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İSTANBUL TEKNİK ÜNİVERSİTESİ ★ FEN BİLİMLERİ ENSTİTÜSÜ

# ŞOK YÜKÜNE MARUZ DAİRESEL SİLİNDİRİK BİR KABUĞUN DİNAMİK ANALİZİ

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Tezin Enstitüye Verildiği Tarih : 16 Ocak 1995

Tezin Savunulduğu Tarih : 3 Şubat 1995

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ŞUBAT 1995

ISTANBUL TECHNICAL UNIVERSITY ★ INSTITUTE OF SCIENCE AND TECHNOLOGY

# DYNAMIC ANALYSIS OF A CIRCULAR CYLINDRICAL SHELL SUBJECTED TO SHOCK LOADING

M.Sc. THESIS

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Date of Submission : January 16, 1995

Date of Approval : February 3, 1995

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FEBRUARY 1995

## ACKNOWLEDGEMENTS

I wish to thank my supervisor Assoc. Prof. Dr. Zahit Mecitoğlu who has given me the opportunity of preparing this thesis. Also I wish to express my gratitude to my friends Research Assistants Abdülkadir Yavuz, Doğan Çelik and Halit S. Türkmen whose supports were invaluable.

Hasan KURTARAN,

January 1995, İSTANBUL

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## ABSTRACT

In this thesis, the dynamic behaviour of a thin circular cylindrical shell subjected to shock loading is studied analytically. The shell theory is established based on the Love's first approximation of elastic thin shells. Shock loading is represented for simplicity as a concentrated moving load at a constant speed along the axial direction but not varying in the circumferential direction. In order to obtain a solution which is also valid for high load speeds, the effects of transverse shear deformation and rotatory inertia, which become increasingly important as the speed is increased, are taken into account. On the other hand, strain and inertial force in the longitudinal directon are neglected. The governing equations of circular cylindrical shell with simply supported edges are derived by the use of Naghdi's theory. The coupled governing equations are solved by the Assumed-Modes Method. The effects of the shock load speed and the diameter of cylindrical shell on the dynamic behaviour are examined. The numerical results are compared with the results of beam under moving load as a special case of the problem and a good agreement is observed.

# HAREKETLİ ŞOK YÜKÜNE MARUZ DAİRESEL SİLİNDİRİK BİR KABUĞUN DİNAMİK ANALİZİ


## ÖZET

Bu tezde, şok yüküne maruz dairesel silindirik bir kabuğun dinamik davranışı analitik olarak incelenmiştir. Basitleştirme için şok yükü sabit bir hızla hareket eden eksenel simetrik bir tekil yük olarak gözönüne alınmıştır ve dairesel silindirik kabuğun ekseni doğrultusundaki hareket çok küçük kabul edildiğinden eksenel doğrultudaki atalet kuvvetleri ve radyal yöndeki deplasmanların küçük kabul edilmesinden dolayı da eksenel doğrultudaki şekil değişimleri ihmal edilmiştir.

Ayrıca yapılan analizin yüksek hızlarda da geçerli olması için kesme ve dönel atalet etkileri de dikkate alınmıştır.

Başlangıçta genel bir kabuk elemanı ele alınmış ve bu eleman için Love'ın ilk kabulleri olarak adlandırılan şu kabuller yapılmıştır:

1) Kabuğun kalınlığı, eğrilik yarıçapı ve uzunluk gibi diğer boyutlar yanında küçük kabul edilmiştir.

2) Şekil değişimleri ve deplasmanlar yeterince küçük olduğundan şekil değişimi-yer değiştirme bağıntılarında ikinci ve daha yüksek mertebeden terimler birinci mertebeden olan terimler yanında ihmal edilmiştir.

3) Radyal doğrultudaki normal gerilmeler diğer doğrultulardaki normal gerilmeler yanında küçük kabul edilmiş ve ihmal edilmiştir.

4) Deformasyondan önce orta düzleme dik olan kesitlerin deformasyondan sonra da orta düzleme dik kaldığı kabul edilmiştir.

Bu kabuller ışığında 3-boyutlu elastisite teorisinin eğrisel koordinatlardaki şekil değiştirme-yer değiştirme bağıntıları aşağıdaki şekilde ifade edilmiştir.

$$e_\alpha = \frac{1}{1+\dfrac{z}{R_\alpha}}\left(\frac{1}{A}\frac{\partial U}{\partial \alpha} + \frac{V}{AB}\frac{\partial A}{\partial \beta} + \frac{W}{R_\alpha}\right)$$

$$e_\beta = \frac{1}{1+\dfrac{z}{R_\beta}}\left(\frac{U}{AB}\frac{\partial B}{\partial \alpha} + \frac{1}{B}\frac{\partial V}{\partial \beta} + \frac{W}{R_\beta}\right)$$

$$e_z = \frac{\partial W}{\partial z}$$

$$\gamma_{\alpha\beta} = \frac{A(1+z/R_\alpha)}{B(1+z/R_\beta)}\frac{\partial}{\partial \beta}\left[\frac{U}{A(1+z/R_\alpha)}\right] + \frac{B(1+z/R_\beta)}{A(1+z/R_\alpha)}\frac{\partial}{\partial \alpha}\left[\frac{V}{B(1+z/R_\beta)}\right]$$

$$\gamma_{\alpha z} = \frac{1}{A(1+z/R_\alpha)}\frac{\partial W}{\partial \alpha} + A(1+z/R_\alpha)\frac{\partial}{\partial z}\left[\frac{U}{A(1+z/R_\alpha)}\right]$$

$$\gamma_{\beta z} = \frac{1}{B(1+z/R_\beta)}\frac{\partial W}{\partial \beta} + B(1+z/R_\beta)\frac{\partial}{\partial z}\left[\frac{V}{B(1+z/R_\beta)}\right]$$

Burada *U, V, W* kabuk elemanı içindeki herhangi bir noktanın yer değiştirme bileşenlerini göstermektedir.

Genel ince bir kabuk elemanı için gerilme-şekil değiştirme bağıntılarının kuvvet ve momentlerin hesabında kullanılmasıyla ve hesaplanan bu ifadelerin genel bir kabuk elemanının dönel ve uzunlamasına dengesinde yerine konmasıyla aşağıdaki denge denklemleri elde edilmiştir.

$$\frac{\partial}{\partial \alpha}(BN_{\alpha})+\frac{\partial}{\partial \beta}(AN_{\beta\alpha})+\frac{\partial A}{\partial \beta}N_{\alpha\beta}-\frac{\partial B}{\partial \alpha}N_{\beta}+\frac{AB}{R_{\alpha}}Q_{\alpha}+ABq_{\alpha}=0$$

$$\frac{\partial}{\partial \beta}(AN_{\beta})+\frac{\partial}{\partial \alpha}(BN_{\alpha\beta})+\frac{\partial B}{\partial \alpha}N_{\beta\alpha}-\frac{\partial A}{\partial \beta}N_{\alpha}+\frac{AB}{R_{\beta}}Q_{\beta}+ABq_{\beta}=0$$

$$-\frac{AB}{R_{\alpha}}N_{\alpha}-\frac{AB}{R_{\beta}}N_{\beta}+\frac{\partial}{\partial \alpha}(BQ_{\alpha})+\frac{\partial}{\partial \beta}(AQ_{\beta})+ABq_{n}=0$$

$$\frac{\partial}{\partial \alpha}(BM_{\alpha})+\frac{\partial}{\partial \beta}(AM_{\beta\alpha})+\frac{\partial A}{\partial \beta}M_{\alpha\beta}-\frac{\partial B}{\partial \alpha}M_{\beta}-ABQ_{\alpha}+ABm_{\beta}=0$$

$$\frac{\partial}{\partial \beta}(AM_{\beta})+\frac{\partial}{\partial \alpha}(BM_{\alpha\beta})+\frac{\partial B}{\partial \alpha}M_{\beta\alpha}-\frac{\partial A}{\partial \beta}M_{\alpha}-ABQ_{\beta}+ABm_{\alpha}=0$$

$$N_{\alpha\beta}-N_{\beta\alpha}+\frac{M_{\alpha\beta}}{R_{\alpha}}-\frac{M_{\beta\alpha}}{R_{\beta}}=0$$

Buradan dairesel silindirik bir kabuk için gerekli dönüşümler yapılarak hareket denklemleri elde edilmiş ve bu denklemler eksenel simetrik bir hareket için bizim problemimizi tanımlayan aşağıdaki denklem takımına dönüştürülmüştür:

$$\kappa Gh\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial \psi_x}{\partial x}\right)-\frac{Eh}{1-\nu^2}\frac{w}{R^2}+f=\rho h\frac{\partial^2 w}{\partial t^2}$$

$$\frac{Eh^3}{12(1-\nu^2)}\frac{\partial^2 \psi_x}{\partial x^2}-\kappa Gh\left(\frac{\partial w}{\partial x}+\psi_x\right)=\frac{\rho h^3}{12}\frac{\partial^2 \psi_x}{\partial t^2}$$

Basit mesnetli uçlarda sınır şartları

$$w(0,t)=0\,,\quad w(L,t)=0$$
$$M_x(0,t)=0\,,\quad M_x(L,t)=0$$

ve başlangıç şartları ise aşağıdaki şekildedir:

$$w(x,0)=0\,,\quad \dot{w}(x,0)=0$$
$$\psi_x(x,0)=0\,,\quad \dot{\psi}_x(x,0)=0$$

Yükleme ise aşağıdaki gibi verilmiştir:

$$f(x,t)=P\,\delta(x-Vt)$$

Bu diferansiyel denklem takımının çözümü için gerek analitik gerekse nümerik olarak birçok çözüm metodu mevcuttur. Bunların başında Sonlu Elemanlar Metodu (Finite Element Method), Galerkin Metodu, Kollokasyon (Collocation) Metodu, Modal Analiz Metodu ve Kabul Edilen Modlar Metodu (Assumed Modes Method) gelmektedir. Diferansiyel denklem takımının çözümünde çözüm sayılan modlar metodu (Assumed Modes Method) kullanılmıştır.

Bu diferansiyel denklem takımının çözümünde çözüm fonksiyonları olarak

$$\psi_x(x,t)=\sum_{n=1}^{N}B_n(t)\cos\frac{n\pi x}{L}$$
$$w(x,t)=\sum_{n=1}^{N}C_n(t)\sin\frac{n\pi x}{L}$$

seçilmiştir. Burada seçilmiş olan trigonometrik fonksiyonlar sınır şartlarını otomatik olarak sağlamaktadır. Bu fonksiyonların kısmi diferansiyel denklem takımına uygulanmasıyla çözüm fonksiyonlarının zamana bağlı kısımlarını ihtiva eden adi diferansiyel denklem takımı elde edilmiştir. Bu denklem takımının çözümü aşağıdaki şekilde bulunmuştur:

$$B_n(t)=c_{1n}e^{s_{1n}t}+c_{2n}e^{-s_{1n}t}+c_{3n}e^{s_{2n}t}+c_{4n}e^{-s_{2n}t}+k_{11n}\sin k_{7n}t$$

$$C_n(t)=-\frac{1}{k_{3n}}\Big[(k_{2n}+s_{1n}^2)(c_{1n}e^{s_{1n}t}+c_{2n}e^{-s_{1n}t})$$
$$+(k_{2n}+s_{2n}^2)(c_{3n}e^{s_{2n}t}+c_{4n}e^{-s_{2n}t})+(k_{2n}-k_{7n})k_{11n}\sin k_{7n}t\Big]$$

Bu denklemlerdeki sabit olan katsayılar başlangıç şartlarının kullanılmasıyla bulunmuştur. Ayrıca buraya kadarki tüm ifadelerde yeralan terimler tez içerisinde yeri geldikçe tanımlanmıştır.

Böylece bulunan çözümler değişik parametreler için tekrarlanmış ve elde edilen sonuçlar grafik olarak sunulmuştur. Bu sonuçlardan başlıca şu yargılara varılmıştır:

1) Dairesel silindirik kabuğun statik çözümü dinamik analizin özel bir hali olarak elde edilmiştir ve bulunan sonuçların dairesel silindirik kabuğun analitik statik çözümüyle uyumlu olduğu gözlemlenmiştir.

2) Çok ince kabukların eğilme çözümleri düşük ve yüksek hızlarda oldukça yaklaşık sonuçlar vermiştir.

3) Hareket eden yükün hızı arttıkça atalet kuvvetleri önem kazanmaya başlamaktadır.

4) Hareket eden yükün çok yüksek olan hızları için -atalet kuvvetleri etkin olduğundan dolayı- silindirik kabuğun radyal doğrultudaki yer değiştirmeleri sıfıra çok yakındır.

# CHAPTER 1

# INTRODUCTION

In this thesis, the dynamic behaviour of a thin circular cylindrical shell subjected to shock loading is studied analytically. Representing shock load as a moving load at a constant speed, the problem is changed to the moving load problem. Moving load problem is a special topic in structural dynamics as it varies in position in contrast to other dynamic loads. The problem has large application areas such as in the field of transportation e. g. bridges, guide ways, cable ways, runways, in the field of military applications e. g. gun barrels and in other areas.

Academical researches related to moving load problem were started in 19$^{th}$ century and widely continued in 20$^{th}$ century. A large number of studies of the moving load problem are referred to Fryba [1]. To solve simple moving load problems many analytical methods have been proposed to solve simple moving load problems [2-8]. Among them there have been ones related to cylindrical shells subjected to the moving load [3, 4, 8]. In general analysis, where analytical treatments are insufficient, numerical methods have to be used. For this purpose Yoshida and Weaver [9] first applied Finite Element Method (FEM) to the moving load problem in 1950's. Since then FEM has been used by many investigators, Olsson [2], Taheri and C. Ting [10].

In this study the shell theory is established based on Love's first approximation of elastic thin shells. In order to obtain a solution which is also valid for high load speeds, the effects of transverse shear deformation and rotatory inertia are taken into account. The governing equations of circular cylindrical shell with simply supported edges are derived by the use of Naghdi's theory. The coupled governing equations are solved by the Assumed-Modes Method. The effects of shock load speed and the diameter of circular cylindrical shell on the dynamic behaviour are examined. The numerical results are compared with the results of beam under moving load as a special case of the problem and a good agreement is observed.

