

ISTANBUL TECHNICAL UNIVERSITY ★ GRADUATE SCHOOL OF SCIENCE
ENGINEERING AND TECHNOLOGY

**NEW EFFICIENT 2-D LATTICE STRUCTURES
FOR GENERAL AUTOREGRESSIVE MODELING**

Ph.D. THESIS

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Department of Electronics and Communications

Telecommunications Programme

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Thesis Advisor: Prof. Dr. A. Hamdi KAYRAN

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**GENEL ÖZBAĞLANIMLI MODELLEME İÇİN
YENİ ETKİN 2-B KAFES YAPILARI**

DOKTORA TEZİ

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MAYIS 2014

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*To my spouse and children,
Tuba, Atakan and Onur*

FOREWORD

After more than 10 years of interruption, it was not easy to complete this Ph.D. program and be a full-time employee at the same time. Which made it even more difficult for me was my yearning for my family. I would like to express my deep appreciation for and thanks to my advisor Prof. Dr. Ahmet Hamdi Kayran, for his patience, assistance and invaluable help throughout the thesis. I would not have finished it without his guidance, his encouragement and his support. I was lucky to have the opportunity and privilege to work with him.

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ABBREVIATIONS

1-D	: One-Dimensional
2-D	: Two-Dimensional
AR	: Autoregressive
ARMA	: Autoregressive Moving Average
ASHP	: Asymmetric Half Plane
FFT	: Fast Fourier Transform
IIR	: Infinite Impulse Response
ISAR	: Inverse Synthetic Aperture Radar
NC	: Noncausal
PES	: Prediction Error Sequence
PEF	: Prediction Error Field
PSR	: Prediction Support Region
QP	: Quarter Plane
SAR	: Synthetic Aperture Radar
SNR	: Signal to Noise Ratio
VLSI	: Very Large Scale Integration

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LIST OF SYMBOLS

$(\cdot)^*$	: Complex conjugate of a complex quantity
$(\cdot)^H$	: Hermitian transpose of a vector or a matrix
$(\cdot)^T$	: Transpose of a vector or a matrix
$(\cdot)^{(p,q)}$	: (p, q) th order of $(\cdot)$
$\mathbf{A}_i^{(p,q)}$	: i th coefficient matrix from (p, q) th order vertical PEFs
$\mathbf{B}_i^{(p,q)}$	: i th coefficient matrix from (p, q) th order horizontal PEFs
$e^{(n)}(k)$	: (n) th order forward prediction error
$\tilde{e}^{(n)}(k)$	: (n) th order backward prediction error
$e_i^{(p,q)}(k, l)$	: i th vertical forward PEF
$\tilde{e}_i^{(p,q)}(k, l)$	: i th vertical backward PEF
$r_i^{(p,q)}(k, l)$	: i th horizontal forward PEF.
$\tilde{r}_i^{(p,q)}(k, l)$	: i th horizontal backward PEF
$\mathbf{e}^{(p,q)}(k, l)$	: Vertical forward PEF vector
$\tilde{\mathbf{e}}^{(p,q)}(k, l)$	: vertical backward PEF vector
$\mathbf{r}^{(p,q)}(k, l)$	: Horizontal forward PEF vector
$\tilde{\mathbf{r}}^{(p,q)}(k, l)$	: Horizontal backward PEF vector
$k_{e_i}^{(p,q)}$	: Vertical lattice reflection coefficient
$k_{r_i}^{(p,q)}$	: Horizontal lattice reflection coefficient
$\mathbf{k}_{e_i}^{(p,q)}$	: Vertical lattice reflection coefficient vector
$\mathbf{k}_{r_i}^{(p,q)}$	: Horizontal lattice reflection coefficient vector
$\mathbf{K}_e^{(p,q)}$	: Vertical lattice reflection coefficient matrix
$\mathbf{K}_r^{(p,q)}$	: Horizontal lattice reflection coefficient matrix
$\mathbf{P}(\cdot)$	: Projection
$\mathbf{P}_{S_{k,l}^{(p,q)}}(y(k, l))$	: Projection of $y(k, l)$ onto subspace $S_{k,l}^{(p,q)}$.
$\mathfrak{R}$	: Autocorrelation matrix
$\mathfrak{R}(i, j)$	: (i, j) th element of autocorrelation matrix
$\Sigma_{\mathbf{x}, \mathbf{x}}^{(p,q)}$	: Covariance matrix
$\Delta_{\mathbf{r}, \mathbf{e}}^{(p,q)}$	: Cross-covariance matrix
H	: Entropy
$I_{n \times n}$	: The $n \times n$ identity matrix
$J_{n \times n}$	: The $n \times n$ exchange matrix
$P_e^{(p,q)}$	: The power propagation for vertical PEFs
$P_r^{(p,q)}$	: The power propagation for horizontal PEFs
Π_j	: The j th row vector of an identity matrix

Ψ_j	: Constant matrix defined by removing the j th row of an identity matrix
$S_k^{(n)}$	: (n)th order 1-D prediction support region
$S_k^{(n)}$	: (n)th order 1-D prediction support region
$S_{k,l}^{(p,q)}$	: (p, q)th order 2-D prediction support region
$\Gamma^{(n)}$	: (n)th order forward reflection coefficient
$\tilde{\Gamma}^{(n)}$	: (n)th order backward reflection coefficient
tr	: trace operator
$det(\cdot)$	: Determinant of matrix
$y(k)$	: 1-D input data
$y(k, l)$	: 2-D input data
$\mathbf{Y}_{k,l}^{(p,q)}$	: Input data vector
$\boldsymbol{\beta}^{(p,q)}(k, l)$	: Auxiliary backward PEFs vector

NEW EFFICIENT 2-D LATTICE STRUCTURES FOR GENERAL AUTOREGRESSIVE MODELING

SUMMARY

In the literature, the theory of 1-D lattice filters is well-known and widely used in the various areas. The theory of two-dimensional lattice filter is developed as a natural extension of 1-D filter theory. The basic fundamental approach to modeling of 2-D fields by the reflection coefficients was made by Marzetta. Lots of researchers realized their studies by using Marzetta's approach.

Although plenty of studies about two-dimensional lattice structures exist in the literature, there is no 2-D lattice structure method to obtain all possible causal and noncausal 2-D models concurrently. In this thesis, a novel 2-D lattice filter structure model is introduced. This method is quite simple and can be generalized as a 2-D lattice structure for modeling of 2-D random fields. There is no complexity, when calculating lattice reflection factors at each stage.

In the 1-D lattice modeling, augmentation can be done only in one direction by adding one more past observation at each stage and there are only one forward and one backward prediction error sequences (PESs). In the proposed 2-D lattice structure, a rectangular prediction support region (PSR) is introduced. It is the same as the projection base in the 1-D case. Additionally, a set of auxiliary horizontal and vertical forward/backward prediction error fields (PEFs) are defined. In other words, with the new proposed method, two forwards and two backwards PEFs are presented. Hence, it is possible to apply the augmentation in either horizontal or vertical direction by adding horizontal or vertical vector consisting of the past observations with respect to the augmentation direction. The augmentation can be repeated until obtaining the desired order filter. The lattice stage order is updated according to the augmentation direction. In the proposed method the lattice stage order is represented as (p, q) where p is used for the order stage of horizontal augmentation and q is used for the order stage of the vertical augmentation.

At the initial first-order stage, from the given data field, the forward and backward prediction error fields are obtained as a set of two 1-D lattice sections. Unlike 1-D case, two auxiliary forward and two auxiliary backward PEFs are obtained. At the *zeroth*-order stage, the PSR is an empty set and two auxiliary forward and two auxiliary backward PEFs are initiated using the input 2-D data. After the initial first-order stage, a set of the auxiliary *vertical forward/backward*, and a set of *horizontal forward/backward* PEFs are obtained. Then, these first-order four PEFs are combined linearly to calculate the auxiliary horizontal and vertical PEFs of successively higher order stages. After initial first order stage, in order to apply a horizontal augmentation, *one* past observation is added to the top of the initial PSR. This horizontal augmentation causes the increase of the auxiliary vertical forward/backward PEFs. All auxiliary horizontal and vertical existing and

new initial PEFs are updated by calculating a linear combination of the first-order stage auxiliary PEFs. At that point, in order to realize a vertical augmentation, *two* additional past observations along the vertical direction are added to the left of the PSR. The consequences of the vertical augmentation, a set of new auxiliary horizontal forward/backward PEFs are introduced.

It is possible to repeat horizontal and vertical augmentation until the desired order stage is reached. After any successive higher order stage, it is possible to define a QP or an ASHP or an NC half-plane models. In other words, after reaching the desired size of the rectangular PSR, all types of causal QP/ASHP and noncausal half-plane 2-D lattice models are obtained by predicting the appropriate PEF from the remaining ones.

In the thesis, it is shown that the proposed auxiliary horizontal and vertical backward prediction error filters inherently generates the mutually orthogonal realization vectors and thus are optimum for all stages. Moreover, it is shown that the entropy nearly preserved after each lattice stage. The information loss between the input data and the backward prediction error vectors is very minimal. The given examples in the thesis demonstrated how efficient the proposed method on modeling AR process, and finding sinusoidal peaks in the given data. Hence, this method can be applied on ISAR/SAR radar data. In addition, the 2-D spectrum estimation with the proposed method gives very satisfactory results.

GENEL ÖZBAĞLANIMLI MODELLEME İÇİN YENİ ETKİN 2-B KAFES YAPILARI

ÖZET

Yapılan çalışmalar sonucu, günümüzde bir-boyutlu kafes süzgeç teorisi çok iyi bilinmekte ve yaygın olarak geniş bir alanda uygulanmaktadır. İki-boyutlu kafes süzgeç teorisi bir-boyutlu kafes süzgeç teorisinin doğal bir uzantısı olarak geliştirilmiştir. İki-boyutlu alanları modellemek için yansıma katsayıları kullanımı yaklaşımı ilk olarak Marzetta tarafından öne sürülmüş ve bu yaklaşım temel alınarak, diğer araştırmacılar tarafından bir çok çalışma gerçekleştirilmiştir.

İki-boyutlu kafes süzgeç yapıları için bir sürü çalışma bulunmasına rağmen, bunların hiçbirisi nedensel ve nedensel olmayan iki-boyutlu kafes süzgeç yapılarını aynı anda modelleyebilmek için genelleştirilememiştir. Bu çalışmada, yeni bir iki-boyutlu kafes süzgeç yapısı önerilmektedir. Önerilen bu metod gayet basit ve iki-boyutlu rastlantı alanlarını modellemek için genelleştirilmiş bir yöntemdir. Bu yöntemle hem nedensel hem de nedensel olmayan iki-boyutlu kafes süzgeç yapıları aynı anda modellenmektedir. Her evre için kafes yansıma katsayıları ileri ve geri yöndeki öngörülen hata alanlarının toplamının ortalama karesel hata yöntemiyle minimize edilmesi ile bulunmaktadır.

Bir-boyutlu kafes modellemede, projeksiyon tabanını büyütme "ekleme işlemi" sadece bir geçmiş gözlem değerini mevcut projeksiyon tabanına eklemekle yapılabilmektedir. Her derece artımında her zaman bir adet ileri yönde ve bir adet geri yönde hata öngörü dizisi elde edilmektedir. Bizim önerdiğimiz iki-boyutlu kafes yapısında, dikdörtgensel bir öngörü destek bölgesi tanımlanmaktadır. Bu öngörü destek bölgesi bir-boyutlu kafes yapılarındaki projeksiyon tabanına eşdeğerdir. Ek olarak da, bir set yardımcı yatay ve dikey ileri/geri yönde öngörü hata alanı tanımlanmaktadır. Diğer bir deyişle, yeni önerilen yöntem ile, iki tip ileri ve iki tip de geri yönde öngörü hata alanı öne sürülmektedir. Böylece, ekleme işlemi, hem yatay hem de dikey yönde, geçmiş gözlem değerlerinden oluşan bir satır ya da sütun vektörünü mevcut öngörü destek bölgesine eklemek ile gerçekleştirilir. Eklenen satır veya sütun vektörüne göre ekleme işlemi adlandırılır. Ekleme işlemi istenen derece evresine ulaşana kadar tekrar edilebilir ve kafes derece evresi ekleme işleminin yönüne göre güncelleştirilir. Önerilen yöntemde, iki-boyutlu kafes derece evresi (p, q) ile gösterilir. Burada p yatay yönde ekleme işleminin derece evresini, q de dikey yönde yapılan ekleme işleminin derece evresini göstermektedir.

Başlangıçta, birinci-derece evrede, ileri ve geri yöndeki öngörülen hata alanları iki adet bir-boyutlu kafes yapıdan elde edilmekte, yani, iki-boyutlu kafes süzgeci, iki adet bir-boyutlu kafes süzgeçten oluşmuş gibi dikkate alınmaktadır. Böylece, bir-boyutlu durumdan farklı olarak, iki adet yardımcı ileri ve iki adet yardımcı geri yönde öngörü hata alanları elde edilmektedir. Sıfırıncı derece evresinde, öngörü destek alanı boş bir kümedir ve iki adet yardımcı ileri ve iki adet yardımcı geri yönde öngörü hata alanları

girdi olarak kullanılan iki-boyutlu veriye eşitlenirler,

$$e_1^{(0,0)} = r_1^{(0,0)} = y(k, l), \quad (1)$$

$$\tilde{e}_1^{(0,0)} = \tilde{r}_1^{(0,0)} = y(k, l), \quad (2)$$

burada e ve r harfleri ileri yönde dikey ve yatay öngörü hata alanlarını, $\tilde{e}$ ve $\tilde{r}$ harfleri de geri yönde dikey ve yatay öngörü hata alanlarını göstermektedir. Ek olarak, üst indis öngörü hata alanının derecesini ve alt indis de öngörü hata alanının nereden hesaplandığını göstermektedir.

Başlangıç evresinden sonra, bir set yardımcı "*dikey ileri/geri yönde*", $e_1^{(0,1)}$, $\tilde{e}_1^{(0,1)}$, ve bir set de "*yatay ileri/geri yönde*", $r_1^{(1,0)}$, $\tilde{r}_1^{(1,0)}$, öngörü hata alanları elde edilmektedir. Elde edilen bu birinci derece dört adet öngörü hata alanı, yardımcı yatay ve dikey öngörü hata alanlarının ardışıl olarak gelen daha yüksek derece evrelerini hesaplayabilmek için doğrusal olarak birleştirilir. Başlangıçtaki birinci-derece evre kafes bölümleri yaygın olarak bilinen formda aşağıdaki gibi verilir:

$$\begin{bmatrix} e_1^{(0,1)}(k, l) \\ \tilde{e}_1^{(0,1)}(k, l) \end{bmatrix} = \begin{bmatrix} \mathbf{1} & -k_{e_1}^{(0,1)*} \\ -k_{e_1}^{(0,1)} & \mathbf{1} \end{bmatrix} \begin{bmatrix} y(k, l) \\ y(k, l-1) \end{bmatrix} \quad (3a)$$

$$\begin{bmatrix} r_1^{(1,0)}(k, l) \\ \tilde{r}_1^{(1,0)}(k, l) \end{bmatrix} = \begin{bmatrix} \mathbf{1} & -k_{r_1}^{(1,0)*} \\ -k_{r_1}^{(1,0)} & \mathbf{1} \end{bmatrix} \begin{bmatrix} y(k, l) \\ y(k-1, l) \end{bmatrix} \quad (3b)$$

burada $k_{e_1}^{(0,1)}$ ve $k_{r_1}^{(1,0)}$ başlangıç birinci-derece evre yansıma katsayılarıdır. Bu yansıma katsayılarını hesaplayabilmek için, öngörü hata vektörleri (3a) ve (3b) ileri ve geri yönde öngörü hata alanlarının toplamalarının ortalama karesel hata yöntemi ile minimize edilmesiyle bulunur.

Başlangıçtaki birinci-derece evreden sonra, bir yatay ekleme işleme yapmak için, *bir* adet geçmiş gözlem değeri başlangıçtaki öngörü destek alanının tepesine eklenir. Bu yatay ekleme işlemi yardımcı dikey ileri/geri yöndeki öngörü hata alanlarının sayısında artışa neden olur. Bütün yardımcı yatay ve dikey önceden mevcut olan ve yeni ortaya çıkan öngörü hata alanları birinci-derece yardımcı öngörü hata alanlarının doğrusal olarak birleşiminden hesaplanarak güncellenir.

Benzer şekilde bu aşamada dikey ekleme yapmak için, *iki* adet geçmiş gözlem noktasından oluşan sütun vektörünü, öngörü destek bölgesinin soluna eklemek yeterli olacaktır. Dikey ekleme işleminin bir sonucu olarak da yeni bir set yatay ileri/geri yönde öngörü hata alanları ortaya çıkmaktadır.

Bir (p, q) -dereceli öngörü destek bölgesi $S_{k,l}^{(p,q)}$ durumunda, bir yatay ekleme işlemi yapmak için, q adet geçmiş gözlem değerlerinden oluşan bir yatay vektör bu öngörü destek bölgesinin tepesine eklenir. Yatay olarak ekleme işlemi şu şekilde de ifade edilebilir:

$$S_{k,l}^{(p+1,q)} = S_{k,l}^{(p,q)} \cup S_{k-p-1,l}^{(0,q)} \quad (4)$$

burada $S_{k-p-1,l}^{(0,q)}$ geçmiş gözlem değerlerinden oluşan bir satır vektörünü ifade etmektedir.

Yatay ekleme işleminin bir sonucu olarak ortaya çıkan yeni yardımcı dikey ileri ve geri yöndeki öngörü hata alanlarını hesaplayabilmek için, yeni iki adet yardımcı ileri ve iki

adet geri yönde öngörü hata alanı tanımlamamız gerekmektedir. Bu işlem de aşağıdaki gibi yapılır.

$$\begin{aligned} e_{p+1}^{(p,q)}(k,l) &= e_p^{(p,q)}(k-1,l) \\ \tilde{e}_{p+1}^{(p,q)}(k,l) &= \tilde{e}_p^{(p,q)}(k,l). \end{aligned} \quad (5)$$

$(p+1, q)$ -inci dereceden yatay ekleme işlemi yapılmış öngörü destek alanı PSR $S_{k,l}^{(p+1,q)}$, ın yardımcı yatay öngörü hata alanları daha düşük dereceli öngörü hata alanlarının doğrusal birleşimlerinden elde edilebilmektedir:

$$\begin{bmatrix} \mathbf{r}^{(p+1,q-1)}(k,l) \\ \tilde{\mathbf{r}}^{(p+1,q-1)}(k,l) \end{bmatrix} = \begin{bmatrix} \mathbf{I}_p & -\mathbf{K}_r^{(p+1,q-1)H} \\ -\mathbf{K}_r^{(p+1,q-1)T} & \mathbf{I}_p \end{bmatrix} \begin{bmatrix} \mathbf{r}^{(p,q-1)}(k,l) \\ \tilde{\mathbf{r}}^{(p,q-1)}(k-1,l) \end{bmatrix} \quad (6)$$

burada $\mathbf{K}_r^{(p+1,q-1)} = \left[\mathbf{k}_{r_1}^{(p+1,q-1)} \mathbf{k}_{r_2}^{(p+1,q-1)} \dots \mathbf{k}_{r_q}^{(p+1,q-1)} \right]^T$ yansıma katsayı matrisidir ve $i = 1, 2, \dots, q$ için $\mathbf{k}_{r_i}^{(p+1,q-1)} = \left[k_{r_i}^{(p+1,q-1)}(1) k_{r_i}^{(p+1,q-1)}(2) \dots k_{r_i}^{(p+1,q-1)}(q) \right]^T$ olarak tanımlanır. Yansıma katsayısı, yatay ileri/geri yöndeki öngörü hata alanlarının ortalama kareler toplamının yansıma katsayısı matrisine göre, ortalama karesel hata yöntemiyle minimize edilmesiyle bulunur.

Yeni oluşan ve mevcut olan yardımcı dikey ileri/geri yöndeki öngörü hata alanlarını ifade eden hata yayılım eşitlikleri tek bir yapısal gösterim altında birleştirilebilir,

$$\begin{aligned} \begin{bmatrix} \mathbf{e}^{(p,q)}(k,l) \\ \tilde{\mathbf{e}}^{(p,q)}(k,l) \end{bmatrix} &= \begin{bmatrix} \mathbf{e}^{(p-1,q)}(k,l) \\ e_p^{(p-1,q)}(k-1,l) \\ \tilde{\mathbf{e}}^{(p-1,q)}(k-1,l) \\ \tilde{e}_p^{(p-1,q)}(k,l) \end{bmatrix} \\ &+ \begin{bmatrix} \mathbf{O}_{(p+1) \times 2q} & \mathbf{O}_{p \times q} & -\hat{\mathbf{K}}_e^{(p,q)H} \\ -\hat{\mathbf{K}}_e^{(p,q)T} & \mathbf{O}_{p \times q} & -\mathbf{k}_{e_{p+1}}^{(p,q)*} \\ \mathbf{0}_q^T & -\mathbf{k}_{e_{p+1}}^{(p,q)T} & \mathbf{0}_q^T \\ \mathbf{O}_{(p+1) \times 2q} \end{bmatrix} \\ &\times \begin{bmatrix} \mathbf{r}^{(p,q-1)}(k,l) \\ \tilde{\mathbf{r}}^{(p,q-1)}(k,l) \\ \mathbf{r}^{(p,q-1)}(k,l-1) \\ \tilde{\mathbf{r}}^{(p,q-1)}(k,l-1) \end{bmatrix} \end{aligned} \quad (7)$$

burada $\hat{\mathbf{K}}_e^{(p,q)} = \left[\mathbf{k}_{e_1}^{(p,q)} \dots \mathbf{k}_{e_p}^{(p,q)} \right]^T$ ve $\mathbf{k}_{e_{p+1}}^{(p,q)}$ sırasıyla yansıma katsayı matrisini ve vektörünü göstermekte, $i = 1, 2, \dots, p+1$ için $\mathbf{k}_{e_i}^{(p,q)} = \left[k_{e_i}^{(p,q)}(1) \dots k_{e_i}^{(p,q)}(q) \right]$ olarak tanımlanmaktadır.

Dikey yansıma katsayı matrisi $\hat{\mathbf{K}}_e^{(p,q)}$ ve vektörü $\mathbf{k}_{e_{p+1}}^{(p,q)}$ dikey ileri/geri yönde öngörü hata alanlarının ortalama karesel hata yöntemi kullanılarak minimize edilmesiyle bulunur.

Benzer şekilde, dikey ekleme yapmak için, p adet geçmiş gözlem noktasından oluşan sütun vektörünü, öngörü destek alanının, $S_{k,l}^{(p,q)}$, soluna eklemek yeterli olacaktır.

Dikey ekleme işlemi şu şekilde ifade edilir:

$$S_{k,l}^{(p,q+1)} = S_{k,l}^{(p,q)} \cup S_{k,l-q-1}^{(p,0)} \quad (8)$$

burada $S_{k,l-q-1}^{(p,0)}$ geçmiş gözlem değerlerinden oluşan bir sütun vektörüdür.

Dikey ekleme işleminin bir sonucu olarak oluşan yeni yatay ileri/geri yöndeki öngörü hata alanlarını hesaplayabilmek için aşağıda ifade edildiği gibi yeni bir set yardımcı yatay ileri/geri yönde öngörü hata alanı tanımlamamız gerekmektedir.

$$\begin{aligned} r_{q+1}^{(p,q-1)}(k,l) &= r_q^{(p,q-1)}(k,l-1) \\ \tilde{r}_{q+1}^{(p,q-1)}(k,l) &= \tilde{r}_q^{(p,q-1)}(k,l). \end{aligned} \quad (9)$$

$(p-1, q+1)$ -inci dereceden dikey öngörü hata alanları aşağıdaki denklemde de gösterildiği gibi daha düşük dereceli öngörü hata alanlarının doğrusal birleşimlerinden elde edilmektedir.

$$\begin{bmatrix} \mathbf{e}^{(p-1,q+1)}(k,l) \\ \tilde{\mathbf{e}}^{(p-1,q+1)}(k,l) \end{bmatrix} = \begin{bmatrix} \mathbf{I}_p & -\mathbf{K}_e^{(p-1,q+1)H} \\ -\mathbf{K}_e^{(p-1,q+1)T} & \mathbf{I}_p \end{bmatrix} \begin{bmatrix} \mathbf{e}^{(p-1,q)}(k,l) \\ \tilde{\mathbf{e}}^{(p-1,q)}(k,l-1) \end{bmatrix} \quad (10)$$

burada $\mathbf{K}_e^{(p-1,q+1)} = \begin{bmatrix} \mathbf{k}_{e_1}^{(p-1,q+1)} & \mathbf{k}_{e_2}^{(p-1,q+1)} & \dots & \mathbf{k}_{e_p}^{(p-1,q+1)} \end{bmatrix}^T$ yansıma katsayı matrisini göstermekte, $i = 1, 2, \dots, p$ için $\mathbf{k}_{e_i}^{(p-1,q+1)} = \begin{bmatrix} k_{e_i}^{(p-1,q+1)}(1) & k_{e_i}^{(p-1,q+1)}(2) & \dots & k_{e_i}^{(p-1,q+1)}(p) \end{bmatrix}^T$ olarak tanımlanmaktadır. Bu yansıma katsayısı da dikey ileri/geri yöndeki öngörü hata alanlarının ortalama karesel hata yöntemi doğrultusunda toplamının minimize edilmesiyle hesaplanmaktadır.

Yardımcı yatay ileri/geri yöndeki öngörü hata alanlarını veren hata yayılım eşitlikleri de aşağıdaki gibi tek bir yapısal denklem altında gösterilebilir.

$$\begin{aligned} \begin{bmatrix} \mathbf{r}^{(p,q)}(k,l) \\ \tilde{\mathbf{r}}^{(p,q)}(k,l) \end{bmatrix} &= \begin{bmatrix} \mathbf{r}^{(p,q-1)}(k,l) \\ r_q^{(p,q-1)}(k,l-1) \\ \tilde{\mathbf{r}}^{(p,q-1)}(k,l-1) \\ \tilde{r}_q^{(p,q-1)}(k,l) \end{bmatrix} \\ &+ \begin{bmatrix} \mathbf{O}_{(q+1) \times 2p} & \mathbf{O}_{q \times p} & -\hat{\mathbf{K}}_r^{(p,q)H} \\ -\hat{\mathbf{K}}_r^{(p,q)T} & \mathbf{O}_{q \times p} & -\mathbf{k}_{r_{q+1}}^{(p,q)*} \\ \mathbf{0}_p^T & -\mathbf{k}_{r_{q+1}}^{(p,q)T} & \mathbf{0}_p^T \\ \mathbf{O}_{(q+1) \times 2p} & \mathbf{O}_{q \times p} & \mathbf{0}_p^T \end{bmatrix} \\ &\times \begin{bmatrix} \mathbf{e}^{(p-1,q)}(k,l) \\ \tilde{\mathbf{e}}^{(p-1,q)}(k,l) \\ \mathbf{e}^{(p-1,q)}(k-1,l) \\ \tilde{\mathbf{e}}^{(p-1,q)}(k-1,l) \end{bmatrix} \quad (11) \end{aligned}$$

burada $\hat{\mathbf{K}}_r^{(p,q)} = \begin{bmatrix} \mathbf{k}_{r_1}^{(p,q)} & \dots & \mathbf{k}_{r_q}^{(p,q)} \end{bmatrix}^T$ ve $\mathbf{k}_{r_{q+1}}^{(p,q)}$ yansıma katsayı matrisi ve vektörünü oluşturmakta ve $i = 1, 2, \dots, q+1$ için $\mathbf{k}_{r_i}^{(p,q)} = \begin{bmatrix} k_{r_i}^{(p,q)}(1) & \dots & k_{r_i}^{(p,q)}(p) \end{bmatrix}$ olarak

tanımlanmaktadır. Yatay yansıma katsayı matrisi de yardımcı yatay ileri/geri yönde öngörü hata alanlarının ortalama karesel hata yöntemi doğrultusunda toplamalarının minimize edilmesiyle bulunur.

İstenilen derece evresine varana kadar yatay ve dikey ekleme işlemi tekrarlanabilir. Herhangi bir yüksek derece evresinde, bir nedensel çeyrek düzlem (Quarter Plain) veya asimetrik yarı düzlem (Asymmetric Half-Plane) ya da bir nedensel olmayan yarı düzlem (Noncausal Half-Plane) kafes modeli elde edilebilir. Diğer bir deyişle, istenilen büyüklükteki bir dikdörtgensel öngörü destek bölgesine erişildiğinde, nedensel "Quarter Plane/Asymmetric Half-Plane" ve nedensel olmayan "Noncausal Half-Plane" 2-D kafes modellerinin her çeşidini uygun öngörü hata alanını diğer geri kalanlarından kestirerek elde etmek mümkündür.

Bu test çalışmasında ayrıca, önerilen yardımcı yatay ve dikey geri yönde öngörü hata süzgeçlerinin doğal olarak karşılıklı birbirine dik vektörler ürettiğini gösterdik ve bu nedenle de bütün evrelerde bu öngörü hata alanlarının optimum olduğunu söyleyebiliriz. Ek olarak, yapılan bu çalışmada her ardışıl kafes evresinde entropinin yaklaşık olarak korunduğunu yani girdi olarak verilen 2-D verisi ile geri yönde öngörü hata vektörleri arasındaki bilgi kaybının minimum olduğunu gözlemledik. Bu çalışmada verilen örnekler, önerilen teoriyi destekleyici sonuçlar vermiştir. AR proses modellemede başarılı sonuçlar elde edilmiştir. Önerilen bu yöntem ile yapılan 2-D spektrum kestirimi de tatmin edici sonuçlar ortaya koymuş ve sinusoidal tepe noktalarının bulunması uygulamalarında klasik FFT yönteminden çok daha başarılı olduğu gözlemlenmiştir. Bu yöntemin ISAR/SAR radar görüntüleri üzerinde de başarı ile uygulanabileceğini düşünmekteyiz. Bu konu ayrı bir çalışma olarak ele alınacaktır. Önerilen bu yöntemin ARMA proses modellemede de kullanılabileceğini düşünmekteyiz. Ek olarak, bu yöntem kullanarak istenilen süzgeç tipine göre uygun yerlere sinusoidler yerleştirilerek 2-D IIR süzgeç tasarımı yapmak da olası gibi gözükmektedir. Bu konu da ayrı bir çalışma olarak ele alınacaktır.

1. INTRODUCTION

The topic of multidimensional signal processing has attracted considerable attention since the beginning of the last quarter of the twentieth century. Nowadays, the interest on this subject still increase rapidly due to being a fast growing and a very challenging field. Specifically, the interest has been directed into the area of two-dimensional (2-D) digital signal processing because of the following main reasons:

- High efficiency due to high speed computations (parallel processing capability)
- Permitting better image processing and analysis
- Great application flexibility and adaptivity
- Less software and hardware cost due to evolution of computers

The theoretical developments in this subject have been expedited by diverse applications spanning areas such as image processing, robotics, target tracking, and so on. The main important operations performed by 2-D digital processing are:

- Two-dimensional digital filtering
- Two-dimensional digital transformations
- Local space processing
- Data compression
- Pattern recognition

Most of the researchers have shown particular interest in 2-D filters, both recursive and non-recursive. There is a big concentration on developing a theory of 2-D lattice filters in order to find an efficient method for modeling 2-D random fields. However, the modeling of 2-D signals with the 2-D lattice filter is not sufficient yet; and until now,

there is no general method to model all kinds of 2-D causal and noncausal filters. The existing 2-D causal filters have either a quarter-plane (QP) or asymmetric half-plane (ASHP) or a noncausal (NC) half-plane support.

In this thesis, the study has been done in the area of 2-D lattice filters and their usage in modeling of 2-D autoregressive (AR) fields and in 2-D spectral estimation.

2-D lattice filter modeling is widely studied and a novel general method is proposed to obtain all possible causal and noncausal filter models. The results are verified by using the several nontrivial examples.

In the following section, a comprehensive survey of multidimensional signal processing and 2-D lattice filters will be presented, and then the purpose of thesis will be explained. Finally, the outline of the thesis will be given in the last section of this chapter.

1.1 Literature Review

Digital signal processing, especially 2-D signal processing, has become a very practical and convenient method for handling signals since digital computational capability is growing at a tremendous rate, with concomitant increase in parallel processing capability.

The application of multidimensional signal processing was initially performed in the area of geophysics to process seismic, gravitational and magnetic data [1]. Later on, studies continued and the multidimensional signal processing has been developed rapidly and applied in various fields [2]. The studies on image processing and parametric modeling of the signal are very widespread approach. Generally, studies are concentrated on 2-D signal processing because of the development of integrated computer technology and 2-D signal processing is widely used in areas such as biomedical engineering and biomedicine, nuclear physics, space imagery, seismic data processing, geophysics. In addition, 2-D digital systems are being used increasingly to replace analogue systems in the important areas such as facsimile, television, sonar, radar, biomedicine, remote sensing, underwater acoustics, moving object recognitions, robotics, etc.

The theory of 1-D lattice filters is well developed and is widely used in the areas such as spectral estimation, maximum entropy signal modeling for spectral analysis, speech synthesis/analysis, system modeling, noise cancellation, channel equalization, system identification [3–10]. In particular, 1-D lattice filter has become a universal method for analysis and synthesis of speech. The lattice-form algorithms are simple to implement, exhibit excellent numerical behavior due to having possibility to estimate coefficients by using a recursive technique. Furthermore, the lattice-form algorithms are very attractive candidates for VLSI implementation owing to having a modular structure.

The lattice approach mainly based on the property of orthogonality that allows the filter to be updated in order, without recalculation of the previous lower order filter coefficients. During the calculation of filter coefficients, due to the orthogonality, only necessary and non-redundant part of information is taken into account for linear prediction. Orthogonality leads to a nice physical structure: Increasing the order of the filter requires adding only one extra module, leaving all other previous modules unchanged. This similar structure of the lattice filter stages makes the filter suitable for efficient hardware or software implementations. Briefly, it can be said the lattice owes its robust numerical behavior to the property of orthogonality [4].

Because of successful application of 1-D lattice filters in various areas, many researchers tried to develop theories for 2-D lattice filters. The theories of 2-D lattice filters are developed as a natural extension of 1-D lattice filter. However, all proposed formulations are capable of implementing only a restricted class of transfer functions and the modeling of 2-D signals with 2-D lattice filter is not efficient enough.

In the literature, the basic fundamental approach to the modeling of 2-D fields by the reflection coefficients was made by Marzetta [11], who developed a class of 2-D minimum phase digital filters involving a sequence of parameters called reflection coefficients by extending some results of 1-D linear prediction to the 2-D case. A half-plane support which is infinite in one of the two dimensions is proposed by Marzetta. Although this approach maintains several of the nice characteristics of 1-D lattices, such as correlation matching and produces a minimum phase filter, it leads to very long delay filters [2]. Marzetta's algorithm has been successfully applied to the 2-D recursive filter design [12] and to linear predictive coding of images [13].

Therrien [14] proposed that there is a relation between the multichannel 1-D AR model and the single-channel 2-D models with quadrant support. He showed that simultaneous computation of parameters of all four quarter-plane filters is possible. This method is generalized to multichannel 2-D models and applied to the problem of estimation of the 2-D auto-spectral and cross-spectral components [15, 16].

Parker and Kayran [17] have introduced a new 2-D lattice filter notion which is known as three-parameter lattice filter, as it introduces only three reflection coefficients at each stage order. Therefore, this model is not suitable for modeling all classes of autoregressive (AR) fields. Only a subset of AR filters of the same order can be represented by using this model. The most important thing is that it lacks the property of orthogonality which is an important characteristic of the lattice structures. Hence, cascading stages may not lead to an optimum filter [2]. If the given data have a certain statistic, a satisfactory result can be obtained only for the first order fields. However, its simplicity makes it attractive and it has been applied to many fields such as spectral estimation, coding, data compression. In the literature, some good results have been reported by using this approach [18–30].

Soon after the three-parameter lattice filter definition, Parker *et al.* [31] have generalized the three-parameter filter to the asymmetric half-plane case. This filter introduces six reflection coefficients calculated from five prediction error fields.

Later, further improvements on extensions the lattice filter from the 2-D to 3-D have been performed by Nam and O'Neill [27], and Kwan and Lui [23, 32].

Afterwards, Ertüzün *et al.* [33] have improved the three-parameter filter and obtained two new structures by taking into account the quadratic growth of the data support. Both of the new structures are quarter-plane filters and the improvement is introduced after the first stage of the three-parameter filter by introducing auxiliary backward prediction error fields. These new filters are called as the extended lattice filter of diagonal form [34–36] and the extended lattice filter of perpendicular form [34, 37, 38]. Ertüzün showed that these proposed methods approximate the maximum entropy more closely compared to the three-parameter filter. It is shown that the increase in entropy naturally leads to a more reliable and better modeling of AR data fields [34]. The structure proposed by Ertüzün can exactly model first and second order AR data

models, but if the order of the AR data fields increases, two additional backward prediction error fields are not enough to introduce sufficient parameters to model all the data points in the support region.

Meanwhile, Hsieh *et al.* [26] have extended the order update 2-D lattice filter model proposed by Parker and Kayran [17] into order update and time update recursion for more efficient computation. They have used 2-D lattice filtering method to extract the prediction parameters of the 2-D linear predictor. It is shown that it is possible to improve the visual quality of reconstructed images.

Kayran [39] has presented a complete and orthogonal 2-D lattice structure for the AR modeling of random fields. The method offers a complete solution for the Levinson type algorithm to compute the prediction error filter coefficients using lattice parameters from the given 2-D augmented normal equation without requiring matrix inversion. 2-D equivalent of Burg's matrix formulation approach is used to derive the Levinson type recursions. The method can be used for any shape and size of prediction support region.

In order to eliminate the missing coefficient problem in case of increased order, Kayran and Parker [40] have introduced a new lattice structure with a complete set of reflection coefficients. In order to obtain growing number of 2-D reflection coefficients at the successive stages, they introduce four auxiliary forward and four auxiliary backward prediction errors after the first stage. Later, Kayran and Küçük [41] presented "2-D Schur recursion" method and applied this method to the 2-D quarter-plane filters for AR modeling. Their method is based on the quarter-plane lattice structure of Parker and Kayran [17]. They presented a quarter-plane auxiliary gapped functions in order to obtain the growing number of reflection coefficients at each stage.

Later on, Kayran and Erer [42] introduced a new lattice structure for asymmetric half-plane 2-D AR modeling. They presented a novel half-plane auxiliary prediction error filter to obtain the growing number of reflection coefficients of the proposed ASHP lattice configuration. In order to obtain an orthogonal ASHP lattice structure, four half-plane prediction error fields at each lattice stage are employed.

Similar to 1-D lattice models, the theory of 2-D lattice filters is used in the areas of cascade filter synthesis for implementation in dedicated VLSI hardware [18, 19], 2-D

filter design [20,21,28], multidimensional spectral estimation [14–16,20,22,23,39,43], image data compression [24, 25, 32], image processing [23–27, 44], noise cancellation [45], adaptive predictive image coding [23, 26, 27], and texture analysis [46–48].

In addition, other potential applications of 2-D lattice filters have been seen in areas such as high-resolution ISAR radar imaging [42, 49–51] and breast cancer detection and classification [52,53].

1.2 Purpose of Thesis

In the literature, existing 2-D lattice filter structures are complex and none of them is a generalized structure for representing all possible causal and noncausal 2-D models. Most of the existing 2-D lattice structures have been investigated for a QP support [14, 17–19, 35, 36, 39–41, 54–56]. On the other hand, ASHP lattice structures have been proposed in a very few studies [11, 12, 22, 39, 42, 57]. It has been shown that ASHP model yields a good power spectrum density compared to the QP models even if it has an infinite region of support. Furthermore, it is shown that the harmonic mean of a combined set of 2-D causal and noncausal prediction models improves the spectral estimation accuracy and the shape of the spectrum [58]. Consequently, it is clear that there is a need for 2-D lattice structures that generate causal and noncausal models concurrently. Moreover, the information loss between the 2-D input data and the backward prediction error vectors will be studied.

The main objective of this thesis is to propose a new and simple method to construct a low complexity and a general purpose 2-D lattice structure for modeling random fields. The proposed 2-D lattice filter has a rectangular shaped prediction support region and it can be used to obtain all possible causal and noncausal 2-D models simultaneously.

In this thesis, a new efficient 2-D lattice filter structures for AR modeling of random fields will be introduced, then a structural and mathematical explanations will be given. It will be shown how such a solution can be obtained by employing the auxiliary vertical and horizontal prediction error fields (PEFs). It is presented that these auxiliary PEFs are obtained by properly ordering the elements of the 2-D prediction support region (PSR). The procedure proposed, unlike all the other configurations [5, 11, 17, 18, 34, 39, 40, 42, 59] , recursively computes all the auxiliary vertical

and horizontal PEFs. The forward and the backward auxiliary PEFs are used to simultaneously obtain all possible causal QP/ASHP and NC half-plane 2-D models for the rectangular-shaped PSR. Since we have an increased number of reflection coefficients depending on the auxiliary PEFs at every stage, it is possible to model all AR fields of higher order as one of the QP, ASHP or NC half-plane filter.

The determination of 2-D transfer function from the lattice coefficients will be established and the stability of the synthesis lattice model will be examined. In addition, spectrum estimation will be investigated on the synthesis lattice model.

The advantages and disadvantages of this new theory will be discussed. The proposed theory will be applied to an 2-D AR fields and the lattice reflection coefficients will be estimated. In numerical examples, 2-D AR field will be used an input signal, and it will be modeled by using the proposed method. The relationships between the lattice reflection parameters and the coefficients of the given 2-D AR data to be modeled will be presented. In addition, this new proposed model will be used to find the sinusoid peaks in the 2-D data generated by sampling some sinusoids buried in white Gaussian noise. Moreover, this method will be used to find the correct spectrum where the normal 2-D FFT failed, especially for the 2-D data field generated as the sum of 2-D complex sinusoids corrupted by additive complex Gaussian noise. In order to confirm the theory, some examples will be realized and simulation results will be given. Some mathematical details will be introduced in the appendices.

1.3 Outline of Thesis

This thesis is organized as follows: A general introduction, literature overview and the purpose of the thesis are given in Chapter 1.

A new 2-D lattice structure model theory is presented in Chapter 2. In this chapter, auxiliary horizontal and vertical forward/backward PEFs are introduced. Structural and mathematical explanations are given in this chapter. Generation methods of all QP, ASHP, and NC half-plane lattice model types employing the auxiliary horizontal and vertical PEFs are explained in Chapter 3. Then in Chapter 4, how to establish the transfer functions from the lattice model parameters is given. Method of the calculation of the transfer functions of all horizontal and vertical PEFs, and all QP, ASHP and NC

half-plane lattice models are explained. In Chapter 5, in order to show the validity and the efficiency of the proposed method, some numerical examples are given. The orthogonality feature of the proposed method is discussed in Chapter 6. In addition, entropies of the input data and the backward prediction error vectors are investigated. Finally, in Chapter 7, the conclusions and the concluding remarks are given.

2. AUXILIARY HORIZONTAL AND VERTICAL PEFs FOR A RECTANGULAR PSR

2.1 Purpose

In this chapter, the novel proposed 2-D lattice structure for 2-D AR random fields will be presented in detail. In the following section, a 1-D lattice structure will be over-viewed and in the next section, some necessary definitions will be given. Herein, the concept of PSR and auxiliary vertical/horizontal PEFs would be explained. Then, a new 2-D lattice structure will be introduced and augmentation methods of the model will be clearly explained. Finally, the method of for the calculation of reflection coefficients will be explained.

2.2 1-D Lattice Structure Overview

The 1-D lattice structures have been developed since last quarter of the twentieth century. The success of 1-D lattice structure modeling diverted the researchers to find 2-D lattice models for random fields. The theory of 1-D lattice structure is well-developed and widely used in numerous applications in system identification and control, spectral estimation, frequency tracking and line enhancement, channel equalization, noise cancellation, and speech analysis and synthesis [3–10]. However, there have been some approaches to the 2-D lattice filtering and its use in 2-D spectral estimation [11, 17, 31, 34, 36–38]. In spite of a small number of approaches to the 2-D lattice structure, there exists number of applications of 2-D lattice filters which are mentioned in the first chapter.

The topic of 1-D lattice structure will not be explained here in detail since the properties of 1-D lattice model, various adaptive techniques for calculating the lattice reflection coefficients, and the related synthesis model for reconstructing the signals are well covered in the literature [5–10]. Hereafter, only a brief introduction the related

to the 1-D will be given. In the literature, the lattice structure is very preferable and widely used because of the following reasons:

- It is the most efficient structure to generate simultaneously forward and backward prediction errors by using orthogonality property.
- The lattice structure is modular since increasing the order of the lattice filter requires adding just only one extra module and keeping all other modules untouched.
- The similar structure of the lattice filter stages makes it very attractive and suitable candidate for VLSI implementation.

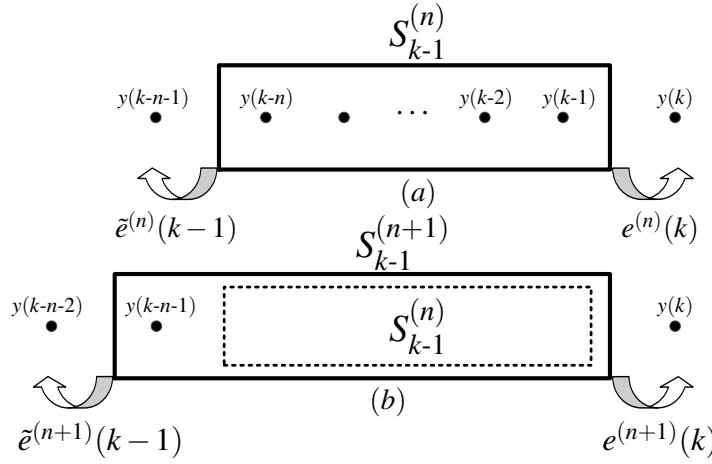


Figure 2.1: The augmentation of the PSR 1-D lattice stages (a) n th order case; (b) $(n+1)$ th order case.

