistent to the way followed in the last section, the realness of the roots of $T_i(\omega)$ is checked firstly. If (3.7) is rewritten as with the change of the variables $A \triangleq L\omega$,

$$T_i(A) = \sin A + \frac{\tau}{L} A \cos A \quad (3.30)$$

$$T_r(A) = \cos A - \frac{\tau}{L} A \sin A + k_p k_s \quad (3.31)$$

and for the roots of $T_i(\omega)$, (3.10) becomes

$$\tan(A) = \frac{-\tau}{L} A \quad (3.32)$$

It is clear that the roots of $T_i(\omega)$ are identical to the solutions of (3.32). However, (3.32) is not an algebraic equation and finding an analytical solution is quite difficult. In Figure 3.2, the graphical illustration of the equation (3.32) is shown.

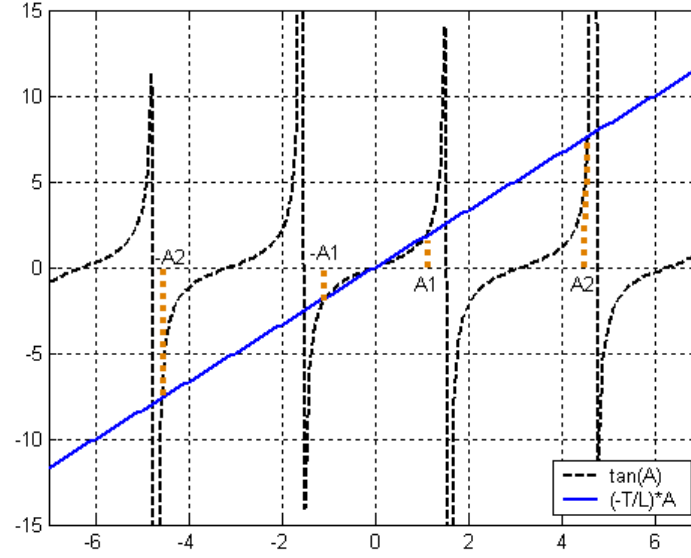


Figure 3.3 The plot of (3.32)

With following the same lines as mentioned in the last section, it can be proved that $T_i(A)$ has $4\ell p + n$ real roots in the interval of

$$-2\ell\pi + \beta \leq \omega \leq 2\ell\pi + \beta, \quad \ell = \ell_0, \ell_0 + 1, \ell_0 + 2, \dots \quad (3.33)$$

starting with sufficiently large ℓ_0 . As can be observed from figure 3.3, these roots are placed at

$$A_0 = 0, A_1 \in \left(0, \frac{\pi}{2}\right), A_2 \in \left(\pi, \frac{3\pi}{2}\right), A_3 \in \left(2\pi, \frac{5\pi}{2}\right) \dots \quad (3.34)$$

For these roots, $T_r(A)$ must satisfy the following:

$$T_r(A_0) < 0, T_r(A_1) > 0, T_r(A_2) < 0, T_r(A_3) > 0 \dots \quad (3.35)$$

for $A_0, A_1, A_2, A_3, \dots$, it can be written respectively (by using (3.31) and (3.32)):

$$T_r(A_0) = 1 + k_p k_s < 0 \Rightarrow k_p < \frac{-1}{k_s} \triangleq M_0 \quad (3.36)$$

$$T_r(A_1) = \frac{-\tau}{L} \sqrt{A_1^2 + \frac{L^2}{\tau^2}} + k_p k_s > 0 \Rightarrow k_p > \frac{\tau}{k_s L} \sqrt{A_1^2 + \frac{L^2}{\tau^2}} \triangleq M_1 \quad (3.37)$$

$$T_r(A_2) = \frac{\tau}{L} \sqrt{A_2^2 + \frac{L^2}{\tau^2}} + k_p k_s < 0 \Rightarrow k_p < \frac{-\tau}{k_s L} \sqrt{A_2^2 + \frac{L^2}{\tau^2}} \triangleq M_2 \quad (3.38)$$

$$T_r(A_3) = \frac{-\tau}{L} \sqrt{A_3^2 + \frac{L^2}{\tau^2}} + k_p k_s > 0 \Rightarrow k_p > \frac{\tau}{k_s L} \sqrt{A_3^2 + \frac{L^2}{\tau^2}} \triangleq M_3 \quad (3.39)$$

and so on. As can be observed from (3.36-3.39), $M_0, M_2, M_4 \dots$ defines an upper bound on k_p whereas $M_1, M_3, M_5 \dots$ defines a lower bound on it. As one might expect, $M_0 < M_2 < M_4 < \dots$ and $M_1 > M_3 > M_5 > \dots$, thus it can be stated that the intersection of all these inequalities stated in (3.36-3.39) give the following inequality:

$$M_1 < k_p < M_0 \Rightarrow \frac{\tau}{k_s L} \sqrt{A_1^2 + \frac{L^2}{\tau^2}} < k_p < \frac{-1}{k_s} \quad (3.40)$$

where A_1 is the solution of equation (3.32) in the interval of $\left(0, \frac{\pi}{2}\right)$.

3.2 Determination via the Method of Walton-Marshall

As mentioned at the begining of the main section, two different cases are going to be dealt in terms of open-loop stability.

3.2.1 Open-loop stable plant

At the first step, as stated in the section 2.2.2, the stability of the delay-free case, i.e. $L=0$ is analyzed. For $L=0$, the system is stable for a k_p range which is between $\frac{-1}{k_s}$ and ∞ , i.e. for

$$\frac{-1}{k_s} < k_p < \infty \quad (3.41)$$

and for the k_p range of

$$-\infty < k_p < \frac{-1}{k_s} \quad (3.42)$$

the system is unstable.

Due to the fact that $\deg D(s) > \deg N(s)$, all of the new roots arise in the left half plane for infinitesimally small L , and the system could be stable for some L . Then, if the polynomial $W(\omega^2)$ is formed:

$$D(j\omega)D(-j\omega) - k_p^2 N(j\omega)N(-j\omega) = 0 \Rightarrow \tau^2 \omega^2 + 1 - k_p^2 k_s^2 = 0 \quad (3.43)$$

which has only one root at $\omega_1^2 = \frac{k_p^2 k_s^2 - 1}{\tau^2}$ and varies with k_p as one might expect.