## CHAPTER 2

# FUNDAMENTAL EQUATIONS OF THIN ELASTIC SHELLS

### 2.1. THIN SHELL THEORY

A thin shell is a three-dimensional body which is bounded by two closely spaced curved surfaces, the distance between the surfaces being small in comparison with the other dimensions. The distance between the surfaces measured along the normal to the middle surface is the thickness of the shell at that point.

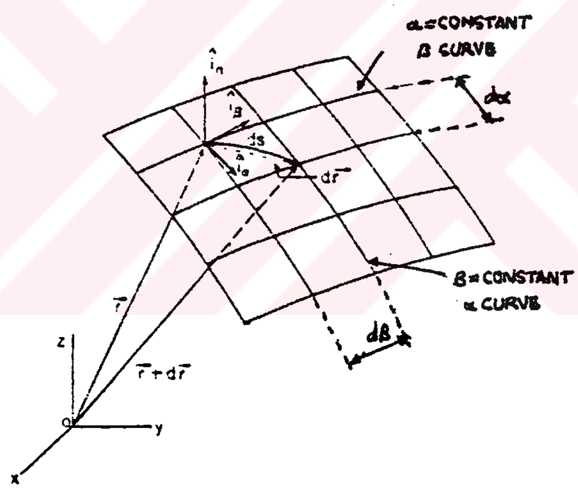


Figure 2.1. Middle surface coordinates.

The deformation of a thin shell will be completely determined by the displacements of its middle surface. Any point on the undeformed middle surface can be determined using parameters $\alpha$ and $\beta$ by the radius vector

$$r = r(\alpha, \beta) \tag{2.1}$$

and the lengths of the vector $r,_{\alpha}$ and $r,_{\beta}$ which are tangent to the $\alpha$ and $\beta$ curves, respectively, are

$$|r,_{\alpha}| = A, \qquad |r,_{\beta}| = B \tag{2.2a}$$

where

$$r_{,\alpha} = \partial r / \partial \alpha \qquad , \qquad r_{,\beta} = \partial r / \partial \beta \tag{2.2b}$$

## 2.2. SHELL COORDINATES AND THE FUNDAMENTAL SHELL ELEMENT

To describe the location of an arbitrary point in the space occupied by a thin shell, the position vector is defined as

$$R(\alpha, \beta, z) = r(\alpha, \beta) + z\, i_n \tag{2.3}$$

where $z$ measures the distance of the point from the corresponding point on the middle surface along $i_n$ and varies over the thickness ( $-h/2 \le z \le h/2$ ).

The magnitude of an arbitrary infinitesimal change in the vector $R(\alpha, \beta, z)$ is determined by

$$(ds)^2 = dR \,.\, dR \tag{2.4}$$

which is equal to

$$(ds)^2 = g_1\, d\alpha^2 + g_2\, d\beta^2 + g_3\, dz^2 \tag{2.5}$$

where

$$g_1 = [A(1 + z/R_\alpha)]^2 \tag{2.6a}$$

$$g_2 = [B(1 + z/R_\beta)]^2 \tag{2.6b}$$

$$g_3 = 1 \tag{2.6c}$$

Having established the coordinate system of the shell space, the fundamental three-dimensional thin shell element will be defined next. The fundamental shell element is the differential element bounded by two surfaces $dz$ apart at a distance $z$ from the middle surface and four ruled surfaces whose generators are the normals to the middle surface along the parametric curves $\alpha=\alpha_0$, $\alpha=\alpha_0+d\alpha$, $\beta=\beta_0$ and $\beta=\beta_0+d\beta$. The assumption that the parametric curves are lines of principal curvature ensures that the ruled surfaces will be plane surfaces and, furthermore, that these planes intersect each other at right angles. The lengths of the edges of this fundamental element are according to equations (2.5) and (2.6) (Figure 2.1).

$$ds_\alpha^{(z)} = A(1 + z/R_\alpha)\, d\alpha \tag{2.7a}$$

$$ds_\beta^{(z)} = B(1 + z/R_\beta)\, d\beta \tag{2.7b}$$

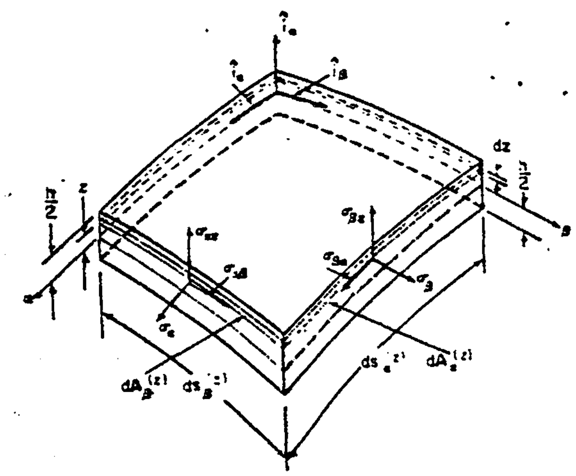

Figure 2.2. Notation and positive directions of stresses in shell coordinates.

The differential areas of the edge faces of the fundamental element are

$$dA_{\alpha}^{(z)} = A\,(\,1 + z\,/\,R_{\alpha}\,)\,d\alpha d\,z \tag{2.8a}$$
$$dA_{\beta}^{(z)} = B\,(\,1 + z\,/\,R_{\beta}\,)\,d\beta d\,z \tag{2.8b}$$

while the volume of the fundamental element is

$$dV^{(z)} = [A\,(\,1 + z\,/\,R_{\alpha}\,)]\,[\,B\,(\,1 + z\,/\,R_{\beta}\,)]d\alpha d\beta d\,z \tag{2.9}$$

## 2.3. ASSUMPTIONS OF CLASSICAL THIN SHELL THEORY

In the classical theory of small displacements of thin shell assumptions are based on Love's first approximation as follows [11]:

1) The shell thickness $h$ is very small in comparison with other shell dimensions such as radius of curvature, length, etc.

2) Strains and displacements are sufficiently small so that the quantities of second and higher order magnitude in the strain-displacement relations may be neglected in comparison with the first-order terms.

3) The transverse normal stress is small compared with the other normal stress components and may be neglected.

4) Normals to the undeformed middle surface remain straigth and normal to the deformed middle surface and suffer no extension.

These four assumptions are the background of any linear theory of thin shells. The first assumption defines the thin shell. Denoting the thickness of the shell by $h$ and the smallest radius of curvature by $R$, then it will be convenient at various places in the derivation of shell theories to neglect higher powers of $z/R$ or $h/R$ in comparison with unity. The second assumption permits one to refer all calculations to the original configuration of the shell and ensures that the differential equations will be linear. The third and fourth assumptions transform the original three-dimensional problem of the mathematical theory of elasticity into a two-dimensional one. The fourth assumption intrinsically expresses Kirchhoff 's hypothesis on cross-section deformation. As a consequence of the assumed geometry of deformation we must have

$$\gamma_{xz} = \gamma_{yz} = \varepsilon_z = 0 \tag{2.10}$$

such that normal stresses in the radial direction become zero ($\sigma_z = 0$).

Different sets of equations have been obtained by various academicians. Differences in the theories are due to the simplifying assumptions and/or the exact point in a derivation where given assumptions is used. For example, $h/R$ or $z/R$ is neglected in comparison with unity in some places instead of expanding $(1 + z/R_i)^{-1}$ into geometric series gives Vlasov's equations [11].

## 2.4. STRAIN-DISPLACEMENT EQUATIONS

Using tensoral notation, strain-displacement equations of the three dimensional theory of elasticity in orthogonal curvilinear coordinates can be written as

$$e_i = \frac{\partial}{\partial \alpha_i}\left(\frac{U_i}{\sqrt{g_i}}\right) + \frac{1}{2g_i}\sum_{k=1}^{3}\frac{\partial g_i}{\partial \alpha_k}\frac{U_k}{\sqrt{g_k}}$$

$$i,j = 1,2,3\ ;\quad i \neq j \tag{2.11}$$

$$\gamma_{ij} = \frac{1}{\sqrt{g_i g_j}}\left[\ g_i \frac{\partial}{\partial \alpha_j}\left(\frac{U_i}{\sqrt{g_i}}\right) + g_j \frac{\partial}{\partial \alpha_i}\left(\frac{U_j}{\sqrt{g_j}}\right)\ \right]$$

here $e_i$ denotes normal strains, $\gamma_{ij}$ shear strains, $U_i$ displacement components at an arbitrary point. In the shell coordinates the indices 1, 2, 3 are replaced by $\alpha, \beta, z$ respectively, except for the displacements $U_1$, $U_2$, $U_3$, which are replaced by $U$, $V$, $W$ respectively.

Using (2.6) in (2.11) yields

$$e_\alpha = \frac{1}{1+\dfrac{z}{R_\alpha}}\left(\frac{1}{A}\frac{\partial U}{\partial \alpha} + \frac{V}{AB}\frac{\partial A}{\partial \beta} + \frac{W}{R_\alpha}\right)$$

$$e_\beta = \frac{1}{1+\dfrac{z}{R_\beta}}\left(\frac{U}{AB}\frac{\partial B}{\partial \alpha} + \frac{1}{B}\frac{\partial V}{\partial \beta} + \frac{W}{R_\beta}\right) \tag{2.12a}$$

$$e_z = \frac{\partial W}{\partial z}$$

$$\gamma_{\alpha\beta} = \frac{A(1+z/R_\alpha)}{B(1+z/R_\beta)}\frac{\partial}{\partial \beta}\left[\frac{U}{A(1+z/R_\alpha)}\right] + \frac{B(1+z/R_\beta)}{A(1+z/R_\alpha)}\frac{\partial}{\partial \alpha}\left[\frac{V}{B(1+z/R_\beta)}\right]$$

$$\gamma_{\alpha z} = \frac{1}{A(1+z/R_\alpha)}\frac{\partial W}{\partial \alpha} + A(1+z/R_\alpha)\frac{\partial}{\partial z}\left[\frac{U}{A(1+z/R_\alpha)}\right] \tag{2.12b}$$

$$\gamma_{\beta z} = \frac{1}{B(1+z/R_\beta)}\frac{\partial W}{\partial \beta} + B(1+z/R_\beta)\frac{\partial}{\partial z}\left[\frac{V}{B(1+z/R_\beta)}\right]$$

According to the Kirchhoff hypothesis the displacements of any point in the shell are as follows:

$$U(\alpha, \beta, z) = u(\alpha, \beta) + z\, \psi_\alpha(\alpha, \beta) \tag{2.13a}$$

$$V(\alpha, \beta, z) = v(\alpha, \beta) + z\, \psi_\beta(\alpha, \beta) \tag{2.13b}$$

$$W(\alpha, \beta, z) = w(\alpha, \beta) \tag{2.13c}$$

where $u$, $v$ and $w$ are the components of displacement at the middle surface in the $\alpha$, $\beta$ and normal direction, respectively. $\psi_\alpha$ and $\psi_\beta$ are the rotations of the normal to the middle surface during deformation about the $\beta$ and $\alpha$ axes, respectively, i.e.

$$\psi_\alpha = \partial U / \partial z \ , \qquad \psi_\beta = \partial V / \partial z \tag{2.14}$$

Using (2.14) in (2.12e) and (2.12f) from the conditions $\gamma_{\alpha z} = \gamma_{\beta z} = 0$ we obtain

$$\psi_\alpha = u / R_\alpha - (1/A)(\partial w / \partial \alpha) \tag{2.15a}$$

$$\psi_\beta = v / R_\beta - (1/B)(\partial w / \partial \beta) \tag{2.15b}$$

Therefore from the preceding equations the strain-displacement relations are expressed as follows:

$$e_\alpha = \frac{1}{1+\dfrac{z}{R_\alpha}}(\varepsilon_\alpha + zk_\alpha)$$

$$e_\beta = \frac{1}{1+\dfrac{z}{R_\beta}}(\varepsilon_\beta + zk_\beta) \tag{2.16}$$

$$\gamma_{\alpha\beta} = \frac{1}{\left(\dfrac{1}{1+\dfrac{z}{R_\alpha}}\right)\left(1+\dfrac{z}{R_\beta}\right)}\left[\ \left(1-\frac{z^2}{R_\alpha R_\beta}\right)\varepsilon_{\alpha\beta} + z\left(1+\frac{z}{2R_\alpha}+\frac{z}{2R_\beta}\right)\tau\ \right]$$

where $\varepsilon_\alpha$, $\varepsilon_\beta$ and $\varepsilon_{\alpha\beta}$ are the normal and shear strains in the middle surface ($z = 0$) given by

$$\varepsilon_\alpha = \frac{1}{A}\frac{\partial u}{\partial \alpha} + \frac{v}{AB}\frac{\partial A}{\partial \beta} + \frac{w}{R_\alpha}$$

$$\varepsilon_\beta = \frac{u}{AB}\frac{\partial B}{\partial \alpha} + \frac{1}{B}\frac{\partial v}{\partial \beta} + \frac{w}{R_\beta} \tag{2.17}$$

$$\varepsilon_{\alpha\beta} = \frac{A}{B}\frac{\partial}{\partial \beta}(\frac{u}{A}) + \frac{B}{A}\frac{\partial}{\partial \alpha}(\frac{v}{B})$$

and $k_\alpha$ and $k_\beta$ are the midsurface changes in curvature and $\tau$ the hardsurface twist, given by

$$k_\alpha = \frac{1}{A}\frac{\partial \psi_\alpha}{\partial \alpha} + \frac{\psi_\beta}{AB}\frac{\partial A}{\partial \beta}$$

$$k_\beta = \frac{\psi_\alpha}{AB}\frac{\partial B}{\partial \alpha} + \frac{1}{B}\frac{\partial \psi_\beta}{\partial \beta} \tag{2.18}$$

$$\tau = \frac{A}{B}\frac{\partial}{\partial \beta}(\frac{\psi_\alpha}{A}) + \frac{B}{A}\frac{\partial}{\partial \alpha}(\frac{\psi_\beta}{B}) + \frac{1}{R_\alpha}(\frac{1}{B}\frac{\partial u}{\partial \beta} - \frac{v}{AB}\frac{\partial B}{\partial \alpha}) + \frac{1}{R_\beta}(\frac{1}{A}\frac{\partial v}{\partial \alpha} - \frac{u}{AB}\frac{\partial A}{\partial \beta})$$

These are the strain-displacement equations used by Byrne, Flügge, Goldenveizer, Lur'ye and Novozhilov.