In the 1-D lattice modeling, it is possible to calculate forward and backward PESs for the higher order models by recursively incrementing the PSR order. This PSR order incrementation is realized by adding only one more past observation point at each stage order. This recursive order incrementation of the PSR is demonstrated in Figure 2.1. Assuming that we have data sequence $\{y(k-n), \dots, y(k-1)\}$ forming a projection basis as $S_{k-1}^{(n)} = \text{span}\{y(k-1), \dots, y(k-n)\}$ that denotes the PSR for the n th order forward and backward PESs, $e^{(n)}(k)$ and $e^{(n)}(k-1)$, respectively. This projection basis is used to calculate forward prediction of $y(k)$, and the backward prediction of $y(k-n-1)$ in the mean-square error sense by using the related forward and backward PESs, $e^{(n)}(k)$ and $e^{(n)}(k-1)$, respectively. For the $(n+1)$ th order lattice model stage

PESs can be obtained [60] from the n th order ones as follows:

$$e^{(n+1)}(k) = e^{(n)}(k) + \Gamma^{(n+1)}\tilde{e}^{(n)}(k-1) \quad (2.1a)$$

$$\tilde{e}^{(n+1)}(k) = \tilde{e}^{(n)}(k-1) + \tilde{\Gamma}^{(n+1)}e^{(n)}(k) \quad (2.1b)$$

where $e^{(n+1)}(k)$ and $\tilde{e}^{(n+1)}(k)$ are the forward and backward prediction error fields at the stage order $(n+1)$ that are calculated from the n th order prediction error fields; $\Gamma^{(n+1)}$ and $\tilde{\Gamma}^{(n+1)}$ are the forward and the backward lattice reflection coefficients, respectively. It is also very important to determine that the initial conditions should be defined as $e^{(0)}(k) = \tilde{e}^{(0)}(k) = y(k)$, for a 1-D lattice filter, where $y(k)$ is the given input data.

In order to calculate the lattice reflection coefficients at each stage, the equation (2.1) can be derived by minimizing the forward and backward prediction errors with respect to the lattice reflection coefficients in the mean-square sense. As a result, the following equations are obtained:

$$\Gamma^{(n+1)} = \frac{E[e^{(n)}(k)\tilde{e}^{(n)}(k-1)]}{E[\tilde{e}^{(n)2}(k-1)]} \quad (2.2a)$$

$$\tilde{\Gamma}^{(n+1)} = \frac{E[e^{(n)}(k)\tilde{e}^{(n)}(k-1)]}{E[e^{(n)2}(k)]} \quad (2.2b)$$

where $E[\cdot]$ denotes the expected value.

We know that in order to formulate the forward and the backward PESs for the stage order (n) , the model order incrementation needs just only one more data of the past at the stage order $(n-1)$. As illustrated in Figure 2.1, $e^{(n)}(k)$ is directly predicted from the projected data, and similarly $\tilde{e}^{(n)}(k-1)$ can be predicted as well. At that moment, in order to calculate the forward PES $e^{(n+1)}(k)$ of the $(n+1)$ th order PSR $S_{k-1}^{(n+1)}$, the backward PES of the orthogonal projection basis of the n th order PSR $S_{k-1}^{(n)}$ is used. As seen that the propagation on the given data is autoregressively realized using only *past* values and the PSR of $e^{(n+1)}(k)$ and $\tilde{e}^{(n+1)}(k)$ is $S_{k-1}^{(n+1)} = S_{k-1}^{(n)} \cup \{y(k-n-1)\}$. Thus, in the 1-D lattice structure, there are only two PESs, -one forward and one backward-, and recursive order incrementation of the PSR can be realized only in one direction towards the past observations. Indeed, for the n th stage order, the predictions of $y(k)$ and $y(k-n-1)$ onto the same subspace $S_{k-1}^{(n)}$ produce the forward and backward PESs, respectively. When stationary autocorrelation sequence

of the random process is assumed, the derivation of the Levinson algorithm can be obtained [60].

Consequently, we can see that equation (2.1) and equation (2.2) are self-sufficient set of equations for generating prediction error sequences and lattice reflection coefficients iteratively. The Figure 2.2 shows a single-stage lattice filter.

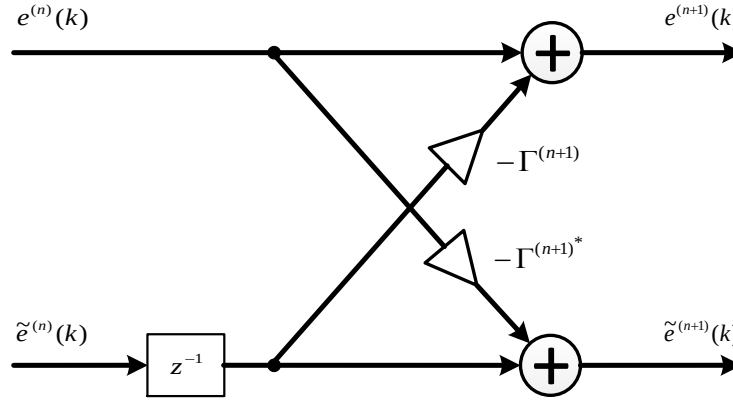


Figure 2.2: Single stage lattice filter from the (n) th order case to the $(n + 1)$ th order case.

It can be said that the essence of the lattice filter for 1-D is to structure an orthogonal basis property so that the update of the reflection coefficients with the order increase is just one addition to the previous calculation. Briefly, 1-D lattice structure is represented by using 1-D prediction model by a linear combination of the past values, named as a *forward prediction*, and future values, named as a *backward prediction*. The lattice reflection coefficients can be calculated by using the minimum mean-square error method.

2.3 Theory of the Proposed 2-D Lattice Structure

In the 1-D lattice structure, line-like PSR is used for linear prediction and recursive order incrementation is possible only along one direction. In the proposed 2-D lattice structure we define a rectangular PSR with a set of auxiliary horizontal and vertical forward/backward PEFs. This newly defined rectangular PSR is used to make prediction for obtaining all horizontal and vertical PEFs. Linear combinations of these PEFs are used for obtaining new PEFs and updating the previous ones after any

augmentation process on vertical or horizontal direction. The stage order of PSR is represented as (p, q) where p is used for the stage order of horizontal augmentation and q is used for the stage order of the vertical augmentation. Thus, for the (p, q) th stage order, the rectangular PSR can be defined as

$$S_{k,l}^{(p,q)} = \text{Span}\{y(k-i, l-j), 0 \leq i \leq p, 0 \leq j \leq q\}. \quad (2.3)$$

In order to define auxiliary vertical and horizontal forward/backward PEFs, we will use the 1-D lattice augmentation method for each direction separately. As seen that in the Figure 2.1, in order to predict $y(k)$, the PSR $S_{k-1}^{(n-1)}$ is used as an orthogonal projection basis.

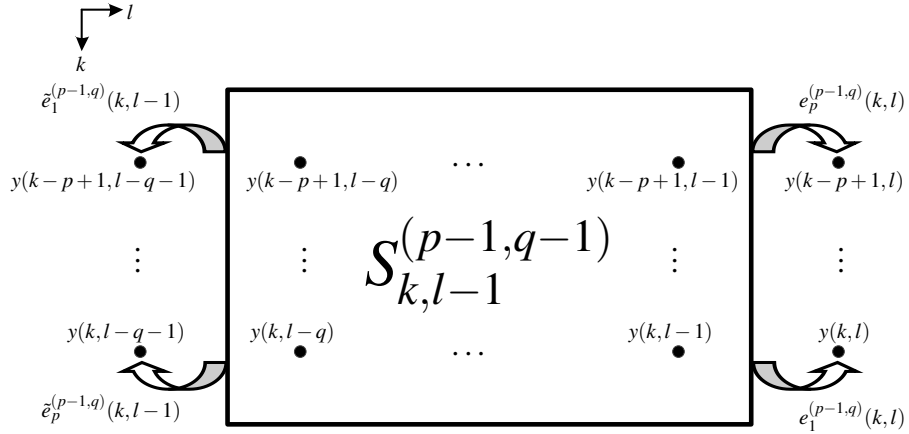


Figure 2.3: Definition of the auxiliary vertical forward/backward PEFs for the rectangular PSR $S_{k,l}^{(p-1,q-1)}$.

Assume that we have a rectangular $(p-1, q-1)$ th stage order PSR $S_{k,l}^{(p-1,q-1)}$. Similar to the 1-D case, we will use the PSR $S_{k,l-1}^{(p-1,q-1)}$ as an orthogonal projection basis in order to define auxiliary vertical forward/backward PEFs as illustrated in Figure 2.3. The PSR $S_{k,l-1}^{(p-1,q-1)}$ can be written as

$$S_{k,l-1}^{(p-1,q-1)} = \text{Span}\{y(k-i, l-j), 0 \leq i \leq p-1, 1 \leq j \leq q\}. \quad (2.4)$$

The prediction of the neighboring points on the right of the PSR $S_{k,l-1}^{(p-1,q-1)}$, from bottom to top, $y(k, l), \dots, y(k-p+1, l)$ will give the auxiliary vertical forward PEFs, $e_1^{(p-1,q)}(k, l), \dots, e_p^{(p-1,q)}(k, l)$, respectively. Similarly, the prediction of the neighboring p points on the left of the PSR $S_{k,l-1}^{(p-1,q-1)}$ from top to bottom, $y(k-p+1, l-q-1), \dots, y(k, l-q-1)$ will be called as the auxiliary vertical backward PEFs, namely, $\tilde{e}_1^{(p-1,q)}(k, l-1), \dots, \tilde{e}_p^{(p-1,q)}(k, l-1)$, respectively.

Similarly, in order to define auxiliary horizontal forward/backward PEFs, the PSR $S_{k-1,l}^{(p-1,q-1)}$ is used as an orthogonal projection basis as shown in Figure 2.4. The PSR $S_{k-1,l}^{(p-1,q-1)}$ can be given as

$$S_{k-1,l}^{(p-1,q-1)} = \text{Span}\{y(k-i, l-j), 1 \leq i \leq p, 0 \leq j \leq q-1\}. \quad (2.5)$$

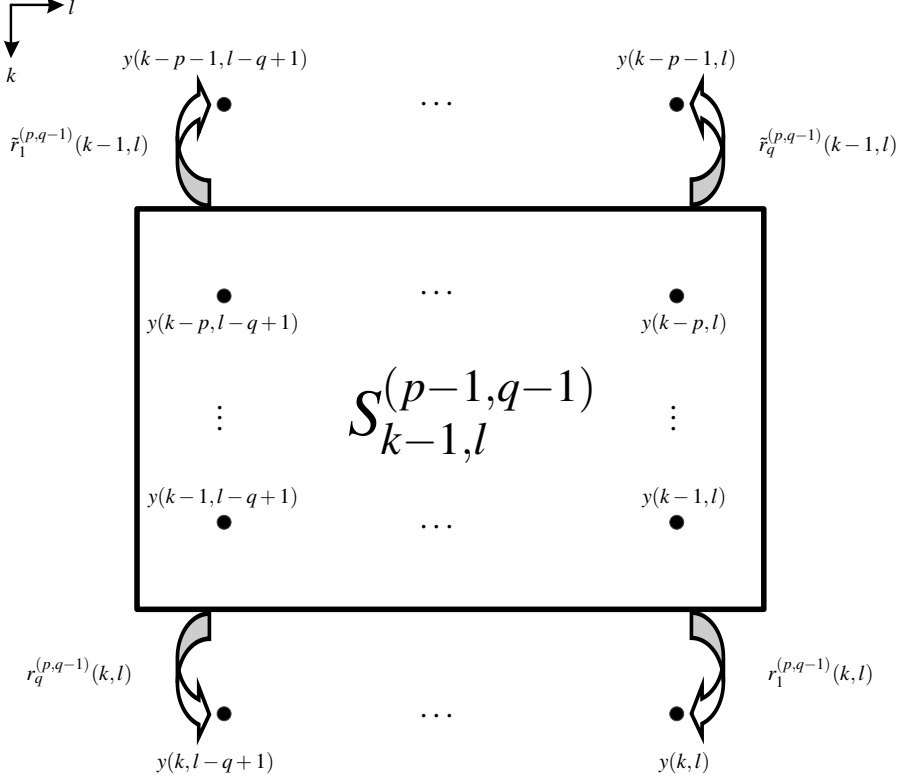


Figure 2.4: Definition of the auxiliary horizontal forward/backward PEFs for the rectangular PSR $S_{k,l}^{(p-1,q-1)}$.

The prediction of the neighboring q points at the bottom of the PSR, from right to left, $y(k, l), \dots, y(k, l-q+1)$ will also give the auxiliary horizontal forward PEFs, $r_1^{(p,q-1)}(k, l), \dots, r_q^{(p,q-1)}(k, l)$, respectively.

The corresponding auxiliary horizontal backward PEFs are defined as the prediction of the neighboring points at the top of this rectangular PSR, from left to right, $y(k-p-1, l-q+1), \dots, y(k-p-1, l)$ will be called as the auxiliary horizontal backward PEFs, $\tilde{r}_1^{(p,q-1)}(k-1, l), \dots, \tilde{r}_q^{(p,q-1)}(k-1, l)$, respectively.

After defining the related auxiliary vertical and horizontal PEFs for the PSRs, $S_{k,l-1}^{(p-1,q-1)}$ and $S_{k-1,l}^{(p-1,q-1)}$, they can be given in the vector form as follows:

$$\begin{aligned} \mathbf{e}^{(p-1,q)}(k, l) &= [e_1^{(p-1,q)}(k, l) \dots e_p^{(p-1,q)}(k, l)]^T, \\ \mathbf{r}^{(p,q-1)}(k, l) &= [r_1^{(p,q-1)}(k, l) \dots r_q^{(p,q-1)}(k, l)]^T \end{aligned} \quad (2.6)$$

and the auxiliary backward PEFs are $\tilde{\mathbf{e}}^{(p-1,q)}(k,l)$ and $\tilde{\mathbf{r}}^{(p,q-1)}(k,l)$ with the same sizes of the corresponding forward ones.

Consequently, it is possible to define totally $2(p+q)$ auxiliary forward and backward PEFs. These auxiliary vertical and horizontal PEFs are used during the augmentation process. Each augmentation makes the PSR grow in opposite of the augmentation direction. For example, in case of vertical augmentation, PSR grows on the horizontal direction. Similarly, horizontal augmentation case, the PSR growth occurs on the vertical direction.

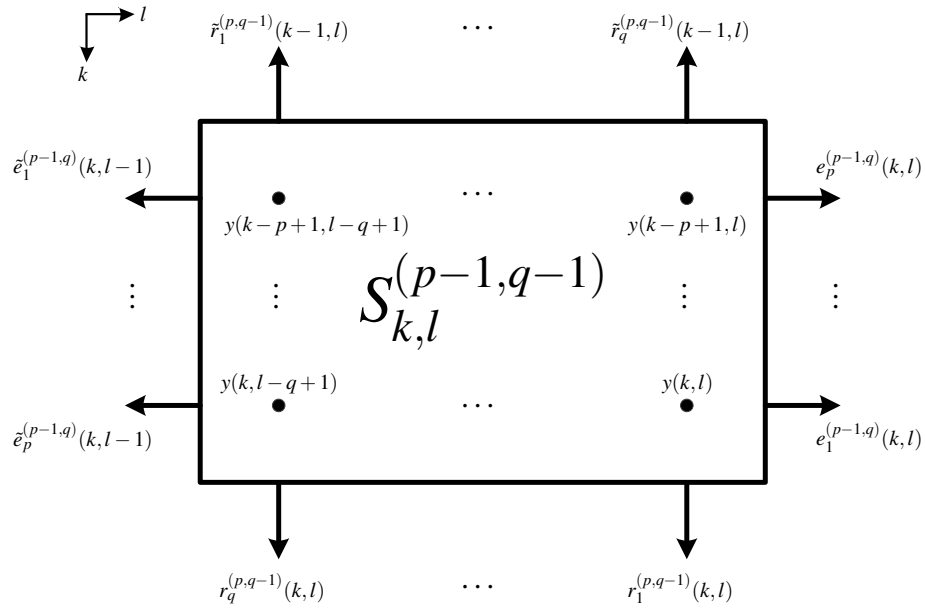


Figure 2.5: Definition of the auxiliary vertical and horizontal forward/backward PEFs for the rectangular PSR $S_{k,l}^{(p-1,q-1)}$.

In order to show the PSRs $S_{k,l-1}^{(p-1,q-1)}$ and $S_{k-1,l}^{(p-1,q-1)}$, and corresponding horizontal and vertical forward/backward PEFs on the PSR $S_{k,l}^{(p-1,q-1)}$, the illustration is given in Figure 2.5. Actually this representation is the visual combination of the definition of auxiliary vertical and horizontal forward/backward PEFs. The illustration of the PSR and its auxiliary vertical and horizontal forward/backward PEFs will be used in the following chapters because of its simplicity. In this representation, backward PEFs are shown as calculated from the PSR $S_{k,l}^{(p-1,q-1)}$, in order to produce $\tilde{\mathbf{e}}^{(p-1,q)}(k,l)$ and $\tilde{\mathbf{r}}^{(p,q-1)}(k,l)$. This calculation method and orthogonal projection of the auxiliary vertical and horizontal forward/backward PEFs are explained in the following subsections in detail.

2.3.1 Auxiliary vertical forward PEFs

The auxiliary vertical forward PEFs are calculated from the orthogonal projection $\mathbf{P}$ of the points on the right side of the PSR $S_{k,l-1}^{(p-1,q-1)}$ as shown in Figure 2.6. In order to calculate auxiliary vertical forward PEFs, the PSR is shifted one step to the left.

$$\begin{aligned} e_1^{(p-1,q)}(k,l) &= y(k,l) - \mathbf{P}_{S_{k,l-1}^{(p-1,q-1)}}(y(k,l)) \\ e_2^{(p-1,q)}(k,l) &= y(k-1,l) - \mathbf{P}_{S_{k,l-1}^{(p-1,q-1)}}(y(k-1,l)) \\ &\vdots \\ e_p^{(p-1,q)}(k,l) &= y(k-p+1,l) - \mathbf{P}_{S_{k,l-1}^{(p-1,q-1)}}(y(k-p+1,l)) \end{aligned}$$

We can rewrite these equations as follows:

$$e_i^{(p-1,q)}(k,l) = y(k-i+1,l) - \mathbf{P}_{S_{k,l-1}^{(p-1,q-1)}}(y(k-i+1,l))$$

for $i = 1, 2, \dots, p$, where $\mathbf{P}_{S_{k,l-1}^{(p-1,q-1)}}(\cdot)$ denotes the orthogonal projections on the PSR $S_{k,l-1}^{(p-1,q-1)}$ that will be defined in the sequel.

These auxiliary vertical forward PEFs can be written in the vector form as follows:

$$\mathbf{e}^{(p-1,q)}(k,l) = [e_1^{(p-1,q)}(k,l) \dots e_p^{(p-1,q)}(k,l)]^T.$$

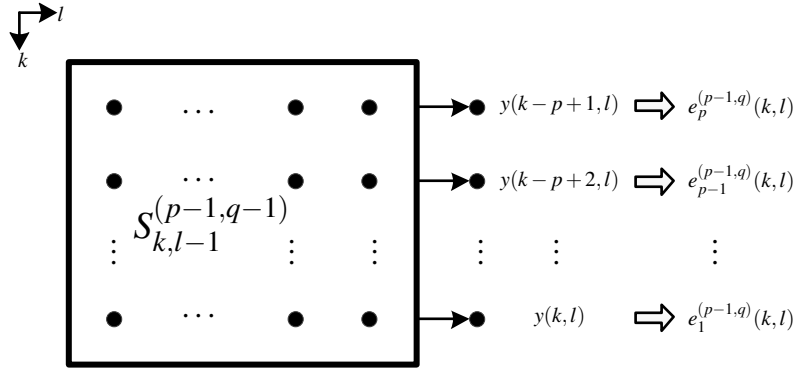


Figure 2.6: Projections of the auxiliary vertical forward PEFs for the PSR $S_{k,l}^{(p-1,q-1)}$.

2.3.2 Auxiliary vertical backward PEFs

The auxiliary vertical backward PEFs are calculated from the orthogonal projection of the points on the left side of the PSR as shown in Figure 2.7.

$$\begin{aligned}
\tilde{e}_1^{(p-1,q)}(k,l) &= y(k-p+1, l-q) - \mathbf{P}_{S_{k,l}^{(p-1,q-1)}}(y(k-p+1, l-q)) \\
\tilde{e}_2^{(p-1,q)}(k,l) &= y(k-p+2, l-q) - \mathbf{P}_{S_{k,l}^{(p-1,q-1)}}(y(k-p+2, l-q)) \\
&\vdots \\
\tilde{e}_p^{(p-1,q)}(k,l) &= y(k, l-q) - \mathbf{P}_{S_{k,l}^{(p-1,q-1)}}(y(k, l-q))
\end{aligned}$$

We can rewrite these equations as follows:

$$\tilde{e}_i^{(p-1,q)}(k,l) = y(k-p+i, l-q) - \mathbf{P}_{S_{k,l}^{(p-1,q-1)}}(y(k-p+i, l-q)) \quad (2.7)$$

for $i = 1, 2, \dots, p$, where $\mathbf{P}_{S_{k,l}^{(p-1,q-1)}}$ denotes the orthogonal projections on the PSR $S_{k,l}^{(p-1,q-1)}$.

These auxiliary vertical backward PEFs can be written in the vector form as follows:

$$\tilde{\mathbf{e}}^{(p-1,q)}(k,l) = [\tilde{e}_1^{(p-1,q)}(k,l) \dots \tilde{e}_p^{(p-1,q)}(k,l)]^T.$$

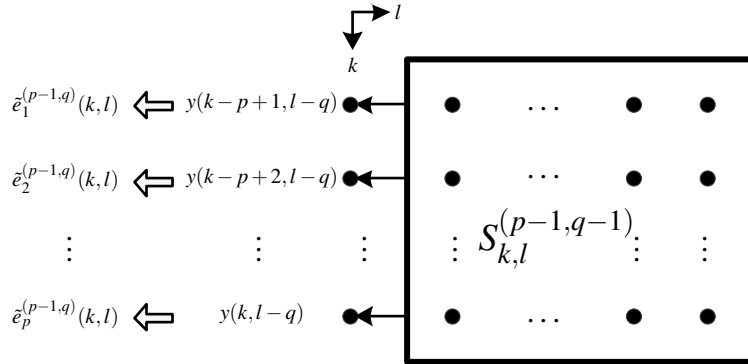


Figure 2.7: Projections of the auxiliary vertical backward PEFs for the PSR $S_{k,l}^{(p-1,q-1)}$.

2.3.3 Auxiliary horizontal forward PEFs

The auxiliary horizontal forward PEFs are calculated from the orthogonal projection of the points at the bottom of the PSR $S_{k-1,l}^{(p-1,q-1)}$ as shown in Figure 2.8. To calculate auxiliary horizontal forward PEFs, the PSR is shifted one step upwards.

$$\begin{aligned}
r_1^{(p,q-1)}(k,l) &= y(k,l) - \mathbf{P}_{S_{k-1,l}^{(p-1,q-1)}}(y(k,l)) \\
r_2^{(p,q-1)}(k,l) &= y(k,l-1) - \mathbf{P}_{S_{k-1,l}^{(p-1,q-1)}}(y(k,l-1)) \\
&\vdots \\
r_q^{(p,q-1)}(k,l) &= y(k,l-q+1) - \mathbf{P}_{S_{k-1,l}^{(p-1,q-1)}}(y(k,l-q+1))
\end{aligned}$$

We can rewrite these equations as follows:

$$r_i^{(p,q-1)}(k,l) = y(k,l-i+1) - \mathbf{P}_{S_{k-1,l}^{(p-1,q-1)}}(y(k,l-i+1))$$

for $i = 1, 2, \dots, q$, where $\mathbf{P}_{S_{k-1,l}^{(p-1,q-1)}}$ denotes the orthogonal projections on the PSR $S_{k-1,l}^{(p-1,q-1)}$.

These auxiliary horizontal forward PEFs can be written in the vector form as follows:

$$\mathbf{r}^{(p,q-1)}(k,l) = [r_1^{(p,q-1)}(k,l) \dots r_q^{(p,q-1)}(k,l)]^T.$$

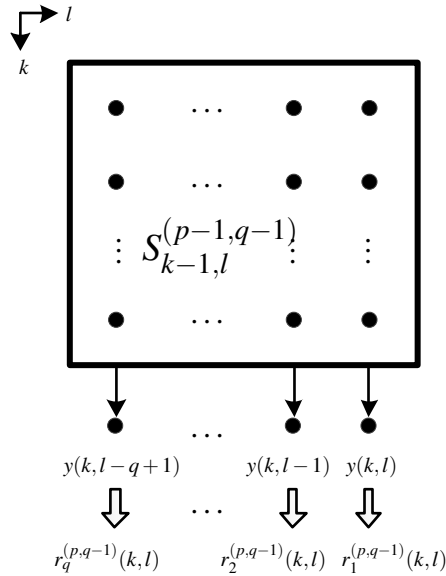


Figure 2.8: Projections of the auxiliary horizontal forward PEFs for the PSR $S_{k,l}^{(p,q)}$.

2.3.4 Auxiliary horizontal backward PEFs

The auxiliary horizontal backward PEFs are calculated from the orthogonal projection of the points at the top of the PSR as shown in Figure 2.9.

$$\begin{aligned}
\tilde{r}_1^{(p,q-1)}(k,l) &= y(k-p, l-q+1) - \mathbf{P}_{S_{k,l}^{(p-1,q-1)}}(y(k-p, l-q+1)) \\
\tilde{r}_2^{(p,q-1)}(k,l) &= y(k-p, l-q+2) - \mathbf{P}_{S_{k,l}^{(p-1,q-1)}}(y(k-p, l-q+2)) \\
&\vdots \\
\tilde{r}_q^{(p,q-1)}(k,l) &= y(k-p, l) - \mathbf{P}_{S_{k,l}^{(p-1,q-1)}}(y(k-p, l))
\end{aligned}$$

We can rewrite these equations as follows:

$$\tilde{r}_i^{(p,q-1)}(k,l) = y(k-p, l-q+i) - \mathbf{P}_{S_{k,l}^{(p-1,q-1)}}(y(k-p, l-q+i))$$

for $i = 1, 2, \dots, q$, where $\mathbf{P}_{S_{k,l}^{(p-1,q-1)}}$ denotes the orthogonal projections on the PSR $S_{k,l}^{(p-1,q-1)}$.

These auxiliary horizontal backward PEFs can be written in the vector form as follows:

$$\tilde{\mathbf{r}}^{(p,q-1)}(k,l) = [\tilde{r}_1^{(p,q-1)}(k,l) \dots \tilde{r}_q^{(p,q-1)}(k,l)]^T.$$

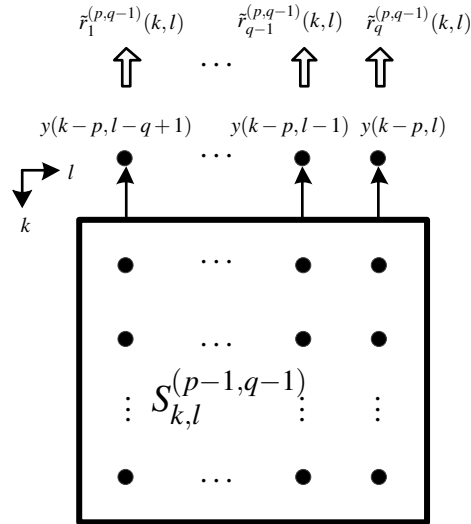


Figure 2.9: Projections of the auxiliary horizontal backward PEFs for the PSR $S_{k,l}^{(p-1,q-1)}$.

2.4 Augmentation Process

In this thesis, we will use the augmentation term for order incrementation of the lattice filter. Augmentation process will cause a growth on the PSR related to the augmentation type. For example, vertical augmentation will cause a horizontal growth. During the order incrementation process, new observation data is added to the initial PSR as a row or column vector. The Figure 2.10 illustrates the definition of row and column vector including new observations by using the PSR notation.

PSR definition	illustration	PSR definition	illustration
$S_{k,l}^{(0,0)} = \text{span}\{y(k,l)\}$		$S_{k-1,l}^{(0,0)} = \text{span}\{y(k-1,l)\}$	
$S_{k,l-1}^{(0,0)} = \text{span}\{y(k,l-1)\}$		$S_{k,l-1}^{(1,0)} = \text{span}\{y(k,l-1), y(k-1,l-1)\}$	
$S_{k-1,l}^{(0,1)} = \text{span}\{y(k-1,l), y(k-1,l-1)\}$		$S_{k-2,l}^{(0,1)} = \text{span}\{y(k-2,l), y(k-2,l-1)\}$	
$S_{k,l-2}^{(2,0)} = \text{span}\{y(k,l-2), y(k-1,l-2), y(k-2,l-2)\}$		$S_{k-2,l}^{(0,2)} = \text{span}\{y(k-2,l), y(k-2,l-1), y(k-2,l-2)\}$	

Figure 2.10: Definition of the row and column vector including new observations demonstrated in black boxes by using the PSR notation.

By using the combination of the shifted prediction support regions given in Figure 2.10, for any (n,n) th degree of PSR can be written as

$$S_{k,l}^{(n,n)} = S_{k,l}^{(0,0)} \bigcup_{i=1}^n \left(S_{k-i,l}^{(0,i-1)} \bigcup S_{k,l-i}^{(i,0)} \right). \quad (2.8)$$

Actually, shifted prediction support region represents the new observed data needed for each order incrementation. If we write the statement (2.8) in the open form for $n = 2$, then we have;

$$S_{k,l}^{(2,2)} = \underbrace{S_{k,l}^{(0,0)}}_{S_0} \bigcup \underbrace{S_{k-1,l}^{(0,0)}}_{S_1} \bigcup \underbrace{S_{k,l-1}^{(1,0)}}_{S_2} \bigcup \underbrace{S_{k-2,l}^{(0,1)}}_{S_3} \bigcup \underbrace{S_{k,l-2}^{(2,0)}}_{S_4} \quad (2.9)$$

This means $S_{k,l}^{(2,2)}$ is the combination of the initial stage and four augmentation processes. If we assign a number for each order incrementation as in (2.9), we get $S_{k,l}^{(2,2)} = S0 \cup S1 \cup S2 \cup S3 \cup S4$, and then we can denotes the total PSR as follows:

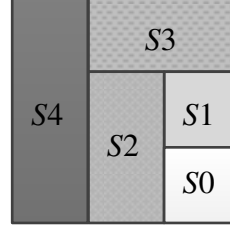


Figure 2.11: The rectangular-shaped PSR augmentation for (2,2)-th stage order.

In the equation (2.8), $S_{k,l}^{(0,0)}$ denotes the initial point, $S_{k-i,l}^{(0,i-1)}$ represents the vertical growth means horizontal augmentation and $S_{k,l-i}^{(i,0)}$ denotes the horizontal growth means vertical augmentation on the given PSR. As seen that we can obtain the desired PSR by making some augmentation processes by adding a new observed data vector horizontally and vertically.

For example, if we have $S_{k,l}^{(p-1,q-1)}$, then we can calculate $S_{k,l}^{(p,q)}$ easily, just by using horizontal and vertical augmentation. First $S_{k,l}^{(p,q-1)}$ is calculated from $S_{k,l}^{(p-1,q-1)}$ by adding a single row of new horizontal observations $S_{k-p,l}^{(0,q-1)}$; $S_{k,l}^{(p,q-1)} = S_{k,l}^{(p-1,q-1)} \cup S_{k-p,l}^{(0,q-1)}$. The $S_{k,l}^{(p,q)}$ is calculated from $S_{k,l}^{(p,q-1)}$ by adding a column of new vertical observations $S_{k,l-q}^{(p,0)}$; $S_{k,l}^{(p,q)} = S_{k,l}^{(p,q-1)} \cup S_{k,l-q}^{(p,0)}$. Augmentation process on a given PSR is depicted in the Figure 2.12.

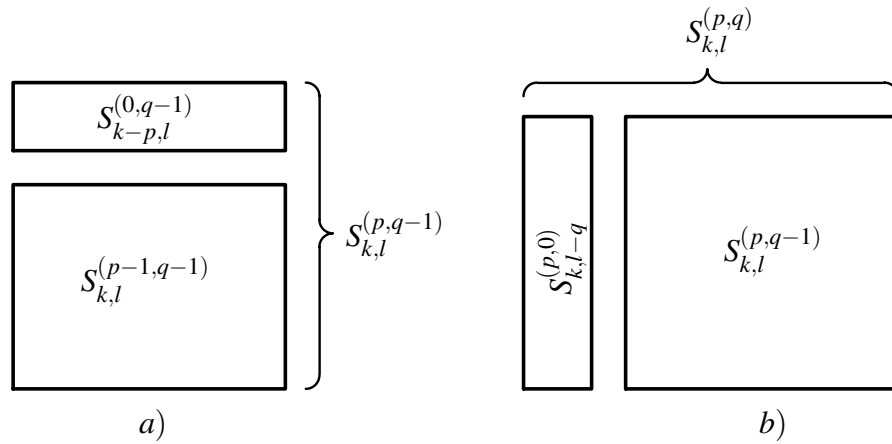


Figure 2.12: Augmentation on the PSR; a) Horizontal; b) Vertical.

2.5 Initial First Order Stage

Initial first order stage is considered as 1-D lattice filter. In order to produce auxiliary forward and backward PEFs, stage-order method will be used like 1-D case. At the *zero*-th order stage, the PSR $S_{k,l}^{(0,0)}$ is empty set and the first order auxiliary vertical and horizontal forward/backward PEFs are initiated by the given 2-D observation data:

$$e_1^{(0,0)}(k,l) = r_1^{(0,0)}(k,l) = y(k,l) \quad (2.10a)$$

$$\tilde{e}_1^{(0,0)}(k,l) = \tilde{r}_1^{(0,0)}(k,l) = y(k,l) \quad (2.10b)$$

where the letters **e** and **r** are used to denote the forward vertical and horizontal PEFs, while the letters **ẽ** and **ř** denote the backward vertical and horizontal PEFs, respectively. Furthermore, the superscript is considered to be the order of the PEF and the subscript indicates where the PEF is initiated.

As seen that unlike 1-D case, we will have two auxiliary forward and two auxiliary backward PEFs. These forward and backward PEFs are calculated as a set of two 1-D lattice sections. In another words, we will have two sets of 1-D lattice section. At the

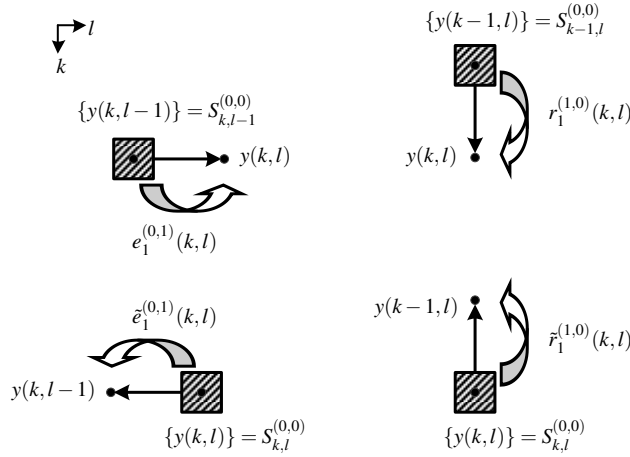


Figure 2.13: Illustration of the auxiliary vertical and horizontal forward/backward PEFs for the initial first-order stage.

initial first order stage, PSR is defined as $S_{k,l}^{(0,0)} = \{y(k,l)\}$ and the first order PEFs $e_1^{(0,1)}(k,l)$ and $r_1^{(1,0)}(k,l)$ are calculated from the orthogonal projection of $y(k,l-1)$ and $y(k-1,l)$ onto the subspace $S_{k,l-1}^{(0,0)} = Span\{y(k,l-1)\}$ and $S_{k-1,l}^{(0,0)} = Span\{y(k-1,l)\}$, respectively. The equations for the auxiliary vertical and horizontal forward

PEFs can be written as follows:

$$e_1^{(0,1)}(k,l) = y(k,l) - \mathbf{P}_{S_{k,l-1}^{(0,0)}}(y(k,l)) \quad (2.11a)$$

$$r_1^{(1,0)}(k,l) = y(k,l) - \mathbf{P}_{S_{k-1,l}^{(0,0)}}(y(k,l)) \quad (2.11b)$$

where $\mathbf{P}$ denotes the orthogonal projection.

Similarly, the backward auxiliary horizontal and vertical PEFs, $\tilde{e}_1^{(0,1)}(k,l)$ and $\tilde{r}_1^{(1,0)}(k,l)$ are obtained by the orthogonal projection of $y(k,l)$ onto the same subspace $S_{k,l}^{(0,0)} = \text{Span}\{y(k,l)\}$. Hence, we can write the following equations for the auxiliary vertical and horizontal backward PEFs:

$$\tilde{e}_1^{(0,1)}(k,l) = y(k,l-1) - \mathbf{P}_{S_{k,l}^{(0,0)}}(y(k,l)) \quad (2.12a)$$

$$\tilde{r}_1^{(1,0)}(k,l) = y(k-1,l) - \mathbf{P}_{S_{k,l}^{(0,0)}}(y(k,l)) \quad (2.12b)$$

where $\mathbf{P}$ denotes the orthogonal projection.

Illustration of the calculation of the auxiliary vertical and horizontal forward/backward PEFs by using orthogonal projection at the initial stage are depicted in Figure 2.13. As seen that we have four auxiliary PEFs on the initial first order stage PSR. For

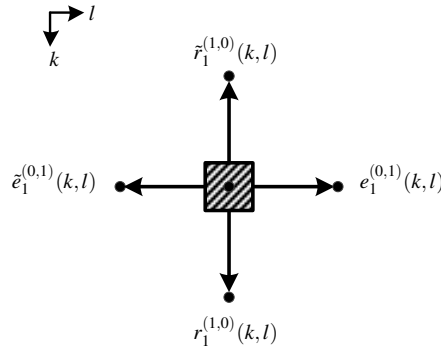


Figure 2.14: Simple illustration of auxiliary vertical and horizontal forward/backward PEFs of initial stage.

simplicity, we will demonstrate these four auxiliary PEFs on the one common PSR as illustrated in Figure 2.14. This demonstration means we have a PSR with its auxiliary vertical and horizontal forward/backward PEFs. In this illustration, auxiliary vertical and horizontal forward PEFs are calculated for $y(k,l)$.

If the random process is stationary, it can be shown that the orthogonal projections can be written as follows [40, 42]

$$\mathbf{P}_{S_{k,l-1}}^{(0,0)}(y(k,l)) = k_{e_1}^{(0,1)} y(k,l-1) \quad (2.13a)$$

$$\mathbf{P}_{S_{k-1,l}}^{(0,0)}(y(k,l)) = k_{r_1}^{(1,0)} y(k-1,l) \quad (2.13b)$$

where $k_{e_1}^{(0,1)}$ and $k_{r_1}^{(1,0)}$ are the first order stage reflection coefficients. For the backward equations, $\mathbf{P}_{S_{k,l}}^{(0,0)}(y(k,l)) = k_{e_1}^{(0,1)} y(k,l)$ is defined for the auxiliary vertical and $\mathbf{P}_{S_{k,l}}^{(0,0)}(y(k,l)) = k_{r_1}^{(0,1)} y(k,l)$ is defined for the auxiliary horizontal backward PEFs. By using the equation (2.13), we can rewrite the auxiliary vertical and horizontal forward/backward PEFs equations as follows:

$$e_1^{(0,1)}(k,l) = y(k,l) - k_{e_1}^{(0,1)*} y(k,l-1) \quad (2.14a)$$

$$r_1^{(1,0)}(k,l) = y(k,l) - k_{r_1}^{(1,0)*} y(k-1,l) \quad (2.14b)$$

$$\tilde{e}_1^{(0,1)}(k,l) = y(k,l-1) - k_{e_1}^{(0,1)} y(k,l) \quad (2.14c)$$

$$\tilde{r}_1^{(1,0)}(k,l) = y(k-1,l) - k_{r_1}^{(1,0)} y(k,l) \quad (2.14d)$$

where $k_{e_1}^{(0,1)*}$ and $k_{r_1}^{(1,0)*}$ are the first order stage reflection coefficients for forward PEFs, and $k_{e_1}^{(0,1)}$ and $k_{r_1}^{(1,0)}$ are the first order stage reflection coefficients for backward PEFs.

From (2.14), in order to predict the lattice coefficients, the initial first order lattice sections as shown in Figure 2.15 can be structured in the following well-known form:

$$\begin{bmatrix} e_1^{(0,1)}(k,l) \\ \tilde{e}_1^{(0,1)}(k,l) \end{bmatrix} = \begin{bmatrix} \mathbf{1} & -k_{e_1}^{(0,1)*} \\ -k_{e_1}^{(0,1)} & \mathbf{1} \end{bmatrix} \begin{bmatrix} y(k,l) \\ y(k,l-1) \end{bmatrix} \quad (2.15a)$$

$$\begin{bmatrix} r_1^{(1,0)}(k,l) \\ \tilde{r}_1^{(1,0)}(k,l) \end{bmatrix} = \begin{bmatrix} \mathbf{1} & -k_{r_1}^{(1,0)*} \\ -k_{r_1}^{(1,0)} & \mathbf{1} \end{bmatrix} \begin{bmatrix} y(k,l) \\ y(k-1,l) \end{bmatrix} \quad (2.15b)$$

So as to calculate the auxiliary reflection coefficients $k_{e_1}^{(0,1)}$ and $k_{r_1}^{(1,0)}$ at the initial first-order stage, the prediction error vectors (2.15a) and (2.15b) should be minimized in the mean-square sense by employing the sum of the forward and backward PEFs,

$$P_{e_1}^{(0,1)} = E \left[e_1^{(0,1)}(k,l) e_1^{(0,1)*}(k,l) \right] + E \left[\tilde{e}_1^{(0,1)}(k,l) \tilde{e}_1^{(0,1)*}(k,l) \right] \quad (2.16)$$

and

$$P_{r_1}^{(1,0)} = E \left[r_1^{(1,0)}(k,l) r_1^{(1,0)*}(k,l) \right] + E \left[\tilde{r}_1^{(1,0)}(k,l) \tilde{r}_1^{(1,0)*}(k,l) \right], \quad (2.17)$$

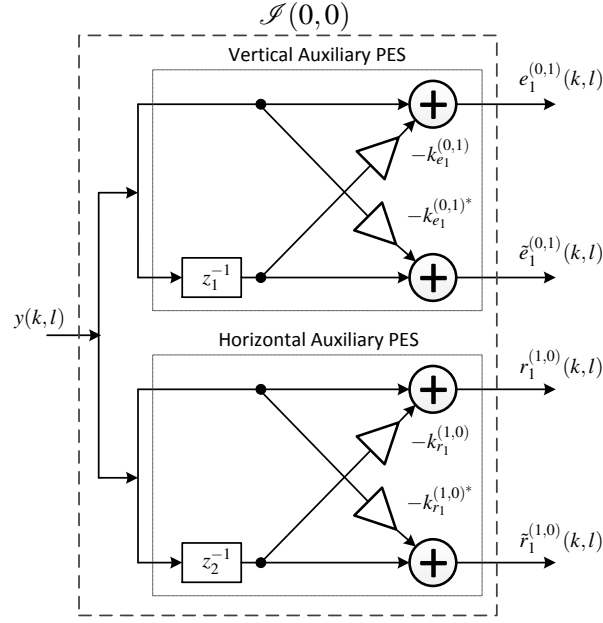


Figure 2.15: Two sets of 1-D lattice filter sections for the initial first-order stage. The block is denoted as $\mathcal{S}(0,0)$.

respectively. The propagation equation (2.16) is minimized with respect to $k_{e_1}^{(0,1)}$ (see Appendix A.1) and we have

$$k_{e_1}^{(1,0)} = \frac{E[y(k, l-1)y^*(k, l)] + E[y(k, l)y^*(k, l-1)]}{E[y(k, l-1)y^*(k, l-1)] + E[y(k, l)y^*(k, l)]}. \quad (2.18)$$

We can write the expected values as a correlation terms,

$$E[y(k, l-1)y^*(k, l)] = \Re_y^*(0, 1) \quad (2.19a)$$

$$E[y(k, l)y^*(k, l-1)] = \Re_y(0, 1) \quad (2.19b)$$

$$E[y(k, l-1)y^*(k, l-1)] = \Re_y(0, 0) \quad (2.19c)$$

$$E[y(k, l)y^*(k, l)] = \Re_y(0, 0). \quad (2.19d)$$

By using the definitions (2.19) and (2.18), it is possible to rewrite the minimized mean-square propagation equation as follows;

$$\min P_{e_1}^{(0,1)} = 2\Re_y(0, 0) - k_{e_1}^{(0,1)*} [\Re_y^*(0, 1) + \Re_y(0, 1)] k_{e_1}^{(0,1)} \quad (2.20)$$

where $2\Re_y(0, 0)$ is considered as $P_{e_1}^{(0,0)}$ which can be calculated as follows:

$$\begin{aligned} P_{e_1}^{(0,0)} &= E[e_1^{(0,0)}(k, l)e_1^{(0,0)*}(k, l)] + E[\tilde{e}_1^{(0,0)}(k, l)\tilde{e}_1^{(0,0)*}(k, l)] \\ &= E[y(k, l)y^*(k, l)] + E[y(k, l)y^*(k, l)] \\ &= 2\Re_y(0, 0) \end{aligned} \quad (2.21)$$

Similarly, the propagation equation (2.17) is minimized with respect to $k_{r_1}^{(1,0)}$ and we obtain

$$k_{r_1}^{(0,1)} = \frac{E[y(k-1, l)y^*(k, l)] + E[y(k, l)y^*(k-1, l)]}{E[y(k-1, l)y^*(k-1, l)] + E[y(k, l)y^*(k, l)]}. \quad (2.22)$$

We know that

$$E[y(k-1, l)y^*(k, l)] = \Re_y^*(1, 0) \quad (2.23a)$$

$$E[y(k, l)y^*(k-1, l)] = \Re_y(1, 0) \quad (2.23b)$$

$$E[y(k-1, l)y^*(k-1, l)] = \Re_y(0, 0). \quad (2.23c)$$

By substituting (2.19d), (2.23) and (2.22) in (2.17), it is possible to rewrite the minimized mean-square propagation equation as follows;

$$\min P_{r_1}^{(1,0)} = 2\Re_y(0, 0) - k_{r_1}^{(1,0)*} [\Re_y^*(1, 0) + \Re_y(1, 0)] k_{r_1}^{(1,0)}. \quad (2.24)$$

2.6 Horizontal Augmentation of the PSR

Horizontal augmentation is to add an horizontal vector containing past observations shown in Figure 2.16 to the top of the PSR to be augmented. In other words, in order to realize horizontal augmentation on the PSR $S_{k,l}^{(p-1,q-1)}$ shown in Figure 2.5, q additional past observations along the horizontal direction are added to the top of this PSR as illustrated in Figure 2.17. The horizontally augmented PSR is $S_{k,l}^{(p,q-1)} = S_{k,l}^{(p-1,q-1)} \cup S_{k-p,l}^{(0,q-1)}$, where $S_{k-p,l}^{(0,q-1)}$ contains the row of the past observations which consists of the points from right to left as $y(k-p, l), \dots, y(k-p, l-q+1)$.