Then, the following observations can be made:

i) In this case, as the given system is of first order, $W'(\omega^2) = \tau^2$, which is positive and invariant with respect to ω_1 , meaning any root pairs which crosses imaginary axis is a “destabilizing pair”. Then, it can be stated that for the interval $-\infty < k_p < \frac{-1}{k_s}$, the delay-free system is unstable and as there is no stabilizing effect of time delay, the system cannot be stabilized for k_p lying in that range. As a result, this interval can safely be discarded.

ii) In order to find a real and positive solution for the equation $W(\omega^2) = 0$, $k_p^2 > k_s^2$ has to be held, i.e. $k_p > \frac{1}{k_s}$ or $k_p < \frac{-1}{k_s}$ (but here, it is worth noting that, the latter one has just been discarded, so only $k_p > \frac{1}{k_s}$ is taken into account). For the interval of $\frac{-1}{k_s} < k_p < \frac{1}{k_s}$, as the equation of $W(\omega^2) = 0$ has no real root, no roots cross the imaginary axis, i.e. there is no stability change in the time-delay system. Then it can be stated that at this interval the stability of the system is that of the delay-free case and as a consequence, for $\frac{-1}{k_s} < k_p < \frac{1}{k_s}$ the system is stable.

After these observations, for the interval of $k_p > \frac{1}{k_s}$, $W(\omega^2) = 0$ has a solution at

$\omega_1^2 = \frac{k_p^2 k_s^2 - 1}{\tau^2}$ and for such a positive real solution of $W(\omega^2)$ the following equations have to be satisfied:

$$\cos L\omega_1 = \frac{-1}{k_p k_s} \quad (3.44)$$

and

$$\sin L\omega_1 = \frac{\tau\omega_1}{k_p k_s} \quad (3.45)$$

and from (3.44)-(3.45), the following can be written

$$L\omega_1 = \arctan(-\tau\omega_1) \quad (3.46)$$

Yet from (3.44)-(3.45), it is worth noting that L is periodic with a period of 2π

instead of π . And also it is noticed that $\omega_1 = \frac{\sqrt{k_p^2 k_s^2 - 1}}{|\tau|}$. As one might expect,

since $L > 0$, $\omega_1 > 0$ and $\tau > 0$ so as to obtain L_0 , $\arctan(-\tau\omega_1)$ has to be in the interval of $\left(\frac{\pi}{2}, \pi\right)$ (for L_k , it has to be in the interval of $\left(2k\pi + \frac{\pi}{2}, (2k+1)\pi\right)$).

For each ω_1 that changes with k_p values, it is possible to find an L_0 value that causes a pair of roots crossing the imaginary axis. With a different value of k_p , roots of $W(\omega^2)$ change and so does the value of L_0 . For the stability of the system given in (3.1), delay term of the system L_s must be smaller than the L_0 value calculated from (3.46), i.e. $L_s < L_0$. Since L_0 is a function of ω_1 and also a function of k_p , if the derivative of L with respect to k_p is taken by using (3.46), the following is obtained:

$$\frac{dL}{dk_p} = \frac{\frac{k_p k_s^2 \tau}{\sqrt{k_p^2 k_s^2 - 1}} \left(\frac{-\sqrt{k_p^2 k_s^2 - 1}}{k_p^2 k_s^2} - \arctan(-\sqrt{k_p^2 k_s^2 - 1}) \right)}{k_p^2 k_s^2 - 1} \quad (3.47)$$

which has a negative sign for $k_p > \frac{1}{k_s}$. Thus it can be stated that the value L_0 that leads the system to instability is decreasing with increasing k_p . Then it can be concluded that, there must be an upper bound of k_p , making the solution of (3.46) equal to L_0 , i.e.

$$L_s = L_0 = \frac{\arctan(-\tau\omega_1)}{\omega_1} \quad (3.48)$$

Let this value of k_p be $k_{p \text{ critical}}$. Then for $k_{p \text{ critical}}$, there is also an ω_{critical} . With the change of the variable $L\omega_{\text{critical}} = A$ in (3.46)

$$A = \arctan\left(\frac{-\tau}{L} A\right) \quad (3.49)$$

Let the solution of this equation in the interval of $\left(\frac{\pi}{2}, \pi\right)$ be A_1 . Remembering the fact that

$$\omega_{\text{critical}} = \frac{\sqrt{k_{\text{critical}}^2 k_s^2 - 1}}{\tau} \quad (3.50)$$

and after some algebraic effort

$$k_{p \text{ critical}} = \frac{\tau}{k_s L_s} \sqrt{A_1^2 + \frac{L_s^2}{\tau^2}} \quad (3.51)$$

then it can be finally stated that, all stabilizing controllers for an open-loop stable first order system can be expressed within the interval of

$$\frac{-1}{k_s} < k_p < \frac{\tau}{k_s L_s} \sqrt{A_1^2 + \frac{L_s^2}{\tau^2}} \quad (3.52)$$

It should be noted that, since L_0 is a decreasing function of k_p as stated above, such a k_p range calculated for a first-order open-loop stable system with time delay L_s will stabilize the same systems possessing smaller time delay.

3.2.2 Open-loop unstable plant

For delay-free case i.e. $L = 0$, the system is stable for a k_p range which is between -

∞ and $\frac{-1}{k_s}$, i.e.

$$-\infty < k_p < \frac{-1}{k_s} \quad (3.53)$$

Since $\deg D(s) > \deg N(s)$, all of the new roots arise in the left half plane for infinitesimally small L , so the system could be stable for some L . Then, if the polynomial $W(\omega^2)$ is formed as below:

$$W(\omega^2) = \tau^2 \omega^2 + 1 - k_p^2 k_s^2 = 0 \quad (3.54)$$

which has only one root at $\omega_1^2 = \frac{k_p^2 k_s^2 - 1}{\tau^2}$.

Then it can be stated that in this case, as in the previous section, the given system is of first order and $W'(\omega^2) = \tau^2 > 0$ for $\forall \omega_1$, meaning that for $\frac{-1}{k_s} < k_p < \infty$, the

system cannot be stabilized. Thus, the interval $\left(\frac{-1}{k_s}, \infty\right)$ can safely be discarded.