If in above equations one neglects the terms $z/R_\alpha$ and $z/R_\beta$ and their products as being small in comparison with unity one obtain strain-displacement equations of Love, Timoshenko, Naghdi, Berry and Reisner as

$$e_\alpha = \varepsilon_\alpha + zk_\alpha$$

$$e_\beta = \varepsilon_\beta + zk_\beta \tag{2.19}$$

$$\gamma_{\alpha\beta} = \varepsilon_{\alpha\beta} + z\tau$$

Recognizing that for a shell $z/R_i$ $(i = \alpha, \beta)$ is less than unity and expanding the quotient $(1+ z/R_i)^{-1}$ into geometric series one obtains the equations of Vlasov.

From the derivations of strain-displacement equations it can be seen that equations are either general or special case of each other.

## 2.5. STRESS-STRAIN RELATIONS

Considering material isotropic and following Hooke's law which gives linear relations between stress and strains for a three-dimensional element the following equations can be written

$$e_\alpha = \frac{1}{E}\left[\sigma_\alpha - \nu(\sigma_\beta + \sigma_z)\right]$$

$$e_\beta = \frac{1}{E}\left[\sigma_\beta - \nu(\sigma_\alpha + \sigma_z)\right]$$

$$e_z = \frac{1}{E}\left[\sigma_z - \nu(\sigma_\beta + \sigma_\alpha)\right]$$

$$\gamma_{\alpha\beta} = \frac{2(1+\nu)}{E}\sigma_{\alpha\beta} \tag{2.20}$$

$$\gamma_{\alpha z} = \frac{2(1+\nu)}{E}\sigma_{\alpha z}$$

$$\gamma_{\beta z} = \frac{2(1+\nu)}{E}\sigma_{\beta z}$$

where $\sigma_\alpha$ and $\sigma_\beta$ are the normal stresses and $\sigma_{\alpha\beta}$ and $\sigma_{\beta\alpha}$ are the shear stresses in the tangential ($\alpha$ and $\beta$) directions and $\sigma_{\alpha z}$ and $\sigma_{\beta z}$ are the transverse (in the $z$ direction) shear stresses, all acting on the transverse faces of a shell element; $E$ is Young's modulus, and $\nu$ is Poisson's ratio (Figure 2.2). Assuming the symmetry of the stress tensor (neglecting body couples), then $\sigma_{\alpha\beta} = \sigma_{\beta\alpha}$.

The Kirchhoff hypothesis yields $e_z = \gamma_{\alpha z} = \gamma_{\beta z} = 0$ and this leads to $\sigma_{\alpha z} = \sigma_{\beta z} = 0$ and $\sigma_z = \nu(\sigma_\alpha + \sigma_\beta)$. Retaining the assumption that $\sigma_z$ is negligibly small reduces three-dimensional problem to one of plane stress problem. In this case equations (2.20) reduce to

$$
\begin{aligned}
e_\alpha &= \frac{1}{E}(\sigma_\alpha - \nu\sigma_\beta) \\
e_\beta &= \frac{1}{E}(\sigma_\beta - \nu\sigma_\alpha) \qquad (2.21) \\
\gamma_{\alpha\beta} &= \frac{2(1+\nu)}{E}\sigma_{\alpha\beta}
\end{aligned}
$$

which, when inverted, give

$$
\begin{aligned}
\sigma_\alpha &= \frac{E}{1-\nu^2}(e_\alpha + \nu e_\beta) \\
\sigma_\beta &= \frac{E}{1-\nu^2}(e_\beta + \nu e_\alpha) \qquad (2.22) \\
\sigma_{\alpha\beta} &= \frac{E}{2(1+\nu)}\gamma_{\alpha\beta}
\end{aligned}
$$

## 2.6. FORCE AND MOMENT RESULTANTS

The arc length of the intercepts at $z$ distance from the middle surface perpendicular to the $\alpha$-axis (Figure 2.2) is $ds_{\beta}{}^{(z)} = B\,(\,1 + z\,/\,R_{\beta}\,)\,d\beta$. The infinitesimal element force acting upon the elemental area of thickness $dz$ on the face is then given by $\sigma_{\alpha}\,ds_{\beta}{}^{(z)}\,dz$. Integrating such force over the shell and dividing $Bd\beta$ yield the force resultant $N_{\alpha}$. Similarly $N_{\alpha\beta}$ and $Q_{\alpha}$ are expressed in units of force per unit length of middle surface. Therefore, the force resultants acting on this face are written as

$$\begin{Bmatrix} N_{\alpha} \\ N_{\alpha\beta} \\ Q_{\alpha} \end{Bmatrix} = \int_{-h/2}^{h/2} \begin{Bmatrix} \sigma_{\alpha} \\ \sigma_{\alpha\beta} \\ \sigma_{\alpha z} \end{Bmatrix} \left(1 + \frac{z}{R_{\beta}}\right) dz \tag{2.23}$$

Similarly those on the face perpendicular to the β axis are

$$\begin{Bmatrix} N_{\beta} \\ N_{\beta\alpha} \\ Q_{\beta} \end{Bmatrix} = \int_{-h/2}^{h/2} \begin{Bmatrix} \sigma_{\beta} \\ \sigma_{\beta\alpha} \\ \sigma_{\beta z} \end{Bmatrix} \left(1 + \frac{z}{R_{\alpha}}\right) dz \tag{2.24}$$

The positive directions of the force resultants are shown in Figure 2.3.

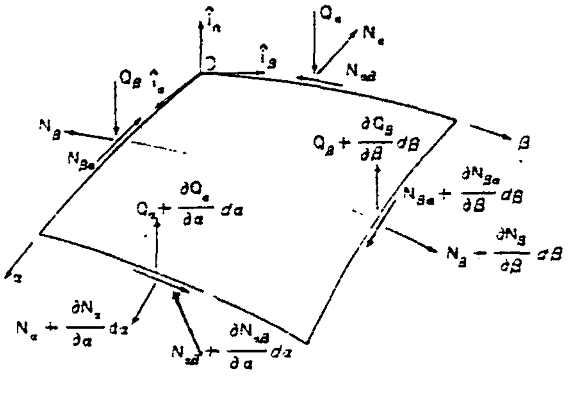

Figure 2.3. Notation and positive directions of force resultants in shell coordinates.

Similarly, the moment of the infinitesimal force $\sigma_\alpha \, ds_\beta{}^{(z)} \, dz$ about the $\beta$-line is simply $z\sigma_\alpha \, ds_\beta{}^{(z)} \, dz$ and the moment resultant $M_\alpha$ is obtained by dividing the total integrated moment over the thickness by $Bd\beta$. Thus, the moment resultants per unit length of middle surface are given by

$$\begin{Bmatrix} M_\alpha \\ M_{\alpha\beta} \end{Bmatrix} = \int_{-h/2}^{h/2} \begin{Bmatrix} \sigma_\alpha \\ \sigma_{\alpha\beta} \end{Bmatrix} \left(1 + \frac{z}{R_\beta}\right) z dz$$

$$\begin{Bmatrix} M_\beta \\ M_{\beta\alpha} \end{Bmatrix} = \int_{-h/2}^{h/2} \begin{Bmatrix} \sigma_\beta \\ \sigma_{\beta\alpha} \end{Bmatrix} \left(1 + \frac{z}{R_\alpha}\right) z dz \tag{2.25}$$

The positive directions of the moment resultants are shown in Figure 2.4.

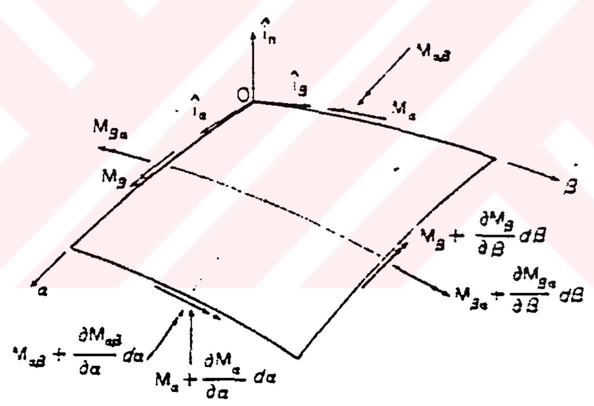


Figure 2.4. Notation and positive directions of moment resultants in shell coordinates.

By using (2.22) in (2.23), (2.24) and (2.25) and expanding $(1+z/R_\alpha)$ and $(1+z/R_\beta)$ into geometric series truncating after terms of the third degree, the following general expressions are arrived at for the force and moment resultants:

$$N_\alpha = \frac{Eh}{1-\nu^2}\left[\ \varepsilon_\alpha + \nu\varepsilon_\beta - \frac{h^2}{12}\left(\frac{1}{R_\alpha} - \frac{1}{R_\beta}\right)\left(k_\alpha - \frac{\varepsilon_\alpha}{R_\alpha}\right)\ \right]$$

$$N_\beta = \frac{Eh}{1-\nu^2}\left[\ \varepsilon_\beta + \nu\varepsilon_\alpha - \frac{h^2}{12}\left(\frac{1}{R_\beta} - \frac{1}{R_\alpha}\right)\left(k_\beta - \frac{\varepsilon_\beta}{R_\beta}\right)\ \right]$$

$$N_{\alpha\beta} = \frac{Eh}{2(1+\nu)}\left[\ \varepsilon_{\alpha\beta} - \frac{h^2}{12}\left(\frac{1}{R_\alpha} - \frac{1}{R_\beta}\right)\left(\frac{\tau}{2} - \frac{\varepsilon_{\alpha\beta}}{R_\alpha}\right)\ \right]$$

$$N_{\beta\alpha} = \frac{Eh}{2(1+\nu)}\left[\ \varepsilon_{\alpha\beta} - \frac{h^2}{12}\left(\frac{1}{R_\beta} - \frac{1}{R_\alpha}\right)\left(\frac{\tau}{2} - \frac{\varepsilon_{\alpha\beta}}{R_\beta}\right)\ \right]$$

$$M_\alpha = \frac{Eh^3}{12(1-\nu^2)}\left[\ k_\alpha + \nu k_\beta - \left(\frac{1}{R_\alpha} - \frac{1}{R_\beta}\right)\varepsilon_\alpha\ \right] \tag{2.26}$$

$$M_\beta = \frac{Eh^3}{12(1-\nu^2)}\left[\ k_\beta + \nu k_\alpha - \left(\frac{1}{R_\beta} - \frac{1}{R_\alpha}\right)\varepsilon_\beta\ \right]$$

$$M_{\alpha\beta} = \frac{Eh^3}{24(1+\nu)}\left(\tau - \frac{\varepsilon_{\alpha\beta}}{R_\alpha}\right)$$

$$M_{\beta\alpha} = \frac{Eh^3}{24(1+\nu)}\left(\tau - \frac{\varepsilon_{\alpha\beta}}{R_\beta}\right)$$

To obtain force and moment resultants in terms of the displacement $u$, $v$ and $w$, it would be necessary to replace $e_\alpha$, $e_\beta$ and $\gamma_{\alpha\beta}$ with the equivalents.

## 2.7. EQUATIONS OF MOTION

There are three distinct methods used in obtaining equations of motion, all depending on the results obtained in the previous sections. The first method is the application of Newton's laws by summing forces and moments which act on a shell element of thickness $h$. The second method begins with the equations of motion of an infinitesimal element of the three-dimensional theory of elasticity and integrates them over the thickness to obtain the equations of motion for a shell element. The third method is actually a class of variational methods.

Now let us consider the equilibrium of the shell element of thickness $h$ in Figure 2.2 under the influence of internal force and moment resultants as shown in Figures 2.3 and 2.4.

The total internal forces acting on the faces defined by $\alpha = constant$ and by $\beta = constant$ denoted by $\vec{F}_\alpha$ and $\vec{F}_\beta$ can be written as

$$\vec{F}_\alpha = (N_\alpha \vec{i}_\alpha + N_{\alpha\beta}\vec{i}_\beta + Q_\alpha \vec{i}_n) B d\beta$$
$$\vec{F}_\beta = (N_{\beta\alpha} \vec{i}_\alpha + N_\beta \vec{i}_\beta + Q_\beta \vec{i}_n) A d\alpha \tag{2.27}$$

The total external force intensity vector $\vec{q}$ which includes body forces, surface loads as well as inertial terms is

$$\vec{q} = q_\alpha \vec{i}_\alpha + q_\beta \vec{i}_\beta + q_n \vec{i}_n \tag{2.28}$$

$\vec{q}$ is force vector per unit area and has three components considered to be acting at the middle surface.