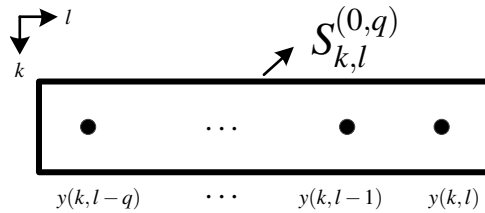


Figure 2.16: Definition of the horizontal line vector PSR $S_{k,l}^{(0,q)}$.

In Figure 2.17, all the auxiliary horizontal and vertical forward/backward PEFs of the augmented PSR are illustrated. Hereafter, each PEFs calculation will be shown step by step in detail.

The definition of the auxiliary horizontal forward and backward PEFs are represented in the Figure 2.18. As seen that the auxiliary horizontal forward PEFs are calculated

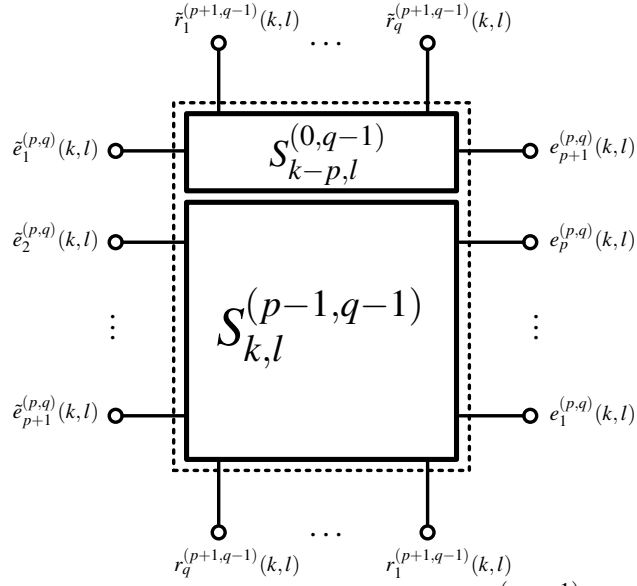


Figure 2.17: The horizontal augmentation of the PSR. $S_{k,l}^{(p,q-1)} = S_{k,l}^{(p-1,q-1)} \cup S_{k-p,l}^{(0,q-1)}$. $e_{p+1}^{(p,q)}(k,l)$ and $\tilde{e}_{p+1}^{(p,q)}(k,l)$ are the new auxiliary vertical forward/backward PEFs, respectively. $S_{k-p,l}^{(0,q-1)} = \text{Span}\{y(k-p, l-i); 0 \leq i \leq q\}$ consists of the new past observations for this stage.

from the shifted PSR $S_{k-1,l}^{(p,q-1)}(k,l)$ and the auxiliary horizontal backward PEFs are computed from the PSR, $S_{k,l}^{(p,q-1)}(k,l)$ by using orthogonal projection. Here, the calculation of the auxiliary horizontal PEFs is just only the update of the previous PEFs according to the new stage order. Similar to the multichannel lattice structure, the auxiliary horizontal PEFs of the $(p+1, q)$ order horizontally augmented PSR can be obtained by a linear combination of the lower order PEFs. Hence, the lattice structure for the auxiliary horizontal PEFs can be defined as:

$$\begin{bmatrix} \mathbf{r}^{(p+1,q-1)}(k,l) \\ \tilde{\mathbf{r}}^{(p+1,q-1)}(k,l) \end{bmatrix} = \begin{bmatrix} \mathbf{I}_p & -\mathbf{K}_r^{(p+1,q-1)H} \\ -\mathbf{K}_r^{(p+1,q-1)T} & \mathbf{I}_p \end{bmatrix} \begin{bmatrix} \mathbf{r}^{(p,q-1)}(k,l) \\ \tilde{\mathbf{r}}^{(p,q-1)}(k-1,l) \end{bmatrix} \quad (2.25)$$

where $\mathbf{K}_r^{(p+1,q-1)} = [\mathbf{k}_{r_1}^{(p+1,q-1)} \mathbf{k}_{r_2}^{(p+1,q-1)} \dots \mathbf{k}_{r_q}^{(p+1,q-1)}]^T$ is the reflection coefficient matrix with $\mathbf{k}_{r_i}^{(p+1,q-1)} = [k_{r_i}^{(p+1,q-1)}(1) k_{r_i}^{(p+1,q-1)}(2) \dots k_{r_i}^{(p+1,q-1)}(q)]^T$, for $i = 1, 2, \dots, q$.

The $\mathbf{r}^{(p+1,q-1)}(k,l) = [r_1^{(p+1,q-1)}(k,l) r_2^{(p+1,q-1)}(k,l) \dots r_q^{(p+1,q-1)}(k,l)]^T$ and $\tilde{\mathbf{r}}^{(p+1,q-1)}(k,l) = [\tilde{r}_1^{(p+1,q-1)}(k,l) \tilde{r}_2^{(p+1,q-1)}(k,l) \dots \tilde{r}_q^{(p+1,q-1)}(k,l)]^T$ are the updated horizontal forward and backward PEFs vectors, respectively.

In order to calculate the horizontal reflection coefficient matrix, the sum of the mean-square values of the auxiliary horizontal forward/backward PEFs will be

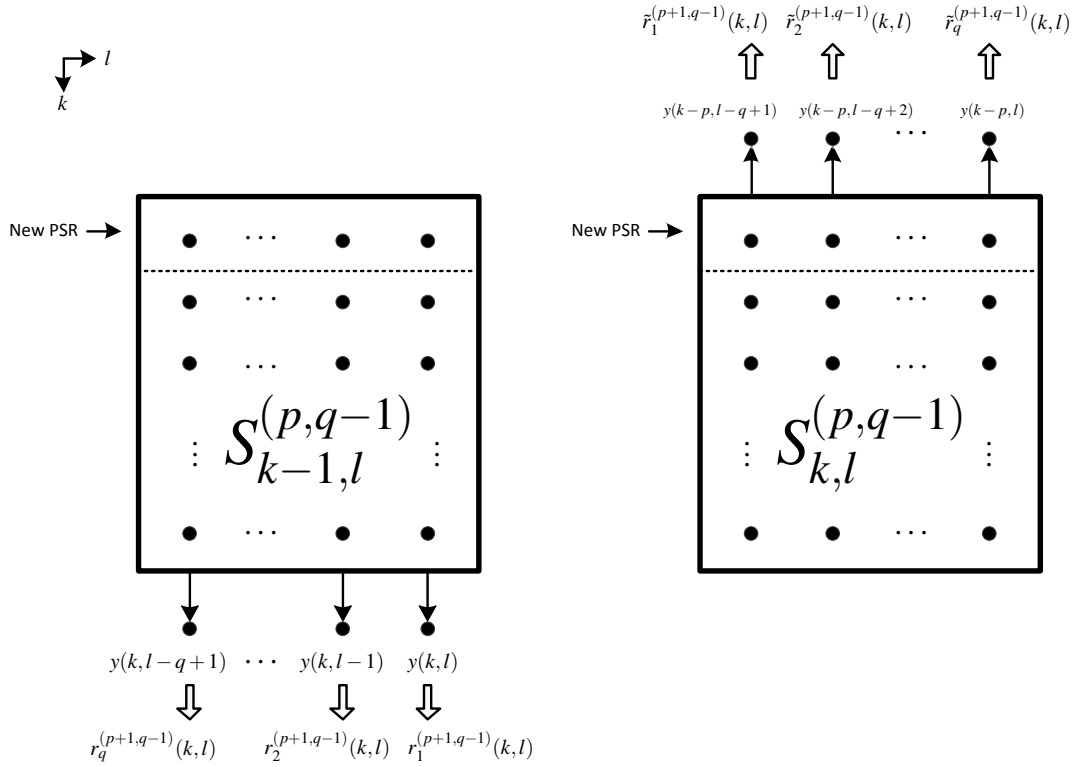


Figure 2.18: Definition of the auxiliary horizontal forward/backward PEFs in case of the horizontal augmentation.

minimized in the mean-square sense, with respect to the reflection coefficient matrix in (2.25). The results of this minimization are summarized in the following equation:

$$\mathbf{K}_{\mathbf{r}}^{(p+1,q-1)} = \left[\boldsymbol{\Sigma}_{\tilde{\mathbf{r}},\tilde{\mathbf{r}}}^{(p,q-1)} + \boldsymbol{\Sigma}_{\mathbf{r},\mathbf{r}}^{(p,q-1)*} \right]^{-1} \left[\boldsymbol{\Delta}_{\tilde{\mathbf{r}},\mathbf{r}}^{(p,q-1)} + \boldsymbol{\Delta}_{\mathbf{r},\tilde{\mathbf{r}}}^{(p,q-1)T} \right]. \quad (2.26)$$

where $\boldsymbol{\Sigma}_{\tilde{\mathbf{r}},\tilde{\mathbf{r}}}^{(p,q-1)}$ and $\boldsymbol{\Sigma}_{\mathbf{r},\mathbf{r}}^{(p,q-1)}$ are the covariance matrices. $\boldsymbol{\Delta}_{\tilde{\mathbf{r}},\mathbf{r}}^{(p,q-1)}$ is the cross-covariance matrix and they are defined as

$$\boldsymbol{\Sigma}_{\tilde{\mathbf{r}},\tilde{\mathbf{r}}}^{(p,q-1)} = E \left[\mathbf{r}^{(p,q-1)}(k, l) \mathbf{r}^{(p,q-1)H}(k, l) \right], \quad (2.27a)$$

$$\boldsymbol{\Sigma}_{\mathbf{r},\mathbf{r}}^{(p,q-1)} = E \left[\tilde{\mathbf{r}}^{(p,q-1)}(k-1, l) \tilde{\mathbf{r}}^{(p,q-1)T}(k-1, l) \right], \quad (2.27b)$$

and

$$\boldsymbol{\Delta}_{\tilde{\mathbf{r}},\mathbf{r}}^{(p,q-1)} = E \left[\tilde{\mathbf{r}}^{(p,q-1)}(k-1, l) \mathbf{r}^{(p,q-1)H}(k, l) \right]. \quad (2.28)$$

It is possible to derive a simple propagation equation for the mean-square the auxiliary horizontal PEFs from $(p, q-1)$ th order to $(p+1, q-1)$ th order by recognizing that the expected value of the sum of the auxiliary horizontal forward and backward prediction errors is $P_r^{(p,q-1)} = \text{tr} \left[\boldsymbol{\Sigma}_{\tilde{\mathbf{r}},\tilde{\mathbf{r}}}^{(p,q-1)} + \boldsymbol{\Sigma}_{\mathbf{r},\mathbf{r}}^{(p,q-1)*} \right]$, where tr represents trace

operator; $\Sigma_{\tilde{\mathbf{r}},\tilde{\mathbf{r}}}^{(p,q-1)}$ and $\Sigma_{\mathbf{r},\mathbf{r}}^{(p,q-1)}$ are defined in (2.27), as follows.

$$\begin{aligned} P_r^{(p+1,q-1)} &= P_r^{(p,q-1)} - \left[\mathbf{K}_r^{(p+1,q-1)H} \left(\Sigma_{\tilde{\mathbf{r}},\tilde{\mathbf{r}}}^{(p,q-1)} + \Sigma_{\mathbf{r},\mathbf{r}}^{(p,q-1)} \right) \mathbf{K}_r^{(p+1,q-1)} \right] \\ &= tr \left[\left(\Sigma_{\tilde{\mathbf{r}},\tilde{\mathbf{r}}}^{(p,q-1)} + \Sigma_{\mathbf{r},\mathbf{r}}^{(p,q-1)} \right) \left(I - \mathbf{K}_r^{(p+1,q-1)} \mathbf{K}_r^{(p+1,q-1)H} \right) \right]. \end{aligned} \quad (2.29)$$

The more explanation of the calculation of the reflection coefficient matrices is given in Appendix A.1.

During the calculation of the auxiliary horizontal forward PEFs, the new augmented PSR is needed, thus it will be defined as

$$S_{k-1,l}^{(p,q-1)} = S_{k-1,l}^{(p-1,q-1)} \cup S_{k-p,l}^{(0,q-1)} \quad (2.30)$$

where $S_{k-1,l}^{(p-1,q-1)}$ represents the previous PSR and $S_{k-p,l}^{(0,q-1)}$ demonstrates the newly added PSR that includes the past observations. For calculation of the auxiliary horizontal backward PEFs, the PSR $S_{k,l}^{(p-1,q-1)}$ will be taken into account and the augmented PSR will be defined as

$$S_{k,l}^{(p,q-1)} = S_{k,l}^{(p-1,q-1)} \cup S_{k-p+1,l}^{(0,q-1)}. \quad (2.31)$$

In the thesis, in case of augmentation we will use just only backward augmentation notation shown in equation (2.31) for representing the general PSR augmentation.

Similar to calculation of the auxiliary horizontal forward PEFs, the left shifted PSR $S_{k,l-1}^{(p-1,q-1)}$ will be used for the auxiliary vertical forward PEFs. In this case we can define the new augmented PSR as

$$S_{k,l-1}^{(p,q-1)} = S_{k,l-1}^{(p-1,q-1)} \cup S_{k-p+1,l-1}^{(0,q-1)} \quad (2.32)$$

where $S_{k,l-1}^{(p-1,q-1)}$ represents previous PSR and $S_{k-p+1,l-1}^{(0,q-1)}$ demonstrates the newly added PSR containing the past observations. For calculation of the auxiliary vertical backward PEFs, the PSR $S_{k,l}^{(p-1,q-1)}$ will be taken into account and the augmented PSR will be defined as

$$S_{k,l}^{(p,q-1)} = S_{k,l}^{(p-1,q-1)} \cup S_{k-p+1,l}^{(0,q-1)}. \quad (2.33)$$

The Figure 2.19 and Figure 2.20 demonstrate the new states of the auxiliary vertical forward and backward PEFs, respectively. It is clearly seen that the consequence of the

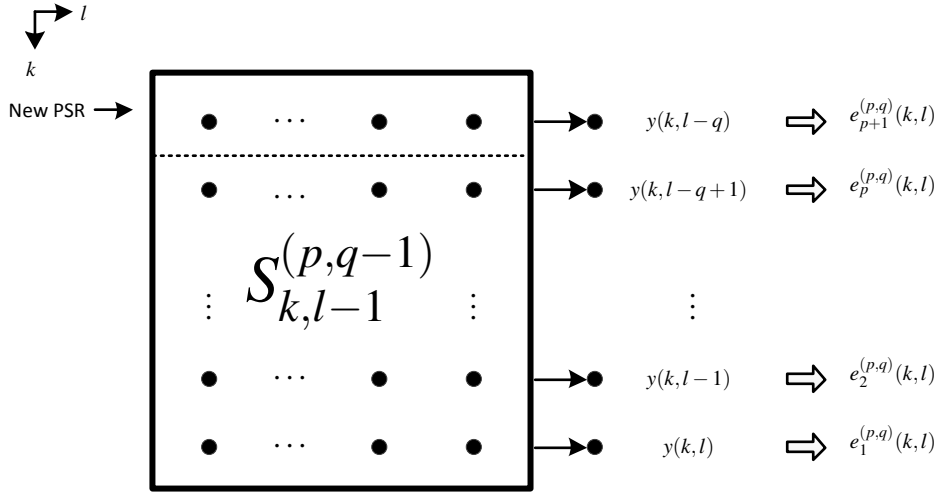


Figure 2.19: Definition of the auxiliary vertical forward PEFs in case of the horizontal augmentation.

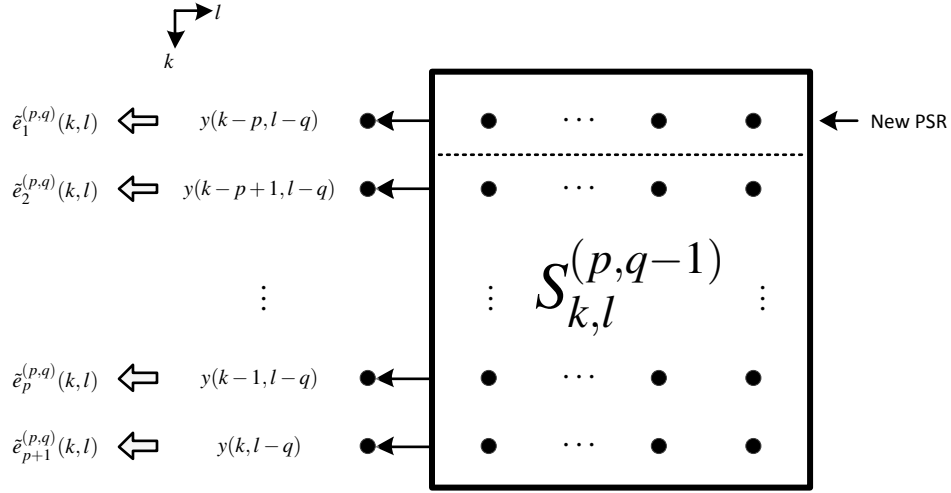


Figure 2.20: Definition of the auxiliary vertical backward PEFs in case of the horizontal augmentation.

horizontal augmentation, a set of new auxiliary vertical forward and backward PEFs, namely $e_{p+1}^{(p,q)}(k, l)$ and $\tilde{e}_{p+1}^{(p,q)}(k, l)$ are introduced, respectively. As a natural result of the horizontal augmentation of the PSR, we can observe that the PSR $S_{k,l}^{(p,q-1)}$ has $p+1$ neighboring points to be predicted in the vertical direction. Therefore, we need to initiate a set of new auxiliary vertical forward and backward PEFs as

$$\begin{aligned} e_{p+1}^{(p,q)}(k, l) &= e_p^{(p,q)}(k-1, l) \\ \tilde{e}_{p+1}^{(p,q)}(k, l) &= \tilde{e}_p^{(p,q)}(k, l). \end{aligned} \quad (2.34)$$

The $\left[e_1^{(p,q)}(k, l) \ e_2^{(p,q)}(k, l) \ \cdots \ e_p^{(p,q)}(k, l) \right]$ and $\left[\tilde{e}_1^{(p,q)}(k, l) \ \tilde{e}_2^{(p,q)}(k, l) \ \cdots \ \tilde{e}_p^{(p,q)}(k, l) \right]$ are the updated auxiliary vertical forward and backward PEFs according to the previous

stage order, respectively. The error propagation equation of the (p, q) order auxiliary vertical forward/backward PEFs including the newly initiated ones can be presented with a single lattice structures as a recursion of the PEFs of the $(p, q - 1)$ order PSR;

$$\begin{aligned}
 \begin{bmatrix} \mathbf{e}^{(p,q)}(k, l) \\ \tilde{\mathbf{e}}^{(p,q)}(k, l) \end{bmatrix} &= \begin{bmatrix} \mathbf{e}^{(p-1,q)}(k, l) \\ e_p^{(p-1,q)}(k-1, l) \\ \tilde{\mathbf{e}}^{(p-1,q)}(k-1, l) \\ \tilde{e}_p^{(p-1,q)}(k, l) \end{bmatrix} \\
 &+ \begin{bmatrix} \mathbf{O}_{(p+1) \times 2q} & \mathbf{O}_{p \times q} & -\hat{\mathbf{K}}_e^{(p,q)H} \\ -\hat{\mathbf{K}}_e^{(p,q)T} & \mathbf{O}_{p \times q} & -\mathbf{k}_{e_{p+1}}^{(p,q)*} \\ \mathbf{0}_q^T & -\mathbf{k}_{e_{p+1}}^{(p,q)T} & \mathbf{O}_{(p+1) \times 2q} \end{bmatrix} \\
 &\times \begin{bmatrix} \mathbf{r}^{(p,q-1)}(k, l) \\ \tilde{\mathbf{r}}^{(p,q-1)}(k, l) \\ \mathbf{r}^{(p,q-1)}(k, l-1) \\ \tilde{\mathbf{r}}^{(p,q-1)}(k, l-1) \end{bmatrix} \quad (2.35)
 \end{aligned}$$

where $\hat{\mathbf{K}}_e^{(p,q)} = [\mathbf{k}_{e_1}^{(p,q)} \dots \mathbf{k}_{e_p}^{(p,q)}]^T$ and $\mathbf{k}_{e_{p+1}}^{(p,q)}$ are the reflection coefficient matrix and vectors, respectively, with $\mathbf{k}_{e_i}^{(p,q)} = [k_{e_i}^{(p,q)}(1) \dots k_{e_i}^{(p,q)}(q)]$, for $i = 1, 2, \dots, p+1$. It is possible to write the (p, q) th order vertical reflection coefficient matrix as $\mathbf{K}_e^{(p,q)} = \begin{bmatrix} \hat{\mathbf{K}}_e^{(p,q)} \\ \mathbf{k}_{e_{p+1}}^{(p,q)*} \end{bmatrix}$. The $\mathbf{e}^{(p,q)}(k, l)$ and $\tilde{\mathbf{e}}^{(p,q)}(k, l)$ include updated previous PEFs and newly initiated ones written as

$$\begin{aligned}
 \mathbf{e}^{(p,q)}(k, l) &= [e_1^{(p,q)}(k, l) \dots e_p^{(p,q)}(k, l) e_{p+1}^{(p,q)}(k, l)] \\
 \tilde{\mathbf{e}}^{(p,q)}(k, l) &= [\tilde{e}_1^{(p,q)}(k, l) \dots \tilde{e}_p^{(p,q)}(k, l) \tilde{e}_{p+1}^{(p,q)}(k, l)]. \quad (2.36)
 \end{aligned}$$

The updated previous PEFs vector can be denoted as $\hat{\mathbf{e}}^{(p,q)}(k, l)$ and it is defined as follows:

$$\begin{aligned}
 \hat{\mathbf{e}}^{(p,q)}(k, l) &= \mathbf{e}^{(p-1,q)}(k, l) \\
 &= [e_1^{(p-1,q)}(k, l) \dots e_p^{(p-1,q)}(k, l)] \\
 \hat{\tilde{\mathbf{e}}}^{(p,q)}(k, l) &= \tilde{\mathbf{e}}^{(p-1,q)}(k, l) \\
 &= [\tilde{e}_1^{(p-1,q)}(k, l) \dots \tilde{e}_p^{(p-1,q)}(k, l)]. \quad (2.37)
 \end{aligned}$$

So as to calculate the vertical reflection coefficient matrix at the (p, q) th order stage, the sum of the mean-square values of the auxiliary vertical forward/backward PEFs should be minimized in the mean-square sense,

$$P_e^{(p,q)} = \text{tr} E [\mathbf{e}^{(p,q)} \mathbf{e}^{(p,q)H} + \tilde{\mathbf{e}}^{(p,q)} \tilde{\mathbf{e}}^{(p,q)H}] \quad (2.38)$$

where $E[\cdot]$ denotes the expected value over the PEFs, tr means trace operator. Substituting (2.35) into (2.38) results in an expression which can be minimized with respect to lattice model reflection coefficient vectors. The results of this minimization are summarized in the following equations for the updated PEFs and the newly initiated ones.

$$\hat{\mathbf{K}}_{\mathbf{e}}^{(p,q)} = \left[\boldsymbol{\Sigma}_{\tilde{\mathbf{r}},\tilde{\mathbf{r}}}^{(p,q-1)} + \boldsymbol{\Sigma}_{\mathbf{r},\mathbf{r}}^{(p,q-1)*} \right]^{-1} \left[\boldsymbol{\Delta}_{\tilde{\mathbf{r}},\mathbf{e}}^{(p,q-1)} + \boldsymbol{\Delta}_{\mathbf{r},\tilde{\mathbf{e}}}^{(p,q-1)*} \right] \quad (2.39)$$

$$\mathbf{k}_{e_{p+1}}^{(p,q)} = \left[\boldsymbol{\Sigma}_{\mathbf{r},\mathbf{r}}^{(p,q-1)} + \boldsymbol{\Sigma}_{\tilde{\mathbf{r}},\tilde{\mathbf{r}}}^{(p,q-1)*} \right]^{-1} \left[\boldsymbol{\Delta}_{\mathbf{r},e_p}^{(p,q-1)} + \boldsymbol{\Delta}_{\tilde{\mathbf{r}},\tilde{e}_p}^{(p,q-1)*} \right] \quad (2.40)$$

where covariance matrix $\boldsymbol{\Sigma}_{\mathbf{r},\mathbf{r}}^{(p,q-1)}$ and $\boldsymbol{\Sigma}_{\tilde{\mathbf{r}},\tilde{\mathbf{r}}}^{(p,q-1)}$ are defined in (2.27) and

$$\boldsymbol{\Delta}_{\tilde{\mathbf{r}},\mathbf{e}}^{(p,q-1)} = E \left[\tilde{\mathbf{r}}^{(p,q-1)} \mathbf{e}^{(p-1,q)H} \right] \quad (2.41)$$

$$\boldsymbol{\Delta}_{\mathbf{r},\tilde{\mathbf{e}}}^{(p,q-1)} = E \left[\mathbf{r}^{(p,q-1)} \tilde{\mathbf{e}}^{(p-1,q)H} \right] \quad (2.42)$$

$$\boldsymbol{\Delta}_{\mathbf{r},e_p}^{(p,q-1)} = E \left[\mathbf{r}^{(p,q-1)} e_p^{(p-1,q)H} \right] \quad (2.43)$$

$$\boldsymbol{\Delta}_{\tilde{\mathbf{r}},\tilde{e}_p}^{(p,q-1)} = E \left[\tilde{\mathbf{r}}^{(p,q-1)} \tilde{e}_p^{(p-1,q)H} \right]. \quad (2.44)$$

The error power propagation equations for the mean-square auxiliary vertical PEFs and the new ones from stage to stage are obtained as follows

$$\hat{P}_e^{(p,q)} = \hat{P}_e^{(p-1,q)} - tr \left[\hat{\mathbf{K}}_{\mathbf{e}}^{(p,q)H} \left(\boldsymbol{\Sigma}_{\tilde{\mathbf{r}},\tilde{\mathbf{r}}}^{(p,q-1)} + \boldsymbol{\Sigma}_{\mathbf{r},\mathbf{r}}^{(p,q-1)T} \right) \hat{\mathbf{K}}_{\mathbf{e}}^{(p,q)} \right] \quad (2.45)$$

$$P_{e_{p+1}}^{(p,q)} = P_{e_p}^{(p-1,q)} - tr \left[\mathbf{k}_{e_{p+1}}^{(p,q)H} \left(\boldsymbol{\Sigma}_{\mathbf{r},\mathbf{r}}^{(p,q-1)} + \boldsymbol{\Sigma}_{\tilde{\mathbf{r}},\tilde{\mathbf{r}}}^{(p,q-1)T} \right) \mathbf{k}_{e_{p+1}}^{(p,q)} \right]. \quad (2.46)$$

It is also possible to combine (2.39) and (2.40) into one equation as follows

$$\mathbf{K}_{\mathbf{e}}^{(p,q)} = \left[\boldsymbol{\Sigma}_{\tilde{\mathbf{r}},\tilde{\mathbf{r}}}^{(p,q-1)} + \boldsymbol{\Sigma}_{\mathbf{r},\mathbf{r}}^{(p,q-1)*} \right]^{-1} \left[\boldsymbol{\Delta}_{\tilde{\mathbf{r}},\mathbf{e}}^{(p,q-1)} + \boldsymbol{\Delta}_{\mathbf{r},\tilde{\mathbf{e}}}^{(p,q-1)*} : \boldsymbol{\Delta}_{\mathbf{r},e_p}^{(p,q-1)} + \boldsymbol{\Delta}_{\tilde{\mathbf{r}},\tilde{e}_p}^{(p,q-1)*} \right] \quad (2.47)$$

and the power propagation equation can be formed as

$$P_e^{(p,q)} = \hat{P}_e^{(p-1,q)} + P_{e_p}^{(p-1,q)} - tr \left[\mathbf{K}_{\mathbf{e}}^{(p,q)H} \left(\boldsymbol{\Sigma}_{\tilde{\mathbf{r}},\tilde{\mathbf{r}}}^{(p,q-1)} + \boldsymbol{\Sigma}_{\mathbf{r},\mathbf{r}}^{(p,q-1)T} \right) \mathbf{K}_{\mathbf{e}}^{(p,q)} \right]. \quad (2.48)$$

The calculation of the reflection coefficient matrices is explained in Appendix A.1.

The generation of the horizontally augmented auxiliary forward/backward PEFs is depicted in Figure 2.21.

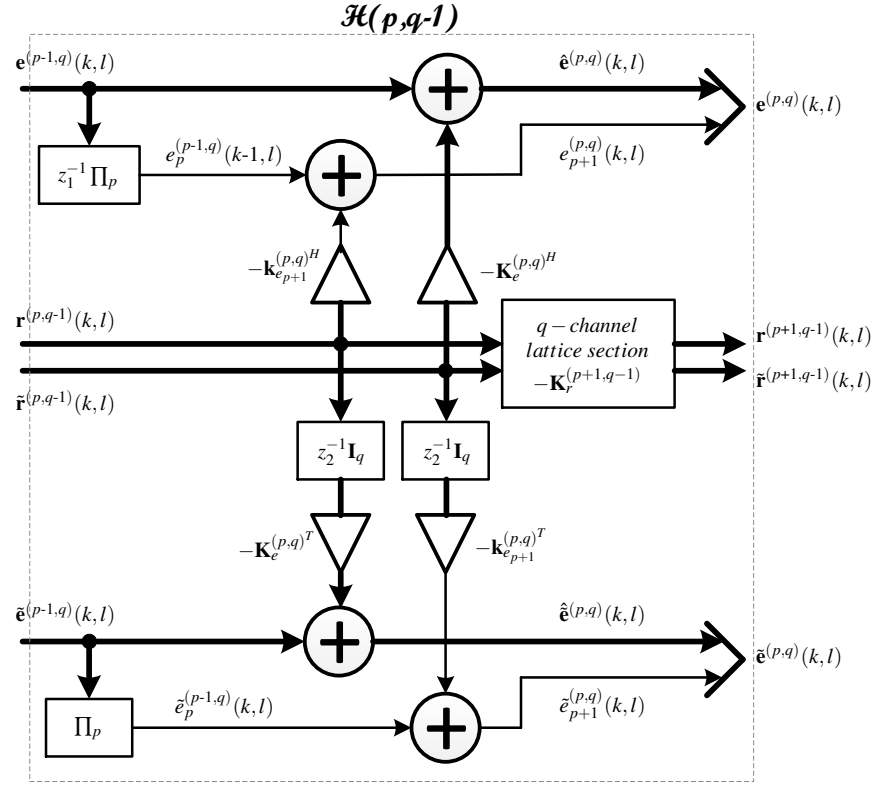


Figure 2.21: Generation of the horizontally augmented auxiliary PEFs $\Pi_p = [0 \cdots 0 \ 1]$ is the pointing vector, and $\mathbf{e}^{(p, q)} = [\hat{\mathbf{e}}^{(p, q)} : e_{p+1}^{(p, q)}]^T$

2.7 Vertical Augmentation of the PSR

In order to realize the vertical augmentation, a vertical vector containing past observations shown in Figure 2.22 is added to the left of the PSR. That means, for the vertical augmentation on the PSR $S_{k,l}^{(p-1, q-1)}$ shown in Figure 2.5, p additional past observations along the vertical direction are added to the left of this PSR as illustrated in Figure 2.23. The vertically augmented PSR is $S_{k,l}^{(p-1, q)} = S_{k,l}^{(p-1, q-1)} \cup S_{k, l-q}^{(p-1, 0)}$, where $S_{k, l-q}^{(p-1, 0)}$ contains the column of the past observations consisting of the points from bottom to top as $y(k, l-q), \dots, y(k-p+1, l-q)$.

All the auxiliary horizontal and vertical forward/backward PEFs of the vertically augmented PSR are illustrated in Figure 2.23. The definition of the auxiliary vertical forward and backward PEFs are illustrated in the Figure 2.24 and 2.25, respectively. Similar to the horizontal augmentation, naturally, the left shifted PSR $S_{k, l-1}^{(p-1, q-1)}$ will be used for the auxiliary vertical forward PEFs. Thus, the newly augmented PSR can

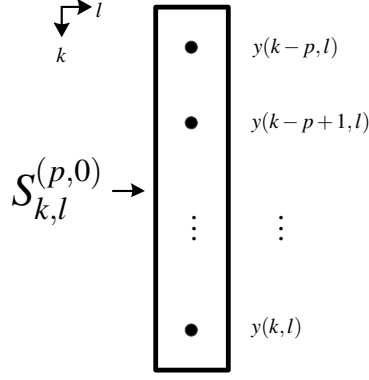


Figure 2.22: Definition of the vertical line vector PSR $S_{k,l}^{(p,0)}$.

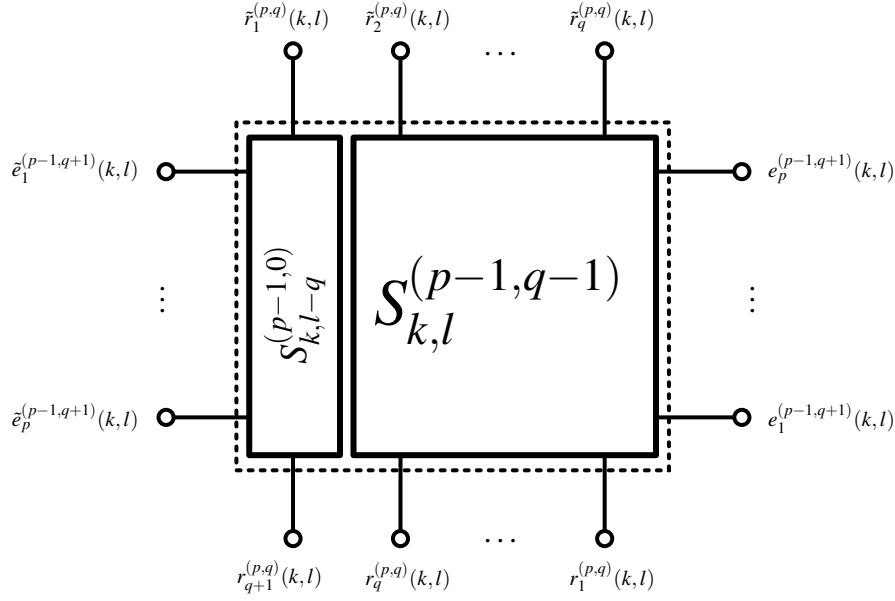


Figure 2.23: The vertical augmentation of the PSR. $S_{k,l}^{(p-1,q)} = S_{k,l}^{(p-1,q-1)} \cup S_{k,l-q}^{(p-1,0)}$. $r_{q+1}^{(p,q)}(k,l)$ and $\tilde{r}_{q+1}^{(p,q)}(k,l)$ are the new auxiliary horizontal forward/backward PEFs, respectively. $S_{k,l-q}^{(p-1,0)} = \text{Span}\{y(k-i, l-q); 0 \leq i \leq p\}$ consists of the new past observations for this stage.

be defined as

$$S_{k,l-1}^{(p-1,q)} = S_{k,l-1}^{(p-1,q-1)} \cup S_{k,l-q}^{(p-1,0)} \quad (2.49)$$

where $S_{k,l-1}^{(p-1,q-1)}$ represents the previous PSR and $S_{k,l-q}^{(p-1,0)}$ shows the new vertically added PSR containing the past observations. For the auxiliary vertical backward PEFs, the PSR $S_{k,l}^{(p-1,q-1)}$ will be taken into account and the augmented PSR will be defined as

$$S_{k,l}^{(p-1,q)} = S_{k,l}^{(p-1,q-1)} \cup S_{k,l-q+1}^{(p-1,0)}. \quad (2.50)$$

Briefly, it can be stated that the auxiliary vertical forward PEFs are calculated from the left-shifted PSR $S_{k-1,l}^{(p-1,q)}(k,l)$ and the auxiliary vertical backward PEFs are computed

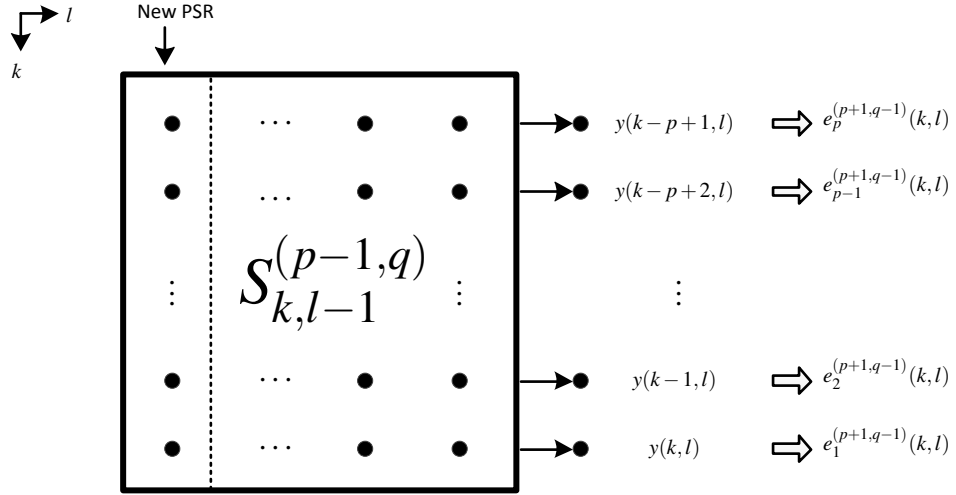


Figure 2.24: Definition of the auxiliary vertical forward PEFs in case of the vertical augmentation.

from the PSR, $S_{k,l}^{(p-1,q)}(k, l)$. Actually, the calculation of the auxiliary vertical PEFs is the update of the previous existing PEFs according to the new stage order. Therefore, the auxiliary horizontal PEFs of the $(p-1, q)$ order vertically augmented PSR can be obtained by a linear combination of the lower order PEFs and the lattice structure of the auxiliary vertical forward/backward PEFs can be written as:

$$\begin{bmatrix} \mathbf{e}^{(p-1,q+1)}(k, l) \\ \tilde{\mathbf{e}}^{(p-1,q+1)}(k, l) \end{bmatrix} = \begin{bmatrix} \mathbf{I}_p & -\mathbf{K}_e^{(p-1,q+1)H} \\ -\mathbf{K}_e^{(p-1,q+1)T} & \mathbf{I}_p \end{bmatrix} \begin{bmatrix} \mathbf{e}^{(p-1,q)}(k, l) \\ \tilde{\mathbf{e}}^{(p-1,q)}(k, l-1) \end{bmatrix} \quad (2.51)$$

where $\mathbf{K}_e^{(p-1,q+1)} = [\mathbf{k}_{e_1}^{(p-1,q+1)} \mathbf{k}_{e_2}^{(p-1,q+1)} \dots \mathbf{k}_{e_p}^{(p-1,q+1)}]^T$ is the reflection coefficient matrix with $\mathbf{k}_{e_i}^{(p-1,q+1)} = [k_{e_i}^{(p-1,q+1)}(1) k_{e_i}^{(p-1,q+1)}(2) \dots k_{e_i}^{(p-1,q+1)}(p)]^T$, for $i = 1, 2, \dots, p$.

The $\mathbf{e}^{(p-1,q+1)}(k, l) = [e_1^{(p-1,q+1)}(k, l) e_2^{(p-1,q+1)}(k, l) \dots e_p^{(p-1,q+1)}(k, l)]^T$ and $\tilde{\mathbf{e}}^{(p-1,q+1)}(k, l) = [\tilde{e}_1^{(p-1,q+1)}(k, l) \tilde{e}_2^{(p-1,q+1)}(k, l) \dots \tilde{e}_p^{(p-1,q+1)}(k, l)]^T$ are the updated auxiliary vertical forward and backward PEFs, respectively.

In order to calculate the vertical reflection coefficient matrix, the sum of the mean-square values of the auxiliary vertical forward/backward PEFs will be minimized in the mean-square sense, with respect to the reflection coefficient matrix in (2.51). The results of this minimization are summarized in the following equation:

$$\mathbf{K}_e^{(p-1,q+1)} = \left[\boldsymbol{\Sigma}_{\tilde{\mathbf{e}}, \tilde{\mathbf{e}}}^{(p-1,q)} + \boldsymbol{\Sigma}_{\mathbf{e}, \mathbf{e}}^{(p-1,q)*} \right]^{-1} \left[\boldsymbol{\Delta}_{\tilde{\mathbf{e}}, \mathbf{e}}^{(p-1,q)} + \boldsymbol{\Delta}_{\tilde{\mathbf{e}}, \mathbf{e}}^{(p-1,q)T} \right]. \quad (2.52)$$

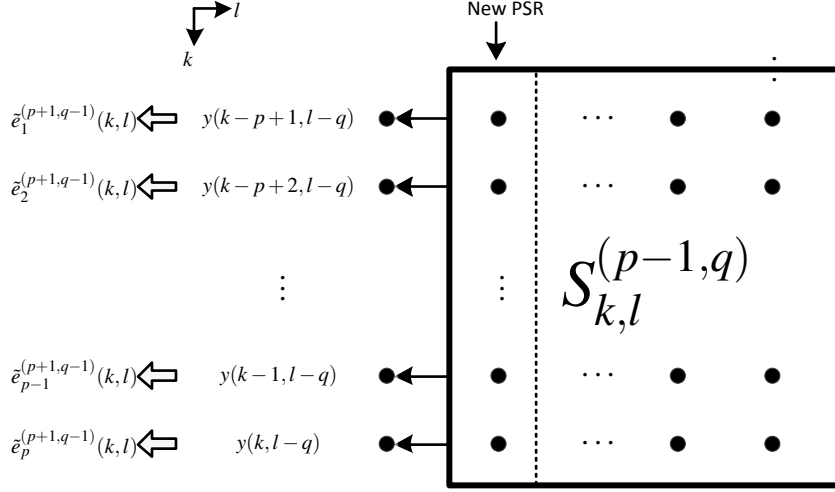


Figure 2.25: Definition of the auxiliary vertical backward PEFs in case of the vertical augmentation.

where $\Sigma_{x,x}^{(p-1,q)}$ and $\Delta_{x,y}^{(p-1,q)}$ are the covariance and the cross-covariance matrices, respectively. And these are defined as follows (detailed explanations are given in Appendix A.1.),

$$\Sigma_{\tilde{\mathbf{e}},\tilde{\mathbf{e}}}^{(p-1,q)} = E \left[\mathbf{e}^{(p-1,q)}(k,l) \mathbf{e}^{(p-1,q)H}(k,l) \right], \quad (2.53a)$$

$$\Sigma_{\mathbf{e},\mathbf{e}}^{(p-1,q)} = E \left[\tilde{\mathbf{e}}^{(p-1,q)}(k,l-1) \tilde{\mathbf{e}}^{(p-1,q)T}(k,l-1) \right], \quad (2.53b)$$

and

$$\Delta_{\tilde{\mathbf{e}},\mathbf{e}}^{(p-1,q)} = E \left[\tilde{\mathbf{e}}^{(p-1,q)}(k-1,l) \mathbf{e}^{(p-1,q)H}(k,l) \right]. \quad (2.54)$$

In order to derive a simple propagation equation for the mean-square the auxiliary vertical PEFs from $(p-1,q)$ th order to $(p-1,q+1)$ th order, the expected value of the sum of the auxiliary vertical forward and backward prediction errors is

$P_e^{(p-1,q)} = tr \left[\Sigma_{\tilde{\mathbf{e}},\tilde{\mathbf{e}}}^{(p-1,q)} + \Sigma_{\mathbf{e},\mathbf{e}}^{(p-1,q)*} \right]$, where tr represents trace operator; $\Sigma_{\tilde{\mathbf{e}},\tilde{\mathbf{e}}}^{(p-1,q)}$ and $\Sigma_{\mathbf{e},\mathbf{e}}^{(p-1,q)}$ are defined in (2.53), as follows.

$$\begin{aligned} P_e^{(p-1,q+1)} &= P_e^{(p-1,q)} - \left[\mathbf{K}_e^{(p-1,q+1)H} \left(\Sigma_{\tilde{\mathbf{e}},\tilde{\mathbf{e}}}^{(p-1,q)} + \Sigma_{\mathbf{e},\mathbf{e}}^{(p-1,q)} \right) \mathbf{K}_e^{(p-1,q+1)} \right] \\ &= tr \left[\left(\Sigma_{\tilde{\mathbf{e}},\tilde{\mathbf{e}}}^{(p-1,q)} + \Sigma_{\mathbf{e},\mathbf{e}}^{(p-1,q)} \right) \left(I - \mathbf{K}_e^{(p-1,q+1)} \mathbf{K}_e^{(p-1,q+1)H} \right) \right]. \end{aligned} \quad (2.55)$$

Similarly, for the calculation of the auxiliary horizontal forward PEFs, the new augmented PSR will be defined as

$$S_{k-1,l}^{(p-1,q)} = S_{k-1,l}^{(p-1,q-1)} \cup S_{k-1,l-q+1}^{(p-1,0)} \quad (2.56)$$

where $S_{k-1,l}^{(p-1,q-1)}$ represents the previous PSR and $S_{k-1,l-q+1}^{(p-1,0)}$ is the new vertically added PSR that includes the past observations. For the auxiliary horizontal backward PEFs, the PSR $S_{k,l}^{(p-1,q-1)}$ will be taken into account and the augmented PSR will be defined as

$$S_{k,l}^{(p-1,q)} = S_{k,l}^{(p-1,q-1)} \cup S_{k,l-q+1}^{(p-1,0)}. \quad (2.57)$$

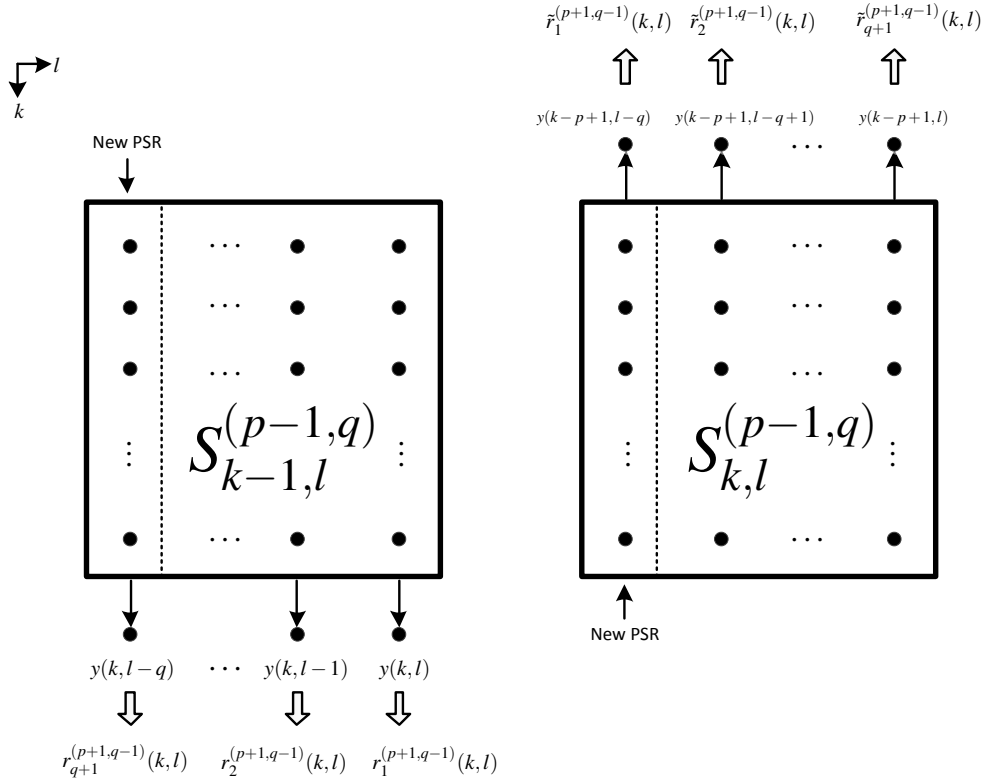


Figure 2.26: Definition of the auxiliary horizontal forward/backward PEFs in case of the vertical augmentation.