For a positive real root of $W(\omega^2)$ remembering that the solution is $\omega_1^2 = \frac{k_p^2 k_s^2 - 1}{\tau^2}$,

the following equations have to be satisfied:

$$\cos L\omega_1 = \frac{-1}{k_p k_s} \quad (3.55)$$

and

$$\sin L\omega_1 = \frac{\tau\omega_1}{k_p k_s} \quad (3.56)$$

and from (3.55)-(3.56), the following can be written

$$L\omega_1 = \arctan(-\tau\omega_1) \quad (3.57)$$

Yet, it is worth noting that L is periodic with a period of 2π instead of π and also it is

noticed that $\omega_1 = \frac{\sqrt{k_p^2 k_s^2 - 1}}{|\tau|}$. As one might expect, since $L > 0$, $\omega_1 > 0$ and $\tau < 0$ in

order to obtain L_0 , $\arctan(-\tau\omega_1)$ has to be in the interval of $\left(0, \frac{\pi}{2}\right)$ (for L_k it has to

be in the interval of $\left(2k\pi, 2k\pi + \frac{\pi}{2}\right)$. For each ω_1 that changes with k_p values, it is

possible to find an L_0 value that causes a pair of roots crossing the imaginary axis.

Since L_0 changes with ω_1 and ω_1 changes with k_p the change of the value of L_0 with respect to k_p can be found. If the derivative of L with respect to k_p is taken, it can be written

$$\frac{dL}{dk_p} = \frac{\frac{k_p k_s^2 \tau}{\sqrt{k_p^2 k_s^2 - 1}} \left(\frac{-\sqrt{k_p^2 k_s^2 - 1}}{k_p^2 k_s^2} + \arctan(\sqrt{k_p^2 k_s^2 - 1}) \right)}{k_p^2 k_s^2 - 1} \quad (3.58)$$

which is positive for $\tau < 0$ and $-\infty < k_p < \frac{-1}{k_s}$ (for details see Appendix 1). Thus it can be stated that the value of L_0 is decreasing with the decreasing value of k_p . Remembering that k_p is negative, there must be a lower bound of k_p . Also there must be an upper bound of ω_1 since ω_1 is increasing with decreasing k_p such that

$$L_s = L_0 = \frac{\arctan(-\tau \omega_1)}{\omega_1} \quad (3.59)$$

and letting the frequency be $\omega_1 = \omega_{critical}$ (and also $k_{p_{critical}}$ for the corresponding value of $\omega_{critical}$) which satisfies the above equation. If these changes take place in (3.57)

$$L_s \omega_{critical} = \arctan(-\tau \omega_{critical}) \quad (3.60)$$

and with the change of variable $L_s \omega_{critical} = A$ in (3.60), the following is obtained:

$$A = \arctan\left(\frac{-\tau}{L} A\right) \quad (3.61)$$

Let the solution of this equation in the interval of $\left(0, \frac{\pi}{2}\right)$ be A_1 . However, it is worth

noting that the inequality of $\frac{-\tau}{L_s} > 1$ has to be satisfied, since otherwise, (3.61) will

have no solution at the interval of $\left(0, \frac{\pi}{2}\right)$ (See Appendix 2).

Rewriting $\omega_{critical}$ in terms of $k_{p_{critical}}$ the following is obtained

$$\omega_{critical} = \frac{\sqrt{k_{critical}^2 k_s^2 - 1}}{|\tau|} \quad (3.62)$$

and after some algebraic effort

$$k_{critical} = \pm \sqrt{\frac{\tau^2 A_1^2 + L_s^2}{k_s^2 L_s^2}} \quad (3.63)$$

Owing to the fact that for a stable system, k_p is $-\infty < k_p < \frac{-1}{k_s}$ as mentioned before,

and so

$$k_{critical} = \frac{-|\tau|}{k_s L_s} \sqrt{A_1^2 + \frac{L_s^2}{\tau^2}} \quad (3.64)$$

then it can be finally stated that, all stabilizing proportional controllers for an open-loop unstable first order system can be expressed within the interval of

$$\frac{-|\tau|}{k_s L_s} \sqrt{A_1^2 + \frac{L_s^2}{\tau^2}} < k_p < \frac{-1}{k_s}, A_1 \in \left(0, \frac{\pi}{2}\right) \quad (3.65)$$

As can be observed, the results are identical to those of obtained via Hermite-Biehler theorem in the last section.

4. CALCULATION OF ALL PROPORTIONAL CONTROLLERS PROVIDING TIME-DELAY INDEPENDENT STABILITY

In this part, an original idea about dispelling the effects of time-delay on stability is going to be developed. To achieve this, an arbitrary LTI system with a single-delay is considered as the treatment. With the help of the stability theorem presented in [9] and some results from the root-locus design, all the proportional controller values that make the stability of the system independent of time delay for an arbitrary plant are calculated. In other words, for these gains, the time delay has no effect on stability; if the system without time delay is stable (unstable) which is equal to assessing the stability of a polynomial, the time-delayed version of the system is stable (unstable) for all positive values of L . At first sight, the proposed method in this paper can be considered as a special case of the method presented in [24] with $L \rightarrow \infty$, however finding a solution is quite difficult with that idea, since the graphical nature of the solution caused by the inverse trigonometric inequality obscures the analytical characterization of a possible solution.

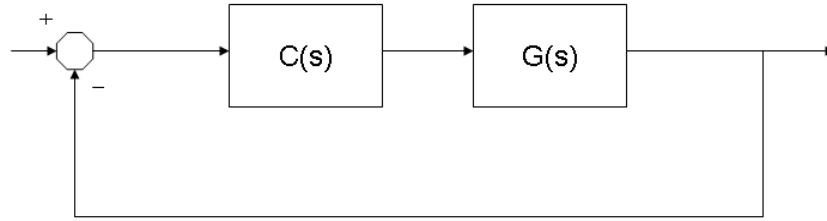


Figure 4.1 A unity feedback control system controlled with a proportional controller

Consider the closed-loop system in figure 4.1, where the system to be controlled $G(s)$ is as given in (2.40) with $m \leq n$, and the controller is of proportional type i.e.