Using (2.27) and (2.28) the vector equation of force equilibrium for the shell element based on Newton's second law is given by

$$\frac{\partial \vec{F}_\alpha}{\partial \alpha} d\alpha + \frac{\partial \vec{F}_\beta}{\partial \beta} d\beta + \vec{q}\, AB d\alpha\, d\beta = 0 \tag{2.29}$$

Substituting (2.27), (2.28) into (2.29) gives the following scalar components

$$\frac{\partial}{\partial\alpha}(BN_{\alpha})+\frac{\partial}{\partial\beta}(AN_{\beta\alpha})+\frac{\partial A}{\partial\beta}N_{\alpha\beta}-\frac{\partial B}{\partial\alpha}N_{\beta}+\frac{AB}{R_{\alpha}}Q_{\alpha}+ABq_{\alpha}=0 \tag{2.30a}$$

$$\frac{\partial}{\partial\beta}(AN_{\beta})+\frac{\partial}{\partial\alpha}(BN_{\alpha\beta})+\frac{\partial B}{\partial\alpha}N_{\beta\alpha}-\frac{\partial A}{\partial\beta}N_{\alpha}+\frac{AB}{R_{\beta}}Q_{\beta}+ABq_{\beta}=0 \tag{2.30b}$$

$$-\frac{AB}{R_{\alpha}}N_{\alpha}-\frac{AB}{R_{\beta}}N_{\beta}+\frac{\partial}{\partial\alpha}(BQ_{\alpha})+\frac{\partial}{\partial\beta}(AQ_{\beta})+ABq_{n}=0 \tag{2.30c}$$

The total moments acting upon the faces defined by $\alpha$ = *constant* and by $\beta$ = *constant* denoted by $\vec{\underline{M}}_{\alpha}$ and $\vec{\underline{M}}_{\beta}$, respectively, are

$$\begin{aligned}\vec{\underline{M}}_{\alpha}&=(-M_{\alpha\beta}\vec{i}_{\alpha}+M_{\alpha}\vec{i}_{\beta})Bd\beta\\ \vec{\underline{M}}_{\beta}&=(-M_{\beta}\vec{i}_{\alpha}+M_{\beta\alpha}\vec{i}_{\beta})Ad\alpha\end{aligned} \tag{2.31}$$

and let the moment intensity vector due to external fields such as gravitational, accelerative, magnetic etc. be given by

$$\vec{m}=m_{\alpha}\vec{i}_{\alpha}+m_{\beta}\vec{i}_{\beta}+m_{n}\vec{i}_{n} \tag{2.32}$$

where $\vec{m}$ denotes moment per unit area.

The vector equation of moment equilibrium for the shell element is given by

$$\begin{aligned}&\frac{\partial\vec{\underline{M}}_{\alpha}}{\partial\alpha}d\alpha+\frac{\partial\vec{\underline{M}}_{\beta}}{\partial\beta}d\beta-(\vec{F}_{\alpha}\times\vec{i}_{\beta})\frac{ds_{\beta}}{2}-(\vec{F}_{\beta}\times\vec{i}_{\alpha})\frac{ds_{\alpha}}{2}\\ &\quad+(\vec{F}_{\alpha}+\frac{\partial\vec{F}_{\alpha}}{\partial\alpha}d\alpha)\times(ds_{\alpha}\vec{i}_{\alpha}+\frac{ds_{\beta}}{2}\vec{i}_{\beta})\\ &\quad+(\vec{F}_{\beta}+\frac{\partial\vec{F}_{\beta}}{\partial\beta}d\beta)\times(ds_{\beta}\vec{i}_{\beta}+\frac{ds_{\alpha}}{2}\vec{i}_{\alpha})+\vec{m}ABd\alpha\, d\beta=0\end{aligned} \tag{2.33}$$

where $ds_\alpha = A \; d\alpha$ and $ds_\beta = A \; d\beta$, and substituting (2.31) and (2.32) into (2.33) yields the following three scalar components

$$\frac{\partial}{\partial\alpha}(BM_{\alpha})+\frac{\partial}{\partial\beta}(AM_{\beta\alpha})+\frac{\partial A}{\partial\beta}M_{\alpha\beta}-\frac{\partial B}{\partial\alpha}M_{\beta}-ABQ_{\alpha}+ABm_{\beta}=0$$

$$\frac{\partial}{\partial\beta}(AM_{\beta})+\frac{\partial}{\partial\alpha}(BM_{\alpha\beta})+\frac{\partial B}{\partial\alpha}M_{\beta\alpha}-\frac{\partial A}{\partial\beta}M_{\alpha}-ABQ_{\beta}+ABm_{\alpha}=0$$

$$N_{\alpha\beta}-N_{\beta\alpha}+\frac{M_{\alpha\beta}}{R_\alpha}-\frac{M_{\beta\alpha}}{R_\beta}=0 \tag{2.34}$$

(2.30) and (2.34) are general equations of motion of a thin shell for translational and rotatory movements.

If the third equilibrium equation of (2.34) is rewritten in terms of the force and moment resultants (eqs. (2.23), (2.24) and (2.25)) then it becomes

$$\int_{-h/2}^{h/2}(\sigma_{\alpha\beta}-\sigma_{\beta\alpha})(1+\frac{z}{R_\alpha})(1+\frac{z}{R_\beta})dz=0 \tag{2.35}$$

which is identically satisfied if the symmetry of the stress tensor is assumed.

In general case the possible set of boundary conditions is given on an edge where $\alpha$= *constant* by

$$
\begin{aligned}
&N_\alpha \quad \text{or} \quad u = 0 \\
&(N_{\alpha\beta} + \frac{M_{\alpha\beta}}{R_\beta}) \quad \text{or} \quad v = 0 \\
&(Q_\alpha + \frac{1}{B}\frac{\partial M_{\alpha\beta}}{\partial \beta}) \quad \text{or} \quad w = 0 \\
&\qquad M_\alpha \quad \text{or} \quad Q_\alpha = 0 \\
&\qquad M_{\alpha\beta} w \Big|_{\beta_1}^{\beta_2} = 0
\end{aligned}
\tag{2.36}
$$

and on an edge where $\beta$= *constant* by

$$
\begin{aligned}
&(N_{\beta\alpha} + \frac{M_{\beta\alpha}}{R_\alpha}) \quad \text{or} \quad u = 0 \\
&N_\beta \quad \text{or} \quad v = 0 \\
&(Q_\beta + \frac{1}{A}\frac{\partial M_{\beta\alpha}}{\partial \alpha}) \quad \text{or} \quad w = 0 \\
&\qquad M_\beta \quad \text{or} \quad Q_\beta = 0 \\
&\qquad M_{\beta\alpha} w \Big|_{\alpha_1}^{\alpha_2} = 0
\end{aligned}
\tag{2.37}
$$

# CHAPTER 3

# FORMULATION OF PROBLEM

## 3.1. EQUATIONS OF MOTION OF CIRCULAR CYLINDRICAL SHELLS

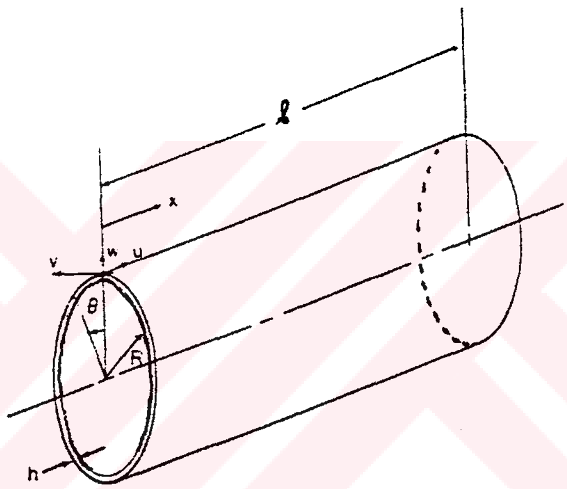


Figure 3.1. Closed circular cylindrical shell and coordinate system.

The basic equations which describe the dynamic behaviour of cylindrical shells under arbitrary loads and moments are derived from the system of equations which has been presented in Chapter 2 by using the change of the following parameters:

$$\begin{aligned} \alpha &= \varphi & \beta &= \theta \\ A &= r_{\varphi} & B &= R \\ R_{\alpha} &= \infty & R_{\beta} &= R \end{aligned} \tag{3.1}$$

in (2.30) and (2.34) equations of motion of a circular cylindrical shell reduce to the equations below

$$
\begin{aligned}
&\frac{1}{r_\varphi}\frac{\partial N_\varphi}{\partial \varphi}+\frac{\partial N_{\theta\varphi}}{R\partial \theta}+q_\varphi=0\\
&\frac{1}{R}\frac{\partial N_\theta}{\partial \theta}+\frac{1}{r_\varphi}\frac{\partial N_{\varphi\theta}}{\partial \varphi}+\frac{Q_\theta}{R}+q_\theta=0\\
&-\frac{N_\theta}{R}+\frac{1}{r_\varphi}\frac{\partial Q_\varphi}{\partial \varphi}+\frac{\partial Q_\theta}{R\partial \theta}+q_n=0\\
&\frac{1}{r_\varphi}\frac{\partial M_\varphi}{\partial \varphi}+\frac{\partial M_{\theta\varphi}}{R\partial \theta}-Q_\varphi+m_\theta=0\\
&\frac{1}{R}\frac{\partial M_\theta}{\partial \theta}+\frac{1}{r_\varphi}\frac{\partial M_{\varphi\theta}}{\partial \varphi}-Q_\theta+m_\varphi=0
\end{aligned}
\tag{3.2}
$$

Making the following change of variable, based on the fact that when the generators are straight the curvature is zero:

$$\lim_{r_\varphi \to \infty}(r_\varphi d\varphi) = dx \tag{3.3}$$

and replacing the variable φ with $x$ , the below general equations of motion for a circular cylindrical shell are arrived at

$$\frac{\partial N_x}{\partial x}+\frac{1}{R}\frac{\partial N_{\theta x}}{\partial \theta}+q_x=0$$

$$\frac{\partial N_{x\theta}}{\partial x}+\frac{1}{R}\frac{\partial N_{\theta}}{\partial \theta}+\frac{Q_\theta}{R}+q_\theta=0$$

$$\frac{\partial Q_x}{\partial x}+\frac{1}{R}\frac{\partial Q_{\theta}}{\partial \theta}-\frac{N_\theta}{R}+q_z=0 \tag{3.4}$$

$$\frac{\partial M_x}{\partial x}+\frac{1}{R}\frac{\partial M_{\theta x}}{\partial \theta}-Q_x+m_\theta=0$$

$$\frac{\partial M_{x\theta}}{\partial x}+\frac{1}{R}\frac{\partial M_{\theta}}{\partial \theta}-Q_\theta+m_x=0$$

where in Naghdi's equations

$$N_x=\frac{Eh}{1-\nu^2}\left\{\left[\frac{\partial u}{\partial x}+\nu\left(\frac{1}{R}\frac{\partial v}{\partial \theta}+\frac{w}{R}\right)\right]+\frac{h^2}{12R}\frac{\partial \psi_x}{\partial x}\right\}$$

$$N_\theta=\frac{Eh}{1-\nu^2}\left\{\left[\left(\frac{1}{R}\frac{\partial v}{\partial \theta}+\frac{w}{R}\right)+\nu\frac{\partial u}{\partial x}\right]-\frac{h^2}{12R}\frac{\partial \psi_\theta}{\partial \theta}\right\}$$

$$N_{x\theta}=\frac{1-\nu}{2}\frac{Eh}{1-\nu^2}\left[\left(\frac{1}{R}\frac{\partial u}{\partial \theta}+\frac{\partial v}{\partial x}\right)+\frac{h^2}{12R}\frac{\partial \psi_\theta}{\partial x}\right]$$

$$N_{\theta x}=\frac{1-\nu}{2}\frac{Eh}{1-\nu^2}\left[\left(\frac{1}{R}\frac{\partial u}{\partial \theta}+\frac{\partial v}{\partial x}\right)+\frac{h^2}{12R}\left(\frac{1}{R^2}\frac{\partial u}{\partial \theta}-\frac{1}{R}\frac{\partial \psi_x}{\partial \theta}\right)\right] \tag{3.5}$$

$$M_x=\frac{Eh^3}{12(1-\nu^2)}\left[\left(\frac{\partial \psi_x}{\partial x}+\frac{\nu}{R}\frac{\partial \psi_\theta}{\partial \theta}\right)+\frac{1}{R}\frac{\partial u}{\partial x}\right]$$

$$M_\theta=\frac{Eh^3}{12(1-\nu^2)}\left[\left(\frac{1}{R}\frac{\partial \psi_\theta}{\partial \theta}+\frac{\partial \psi_x}{\partial x}\right)-\frac{1}{R}\left(\frac{1}{R}\frac{\partial v}{\partial \theta}+\frac{w}{R}\right)\right]$$

$$M_{x\theta}=\frac{1-\nu}{2}\frac{Eh^3}{12(1-\nu^2)}\left[\left(\frac{1}{R}\frac{\partial \psi_x}{\partial \theta}+\frac{\partial \psi_\theta}{\partial x}\right)+\frac{1}{R}\frac{\partial v}{\partial x}\right]$$

$$M_{\theta x}=\frac{1-\nu}{2}\frac{Eh^3}{12(1-\nu^2)}\left[\left(\frac{1}{R}\frac{\partial \psi_x}{\partial \theta}+\frac{\partial \psi_\theta}{\partial x}\right)-\frac{1}{R^2}\frac{\partial v}{\partial \theta}\right]$$

$$Q_x=\kappa Gh\left(\frac{\partial w}{\partial x}+\psi_x\right)$$

$$Q_x=\kappa Gh\left(\frac{1}{R}\frac{\partial w}{\partial x}-(\frac{\nu}{R}-\psi_\theta)\right)$$

where $\kappa$ is a constant used to modify shear stress-strain relation whose value depends on cross section geometry [12]. Similar expressions were derived by Lin and Morgan [13], Herrmann and Mirsky [14].

## 3.2. AXISYMMETRIC MOTIONS OF CIRCULAR CYLINDRICAL SHELLS

In the preceding equations, (3.4) and (3.5), equations were written for nonaxially symmetric i.e. for general motions of a circular cylindrical shell.

Since axisymmetric motions are of importance in the problem, which means

$$v = \psi_\theta = \partial(\ )/\partial\theta = 0 \tag{3.6}$$

and because of the assumption of small displacements, the displacement $u$ and its derivatives can be neglected in order to admit analytical treatment including rotatory and translational motions. In the loading case there are only $q_z$ and $m_\theta$ as nonzero terms while others are equal to zero.