The Figure 2.26 demonstrates the new states of the auxiliary horizontal forward and backward PEFs, respectively. As a result of the vertical augmentation, a set of new auxiliary horizontal forward and backward PEFs, namely $r_{q+1}^{(p,q)}(k,l)$ and $\tilde{r}_{q+1}^{(p,q)}(k,l)$ are introduced, respectively. Consequently, we can observe that the PSR $S_{k,l}^{(p-1,q)}$ has $q+1$ neighboring points to be predicted in the horizontal direction. Hence, we need to initiate a set of new auxiliary horizontal forward and backward PEFs as

$$\begin{aligned} r_{q+1}^{(p,q-1)}(k,l) &= r_q^{(p,q-1)}(k,l-1) \\ \tilde{r}_{q+1}^{(p,q-1)}(k,l) &= \tilde{r}_q^{(p,q-1)}(k,l). \end{aligned} \quad (2.58)$$

The $\begin{bmatrix} r_1^{(p,q)}(k,l) & r_2^{(p,q)}(k,l) & \dots & r_q^{(p,q)}(k,l) \end{bmatrix}$ and $\begin{bmatrix} \tilde{r}_1^{(p,q)}(k,l) & \tilde{r}_2^{(p,q)}(k,l) & \dots & \tilde{r}_q^{(p,q)}(k,l) \end{bmatrix}$ are the update of existing auxiliary vertical forward and backward PEFs according to the previous stage order, respectively.

The error propagation equation of the (p,q) th order auxiliary existing and the newly initiated horizontal forward/backward PEFs can be combined into a single lattice structures as a recursion of the PEFs of the $(p-1,q)$ th order PSR;

$$\begin{aligned} \begin{bmatrix} \mathbf{r}^{(p,q)}(k,l) \\ \tilde{\mathbf{r}}^{(p,q)}(k,l) \end{bmatrix} &= \begin{bmatrix} \mathbf{r}^{(p,q-1)}(k,l) \\ r_q^{(p,q-1)}(k,l-1) \\ \tilde{\mathbf{r}}^{(p,q-1)}(k,l-1) \\ \tilde{r}_q^{(p,q-1)}(k,l) \end{bmatrix} \\ &+ \begin{bmatrix} \mathbf{O}_{(q+1) \times 2p} & \mathbf{O}_{q \times p} & -\hat{\mathbf{K}}_r^{(p,q)H} \\ -\hat{\mathbf{K}}_r^{(p,q)T} & \mathbf{O}_{q \times p} & -\mathbf{k}_{r_{q+1}}^{(p,q)*} \\ \mathbf{0}_p^T & -\mathbf{k}_{r_{q+1}}^{(p,q)T} & \mathbf{0}_p^T \\ \mathbf{O}_{(q+1) \times 2p} \end{bmatrix} \\ &\times \begin{bmatrix} \mathbf{e}^{(p-1,q)}(k,l) \\ \tilde{\mathbf{e}}^{(p-1,q)}(k,l) \\ \mathbf{e}^{(p-1,q)}(k-1,l) \\ \tilde{\mathbf{e}}^{(p-1,q)}(k-1,l) \end{bmatrix} \end{aligned} \quad (2.59)$$

where $\hat{\mathbf{K}}_r^{(p,q)} = \begin{bmatrix} \mathbf{k}_{r_1}^{(p,q)} : \mathbf{k}_{r_q}^{(p,q)} \end{bmatrix}^T$ and $\mathbf{k}_{r_{q+1}}^{(p,q)}$ are the reflection coefficient matrix and vectors, respectively, with $\mathbf{k}_{r_i}^{(p,q)} = \begin{bmatrix} k_{r_i}^{(p,q)}(1) & \dots & k_{r_i}^{(p,q)}(p) \end{bmatrix}$, for $i = 1, 2, \dots, q+1$. It is possible to write the (p,q) th order vertical reflection coefficient matrix as $\mathbf{K}_r^{(p,q)} = \begin{bmatrix} \hat{\mathbf{K}}_r^{(p,q)} : \mathbf{k}_{r_{p+1}}^{(p,q)*} \end{bmatrix}$.

The $\mathbf{r}^{(p,q)}(k,l)$ and $\tilde{\mathbf{r}}^{(p,q)}(k,l)$ include the updated existing PEFs and the newly initiated ones.

$$\begin{aligned} \mathbf{r}^{(p,q)}(k,l) &= \begin{bmatrix} r_1^{(p,q)}(k,l) & \dots & r_q^{(p,q)}(k,l) & r_{q+1}^{(p,q)}(k,l) \end{bmatrix} \\ \tilde{\mathbf{r}}^{(p,q)}(k,l) &= \begin{bmatrix} \tilde{r}_1^{(p,q)}(k,l) & \dots & \tilde{r}_q^{(p,q)}(k,l) & \tilde{r}_{q+1}^{(p,q)}(k,l) \end{bmatrix} \end{aligned} \quad (2.60)$$

The updated previous PEFs vector can be denoted as $\hat{\mathbf{r}}^{(p,q)}(k,l)$ and it is defined as follows:

$$\begin{aligned} \hat{\mathbf{r}}^{(p,q)}(k,l) &= \mathbf{r}^{(p,q-1)}(k,l) \\ &= \begin{bmatrix} r_1^{(p,q-1)}(k,l) & \dots & r_q^{(p,q-1)}(k,l) \end{bmatrix} \\ \hat{\tilde{\mathbf{r}}}^{(p,q)}(k,l) &= \tilde{\mathbf{r}}^{(p,q-1)}(k,l) \\ &= \begin{bmatrix} \tilde{r}_1^{(p,q-1)}(k,l) & \dots & \tilde{r}_q^{(p,q-1)}(k,l) \end{bmatrix}. \end{aligned} \quad (2.61)$$

As we represented in section 2.6, the horizontal reflection coefficient matrix at the (p, q) th order stage can be calculated by minimizing the sum of the mean-square values of the auxiliary horizontal forward/backward PEFs in the mean-square sense,

$$P_r^{(p,q)} = tr E \left[\mathbf{r}^{(p,q)} \mathbf{r}^{(p,q)H} + \tilde{\mathbf{r}}^{(p,q)} \tilde{\mathbf{r}}^{(p,q)H} \right] \quad (2.62)$$

where tr is trace operator and $E[\cdot]$ is the expected value over the PEFs. Substituting (2.59) into (2.62) results in an expression which can be minimized with respect to the lattice model reflection coefficient vectors. The results of this minimization are summarized in the following equations for the updated PEFs and the newly initiated ones.

$$\hat{\mathbf{K}}_{\mathbf{r}}^{(p,q)} = \left[\boldsymbol{\Sigma}_{\tilde{\mathbf{e}},\tilde{\mathbf{e}}}^{(p-1,q)} + \boldsymbol{\Sigma}_{\mathbf{e},\mathbf{e}}^{(p-1,q)*} \right]^{-1} \left[\boldsymbol{\Delta}_{\tilde{\mathbf{e}},\mathbf{r}}^{(p-1,q)} + \boldsymbol{\Delta}_{\mathbf{e},\tilde{\mathbf{r}}}^{(p-1,q)*} \right] \quad (2.63)$$

$$\mathbf{k}_{r_{q+1}}^{(p,q)} = \left[\boldsymbol{\Sigma}_{\mathbf{e},\mathbf{e}}^{(p-1,q)} + \boldsymbol{\Sigma}_{\tilde{\mathbf{e}},\tilde{\mathbf{e}}}^{(p-1,q)*} \right]^{-1} \left[\boldsymbol{\Delta}_{\mathbf{e},r_q}^{(p-1,q)} + \boldsymbol{\Delta}_{\tilde{\mathbf{e}},\tilde{r}_q}^{(p-1,q)*} \right] \quad (2.64)$$

where covariance matrix $\boldsymbol{\Sigma}_{\mathbf{e},\mathbf{e}}^{(p-1,q)}$ and $\boldsymbol{\Sigma}_{\tilde{\mathbf{e}},\tilde{\mathbf{e}}}^{(p-1,q)}$ are defined in (2.27) and

$$\boldsymbol{\Delta}_{\tilde{\mathbf{e}},\mathbf{r}}^{(p-1,q)} = E \left[\tilde{\mathbf{e}}^{(p-1,q)} \mathbf{r}^{(p,q-1)H} \right] \quad (2.65)$$

$$\boldsymbol{\Delta}_{\mathbf{e},\tilde{\mathbf{r}}}^{(p-1,q)} = E \left[\mathbf{e}^{(p-1,q)} \tilde{\mathbf{r}}^{(p,q-1)H} \right] \quad (2.66)$$

$$\boldsymbol{\Delta}_{\mathbf{e},r_q}^{(p-1,q)} = E \left[\mathbf{e}^{(p-1,q)} r_q^{(p,q-1)H} \right] \quad (2.67)$$

$$\boldsymbol{\Delta}_{\tilde{\mathbf{e}},\tilde{r}_q}^{(p-1,q)} = E \left[\tilde{\mathbf{e}}^{(p-1,q)} \tilde{r}_q^{(p,q-1)H} \right]. \quad (2.68)$$

The propagation equations for the mean-square auxiliary horizontal previous PEFs and the new ones from stage to stage can be stated as follows

$$\hat{P}_r^{(p,q)} = \hat{P}_r^{(p-1,q)} - tr \left[\hat{\mathbf{K}}_{\mathbf{r}}^{(p,q)H} \left(\boldsymbol{\Sigma}_{\tilde{\mathbf{e}},\tilde{\mathbf{e}}}^{(p-1,q)} + \boldsymbol{\Sigma}_{\mathbf{e},\mathbf{e}}^{(p-1,q)T} \right) \hat{\mathbf{K}}_{\mathbf{e}}^{(p,q)} \right] \quad (2.69)$$

$$P_{r_{q+1}}^{(p,q)} = P_{r_q}^{(p-1,q)} - tr \left[\mathbf{k}_{r_{q+1}}^{(p,q)H} \left(\boldsymbol{\Sigma}_{\mathbf{e},\mathbf{e}}^{(p-1,q)} + \boldsymbol{\Sigma}_{\tilde{\mathbf{e}},\tilde{\mathbf{e}}}^{(p-1,q)T} \right) \mathbf{k}_{r_{q+1}}^{(p,q)} \right]. \quad (2.70)$$

We can also combine (2.39) and (2.40) into one equation as follows

$$\mathbf{K}_{\mathbf{r}}^{(p,q)} = \left[\boldsymbol{\Sigma}_{\tilde{\mathbf{e}},\tilde{\mathbf{e}}}^{(p-1,q)} + \boldsymbol{\Sigma}_{\mathbf{e},\mathbf{e}}^{(p-1,q)*} \right]^{-1} \left[\boldsymbol{\Delta}_{\tilde{\mathbf{e}},\mathbf{r}}^{(p-1,q)} + \boldsymbol{\Delta}_{\mathbf{e},\tilde{\mathbf{r}}}^{(p-1,q)*} : \boldsymbol{\Delta}_{\mathbf{e},r_q}^{(p-1,q)} + \boldsymbol{\Delta}_{\tilde{\mathbf{e}},\tilde{r}_q}^{(p-1,q)*} \right] \quad (2.71)$$

and the propagation equation can be formed as

$$P_r^{(p,q)} = \hat{P}_r^{(p-1,q)} + P_{r_q}^{(p-1,q)} - tr \left[\mathbf{K}_{\mathbf{r}}^{(p,q)H} \left(\boldsymbol{\Sigma}_{\tilde{\mathbf{e}},\tilde{\mathbf{e}}}^{(p-1,q)} + \boldsymbol{\Sigma}_{\mathbf{e},\mathbf{e}}^{(p-1,q)T} \right) \mathbf{K}_{\mathbf{r}}^{(p,q)} \right]. \quad (2.72)$$

The calculation of the reflection coefficient matrices is given in Appendix A.1.

The generation of the vertically augmented auxiliary forward/backward PEFs is depicted in Figure 2.27.

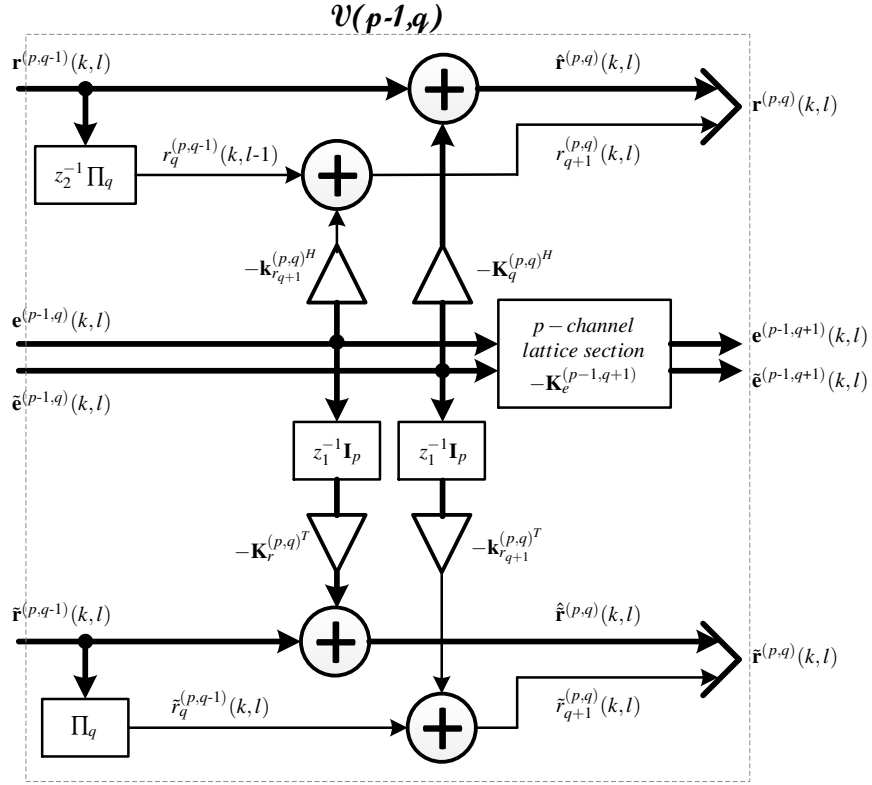


Figure 2.27: Generation of the vertically augmented auxiliary PEFs $\Pi_q = [0 \cdots 0 \ 1]$ is the pointing vector, and $\mathbf{r}^{(p,q)} = [\hat{\mathbf{r}}^{(p,q)} : \mathbf{r}_{q+1}^{(p,q)}]^T$.

2.8 Application of the Vertical and Horizontal Augmentation Methods

At the initial stage, we have the first order four PEFs in (2.15), and it is possible to calculate the auxiliary vertical and horizontal PEFs of successively higher order stages by combining linearly. This calculation is realized by using vertical and horizontal augmentations. The basic augmentation schemes for the PSR are given in Figure 2.28. In this figure, $\mathcal{J}(0,0)$ represents the initial stage, $\mathcal{V}(*,*)$ shows the vertical augmentation and $\mathcal{H}(*,*)$ demonstrates the horizontal augmentation. If the PSR is horizontally augmented, the PEFs are calculated by the method to be explained in section 2.6. On the other hand, if the PSR is vertically augmented, the PEFs are computed by the method to be explained in section 2.7.

After any successive higher order stage, it is possible to define a QP or an ASHP or an NC asymmetric half-plane models as illustrated in Figure 2.29.

The most important thing is to take into account only the given data when calculating the lattice coefficients and updating the auxiliary vertical/horizontal forward/backward

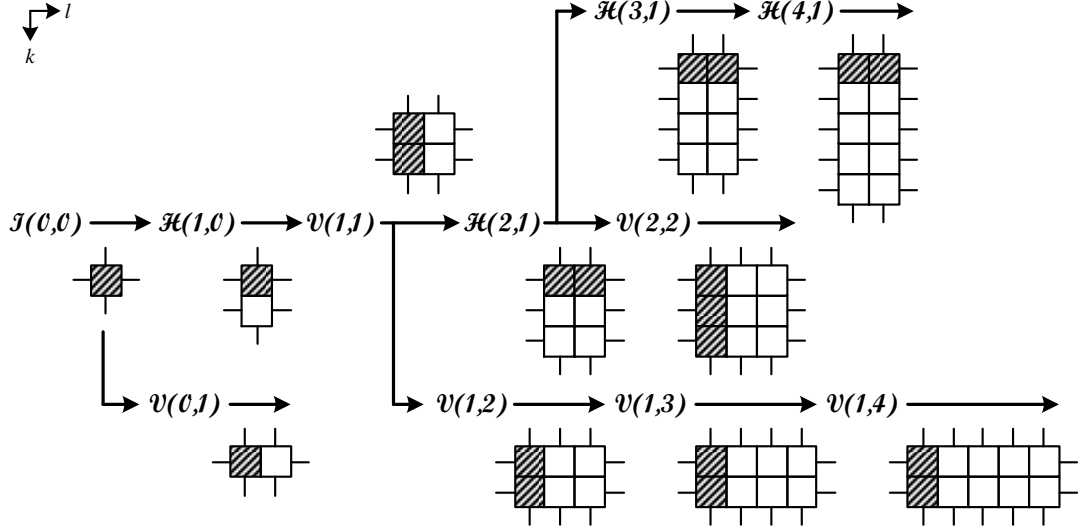


Figure 2.28: The basic augmentation schemes for the 2-D PSR.

PEFs using Burg's method [60]. That means we are not using any boundary conditions for the calculation of correlation values in (2.73). After the computation of lattice parameters, the PEFs for the new stage order are updated. As a result, the updated PEFs support region will get smaller depending on the stage order. For the initial case, if the support data size is $M \times N$, then the size of auxiliary vertical forward/backward PEFs will be $M \times (N - 1)$, and the size of auxiliary horizontal forward/backward PEFs will be $(M - 1) \times N$. Actually, this reduction is related to the order of the proposed algorithm. For the (p, q) th order stage, the size of the forward/backward PEFs will be $(M - p) \times (N - q)$. It is clear that the orders of the predicted 2-D lattice filters are bounded with the size of the observed data. In other words, the data in the PEFs get smaller if the order p, q increases. For the given $M \times N$ data field, the correlation between $e_x^{(p,q)}(k, l)$ and $e_y^{(p,q)}(k, l)$ is calculated as follows:

$$\begin{aligned} \Re_{xy}(i, j) &= E \left[e_x^{(p,q)}(k, l) e_y^{(p,q)*}(k - i, l - j) \right] \\ &= \frac{1}{(M - p)(N - q)} \sum_{k=p+1}^M \sum_{l=q+1}^N e_x^{(p,q)}(k, l) e_y^{(p,q)}(k - i, l - j)^* \quad (2.73) \end{aligned}$$

where $E[\cdot]$ denotes the expected value over the field of dimension $(M - p) \times (N - q)$.

For example for the initial stage, if we have an $M \times N$ observed data, the calculated PEF $e_1^{(0,1)}(k, l)$ is valid for $0 \leq k < M$ and $1 \leq l < N$. The calculated data in case of $l = 0$ will be zero because there is no data to calculate this value. In other words, first column of the auxiliary vertical forward/backward PEFs will be zero. Similarly,

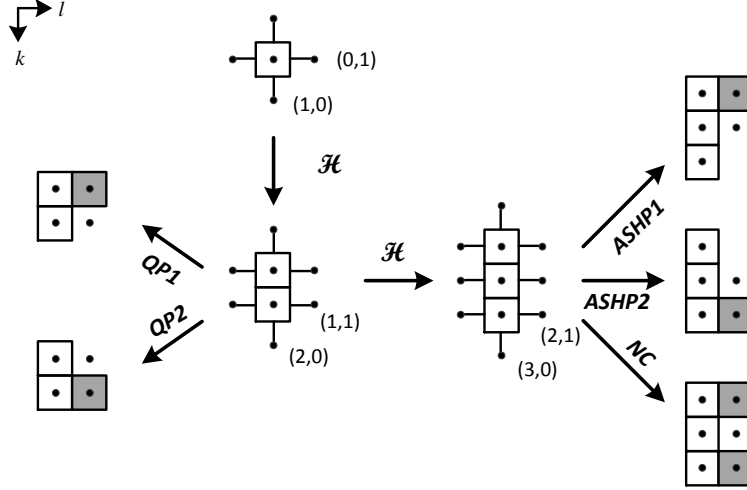


Figure 2.29: The basic example of obtaining QP, ASHP and NC half-plane models from the 2-D PSR.

first row of the auxiliary horizontal forward/backward of the PEFs will be zero for the auxiliary PEFs $r_1^{(1,0)}(k,l)$ and $\bar{e}_1^{(1,0)}(k,l)$. In the Appendix A.2, a clear example will be given to explain this reduction process.

2.8.1 Application of horizontal augmentation

The calculation of the first order stage PEFs is already explained in detail in the Section 2.7. After the initial stage, it is possible to make an augmentation on the vertical or horizontal direction. After each augmentation, we have two possibilities to continue to scan the data. After the initial stage, we assume that an augmentation is applied to the PSR $S_{k,l}^{(0,0)}$ by adding one point in the vertical direction as depicted in Figure 2.30. This augmentation is called as *Horizontal Augmentation*. As seen that the horizontal augmentation causes the increase of the number of auxiliary vertical forward and backward PEFs. Due to the horizontally augmented PSR, $S_{k,l}^{(1,0)}$ has two new neighboring points to be predicted in the vertical direction, a set of new auxiliary vertical forward and backward PEFs $e_2^{(0,1)}(k,l) = e_1^{(0,1)}(k,l)$ and $\bar{e}_2^{(0,1)}(k,l) = \bar{e}_1^{(0,1)}(k,l)$ must be predefined. The calculation of existing and the newly initiated auxiliary vertical forward and backward PEFs are illustrated in Figure 2.31.

The update of existing auxiliary vertical forward PEF, $e_1^{(1,1)}(k,l)$, is calculated by spanning the points $\{y(k,l), y(k,l-1), y(k-1,l-1)\}$ and the newly initiated one, $e_2^{(1,1)}(k,l)$ is computed by spanning the points $\{y(k-1,l), y(k,l-1), y(k-1,l-1)\}$.

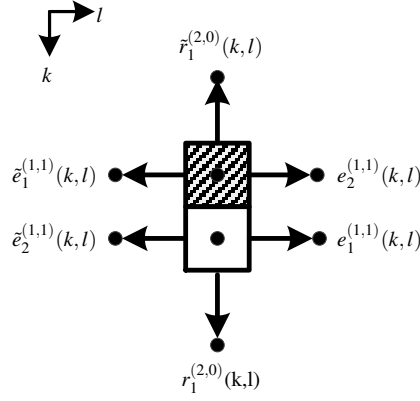


Figure 2.30: Auxiliary vertical and horizontal forward/backward PEFs of the PSR $S_{k,l}^{(1,0)}$.

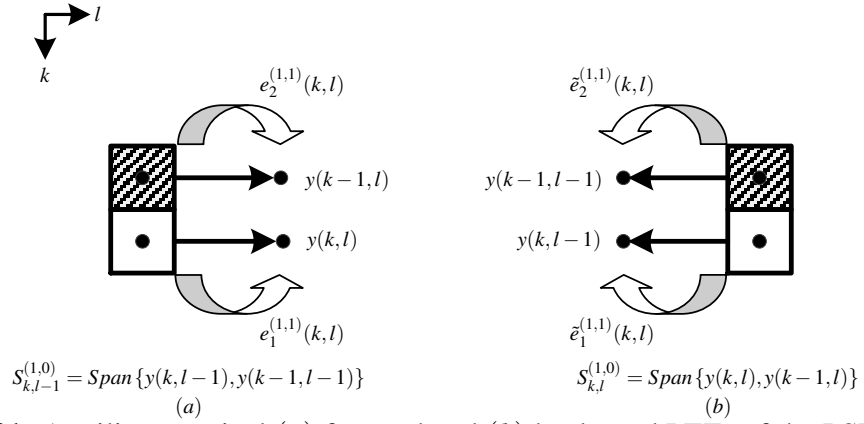


Figure 2.31: Auxiliary vertical (a) forward and (b) backward PEFs of the PSR $S_{k,l}^{(1,0)}$.

In other words, the update of existing auxiliary vertical forward PEF, $e_1^{(1,1)}(k,l)$ and the newly initiated one, $e_2^{(1,1)}(k,l)$ are linearly predicted by using the previous stage order PEFs, $e_1^{(0,1)}(k,l)$, $\tilde{e}_1^{(0,1)}(k,l)$, $r_1^{(1,0)}(k,l)$, and $\tilde{r}_1^{(1,0)}(k,l)$ which are already calculated at the initial stage. For the updated auxiliary vertical forward PEF, we need the projections from the point $y(k,l)$ to $y(k,l-1)$, which equals to $e_1^{(0,1)}(k,l)$, and from the point $y(k-1,l-1)$ to $y(k,l-1)$, which equals to $\tilde{r}_1^{(1,0)}(k,l-1)$. The equation for the updated auxiliary vertical forward PEF can be written as

$$e_1^{(1,1)}(k,l) = \left(y(k,l) - k_{e_1}^{(0,1)*} y(k,l-1) \right) - k_{\tilde{e}_1}^{(1,1)*} \left(y(k-1,l-1) - k_{r_1}^{(1,0)} y(k,l-1) \right). \quad (2.74)$$

We have already found at the initial stage

$$e_1^{(0,1)}(k, l) = y(k, l) - k_{e_1}^{(0,1)*} y(k, l-1) \quad (2.75)$$

$$\tilde{r}_1^{(1,0)}(k, l) = y(k-1, l) - k_{r_1}^{(1,0)} y(k, l). \quad (2.76)$$

If we realize a delay on l axis for $\tilde{r}_1^{(1,0)}(k, l)$, we obtain

$$\tilde{r}_1^{(1,0)}(k, l-1) = y(k-1, l-1) - k_{r_1}^{(1,0)} y(k, l-1). \quad (2.77)$$

So, we can rephrase the equation (2.75) as follows:

$$e_1^{(1,1)}(k, l) = e_1^{(0,1)}(k, l) - k_{e_1}^{(1,1)*} \tilde{r}_1^{(1,0)}(k, l-1). \quad (2.78)$$

For the newly initiated auxiliary vertical forward PEF, the projections from the point $y(k-1, l)$ to $y(k-1, l-1)$, which equals to $e_1^{(0,1)}(k-1, l)$, and from the point $y(k, l-1)$ to $y(k-1, l-1)$, which equals to $r_1^{(1,0)}(k, l-1)$, can be needed. The equation for the newly initiated auxiliary vertical forward PEF can be formed as

$$\begin{aligned} e_2^{(1,1)}(k, l) = & \left(y(k-1, l) - k_{e_1}^{(0,1)*} y(k-1, l-1) \right) \\ & - k_{e_2}^{(1,1)*} \left(y(k, l-1) - k_{r_1}^{(1,0)} y(k-1, l-1) \right). \end{aligned} \quad (2.79)$$

After making necessary time delay on the initial stage PEFs, we can obtain

$$e_1^{(0,1)}(k, l) \Rightarrow e_1^{(0,1)}(k-1, l) = y(k-1, l) - k_{e_1}^{(0,1)*} y(k-1, l-1) \quad (2.80)$$

$$r_1^{(1,0)}(k, l) \Rightarrow r_1^{(1,0)}(k, l-1) = y(k, l-1) - k_{r_1}^{(1,0)*} y(k-1, l-1) \quad (2.81)$$

and the equation (2.79) can be reformed as

$$e_2^{(1,1)}(k, l) = e_1^{(0,1)}(k-1, l) - k_{e_2}^{(1,1)*} r_1^{(1,0)}(k, l-1). \quad (2.82)$$

In the same manner, the updated auxiliary vertical backward PEF and the newly initiated one can be calculated from the defined PSR shown in Figure 2.31. The update of existing auxiliary vertical backward PEF, $\tilde{e}_1^{(1,1)}(k, l)$, and the newly initiated one, $\tilde{e}_2^{(1,1)}(k, l)$ can be defined as

$$\begin{aligned} \tilde{e}_1^{(1,1)}(k, l) = & \left(y(k-1, l-1) - k_{e_1}^{(0,1)*} y(k-1, l) \right) \\ & - k_{e_1}^{(1,1)*} \left(y(k, l) - k_{r_1}^{(1,0)*} y(k-1, l) \right) \end{aligned} \quad (2.83)$$

$$\begin{aligned} \tilde{e}_2^{(1,1)}(k, l) = & \left(y(k, l-1) - k_{e_1}^{(0,1)*} y(k, l) \right) \\ & - k_{e_2}^{(1,1)*} \left(y(k-1, l) - k_{r_1}^{(1,0)*} y(k, l) \right). \end{aligned} \quad (2.84)$$

By applying the necessary definitions, we can rewritten the equations (2.83) and (2.84) as follows

$$\tilde{e}_1^{(1,1)}(k, l) = \tilde{e}_1^{(0,1)}(k-1, l) - k_{e_1}^{(1,1)} r_1^{(1,0)}(k, l) \quad (2.85)$$

$$\tilde{e}_2^{(1,1)}(k, l) = \tilde{e}_1^{(0,1)}(k, l) - k_{e_2}^{(1,1)} \tilde{r}_1^{(1,0)}(k, l). \quad (2.86)$$

The sum of the mean-square values of the auxiliary existing and newly initiated vertical forward/backward PEFs can be minimized in the mean-square sense, with respect to the reflection coefficient vectors $k_{e_1}^{(0,1)}$ and $k_{e_2}^{(0,1)}$, respectively. The following equation (2.90) will be minimized for existing vertical forward/backward PEFs, after substituting the following definitions

$$\hat{\mathbf{K}}_{\mathbf{e}}^{(1,1)} = \left[k_{e_1}^{(1,1)} \right], \quad (2.87)$$

$$\hat{\mathbf{e}}^{(1,1)}(k, l) = \left[e_1^{(1,1)}(k, l) \right], \quad (2.88)$$

$$\tilde{\mathbf{e}}^{(1,1)}(k, l) = \left[\tilde{e}_1^{(1,1)}(k, l) \right], \quad (2.89)$$

$$P_{\mathbf{e}}^{(1,1)} = \text{tr} \left[E \left[\mathbf{e}^{(1,1)}(k, l) \mathbf{e}^{(1,1)H}(k, l) \right] + E \left[\tilde{\mathbf{e}}^{(1,1)}(k, l) \tilde{\mathbf{e}}^{(1,1)H}(k, l) \right] \right]. \quad (2.90)$$

The lattice reflection vector for existing auxiliary vertical forward/backward PEFs is found as

$$\hat{\mathbf{K}}_{\mathbf{e}}^{(1,1)} = \left[\boldsymbol{\Sigma}_{\tilde{\mathbf{r}}, \tilde{\mathbf{r}}}^{(1,0)} + \boldsymbol{\Sigma}_{\mathbf{r}, \mathbf{r}}^{(1,0)*} \right]^{-1} \left[\boldsymbol{\Delta}_{\tilde{\mathbf{r}}, \mathbf{e}}^{(0,1)} + \boldsymbol{\Delta}_{\mathbf{r}, \tilde{\mathbf{e}}}^{(0,1)*} \right] \quad (2.91)$$

where

$$\begin{aligned} \boldsymbol{\Sigma}_{\mathbf{r}, \mathbf{r}}^{(1,0)} &= E \left[\mathbf{r}^{(1,0)}(k, l) \mathbf{r}^{(1,0)H}(k, l) \right], \\ \boldsymbol{\Sigma}_{\tilde{\mathbf{r}}, \tilde{\mathbf{r}}}^{(1,0)} &= E \left[\tilde{\mathbf{r}}^{(1,0)}(k, l-1) \tilde{\mathbf{r}}^{(1,0)H}(k, l-1) \right], \\ \boldsymbol{\Delta}_{\tilde{\mathbf{r}}, \mathbf{e}}^{(0,1)} &= E \left[\tilde{\mathbf{r}}^{(1,0)}(k, l-1) \mathbf{e}^{(0,1)H}(k, l) \right], \\ \boldsymbol{\Delta}_{\mathbf{r}, \tilde{\mathbf{e}}}^{(0,1)} &= E \left[\mathbf{r}^{(1,0)}(k, l) \tilde{\mathbf{e}}^{(0,1)H}(k-1, l) \right]. \end{aligned}$$

For the newly initiated vertical forward/backward PEFs,

$$P_{e_2}^{(1,1)} = E \left[e_2^{(1,1)}(k, l) e_2^{(1,1)H}(k, l) \right] + E \left[\tilde{e}_2^{(1,1)}(k, l) \tilde{e}_2^{(1,1)H}(k, l) \right] \quad (2.92)$$

The lattice reflection vector for the newly initiated auxiliary vertical forward/backward PEFs is calculated as

$$k_{e_2}^{(1,1)} = \left[\boldsymbol{\Sigma}_{\mathbf{r}, \mathbf{r}}^{(1,0)} + \boldsymbol{\Sigma}_{\tilde{\mathbf{r}}, \tilde{\mathbf{r}}}^{(1,0)*} \right]^{-1} \left[\boldsymbol{\Delta}_{\mathbf{r}, e_1}^{(0,1)} + \boldsymbol{\Delta}_{\tilde{\mathbf{r}}, \tilde{e}_1}^{(0,1)*} \right] \quad (2.93)$$

where

$$\begin{aligned}\Delta_{\mathbf{r},e_1}^{(0,1)} &= E \left[\mathbf{r}^{(1,0)}(k, l-1) e_1^{(0,1)H}(k-1, l) \right], \\ \Delta_{\tilde{\mathbf{r}},\tilde{e}_1}^{(0,1)} &= E \left[\tilde{\mathbf{r}}^{(1,0)}(k, l) \tilde{e}_1^{(0,1)H}(k, l) \right].\end{aligned}$$

It is possible to show lattice reflection values in the one vector as

$$\mathbf{K}_e^{(1,1)} = \left[k_{e_1}^{(1,1)} : k_{e_2}^{(1,1)} \right], \quad (2.94)$$

and both vertical forward/backward PEFs can be combined as

$$\mathbf{e}^{(1,1)} = \left[e_1^{(1,1)} : e_2^{(1,1)} \right]^T \quad (2.95)$$

$$\tilde{\mathbf{e}}^{(1,1)} = \left[\tilde{e}_1^{(1,1)} : \tilde{e}_2^{(1,1)} \right]^T. \quad (2.96)$$

In case of horizontal augmentation, the new order auxiliary horizontal forward and backward PEFs can be obtained by linear combination of the initial stage order PEFs. Thus, it will be enough only to update of the initial stage order auxiliary horizontal forward/backward PEFs. The necessary PSR and the projections for updating the auxiliary horizontal forward/backward PEFs are illustrated in Figure 2.32.

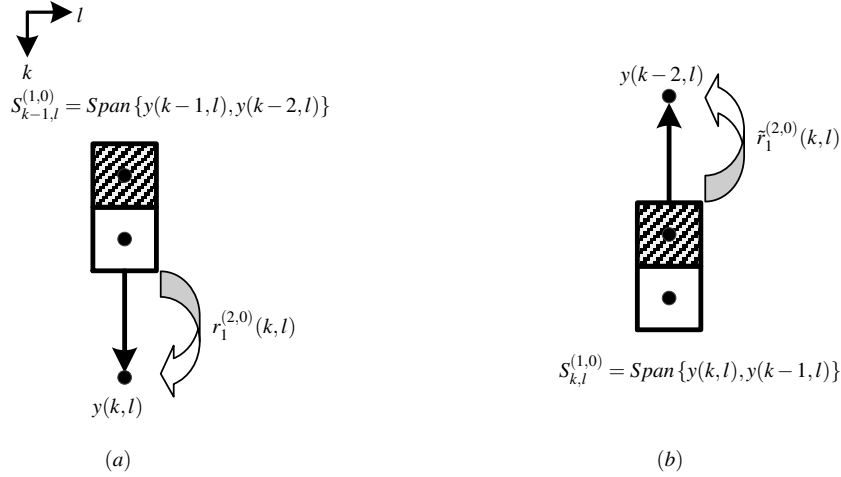


Figure 2.32: Auxiliary horizontal (a) forward and (b) backward PEFs of the PSR $S_{k,l}^{(1,0)}$.

By using projections, we can write the equations for the update of existing auxiliary horizontal forward and backward PEFs which are $r_1^{(2,0)}(k, l)$ and $\tilde{r}_1^{(2,0)}(k, l)$,

respectively.

$$\begin{aligned} r_1^{(2,0)}(k,l) &= \left(y(k,l) - k_{r_1}^{(1,0)*} y(k-1,l) \right) \\ &\quad - k_{r_1}^{(2,0)*} \left(y(k-2,l) - k_{r_1}^{(1,0)} y(k-1,l) \right) \end{aligned} \quad (2.97)$$

$$\begin{aligned} \tilde{r}_1^{(2,0)}(k,l) &= \left(y(k-2,l) - k_{r_1}^{(1,0)} y(k-1,l) \right) \\ &\quad - k_{r_1}^{(2,0)} \left(y(k,l) - k_{r_1}^{(1,0)*} y(k-1,l) \right). \end{aligned} \quad (2.98)$$

After making the necessary simplifications, these equations can be rewritten as

$$r_1^{(2,0)}(k,l) = r_1^{(1,0)}(k,l) - k_{r_1}^{(2,0)*} \tilde{r}_1^{(1,0)}(k-1,l) \quad (2.99)$$

$$\tilde{r}_1^{(2,0)}(k,l) = \tilde{r}_1^{(1,0)}(k-1,l) - k_{r_1}^{(2,0)} r_1^{(1,0)}(k,l). \quad (2.100)$$

For calculating lattice reflection value $k_{r_1}^{(2,0)}$, the sum of the mean-square values of the auxiliary horizontal forward/backward PEFs are minimized in the mean-square sense, with respect to the reflection coefficient vectors $k_{r_1}^{(1,0)}$. If we define,

$$\mathbf{K}_r^{(2,0)} = \begin{bmatrix} k_{r_1}^{(2,0)} \end{bmatrix}, \quad (2.101)$$

$$\mathbf{r}^{(1,0)}(k,l) = \begin{bmatrix} r_1^{(1,0)}(k,l) \end{bmatrix}, \quad (2.102)$$

$$\tilde{\mathbf{r}}^{(1,0)}(k,l) = \begin{bmatrix} \tilde{r}_1^{(1,0)}(k,l) \end{bmatrix}, \quad (2.103)$$

then we can write

$$P_r^{(2,0)} = \text{tr} \left[E \left[\mathbf{r}^{(2,0)}(k,l) \mathbf{r}^{(2,0)H}(k,l) \right] + E \left[\tilde{\mathbf{r}}^{(2,0)}(k,l) \tilde{\mathbf{r}}^{(2,0)H}(k,l) \right] \right] \quad (2.104)$$

and after the minimization with respect to the lattice reflection value,

$$\mathbf{K}_r^{(2,0)} = \left[\mathbf{\Sigma}_{\tilde{\mathbf{r}},\tilde{\mathbf{r}}}^{(1,0)} + \mathbf{\Sigma}_{\mathbf{r},\mathbf{r}}^{(1,0)*} \right]^{-1} \left[\mathbf{\Delta}_{\tilde{\mathbf{r}},\mathbf{r}}^{(1,0)} + \mathbf{\Delta}_{\tilde{\mathbf{r}},\mathbf{r}}^{(1,0)T} \right] \quad (2.105)$$

where $\mathbf{\Delta}_{\tilde{\mathbf{r}},\mathbf{r}}^{(1,0)} = E \left[\tilde{\mathbf{r}}^{(1,0)}(k-1,l) \mathbf{r}^{(1,0)H}(k,l) \right]$.

At that point, a horizontal augmentation is applied and all the related lattice reflection values are calculated and the update of existing of auxiliary vertical and horizontal, and the newly initiated auxiliary vertical forward/backward PEFs equations are explained.

2.8.2 Application of vertical augmentation

Now, after the horizontal augmentation, we have (1,1)th order auxiliary vertical forward/backward PEFs, which are $e_1^{(1,1)}(k,l)$, $e_2^{(1,1)}(k,l)$, $\tilde{e}_1^{(1,1)}(k,l)$, and $\tilde{e}_2^{(1,1)}(k,l)$; and (2,0)th order auxiliary horizontal forward/backward PEFs, which are $r_1^{(2,0)}(k,l)$, $\tilde{r}_1^{(2,0)}(k,l)$. From this point, it is possible to make a horizontal augmentation again or a vertical augmentation causing a growth horizontally. In order to come to the desired stage order, the augmentation sequence is not very important. In other words, these augmentation steps can be repeated freely until the desired rectangular PSR is obtained. At that point, we will choose the vertical augmentation to continue.

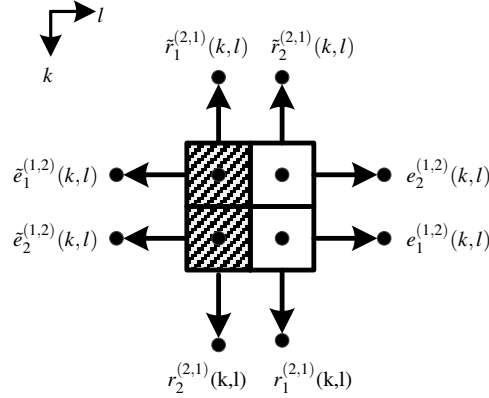


Figure 2.33: Auxiliary vertical and horizontal forward/backward PEFs of the PSR $S_{k,l}^{(1,1)}$.

In order to vertically augment the PSR, $S_{k,l}^{(1,0)}$, two new points are added vertically in the horizontal direction as shown in Figure 2.33. The obtained square PSR, $S_{k,l}^{(1,1)}$ has four neighboring points. Due to the vertically augmented PSR, $S_{k,l}^{(1,1)}$ has new introduced auxiliary horizontal forward and backward PEFs to be predicted. Therefore, similar to the horizontal augmentation, a set of new PEFs, $r_2^{(2,0)}(k,l) = r_1^{(2,0)}(k,l)$ and $\tilde{r}_2^{(2,0)}(k,l) = \tilde{r}_1^{(2,0)}(k,l)$ must be predefined. Projections of the update of existing auxiliary vertical forward/backward and all the update of existing and the newly initiated auxiliary horizontal forward/backward PEFs are illustrated in Figure 2.34 and Figure 2.35, respectively.

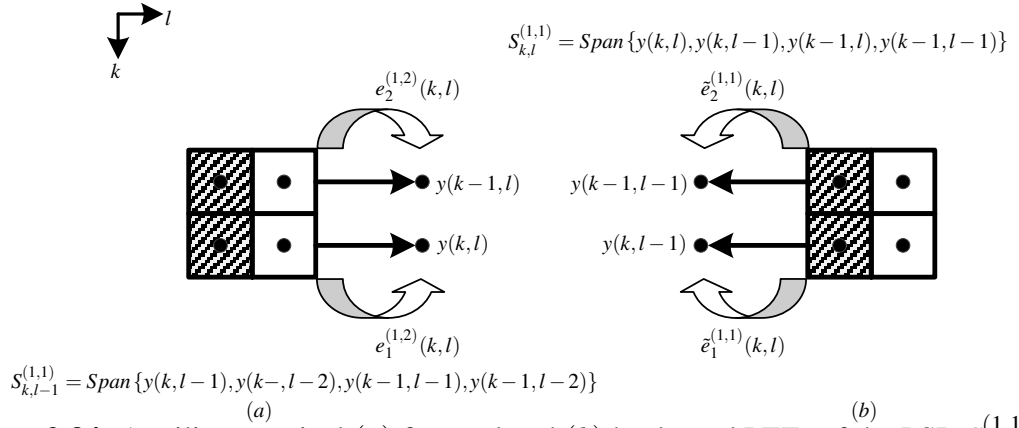


Figure 2.34: Auxiliary vertical (a) forward and (b) backward PEFs of the PSR $S_{k,l}^{(1,1)}$.