$$C(s) = k_p \quad (4.1)$$

Let polynomials $A(v)$ and $B(v)$ are defined as

$$A(v) \triangleq D(j\omega)D(-j\omega) \quad (4.2)$$

$$B(v) \triangleq N(j\omega)N(-j\omega) \quad (4.3)$$

where $v \triangleq \omega^2$. Denote the positive real roots of $\frac{d}{dv} \left(\frac{B(v)}{A(v)} \right)$ as $v_1, v_2 \dots v_k$, and define

the corresponding values of $\alpha_i \triangleq \frac{A(v_i)}{B(v_i)}$ ($\alpha_i = \alpha_1, \alpha_2 \dots \alpha_k$). Additionally

define $\alpha_{k+1} \triangleq \frac{d_0^2}{n_0^2}$ (if $n_0 \neq 0$) and $\alpha_{k+2} \triangleq \frac{d_n^2}{n_n^2}$ (if $n_n \neq 0$). Then, select positive real α_i

(say $\alpha_1, \alpha_2 \dots \alpha_r$) and rearrange them in ascending order while adding $\alpha_0 = 0$

and $\alpha_{r+1} = \infty$. If the system is biproper, consider only those values of $\alpha_i \leq \frac{d_n^2}{n_n^2}$.

Then, the following theorem can be given.

Theorem 1 (Main result): The stability of the closed-loop system shown in figure 4.1 is independent of time delay, only for gain intervals $k_p \in \left(-\sqrt{\alpha_{i+1}}, -\sqrt{\alpha_i} \right) \cup \left(\sqrt{\alpha_i}, \sqrt{\alpha_{i+1}} \right)$ where α_i and α_{i+1} are such that the polynomial $W(v) \triangleq A(v) - \alpha B(v)$ does not have any positive real roots for a given $\alpha \in (\alpha_i, \alpha_{i+1})$. Moreover, the number of positive real roots of $W(v)$ is constant for the interval (α_i, α_{i+1}) .

Proof: If the polynomial $W(\omega^2)$ is formed for the system given in the figure 2 as described in section 2.2.2, the following is obtained:

$$W(\omega^2) = D(j\omega)D(-j\omega) - k_p^2 N(j\omega)N(-j\omega) \quad (4.4)$$

Using the definitions of $A(v)$ and $B(v)$ in (4.2) and (4.3) in (4.4) and with the change of variables $v = \omega^2$ and $\alpha \triangleq k_p^2$, it is possible to show that (4.4) is equivalent to (4.5):

$$W(v) = A(v) - \alpha B(v) \quad (4.5)$$

Note that $\alpha \in (0, \infty)$ in (4.5). As mentioned in section 2, the poles of the closed-loop system cross the imaginary axis only at the frequencies calculated from the following equation:

$$W(\omega^2) = 0 \quad (4.6)$$

As one might expect, for a relevant solution, (4.6) must have at least one real and positive solution. As the roots of (4.6) vary with α values, there may be a range or ranges of α for which (4.6) has no positive real roots. If such an α range is calculated, then it means that time-delay has no effect on changing the stability, i.e. if the system is stable (unstable) for $L = 0$, then for $\forall L > 0$, the system is stable (unstable). Yet here, it should be noted that this statement is true if the following condition holds:

$$\deg N(s) \leq \deg D(s) \quad (4.7)$$

In addition to the condition stated above, for the case of $\deg N(s) = \deg D(s)$ (i.e. $m = n$), due to the requirement of (2.41), the statement is true if the following condition holds:

$$\frac{|d_n|}{|k_p n_n|} > 1 \Rightarrow \frac{|d_n|}{|n_n|} > |k_p| \quad (4.8)$$

Otherwise the system has infinite number of new roots in the right half plane with the emergence of the time-delay and such a system is unstable for $\forall L > 0$.

Then (4.5) can be written as:

$$W(v) = A(v) + \alpha P(v) = 0 \quad (4.9)$$

where $P(v) \triangleq -B(v)$. However, this is a root locus problem. Remembering the root locus theorem, the following observations can be made:

i) This problem is the same as the problem of determining the roots of

$$1 + \alpha \frac{P(v)}{A(v)} = 0 \quad (4.10)$$

ii) For the root locus of (4.10), since $P(v)$ and $A(v)$ are real polynomials of v , when a pair of roots leave the real axis, or reach at it, that pair forms multiple roots (breakaway points).

iii) In order to find such values of v_i , the following equation can be used:

$$\frac{d}{dv} \left(\frac{P(v)}{A(v)} \right) = \frac{d}{dv} \left(\frac{B(v)}{A(v)} \right) = 0 \quad (4.11)$$

Since this last equation is independent of α , it is possible to calculate square frequencies at which breakaway points occur. It should be noted that, only the positive real roots of (4.11) are of interest as $v = \omega^2$.

iv) Suppose that (4.11) has k positive real solutions, i.e. $v_1, v_2 \dots v_k$. Simple root-locus arguments imply that there exist a crossing from positive real axis to negative real axis (or vice versa) at $v_{k+1} = 0$, if the open-loop system does not have a zero at the origin (i.e. $n_0 \neq 0$), and at $v_{k+2} = \infty$, if the system is biproper (i.e. $n_n \neq 0$). It is remarked that the positive real roots of (4.5) (or (4.10) at the same time) increase or decrease by 1 for v_{k+1} (for $n_0 \neq 0$) and v_{k+2} (for $n_n \neq 0$). The corresponding values of α_i can then be determined from (4.10) and the definition of $P(v_i)$ as follows.

$$\alpha_i = \frac{A(v_i)}{B(v_i)} \quad (4.12)$$

It is remarked that $\alpha_{k+1} = \frac{d_0^2}{n_0^2}$ and $\alpha_{k+2} = \frac{d_n^2}{n_n^2}$. Additionally, only positive α_i are of interest as $\alpha = k_p^2$.

Suppose that there exist r positive α_i . If these α_i are arranged in an ascending order, such that $\alpha_i \leq \alpha_{i+1}$ $i = (0, 1, 2, \dots, r)$, with $\alpha_0 = 0$ and $\alpha_{r+1} = \infty$, it is concluded that for the interval (α_i, α_{i+1}) , the number of real roots of (4.5) remains constant. Moreover,

it is worth noting that, the α values satisfying $\alpha_i > \frac{d_n^2}{n_n^2}$ should not be considered for

biproper systems due to (4.8). It is then possible to argue that for those intervals $\alpha \in (\alpha_i, \alpha_{i+1})$ where (4.5) does not have any positive real roots; the time-delay system behaves as if it is a delay-free system from the stability point of view.