As the loads include surface loads and inertial loads, they can be separated as below

$$q_z = f - \rho h\,(\partial^2 w / \partial t^2) \tag{3.7}$$
$$m_\theta = I\ (\partial^2 \psi_x / \partial t^2) \tag{3.8}$$

where $I = \rho h^3/12$, moment of inertia and $f$ denotes external (surface) load per unit area. Substituting (3.6), (3.7) and (3.8) into (3.4) and (3.5) one can obtain the following equations of motion

$$\begin{gathered} k_1\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial \psi_x}{\partial x}\right) - \frac{w}{R^2} + \frac{1-\nu^2}{hE} f = \gamma^2 \frac{\partial^2 w}{\partial t^2} \\ \frac{h^2}{12}\frac{\partial^2 \psi_x}{\partial x^2} - k_1\left(\frac{\partial w}{\partial x} + \psi_x\right) = \frac{h^2}{12}\frac{\partial^2 \psi_x}{\partial t^2} \end{gathered} \tag{3.9}$$

where $\gamma^2 = \dfrac{1-\nu^2}{E}\rho$ and $k_1 = \dfrac{1-\nu}{2}\kappa$

If the effects of transverse shear deformation and rotatory inertia are not important, then, $\psi_x = -\partial w / \partial x$ and equations of motion, (3.9), reduce to the following equation

$$-\frac{Eh^3}{12(1-\nu^2)}\frac{\partial^4 w}{\partial x^4} - Eh\frac{w}{R^2} + f = \rho h\frac{\partial^2 w}{\partial t^2} \tag{3.10}$$

## 3.3. BOUNDARY AND INITIAL CONDITIONS

In the problem, the boundary conditions at simply supported edges become

$$w(0,t) = 0\,, \quad w(L,t) = 0 \tag{3.11a}$$

$$M_x(0,t) = 0\,, \quad M_x(L,t) = 0 \tag{3.11b}$$

and the following initial conditions

$$w(x,0) = 0\,, \quad \dot{w}(x,0) = 0 \tag{3.12a}$$

$$\psi_x(x,0) = 0\,, \quad \dot{\psi}_x(x,0) = 0 \tag{3.12b}$$

which mean displacement, rotation and their derivatives with respect to time at $t = 0$.

## 3.4. LOADING

In this study the shock load is considered. For simplicity in the analysis this load is modeled as a concentrated load moving at a constant speed $V$ in the $x$-direction but not varying in the circumferential direction in magnitude.

A moving load can generally be described by

$$f(x,t) = F(t)\,\delta(x-Vt) \qquad \text{for} \quad 0 \le Vt \le L$$

$$f(x,t) = 0 \qquad \text{for} \quad Vt > L$$

where $F(t)$ describes the time dependence of the load. In this case $F(t)$ is constant, which means load position varies while its magnitude remains the same, i.e.

$$f(x,t) = P\,\delta(x-Vt)$$

where $\delta$ is Dirac delta function and the time t is set to zero when the moving load enters the cylindrical shell [15].

# CHAPTER 4

# METHODS OF SOLUTION

There are many methods in order to solve the equations of motion both analytically and numerically. Among the analytical methods are modal analysis method, assumed modes method, Galerkin's method, Collocation method and so on. Development of computers has enabled us to use numerical methods efficiently in the dynamic analysis of structures which are impossible to treat analytically. Most widely used of numerical methods is finite element method. In this chapter it will be given brief information to the solution of equations of motion except for assumed modes method. Assumed modes method will be explained in detail for the equations of motion will be solved by this method.

## 4.1. FINITE ELEMENT METHOD

The increasing complexity of machine and structural systems and sophistication of digital computers have been instrumental in the development of finite element method. This method is a numerical procedure in which a complex structure is considered as an assemblage of a number of smaller elements, where each element is a continuous structural member called a finite element. The elements are assumed to be interconnected at certain points called nodes. Since it is very difficult to find the exact solution (such as displacements) of the original structure under the specified loads, a convenient approximate solution is assumed in each finite element. By requiring that the displacements be compatible and the forces balance at the joints, the entire structure is compelled to behave as a single entity.

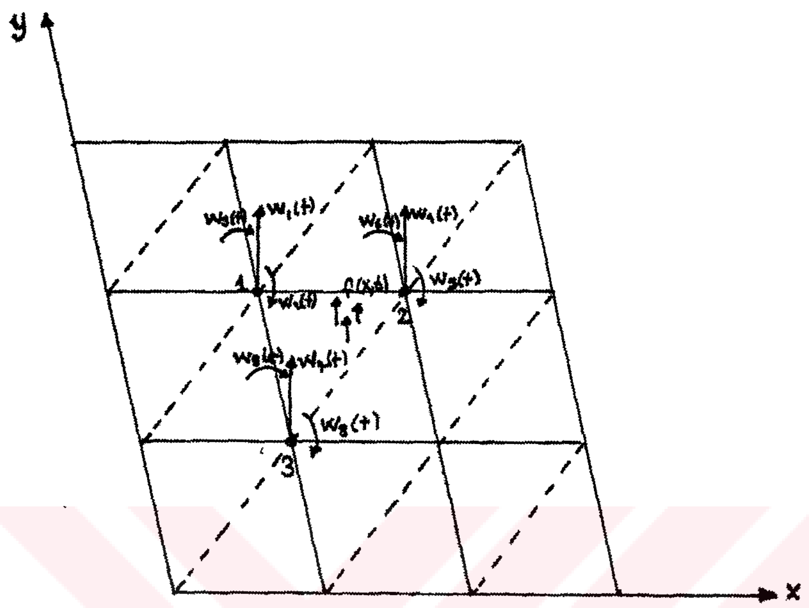


Figure 4.1.

The displacement $w(x,y,t)$ for Figure 4.1. can be expressed in terms of the unknown nodal displacements $W_i(t)$ in the form

$$w(x,y,t)=\sum_{i=1}^{n}N_i(x,y)W_i(t) \tag{4.1}$$

where $N_i(x,y)$ is called the shape function corresponding to the nodal displacement $W_i(t)$ and $n$ is the number of unknown nodal displacements ($n=9$ in Figure 4.1.).

In finite element method everything including mass, distributed force, stiffness etc. is converted into nodal values then the equation of motion for complete structure is obtained as follow

$$[M]_{n\times n}\ddot{\vec{U}}_{n\times 1}+[K]_{n\times n}\vec{U}_{n\times 1}=\vec{F}_{n\times 1} \tag{4.2}$$

where $n$ denotes the number of free node displacements, $[M]$ mass matrix, $[K]$ stiffness matrix, $\vec{F}$ force vector, $\vec{U}$ displacement vector of the structure.

The solution of (4.2) yields the wanted values at nodes [4].

## 4.2. MODAL ANALYSIS METHOD

In this method, system behaviour is determined as a superposition of normal modes multiplying corresponding time-dependent generalized coordinates. Therefore in order to obtain system response it is necessary to find eigenfunctions of free vibration.

Let us consider a continuous system described by the partial differential equation

$$L[w(P,t)] + M(P)\partial^2 w(P,t)/\partial t^2 = f(P,t) \tag{4.3}$$

over the domain D. In the above $L$ is a linear homogenous self-adjoint differential operator consisting of derivatives with respect to time, $f(P,t)$ is a distributed force.

The normal modes analysis calls for the solution of the special eigenvalue problem consisting of the differential equation

$$L[w] = \lambda Mw = \omega^2 Mw \tag{4.4}$$

to be satisfied over the domain $D$.

The solution of the eigenvalue problem (4.4) consists of an infinite set of denumerable eigenfunctions $W_r(P)$ with corresponding natural frequencies $\omega_r$.

The eigenfunctions are orthogonal, and if they are normalized such that

$$\int_D M(P)\, W_r(P)\, W_s(P)\, dD(P) = \delta_{rs} \tag{4.5a}$$

it follows

$$\int_D W_s(P)\, L[W_r(P)]\, dD(P) = \omega_r^2\, \delta_{rs} \tag{4.5b}$$

Using the expansion theorem the solution of (4.3) can be written as a superposition of the normal modes $W_r(P)$ multiplying corresponding time-dependent generalized coordinates $\eta_r(t)$. Therefore $w(P,t)$ is

$$w(P,t) = \sum_{r=1}^{\infty} W_r(P)\, \eta_r(t) \tag{4.6}$$

Substituting (4.6) into (4.3) gives

$$L\left[\sum_{r=1}^{\infty} W_r(P)\,\eta_r(t)\right] + M(P)\frac{\partial^2}{\partial t^2}\sum_{r=1}^{\infty} W_r(P)\,\eta_r(t) = f(P,t) \tag{4.7}$$

multiplying (4.7) by $W_s(P)$ and integrating over the domain $D$ with the help of (4.5) one obtains

$$\eta_r(t) + \omega_r^2\,\eta_r(t) = N_r(t) \qquad r = 1,2,\ldots \tag{4.8}$$

which represents an infinite set of ordinary differential equations easy to solve where

$$N_r(t) = \int_D W_r(P)\,f(P,t)\,dD(P) \tag{4.9}$$

## 4.3. GALERKIN'S METHOD

This method is based on the error concept. First, an approximate solution satisfying the associated boundary conditions is assumed and then the weighted error integrated over the domain is made equal to zero. After this procedure ordinary differential equations are obtained which can be solved easily.

Let us consider a continuous system described before by the differential equation (4.3) as

$$L[w(P,t)] + M(P)\partial^2 w(P,t)/\partial t^2 = f(P,t) \tag{4.10}$$

and let the approximate solution of the above equation be

$$w_n(P,t) = \sum_{j=1}^{n} \Phi_j(P)\,q_j(t) \qquad j = 1,2,\ldots,n \tag{4.11}$$

such that it satisfies the associated boundary conditions. Here $\phi_j(P)$ are comparison functions depending on the spatial coordinates and $q_j(t)$ are time-dependent generalized coordinates. Since $W_n(P,t)$ is only an approximate solution it will not satisfy (4.10) exactly and therefore there will be a small difference between the approximate and the exact solution. Denoting this difference by $\varepsilon(P,t)$, we have

$$\varepsilon(P,t) = L[w(P,t)] + M(P)\partial^2 w(P,t)/\partial t^2 - f(P,t) \tag{4.12}$$

where $\varepsilon(P,t)$ is referred to as the error. According to the Galerkin's method the weighted error integrated over the domain must be zero. The weighting functions are the comparison functions $\phi_j(P)$ such that

$$\int_D \varepsilon(P,t)\, \phi_r(P)\, dD(P) = 0 \qquad r = 1,2, \ldots , n \tag{4.13}$$

This procedure gives us ordinary differential equations which can be solved easily by other methods.

## 4.4. COLLOCATION METHOD

Another method based on the error concept is the collocation method. The collocation method also assumes an approximate solution as a combination of some functions and associated coefficients. The method consists of selecting a function and a set of locations called stations. The coefficients are determined by insisting that at the selected stations approximate solution satisfies the differential equation.

As an example let us consider the equation (4.10)

$$L[w(P,t)] + M(P)\partial^2 w(P,t) / \partial t^2 = f(P,t) \tag{4.10}$$

with the assumed solution (4.11) as below

$$w_n(P,t) = \sum_{j=1}^{n} \Phi_j(P) q_j(t) \qquad j = 1,2,\ldots,n \tag{4.11}$$

where $\phi_j(P)$ are a set of comparison functions.

Substituting (4.11) into (4.10) gives (4.12) as mentioned before as

$$\varepsilon(P,t) = L[w(P,t)] + M(P)\partial^2 w(P,t) / \partial t^2 - f(P,t) \tag{4.12}$$

where $\varepsilon(P,t)$ denotes the difference between the approximate and exact solution as mentioned before.

Letting the error be zero at $n$ such points, we obtain a set of $n$ ordinary differential equations as

$$\sum_{j=1}^{n} \ddot{q}_j(t)\left[M(P_r)\Phi_j(P_r)\right] + \sum_{j=1}^{n} q_j(t)L\left[\Phi_j(P_r)\right] = f(P_r,t) \ , \qquad r = 1,2,\ldots,n \tag{4.14}$$

which can be solved using other methods.

## 4.5. ASSUMED-MODES METHOD

This method resembles modal analysis method. In this method the response of a system is assumed in the form

$$w(P,t) = \sum_{i=1}^{n} \Phi_i(P)q_i(t) \tag{4.15}$$

where $\phi_i(P)$ are admissable functions, which are functions of spatial coordinates $P$ and satisfy the geometric boundary conditions of the system and $q_i(t)$ are time-dependent generalized coordinates. In this manner a continuous system is approximated by an $n$-degree of freedom system.

Now, the governing equations of motion of the problem given by (3.9) are solved by means of this method, although many other methods mentioned before are possible for the solution. The solution is assumed as

$$\psi_x(x,t) = \sum_{n=1}^{N} B_n(t)\cos\frac{n\pi x}{L}$$

$$w(x,t) = \sum_{n=1}^{N} C_n(t)\sin\frac{n\pi x}{L} \tag{4.16}$$

where $\cos\dfrac{n\pi x}{L}$ and $\sin\dfrac{n\pi x}{L}$ are admissible functions satisfying the following boundary conditions:

$$w(0,t) = 0\ , \quad w(L,t) = 0$$
$$M_x(0,t) = 0\ , \quad M_x(L,t) = 0$$

Substituting (4.16) into (3.9) and multiplying (3.9a) by $\sin(m\pi x/L)$, (3.9b) by $\cos(m\pi x/L)$ and integrating from zero to $L$ and using the orthogonality relations of trigonometric functions give the following system of ordinary coupled differential equations:

$$\ddot{B}_n(t) = -k_{2n}B_n(t) - k_{3n}C_n(t)$$

$$\ddot{C}_n(t) = -k_{4n}B_n(t) - k_{5n}C_n(t) + k_6 \sin(k_{7n}t) \tag{4.17}$$

where

$$k_{2n} = \frac{12}{h^2\gamma^2}(k_1 + \frac{h^2}{12}\frac{n^2\pi^2}{L^2})$$
$$k_{3n} = \frac{12k_1}{h^2\gamma^2}\frac{n\pi}{L}$$
$$k_{4n} = \frac{k_1}{\gamma^2}\frac{n\pi}{L}$$
$$k_{5n} = \frac{1}{\gamma^2}(k_1\frac{n^2\pi^2}{L^2} + \frac{1}{R^2})$$
$$k_6 = \frac{2P}{\rho hL}$$
$$k_{7n} = \frac{n\pi V}{L}$$

and $V$ denotes the velocity of the moving load.