Auxiliary vertical forward/backward PEFS are only updated by combining linearly of the previous stage order auxiliary vertical forward/backward PEFs and the lattice structure equations can be written as

$$\mathbf{e}^{(1,2)}(k, l) = \mathbf{e}^{(1,1)}(k, l) - \mathbf{K}_{\mathbf{e}}^{(1,2)H} \tilde{\mathbf{e}}^{(1,1)}(k, l-1) \quad (2.106)$$

$$\tilde{\mathbf{e}}^{(1,2)}(k, l) = \tilde{\mathbf{e}}^{(1,1)}(k, l-1) - \mathbf{K}_{\mathbf{e}}^{(1,2)T} \mathbf{e}^{(1,1)}(k, l). \quad (2.107)$$

where

$$\mathbf{e}^{(1,2)}(k, l) = \begin{bmatrix} e_1^{(1,2)}(k, l) & e_2^{(1,2)}(k, l) \end{bmatrix}^T,$$

$$\tilde{\mathbf{e}}^{(1,2)}(k, l) = \begin{bmatrix} \tilde{e}_1^{(1,2)}(k, l) & \tilde{e}_2^{(1,2)}(k, l) \end{bmatrix}^T,$$

$$\mathbf{e}^{(1,1)}(k, l) = \begin{bmatrix} e_1^{(1,1)}(k, l) & e_2^{(1,1)}(k, l) \end{bmatrix}^T,$$

$$\tilde{\mathbf{e}}^{(1,1)}(k, l) = \begin{bmatrix} \tilde{e}_1^{(1,1)}(k, l) & \tilde{e}_2^{(1,1)}(k, l) \end{bmatrix}^T,$$

$$\begin{aligned} \mathbf{K}_{\mathbf{e}}^{(1,2)} &= \begin{bmatrix} \mathbf{k}_{e_1}^{(1,2)} & \mathbf{k}_{e_2}^{(1,2)} \end{bmatrix} \\ &= \begin{bmatrix} k_{e_1}^{(1,2)}(1) & k_{e_2}^{(1,2)}(1) \\ k_{e_1}^{(1,2)}(2) & k_{e_2}^{(1,2)}(2) \end{bmatrix}. \end{aligned}$$

In order to calculate the vertical reflection coefficient matrix at the (1, 2)th order stage, the sum of the mean-square values of the auxiliary vertical forward and backward PEFs will be minimized in the mean-square sense,

$$P_{\mathbf{e}}^{(1,2)} = \text{tr} \left[E \left[\mathbf{e}^{(1,2)}(k, l) \mathbf{e}^{(1,2)H}(k, l) \right] + E \left[\tilde{\mathbf{e}}^{(1,2)}(k, l) \tilde{\mathbf{e}}^{(1,2)H}(k, l) \right] \right]. \quad (2.108)$$

If we minimize the equation (2.108) with respect to the lattice model reflection coefficient vectors, $\mathbf{K}_e^{(1,2)}$:

$$\mathbf{K}_e^{(1,2)} = \left[\boldsymbol{\Sigma}_{\tilde{\mathbf{e}}, \tilde{\mathbf{e}}}^{(1,1)} + \boldsymbol{\Sigma}_{\mathbf{e}, \mathbf{e}}^{(1,1)*} \right]^{-1} \left[\boldsymbol{\Delta}_{\tilde{\mathbf{e}}, \mathbf{e}}^{(1,1)} + \boldsymbol{\Delta}_{\tilde{\mathbf{e}}, \mathbf{e}}^{(1,1)T} \right] \quad (2.109)$$

where

$$\begin{aligned} \boldsymbol{\Sigma}_{\tilde{\mathbf{e}}, \tilde{\mathbf{e}}}^{(1,1)} &= E \left[\mathbf{e}^{(1,1)}(k, l) \mathbf{e}^{(1,1)H}(k, l) \right], \\ \boldsymbol{\Sigma}_{\mathbf{e}, \mathbf{e}}^{(1,1)} &= E \left[\tilde{\mathbf{e}}^{(1,1)}(k, l-1) \tilde{\mathbf{e}}^{(1,1)H}(k, l-1) \right], \\ \boldsymbol{\Delta}_{\tilde{\mathbf{e}}, \mathbf{e}}^{(1,1)} &= E \left[\tilde{\mathbf{e}}^{(1,1)}(k, l-1) \mathbf{e}^{(1,1)H}(k, l) \right]. \end{aligned}$$

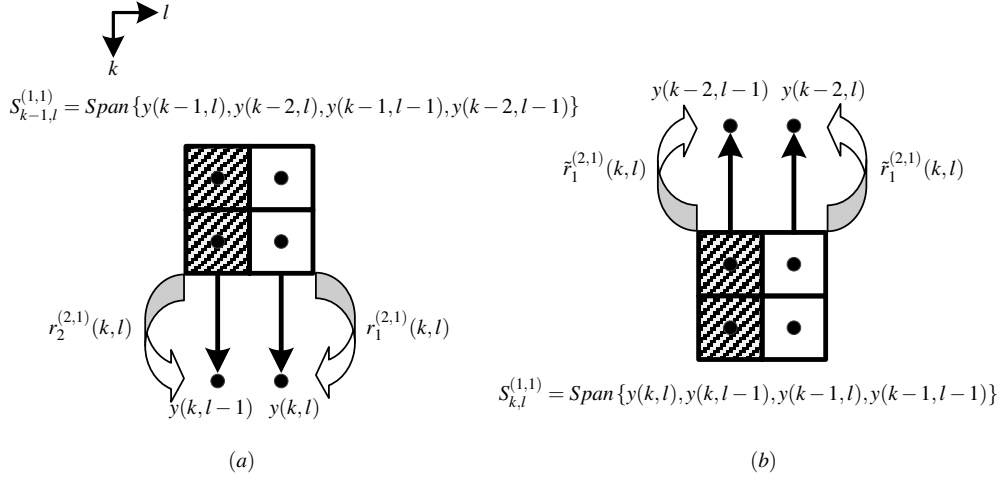


Figure 2.35: Auxiliary horizontal (a) forward and (b) backward PEFs of the PSR $S_{k,l}^{(1,1)}$.

Similarly, the error propagation equations for the update of existing and the newly initiated auxiliary horizontal forward and backward PEFs can be given as follows for the update of existing ones

$$\hat{\mathbf{r}}^{(2,1)}(k, l) = \mathbf{r}^{(2,0)}(k, l) - \hat{\mathbf{K}}_{\mathbf{r}}^{(2,1)H} \tilde{\mathbf{e}}^{(1,1)}(k-1, l) \quad (2.110)$$

$$\tilde{\mathbf{r}}^{(2,1)}(k, l) = \tilde{\mathbf{r}}^{(2,0)}(k, l-1) - \hat{\mathbf{K}}_{\mathbf{r}}^{(2,1)T} \mathbf{e}^{(1,1)}(k, l) \quad (2.111)$$

and for newly initiated ones

$$r_2^{(2,1)}(k, l) = r_2^{(2,0)}(k, l-1) - \mathbf{k}_{r_2}^{(2,1)H} \mathbf{e}^{(1,1)}(k-1, l) \quad (2.112)$$

$$\tilde{r}_2^{(2,1)}(k, l) = \tilde{r}_2^{(2,0)}(k, l) - \mathbf{k}_{r_2}^{(2,1)T} \tilde{\mathbf{e}}^{(1,1)}(k, l) \quad (2.113)$$

where

$$\begin{aligned}\hat{\mathbf{r}}^{(2,1)}(k, l) &= \left[r_1^{(2,1)}(k, l) \right], \\ \tilde{\mathbf{r}}^{(2,1)}(k, l) &= \left[\tilde{r}_1^{(2,1)}(k, l) \right], \\ \mathbf{r}^{(2,0)}(k, l) &= \left[r_1^{(2,0)}(k, l) \ r_2^{(2,0)}(k, l) \right]^T, \\ \tilde{\mathbf{r}}^{(2,0)}(k, l) &= \left[\tilde{r}_1^{(2,0)}(k, l) \ \tilde{r}_2^{(2,0)}(k, l) \right]^T,\end{aligned}$$

$$\begin{aligned}\hat{\mathbf{K}}_{\mathbf{r}}^{(2,1)} &= \left[\mathbf{k}_{r_1}^{(2,1)} \right] = \left[k_{r_1}^{(2,1)}(1) \ k_{r_1}^{(2,1)}(2) \right]^T, \\ \mathbf{k}_{r_2}^{(2,1)} &= \left[k_{r_2}^{(2,1)}(1) \ k_{r_2}^{(2,1)}(2) \right]^T.\end{aligned}$$

We know that $\hat{\mathbf{K}}_{\mathbf{r}}^{(2,1)}$ and $\mathbf{k}_{r_2}^{(2,1)}$ are the reflection coefficient vectors. It is possible to write the $(2, 1)$ th order horizontal reflection coefficient matrix as

$$\mathbf{K}_{\mathbf{r}}^{(2,1)} = \left[\hat{\mathbf{K}}_{\mathbf{r}}^{(2,1)} : \mathbf{k}_{r_2}^{(2,1)} \right]. \quad (2.114)$$

In addition, it is also possible to combine existing auxiliary horizontal forward/backward vectors and the newly initiated ones into the one vector as follows

$$\mathbf{r}^{(2,1)}(k, l) = \left[\hat{\mathbf{r}}^{(2,1)}(k, l) : r_2^{(2,1)}(k, l) \right]^T, \quad (2.115)$$

$$\tilde{\mathbf{r}}^{(2,1)}(k, l) = \left[\tilde{\mathbf{r}}^{(2,1)}(k, l) : \tilde{r}_2^{(2,1)}(k, l) \right]^T. \quad (2.116)$$

The reflection coefficient vector for the update of existing auxiliary horizontal forward/backward PEFs is calculated by minimizing the sum of the mean-square values of the existing auxiliary horizontal forward/backward PEFs in the mean-square sense,

$$\hat{P}_{\mathbf{r}}^{(2,1)} = \text{tr} \left[E \left[\hat{\mathbf{r}}^{(2,1)}(k, l) \hat{\mathbf{r}}^{(2,1)H}(k, l) \right] + E \left[\tilde{\mathbf{r}}^{(2,1)}(k, l) \tilde{\mathbf{r}}^{(2,1)H}(k, l) \right] \right]. \quad (2.117)$$

If the equation (2.117) is minimized with respect to the lattice model reflection coefficient vectors, $\hat{\mathbf{K}}_{\mathbf{r}}^{(2,1)}$,

$$\hat{\mathbf{K}}_{\mathbf{r}}^{(2,1)} = \left[\boldsymbol{\Sigma}_{\tilde{\mathbf{e}}, \tilde{\mathbf{e}}}^{(1,1)} + \boldsymbol{\Sigma}_{\mathbf{e}, \mathbf{e}}^{(1,1)*} \right]^{-1} \left[\boldsymbol{\Delta}_{\tilde{\mathbf{e}}, \mathbf{r}}^{(2,0)} + \boldsymbol{\Delta}_{\mathbf{e}, \tilde{\mathbf{r}}}^{(2,0)*} \right] \quad (2.118)$$

where

$$\begin{aligned}\boldsymbol{\Delta}_{\tilde{\mathbf{e}}, \mathbf{r}}^{(2,0)} &= E \left[\tilde{\mathbf{e}}^{(1,1)}(k-1, l) \mathbf{r}^{(2,0)H}(k, l) \right], \\ \boldsymbol{\Delta}_{\mathbf{e}, \tilde{\mathbf{r}}}^{(2,0)} &= E \left[\mathbf{e}^{(1,1)}(k, l) \tilde{\mathbf{r}}^{(2,0)H}(k, l-1) \right].\end{aligned}$$

The reflection coefficient vector for the newly initiated one can be calculated by minimizing the sum of the mean-square values of the newly initiated auxiliary horizontal forward/backward PEFs in the mean-square sense,

$$P_{r_2}^{(2,1)} = \text{tr} \left[E \left[r_2^{(2,1)}(k, l) r_2^{(2,1)H}(k, l) \right] + E \left[\tilde{r}_2^{(2,1)}(k, l) \tilde{r}_2^{(2,1)H}(k, l) \right] \right]. \quad (2.119)$$

By minimizing with respect to the $\mathbf{k}_{r_2}^{(2,1)}$,

$$\mathbf{k}_{r_2}^{(2,1)} = \left[\boldsymbol{\Sigma}_{\mathbf{e}, \mathbf{e}}^{(1,1)} + \boldsymbol{\Sigma}_{\tilde{\mathbf{e}}, \tilde{\mathbf{e}}}^{(1,1)*} \right]^{-1} \left[\boldsymbol{\Delta}_{\mathbf{e}, r_1}^{(2,0)} + \boldsymbol{\Delta}_{\tilde{\mathbf{e}}, \tilde{r}_1}^{(2,0)*} \right] \quad (2.120)$$

where

$$\begin{aligned} \boldsymbol{\Delta}_{\mathbf{e}, r_1}^{(2,0)} &= E \left[\mathbf{e}^{(1,1)}(k-1, l) r_1^{(2,0)H}(k, l) \right], \\ \boldsymbol{\Delta}_{\tilde{\mathbf{e}}, \tilde{r}_1}^{(2,0)} &= E \left[\tilde{\mathbf{e}}^{(1,1)}(k, l) \tilde{r}_1^{(2,0)H}(k, l) \right]. \end{aligned}$$

2.9 Summarize of Augmentation Method

As seen in Section 2.8.1 and 2.8.2, after each augmentation, we have one set of auxiliary vertical and one set of auxiliary horizontal forward/backward PEFs lattice propagation equations for the update of existing auxiliary vertical and horizontal forward/backward PEFs. Additionally, we have one set of newly initiated lattice propagation equations for auxiliary vertical forward/backward PEFs in case of horizontal augmentation. Similarly, at the vertical augmentation, a set of newly initiated lattice propagation equations for auxiliary horizontal forward/backward PEFs are introduced.

We can easily formulate the general lattice equations for existing and the newly initiated ones of the both auxiliary vertical and horizontal forward/backward PEFs, which are already defined in Section 2.7 and 2.7, respectively. We know that, at the $(p-1, q-1)$ th order stage, the rectangular PSR, $S_{k,l}^{(p-1, q-1)}$ has $p \times q$ points. Hence, two sets of auxiliary vertical and horizontal forward PEFs can be defined

$$\mathbf{e}^{(p-1, q)}(k, l) = \left[e_1^{(p-1, q)}(k, l) \cdots e_p^{(p-1, q)}(k, l) \right]^T, \quad (2.121)$$

$$\mathbf{r}^{(p, q-1)}(k, l) = \left[r_1^{(p, q-1)}(k, l) \cdots r_q^{(p, q-1)}(k, l) \right]^T. \quad (2.122)$$

The related backward versions of them are given $\tilde{\mathbf{e}}^{(p-1, q)}$ and $\tilde{\mathbf{r}}^{(p, q-1)}$. If an augmentation is applied on $S_{k,l}^{(p-1, q-1)}$ in the vertical direction which means

horizontal augmentation, the augmented PSR will be stated as $S_{k,l}^{(p,q-1)} = S_{k,l}^{(p-1,q-1)} \cup S_{k-p,l}^{(0,q-1)}$ which is depicted in 2.12. Consequently, the recursion of the PEFs for the $(p, q-1)$ th order stage PSR $S_{k,l}^{(p,q-1)}$ can be presented in the equations (2.123) and (2.124) as a linear combination of $(2(p+q))$ lower order PEFs. Detailed explanations are already given in Section 2.6.

$$\begin{bmatrix} \hat{\mathbf{e}}^{(p,q)}(k,l) \\ e_{p+1}^{(p,q)}(k-1,l) \\ \tilde{\mathbf{e}}^{(p,q)}(k,l) \\ \tilde{e}_{p+1}^{(p,q)}(k,l) \end{bmatrix} = \begin{bmatrix} \mathbf{e}^{(p-1,q)}(k,l) \\ e_p^{(p-1,q)}(k-1,l) \\ \tilde{\mathbf{e}}^{(p-1,q)}(k-1,l) \\ \tilde{e}_p^{(p-1,q)}(k,l) \end{bmatrix} + \begin{bmatrix} \mathbf{O}_{(p+1) \times 2q} & \mathbf{O}_{p \times q} & -\hat{\mathbf{K}}_e^{(p,q)H} \\ -\hat{\mathbf{K}}_e^{(p,q)T} & \mathbf{O}_{p \times q} & -\mathbf{k}_{e_{p+1}}^{(p,q)*} \\ \mathbf{0}_q^T & -\mathbf{k}_{e_{p+1}}^{(p,q)T} & \mathbf{0}_q^T \\ \mathbf{O}_{(p+1) \times 2q} & \mathbf{O}_{p \times q} & \mathbf{0}_q^T \end{bmatrix} \times \begin{bmatrix} \mathbf{r}^{(p,q-1)}(k,l) \\ \tilde{\mathbf{r}}^{(p,q-1)}(k,l) \\ \mathbf{r}^{(p,q-1)}(k,l-1) \\ \tilde{\mathbf{r}}^{(p,q-1)}(k,l-1) \end{bmatrix}, \quad (2.123)$$

$$\begin{bmatrix} \mathbf{r}^{(p+1,q-1)}(k,l) \\ \tilde{\mathbf{r}}^{(p+1,q-1)}(k,l) \end{bmatrix} = \begin{bmatrix} \mathbf{I}_p & -\mathbf{K}_r^{(p+1,q-1)H} \\ -\mathbf{K}_r^{(p+1,q-1)T} & \mathbf{I}_p \end{bmatrix} \times \begin{bmatrix} \mathbf{r}^{(p,q-1)}(k,l) \\ \tilde{\mathbf{r}}^{(p,q-1)}(k-1,l) \end{bmatrix} \quad (2.124)$$

The auxiliary vertical forward/backward PEFs vectors, $\mathbf{e}^{(p,q)}(k,l)$ and $\tilde{\mathbf{e}}^{(p,q)}(k,l)$, at the (p,q) th order stage are represented with the combination of the update of existing and the newly initiated auxiliary vertical forward/backward PEFs.

$$\begin{bmatrix} \mathbf{e}^{(p,q)}(k,l) \\ \tilde{\mathbf{e}}^{(p,q)}(k,l) \end{bmatrix} = \begin{bmatrix} \hat{\mathbf{e}}^{(p,q)}(k,l) \\ e_{p+1}^{(p,q)}(k,l) \\ \tilde{\mathbf{e}}^{(p,q)}(k,l) \\ \tilde{e}_{p+1}^{(p,q)}(k,l) \end{bmatrix} \quad (2.125)$$

After the horizontal augmentation, the PSR $S_{k,l}^{(p,q-1)}$ can be augmented in the horizontal direction which means *vertical augmentation*, and the augmented PSR becomes $S_{k,l}^{(p,q)} = S_{k,l}^{(p,q-1)} \cup S_{k,l-q}^{(p,0)}$ as shown in Figure 2.12. The propagation equations for the update of existing and the newly initiated auxiliary horizontal, and the update of existing auxiliary vertical forward/backward PEFs can be written as in equation (2.126)

and (2.127), respectively. Detail explanations for the vertical augmentation are already given in Section 2.7.

$$\begin{aligned}
\begin{bmatrix} \hat{\mathbf{r}}^{(p,q)}(k,l) \\ \mathbf{r}_{q+1}^{(p,q)}(k,l-1) \\ \tilde{\mathbf{r}}^{(p,q)}(k,l) \\ \tilde{\mathbf{r}}_{q+1}^{(p,q)}(k,l) \end{bmatrix} &= \begin{bmatrix} \mathbf{r}^{(p,q-1)}(k,l) \\ \mathbf{r}_q^{(p,q-1)}(k,l-1) \\ \tilde{\mathbf{r}}^{(p,q-1)}(k,l-1) \\ \tilde{\mathbf{r}}_q^{(p,q-1)}(k,l) \end{bmatrix} \\
&+ \begin{bmatrix} \mathbf{O}_{(q+1) \times 2p} & \mathbf{O}_{q \times p} & -\hat{\mathbf{K}}_r^{(p,q)H} \\ -\hat{\mathbf{K}}_r^{(p,q)T} & \mathbf{O}_{q \times p} & -\mathbf{k}_{r_{q+1}}^{(p,q)*} \\ \mathbf{0}_p^T & -\mathbf{k}_{r_{q+1}}^{(p,q)T} & \mathbf{0}_p^T \\ \mathbf{O}_{(q+1) \times 2p} & \mathbf{O}_{q \times p} & \mathbf{0}_p^T \end{bmatrix} \\
&\times \begin{bmatrix} \mathbf{e}^{(p-1,q)}(k,l) \\ \tilde{\mathbf{e}}^{(p-1,q)}(k,l) \\ \mathbf{e}^{(p-1,q)}(k-1,l) \\ \tilde{\mathbf{e}}^{(p-1,q)}(k-1,l) \end{bmatrix}, \tag{2.126}
\end{aligned}$$

$$\begin{aligned}
\begin{bmatrix} \mathbf{e}^{(p-1,q+1)}(k,l) \\ \tilde{\mathbf{e}}^{(p-1,q+1)}(k,l) \end{bmatrix} &= \begin{bmatrix} \mathbf{I}_p & -\mathbf{K}_e^{(p-1,q+1)H} \\ -\mathbf{K}_e^{(p-1,q+1)T} & \mathbf{I}_p \end{bmatrix} \\
&\times \begin{bmatrix} \mathbf{e}^{(p-1,q)}(k,l) \\ \tilde{\mathbf{e}}^{(p-1,q)}(k,l-1) \end{bmatrix} \tag{2.127}
\end{aligned}$$

The auxiliary horizontal forward/backward PEFs vectors, $\mathbf{r}^{(p,q)}(k,l)$ and $\tilde{\mathbf{r}}^{(p,q)}(k,l)$, at the (p,q) th order stage are represented with the combination of the update of existing and the newly initiated auxiliary horizontal forward/backward PEFs.

$$\begin{bmatrix} \mathbf{r}^{(p,q)}(k,l) \\ \tilde{\mathbf{r}}^{(p,q)}(k,l) \end{bmatrix} = \begin{bmatrix} \hat{\mathbf{r}}^{(p,q)}(k,l) \\ \mathbf{r}_{q+1}^{(p,q)}(k,l) \\ \tilde{\mathbf{r}}^{(p,q)}(k,l) \\ \tilde{\mathbf{r}}_{q+1}^{(p,q)}(k,l) \end{bmatrix} \tag{2.128}$$

3. PROPOSED GENERAL 2-D LATTICE FILTERS

3.1 Purpose

In this chapter, it will be explained how to obtain all types of 2-D lattice prediction models, concurrently. After the calculation of all the auxiliary horizontal and vertical PEFs introduced in Chapter 2, it is possible to obtain QP, ASHP, and NC half-plane lattice prediction models by employing the calculated auxiliary vertical and horizontal forward/backward PEFs. It will be shown that *eight* causal QP models, *two* types causal AR models and one noncausal half-plane prediction model for each of the auxiliary horizontal and vertical forward/backward PEFs can be structured. After explaining all model extractions, it will be stated how to unify all of the proposed prediction models. Then, stability of the predicted models will be studied and finally, the proposed method will be compared with the existing 2-D lattice models.

3.2 Quarter-Plane Models

For a $(M - 1, N - 1)$ th order rectangular, $S_{k,l}^{(M-1,N-1)}$, *eight* QP AR models with different recurse directions can be obtained by using the appropriate auxiliary PEFs shown in Figure 3.1. In order to generate the first- and third-quadrant QP PEFs, a pair of $e_1^{(M-1,N)}(k,l)$ and $\tilde{e}_1^{(M-1,N)}(k,l)$, or a pair of $r_1^{(M,N-1)}(k,l)$ and $\tilde{r}_1^{(M,N-1)}(k,l)$ are predicted by the linear combination of the other PEFs, in the same column or in the same row, respectively.

For example, the $(M - 1, N - 1)$ th order rectangular PSR, $S_{k,l}^{(M-1,N-1)}$ has $M \times N$ points and its auxiliary vertical forward and backward PEFs can be defined as in Figure 3.2. The vertical first- and second quadrant QP PEFs, $e_{QP_1}^{(M-1,N)}(k,l)$ and $e_{QP_2}^{(M-1,N)}(k,l)$ are obtained by the linear prediction of $e_1^{(M-1,N)}(k,l)$ from $e_2^{(M-1,N)}(k,l), \dots, e_M^{(M-1,N)}(k,l)$ and $e_M^{(M-1,N)}(k,l)$ from $e_1^{(M-1,N)}(k,l), \dots, e_{M-1}^{(M-1,N)}(k,l)$, respectively. The vertical third- and fourth-quadrant QP models $\tilde{e}_{QP_1}^{(M-1,N)}(k,l)$ and $\tilde{e}_{QP_2}^{(M-1,N)}(k,l)$ considered as corresponding backward

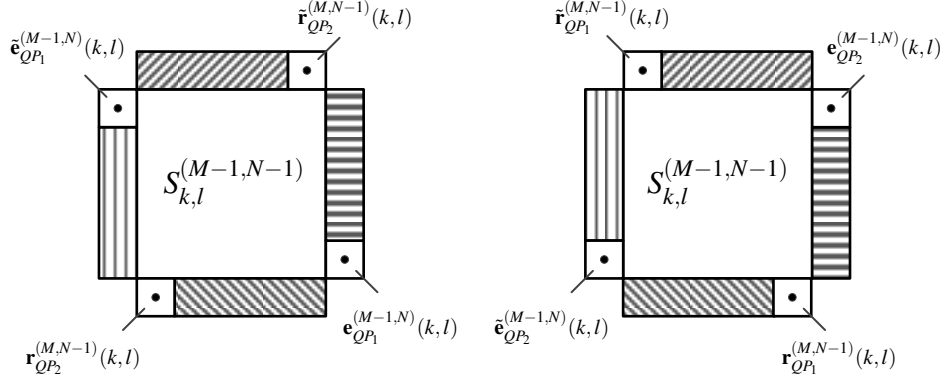


Figure 3.1: Generalized QP model 2-D filters for the PSR $S_{k,l}^{(M-1,N-1)}$.

versions of the vertical first- second quadrant QP models are generated by the prediction of $\tilde{e}_M^{(M-1,N)}(k,l)$ from $\tilde{e}_1^{(M-1,N)}(k,l), \dots, \tilde{e}_{M-1}^{(M-1,N)}(k,l)$ and $\tilde{e}_1^{(M-1,N)}(k,l)$ from $\tilde{e}_2^{(M-1,N)}(k,l), \dots, \tilde{e}_M^{(M-1,N)}(k,l)$, respectively.

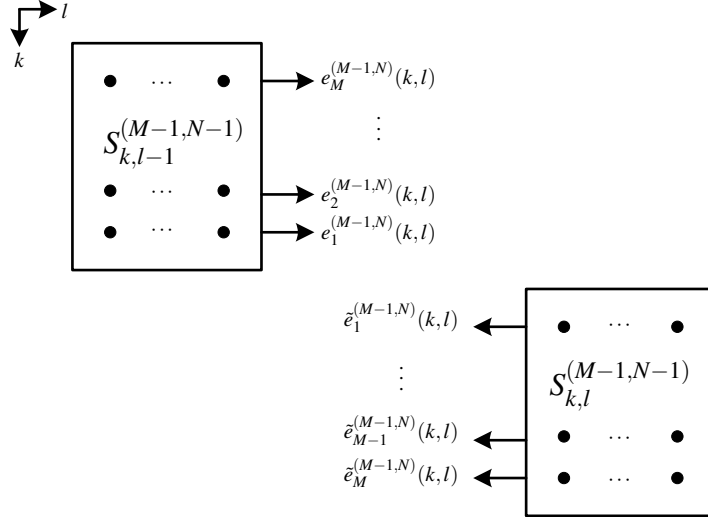


Figure 3.2: The vertical PEFs for the first- and second-quadrant QP models of the PSR $S_{k,l}^{(M-1,N-1)}$.

Therefore, the vertical first- and third-quadrant QP PEFs, $e_{QP1}^{(M-1,N)}(k,l)$ and $\tilde{e}_{QP1}^{(M-1,N)}(k,l)$, and the second- and fourth-quadrant QP PEFs, $e_{QP2}^{(M-1,N)}(k,l)$ and $\tilde{e}_{QP2}^{(M-1,N)}(k,l)$, can be written as

$$\begin{aligned} e_{QP1}^{(M-1,N)}(k,l) &= e_1^{(M-1,N)}(k,l) - \mathbf{k}_{e_{QP1}}^{(M-1,N)H} \mathbf{\Psi}_1 \mathbf{e}^{(M-1,N)}(k,l) \\ \tilde{e}_{QP1}^{(M-1,N)}(k,l) &= \tilde{e}_1^{(M-1,N)}(k,l) - \mathbf{k}_{e_{QP1}}^{(M-1,N)T} \mathbf{\Psi}_1 \tilde{\mathbf{e}}^{(M-1,N)}(k,l) \end{aligned} \quad (3.1)$$

and

$$\begin{aligned} e_{QP_2}^{(M-1,N)}(k,l) &= e_M^{(M-1,N)}(k,l) - \mathbf{k}_{e_{QP_2}}^{(M-1,N)H} \Psi_M \mathbf{e}^{(M-1,N)}(k,l) \\ \tilde{e}_{QP_2}^{(M-1,N)}(k,l) &= \tilde{e}_M^{(M-1,N)}(k,l) - \mathbf{k}_{e_{QP_2}}^{(M-1,N)T} \Psi_M \tilde{\mathbf{e}}^{(M-1,N)}(k,l) \end{aligned} \quad (3.2)$$

where the constant matrix Ψ_j is defined by removing the j th row of an identity matrix with an appropriate order. $\mathbf{k}_{e_{QP_1}}^{(M-1,N)}$ and $\mathbf{k}_{e_{QP_2}}^{(M-1,N)}$ are the vertical first- and second-quadrant QP lattice parameter reflection vectors, respectively. The reflection vectors $\mathbf{k}_{e_{QP_1}}^{(M-1,N)}$ and $\mathbf{k}_{e_{QP_2}}^{(M-1,N)}$ can be calculated by minimizing the sum of the mean-square values of $e_{QP_1}^{(M-1,N)}$ and $\tilde{e}_{QP_1}^{(M-1,N)}$ in (3.3), and $e_{QP_2}^{(M-1,N)}$ and $\tilde{e}_{QP_2}^{(M-1,N)}$ in (3.4), respectively.

$$P_{e_{QP_1}}^{(M-1,N)} = E \left[\left| e_{QP_1}^{(M-1,N)}(k,l) \right|^2 \right] + E \left[\left| \tilde{e}_{QP_1}^{(M-1,N)}(k,l) \right|^2 \right] \quad (3.3)$$

$$P_{e_{QP_2}}^{(M-1,N)} = E \left[\left| e_{QP_2}^{(M-1,N)}(k,l) \right|^2 \right] + E \left[\left| \tilde{e}_{QP_2}^{(M-1,N)}(k,l) \right|^2 \right] \quad (3.4)$$

If these equations are minimized with respect to the $\mathbf{k}_{e_{QP_1}}^{(M-1,N)}$ and $\mathbf{k}_{e_{QP_2}}^{(M-1,N)}$, respectively, the results of these minimizations are summarized in the following equations:

$$\begin{aligned} \mathbf{k}_{e_{QP_1}}^{(M-1,N)} &= \left[\Psi_1 \left(\Sigma_{\mathbf{e},\mathbf{e}}^{(M-1,N)} + \Sigma_{\tilde{\mathbf{e}},\tilde{\mathbf{e}}}^{(M-1,N)*} \right) \Psi_1^T \right]^{-1} \\ &\quad \times \Psi_1 \left(\Sigma_{\mathbf{e},\mathbf{e}}^{(M-1,N)} + \Sigma_{\tilde{\mathbf{e}},\tilde{\mathbf{e}}}^{(M-1,N)*} \right) \Pi_1^T \end{aligned} \quad (3.5)$$

$$\begin{aligned} \mathbf{k}_{e_{QP_2}}^{(M-1,N)} &= \left[\Psi_M \left(\Sigma_{\mathbf{e},\mathbf{e}}^{(M-1,N)} + \Sigma_{\tilde{\mathbf{e}},\tilde{\mathbf{e}}}^{(M-1,N)*} \right) \Psi_M^T \right]^{-1} \\ &\quad \times \Psi_M \left(\Sigma_{\mathbf{e},\mathbf{e}}^{(M-1,N)} + \Sigma_{\tilde{\mathbf{e}},\tilde{\mathbf{e}}}^{(M-1,N)*} \right) \Pi_M^T \end{aligned} \quad (3.6)$$

where $\Sigma_{\mathbf{e},\mathbf{e}}^{(M-1,N)}$ and $\Sigma_{\tilde{\mathbf{e}},\tilde{\mathbf{e}}}^{(M-1,N)}$ are the sum of the covariance matrices of forward and backward vertical auxiliary PEF vectors, respectively. The constant Π_j is the j th row vector of an identity matrix with an appropriate order.

It is clear that Π_j and Ψ_j are strongly related to the identity matrix with the appropriate order. For the vertical first- and third-quadrant QP PEFs, these are defined as

$$\mathbf{I}_{M \times M} = \begin{bmatrix} \Pi_1 \\ \Psi_1 \end{bmatrix} \quad (3.7)$$

and for the vertical second- and fourth-quadrant QP PEFs, the definition will be

$$\mathbf{I}_{M \times M} = \begin{bmatrix} \Psi_M \\ \Pi_M \end{bmatrix}. \quad (3.8)$$

The power propagation equations for the mean-square values of the vertical first- and second-quadrant QP PEFs from stage to stage can be built up as follows

$$P_{e_{QP_1}}^{(M-1,N)} = P_{e_1}^{(M-1,N)} - \mathbf{k}_{e_{QP_1}}^{(M-1,N)} \Psi_1 \left(\Sigma_{\mathbf{e},\mathbf{e}}^{(M-1,N)} + \Sigma_{\tilde{\mathbf{e}},\tilde{\mathbf{e}}}^{(M-1,N)*} \right) \Psi_1^T \mathbf{k}_{e_{QP_1}}^{(M-1,N)} \quad (3.9)$$

$$P_{e_{QP_2}}^{(M-1,N)} = P_{e_M}^{(M-1,N)} - \mathbf{k}_{e_{QP_2}}^{(M-1,N)} \Psi_M \left(\Sigma_{\mathbf{e},\mathbf{e}}^{(M-1,N)} + \Sigma_{\tilde{\mathbf{e}},\tilde{\mathbf{e}}}^{(M-1,N)*} \right) \Psi_M^T \mathbf{k}_{e_{QP_2}}^{(M-1,N)} \quad (3.10)$$

where

$$P_{e_1}^{(M-1,N)} = E \left[\left| e_1^{(M-1,N)}(k,l) \right|^2 \right] + E \left[\left| \tilde{e}_1^{(M-1,N)}(k,l) \right|^2 \right]$$

and

$$P_{e_M}^{(M-1,N)} = E \left[\left| e_M^{(M-1,N)}(k,l) \right|^2 \right] + E \left[\left| \tilde{e}_M^{(M-1,N)}(k,l) \right|^2 \right].$$

Similarly, auxiliary horizontal forward and backward PEFs of the $(M-1, N-1)$ th order rectangular PSR, $S_{k,l}^{(M-1,N-1)}$ can be defined as in Figure 3.3. The horizontal first- and fourth-quadrant QP PEFs $r_{QP_1}^{(M,N-1)}(k,l)$ and $r_{QP_2}^{(M,N-1)}(k,l)$ are obtained by predicting the first auxiliary PEFs, $r_1^{(M,N-1)}(k,l)$ from $r_2^{(M,N-1)}(k,l), \dots, r_N^{(M,N-1)}(k,l)$, and the final auxiliary PEFs, $r_N^{(M,N-1)}(k,l)$ from $r_2^{(M,N-1)}(k,l), \dots, r_{N-1}^{(M,N-1)}(k,l)$, respectively. In the same manner, the horizontal third- and second-quadrant QP PEFs $\tilde{r}_{QP_1}^{(M,N-1)}(k,l)$ and $\tilde{r}_{QP_2}^{(M,N-1)}(k,l)$ considered as corresponding backward versions of the horizontal first- and fourth-quadrant QP models are generated by the linear prediction of $\tilde{r}_1^{(M,N-1)}(k,l)$ from $\tilde{r}_2^{(M,N-1)}(k,l), \dots, \tilde{r}_N^{(M,N-1)}(k,l)$ and $\tilde{r}_N^{(M,N-1)}(k,l)$ from $\tilde{r}_1^{(M,N-1)}(k,l), \dots, \tilde{r}_{N-1}^{(M,N-1)}(k,l)$, respectively.

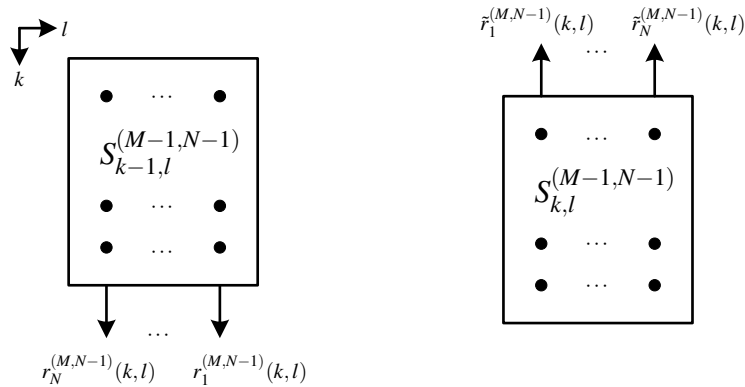


Figure 3.3: The horizontal PEFs for the first- and second QP models of the PSR $S_{k,l}^{(M-1,N-1)}$.

Consequently, the horizontal first- and third-quadrant QP PEFs, $r_{QP_1}^{(M,N-1)}(k,l)$ and $\tilde{r}_{QP_1}^{(M,N-1)}(k,l)$, respectively, and the fourth- and second-quadrant QP PEFs, $r_{QP_2}^{(M,N-1)}(k,l)$ and $\tilde{r}_{QP_2}^{(M,N-1)}(k,l)$, respectively, can be written as

$$\begin{aligned} r_{QP_1}^{(M,N-1)}(k,l) &= r_1^{(M,N-1)}(k,l) - \mathbf{k}_{r_{QP_1}}^{(M,N-1)H} \Psi_1 \mathbf{r}^{(M,N-1)}(k,l) \\ \tilde{r}_{QP_1}^{(M,N-1)}(k,l) &= \tilde{r}_1^{(M,N-1)}(k,l) - \mathbf{k}_{r_{QP_1}}^{(M,N-1)T} \Psi_1 \tilde{\mathbf{r}}^{(M,N-1)}(k,l) \end{aligned} \quad (3.11)$$

and

$$\begin{aligned} r_{QP_2}^{(M,N-1)}(k,l) &= r_N^{(M,N-1)}(k,l) - \mathbf{k}_{r_{QP_2}}^{(M,N-1)H} \Psi_N \mathbf{r}^{(M,N-1)}(k,l) \\ \tilde{r}_{QP_2}^{(M,N-1)}(k,l) &= \tilde{r}_N^{(M,N-1)}(k,l) - \mathbf{k}_{r_{QP_2}}^{(M,N-1)T} \Psi_N \tilde{\mathbf{r}}^{(M,N-1)}(k,l) \end{aligned} \quad (3.12)$$

where $\mathbf{k}_{r_{QP_1}}^{(M,N-1)}$ and $\mathbf{k}_{r_{QP_2}}^{(M,N-1)}$ are the horizontal first- and second-quadrant QP lattice parameter reflection vectors, respectively, and can be calculated by minimizing the sum of the mean-square values of $r_{QP_1}^{(M,N-1)}$ and $\tilde{r}_{QP_1}^{(M,N-1)}$ in (3.13), and $r_{QP_2}^{(M,N-1)}$ and $\tilde{r}_{QP_2}^{(M,N-1)}$ in (3.14), respectively.

$$P_{r_{QP_1}}^{(M,N-1)} = E \left[\left| r_{QP_1}^{(M,N-1)}(k,l) \right|^2 \right] + E \left[\left| \tilde{r}_{QP_1}^{(M,N-1)}(k,l) \right|^2 \right] \quad (3.13)$$

$$P_{r_{QP_2}}^{(M,N-1)} = E \left[\left| r_{QP_2}^{(M,N-1)}(k,l) \right|^2 \right] + E \left[\left| \tilde{r}_{QP_2}^{(M,N-1)}(k,l) \right|^2 \right] \quad (3.14)$$

The minimization of the sum of the mean-square values of $r_{QP_1}^{(M,N-1)}$ and $\tilde{r}_{QP_1}^{(M,N-1)}$ with respect to the horizontal lattice coefficient vector $\mathbf{k}_{r_{QP_1}}^{(M,N-1)}$, and similarly, the minimization of the mean-square values of $r_{QP_2}^{(M,N-1)}$ and $\tilde{r}_{QP_2}^{(M,N-1)}$ with respect to the horizontal lattice coefficient vector $\mathbf{k}_{r_{QP_2}}^{(M,N-1)}$ yields the following equations:

$$\begin{aligned} \mathbf{k}_{r_{QP_1}}^{(M,N-1)} &= \left[\Psi_1 \left(\Sigma_{\mathbf{r},\mathbf{r}}^{(M,N-1)} + \Sigma_{\tilde{\mathbf{r}},\tilde{\mathbf{r}}}^{(M,N-1)*} \right) \Psi_1^T \right]^{-1} \\ &\quad \times \Psi_1 \left(\Sigma_{\mathbf{r},\mathbf{r}}^{(M,N-1)} + \Sigma_{\tilde{\mathbf{r}},\tilde{\mathbf{r}}}^{(M,N-1)*} \right) \Pi_1^T \end{aligned} \quad (3.15)$$

$$\begin{aligned} \mathbf{k}_{r_{QP_2}}^{(M,N-1)} &= \left[\Psi_N \left(\Sigma_{\mathbf{r},\mathbf{r}}^{(M,N-1)} + \Sigma_{\tilde{\mathbf{r}},\tilde{\mathbf{r}}}^{(M,N-1)*} \right) \Psi_N^T \right]^{-1} \\ &\quad \times \Psi_N \left(\Sigma_{\mathbf{r},\mathbf{r}}^{(M,N-1)} + \Sigma_{\tilde{\mathbf{r}},\tilde{\mathbf{r}}}^{(M,N-1)*} \right) \Pi_N^T \end{aligned} \quad (3.16)$$

where $\Sigma_{\mathbf{r},\mathbf{r}}^{(M,N-1)}$ and $\Sigma_{\tilde{\mathbf{r}},\tilde{\mathbf{r}}}^{(M,N-1)}$ are the sum of the covariance matrices of forward and backward horizontal auxiliary PEF vectors, respectively.

The power propagation equations for the mean-square values of the horizontal first- and second-quadrant QP PEFs from stage to stage can be formed as follows

$$P_{r_{QP_1}}^{(M,N-1)} = P_{r_1}^{(M,N-1)} - \mathbf{k}_{r_{QP_1}}^{(M,N-1)H} \Psi_1 \left(\Sigma_{\mathbf{r},\mathbf{r}}^{(M,N-1)} + \Sigma_{\tilde{\mathbf{r}},\tilde{\mathbf{r}}}^{(M,N-1)*} \right) \Psi_1^T \mathbf{k}_{r_{QP_1}}^{(M,N-1)} \quad (3.17)$$

$$P_{r_{QP_2}}^{(M,N-1)} = P_{r_N}^{(M,N-1)} - \mathbf{k}_{r_{QP_2}}^{(M,N-1)H} \Psi_N \left(\Sigma_{\mathbf{r},\mathbf{r}}^{(M,N-1)} + \Sigma_{\tilde{\mathbf{r}},\tilde{\mathbf{r}}}^{(M,N-1)*} \right) \Psi_N^T \mathbf{k}_{r_{QP_2}}^{(M,N-1)} \quad (3.18)$$

where

$$P_{r_1}^{(M,N-1)} = E \left[\left| r_1^{(M,N-1)}(k,l) \right|^2 \right] + E \left[\left| \tilde{r}_1^{(M,N-1)}(k,l) \right|^2 \right]$$

and

$$P_{r_N}^{(M,N-1)} = E \left[\left| r_N^{(M,N-1)}(k,l) \right|^2 \right] + E \left[\left| \tilde{r}_N^{(M,N-1)}(k,l) \right|^2 \right].$$

3.2.1 Quarter-Plane 2-D lattice models of the PSR $S_{k,l}^{(1,0)}$

For the PSR $S_{k,l}^{(1,0)}$, it is possible to define *four* QP models by using the vertical auxiliary forward and backward PEFs. These are pair of the vertical first- and third-quadrant QP PEFs $e_{QP_1}^{(1,1)}(k,l)$ and $\tilde{e}_{QP_1}^{(1,1)}(k,l)$, respectively, and the pair of the vertical second- and fourth-quadrant QP PEFs $e_{QP_2}^{(1,1)}(k,l)$ and $\tilde{e}_{QP_2}^{(1,1)}(k,l)$, respectively. As shown in Figure 3.4, the PSR $S_{k,l}^{(1,0)}$ has two vertical forward and two vertical backward PEFs. As seen that it is not possible to define any QP models from the horizontal forward and backward PEFs.

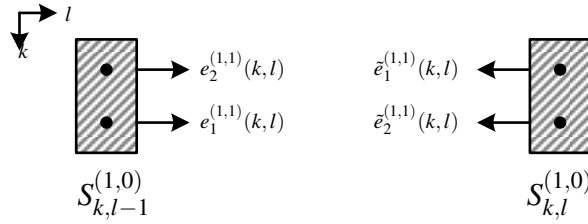


Figure 3.4: The vertical PEFs of the PSR $S_{k,l}^{(1,0)}$.

The first- and third-quadrant QP PEFs, $e_{QP_1}^{(1,1)}(k,l)$ and $\tilde{e}_{QP_1}^{(1,1)}(k,l)$ can be written as

$$\begin{aligned} e_{QP_1}^{(1,1)}(k,l) &= e_1^{(1,1)}(k,l) - \mathbf{k}_{e_{QP_1}}^{(1,1)H} \mathbf{\Psi}_1 \mathbf{e}^{(1,1)}(k,l) \\ \tilde{e}_{QP_1}^{(1,1)}(k,l) &= \tilde{e}_1^{(1,1)}(k,l) - \mathbf{k}_{e_{QP_1}}^{(1,1)T} \mathbf{\Psi}_1 \tilde{\mathbf{e}}^{(1,1)}(k,l) \end{aligned} \quad (3.19)$$

where the constant matrix $\mathbf{\Psi}_1$ is defined by deleting the first row of an identity matrix $\mathbf{I}_{2 \times 2} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. So, it is equal to $\mathbf{\Psi}_1 = \begin{bmatrix} 0 & 1 \end{bmatrix}$. If we put this value in the equation (3.19), then we have

$$\begin{aligned} e_{QP_1}^{(1,1)}(k,l) &= e_1^{(1,1)}(k,l) - k_{e_{QP_1}}^{(1,1)*} e_2^{(1,1)}(k,l) \\ \tilde{e}_{QP_1}^{(1,1)}(k,l) &= \tilde{e}_1^{(1,1)}(k,l) - k_{e_{QP_1}}^{(1,1)} \tilde{e}_2^{(1,1)}(k,l). \end{aligned} \quad (3.20)$$

The sum of the mean-square values of $e_{QP_1}^{(1,1)}(k, l)$ and $\tilde{e}_{QP_1}^{(1,1)}(k, l)$ can be written as follows

$$P_{e_{QP_1}}^{(1,1)} = E \left[\left| e_{QP_1}^{(1,1)}(k, l) \right|^2 \right] + E \left[\left| \tilde{e}_{QP_1}^{(1,1)}(k, l) \right|^2 \right]. \quad (3.21)$$

If we minimize the equation (3.21) with respect to the vertical lattice coefficient vector $k_{e_{QP_1}}^{(1,1)}$, then we will have

$$\begin{aligned} k_{e_{QP_1}}^{(1,1)} &= \left[E \left[e_2^{(1,1)}(k, l) e_2^{(1,1)*}(k, l) \right] + E \left[\tilde{e}_2^{(1,1)}(k, l) \tilde{e}_2^{(1,1)*}(k, l) \right]^* \right]^{-1} \\ &\times \left[E \left[e_2^{(1,1)}(k, l) e_1^{(1,1)*}(k, l) \right] + E \left[\tilde{e}_2^{(1,1)}(k, l) \tilde{e}_1^{(1,1)*}(k, l) \right]^* \right]. \end{aligned} \quad (3.22)$$

It is clearly seen that the reflection coefficient vector $k_{e_{QP_1}}^{(1,1)}$ is a scalar for this order.