Remembering $\alpha = k_p^2$ corresponding gain intervals are given as $k_p \in (-\sqrt{\alpha_{i+1}}, -\sqrt{\alpha_i}) \cup (\sqrt{\alpha_i}, \sqrt{\alpha_{i+1}})$. This completes the proof.

The number of positive real roots of the equation (4.5) for the interval (α_i, α_{i+1}) can be determined easily by selecting a sample value, say α_{i+1}^* such that $\alpha_{i+1}^* \in (\alpha_i, \alpha_{i+1})$.

It is remarked that the gain intervals given by Theorem 1 form all possible gain

values for which the closed-loop system is insensitive to time-delay with respect to stability. These concepts are illustrated with a couple of examples:

4.1 Example 4.1

Consider the system with the open-loop transfer function

$$G(s) = \frac{s^2 + 5s + 60}{s^3 + 36s^2 + 350s + 2750} e^{-sL} \quad (4.13)$$

controlled with a proportional controller, i.e.

$$C(s) = k_p \quad (4.14)$$

Let all the proportional controllers providing time delay independent stability for the system given in (4.13) be found.

For the delay-free case, the system is stable for $-31.5 < k_p < \infty$ and unstable for $-\infty < k_p < -31.5$. The polynomial $W(\omega^2)$ is formed as

$$W(\omega^2) = A(\omega^2) - k_p^2 B(\omega^2) = 0 \quad (4.15)$$

where

$$A(\omega^2) = \omega^6 + 596\omega^4 - 75500\omega^2 + 7562500 \quad (4.16)$$

and

$$B(\omega^2) = \omega^4 - 95\omega^2 + 3600 \quad (4.17)$$

Since $\deg N(s) < \deg D(s)$, the condition given by (4.7) is satisfied. Then, with the change of the variables of $v = \omega^2$ and $\alpha \triangleq k_p^2$, the following is obtained:

$$A(v) - \alpha B(v) = 0 \quad (4.18)$$

where

$$A(v) = v^3 + 596v^2 - 75500v + 7562500 \quad (4.19)$$

and

$$B(v) = v^2 - 95v + 3600 \quad (4.20)$$

If the procedure described above is employed, the expression (4.11) is satisfied for $v_1 = 45.6716$ and $v_2 = 229.453$. After adding $v_3 = 0$ to this list, if the corresponding α_i are calculated, $\alpha_1 = 4047.8$, $\alpha_2 = 978.148$ and $\alpha_3 = 2100.69$ are found. Let $\alpha_1 = 978.148$, $\alpha_2 = 2100.69$ and $\alpha_3 = 4047.8$, i.e. rearrange them in an ascending order with $\alpha_0 = 0$ and $\alpha_4 = \infty$. Then, it can be stated that there are four intervals of α for which the number of positive real roots of $A(v) - \alpha B(v) = 0$ have to be checked. If those positive real root numbers are determined, the following is obtained:

For $(\alpha_0, \alpha_1) = (0, 978.148)$: 0 positive real roots

For $(\alpha_1, \alpha_2) = (978.148, 2100.69)$: 2 positive real roots

For $(\alpha_2, \alpha_3) = (2100.69, 4047.8)$: 3 positive real roots

For $(\alpha_3, \alpha_4) = (4047.8, \infty)$: 1 positive real root

Then it can be finally stated that, for $(-31.275, 0) \cup (0, 31.275)$ (as $\sqrt{978.148} = 31.275$), the system behaves as if it is a delay-free system from the stability point of view. (In this case, the system is stable for $\forall L > 0$, since the delay-free system is stable for the intervals which are of interest). This can be observed in figures 4.1-4.2 and 4.3 which are obtained for some values of L and k_p for unit step input. As can be seen, the system is stable for the case of $L = 10$ seconds when $k_p = 29$ which remains in the border of the controller values providing time-delay independent stability, whereas same system possessing an equivalent time delay is not, for $k_p = 34$, which is out of the borderline. And also for another value of $k_p = 20$ remaining inside of the borderline, since the time delay does not affect the stability, the system is stable under a huge value of time delay, $L = 25$ seconds.

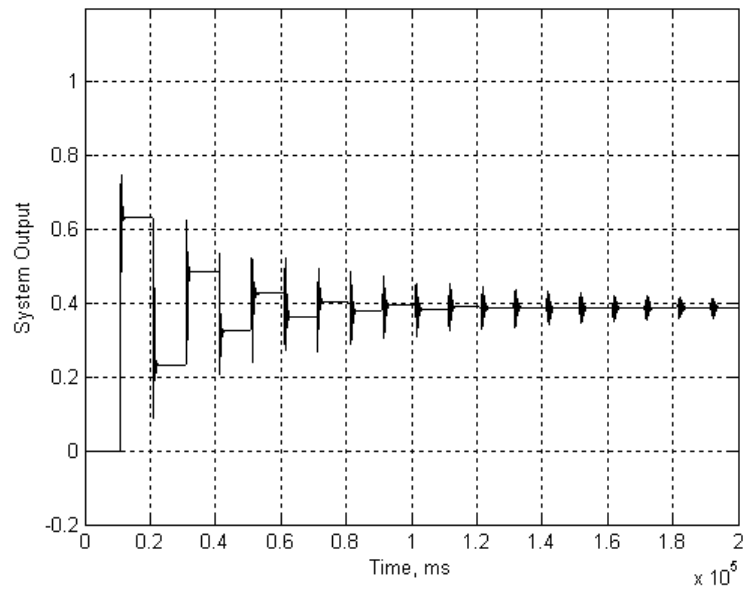


Figure 4.2 The output response of the system in example 1 for $L = 10$, $k_p = 29$