Solutions of (4.17) are

$$B_n(t) = c_{1n}e^{s_{1n}t} + c_{2n}e^{-s_{1n}t} + c_{3n}e^{s_{2n}t} + c_{4n}e^{-s_{2n}t} + k_{11n}\sin k_{7n}t$$

$$C_n(t) = -\frac{1}{k_{3n}}\Big[(k_{2n} + s_{1n}^2)(c_{1n}e^{s_{1n}t} + c_{2n}e^{-s_{1n}t}) + (k_{2n} + s_{2n}^2)(c_{3n}e^{s_{2n}t} + c_{4n}e^{-s_{2n}t}) + (k_{2n} - k_{7n})k_{11n}\sin k_{7n}t\Big] \tag{4.18}$$

where

$$s_{1n} = \sqrt{\frac{-k_{8n} + \sqrt{k_{8n}^2 - 4k_{9n}}}{2}}$$

$$s_{2n} = \sqrt{\frac{-k_{8n} - \sqrt{k_{8n}^2 - 4k_{9n}}}{2}}$$

and

$$k_{8n} = k_{2n} + k_{5n}$$
$$k_{9n} = k_{2n}k_{5n} - k_{3n}k_{4n}$$
$$k_{10n} = -k_{3n}k_6$$
$$k_{11n} = \frac{k_{10n}}{k_{7n}^4 - k_{8n}k_{7n}^2 + k_{9n}}$$

The constants $c_{1n}$, $c_{2n}, c_{3n}, c_{4n}$ are determined from the initial conditions. Then substituting (4.18) into (4.16) gives the analytical solution of the coupled governing equations of motion given by (3.9). With this analytical solution, some parametric studies are performed and represented graphically in the following chapter.

# CHAPTER 5

## RESULTS AND DISCUSSION

The system parameters indicating the cylindrical shell and the moving load are

| | | | |
|---|---|---|---|
| $h = 3$ mm | $R = 6$ cm | $L = 80$ cm | $P = 50$ bar |
| $\rho = 7860$ kg/m$^3$ | | $\nu = 0.25$ | $E = 20000$ kg/mm$^2$ |

For various speed values the following graphics for mid-span displacement are obtined. From these results it is observed that

1) As the speed of the moving load increases, the behaviour of the elastic circular shell changes. The shell exhibits some elastic waves with increases in amplitudes before the load reaches and after it passes the mid-span of the shell up to speed about 1000 m/s (Figure 5.1-6).

2) The speed about 1000 m/s is special value for the problem because at speeds greater than this value, amplitudes of the points before and after mid-span reach their maximum with decreases in frequency. For the speed about 1000 m/s mid-span seems to reach its maximum after the load exits the circular shell (Figure 5.6).

3) From the speed 1250 to 2000 m/s both the amplitude and the frequency decrease before and after the mid-span (Figure 5.7-10).

4) For very high speeds the amplitudes of the points decrease and become almost zero (Figure 5.10-12).

5) In the problem, as the shell is very thin there is no big difference between the solutions obtained by taking into account the effects of transverse shear deformation and rotatory inertia and that obtained by pure bending.

6) The results obtained by the use of Naghdi's equations, as $(h/R) \ll 1$, are in good agreement with the beam solution by Olsson [2].

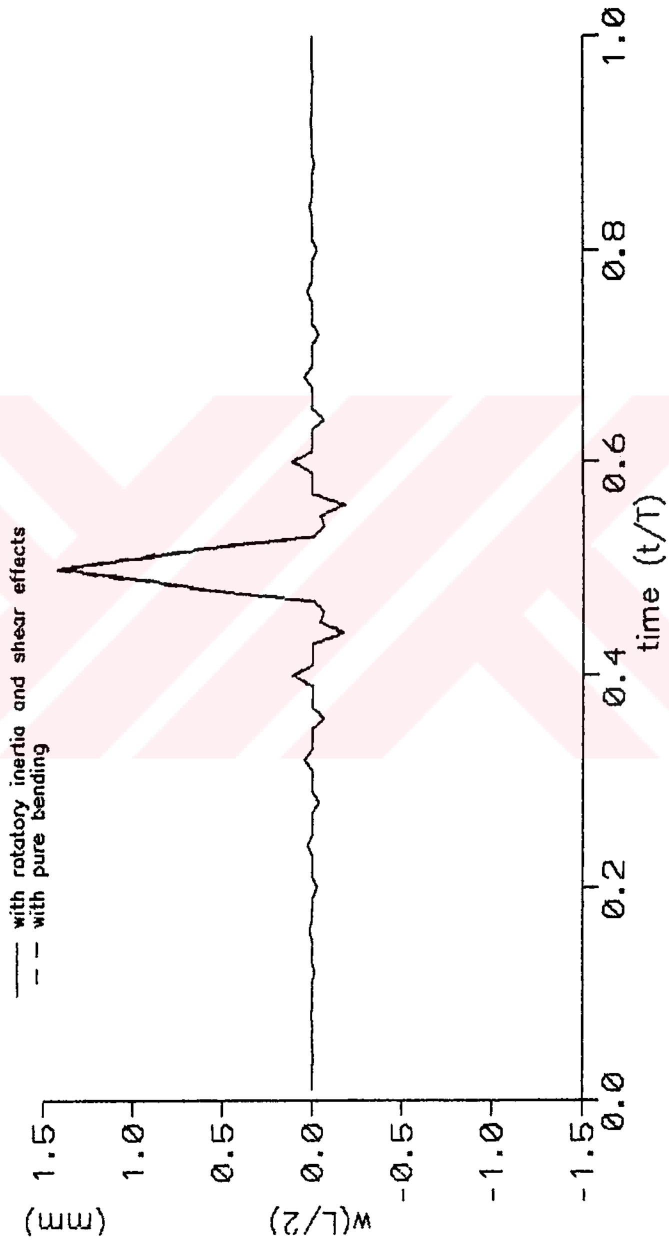


Figure 5.1. Time history of mid-span for V = 0.005 m/s.

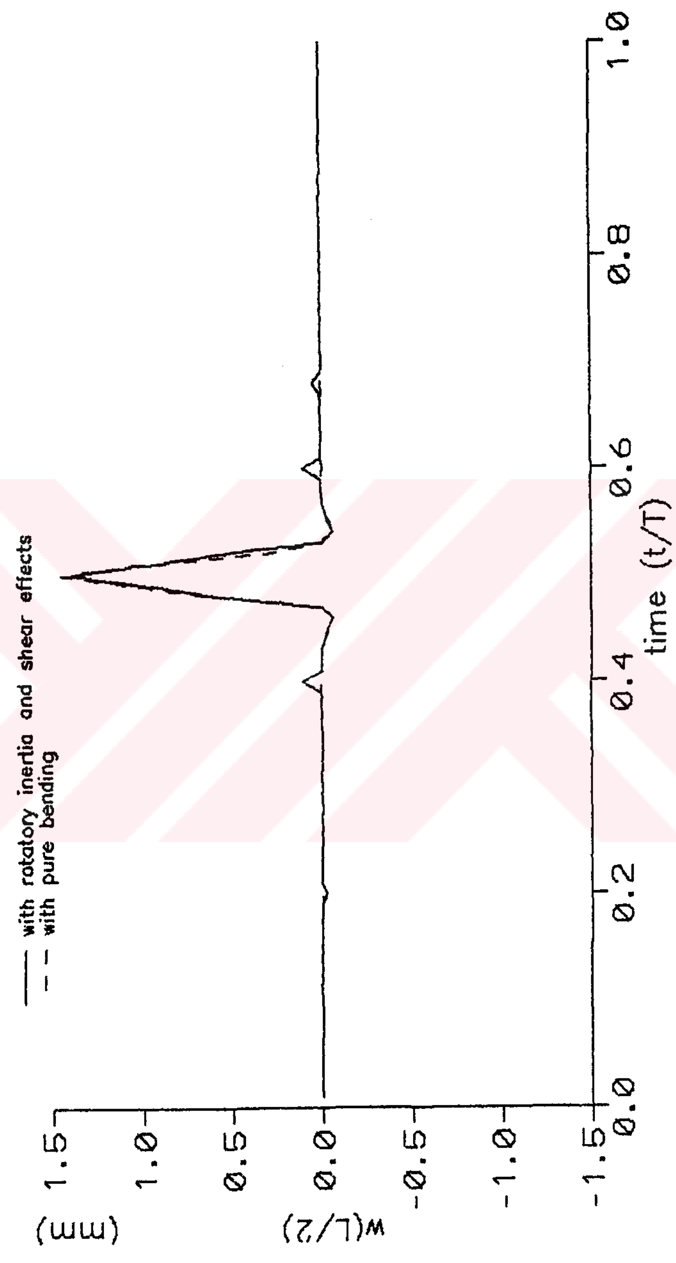


Figure 5.2. Time history of mid-span for V = 10 m/s.

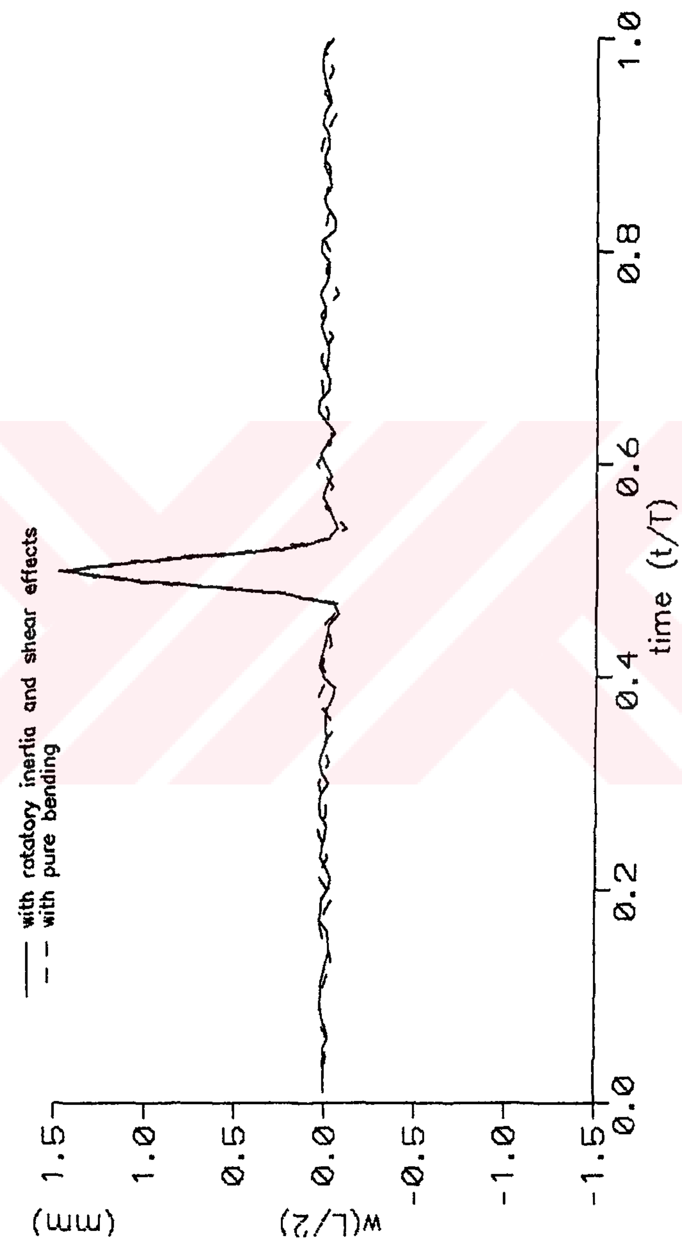


Figure 5.3. Time history of mid-span for V = 150 m/s.

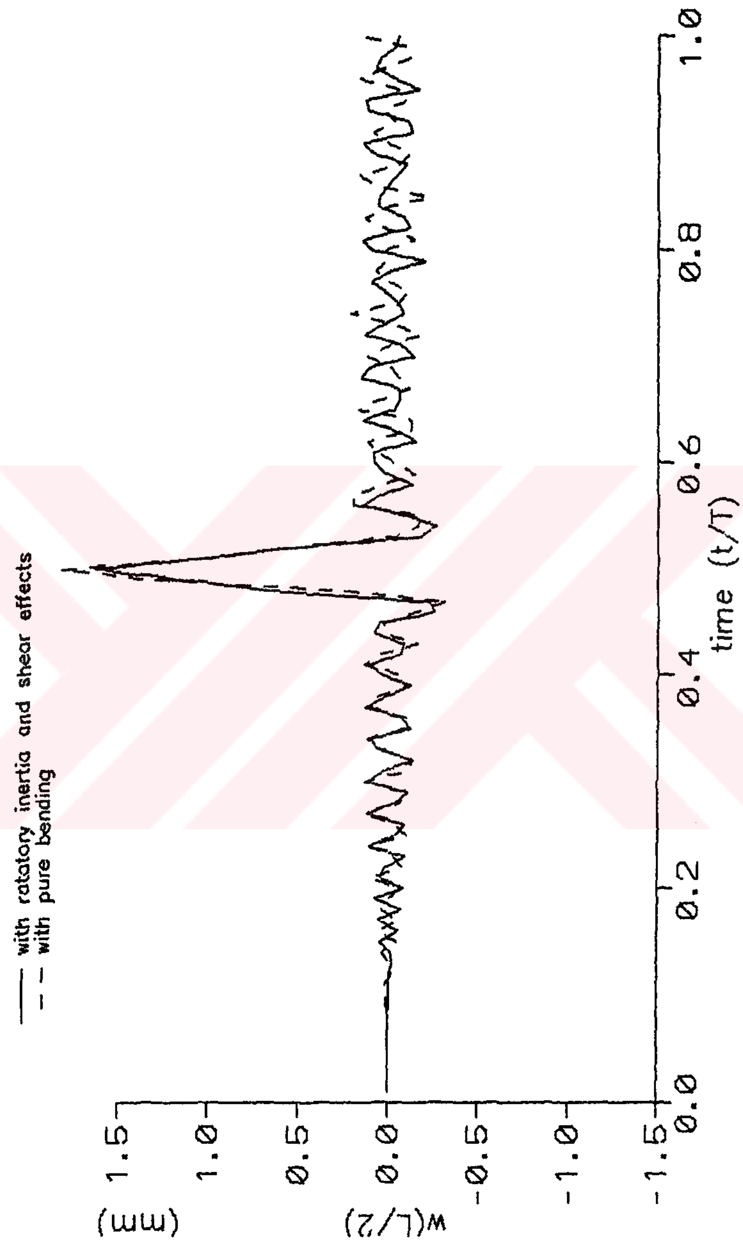


Figure 5.4. Time history of mid-span for V = 500 m/s.