In Section 3.2, we have already defined the vertical lattice coefficient vector $k_{e_{QP_1}}^{(M-1, N)}$, and we can write it for the $(1, 1)$ th order stage as

$$\mathbf{k}_{e_{QP_1}}^{(1,1)} = \left[\mathbf{\Psi}_1 \left(\mathbf{\Sigma}_{\mathbf{e}, \mathbf{e}}^{(1,1)} + \mathbf{\Sigma}_{\tilde{\mathbf{e}}, \tilde{\mathbf{e}}}^{(1,1)*} \right) \mathbf{\Psi}_1^T \right]^{-1} \mathbf{\Psi}_1 \left(\mathbf{\Sigma}_{\mathbf{e}, \mathbf{e}}^{(1,1)} + \mathbf{\Sigma}_{\tilde{\mathbf{e}}, \tilde{\mathbf{e}}}^{(1,1)*} \right) \mathbf{\Pi}_1^T. \quad (3.23)$$

Now, we can prove that the equation (3.23) is equal to the equation (3.22). We know that $\mathbf{\Psi}_1 = \begin{bmatrix} 0 & 1 \end{bmatrix}$ and

$$\begin{aligned} \mathbf{\Sigma}_{\mathbf{e}, \mathbf{e}}^{(1,1)} &= E \left[\mathbf{e}^{(1,1)}(k, l) \mathbf{e}^{(1,1)H} \right] \\ &= \begin{bmatrix} E \left[e_1^{(1,1)}(k, l) e_1^{(1,1)*}(k, l) \right] & E \left[e_1^{(1,1)}(k, l) e_2^{(1,1)*}(k, l) \right] \\ E \left[e_2^{(1,1)}(k, l) e_1^{(1,1)*}(k, l) \right] & E \left[e_2^{(1,1)}(k, l) e_2^{(1,1)*}(k, l) \right] \end{bmatrix}. \end{aligned} \quad (3.24)$$

Hence, we can write that

$$\mathbf{\Psi}_1 \mathbf{\Sigma}_{\mathbf{e}, \mathbf{e}}^{(1,1)} \mathbf{\Psi}_1^T = E \left[e_2^{(1,1)}(k, l) e_2^{(1,1)*}(k, l) \right]. \quad (3.25)$$

In the same manner, for the backward covariance matrix, $\mathbf{\Sigma}_{\tilde{\mathbf{e}}, \tilde{\mathbf{e}}}^{(1,1)}$, it can be defined as

$$\begin{aligned} \mathbf{\Sigma}_{\tilde{\mathbf{e}}, \tilde{\mathbf{e}}}^{(1,1)} &= E \left[\tilde{\mathbf{e}}^{(1,1)}(k, l) \tilde{\mathbf{e}}^{(1,1)H} \right] \\ &= \begin{bmatrix} E \left[\tilde{e}_1^{(1,1)}(k, l) \tilde{e}_1^{(1,1)*}(k, l) \right] & E \left[\tilde{e}_1^{(1,1)}(k, l) \tilde{e}_2^{(1,1)*}(k, l) \right] \\ E \left[\tilde{e}_2^{(1,1)}(k, l) \tilde{e}_1^{(1,1)*}(k, l) \right] & E \left[\tilde{e}_2^{(1,1)}(k, l) \tilde{e}_2^{(1,1)*}(k, l) \right] \end{bmatrix} \end{aligned} \quad (3.26)$$

and we can find that

$$\mathbf{\Psi}_1 \mathbf{\Sigma}_{\tilde{\mathbf{e}}, \tilde{\mathbf{e}}}^{(1,1)} \mathbf{\Psi}_1^T = E \left[\tilde{e}_2^{(1,1)}(k, l) \tilde{e}_2^{(1,1)*}(k, l) \right]. \quad (3.27)$$

By using (3.25) and (3.27), we can define that

$$\mathbf{\Psi}_1 \left[\mathbf{\Sigma}_{\mathbf{e},\mathbf{e}}^{(1,1)} + \mathbf{\Sigma}_{\tilde{\mathbf{e}},\tilde{\mathbf{e}}}^{(1,1)*} \right] \mathbf{\Psi}_1^T = E \left[e_2^{(1,1)}(k,l) e_2^{(1,1)*}(k,l) \right] + E \left[\tilde{e}_2^{(1,1)}(k,l) \tilde{e}_2^{(1,1)*}(k,l) \right]^* \quad (3.28)$$

In addition, by using the definition the constant vector $\mathbf{\Pi}_1 = \begin{bmatrix} 1 & 0 \end{bmatrix}$ which is the 1st row of the identity matrix $\mathbf{I}_{2 \times 2}$, It can be written the following equation:

$$\mathbf{\Psi}_1 \left[\mathbf{\Sigma}_{\mathbf{e},\mathbf{e}}^{(1,1)} + \mathbf{\Sigma}_{\tilde{\mathbf{e}},\tilde{\mathbf{e}}}^{(1,1)*} \right] \mathbf{\Pi}_1^T = E \left[e_2^{(1,1)}(k,l) e_1^{(1,1)*}(k,l) \right] + E \left[\tilde{e}_2^{(1,1)}(k,l) \tilde{e}_1^{(1,1)*}(k,l) \right]^* \quad (3.29)$$

If we substitute (3.28) and (3.29) into (3.23), then we have the equation (3.22).

Similarly, the second- and fourth-quadrant QP PEFs, $e_{QP_2}^{(1,1)}(k,l)$ and $\tilde{e}_{QP_2}^{(1,1)}(k,l)$ can be formed as

$$\begin{aligned} e_{QP_2}^{(1,1)}(k,l) &= e_2^{(1,1)}(k,l) - \mathbf{k}_{e_{QP_2}}^{(1,1)H} \mathbf{\Psi}_2 \mathbf{e}^{(1,1)}(k,l) \\ \tilde{e}_{QP_2}^{(1,1)}(k,l) &= \tilde{e}_2^{(1,1)}(k,l) - \mathbf{k}_{e_{QP_2}}^{(1,1)T} \mathbf{\Psi}_2 \tilde{\mathbf{e}}^{(1,1)}(k,l) \end{aligned} \quad (3.30)$$

The sum of the mean-square values of $e_{QP_1}^{(1,1)}(k,l)$ and $\tilde{e}_{QP_1}^{(1,1)}(k,l)$ are defined as

$$P_{e_{QP_2}}^{(1,1)} = E \left[\left| e_{QP_2}^{(1,1)}(k,l) \right|^2 \right] + E \left[\left| \tilde{e}_{QP_2}^{(1,1)}(k,l) \right|^2 \right]. \quad (3.31)$$

If we minimize (3.31) with respect to the lattice coefficient vector $\mathbf{k}_{e_{QP_2}}^{(1,1)}$, then we obtain

$$\mathbf{k}_{e_{QP_2}}^{(1,1)} = \left[\mathbf{\Psi}_2 \left(\mathbf{\Sigma}_{\mathbf{e},\mathbf{e}}^{(1,1)} + \mathbf{\Sigma}_{\tilde{\mathbf{e}},\tilde{\mathbf{e}}}^{(1,1)*} \right) \mathbf{\Psi}_2^T \right]^{-1} \mathbf{\Psi}_2 \left(\mathbf{\Sigma}_{\mathbf{e},\mathbf{e}}^{(1,1)} + \mathbf{\Sigma}_{\tilde{\mathbf{e}},\tilde{\mathbf{e}}}^{(1,1)*} \right) \mathbf{\Pi}_2^T. \quad (3.32)$$

By using the definitions $\mathbf{\Psi}_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ obtained from removing the second row of the identity matrix $\mathbf{I}_{2 \times 2} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, and $\mathbf{\Pi}_2 = \begin{bmatrix} 0 & 1 \end{bmatrix}$ formed by getting the second row of $\mathbf{I}_{2 \times 2}$, we can rewrite the equation (3.30) and (3.32) as follows

$$\begin{aligned} e_{QP_2}^{(1,1)}(k,l) &= e_2^{(1,1)}(k,l) - k_{e_{QP_2}}^{(1,1)*} e_1^{(1,1)}(k,l) \\ \tilde{e}_{QP_2}^{(1,1)}(k,l) &= \tilde{e}_2^{(1,1)}(k,l) - k_{e_{QP_2}}^{(1,1)} \tilde{e}_1^{(1,1)}(k,l). \end{aligned} \quad (3.33)$$

and

$$\begin{aligned} k_{e_{QP_2}}^{(1,1)} &= \left[E \left[e_1^{(1,1)}(k,l) e_1^{(1,1)*}(k,l) \right] + E \left[\tilde{e}_1^{(1,1)}(k,l) \tilde{e}_1^{(1,1)*}(k,l) \right]^* \right]^{-1} \\ &\quad \times \left[E \left[e_1^{(1,1)}(k,l) e_2^{(1,1)*}(k,l) \right] + E \left[\tilde{e}_1^{(1,1)}(k,l) \tilde{e}_2^{(1,1)*}(k,l) \right]^* \right], \end{aligned} \quad (3.34)$$

respectively. As the lattice coefficient vector $k_{e_{QP_1}}^{(1,1)}$, this lattice coefficient vector $k_{e_{QP_2}}^{(1,1)}$ is also a scalar.

3.2.2 Quarter-Plane 2-D lattice models of the PSR $S_{k,l}^{(3,2)}$

In this example, we will introduce all the possible vertical and horizontal QP PEFs for the PSR $S_{k,l}^{(3,2)}$. It is possible to define *four* pairs of QP models by using the vertical and horizontal auxiliary forward and backward PEFs. These are the pair of the vertical first- and third-quadrant QP PEFs $e_{QP_1}^{(3,3)}(k,l)$ and $\tilde{e}_{QP_1}^{(3,3)}(k,l)$, respectively, and the pair of the vertical second- and fourth-quadrant QP PEFs $e_{QP_2}^{(3,3)}(k,l)$ and $\tilde{e}_{QP_2}^{(3,3)}(k,l)$, respectively, and similarly, the pair of the horizontal first- and third-quadrant $r_{QP_1}^{(4,2)}(k,l)$ and $\tilde{r}_{QP_1}^{(4,2)}(k,l)$, respectively, and the pair of the horizontal second- and fourth-quadrant QP PEFs $r_{QP_2}^{(4,2)}(k,l)$ and $\tilde{r}_{QP_2}^{(4,2)}(k,l)$, respectively. The PSR $S_{k,l}^{(3,2)}$ has *four* vertical forward/backward PEFs and *three* horizontal forward/backward PEFs as shown in Figure 3.5.

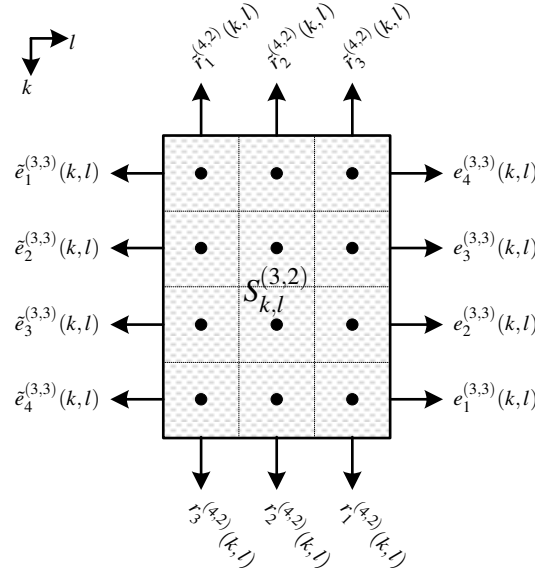


Figure 3.5: The auxiliary vertical and horizontal forward/backward PEFs of the PSR $S_{k,l}^{(3,2)}$.

From the definition given in the Section 3.2, we know that the vertical first-quadrant QP PEFs, $e_{QP_1}^{(3,3)}(k,l)$ is linearly prediction of $e_1^{(3,3)}(k,l)$ from $e_2^{(3,3)}(k,l)$, $e_3^{(3,3)}(k,l)$, and $e_4^{(3,3)}(k,l)$ as shown in the Figure 3.6. Naturally, the vertical third-quadrant QP PEFs $\tilde{e}_{QP_1}^{(3,3)}(k,l)$ considered as a backward of $e_{QP_1}^{(3,3)}(k,l)$ is obtained by the linear prediction of $\tilde{e}_1^{(3,3)}(k,l)$ from $\tilde{e}_2^{(3,3)}(k,l)$, $\tilde{e}_3^{(3,3)}(k,l)$, and $\tilde{e}_4^{(3,3)}(k,l)$

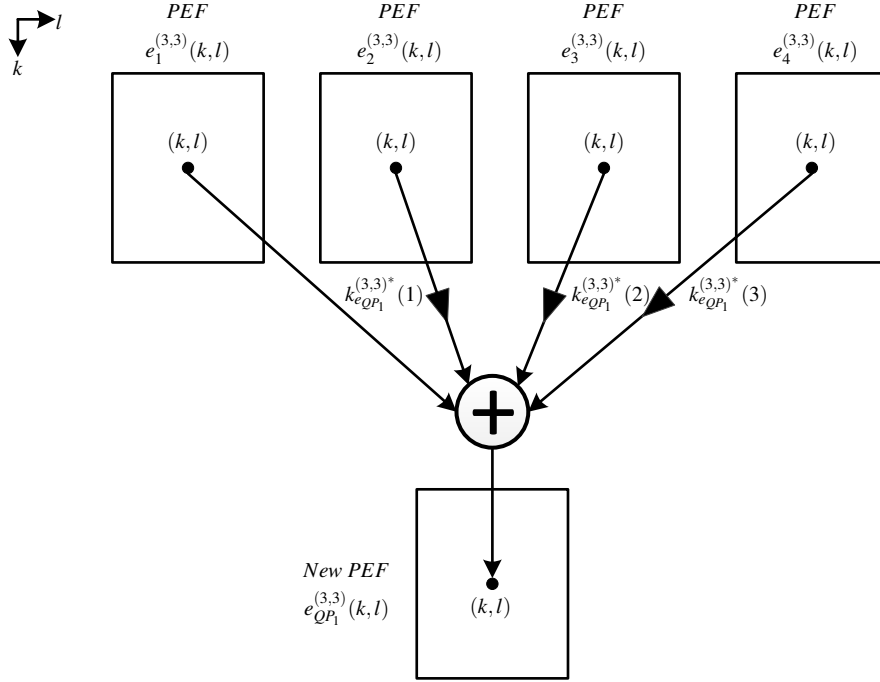


Figure 3.6: The construction of auxiliary vertical first-quadrant QP PEF $e_{QP_1}^{(3,3)}(k, l)$ by predicting the first auxiliary vertical PEF $e_1^{(3,3)}(k, l)$ from the remaining vertical PEFs, $e_2^{(3,3)}(k, l)$, $e_3^{(3,3)}(k, l)$, and $e_4^{(3,3)}(k, l)$.

The vertical first- and third quadrant QP model equations can be written as follows,

$$\begin{aligned}
 e_{QP_1}^{(3,3)}(k, l) &= e_1^{(3,3)}(k, l) - k_{e_{QP_1}}^{(3,3)*}(1)e_2^{(3,3)}(k, l) \\
 &\quad - k_{e_{QP_1}}^{(3,3)*}(2)e_3^{(3,3)}(k, l) - k_{e_{QP_1}}^{(3,3)*}(3)e_4^{(3,3)}(k, l) \\
 \tilde{e}_{QP_1}^{(3,3)}(k, l) &= \tilde{e}_1^{(3,3)}(k, l) - k_{e_{QP_1}}^{(3,3)}(1)\tilde{e}_2^{(3,3)}(k, l) \\
 &\quad - k_{e_{QP_1}}^{(3,3)}(2)\tilde{e}_3^{(3,3)}(k, l) - k_{e_{QP_1}}^{(3,3)}(3)\tilde{e}_4^{(3,3)}(k, l). \tag{3.35}
 \end{aligned}$$

If we constitute $\mathbf{\Pi}_1$ and $\mathbf{\Psi}_1$ from the identity matrix $\mathbf{I}_{4 \times 4}$, then we have

$$\mathbf{\Pi}_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}, \tag{3.36}$$

$$\mathbf{\Psi}_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{3.37}$$

The vertical forward and backward PEFs can be represented as a vector based as,

$$\mathbf{e}^{(3,3)} = \begin{bmatrix} e_1^{(3,3)}(k,l) \\ e_2^{(3,3)}(k,l) \\ e_3^{(3,3)}(k,l) \\ e_4^{(3,3)}(k,l) \end{bmatrix}, \quad (3.38)$$

$$\tilde{\mathbf{e}}^{(3,3)} = \begin{bmatrix} \tilde{e}_1^{(3,3)}(k,l) \\ \tilde{e}_2^{(3,3)}(k,l) \\ \tilde{e}_3^{(3,3)}(k,l) \\ \tilde{e}_4^{(3,3)}(k,l) \end{bmatrix}. \quad (3.39)$$

In addition we can state the vertical lattice coefficient vector as

$$\mathbf{k}_{e_{QP_1}}^{(3,3)} = \begin{bmatrix} k_{e_{QP_1}}^{(3,3)}(1) \\ k_{e_{QP_1}}^{(3,3)}(2) \\ k_{e_{QP_1}}^{(3,3)}(3) \end{bmatrix}. \quad (3.40)$$

By using (3.36) and (3.37) on the vertical forward and backward vectors, we can get the following results:

$$e_1^{(3,3)}(k,l) = \mathbf{\Pi}_1 \mathbf{e}^{(3,3)}(k,l) \quad (3.41)$$

$$\begin{bmatrix} e_2^{(3,3)}(k,l) \\ e_3^{(3,3)}(k,l) \\ e_4^{(3,3)}(k,l) \end{bmatrix} = \mathbf{\Psi}_1 \mathbf{e}^{(3,3)}(k,l) \quad (3.42)$$

$$\tilde{e}_1^{(3,3)}(k,l) = \mathbf{\Pi}_1 \tilde{\mathbf{e}}^{(3,3)}(k,l) \quad (3.43)$$

$$\begin{bmatrix} \tilde{e}_2^{(3,3)}(k,l) \\ \tilde{e}_3^{(3,3)}(k,l) \\ \tilde{e}_4^{(3,3)}(k,l) \end{bmatrix} = \mathbf{\Psi}_1 \tilde{\mathbf{e}}^{(3,3)}(k,l).. \quad (3.44)$$

Now, we can rewrite the equation (3.35) by using (3.40), (3.42), and (3.44) as follows,

$$\begin{aligned} e_{QP_1}^{(3,3)}(k,l) &= e_1^{(3,3)}(k,l) - \mathbf{k}_{e_{QP_1}}^{(3,3)H} \mathbf{\Psi}_1 \mathbf{e}^{(3,3)}(k,l) \\ \tilde{e}_{QP_1}^{(3,3)}(k,l) &= \tilde{e}_1^{(3,3)}(k,l) - \mathbf{k}_{e_{QP_1}}^{(3,3)T} \mathbf{\Psi}_1 \tilde{\mathbf{e}}^{(3,3)}(k,l). \end{aligned} \quad (3.45)$$

The sum of the mean-square values of $e_{QP_1}^{(3,3)}(k,l)$ and $\tilde{e}_{QP_1}^{(3,3)}(k,l)$ are given as

$$P_{e_{QP_1}}^{(3,3)} = E \left[\left| e_{QP_1}^{(3,3)}(k,l) \right|^2 \right] + E \left[\left| \tilde{e}_{QP_1}^{(3,3)}(k,l) \right|^2 \right]. \quad (3.46)$$

The minimization of (3.46) with respect to the lattice coefficient vector $\mathbf{k}_{e_{QP_1}}^{(3,3)}$ yields

$$\mathbf{k}_{e_{QP_1}}^{(3,3)} = \left[\mathbf{\Psi}_1 \left(\mathbf{\Sigma}_{\mathbf{e},\mathbf{e}}^{(3,3)} + \mathbf{\Sigma}_{\tilde{\mathbf{e}},\tilde{\mathbf{e}}}^{(3,3)*} \right) \mathbf{\Psi}_1^T \right]^{-1} \mathbf{\Psi}_1 \left(\mathbf{\Sigma}_{\mathbf{e},\mathbf{e}}^{(3,3)} + \mathbf{\Sigma}_{\tilde{\mathbf{e}},\tilde{\mathbf{e}}}^{(3,3)*} \right) \mathbf{\Pi}_1^T. \quad (3.47)$$

Likewise, the vertical second- and fourth quadrant QP PEFs can be formed as a pair of QP model equations,

$$\begin{aligned}
e_{QP_2}^{(3,3)}(k, l) &= e_4^{(3,3)}(k, l) - k_{e_{QP_2}}^{(3,3)*}(1)e_1^{(3,3)}(k, l) \\
&\quad - k_{e_{QP_2}}^{(3,3)*}(2)e_2^{(3,3)}(k, l) - k_{e_{QP_2}}^{(3,3)*}(3)e_3^{(3,3)}(k, l) \\
\tilde{e}_{QP_2}^{(3,3)}(k, l) &= \tilde{e}_4^{(3,3)}(k, l) - k_{e_{QP_2}}^{(3,3)}(1)\tilde{e}_1^{(3,3)}(k, l) \\
&\quad - k_{e_{QP_2}}^{(3,3)}(2)\tilde{e}_2^{(3,3)}(k, l) - k_{e_{QP_2}}^{(3,3)}(3)\tilde{e}_3^{(3,3)}(k, l)
\end{aligned} \tag{3.48}$$

and it can be reformed as

$$\begin{aligned}
e_{QP_2}^{(3,3)}(k, l) &= e_4^{(3,3)}(k, l) - \mathbf{k}_{e_{QP_2}}^{(3,3)H} \mathbf{\Psi}_4 \mathbf{e}^{(3,3)}(k, l) \\
\tilde{e}_{QP_2}^{(3,3)}(k, l) &= \tilde{e}_4^{(3,3)}(k, l) - \mathbf{k}_{e_{QP_2}}^{(3,3)T} \mathbf{\Psi}_4 \tilde{\mathbf{e}}^{(3,3)}(k, l)
\end{aligned} \tag{3.49}$$

where $\mathbf{k}_{e_{QP_2}}^{(3,3)} = \begin{bmatrix} k_{e_{QP_2}}^{(3,3)}(1) & k_{e_{QP_2}}^{(3,3)}(2) & k_{e_{QP_2}}^{(3,3)}(3) \end{bmatrix}^T$.

The sum of the mean-square values of $e_{QP_2}^{(3,3)}(k, l)$ and $\tilde{e}_{QP_2}^{(3,3)}(k, l)$ are

$$P_{e_{QP_2}}^{(3,3)} = E \left[\left| e_{QP_2}^{(3,3)}(k, l) \right|^2 \right] + E \left[\left| \tilde{e}_{QP_2}^{(3,3)}(k, l) \right|^2 \right]. \tag{3.50}$$

and the minimization of the equation (3.50) with respect to the vertical lattice coefficient vector $\mathbf{k}_{e_{QP_2}}^{(3,3)}$ gives

$$\mathbf{k}_{e_{QP_2}}^{(3,3)} = \left[\mathbf{\Psi}_4 \left(\mathbf{\Sigma}_{\mathbf{e}, \mathbf{e}}^{(3,3)} + \mathbf{\Sigma}_{\tilde{\mathbf{e}}, \tilde{\mathbf{e}}}^{(3,3)*} \right) \mathbf{\Psi}_4^T \right]^{-1} \mathbf{\Psi}_4 \left(\mathbf{\Sigma}_{\mathbf{e}, \mathbf{e}}^{(3,3)} + \mathbf{\Sigma}_{\tilde{\mathbf{e}}, \tilde{\mathbf{e}}}^{(3,3)*} \right) \mathbf{\Pi}_4^T. \tag{3.51}$$

Similarly, the horizontal first- and third quadrant, QP PEFs equations can be written as

$$\begin{aligned}
r_{QP_1}^{(4,2)}(k, l) &= r_1^{(4,2)}(k, l) - k_{r_{QP_1}}^{(4,2)*}(1)r_2^{(4,2)}(k, l) - k_{r_{QP_1}}^{(4,2)*}(2)r_3^{(4,2)}(k, l) \\
\tilde{r}_{QP_1}^{(4,2)}(k, l) &= \tilde{r}_1^{(4,2)}(k, l) - k_{r_{QP_1}}^{(4,2)}(1)\tilde{r}_2^{(4,2)}(k, l) - k_{r_{QP_1}}^{(4,2)}(2)\tilde{r}_3^{(4,2)}(k, l).
\end{aligned} \tag{3.52}$$

Concurrently, the horizontal second- and fourth-quadrant QP PEFs equations can be built up as

$$\begin{aligned}
r_{QP_2}^{(4,2)}(k, l) &= r_3^{(4,2)}(k, l) - k_{r_{QP_2}}^{(4,2)*}(1)r_1^{(4,2)}(k, l) - k_{r_{QP_2}}^{(4,2)*}(2)r_2^{(4,2)}(k, l) \\
\tilde{r}_{QP_2}^{(4,2)}(k, l) &= \tilde{r}_3^{(4,2)}(k, l) - k_{r_{QP_2}}^{(4,2)}(1)\tilde{r}_1^{(4,2)}(k, l) - k_{r_{QP_2}}^{(4,2)}(2)\tilde{r}_2^{(4,2)}(k, l).
\end{aligned} \tag{3.53}$$

In order to restructure the horizontal QP model equations, we need the identity matrix $\mathbf{I}_{3 \times 3}$ to define $\mathbf{\Pi}_1 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$, $\mathbf{\Psi}_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $\mathbf{\Pi}_3 = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$,

$\Psi_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$. The horizontal lattice coefficient vectors for $\mathbf{k}_{r_{QP_1}}^{(4,2)}$ and $\mathbf{k}_{r_{QP_2}}^{(4,2)}$ are defined as

$$\mathbf{k}_{r_{QP_1}}^{(4,2)} = \begin{bmatrix} k_{r_{QP_1}}^{(4,2)}(1) \\ k_{r_{QP_1}}^{(4,2)}(2) \end{bmatrix}, \quad (3.54)$$

and

$$\mathbf{k}_{r_{QP_2}}^{(4,2)} = \begin{bmatrix} k_{r_{QP_2}}^{(4,2)}(1) \\ k_{r_{QP_2}}^{(4,2)}(2) \end{bmatrix}. \quad (3.55)$$

It can easily be rewritten the horizontal QP model equations (3.52) and (3.53) by means of the definitions of $\mathbf{k}_{r_{QP_1}}^{(4,2)}$, $\mathbf{k}_{r_{QP_2}}^{(4,2)}$, Π_1 , Π_2 , Ψ_3 , and Ψ_3 . The restructured equations are shown in (3.56) and (3.57), respectively.

$$\begin{aligned} r_{QP_1}^{(4,2)}(k, l) &= r_1^{(4,2)}(k, l) - \mathbf{k}_{r_{QP_1}}^{(4,2)H} \Psi_1 \mathbf{r}^{(4,2)}(k, l) \\ \tilde{r}_{QP_1}^{(4,2)}(k, l) &= \tilde{r}_1^{(4,2)}(k, l) - \mathbf{k}_{r_{QP_1}}^{(4,2)T} \Psi_1 \tilde{\mathbf{r}}^{(4,2)}(k, l). \end{aligned} \quad (3.56)$$

$$\begin{aligned} r_{QP_2}^{(4,2)}(k, l) &= r_3^{(4,2)}(k, l) - \mathbf{k}_{r_{QP_2}}^{(4,2)H} \Psi_3 \mathbf{r}^{(4,2)}(k, l) \\ \tilde{r}_{QP_2}^{(4,2)}(k, l) &= \tilde{r}_3^{(4,2)}(k, l) - \mathbf{k}_{r_{QP_2}}^{(4,2)T} \Psi_3 \tilde{\mathbf{r}}^{(4,2)}(k, l). \end{aligned} \quad (3.57)$$

In order to calculate the horizontal lattice coefficient vectors $\mathbf{k}_{r_{QP_1}}^{(4,2)}$, and $\mathbf{k}_{r_{QP_2}}^{(4,2)}$, the sum of the mean-square values of $r_{QP_1}^{(4,2)}(k, l)$ and $\tilde{r}_{QP_1}^{(4,2)}(k, l)$, and $r_{QP_2}^{(4,2)}(k, l)$ and $\tilde{r}_{QP_2}^{(4,2)}(k, l)$ can be defined as

$$P_{r_{QP_1}}^{(3,3)} = E \left[\left| r_{QP_1}^{(4,2)}(k, l) \right|^2 \right] + E \left[\left| \tilde{r}_{QP_1}^{(4,2)}(k, l) \right|^2 \right], \quad (3.58)$$

$$P_{r_{QP_2}}^{(4,2)} = E \left[\left| r_{QP_2}^{(4,2)}(k, l) \right|^2 \right] + E \left[\left| \tilde{r}_{QP_2}^{(4,2)}(k, l) \right|^2 \right], \quad (3.59)$$

respectively. After the minimization of each one of (3.58) and (3.59) with respect to the related lattice coefficient vectors $\mathbf{k}_{r_{QP_1}}^{(4,2)}$ and $\mathbf{k}_{r_{QP_2}}^{(4,2)}$, respectively, then the following results are obtained:

$$\begin{aligned} \mathbf{k}_{r_{QP_1}}^{(4,2)} &= \left[\Psi_1 \left(\Sigma_{\mathbf{r}, \mathbf{r}}^{(4,2)} + \Sigma_{\tilde{\mathbf{r}}, \tilde{\mathbf{r}}}^{(4,2)*} \right) \Psi_1^T \right]^{-1} \Psi_1 \left(\Sigma_{\mathbf{r}, \mathbf{r}}^{(4,2)} + \Sigma_{\tilde{\mathbf{r}}, \tilde{\mathbf{r}}}^{(4,2)*} \right) \Pi_1^T, \\ \mathbf{k}_{r_{QP_2}}^{(4,2)} &= \left[\Psi_3 \left(\Sigma_{\mathbf{r}, \mathbf{r}}^{(4,2)} + \Sigma_{\tilde{\mathbf{r}}, \tilde{\mathbf{r}}}^{(4,2)*} \right) \Psi_3^T \right]^{-1} \Psi_3 \left(\Sigma_{\mathbf{r}, \mathbf{r}}^{(4,2)} + \Sigma_{\tilde{\mathbf{r}}, \tilde{\mathbf{r}}}^{(4,2)*} \right) \Pi_3^T. \end{aligned} \quad (3.60)$$

3.3 Asymmetric Half-Plane Models

The proposed method can be used to obtain causal ASHP lattice modeling with *eight* different recurse directions by using the vertical and horizontal forward/backward PEFs of the PSR as shown in Figure 3.7. It is possible to define two types ASHP models PEFs called as type-1 and type-2 ASHP model PEFs.

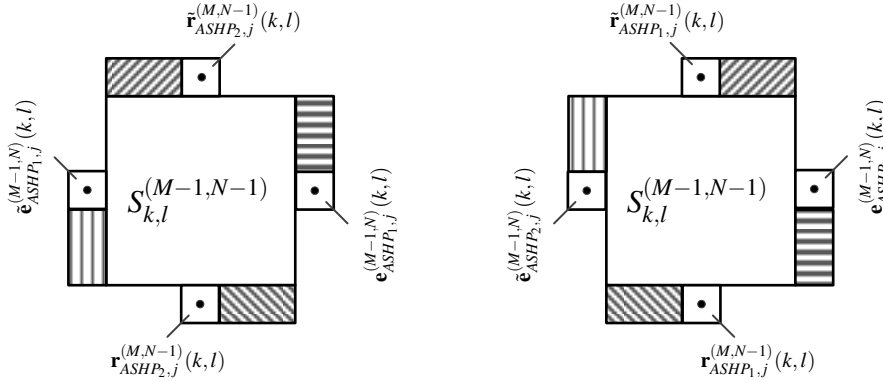


Figure 3.7: Generalized ASHP Type-1 and Type-2 model 2-D filters for the PSR $S_{k,l}^{(M-1,N-1)}$.

If the predictions of $e_j^{(M-1,N)}(k,l)$ and $\tilde{e}_j^{(M-1,N)}(k,l)$ are generated from the linear predictions $e_i^{(M-1,N)}(k,l)$ and $\tilde{e}_i^{(M-1,N)}(k,l)$ for $i = j+1, \dots, M$, respectively. They are called as type-1 ASHP model PEFs, namely, $e_{ASHP1,j}^{(M-1,N)}(k,l)$ and $\tilde{e}_{ASHP1,j}^{(M-1,N)}(k,l)$ for $j = 2, \dots, M-1$, where j indicates the place where the ASHP model is initiated.

On the other hand, if $e_j^{(M-1,N)}(k,l)$ and $\tilde{e}_j^{(M-1,N)}(k,l)$ are predicted by the linear combinations of $e_i^{(M-1,N)}(k,l)$ and $\tilde{e}_i^{(M-1,N)}(k,l)$ for $i = 1, \dots, j-1$, respectively. Then, type-2 ASHP model PEFs can be obtained, namely, $e_{ASHP2,j}^{(M-1,N)}(k,l)$ and $\tilde{e}_{ASHP2,j}^{(M-1,N)}(k,l)$ for $j = 2, \dots, M-1$. Thus, $4(M-2)$ different ASHP model causal 2-D lattice filters can be generated by using the vertical auxiliary PEFs. The outputs of their synthesis models can be computed column by column from top to bottom or vice versa.

Moreover, the horizontal auxiliary forward/backward PEFs, $r_j^{(M,N-1)}(k,l)$ and $\tilde{r}_j^{(M,N-1)}(k,l)$, can also be employed to produce $4(N-2)$ ASHP model lattice filters and their recurse directions are row by row from left to right or vice versa.

For both type-1 and type-2 ASHP lattice filter models, j is defined as starting from 2 to $(M-1)$ or $(N-1)$. Because, for $j=1$, type-1 ASHP lattice model is equal to the first-quadrant QP model, and for $j=M$, the type-1 ASHP lattice model is equal to the M th vertical forward PEFs itself for the auxiliary vertical forward PEFs.

3.3.1 General case of ASHP models using vertical PEFs

3.3.1.1 Type-1 ASHP models

Similar to the QP case, for a given $(M-1, N-1)$ th order rectangular, $S_{k,l}^{(M-1, N-1)}$, by using the auxiliary vertical forward/backward PEFs as shown in Figure 3.8, for $j=2, \dots, M-1$, the type-1 ASHP model equations can be defined as,

$$\begin{aligned} e_{ASHP_{1,j}}^{(M-1,N)}(k,l) &= e_j^{(M-1,N)}(k,l) - \mathbf{k}_{e_{ASHP_{1,j}}}^{(M-1,N)H} \Psi_{1,\dots,j} \mathbf{e}^{(M-1,N)}(k,l) \\ \tilde{e}_{ASHP_{1,j}}^{(M-1,N)}(k,l) &= \tilde{e}_j^{(M-1,N)}(k,l) - \mathbf{k}_{e_{ASHP_{1,j}}}^{(M-1,N)T} \Psi_{1,\dots,j} \tilde{\mathbf{e}}^{(M-1,N)}(k,l) \end{aligned} \quad (3.61)$$

where $\mathbf{k}_{e_{ASHP_{1,j}}}^{(M-1,N)}$ is the vertical lattice coefficient vectors for the type-1 ASHP and the constant matrix $\Psi_{1,\dots,j} = \Psi_{ASHP_{1,j}}$ is defined by deleting the 1st, 2nd, $\dots$, j th rows of an identity matrix with an appropriate order. The general definition is given in the Appendix A.3. For the $(M-1, N-1)$ th order case, the size of the matrix $\Psi_{1,\dots,j}$ is $M-1$ by M . Hence, we can write the following definitions:

$$\Psi_{1,\dots,j} \mathbf{e}^{(M-1,N)}(k,l) = \begin{bmatrix} e_{j+1}^{(M-1,N)} & \dots & e_M^{(M-1,N)} \end{bmatrix}^T \quad (3.62)$$

$$\Psi_{1,\dots,j} \tilde{\mathbf{e}}^{(M-1,N)}(k,l) = \begin{bmatrix} \tilde{e}_{j+1}^{(M-1,N)} & \dots & \tilde{e}_M^{(M-1,N)} \end{bmatrix}^T \quad (3.63)$$

The vertical lattice coefficient vector $\mathbf{k}_{e_{ASHP_{1,j}}}^{(M-1,N)}$ is defined as

$$\mathbf{k}_{e_{ASHP_{1,j}}}^{(M-1,N)} = \begin{bmatrix} k_{e_{ASHP_{1,j}}}^{(M-1,N)}(1) & \dots & k_{e_{ASHP_{1,j}}}^{(M-1,N)}(M-j) \end{bmatrix}^T. \quad (3.64)$$

The sum of the mean-square values of $e_{ASHP_{1,j}}^{(M-1,N)}(k,l)$ and $\tilde{e}_{ASHP_{1,j}}^{(M-1,N)}(k,l)$ can be written as

$$P_{e_{ASHP_{1,j}}}^{(M-1,N)} = E \left[\left| e_{ASHP_{1,j}}^{(M-1,N)}(k,l) \right|^2 \right] + E \left[\left| \tilde{e}_{ASHP_{1,j}}^{(M-1,N)}(k,l) \right|^2 \right]. \quad (3.65)$$

The result of the minimization of (3.65) with respect to the vertical lattice coefficient vector $\mathbf{k}_{e_{ASHP_{1,j}}}^{(M-1,N)}$ is found as

$$\mathbf{k}_{e_{ASHP_{1,j}}}^{(M-1,N)} = \left(\Psi_{1,\dots,j} \Phi_{\mathbf{e},\tilde{\mathbf{e}}}^{(M-1,N)} \Psi_{1,\dots,j}^T \right)^{-1} \Psi_{1,\dots,j} \Phi_{\mathbf{e},\tilde{\mathbf{e}}}^{(M-1,N)} \Pi_j \quad (3.66)$$

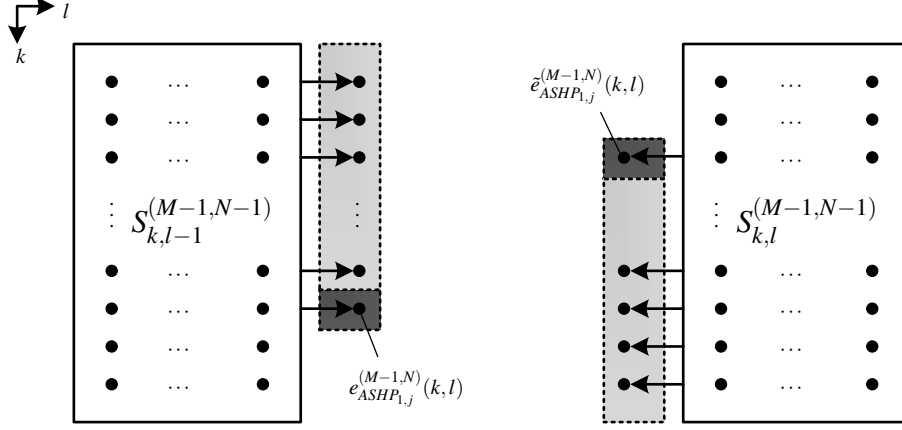


Figure 3.8: Type-1 ASHP model construction by using auxiliary vertical PEFs of the PSR $S_{k,l}^{(M-1,N-1)}$.

where

$$\Phi_{\mathbf{e},\tilde{\mathbf{e}}}^{(M-1,N)} = \Sigma_{\mathbf{e},\mathbf{e}}^{(M-1,N)} + \Sigma_{\tilde{\mathbf{e}},\tilde{\mathbf{e}}}^{(M-1,N)*} \quad (3.67)$$

and covariance matrices $\Sigma_{\mathbf{e},\mathbf{e}}^{(M-1,N)}$ and $\Sigma_{\tilde{\mathbf{e}},\tilde{\mathbf{e}}}^{(M-1,N)}$ are already defined in Section 3.2.

If we rewrite the equation (3.65) from stage to stage, it can be stated as follows

$$P_{e_{ASHP_{1,j}}}^{(M-1,N)} = P_{e_j}^{(M-1,N)} - \mathbf{k}_{e_{ASHP_{1,j}}}^{(M-1,N)H} \Psi_{1,\dots,j} \Phi_{\mathbf{e},\tilde{\mathbf{e}}}^{(M-1,N)} \Psi_{1,\dots,j}^T \mathbf{k}_{e_{ASHP_{1,j}}}^{(M-1,N)} \quad (3.68)$$

where

$$\begin{aligned} P_{e_j}^{(M-1,N)} &= E \left[\left| e_j^{(M-1,N)} \right|^2 \right] + E \left[\left| \tilde{e}_j^{(M-1,N)} \right|^2 \right] \\ &= \Pi_j \Phi_{\mathbf{e},\tilde{\mathbf{e}}}^{(M-1,N)} \Pi_j, \end{aligned} \quad (3.69)$$

for $j = 2, \dots, M-1$.

For a given PSR $S_{k,l}^{(M-1,N-1)}$, it is possible to define Type-1 ASHP 2-D lattice coefficient vectors of all possible type-1 ASHP lattice filters into a single matrix defined as

$$\mathbf{K}_{ASHP_1}^{(M-1,N)} = \begin{bmatrix} \mathbf{k}_{e_{ASHP_{1,2}}}^{(M-1,N)} & \mathbf{k}_{e_{ASHP_{1,3}}}^{(M-1,N)} & \dots & \mathbf{k}_{e_{ASHP_{1,M-2}}}^{(M-1,N)} & \mathbf{k}_{e_{ASHP_{1,M-1}}}^{(M-1,N)} \end{bmatrix}_{(M-2) \times (M-2)}. \quad (3.70)$$

If we put all lattice vectors into the matrix, then it can be written as

$$\mathbf{K}_{ASHP_1}^{(M-1,N)} = \begin{bmatrix} k_{e_{ASHP_{1,2}}}^{(M-1,N)}(1) & 0 & \cdots & 0 & 0 \\ k_{e_{ASHP_{1,2}}}^{(M-1,N)}(2) & k_{e_{ASHP_{1,3}}}^{(M-1,N)}(1) & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ k_{e_{ASHP_{1,2}}}^{(M-1,N)}(M-3) & k_{e_{ASHP_{1,3}}}^{(M-1,N)}(M-4) & \cdots & k_{e_{ASHP_{1,M-2}}}^{(M-1,N)}(1) & 0 \\ k_{e_{ASHP_{1,2}}}^{(M-1,N)}(M-2) & k_{e_{ASHP_{1,3}}}^{(M-1,N)}(M-3) & \cdots & k_{e_{ASHP_{1,M-2}}}^{(M-1,N)}(2) & k_{e_{ASHP_{1,M-1}}}^{(M-1,N)} \end{bmatrix} \quad (3.71)$$

In the same manner, we can define all type-1 ASHP lattice models as

$$\mathbf{e}_{ASHP_1}^{(M-1,N)}(k, l) = \begin{bmatrix} e_{ASHP_{1,2}}^{(M-1,N)}(k, l) & e_{ASHP_{1,3}}^{(M-1,N)}(k, l) & \cdots & e_{ASHP_{1,M-1}}^{(M-1,N)}(k, l) \end{bmatrix}^T \quad (3.72)$$

and

$$\tilde{\mathbf{e}}_{ASHP_1}^{(M-1,N)}(k, l) = \begin{bmatrix} \tilde{e}_{ASHP_{1,2}}^{(M-1,N)}(k, l) & \tilde{e}_{ASHP_{1,3}}^{(M-1,N)}(k, l) & \cdots & \tilde{e}_{ASHP_{1,M-1}}^{(M-1,N)}(k, l) \end{bmatrix}^T. \quad (3.73)$$

Now, we define the following constant matrix and vectors derived from the identity matrix $\mathbf{I}_M$ as follows

$$\mathbf{I}_M = \begin{bmatrix} \mathbf{\Pi}_1^T \\ \mathbf{\Psi}_{1,M} \\ \mathbf{\Pi}_M^T \end{bmatrix}, \quad (3.74)$$

hence,

$$\mathbf{\Psi}_{1,M} = \begin{bmatrix} \mathbf{O}_{(M-2) \times 1} & \vdots & \mathbf{I}_{M-2} & \vdots & \mathbf{O}_{(M-2) \times 1} \end{bmatrix}. \quad (3.75)$$

And, for obtaining $\mathbf{\Psi}_{1,2}$, identity matrix is defined as

$$\mathbf{I}_M = \begin{bmatrix} \mathbf{\Pi}_1^T \\ \mathbf{\Pi}_2^T \\ \mathbf{\Psi}_{1,2} \end{bmatrix}, \quad (3.76)$$

so,

$$\mathbf{\Psi}_{1,2} = \begin{bmatrix} \mathbf{O}_{(M-2) \times 1} & \vdots & \mathbf{O}_{(M-2) \times 1} & \vdots & \mathbf{I}_{M-2} \end{bmatrix}. \quad (3.77)$$

By using (3.75) and (3.77), we can write the lattice filter equations as

$$\begin{aligned} \mathbf{e}_{ASHP_1}^{(M-1,N)}(k, l) &= \mathbf{\Psi}_{1,M} \mathbf{e}^{(M-1,N)}(k, l) - \mathbf{K}_{e_{ASHP_1}}^{(M-1,N)H} \mathbf{\Psi}_{1,2} \mathbf{e}^{(M-1,N)}(k, l) \\ \tilde{\mathbf{e}}_{ASHP_1}^{(M-1,N)}(k, l) &= \mathbf{\Psi}_{1,M} \tilde{\mathbf{e}}^{(M-1,N)}(k, l) - \mathbf{K}_{e_{ASHP_1}}^{(M-1,N)T} \mathbf{\Psi}_{1,2} \tilde{\mathbf{e}}^{(M-1,N)}(k, l). \end{aligned} \quad (3.78)$$

The sum of the mean-square values of 3.78 can be written as

$$P_{e_{ASHP_1}}^{(M-1,N)} = \text{tr } E \left[\mathbf{e}_{ASHP_1}^{(M-1,N)}(k,l) \mathbf{e}_{ASHP_1}^{(M-1,N)H}(k,l) \right] + \text{tr } E \left[\tilde{\mathbf{e}}_{ASHP_1}^{(M-1,N)}(k,l) \tilde{\mathbf{e}}_{ASHP_1}^{(M-1,N)H}(k,l) \right]. \quad (3.79)$$

By minimizing (3.79) with respect to the lattice coefficient matrix $\mathbf{K}_{e_{ASHP_1}}^{(M-1,N)}$ as follows

$$\frac{\partial}{\partial \mathbf{K}_{e_{ASHP_1}}^{(M-1,N)}} P_{e_{ASHP_1}}^{(M-1,N)} = 0,$$

we get the following result

$$\mathbf{K}_{e_{ASHP_1}}^{(M-1,N)} = \left[\Psi_{1,2} \Phi_{\mathbf{e},\tilde{\mathbf{e}}}^{(M-1,N)} \Psi_{1,2}^T \right]^{-1} \Psi_{1,2} \Phi_{\mathbf{e},\tilde{\mathbf{e}}}^{(M-1,N)} \Psi_{1,M}^T, \quad (3.80)$$

where $\Phi_{\mathbf{e},\tilde{\mathbf{e}}}^{(M-1,N)}$ is defined in (3.67).