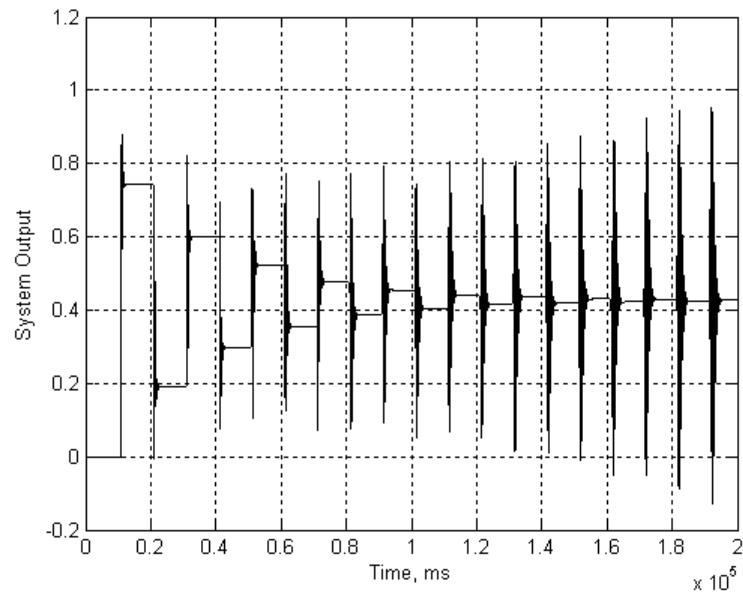


Figure 4.3 The output response of the system in example 1 for $L = 10$, $k_p = 34$

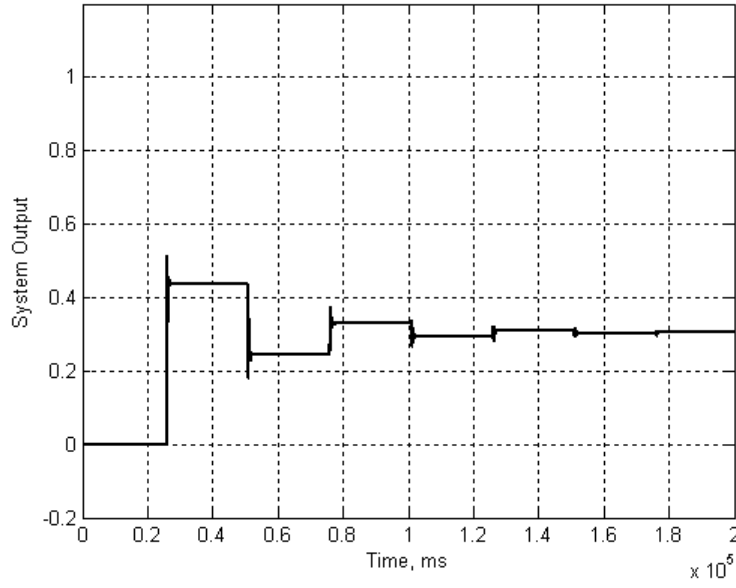


Figure 4.4 The output response of the system in example 1 for $L = 25$, $k_p = 20$

4.2 Example 4.2

Consider the system with the open-loop transfer function

$$G(s) = \frac{0.01s^4 + 0.15s^3 + s^2 + 3s + 4}{s^4 + 20s^3 + 152s^2 + 515s + 660} e^{-sL} \quad (4.21)$$

controlled with a proportional controller, i.e.

$$C(s) = k_p \quad (4.22)$$

Then let all the proportional controllers providing time delay independent stability for the system given in (4.21) be found.

For delay-free case, the system is stable for $k_p \in (-\infty, -176) \cup (-100, \infty)$, and unstable for $k_p \in (-176, -100)$. If the procedure described in the last section is applied, it is possible to show that equation (4.11) is satisfied only for one positive real square frequency: $\nu_1 = 6.58536$. After adding $\nu_2 = 0$ and $\nu_3 = \infty$ (since the system is biproper, i.e. $\deg D(s) = \deg N(s)$) to this frequency, the corresponding α_i are found as $\alpha_1 = 33746.6$, $\alpha_2 = \frac{d_0^2}{n_0^2} = 27225$

and $\alpha_3 = \frac{d_n^2}{n_n^2} = 10000$. Since the system is biproper, only α_i that satisfy

$\alpha_i \leq \frac{d_n^2}{n_n^2} = 10000$ should be considered. Hence, only the interval $(0, 10000)$ needs

to be checked. It is possible to show that the number of positive real roots of $W(v) = A(v) - \alpha B(v)$ is zero in this interval. It can be concluded that for $k_p \in (-100, 100)$ the system behaves as if it is a delay-free system from the stability point of view, i.e. time delay does not affect the stability. (In this case, the delay-free system is stable for this range of gain values, so the time-delay system is stable for $\forall L > 0$ as $k_p \in (-100, 100)$). The results are illustrated in figures 4.5-4.6 where step responses of the closed-loop system are drawn for some k_p and L values. As observed from the figures, the system is stable under the delay of $L = 8$ seconds for $k_p = 90$, which lies in the resulting set calculated above. On the other hand, the system possessing the same delay is unstable for $k_p = 103$ which is placed outside of the borderline.

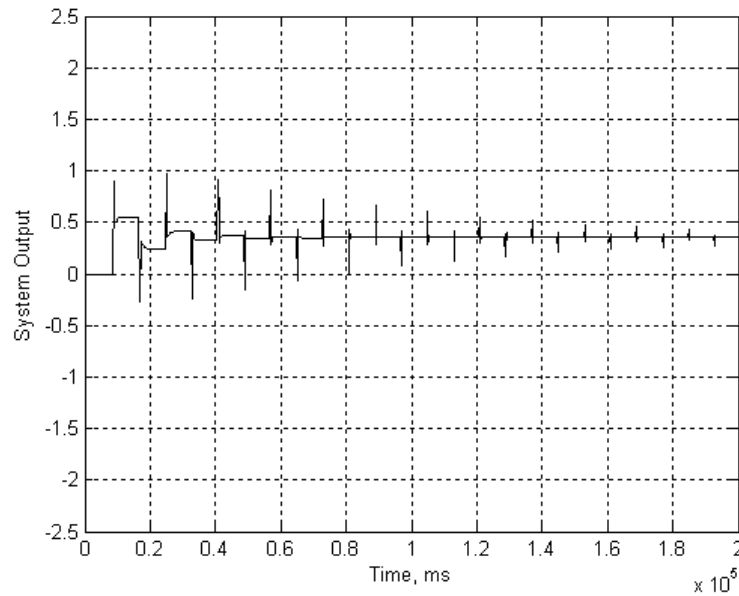


Figure 4.5 The output response of the system in example 2 for $L = 8$, $k_p = 90$