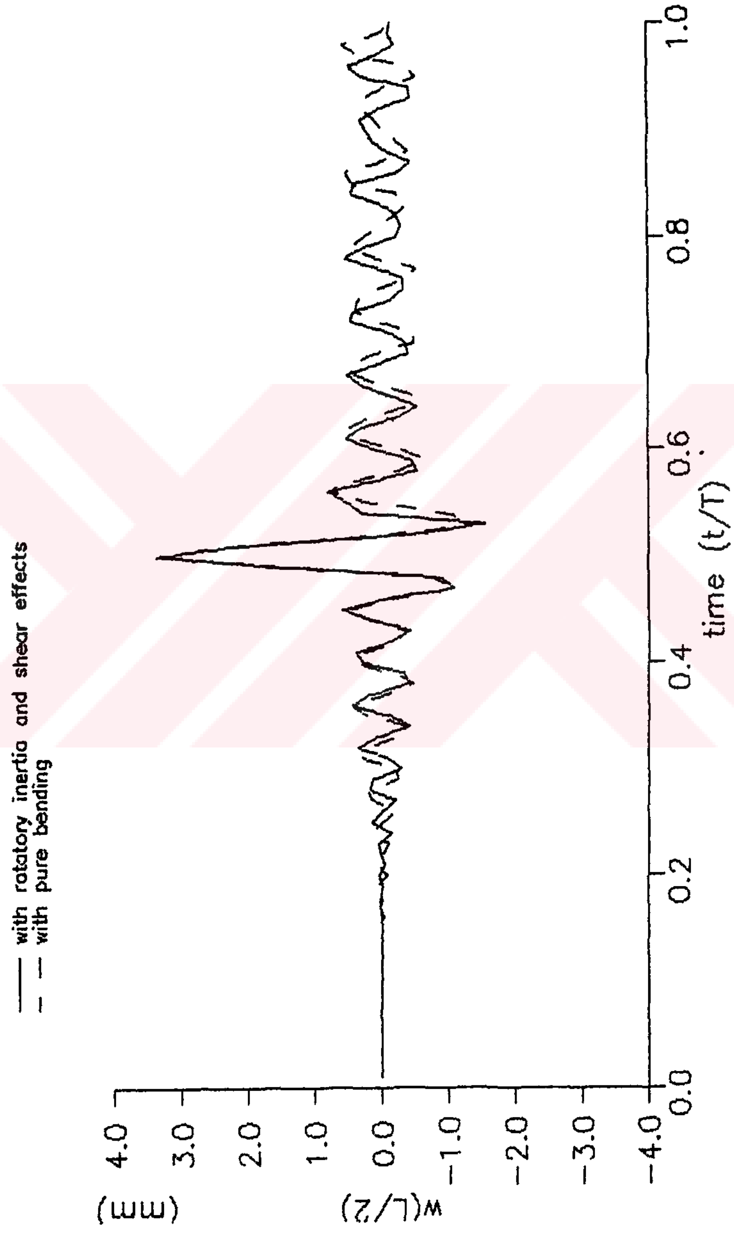


Figure 5.5. Time history of mid-span for V = 750 m/s.

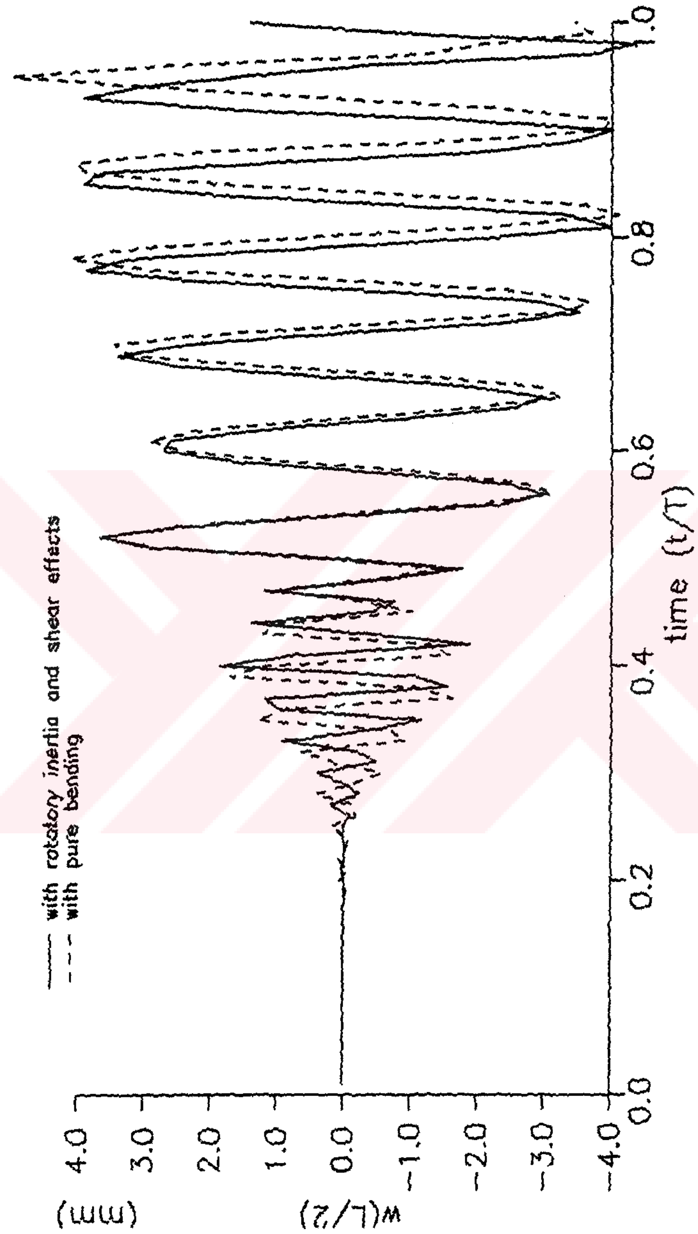


Figure 5.6. Time history of mid-span for V = 1000 m/s.

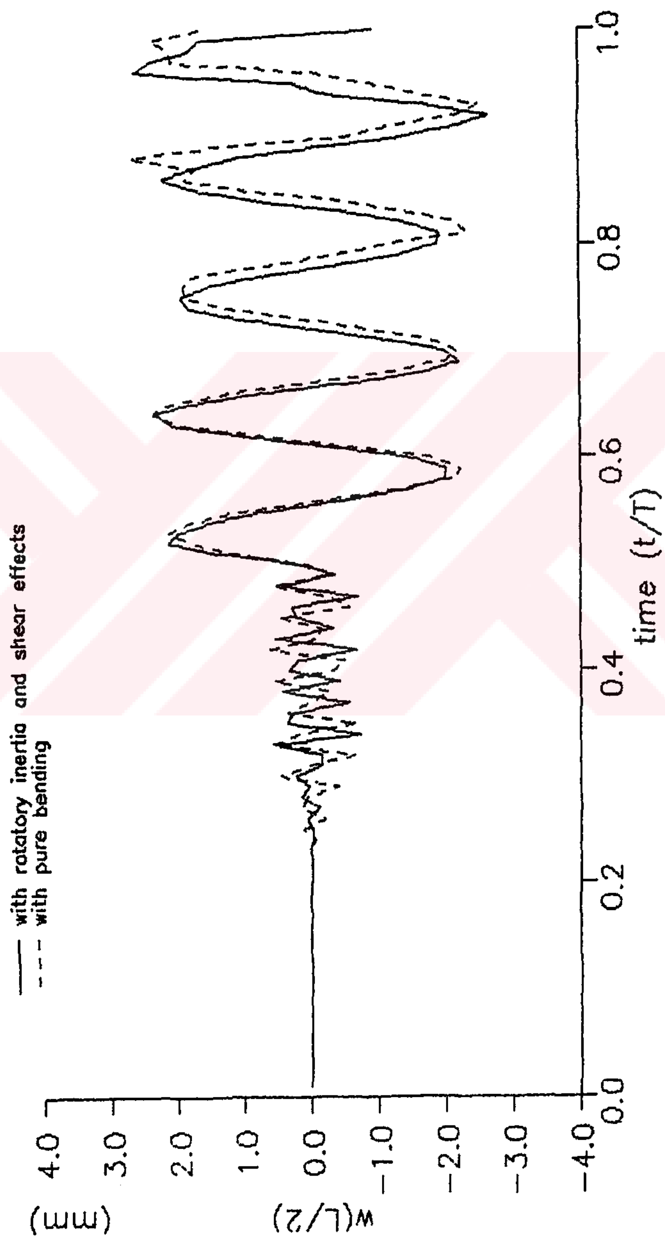


Figure 5.7. Time history of mid-span for V = 1250 m/s.

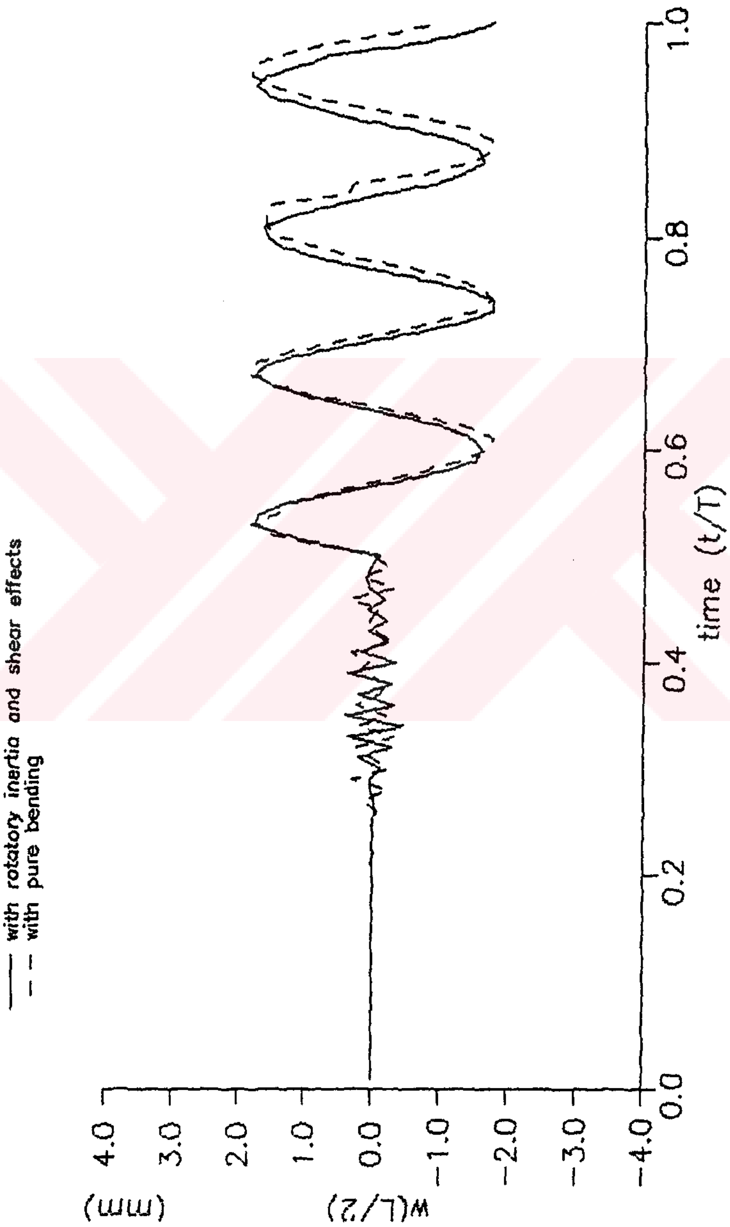


Figure 5.8. Time history of mid-span for V = 1500 m/s.

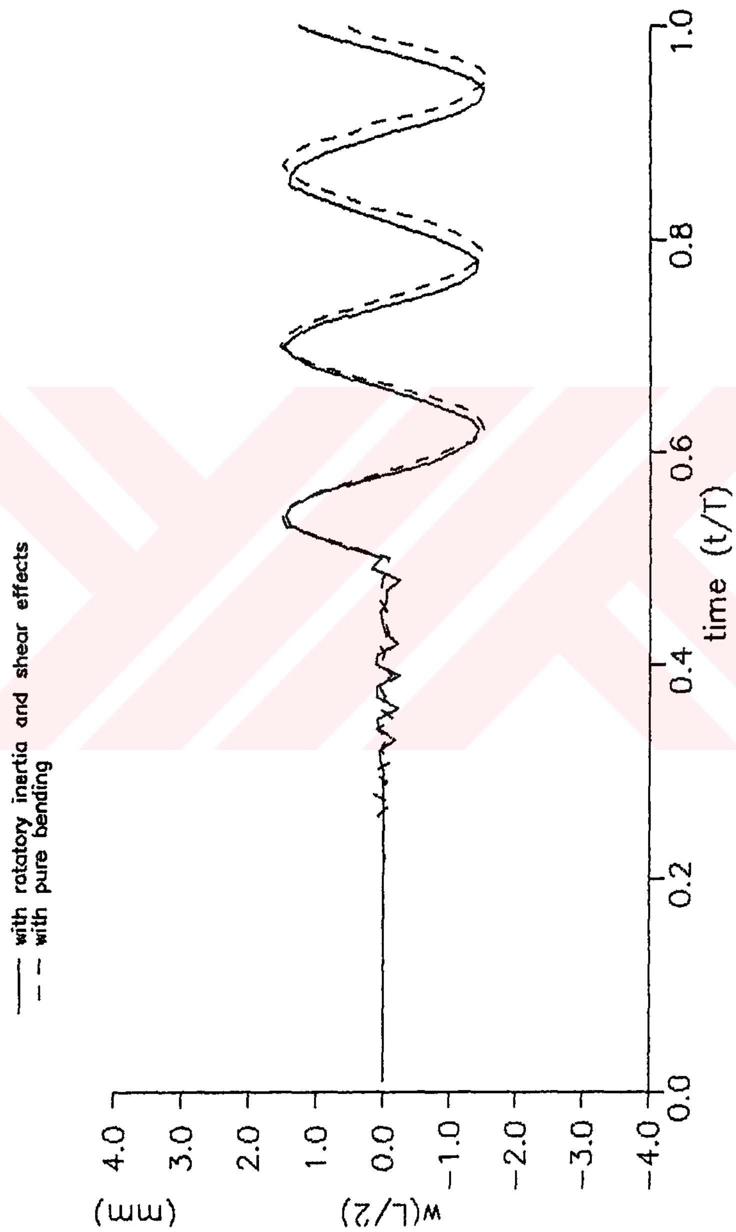


Figure 5.9. Time history of mid-span for V = 1750 m/s.