3.3.1.2 Type-2 ASHP models

For type-2 ASHP models, opposite recurse directions according to the type-1 case can be used. That means, the type-2 ASHP model PEFs will be generated from the auxiliary vertical forward/backward PEFs that are not used for the type-1 case. For the same PSR $S_{k,l}^{(M-1,N-1)}$, the construction of the type-2 ASHP model lattice filters are demonstrated in Figure 3.9.

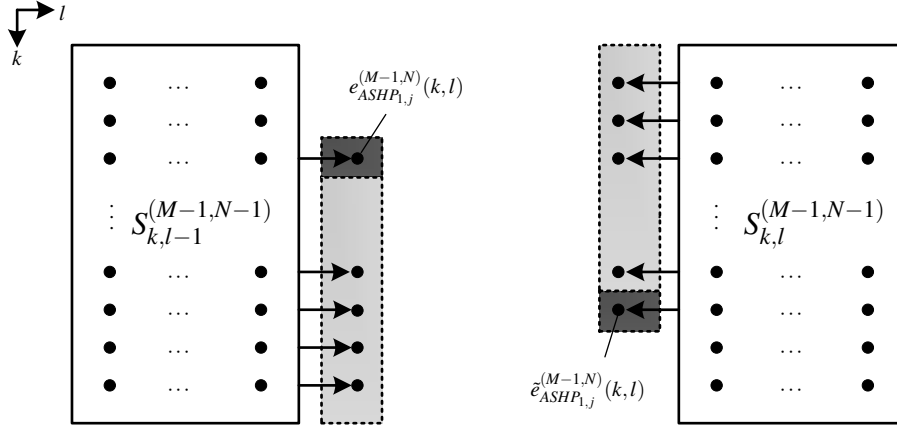


Figure 3.9: Type-2 ASHP model construction by using auxiliary vertical PEFs of the PSR $S_{k,l}^{(M-1,N-1)}$.

For $j = 2, \dots, M-1$, the type-2 ASHP model equations can be written as,

$$\begin{aligned} e_{ASHP_{2,j}}^{(M-1,N)}(k,l) &= e_j^{(M-1,N)}(k,l) - \mathbf{k}_{e_{ASHP_{2,j}}}^{(M-1,N)H} \Psi_{j,\dots,M} \mathbf{e}^{(M-1,N)}(k,l) \\ \tilde{e}_{ASHP_{2,j}}^{(M-1,N)}(k,l) &= \tilde{e}_j^{(M-1,N)}(k,l) - \mathbf{k}_{e_{ASHP_{2,j}}}^{(M-1,N)T} \Psi_{j,\dots,M} \tilde{\mathbf{e}}^{(M-1,N)}(k,l) \end{aligned} \quad (3.81)$$

where $\mathbf{k}_{e_{ASHP_{2,j}}}^{(M-1,N)}$ is the vertical lattice coefficient vectors for the type-2 ASHP and the constant matrix $\Psi_{j,\dots,M} = \Psi_{ASHP_{2,j}}$ is defined by deleting the rows, from j th to (M) th, of an identity matrix with an appropriate order. The general definition is given in the Appendix A.3. Thus, the employed auxiliary vertical forward/backward PEFs can be stated as

$$\Psi_{j,\dots,M} \mathbf{e}^{(M-1,N)}(k,l) = \begin{bmatrix} e_1^{(M-1,N)} & \dots & e_{j-1}^{(M-1,N)} \end{bmatrix}^T \quad (3.82)$$

$$\Psi_{j,\dots,M} \tilde{\mathbf{e}}^{(M-1,N)}(k,l) = \begin{bmatrix} \tilde{e}_1^{(M-1,N)} & \dots & \tilde{e}_{j-1}^{(M-1,N)} \end{bmatrix}^T. \quad (3.83)$$

The vertical lattice coefficient vector $\mathbf{k}_{e_{ASHP_{2,j}}}^{(M-1,N)}$ is defined as

$$\mathbf{k}_{e_{ASHP_{2,j}}}^{(M-1,N)} = \begin{bmatrix} k_{e_{ASHP_{2,j}}}^{(M-1,N)}(1) & \dots & k_{e_{ASHP_{2,j}}}^{(M-1,N)}(M-j) \end{bmatrix}^T. \quad (3.84)$$

As a result of the mean-square minimization of the ASHP PEFs in (3.81), the reflection coefficient vector yields

$$\mathbf{k}_{e_{ASHP_{2,j}}}^{(M-1,N)} = \left(\Psi_{j,\dots,M} \Phi_{\mathbf{e},\tilde{\mathbf{e}}}^{(M-1,N)} \Psi_{j,\dots,M}^T \right)^{-1} \Psi_{j,\dots,M} \Phi_{\mathbf{e},\tilde{\mathbf{e}}}^{(M-1,N)} \Pi_j, \quad (3.85)$$

where $\Phi_{\mathbf{e},\tilde{\mathbf{e}}}^{(M-1,N)}$ is defined in (3.67).

The power of the error propagation equation from stage to stage can be written as follows

$$P_{e_{ASHP_{2,j}}}^{(M-1,N)} = P_{e_j}^{(M-1,N)} - \mathbf{k}_{e_{ASHP_{2,j}}}^{(M-1,N)H} \Psi_{j,\dots,M} \Phi_{\mathbf{e},\tilde{\mathbf{e}}}^{(M-1,N)} \Psi_{j,\dots,M}^T \mathbf{k}_{e_{ASHP_{2,j}}}^{(M-1,N)} \quad (3.86)$$

where $P_{e_j}^{(M-1,N)}$ is defined in (3.69).

Consequently, similar to the type-1 ASHP lattice model case, it is possible to define Type-2 ASHP 2-D lattice coefficient vectors of all possible type-2 ASHP lattice filters into a single matrix defined as

$$\mathbf{K}_{ASHP_2}^{(M-1,N)} = \begin{bmatrix} \mathbf{k}_{e_{ASHP_{2,2}}}^{(M-1,N)} & \mathbf{k}_{e_{ASHP_{2,3}}}^{(M-1,N)} & \dots & \mathbf{k}_{e_{ASHP_{2,M-2}}}^{(M-1,N)} & \mathbf{k}_{e_{ASHP_{2,M-1}}}^{(M-1,N)} \end{bmatrix}_{(M-2) \times (M-2)}. \quad (3.87)$$

By writing all lattice vectors into the matrix, then we have

$$\mathbf{K}_{ASHP_2}^{(M-1,N)} = \begin{bmatrix} k_{e_{ASHP_2,2}}^{(M-1,N)} & k_{e_{ASHP_2,3}}^{(M-1,N)}(1) & \cdots & k_{e_{ASHP_2,M-2}}^{(M-1,N)}(1) & k_{e_{ASHP_2,M-1}}^{(M-1,N)}(1) \\ 0 & k_{e_{ASHP_2,3}}^{(M-1,N)}(2) & \cdots & k_{e_{ASHP_2,M-2}}^{(M-1,N)}(2) & k_{e_{ASHP_2,M-1}}^{(M-1,N)}(2) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & k_{e_{ASHP_2,M-2}}^{(M-1,N)}(M-3) & k_{e_{ASHP_2,M-1}}^{(M-1,N)}(M-3) \\ 0 & 0 & \cdots & 0 & k_{e_{ASHP_2,M-1}}^{(M-1,N)}(M-2) \end{bmatrix} \quad (3.88)$$

All type-2 ASHP lattice models can be defined as

$$\mathbf{e}_{ASHP_2}^{(M-1,N)}(k,l) = \begin{bmatrix} e_{ASHP_2,2}^{(M-1,N)}(k,l) & e_{ASHP_2,3}^{(M-1,N)}(k,l) & \cdots & e_{ASHP_2,M-1}^{(M-1,N)}(k,l) \end{bmatrix}^T \quad (3.89)$$

and

$$\tilde{\mathbf{e}}_{ASHP_2}^{(M-1,N)}(k,l) = \begin{bmatrix} \tilde{e}_{ASHP_2,2}^{(M-1,N)}(k,l) & \tilde{e}_{ASHP_2,3}^{(M-1,N)}(k,l) & \cdots & \tilde{e}_{ASHP_2,M-1}^{(M-1,N)}(k,l) \end{bmatrix}^T. \quad (3.90)$$

We have already defined $\Psi_{1,M}$ in (3.75), now we need to define $\Psi_{1,2}$ as

$$\mathbf{I}_M = \begin{bmatrix} \Psi_{M-1,M} \\ \Pi_{M-1}^T \\ \Pi_M^T \end{bmatrix}, \quad (3.91)$$

so,

$$\Psi_{M-1,M} = \begin{bmatrix} \mathbf{I}_{M-2} & \vdots & \mathbf{O}_{(M-2) \times 1} & \vdots & \mathbf{O}_{(M-2) \times 1} \end{bmatrix}. \quad (3.92)$$

By using (3.75) and (3.92), all lattice filter equations can be demonstrated with one pair of lattice equation as

$$\begin{aligned} \mathbf{e}_{ASHP_2}^{(M-1,N)}(k,l) &= \Psi_{1,M} \mathbf{e}^{(M-1,N)}(k,l) - \mathbf{K}_{e_{ASHP_2}}^{(M-1,N)H} \Psi_{M-1,M} \mathbf{e}^{(M-1,N)}(k,l) \\ \tilde{\mathbf{e}}_{ASHP_2}^{(M-1,N)}(k,l) &= \Psi_{1,M} \tilde{\mathbf{e}}^{(M-1,N)}(k,l) - \mathbf{K}_{e_{ASHP_2}}^{(M-1,N)T} \Psi_{M-1,M} \tilde{\mathbf{e}}^{(M-1,N)}(k,l). \end{aligned} \quad (3.93)$$

The sum of the mean-square values of 3.93 can be written as

$$\begin{aligned} P_{e_{ASHP_2}}^{(M-1,N)} &= tr E \left[\mathbf{e}_{ASHP_2}^{(M-1,N)}(k,l) \mathbf{e}_{ASHP_2}^{(M-1,N)H}(k,l) \right] \\ &+ tr E \left[\tilde{\mathbf{e}}_{ASHP_2}^{(M-1,N)}(k,l) \tilde{\mathbf{e}}_{ASHP_2}^{(M-1,N)H}(k,l) \right] \end{aligned} \quad (3.94)$$

where tr denotes trace operator.

By minimizing (3.79) with respect to the lattice coefficient matrix $\mathbf{K}_{e_{ASHP_2}}^{(M-1,N)}$ as follows

$$\frac{\partial}{\partial \mathbf{K}_{e_{ASHP_2}}^{(M-1,N)}} P_{e_{ASHP_2}}^{(M-1,N)} = 0,$$

we get the following result

$$\mathbf{K}_{e_{ASHP_2}}^{(M-1,N)} = \left[\Psi_{M-1,M} \Phi_{\mathbf{e},\tilde{\mathbf{e}}}^{(M-1,N)} \Psi_{M-1,M}^T \right]^{-1} \Psi_{M-1,M} \Phi_{\mathbf{e},\tilde{\mathbf{e}}}^{(M-1,N)} \Psi_{1,M}^T. \quad (3.95)$$

3.3.2 General case of ASHP models using horizontal PEFs

Likewise the constructions of type-1 and type-2 ASHP lattice filters, it is also possible to define type-1 and type-2 ASHP lattice filter by employing the auxiliary horizontal forward/backward PEFs. Type-1 ASHP lattice construction from the auxiliary forward/backward PEFs is illustrated in Figure 3.10. By getting the opposite recurse directions used for the type-1 ASHP lattice filter case, we can generate type-2 ASHP lattice filter models from the shaded horizontal PEFs as shown in Figure 3.11.

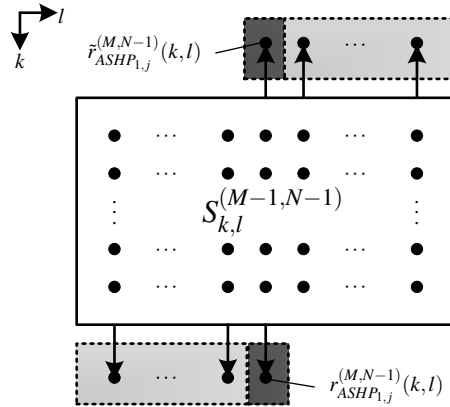


Figure 3.10: Type-1 ASHP model construction by using auxiliary horizontal PEFs of the PSR $S_{k,l}^{(M-1,N-1)}$.

For $j = 2, \dots, N-1$, the type-1 ASHP model equations from the related auxiliary horizontal forward/backward PEFs, which are shaded as shown in Figure 3.10, can be defined as,

$$\begin{aligned} r_{ASHP_{1,j}}^{(M,N-1)}(k,l) &= r_j^{(M,N-1)}(k,l) - \mathbf{k}_{r_{ASHP_{1,j}}}^{(M,N-1)H} \Psi_{1,\dots,j} \mathbf{r}^{(M,N-1)}(k,l) \\ \tilde{r}_{ASHP_{1,j}}^{(M,N-1)}(k,l) &= \tilde{r}_j^{(M,N-1)}(k,l) - \mathbf{k}_{r_{ASHP_{1,j}}}^{(M,N-1)T} \Psi_{1,\dots,j} \tilde{\mathbf{r}}^{(M,N-1)}(k,l) \end{aligned} \quad (3.96)$$

where $\mathbf{k}_{r_{ASHP_{1,j}}}^{(M,N-1)}$ is the horizontal lattice coefficient vectors for the type-1 ASHP and the constant matrix $\Psi_{1,\dots,j}$ is defined in the Section 3.3.1. The auxiliary horizontal forward/backward PEFs to be employed for type-1 ASHP lattice filter can be shown as follows

$$\Psi_{1,\dots,j} \mathbf{r}^{(M,N-1)}(k,l) = \begin{bmatrix} r_{j+1}^{(M,N-1)} & \dots & r_N^{(M,N-1)} \end{bmatrix}^T \quad (3.97)$$

$$\Psi_{1,\dots,j} \tilde{\mathbf{r}}^{(M,N-1)}(k,l) = \begin{bmatrix} \tilde{r}_{j+1}^{(M,N-1)} & \dots & \tilde{r}_N^{(M,N-1)} \end{bmatrix}^T. \quad (3.98)$$

The horizontal lattice coefficient vector of type-1 ASHP lattice filter from the related auxiliary horizontal forward/backward PEFs, $\mathbf{k}_{r_{ASHP_{1,j}}}^{(M,N-1)}$, is defined as

$$\mathbf{k}_{r_{ASHP_{1,j}}}^{(M,N-1)} = \begin{bmatrix} k_{r_{ASHP_{1,j}}}^{(M,N-1)}(1) & \dots & k_{r_{ASHP_{1,j}}}^{(M,N-1)}(N-j) \end{bmatrix}^T. \quad (3.99)$$

The sum of the mean-square values of $r_{ASHP_{1,j}}^{(M,N-1)}(k,l)$ and $\tilde{r}_{ASHP_{1,j}}^{(M,N-1)}(k,l)$ can be written as

$$P_{r_{ASHP_{1,j}}}^{(M,N-1)} = E \left[\left| r_{ASHP_{1,j}}^{(M,N-1)}(k,l) \right|^2 \right] + E \left[\left| \tilde{r}_{ASHP_{1,j}}^{(M,N-1)}(k,l) \right|^2 \right]. \quad (3.100)$$

The result of the minimization of (3.100) with respect to the vertical lattice coefficient vector $\mathbf{k}_{r_{ASHP_{1,j}}}^{(M,N-1)}$ is found as

$$\mathbf{k}_{r_{ASHP_{1,j}}}^{(M,N-1)} = \left(\Psi_{1,\dots,j} \Phi_{\mathbf{r},\tilde{\mathbf{r}}}^{(M,N-1)} \Psi_{1,\dots,j}^T \right)^{-1} \Psi_{1,\dots,j} \Phi_{\mathbf{r},\tilde{\mathbf{r}}}^{(M,N-1)} \Pi_j \quad (3.101)$$

where

$$\Phi_{\mathbf{r},\tilde{\mathbf{r}}}^{(M,N-1)} = \Sigma_{\mathbf{r},\mathbf{r}}^{(M,N-1)} + \Sigma_{\tilde{\mathbf{r}},\tilde{\mathbf{r}}}^{(M,N-1)*} \quad (3.102)$$

and covariance matrices $\Sigma_{\mathbf{r},\mathbf{r}}^{(M,N-1)}$ and $\Sigma_{\tilde{\mathbf{r}},\tilde{\mathbf{r}}}^{(M,N-1)}$ are already defined in Section 3.2.

If we rewrite the equation (3.100) from stage to stage, we get

$$P_{r_{ASHP_{1,j}}}^{(M,N-1)} = P_{r_j}^{(M,N-1)} - \mathbf{k}_{r_{ASHP_{1,j}}}^{(M,N-1)H} \Psi_{1,\dots,j} \Phi_{\mathbf{r},\tilde{\mathbf{r}}}^{(M,N-1)} \Psi_{1,\dots,j}^T \mathbf{k}_{r_{ASHP_{1,j}}}^{(M,N-1)} \quad (3.103)$$

where

$$\begin{aligned} P_{r_j}^{(M,N-1)} &= E \left[\left| r_j^{(M,N-1)} \right|^2 \right] + E \left[\left| \tilde{r}_j^{(M,N-1)} \right|^2 \right] \\ &= \Pi_j \Phi_{\mathbf{r},\tilde{\mathbf{r}}}^{(M,N-1)} \Pi_j. \end{aligned} \quad (3.104)$$

for $j = 2, \dots, N-1$.

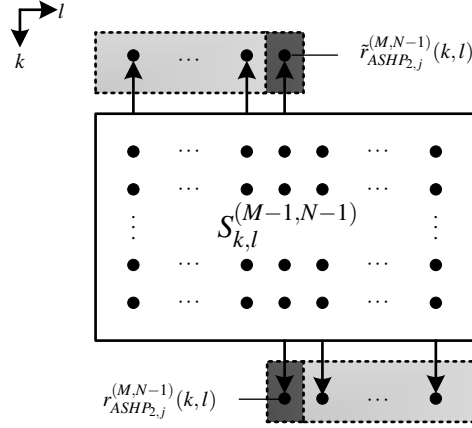


Figure 3.11: Type-2 ASHP model construction by using auxiliary horizontal PEFs of the PSR $S_{k,l}^{(M-1,N-1)}$.

Similarly, by using the corresponding auxiliary horizontal forward/backward PEFs as shown in Figure 3.11, we can write the type-2 ASHP lattice model equations, for $j = 2, \dots, N-1$, as follows

$$\begin{aligned} r_{ASHP_{2,j}}^{(M,N-1)}(k,l) &= r_j^{(M,N-1)}(k,l) - \mathbf{k}_{r_{ASHP_{2,j}}}^{(M,N-1)H} \Psi_{j,\dots,N} \mathbf{r}^{(M,N-1)}(k,l) \\ \tilde{r}_{ASHP_{2,j}}^{(M,N-1)}(k,l) &= \tilde{r}_j^{(M,N-1)}(k,l) - \mathbf{k}_{r_{ASHP_{1,j}}}^{(M,N-1)T} \Psi_{j,\dots,N} \tilde{\mathbf{r}}^{(M,N-1)}(k,l) \end{aligned} \quad (3.105)$$

where $\mathbf{k}_{r_{ASHP_{2,j}}}^{(M,N-1)}$ is the vertical lattice coefficient vectors for the type-2 ASHP and the constant matrix $\Psi_{j,\dots,N}$ is defined by deleting the rows, from j th to N th, of an identity matrix with an appropriate order. Thus, we can show the auxiliary horizontal forward/backward PEFs to be taken into account as follows:

$$\Psi_{j,\dots,N} \mathbf{r}^{(M,N-1)}(k,l) = \begin{bmatrix} r_1^{(M,N-1)} & \dots & r_{j-1}^{(M,N-1)} \end{bmatrix}^T \quad (3.106)$$

$$\Psi_{j,\dots,N} \tilde{\mathbf{r}}^{(M,N-1)}(k,l) = \begin{bmatrix} \tilde{r}_1^{(M,N-1)} & \dots & \tilde{r}_{j-1}^{(M,N-1)} \end{bmatrix}^T \quad (3.107)$$

The vertical lattice coefficient vector $\mathbf{k}_{r_{ASHP_{2,j}}}^{(M,N-1)}$ is defined as

$$\mathbf{k}_{r_{ASHP_{2,j}}}^{(M,N-1)} = \begin{bmatrix} k_{r_{ASHP_{2,j}}}^{(M,N-1)}(1) & \dots & k_{r_{ASHP_{2,j}}}^{(M,N-1)}(N-j) \end{bmatrix}^T. \quad (3.108)$$

As a result of the mean-square minimization of the type-2 ASHP PEFs in (3.105), the reflection coefficient vector is computed as

$$\mathbf{k}_{r_{ASHP_{2,j}}}^{(M,N-1)} = \left(\Psi_{j,\dots,N} \Phi_{\mathbf{r},\tilde{\mathbf{r}}}^{(M,N-1)} \Psi_{j,\dots,N}^T \right)^{-1} \Psi_{j,\dots,N} \Phi_{\mathbf{r},\tilde{\mathbf{r}}}^{(M,N-1)} \Pi_j, \quad (3.109)$$

where $\Phi_{\mathbf{r},\tilde{\mathbf{r}}}^{(M,N-1)}$ is defined in (3.102).

The power of the error propagation equation from stage to stage can be written as follows

$$P_{r_{ASHP_{2,j}}}^{(M,N-1)} = P_{r_j}^{(M,N-1)} - \mathbf{k}_{r_{ASHP_{2,j}}}^{(M,N-1)H} \Psi_{j,\dots,N} \Phi_{\mathbf{r},\tilde{\mathbf{r}}}^{(M,N-1)} \Psi_{j,\dots,N}^T \mathbf{k}_{r_{ASHP_{2,j}}}^{(M,N-1)} \quad (3.110)$$

where $P_{r_j}^{(M,N-1)}$ is defined in (3.104).

Compact representations and calculations of type-1 and type-2 ASHP 2-D lattice models' coefficient vectors obtained from the auxiliary horizontal forward/backward PEFs are the same as for the type-1 and type-2 ASHP lattice models produced from the auxiliary vertical PEFs. The lattice coefficient matrix for type-1 and type-2 ASHP lattice filters are defined as

$$\mathbf{K}_{r_{ASHP_1}}^{(M,N-1)} = \begin{bmatrix} \mathbf{k}_{r_{ASHP_{1,2}}}^{(M,N-1)} & \mathbf{k}_{r_{ASHP_{1,3}}}^{(M,N-1)} & \dots & \mathbf{k}_{r_{ASHP_{1,N-2}}}^{(M,N-1)} & \mathbf{k}_{r_{ASHP_{1,N-1}}}^{(M,N-1)} \end{bmatrix}_{(N-2) \times (N-2)}, \quad (3.111)$$

and

$$\mathbf{K}_{r_{ASHP_2}}^{(M,N-1)} = \begin{bmatrix} \mathbf{k}_{r_{ASHP_{2,2}}}^{(M,N-1)} & \mathbf{k}_{r_{ASHP_{2,3}}}^{(M,N-1)} & \dots & \mathbf{k}_{r_{ASHP_{2,N-2}}}^{(M,N-1)} & \mathbf{k}_{r_{ASHP_{2,N-1}}}^{(M,N-1)} \end{bmatrix}_{(N-2) \times (N-2)}, \quad (3.112)$$

respectively.

In the same manner, we can define all type-1 ASHP lattice models as

$$\begin{aligned} \mathbf{r}_{ASHP_1}^{(M,N-1)}(k,l) &= \begin{bmatrix} r_{ASHP_{1,2}}^{(M,N-1)}(k,l) & r_{ASHP_{1,3}}^{(M,N-1)}(k,l) & \dots & r_{ASHP_{1,N-1}}^{(M,N-1)}(k,l) \end{bmatrix}^T \\ \tilde{\mathbf{r}}_{ASHP_1}^{(M,N-1)}(k,l) &= \begin{bmatrix} \tilde{r}_{ASHP_{1,2}}^{(M,N-1)}(k,l) & \tilde{r}_{ASHP_{1,3}}^{(M,N-1)}(k,l) & \dots & \tilde{r}_{ASHP_{1,N-1}}^{(M,N-1)}(k,l) \end{bmatrix}^T \end{aligned} \quad (3.113)$$

and all type-2 ASHP lattice models as

$$\begin{aligned} \mathbf{r}_{ASHP_2}^{(M,N-1)}(k,l) &= \begin{bmatrix} r_{ASHP_{2,2}}^{(M,N-1)}(k,l) & r_{ASHP_{2,3}}^{(M,N-1)}(k,l) & \dots & r_{ASHP_{2,N-1}}^{(M,N-1)}(k,l) \end{bmatrix}^T \\ \tilde{\mathbf{r}}_{ASHP_2}^{(M,N-1)}(k,l) &= \begin{bmatrix} \tilde{r}_{ASHP_{2,2}}^{(M,N-1)}(k,l) & \tilde{r}_{ASHP_{2,3}}^{(M,N-1)}(k,l) & \dots & \tilde{r}_{ASHP_{2,N-1}}^{(M,N-1)}(k,l) \end{bmatrix}^T \end{aligned} \quad (3.114)$$

We can write all lattice filter equations of type-1 and type-2 ASHP lattice models in the one pair of lattice equations as

$$\begin{aligned} \mathbf{r}_{ASHP_1}^{(M,N-1)}(k,l) &= \Psi_{1,N} \mathbf{r}^{(M,N-1)}(k,l) - \mathbf{K}_{r_{ASHP_1}}^{(M,N-1)H} \Psi_{1,2} \mathbf{r}^{(M,N-1)}(k,l) \\ \tilde{\mathbf{r}}_{ASHP_1}^{(M,N-1)}(k,l) &= \Psi_{1,N} \tilde{\mathbf{r}}^{(M,N-1)}(k,l) - \mathbf{K}_{r_{ASHP_1}}^{(M,N-1)T} \Psi_{1,2} \tilde{\mathbf{r}}^{(M,N-1)}(k,l), \end{aligned} \quad (3.115)$$

and

$$\begin{aligned} \mathbf{r}_{ASHP_2}^{(M,N-1)}(k,l) &= \mathbf{\Psi}_{1,N} \mathbf{r}^{(M,N-1)}(k,l) - \mathbf{K}_{r_{ASHP_2}}^{(M,N-1)H} \mathbf{\Psi}_{N-1,N} \mathbf{r}^{(M,N-1)}(k,l) \\ \tilde{\mathbf{r}}_{ASHP_2}^{(M,N-1)}(k,l) &= \mathbf{\Psi}_{1,N} \tilde{\mathbf{r}}^{(M,N-1)}(k,l) - \mathbf{K}_{r_{ASHP_2}}^{(M,N-1)T} \mathbf{\Psi}_{N-1,N} \tilde{\mathbf{r}}^{(M,N-1)}(k,l) \end{aligned} \quad (3.116)$$

respectively.

By minimizing the sum of the mean-square values of (3.115) and (3.116) with respect to the lattice coefficient matrices $\mathbf{K}_{r_{ASHP_1}}^{(M,N-1)}$ and $\mathbf{K}_{r_{ASHP_2}}^{(M,N-1)}$, respectively. We can find the following results as follows

$$\mathbf{K}_{r_{ASHP_1}}^{(M,N-1)} = \left[\mathbf{\Psi}_{1,2} \mathbf{\Phi}_{\mathbf{r},\tilde{\mathbf{r}}}^{(M,N-1)} \mathbf{\Psi}_{1,2}^T \right]^{-1} \mathbf{\Psi}_{1,2} \mathbf{\Phi}_{\mathbf{r},\tilde{\mathbf{r}}}^{(M,N-1)} \mathbf{\Psi}_{1,N}^T, \quad (3.117)$$

and

$$\mathbf{K}_{r_{ASHP_2}}^{(M,N-1)} = \left[\mathbf{\Psi}_{N-1,N} \mathbf{\Phi}_{\mathbf{r},\tilde{\mathbf{r}}}^{(M,N-1)} \mathbf{\Psi}_{N-1,N}^T \right]^{-1} \mathbf{\Psi}_{N-1,N} \mathbf{\Phi}_{\mathbf{r},\tilde{\mathbf{r}}}^{(M,N-1)} \mathbf{\Psi}_{1,N}^T. \quad (3.118)$$

respectively.

3.3.3 Application examples of the ASHP 2-D lattice models

In this section, some examples of the type-1 and type-2 ASHP 2-D lattice models obtained from vertical and horizontal forward/backward PEFs will be introduced. For each case, all possible type-1 and type-2 ASHP 2-D lattice models will be elaborated.

According to the proposed ASHP definition, it is not possible to define any ASHP lattice model from the PSRs $S_{k,l}^{(1,0)}$ and $S_{k,l}^{(0,1)}$. Moreover, the final PEFs for the type-1 and the first PEFs for the type-2 might be considered as a kind of ASHP lattice model themselves. For example, in case of the PSR $S_{k,l}^{(1,0)}$, the final PEFs $e_2^{(1,1)}$ and $\tilde{e}_2^{(1,1)}$ might be regarded as type-1 ASHP lattice model. Similarly, the first PEFs $e_1^{(1,1)}$ and $\tilde{e}_1^{(1,1)}$ might be taken into consideration as type-2 ASHP model. But, these kind of ASHP lattice model definitions will not be taken into account in the thesis. That is to say, only one PEF will not be considered as ASHP lattice model.

In order to define any ASHP model, the order of the PSR should be higher than (1,0) and (0,1).

3.3.3.1 ASHP 2-D lattice models of the PSR $S_{k,l}^{(2,0)}$

Natural structure of the PSR, only type-1 and type-2 ASHP lattice model using auxiliary vertical forward/backward PEFs can be generated. It is clear that just only one pair of type-1 and one pair of type-2 ASHP lattice filter can be produced. Here, M is found as 3 from the PSR $S_{k,l}^{(2,0)}$. If we say the PSR is $S_{k,l}^{(p,q)}$, then the number vertical forward PEFs, M , is equal to $(p+1)$, and the number of forward horizontal PEFs, N , is equal to $(q+1)$. Since type-1 or type-2 ASHP lattice models are defined for $j = 2, \dots, M-1$, it is possible to construct type-1 ASHP models, namely, $e_{ASHP_{1,2}}^{(2,1)}(k,l)$ and $\tilde{e}_{ASHP_{1,2}}^{(2,1)}(k,l)$, and type-2 ASHP models, namely, $e_{ASHP_{2,2}}^{(2,1)}(k,l)$ and $\tilde{e}_{ASHP_{2,2}}^{(2,1)}(k,l)$.

In section 3.2.1, we illustrated auxiliary vertical forward and backward PEFs of $S_{k,l}^{(1,0)}$ in two PSRs, $S_{k,l-1}^{(1,0)}$ and $S_{k,l}^{(1,0)}$, as shown in Figure 3.4. In this example and so on, all auxiliary vertical/horizontal forward and backward PEFs will be demonstrated on the one arbitrary PSR with the given order $(M-1, N-1)$, $S_{k,l}^{(M-1, N-1)}$ as defined in Figure 2.10.

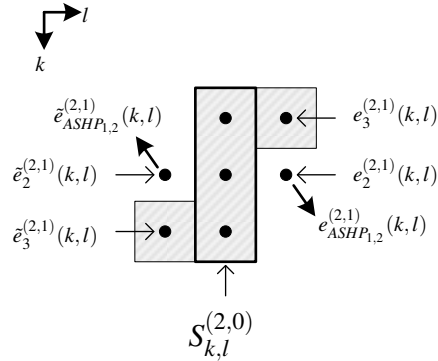


Figure 3.12: Type-1 ASHP lattice model from the corresponding auxiliary vertical PEFs of the PSR $S_{k,l}^{(2,0)}$.

Figure 3.12 shows the illustration of the type-1 ASHP lattice model construction of the PSR $S_{k,l}^{(2,0)}$. The type-1 ASHP lattice model equations can be written as,

$$\begin{aligned} e_{ASHP_{1,2}}^{(2,1)}(k,l) &= e_2^{(2,1)}(k,l) - \mathbf{k}_{e_{ASHP_{1,2}}}^{(2,1)H} \Psi_{1,2} \mathbf{e}^{(2,1)}(k,l) \\ \tilde{e}_{ASHP_{1,2}}^{(2,1)}(k,l) &= \tilde{e}_2^{(2,1)}(k,l) - \mathbf{k}_{e_{ASHP_{1,2}}}^{(2,1)T} \Psi_{1,2} \tilde{\mathbf{e}}^{(2,1)}(k,l) \end{aligned} \quad (3.119)$$

From the definition of the $\Psi_{1,\dots,j}$, we can get $\Psi_{1,2} = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$ from removing the first and second row of the identity matrix $\mathbf{I}_{3 \times 3}$ that will be necessary for this order due

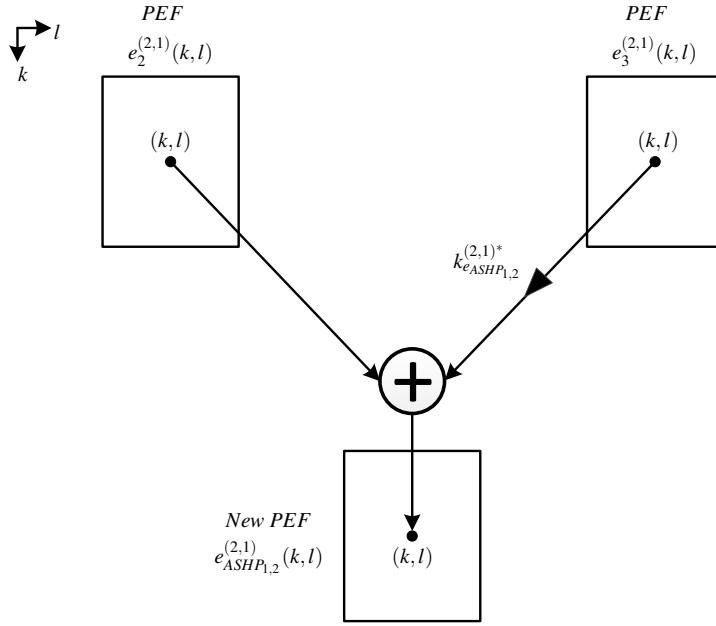


Figure 3.13: The construction of type-1 ASHP PEF $e_{ASHP_{1,2}}^{(2,1)}(k,l)$ by predicting the second auxiliary vertical PEF $e_2^{(2,1)}(k,l)$ from the vertical PEF $e_3^{(2,1)}(k,l)$.

to the number of the vertical forward/backward PEFs. Thus, we can find that

$$\begin{aligned} \Psi_{1,2} \mathbf{e}^{(2,1)}(k,l) &= \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} e_1^{(2,1)}(k,l) \\ e_2^{(2,1)}(k,l) \\ e_3^{(2,1)}(k,l) \end{bmatrix} \\ &= e_3^{(2,1)}(k,l) \end{aligned} \quad (3.120)$$

$$\begin{aligned} \Psi_{1,2} \tilde{\mathbf{e}}^{(2,1)}(k,l) &= \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \tilde{e}_1^{(2,1)}(k,l) \\ \tilde{e}_2^{(2,1)}(k,l) \\ \tilde{e}_3^{(2,1)}(k,l) \end{bmatrix} \\ &= \tilde{e}_3^{(2,1)}(k,l). \end{aligned} \quad (3.121)$$

By substituting (3.120) and (3.121) into (3.119), the type-1 ASHP lattice model equations will get the following form:

$$\begin{aligned} e_{ASHP_{1,2}}^{(2,1)}(k,l) &= e_2^{(2,1)}(k,l) - k_{e_{ASHP_{1,2}}}^{(2,1)*} e_3^{(2,1)}(k,l) \\ \tilde{e}_{ASHP_{1,2}}^{(2,1)}(k,l) &= \tilde{e}_2^{(2,1)}(k,l) - k_{e_{ASHP_{1,2}}}^{(2,1)} \tilde{e}_3^{(2,1)}(k,l). \end{aligned} \quad (3.122)$$

The Figure 3.13 demonstrates the calculation of the type-1 ASHP lattice model $e_{ASHP_{1,2}}^{(2,1)}(k,l)$ by using $e_2^{(2,1)}(k,l)$ and $e_3^{(2,1)}(k,l)$. Of course, for the backward case $\tilde{e}_{ASHP_{1,2}}^{(2,1)}(k,l)$, a similar figure can be delineated.

The power of the error propagation equation from the lattice form equations (3.122) can be written as

$$P_{e_{ASHP_{1,2}}}^{(2,1)} = E \left[\left| e_{ASHP_{1,2}}^{(2,1)}(k, l) \right|^2 \right] + E \left[\left| \tilde{e}_{ASHP_{1,2}}^{(2,1)}(k, l) \right|^2 \right]. \quad (3.123)$$

As a result of the mean-square minimization of (3.123), the reflection coefficient vector is calculated as

$$\begin{aligned} \mathbf{k}_{e_{ASHP_{1,2}}}^{(2,1)} &= \left(\Psi_{1,2} \Phi_{\mathbf{e}, \tilde{\mathbf{e}}}^{(2,1)} \Psi_{1,2}^T \right)^{-1} \Psi_{1,2} \Phi_{\mathbf{e}, \tilde{\mathbf{e}}}^{(2,1)} \Pi_2 \\ &= \left(E \left[\left| e_3^{(2,1)}(k, l) \right|^2 \right] + E \left[\left| \tilde{e}_3^{(2,1)}(k, l) \right|^2 \right] \right)^{-1} \\ &\quad \left(E \left[e_2^{(2,1)}(k, l) \tilde{e}_3^{(2,1)*}(k, l) \right] + E \left[\tilde{e}_3^{(2,1)}(k, l) e_2^{(2,1)*}(k, l) \right] \right). \end{aligned} \quad (3.124)$$

We can restructure the power of the error propagation equation from stage to stage by using (3.124) in (3.123),

$$\begin{aligned} P_{e_{ASHP_{1,2}}}^{(2,1)} &= P_{e_2}^{(2,1)} - \mathbf{k}_{e_{ASHP_{1,2}}}^{(2,1)*} \Psi_{1,2} \Phi_{\mathbf{e}, \tilde{\mathbf{e}}}^{(2,1)} \Psi_{1,2}^T \mathbf{k}_{e_{ASHP_{1,2}}}^{(2,1)} \\ &= P_{e_2}^{(2,1)} - \mathbf{k}_{e_{ASHP_{1,2}}}^{(2,1)*} \left(E \left[\left| e_3^{(2,1)} \right|^2 \right] + E \left[\left| \tilde{e}_3^{(2,1)} \right|^2 \right] \right) \mathbf{k}_{e_{ASHP_{1,2}}}^{(2,1)} \\ &= P_{e_2}^{(2,1)} - \left| \mathbf{k}_{e_{ASHP_{1,2}}}^{(2,1)} \right|^2 P_{e_3}^{(2,1)} \end{aligned} \quad (3.125)$$

where

$$P_{e_2}^{(2,1)} = E \left[\left| e_2^{(2,1)}(k, l) \right|^2 \right] + E \left[\left| \tilde{e}_2^{(2,1)}(k, l) \right|^2 \right]$$

and

$$P_{e_3}^{(2,1)} = E \left[\left| e_3^{(2,1)}(k, l) \right|^2 \right] + E \left[\left| \tilde{e}_3^{(2,1)}(k, l) \right|^2 \right].$$

Similarly, type-2 ASHP PEFs lattice filter model construction can be demonstrated as in Figure 3.14, and the type-2 ASHP lattice model equations can be formed as

$$\begin{aligned} e_{ASHP_{2,2}}^{(2,1)}(k, l) &= e_2^{(2,1)}(k, l) - \mathbf{k}_{e_{ASHP_{2,2}}}^{(2,1)H} \Psi_{2,3} \mathbf{e}^{(2,1)}(k, l) \\ \tilde{e}_{ASHP_{2,2}}^{(2,1)}(k, l) &= \tilde{e}_2^{(2,1)}(k, l) - \mathbf{k}_{e_{ASHP_{2,2}}}^{(2,1)T} \Psi_{2,3} \tilde{\mathbf{e}}^{(2,1)}(k, l). \end{aligned} \quad (3.126)$$

Clearly the equations given in (3.126) can be rewritten as

$$\begin{aligned} e_{ASHP_{2,2}}^{(2,1)}(k, l) &= e_2^{(2,1)}(k, l) - k_{e_{ASHP_{2,2}}}^{(2,1)*} e_1^{(2,1)}(k, l) \\ \tilde{e}_{ASHP_{2,2}}^{(2,1)}(k, l) &= \tilde{e}_2^{(2,1)}(k, l) - k_{e_{ASHP_{2,2}}}^{(2,1)} \tilde{e}_1^{(2,1)}(k, l). \end{aligned} \quad (3.127)$$

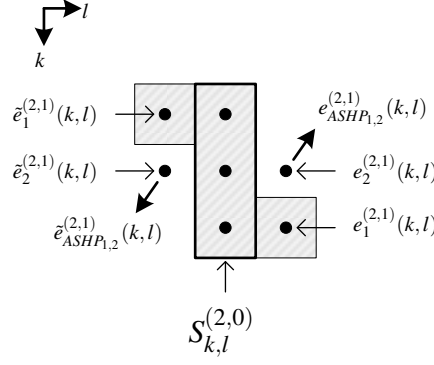


Figure 3.14: Type-2 ASHP lattice model from the corresponding auxiliary vertical PEFs of the PSR $S_{k,l}^{(2,0)}$.

As explained for type-1 case, the power of the error propagation equation from the equations given in (3.127) can be defined as

$$P_{e_{ASHP_{2,2}}}^{(2,1)} = E \left[\left| e_{ASHP_{2,2}}^{(2,1)}(k,l) \right|^2 \right] + E \left[\left| \tilde{e}_{ASHP_{2,2}}^{(2,1)}(k,l) \right|^2 \right]. \quad (3.128)$$

The reflection coefficient vector is calculated by minimizing of (3.128) in the mean-square sense:

$$\begin{aligned} \mathbf{k}_{e_{ASHP_{2,2}}}^{(2,1)} &= \left(\Psi_{2,3} \Phi_{\mathbf{e}, \tilde{\mathbf{e}}}^{(2,1)} \Psi_{2,3}^T \right)^{-1} \Psi_{2,3} \Phi_{\mathbf{e}, \tilde{\mathbf{e}}}^{(2,1)} \Pi_2 \\ &= \left(E \left[\left| e_1^{(2,1)}(k,l) \right|^2 \right] + E \left[\left| \tilde{e}_1^{(2,1)}(k,l) \right|^2 \right] \right)^{-1} \\ &\quad \left(E \left[e_2^{(2,1)}(k,l) \tilde{e}_1^{(2,1)*}(k,l) \right] + E \left[\tilde{e}_1^{(2,1)}(k,l) e_2^{(2,1)*}(k,l) \right] \right). \end{aligned} \quad (3.129)$$

The power of the error propagation equation from stage to stage by using (3.129) can be stated as

$$\begin{aligned} P_{e_{ASHP_{2,2}}}^{(2,1)} &= P_{e_2}^{(2,1)} - \mathbf{k}_{e_{ASHP_{2,2}}}^{(2,1)*} \Psi_{2,3} \Phi_{\mathbf{e}, \tilde{\mathbf{e}}}^{(2,1)} \Psi_{2,3}^T \mathbf{k}_{e_{ASHP_{2,2}}}^{(2,1)} \\ &= P_{e_2}^{(2,1)} - \mathbf{k}_{e_{ASHP_{2,2}}}^{(2,1)*} \left(E \left[\left| e_1^{(2,1)} \right|^2 \right] + E \left[\left| \tilde{e}_1^{(2,1)} \right|^2 \right] \right) \mathbf{k}_{e_{ASHP_{2,2}}}^{(2,1)} \\ &= P_{e_2}^{(2,1)} - \left| \mathbf{k}_{e_{ASHP_{2,2}}}^{(2,1)} \right|^2 P_{e_1}^{(2,1)} \end{aligned} \quad (3.130)$$

where

$$P_{e_1}^{(2,1)} = E \left[\left| e_1^{(2,1)}(k,l) \right|^2 \right] + E \left[\left| \tilde{e}_1^{(2,1)}(k,l) \right|^2 \right].$$

3.3.3.2 ASHP 2-D lattice models of the PSR $S_{k,l}^{(0,2)}$

Naturally, only type-1 and type-2 ASHP lattice model using auxiliary horizontal forward/backward PEFs can be generated from the PSR $S_{k,l}^{(0,2)}$. It is clearly seen that just only one pair of type-1 and one pair of type-2 ASHP lattice filter can be produced by utilizing the auxiliary horizontal forward/backward PEFs. Since type-1 or type-2 ASHP lattice models are defined for $j = 2, \dots, N-1$, from the PSR order N is found as 3 which is equal to number of the horizontal forward or backward PEFs, it is possible to construct type-1 ASHP models, namely, $r_{ASHP_{1,2}}^{(1,2)}(k,l)$ and $\tilde{r}_{ASHP_{1,2}}^{(1,2)}(k,l)$, and type-2 ASHP models, namely, $r_{ASHP_{2,2}}^{(1,2)}(k,l)$ and $\tilde{e}_{ASHP_{2,2}}^{(1,2)}(k,l)$.

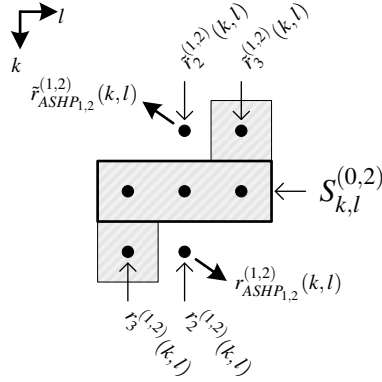


Figure 3.15: Type-1 ASHP lattice model from the corresponding auxiliary horizontal PEFs of the PSR $S_{k,l}^{(0,2)}$.