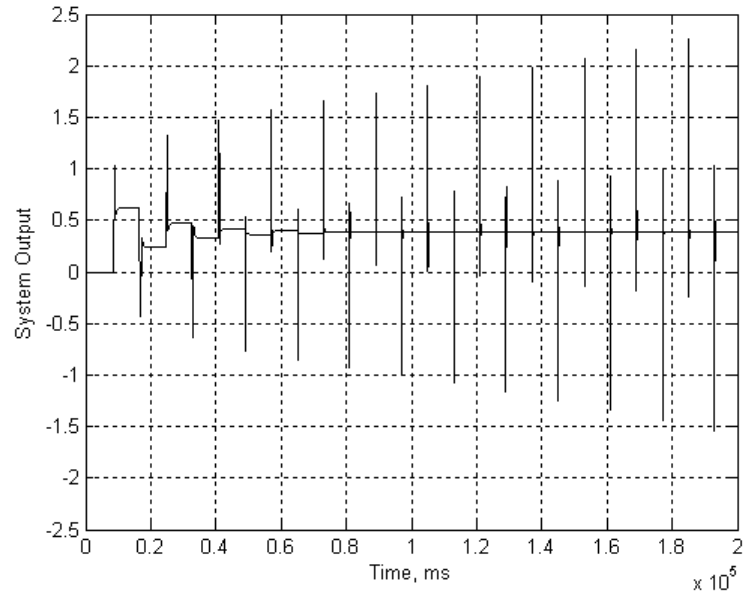


Figure 4.6 The output response of the system in example 2 for $L = 8$, $k_p = 103$

5. CONCLUSIONS

In this thesis, detailed examinations and the achieved results about the low-order controllers for time-delay systems are given. As can be understood from the previous sections, the systems possessing time delays have always much more complicated nature than that of delay-free systems. As proved in the section 2, even determination of the stability of a time-delay system, the foremost important requirement of a possible design is not straightforward.

Therefore, designing a low-order controller that stabilizes a time-delay system is a superfluous one in terms of difficulty. To facilitate the matters, in the literature, some results are derived for plants with some restrictions. For instance, since the most processes in industry are modelled as first or second order systems, deriving necessary conditions for P, PI and PID type controllers that stabilize the given system belongs to a significant research.

In this thesis, one of the works is focused on deriving all stabilizing proportional controllers for first-order systems simpler than offered in [2]. The derivation achieved here is shorter and more efficiently in terms of being comprehended by the reader. The success of this derivation exactly comes from the superiority of the stability method presented in [9] over the Hermite-Biehler theorem, when the case is dealing with systems possessing single time delay. In order that these facts to be evidenced, both of these methods are given and those are applied to the same systems as examples, in section 2.

The other and the most important work achieved in this thesis is proving that there may be a range or ranges of proportional controllers for time-delay systems, for which the time delay has absolutely no effect on stability, i.e. stabilizing or instabilizing effect of the time delay (the value of time delay does not matter) cannot be observed for these values of proportional controller values. This amazing fact is based on the calculation of these controller values for which a root crossing from the imaginary axis cannot occur. If such a value range of controllers can be found, then it

means that the stability of the delay-free system does not change for these values of controller, under any value of the time delay.

Whereas finding low-order controllers for time-delay systems is not straightforward, designing such controllers in order to meet a prescribed objective(s) for such systems is extremely difficult. Thus, the future possible work(s) will focus on deriving such controllers to meet different criteria more than to be stabilizing ones.

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APPENDICES

A1. The Sign of the Derivative of $\frac{dk_p}{dL}$ in the Section 3.2.2

Since $\tau < 0$, then $\frac{\tau k_p k_s^2}{\sqrt{k_p^2 k_s^2 - 1}} > 0$ and $k_p^2 k_s^2 - 1 > 0$ and only the sign of $\frac{dk_p}{dL}$ is of interest then it is sufficient to show

$$\left[\arctan(\sqrt{k_p^2 k_s^2 - 1}) - \frac{\sqrt{k_p^2 k_s^2 - 1}}{k_p^2 k_s^2} \right] > 0 \quad (\text{A.1})$$

for

$$k_p < \frac{-1}{k_s}. \quad (\text{A.2})$$

With the change of variables $\sqrt{k_p^2 k_s^2 - 1} = b$, the following is obtained:

$$\arctan b - \frac{b}{b^2 + 1} = f(b) \quad (\text{A.3})$$

for $k_p = \frac{-1}{k_s}$, $b = 0$. Thus only the range of $b > 0$ is of interest where $k_p < \frac{-1}{k_s}$. As it can be seen, for $b = 0$, $f(b) = 0$. If the derivative of $f(b)$ with respect to b is taken:

$$\frac{d(f(b))}{db} = \frac{2b^2}{(b^2 + 1)^2} \quad (\text{A.4})$$

which is always positive. Since $f(b) = 0$ for $b = 0$ and $\frac{d(f(b))}{db} > 0$, then it can be concluded that $f(b) > 0$ for $b > 0$.

A2. The Condition of $\frac{-\tau}{L_s} > 1$ in the Section 3.2.2

As mentioned in the section 3.b, in order that (3.61) has a solution for $A_1 \in \left(0, \frac{\pi}{2}\right)$,

the condition $\frac{-\tau}{L_s} > 1$ must be satisfied. To prove this, define the function

$$f(A) = \arctan\left(\frac{-\tau}{L} A\right) - A \quad (\text{A.5})$$

which is equal to zero when (3.61) is satisfied. Then the following observations can be made:

i) For $A = 0$, $f(A) = 0$ i.e. (3.61) is satisfied.

ii) For $A_1 \in \left(0, \frac{\pi}{2}\right)$ both of $\arctan\left(\frac{-\tau}{L} A\right)$ and A have a positive derivative with respect to A , i.e.:

$$\frac{d(A)}{dA} = 1 \quad (\text{A.6})$$

and

$$\frac{d\left(\arctan\left(\frac{-\tau}{L} A\right)\right)}{dA} = \frac{\frac{-\tau}{L}}{1 + \frac{\tau^2}{L^2} A^2} \quad (\text{A.7})$$