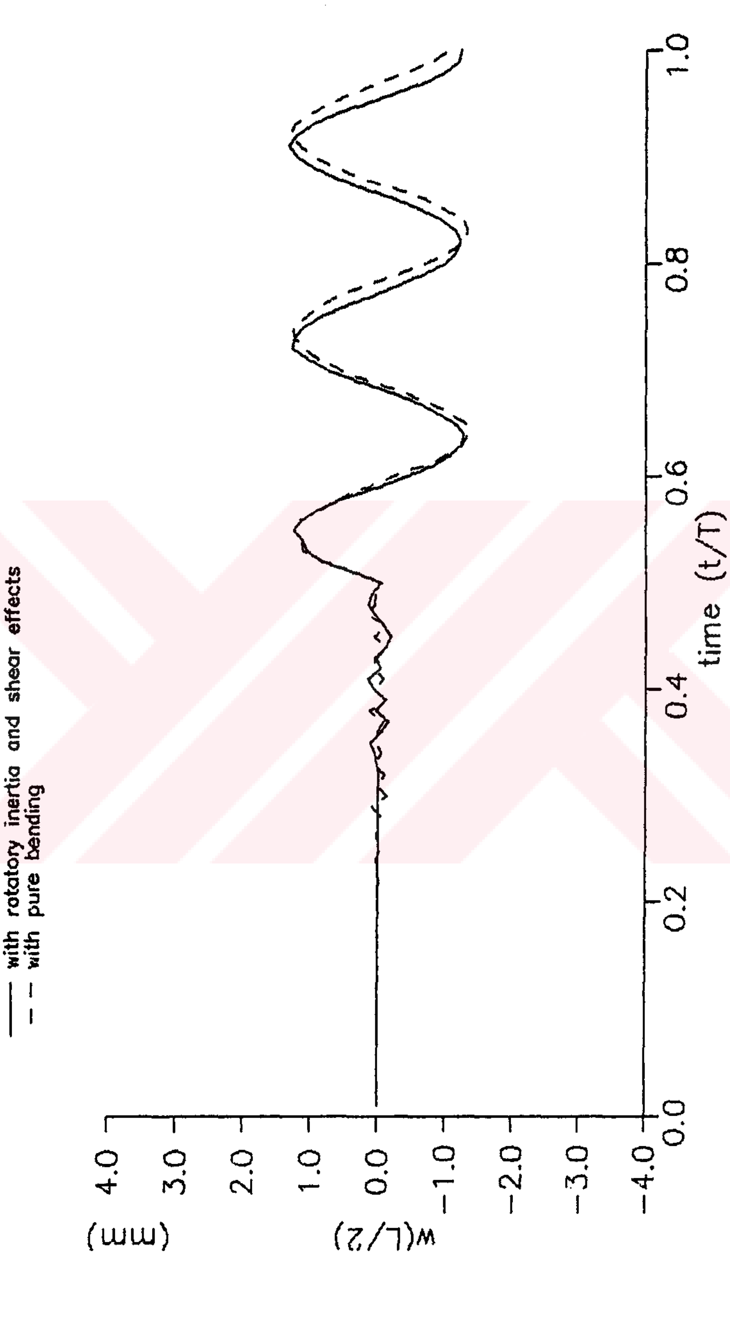


Figure 5.10. Time history of mid-span for V = 2000 m/s.

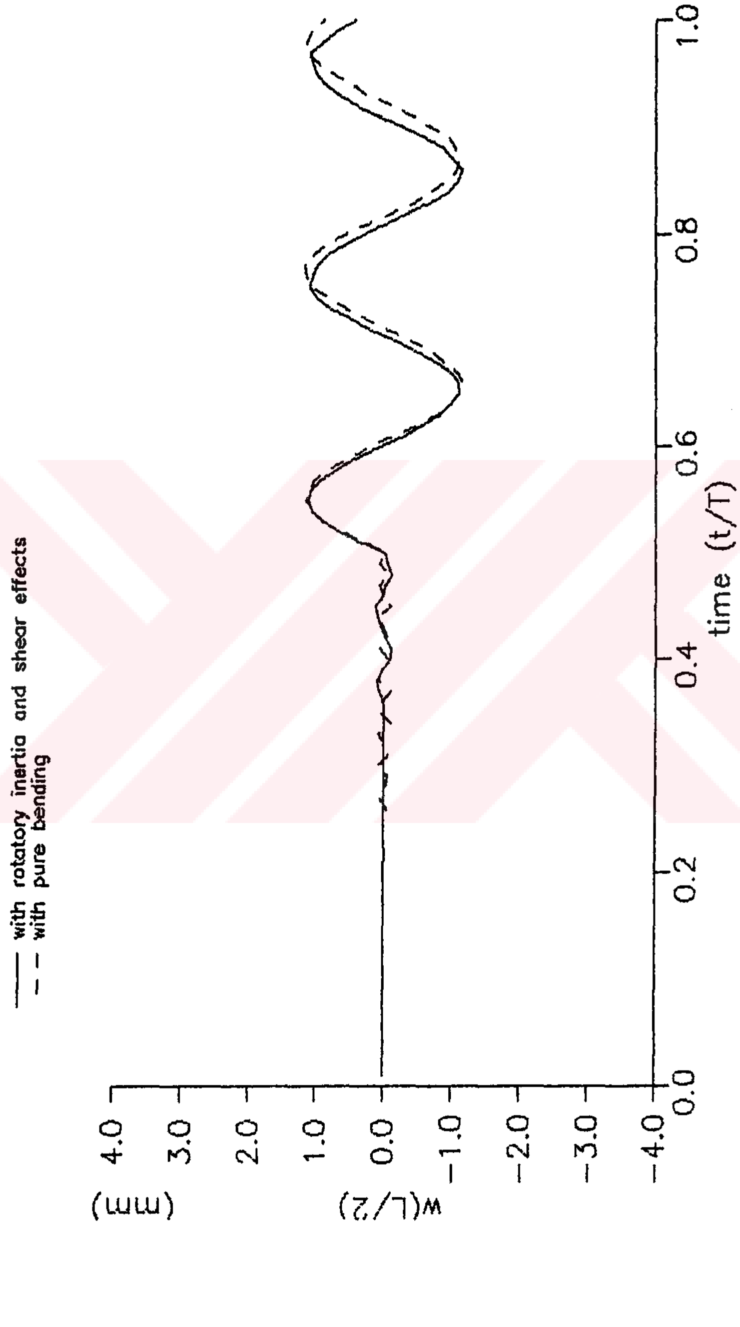


Figure 5.11. Time history of mid-span for V = 2250 m/s.

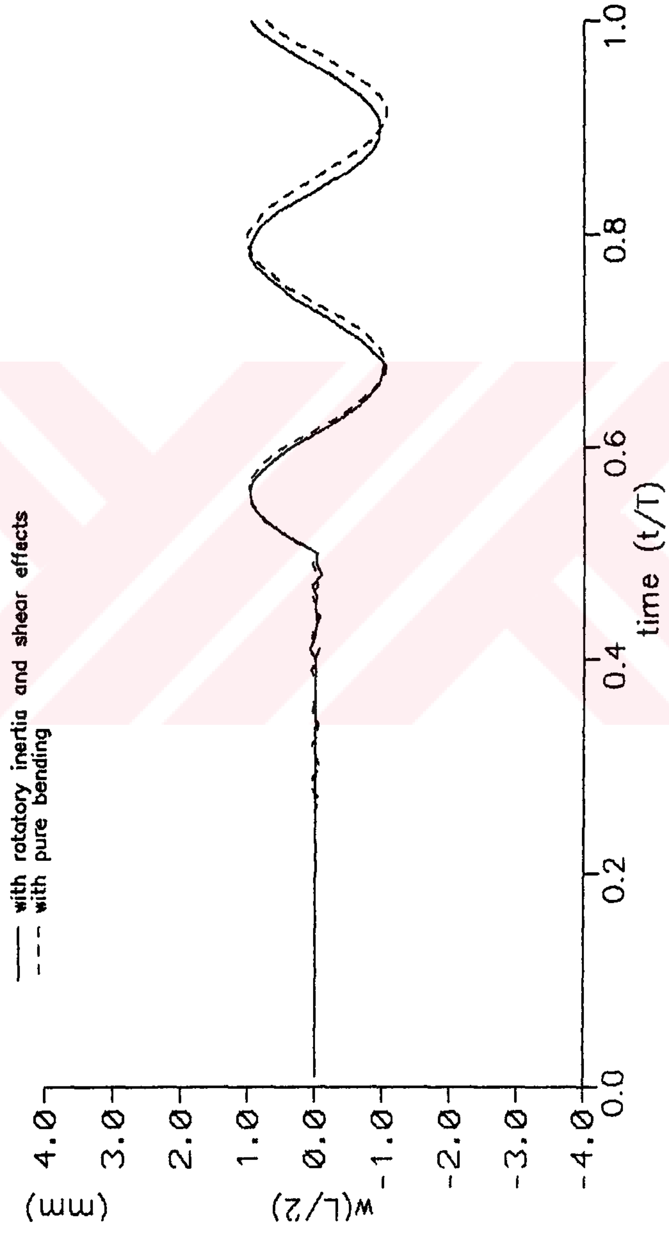


Figure 5.12. Time history of mid-span for V = 2500 m/s.

# CHAPTER 6

# CONCLUSIONS

The dynamic analysis of a thin circular cylindrical shell subjected to shock loading is done. To simplify the complete three-dimensional equations to two-dimensional ones, some assumptions are made: The shock loading is represented as a concentrated moving load at a constant speed along the axial direction but not varying in the circumferential direction and the small displacement theory is used. In order to make the analytical solution valid for high frequencies, the effects of transverse shear deformation and rotatory inertia, which become increasingly important as the frequency is increased, are taken into account. However, the longitudinal inertia force is neglected. Then the obtained two-dimensional partial differential equations are reduced to two-ordinary coupled differential equations by means of The Assumed Modes Method with application of the orthogonalization procedure. These equations are solved analytically. Finally, from the obtained results in the previous chapter the following conclusions are arrived at:

1) The static solution of the circular cylindrical shell is obtained as a special case of the dynamic analysis. It is consistent with the analytical static solution of the circular cylindrical shell [16].

2) For very thin shells pure bending solutions are in good accuracy at low and high speeds.

3) As the speed increases, inertial forces begin to become important.

4) For very high speeds, as the inertial forces are completely active, displacements of the points of the cylindrical shell are near zero.

## REFERENCES


[1] FRYBA, L. "Vibration of Solids and Structures Under Moving Load", (Noordhoff, Groningen, 1972).

[2] OLSSON, M. "On the Fundamental Moving Load Problem", Journal of Sound and Vibration, Vol. 45, No. 2, 1991.

[3] JONES, P. and BHUTA, P. G., "Response of Cylindrical Shells to Moving Loads", Vol. 31, No.1, Tras. of the ASME Journal of Applied Mechanics, Ser.E: March 1964, pp. 105-111.

[4] HERRMANN, G. and BAKER, E. H., "Response of Cylindrical Shells to Moving Loads", Vol. 34, No.1, Tras. of the ASME Journal of Applied Mechanics, Ser.E: March 1967, pp. 81-96.

[5] STEELE, C. R., "The Timoshenko Beam with a Moving Load", Vol. 35, No.3, Tras. of the ASME Journal of Applied Mechanics, Ser.E: Sept. 1968, pp. 481-488.

[6] STEELE, C. R., "The Finite Beam with a Moving Load", Vol. 34, No.1, Tras. of the ASME Journal of Applied Mechanics, Ser.E: March 1967, pp. 111-118.

[7] PIERUCCI, M. "The Flexural Wave-number Response of a String and a Beam Subjected to a Moving Harmonic Force", Journal of Acoustical Society of America, Vol. 93, No. 4, April 1993, pp. 1908-1917.

[8] SCHIFFNER, K. and STEELE, C. R., "Cylindrical Shell with an Axisymmetric Moving Load", AIAA Journal, Vol. 9, No. 1, Jan. 1971, pp. 37-47.

[9] YOSHIDA, D. M. and WEAVER, W., "Finite Element Analysis of Beams and Plates with Moving Loads", Publ. Int. Ass. Bridge Struct. Engr., Vol. 31, 1971, pp. 179-195.

[10] TAHERI, M. R. and TING, E. C., "Dynamic Response of Plates to Moving Loads", Computers and Structures, Vol. 34, No. 3, 1990, pp. 509-521.

[11] LEISSA, A. W. "Vibration of Shells", Washington D. C. 1973.

[12] NAGHDI, P. M. and BERRY, J. G., "On the Equations of Motion of Cylindrical Shells", Vol. 76, Tras. of the ASME Journal of Applied Mechanics, 1954, pp. 160-166.

[13] LIN, T. C. and MORGAN, G. W., "A Study of Axisymmetric Vibrations of Cylindrical Shells as Affected by Rotatory Inertia and Transverse Shear", Journal of Applied Mechanics, June 1956, pp. 255-261.

[14] HERRMANN, G. and MIRSKY, I., "Three-dimensional and Shell-Theory Analysis of Axially Symmetric Motions of Cylinders", Journal of Applied Mechanics, Dec. 1956, pp. 563-568.

[15] MEIROVITCH, L., "Analytical Methods in Vibration", New York, 1967.

[16] TIMOSHENKO, S. and WOINOWSKY-KRIEGER, S. " Theory of Plates and Shells", McGraw-Hill Book Company, 1959.

# CURRICULUM VITAE

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