Since A has a constant derivative whereas $\arctan\left(\frac{-\tau}{L} A\right)$ has a decreasing derivative, meaning that it has a negative second derivative as below:

$$\left\{ \frac{d^2\left(\arctan\left(\frac{-\tau}{L} A\right)\right)}{dA^2} = \frac{\frac{2\tau^3}{L^3} A}{\left(1 + \frac{\tau^2}{L^2} A^2\right)^2} \right\} \quad (\text{A.8})$$

for $A_1 \in \left(0, \frac{\pi}{2}\right)$, and $f(A) = 0$ for $A = 0$ as stated in (i), $f(A)$ can be equal to zero for $A_1 \in \left(0, \frac{\pi}{2}\right)$, if and only if $\frac{df(A)}{dA} > 0$ for an interval of $(0, A_k)$ where $A_k \in \left(0, \frac{\pi}{2}\right)$. Then for a infinitesimally small positive A , i.e.

$$A \cong 0 \tag{A.9}$$

the derivative of $f(A)$ must be positive. If the derivative of $f(A)$ is taken for such an A , the following is obtained:

$$\frac{df(A)}{dA} = \frac{\frac{-\tau}{L}}{1 + \frac{\tau^2}{L^2} A^2} - 1 > 0 \tag{A.10}$$

Since A approaches to zero, then $\frac{\tau^2}{L^2} A^2 \approx 0$. Hence, the following is obtained

$$\frac{-\tau}{L} - 1 > 0 \tag{A.11}$$

and finally

$$\frac{-\tau}{L} > 1 \tag{A.12}$$

These results are depicted in Figures A.1-A.5 drawn for A and $\arctan\left(\frac{-\tau}{L} A\right)$ for

$0 \leq A \leq \frac{\pi}{2}$ with different values of $\frac{-\tau}{L}$.

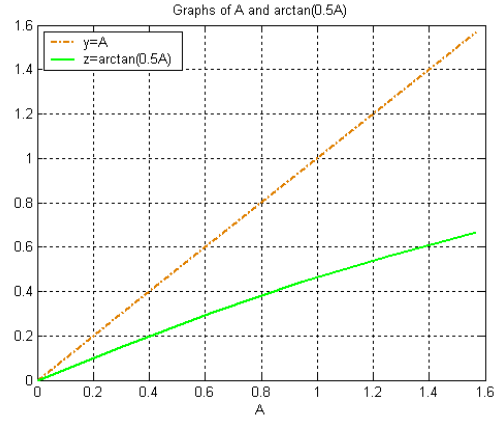


Figure A.1. Plots of A and $\arctan(\frac{-\tau}{L}A)$ for $\frac{-\tau}{L} = 0.5$

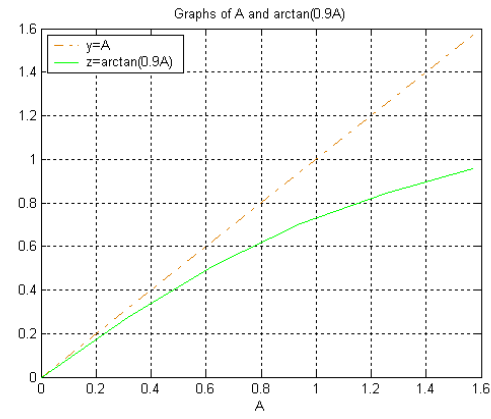


Figure A.2. Plots of A and $\arctan(\frac{-\tau}{L}A)$ for $\frac{-\tau}{L} = 0.9$

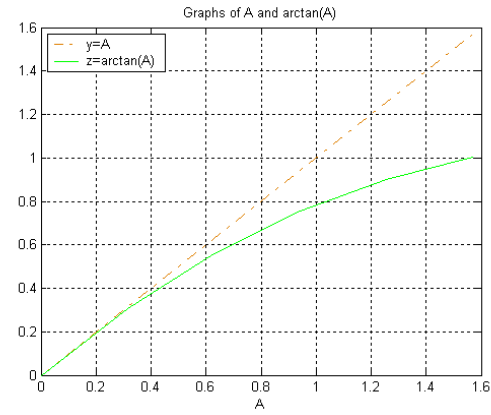


Figure A.3. Plots of A and $\arctan(\frac{-\tau}{L}A)$ for $\frac{-\tau}{L} = 1$

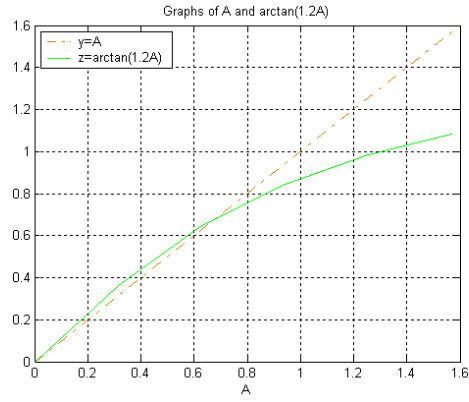


Figure A.4. Plots of A and $\arctan(\frac{-\tau}{L}A)$ for $\frac{-\tau}{L}=1.2$

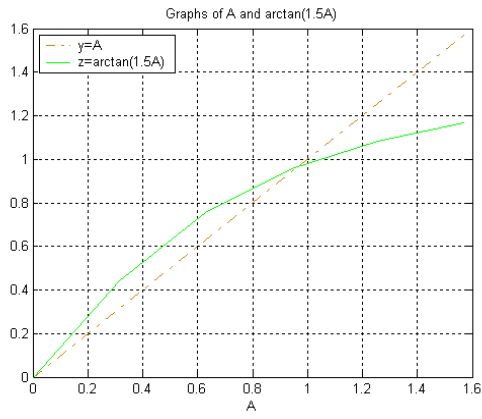
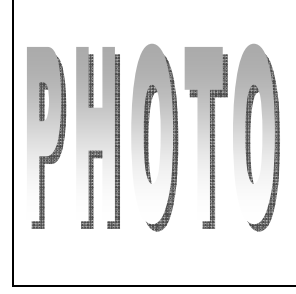


Figure A.5. Plots of A and $\arctan(\frac{-\tau}{L}A)$ for $\frac{-\tau}{L}=1.5$

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