Figure 3.15 shows the illustration of the type-1 ASHP lattice model construction of the PSR $S_{k,l}^{(0,2)}$ from the horizontal PEFs. The type-1 ASHP lattice model equations can be given as,

$$\begin{aligned} r_{ASHP_{1,2}}^{(1,2)}(k,l) &= r_2^{(1,2)}(k,l) - \mathbf{k}_{r_{ASHP_{1,2}}}^{(1,2)H} \Psi_{1,2} \mathbf{r}^{(1,2)}(k,l) \\ \tilde{r}_{ASHP_{1,2}}^{(1,2)}(k,l) &= \tilde{r}_2^{(1,2)}(k,l) - \mathbf{k}_{r_{ASHP_{1,2}}}^{(1,2)T} \Psi_{1,2} \tilde{\mathbf{r}}^{(1,2)}(k,l) \end{aligned} \quad (3.131)$$

From the definition of the $\Psi_{1,\dots,j}$, we can get $\Psi_{1,2} = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$. Hence, the following definitions can be stated as

$$\begin{aligned}\Psi_{1,2}\mathbf{r}^{(1,2)}(k,l) &= \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} r_1^{(1,2)}(k,l) \\ r_2^{(1,2)}(k,l) \\ r_3^{(1,2)}(k,l) \end{bmatrix} \\ &= r_3^{(1,2)}(k,l)\end{aligned}\quad (3.132)$$

$$\begin{aligned}\Psi_{1,2}\tilde{\mathbf{r}}^{(1,2)}(k,l) &= \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \tilde{r}_1^{(1,2)}(k,l) \\ \tilde{r}_2^{(1,2)}(k,l) \\ \tilde{r}_3^{(1,2)}(k,l) \end{bmatrix} \\ &= \tilde{r}_3^{(1,2)}(k,l).\end{aligned}\quad (3.133)$$

By substituting (3.132) and (3.133) into (3.131), the type-1 ASHP lattice model equations will be obtained as

$$\begin{aligned}r_{ASHP_{1,2}}^{(1,2)}(k,l) &= r_2^{(1,2)}(k,l) - k_{r_{ASHP_{1,2}}}^{(1,2)*} r_3^{(1,2)}(k,l) \\ \tilde{r}_{ASHP_{1,2}}^{(1,2)}(k,l) &= \tilde{r}_2^{(1,2)}(k,l) - k_{r_{ASHP_{1,2}}}^{(1,2)} \tilde{r}_3^{(1,2)}(k,l).\end{aligned}\quad (3.134)$$

The power of the error propagation equation from the lattice model equations given in (3.134) can be written as

$$P_{r_{ASHP_{1,2}}}^{(1,2)} = E \left[\left| r_{ASHP_{1,2}}^{(1,2)}(k,l) \right|^2 \right] + E \left[\left| \tilde{r}_{ASHP_{1,2}}^{(1,2)}(k,l) \right|^2 \right]. \quad (3.135)$$

As a result of the mean-square minimization of (3.135), the reflection coefficient vector is calculated as

$$\begin{aligned}\mathbf{k}_{r_{ASHP_{1,2}}}^{(1,2)} &= \left(\Psi_{1,2} \Phi_{\mathbf{r},\tilde{\mathbf{r}}}^{(1,2)} \Psi_{1,2}^T \right)^{-1} \Psi_{1,2} \Phi_{\mathbf{r},\tilde{\mathbf{r}}}^{(1,2)} \Pi_2 \\ &= \left(E \left[\left| r_3^{(1,2)}(k,l) \right|^2 \right] + E \left[\left| \tilde{r}_3^{(1,2)}(k,l) \right|^2 \right] \right)^{-1} \\ &\quad \times \left(E \left[r_2^{(1,2)}(k,l) \tilde{r}_3^{(1,2)*}(k,l) \right] + E \left[\tilde{r}_3^{(1,2)}(k,l) e_2^{(1,2)*}(k,l) \right] \right)\end{aligned}\quad (3.136)$$

The power of the error propagation equation from stage to stage by using (3.136) can be defined as

$$\begin{aligned}P_{r_{ASHP_{1,2}}}^{(1,2)} &= P_{r_2}^{(1,2)} - \mathbf{k}_{r_{ASHP_{1,2}}}^{(1,2)*} \Psi_{1,2} \Phi_{\mathbf{r},\tilde{\mathbf{r}}}^{(1,2)} \Psi_{1,2}^T \mathbf{k}_{r_{ASHP_{1,2}}}^{(1,2)} \\ &= P_{e_2}^{(1,2)} - \mathbf{k}_{r_{ASHP_{1,2}}}^{(1,2)*} \left(E \left[\left| r_3^{(1,2)} \right|^2 \right] + E \left[\left| \tilde{r}_3^{(1,2)} \right|^2 \right] \right) \mathbf{k}_{r_{ASHP_{1,2}}}^{(1,2)} \\ &= P_{r_2}^{(1,2)} - \left| \mathbf{k}_{r_{ASHP_{1,2}}}^{(1,2)} \right|^2 P_{r_3}^{(1,2)}\end{aligned}\quad (3.137)$$

where

$$P_{r_2}^{(1,2)} = E \left[\left| r_2^{(1,2)}(k, l) \right|^2 \right] + E \left[\left| \tilde{r}_2^{(1,2)}(k, l) \right|^2 \right]$$

and

$$P_{r_3}^{(1,2)} = E \left[\left| r_3^{(1,2)}(k, l) \right|^2 \right] + E \left[\left| \tilde{r}_3^{(1,2)}(k, l) \right|^2 \right].$$

Similar to the type-1 ASHP case, type-2 ASHP PEFs lattice filter model construction can be demonstrated as in Figure 3.16, and the type-2 ASHP lattice model equations can be built up as

$$\begin{aligned} r_{ASHP_{2,2}}^{(1,2)}(k, l) &= r_2^{(1,2)}(k, l) - \mathbf{k}_{r_{ASHP_{2,2}}}^{(1,2)H} \mathbf{\Psi}_{2,3} \mathbf{r}^{(1,2)}(k, l) \\ \tilde{r}_{ASHP_{2,2}}^{(1,2)}(k, l) &= \tilde{r}_2^{(1,2)}(k, l) - \mathbf{k}_{r_{ASHP_{2,2}}}^{(1,2)T} \mathbf{\Psi}_{2,3} \tilde{\mathbf{r}}^{(1,2)}(k, l). \end{aligned} \quad (3.138)$$

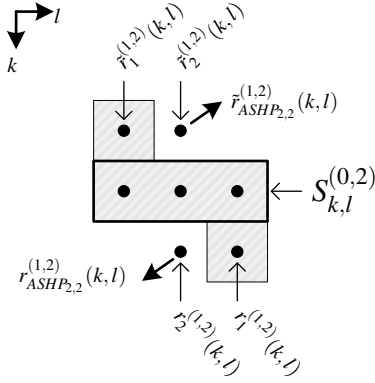


Figure 3.16: Type-2 ASHP lattice model from the corresponding auxiliary horizontal PEFs of the PSR $S_{k,l}^{(0,2)}$.

By using the definition of $\mathbf{\Psi}_{2,3}$, the equations given in (3.138) can be rewritten as

$$\begin{aligned} r_{ASHP_{2,2}}^{(1,2)}(k, l) &= r_2^{(1,2)}(k, l) - k_{r_{ASHP_{2,2}}}^{(1,2)*} r_1^{(1,2)}(k, l) \\ \tilde{r}_{ASHP_{2,2}}^{(1,2)}(k, l) &= \tilde{r}_2^{(1,2)}(k, l) - k_{r_{ASHP_{2,2}}}^{(1,2)} \tilde{r}_1^{(1,2)}(k, l). \end{aligned} \quad (3.139)$$

The power of the error propagation equation from (3.139) can be defined as

$$P_{r_{ASHP_{2,2}}}^{(1,2)} = E \left[\left| r_{ASHP_{2,2}}^{(1,2)}(k, l) \right|^2 \right] + E \left[\left| \tilde{r}_{ASHP_{2,2}}^{(1,2)}(k, l) \right|^2 \right]. \quad (3.140)$$

By minimizing of (3.140) in the mean-square sense, the reflection coefficient vector is calculated as

$$\begin{aligned}
\mathbf{k}_{r_{ASHP_{2,2}}}^{(1,2)} &= \left(\Psi_{2,3} \Phi_{\mathbf{r}, \tilde{\mathbf{r}}}^{(1,2)} \Psi_{2,3}^T \right)^{-1} \Psi_{2,3} \Phi_{\mathbf{r}, \tilde{\mathbf{r}}}^{(1,2)} \Pi_2 \\
&= \left(E \left[\left| r_1^{(1,2)}(k, l) \right|^2 \right] + E \left[\left| \tilde{r}_1^{(1,2)}(k, l) \right|^2 \right] \right)^{-1} \\
&\quad \times \left(E \left[r_2^{(1,2)}(k, l) \tilde{r}_1^{(1,2)*}(k, l) \right] + E \left[\tilde{r}_1^{(1,2)}(k, l) r_2^{(1,2)*}(k, l) \right] \right) \quad (3.141)
\end{aligned}$$

The power of the error propagation equation from stage to stage by using (3.129) can be defined as

$$\begin{aligned}
P_{r_{ASHP_{2,2}}}^{(1,2)} &= P_{r_2}^{(1,2)} - \mathbf{k}_{r_{ASHP_{2,2}}}^{(1,2)*} \Psi_{2,3} \Phi_{\mathbf{r}, \tilde{\mathbf{r}}}^{(1,2)} \Psi_{2,3}^T \mathbf{k}_{r_{ASHP_{2,2}}}^{(1,2)} \\
&= P_{r_2}^{(1,2)} - \mathbf{k}_{r_{ASHP_{2,2}}}^{(1,2)*} \left(E \left[\left| r_1^{(1,2)} \right|^2 \right] + E \left[\left| \tilde{r}_1^{(1,2)} \right|^2 \right] \right) \mathbf{k}_{r_{ASHP_{2,2}}}^{(1,2)} \\
&= P_{r_2}^{(1,2)} - \left| \mathbf{k}_{r_{ASHP_{2,2}}}^{(1,2)} \right|^2 P_{r_1}^{(1,2)} \quad (3.142)
\end{aligned}$$

where

$$P_{r_1}^{(1,2)} = E \left[\left| r_1^{(1,2)}(k, l) \right|^2 \right] + E \left[\left| \tilde{r}_1^{(1,2)}(k, l) \right|^2 \right].$$

3.3.3.3 ASHP 2-D lattice models of the PSR $S_{k,l}^{(1,4)}$

In this section, we will show how to obtain all possible type-1 and type-2 ASHP lattice models by utilizing auxiliary horizontal forward/backward PEFs. The generation of the whole type-1 ASHP lattice models is illustrated in Figure 3.17, 3.18, and 3.19. It is not possible to produce any type-1 and type-2 ASHP lattice filter from the auxiliary vertical forward/backward PEFs since there are only two pairs of forward/backward auxiliary vertical PEFs. According to our definition, it is possible to define ASHP lattice model for $j = 2, \dots, M - 1$. In our example M is 2. Hence, there is no possibility to generate any ASHP lattice model from the vertical PEFs. In order to build up any ASHP lattice model, $M - 1$ must be equal or greater than 2.

The type-1 ASHP lattice models' equations constructed from the horizontal forward/backward PEFs of the PSR $S_{k,l}^{(1,4)}$ can be written for $j = 2, 3, 4$ as

$$\begin{aligned}
r_{ASHP_{1,2}}^{(2,4)}(k, l) &= r_2^{(2,4)}(k, l) - \mathbf{k}_{r_{ASHP_{1,2}}}^{(2,4)H} \Psi_{1,2} \mathbf{r}^{(2,4)}(k, l) \\
\tilde{r}_{ASHP_{1,2}}^{(2,4)}(k, l) &= \tilde{r}_2^{(2,4)}(k, l) - \mathbf{k}_{r_{ASHP_{1,2}}}^{(2,4)T} \Psi_{1,2} \tilde{\mathbf{r}}^{(2,4)}(k, l), \quad (3.143)
\end{aligned}$$

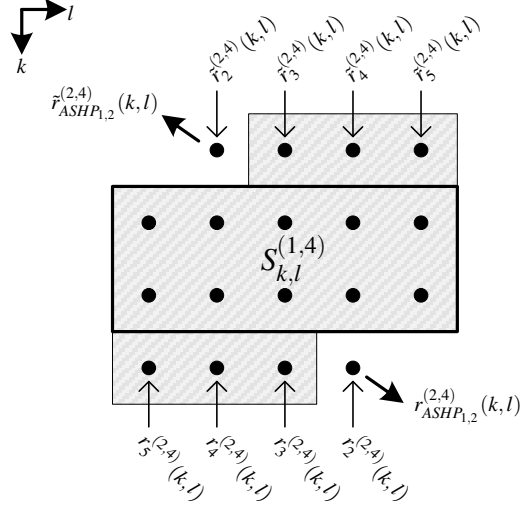


Figure 3.17: Type-1 ASHP lattice models of the PSR $S_{k,l}^{(1,4)}$: $r_{ASHP_{1,2}}^{(2,4)}(k,l)$ and $\tilde{r}_{ASHP_{1,2}}^{(2,4)}(k,l)$.

$$\begin{aligned} r_{ASHP_{1,3}}^{(2,4)}(k,l) &= r_3^{(2,4)}(k,l) - \mathbf{k}_{r_{ASHP_{1,3}}}^{(2,4)H} \Psi_{1,2,3} \mathbf{r}^{(2,4)}(k,l) \\ \tilde{r}_{ASHP_{1,3}}^{(2,4)}(k,l) &= \tilde{r}_3^{(2,4)}(k,l) - \mathbf{k}_{r_{ASHP_{1,3}}}^{(2,4)T} \Psi_{1,2,3} \tilde{\mathbf{r}}^{(2,4)}(k,l), \end{aligned} \quad (3.144)$$

and

$$\begin{aligned} r_{ASHP_{1,4}}^{(2,4)}(k,l) &= r_4^{(1,2)}(k,l) - \mathbf{k}_{r_{ASHP_{1,4}}}^{(2,4)H} \Psi_{1,2,3,4} \mathbf{r}^{(2,4)}(k,l) \\ \tilde{r}_{ASHP_{1,4}}^{(2,4)}(k,l) &= \tilde{r}_4^{(2,4)}(k,l) - \mathbf{k}_{r_{ASHP_{1,4}}}^{(2,4)T} \Psi_{1,2,3,4} \tilde{\mathbf{r}}^{(2,4)}(k,l), \end{aligned} \quad (3.145)$$

where

$$\mathbf{k}_{r_{ASHP_{1,2}}}^{(2,4)} = \begin{bmatrix} k_{r_{ASHP_{1,2}}}^{(2,4)}(1) \\ k_{r_{ASHP_{1,2}}}^{(2,4)}(2) \\ k_{r_{ASHP_{1,2}}}^{(2,4)}(3) \end{bmatrix}, \quad (3.146)$$

and

$$\mathbf{k}_{r_{ASHP_{1,3}}}^{(2,4)} = \begin{bmatrix} k_{r_{ASHP_{1,3}}}^{(2,4)}(1) \\ k_{r_{ASHP_{1,3}}}^{(2,4)}(2) \end{bmatrix}, \quad (3.147)$$

and

$$\mathbf{k}_{r_{ASHP_{1,4}}}^{(2,4)} = \begin{bmatrix} k_{r_{ASHP_{1,4}}}^{(2,4)} \end{bmatrix}, \quad (3.148)$$

respectively.

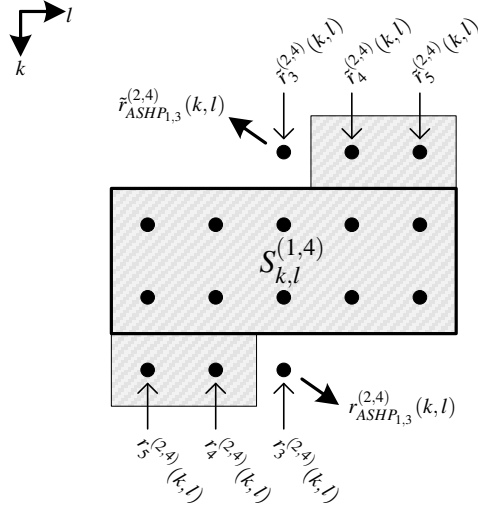


Figure 3.18: Type-1 ASHP lattice models of the PSR $S_{k,l}^{(1,4)}$: $r_{ASHP_{1,3}}^{(2,4)}(k,l)$ and $\tilde{r}_{ASHP_{1,3}}^{(2,4)}(k,l)$.

For this example, we need the identity matrix $\mathbf{I}_{5 \times 5}$ and from the definition of the $\Psi_{1,\dots,j}$, we can get for $j = 2, \dots, 4$,

$$\begin{aligned} \Psi_{1,2} &= \Psi_{ASHP_{1,2}} \\ &= \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \end{aligned} \quad (3.149)$$

$$\begin{aligned} \Psi_{1,2,3} &= \Psi_{ASHP_{1,3}} \\ &= \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \end{aligned} \quad (3.150)$$

$$\begin{aligned} \Psi_{1,2,3,4} &= \Psi_{ASHP_{1,4}} \\ &= \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \end{aligned} \quad (3.151)$$

respectively.

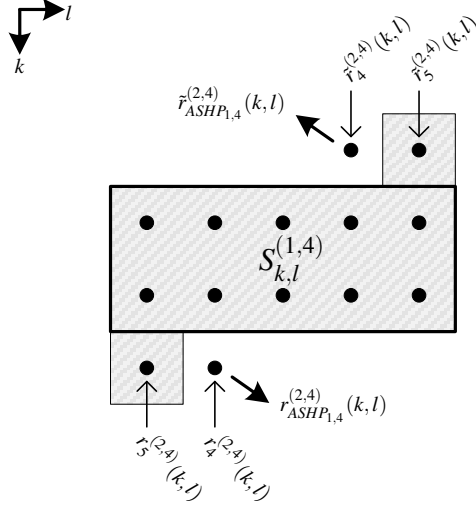


Figure 3.19: Type-1 ASHP lattice models of the PSR $S_{k,l}^{(1,4)}$: $r_{ASHP_{1,4}}^{(2,4)}(k,l)$ and $\tilde{r}_{ASHP_{1,4}}^{(2,4)}(k,l)$.

By substituting (3.149), (3.150), and (3.151) into (3.143), (3.144), and (3.145), respectively, the type-1 ASHP model equations will be as follows

$$\begin{aligned}
 r_{ASHP_{1,2}}^{(2,4)}(k,l) &= r_2^{(2,4)}(k,l) - k_{r_{ASHP_{1,2}}}^{(2,4)*}(1)r_3^{(2,4)}(k,l) \\
 &\quad - k_{r_{ASHP_{1,2}}}^{(2,4)*}(2)r_4^{(2,4)}(k,l) - k_{r_{ASHP_{1,2}}}^{(2,4)*}(3)r_5^{(2,4)}(k,l) \\
 \tilde{r}_{ASHP_{1,2}}^{(2,4)}(k,l) &= \tilde{r}_2^{(2,4)}(k,l) - k_{r_{ASHP_{1,2}}}^{(2,4)}(1)\tilde{r}_3^{(2,4)}(k,l) \\
 &\quad - k_{r_{ASHP_{1,2}}}^{(2,4)}(2)\tilde{r}_4^{(2,4)}(k,l) - k_{r_{ASHP_{1,2}}}^{(2,4)}(3)\tilde{r}_5^{(2,4)}(k,l), \quad (3.152)
 \end{aligned}$$

$$\begin{aligned}
 r_{ASHP_{1,3}}^{(2,4)}(k,l) &= r_3^{(2,4)}(k,l) - k_{r_{ASHP_{1,3}}}^{(2,4)*}(1)r_4^{(2,4)}(k,l) \\
 &\quad - k_{r_{ASHP_{1,3}}}^{(2,4)*}(2)r_5^{(2,4)}(k,l) \\
 \tilde{r}_{ASHP_{1,3}}^{(2,4)}(k,l) &= \tilde{r}_3^{(2,4)}(k,l) - k_{r_{ASHP_{1,3}}}^{(2,4)}(1)\tilde{r}_4^{(2,4)}(k,l) \\
 &\quad - k_{r_{ASHP_{1,3}}}^{(2,4)}(2)\tilde{r}_5^{(2,4)}(k,l), \quad (3.153)
 \end{aligned}$$

$$\begin{aligned}
 r_{ASHP_{1,4}}^{(2,4)}(k,l) &= r_4^{(2,4)}(k,l) - k_{r_{ASHP_{1,4}}}^{(2,4)*}r_5^{(2,4)}(k,l) \\
 \tilde{r}_{ASHP_{1,4}}^{(2,4)}(k,l) &= \tilde{r}_4^{(2,4)}(k,l) - k_{r_{ASHP_{1,4}}}^{(2,4)}\tilde{r}_5^{(2,4)}(k,l). \quad (3.154)
 \end{aligned}$$

The power of the error propagation equations from the lattice model equations given in (3.152), (3.153), and (3.154) can be written as

$$P_{r_{ASHP_{1,j}}}^{(2,4)} = E \left[\left| r_{ASHP_{1,j}}^{(2,4)}(k,l) \right|^2 \right] + E \left[\left| \tilde{r}_{ASHP_{1,j}}^{(2,4)}(k,l) \right|^2 \right], \quad (3.155)$$

for $j = 2, 3, 4$.

As a result of the mean-square minimization of (3.155), the reflection coefficient vectors are calculated for $j = 2, 3, 4$ as

$$\begin{aligned} \mathbf{k}_{r_{ASHP_{1,2}}}^{(2,4)} &= \left(\Psi_{1,2} \Phi_{\mathbf{r},\tilde{\mathbf{r}}}^{(2,4)} \Psi_{1,2}^T \right)^{-1} \Psi_{1,2} \Phi_{\mathbf{r},\tilde{\mathbf{r}}}^{(2,4)} \Pi_2 \\ &= \begin{bmatrix} k_{r_{ASHP_{1,2}}}^{(2,4)*} (1) \\ k_{r_{ASHP_{1,2}}}^{(2,4)*} (2) \\ k_{r_{ASHP_{1,2}}}^{(2,4)*} (3) \end{bmatrix}, \end{aligned} \quad (3.156)$$

$$\begin{aligned} \mathbf{k}_{r_{ASHP_{1,3}}}^{(2,4)} &= \left(\Psi_{1,2,3} \Phi_{\mathbf{r},\tilde{\mathbf{r}}}^{(2,4)} \Psi_{1,2,3}^T \right)^{-1} \Psi_{1,2,3} \Phi_{\mathbf{r},\tilde{\mathbf{r}}}^{(2,4)} \Pi_3, \\ &= \begin{bmatrix} k_{r_{ASHP_{1,3}}}^{(2,4)*} (1) \\ k_{r_{ASHP_{1,3}}}^{(2,4)*} (2) \end{bmatrix}, \end{aligned} \quad (3.157)$$

and

$$\begin{aligned} \mathbf{k}_{r_{ASHP_{1,4}}}^{(2,4)} &= \left(\Psi_{1,2,3,4} \Phi_{\mathbf{r},\tilde{\mathbf{r}}}^{(2,4)} \Psi_{1,2,3,4}^T \right)^{-1} \Psi_{1,2,3,4} \Phi_{\mathbf{r},\tilde{\mathbf{r}}}^{(2,4)} \Pi_4, \\ &= \begin{bmatrix} k_{r_{ASHP_{1,2}}}^{(2,4)*} \end{bmatrix}, \end{aligned} \quad (3.158)$$

where

$$\Phi_{\mathbf{r},\tilde{\mathbf{r}}}^{(2,4)} = \Sigma_{\mathbf{r},\mathbf{r}}^{(2,4)} + \Sigma_{\tilde{\mathbf{r}},\tilde{\mathbf{r}}}^{(2,4)*},$$

and

$$\begin{aligned} \Pi_2 &= \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \end{bmatrix}^T, \\ \Pi_3 &= \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \end{bmatrix}^T, \\ \Pi_4 &= \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \end{bmatrix}^T. \end{aligned}$$

In order to show the results of the lattice coefficient vectors for each type-1 ASHP lattice model, first we compute $\Sigma_{\mathbf{r},\mathbf{r}}$ and $\Sigma_{\tilde{\mathbf{r}},\tilde{\mathbf{r}}}^*$ as

$$\begin{aligned} \Sigma_{\mathbf{r},\mathbf{r}}^{(2,4)} &= E \left[\mathbf{r}^{(2,4)}(k, l) \mathbf{r}^{(2,4)H}(k, l) \right] \\ &= \begin{bmatrix} E[r_1 r_1^*] & E[r_1 r_2^*] & E[r_1 r_3^*] & E[r_1 r_4^*] & E[r_1 r_5^*] \\ E[r_2 r_1^*] & E[r_2 r_2^*] & E[r_2 r_3^*] & E[r_2 r_4^*] & E[r_2 r_5^*] \\ E[r_3 r_1^*] & E[r_3 r_2^*] & E[r_3 r_3^*] & E[r_3 r_4^*] & E[r_3 r_5^*] \\ E[r_4 r_1^*] & E[r_4 r_2^*] & E[r_4 r_3^*] & E[r_4 r_4^*] & E[r_4 r_5^*] \\ E[r_5 r_1^*] & E[r_5 r_2^*] & E[r_5 r_3^*] & E[r_5 r_4^*] & E[r_5 r_5^*] \end{bmatrix}, \end{aligned} \quad (3.159)$$

$$\begin{aligned}
\mathbf{\Sigma}_{\tilde{\mathbf{r}}, \tilde{\mathbf{r}}}^{(2,4)} &= E \left[\tilde{\mathbf{r}}^{(2,4)}(k, l) \tilde{\mathbf{r}}^{(2,4)H}(k, l) \right] \\
&= \begin{bmatrix} E[\tilde{r}_1 \tilde{r}_1^*] & E[\tilde{r}_1 \tilde{r}_2^*] & E[\tilde{r}_1 \tilde{r}_3^*] & E[\tilde{r}_1 \tilde{r}_4^*] & E[\tilde{r}_1 \tilde{r}_5^*] \\ E[\tilde{r}_2 \tilde{r}_1^*] & E[\tilde{r}_2 \tilde{r}_2^*] & E[\tilde{r}_2 \tilde{r}_3^*] & E[\tilde{r}_2 \tilde{r}_4^*] & E[\tilde{r}_2 \tilde{r}_5^*] \\ E[\tilde{r}_3 \tilde{r}_1^*] & E[\tilde{r}_3 \tilde{r}_2^*] & E[\tilde{r}_3 \tilde{r}_3^*] & E[\tilde{r}_3 \tilde{r}_4^*] & E[\tilde{r}_3 \tilde{r}_5^*] \\ E[\tilde{r}_4 \tilde{r}_1^*] & E[\tilde{r}_4 \tilde{r}_2^*] & E[\tilde{r}_4 \tilde{r}_3^*] & E[\tilde{r}_4 \tilde{r}_4^*] & E[\tilde{r}_4 \tilde{r}_5^*] \\ E[\tilde{r}_5 \tilde{r}_1^*] & E[\tilde{r}_5 \tilde{r}_2^*] & E[\tilde{r}_5 \tilde{r}_3^*] & E[\tilde{r}_5 \tilde{r}_4^*] & E[\tilde{r}_5 \tilde{r}_5^*] \end{bmatrix}, \quad (3.160)
\end{aligned}$$

where for the simplicity (k, l) and superscript representing stage order are omitted.

Now, for $j = 2$, we can calculate

$$\begin{aligned}
& \left(\mathbf{\Psi}_{1,2} \left[\mathbf{\Sigma}_{\mathbf{r}, \mathbf{r}}^{(2,4)} + \mathbf{\Sigma}_{\tilde{\mathbf{r}}, \tilde{\mathbf{r}}}^{(2,4)*} \right] \mathbf{\Psi}_{1,2}^T \right) \\
&= \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \left[\mathbf{\Sigma}_{\mathbf{r}, \mathbf{r}}^{(2,4)} + \mathbf{\Sigma}_{\tilde{\mathbf{r}}, \tilde{\mathbf{r}}}^{(2,4)*} \right] \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} E[r_3 r_3^*] + E[\tilde{r}_3 \tilde{r}_3^*] & E[r_3 r_4^*] + E[\tilde{r}_4 \tilde{r}_3^*] & E[r_3 r_5^*] + E[\tilde{r}_5 \tilde{r}_3^*] \\ E[r_4 r_3^*] + E[\tilde{r}_3 \tilde{r}_4^*] & E[r_4 r_4^*] + E[\tilde{r}_4 \tilde{r}_4^*] & E[r_4 r_5^*] + E[\tilde{r}_5 \tilde{r}_4^*] \\ E[r_5 r_3^*] + E[\tilde{r}_3 \tilde{r}_5^*] & E[r_5 r_4^*] + E[\tilde{r}_4 \tilde{r}_5^*] & E[r_5 r_5^*] + E[\tilde{r}_5 \tilde{r}_5^*] \end{bmatrix} \quad (3.161)
\end{aligned}$$

and

$$\begin{aligned}
\left(\mathbf{\Psi}_{1,2} \left[\mathbf{\Sigma}_{\mathbf{r}, \mathbf{r}}^{(2,4)} + \mathbf{\Sigma}_{\tilde{\mathbf{r}}, \tilde{\mathbf{r}}}^{(2,4)*} \right] \mathbf{\Pi}_2 \right) &= \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \left[\mathbf{\Sigma}_{\mathbf{r}, \mathbf{r}}^{(2,4)} + \mathbf{\Sigma}_{\tilde{\mathbf{r}}, \tilde{\mathbf{r}}}^{(2,4)*} \right] \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} E[r_3 r_2^*] + E[\tilde{r}_2 \tilde{r}_3^*] \\ E[r_4 r_2^*] + E[\tilde{r}_2 \tilde{r}_4^*] \\ E[r_5 r_2^*] + E[\tilde{r}_2 \tilde{r}_5^*] \end{bmatrix}, \quad (3.162)
\end{aligned}$$

By substituting (3.161) and (3.162) into (3.156), we can rewrite the lattice coefficient vector for $j = 2$ as

$$\begin{aligned}
\mathbf{k}_{r_{ASHP1,2}}^{(2,4)} &= \begin{bmatrix} E[r_3 r_3^*] + E[\tilde{r}_3 \tilde{r}_3^*] & E[r_3 r_4^*] + E[\tilde{r}_4 \tilde{r}_3^*] & E[r_3 r_5^*] + E[\tilde{r}_5 \tilde{r}_3^*] \\ E[r_4 r_3^*] + E[\tilde{r}_3 \tilde{r}_4^*] & E[r_4 r_4^*] + E[\tilde{r}_4 \tilde{r}_4^*] & E[r_4 r_5^*] + E[\tilde{r}_5 \tilde{r}_4^*] \\ E[r_5 r_3^*] + E[\tilde{r}_3 \tilde{r}_5^*] & E[r_5 r_4^*] + E[\tilde{r}_4 \tilde{r}_5^*] & E[r_5 r_5^*] + E[\tilde{r}_5 \tilde{r}_5^*] \end{bmatrix}^{-1} \\
&\quad \times \begin{bmatrix} E[r_3 r_2^*] + E[\tilde{r}_2 \tilde{r}_3^*] \\ E[r_4 r_2^*] + E[\tilde{r}_2 \tilde{r}_4^*] \\ E[r_5 r_2^*] + E[\tilde{r}_2 \tilde{r}_5^*] \end{bmatrix}. \quad (3.163)
\end{aligned}$$

It is possible to do same calculations for $j = 3$ and $j = 4$, and the following results can be found as

$$\begin{aligned}
\mathbf{k}_{r_{ASHP1,3}}^{(2,4)} &= \begin{bmatrix} E[r_4 r_4^*] + E[\tilde{r}_4 \tilde{r}_4^*] & E[r_4 r_5^*] + E[\tilde{r}_5 \tilde{r}_4^*] \\ E[r_5 r_4^*] + E[\tilde{r}_4 \tilde{r}_5^*] & E[r_5 r_5^*] + E[\tilde{r}_5 \tilde{r}_5^*] \end{bmatrix}^{-1} \\
&\quad \times \begin{bmatrix} E[r_4 r_3^*] + E[\tilde{r}_3 \tilde{r}_4^*] \\ E[r_5 r_3^*] + E[\tilde{r}_3 \tilde{r}_5^*] \end{bmatrix}, \quad (3.164)
\end{aligned}$$

and

$$\mathbf{k}_{r_{ASHP_{1,4}}}^{(2,4)} = \left[E[r_5 r_5^*] + E[\tilde{r}_5 \tilde{r}_5^*] \right]^{-1} \left[E[r_5 r_4^*] + E[\tilde{r}_4 \tilde{r}_5^*] \right], \quad (3.165)$$

respectively.

Similarly, it is possible to obtain the type-2 ASHP lattice models generated from the auxiliary horizontal forward/backward PEFs as illustrated in Figure 3.20, Figure 3.21, and Figure 3.22. The type-2 ASHP lattice model equations, for $j = 2, 3, 4$, can be written as

$$\begin{aligned} r_{ASHP_{2,j}}^{(2,4)}(k, l) &= r_j^{(2,4)}(k, l) - \mathbf{k}_{r_{ASHP_{2,j}}}^{(2,4)H} \boldsymbol{\Psi}_{ASHP_{2,j}} \mathbf{r}^{(2,4)}(k, l) \\ \tilde{r}_{ASHP_{2,j}}^{(2,4)}(k, l) &= \tilde{r}_j^{(2,4)}(k, l) - \mathbf{k}_{r_{ASHP_{2,j}}}^{(2,4)T} \boldsymbol{\Psi}_{ASHP_{2,j}} \tilde{\mathbf{r}}^{(2,4)}(k, l) \end{aligned} \quad (3.166)$$

where for $j = 2, 3, 4$,

$$\mathbf{k}_{r_{ASHP_{2,2}}}^{(2,4)} = \left[k_{r_{ASHP_{2,2}}}^{(2,4)}, \right], \quad (3.167)$$

$$\mathbf{k}_{r_{ASHP_{2,3}}}^{(2,4)} = \begin{bmatrix} k_{r_{ASHP_{2,3}}}^{(2,4)}(1) \\ k_{r_{ASHP_{2,3}}}^{(2,4)}(2) \end{bmatrix}, \quad (3.168)$$

and

$$\mathbf{k}_{r_{ASHP_{2,4}}}^{(2,4)} = \begin{bmatrix} k_{r_{ASHP_{2,4}}}^{(2,4)}(1) \\ k_{r_{ASHP_{2,4}}}^{(2,4)}(2) \\ k_{r_{ASHP_{2,4}}}^{(2,4)}(3) \end{bmatrix}, \quad (3.169)$$

respectively.

For the type-2 ASHP case, we need the determination of the $\boldsymbol{\Psi}_{j, \dots, N} = \boldsymbol{\Psi}_{ASHP_{2,j}}$, we can write it for $j = 2, 3, 4$ as

$$\begin{aligned} \boldsymbol{\Psi}_{2,3,4,5} &= \boldsymbol{\Psi}_{ASHP_{2,2}} \\ &= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \end{bmatrix}, \end{aligned} \quad (3.170)$$

$$\begin{aligned} \boldsymbol{\Psi}_{3,4,5} &= \boldsymbol{\Psi}_{ASHP_{2,3}} \\ &= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}, \end{aligned} \quad (3.171)$$

$$\begin{aligned} \boldsymbol{\Psi}_{4,5} &= \boldsymbol{\Psi}_{ASHP_{2,4}} \\ &= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}, \end{aligned} \quad (3.172)$$

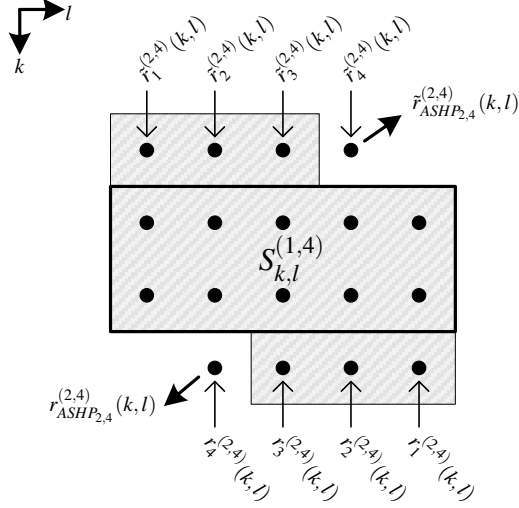


Figure 3.20: Type-2 ASHP lattice models of the PSR $S_{k,l}^{(1,4)}$: $r_{ASHP_{2,4}}^{(2,4)}(k,l)$ and $\tilde{r}_{ASHP_{2,4}}^{(2,4)}(k,l)$.

respectively.

By substituting (3.170), (3.171), and (3.172) into (3.166) for $j = 2, 3, 4$, respectively, the type-2 ASHP lattice model equations will be as follows

$$\begin{aligned} r_{ASHP_{2,2}}^{(2,4)}(k,l) &= r_2^{(2,4)}(k,l) - k_{r_{ASHP_{2,2}}}^{(2,4)*} r_1^{(2,4)}(k,l) \\ \tilde{r}_{ASHP_{2,2}}^{(2,4)}(k,l) &= \tilde{r}_2^{(2,4)}(k,l) - k_{r_{ASHP_{2,2}}}^{(2,4)} \tilde{r}_1^{(2,4)}(k,l), \end{aligned} \quad (3.173)$$

$$\begin{aligned} r_{ASHP_{2,3}}^{(2,4)}(k,l) &= r_3^{(2,4)}(k,l) - k_{r_{ASHP_{2,3}}}^{(2,4)*} (1) r_1^{(2,4)}(k,l) \\ &\quad - k_{r_{ASHP_{2,3}}}^{(2,4)*} (2) r_2^{(2,4)}(k,l) \\ \tilde{r}_{ASHP_{2,3}}^{(2,4)}(k,l) &= \tilde{r}_3^{(2,4)}(k,l) - k_{r_{ASHP_{2,3}}}^{(2,4)} (1) \tilde{r}_1^{(2,4)}(k,l) \\ &\quad - k_{r_{ASHP_{2,3}}}^{(2,4)} (2) \tilde{r}_2^{(2,4)}(k,l), \end{aligned} \quad (3.174)$$

$$\begin{aligned} r_{ASHP_{2,4}}^{(2,4)}(k,l) &= r_4^{(2,4)}(k,l) - k_{r_{ASHP_{2,4}}}^{(2,4)*} (1) r_1^{(2,4)}(k,l) \\ &\quad - k_{r_{ASHP_{2,4}}}^{(2,4)*} (2) r_2^{(2,4)}(k,l) - k_{r_{ASHP_{2,4}}}^{(2,4)*} (3) r_3^{(2,4)}(k,l) \\ \tilde{r}_{ASHP_{2,4}}^{(2,4)}(k,l) &= \tilde{r}_4^{(2,4)}(k,l) - k_{r_{ASHP_{2,4}}}^{(2,4)} (1) \tilde{r}_1^{(2,4)}(k,l) \\ &\quad - k_{r_{ASHP_{2,4}}}^{(2,4)} (2) \tilde{r}_2^{(2,4)}(k,l) - k_{r_{ASHP_{2,4}}}^{(2,4)} (3) \tilde{r}_3^{(2,4)}(k,l). \end{aligned} \quad (3.175)$$

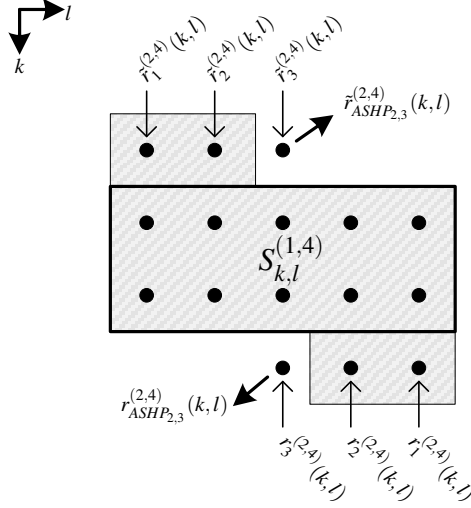


Figure 3.21: Type-2 ASHP lattice models of the PSR $S_{k,l}^{(1,4)}$: $r_{ASHP_{2,3}}^{(2,4)}(k,l)$ and $\tilde{r}_{ASHP_{2,3}}^{(2,4)}(k,l)$.

The power of the error propagation equations from the lattice model equations given in (3.152), (3.153), and (3.154) can be written as

$$P_{r_{ASHP_{2,j}}}^{(2,4)} = E \left[\left| r_{ASHP_{2,j}}^{(2,4)}(k,l) \right|^2 \right] + E \left[\left| \tilde{r}_{ASHP_{2,j}}^{(2,4)}(k,l) \right|^2 \right], \quad (3.176)$$

for $j = 2, 3, 4$.

As a result of the mean-square minimization of (3.176), the reflection coefficient vectors are computed for $j = 2, 3, 4$ as

$$\mathbf{k}_{r_{ASHP_{2,j}}}^{(2,4)} = \left(\Psi_{ASHP_{2,j}} \Phi_{\mathbf{r}, \tilde{\mathbf{r}}}^{(2,4)} \Psi_{ASHP_{2,j}}^T \right)^{-1} \Psi_{ASHP_{2,j}} \Phi_{\mathbf{r}, \tilde{\mathbf{r}}}^{(2,4)} \Pi_j, \quad (3.177)$$

and the minimum mean-square error equation, from stage to stage, can be written as

$$P_{r_{ASHP_{2,j}}}^{(2,4)} = P_{r_j}^{(2,4)} - \mathbf{k}_{r_{ASHP_{2,j}}}^{(2,4)H} \Psi_{ASHP_{2,j}} \Phi_{\mathbf{r}, \tilde{\mathbf{r}}}^{(2,4)} \Psi_{ASHP_{2,j}}^T \mathbf{k}_{r_{ASHP_{2,j}}}^{(2,4)}, \quad (3.178)$$

where

$$\Phi_{\mathbf{r}, \tilde{\mathbf{r}}}^{(2,4)} = \Sigma_{\mathbf{r}, \mathbf{r}}^{(2,4)} + \Sigma_{\tilde{\mathbf{r}}, \tilde{\mathbf{r}}}^{(2,4)*},$$

and

$$P_{r_j}^{(2,4)} = E \left[\left| r_j^{(2,4)}(k,l) \right|^2 \right] + E \left[\left| \tilde{r}_j^{(2,4)}(k,l) \right|^2 \right], \quad (3.179)$$

and

$$\Pi_2 = [0 \ 1 \ 0 \ 0 \ 0]^T,$$

$$\Pi_3 = [0 \ 0 \ 1 \ 0 \ 0]^T,$$

$$\Pi_4 = [0 \ 0 \ 0 \ 1 \ 0]^T.$$

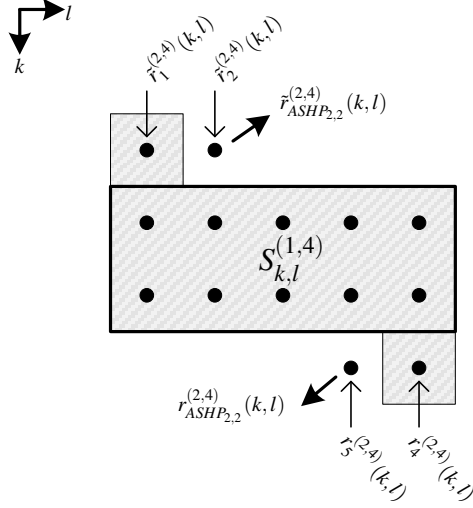


Figure 3.22: Type-2 ASHP lattice models of the PSR $S_{k,l}^{(1,4)}$: $r_{ASHP_{2,2}}^{(2,4)}(k,l)$ and $\tilde{r}_{ASHP_{2,2}}^{(2,4)}(k,l)$.

We know the calculation results of $\mathbf{\Sigma}_{\mathbf{r},\mathbf{r}}$ and $\mathbf{\Sigma}_{\tilde{\mathbf{r}},\tilde{\mathbf{r}}}^*$ from the type-1 ASHP lattice model case given in (3.159) and (3.160).

Now, for $j = 4$, let us calculate

$$\begin{aligned}
 & \left(\mathbf{\Psi}_{4,5} \left[\mathbf{\Sigma}_{\mathbf{r},\mathbf{r}}^{(2,4)} + \mathbf{\Sigma}_{\tilde{\mathbf{r}},\tilde{\mathbf{r}}}^{(2,4)*} \right] \mathbf{\Psi}_{4,5}^T \right) \\
 &= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} \left[\mathbf{\Sigma}_{\mathbf{r},\mathbf{r}}^{(2,4)} + \mathbf{\Sigma}_{\tilde{\mathbf{r}},\tilde{\mathbf{r}}}^{(2,4)*} \right] \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\
 &= \begin{bmatrix} E[r_1 r_1^*] + E[\tilde{r}_1 \tilde{r}_1^*] & E[r_1 r_2^*] + E[\tilde{r}_2 \tilde{r}_1^*] & E[r_1 r_3^*] + E[\tilde{r}_3 \tilde{r}_1^*] \\ E[r_2 r_1^*] + E[\tilde{r}_1 \tilde{r}_2^*] & E[r_2 r_2^*] + E[\tilde{r}_2 \tilde{r}_2^*] & E[r_2 r_3^*] + E[\tilde{r}_3 \tilde{r}_2^*] \\ E[r_3 r_1^*] + E[\tilde{r}_1 \tilde{r}_3^*] & E[r_3 r_2^*] + E[\tilde{r}_2 \tilde{r}_3^*] & E[r_3 r_3^*] + E[\tilde{r}_3 \tilde{r}_3^*] \end{bmatrix} \quad (3.180)
 \end{aligned}$$

and

$$\begin{aligned}
 \left(\mathbf{\Psi}_{4,5} \left[\mathbf{\Sigma}_{\mathbf{r},\mathbf{r}}^{(2,4)} + \mathbf{\Sigma}_{\tilde{\mathbf{r}},\tilde{\mathbf{r}}}^{(2,4)*} \right] \mathbf{\Pi}_4 \right) &= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} \left[\mathbf{\Sigma}_{\mathbf{r},\mathbf{r}}^{(2,4)} + \mathbf{\Sigma}_{\tilde{\mathbf{r}},\tilde{\mathbf{r}}}^{(2,4)*} \right] \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \\
 &= \begin{bmatrix} E[r_1 r_4^*] + E[\tilde{r}_4 \tilde{r}_1^*] \\ E[r_2 r_4^*] + E[\tilde{r}_4 \tilde{r}_2^*] \\ E[r_3 r_4^*] + E[\tilde{r}_4 \tilde{r}_3^*] \end{bmatrix}, \quad (3.181)
 \end{aligned}$$

By substituting (3.180) and (3.181) into (3.177) for $j = 4$, we can rewrite the lattice coefficient vector as

$$\mathbf{k}_{r_{ASHP_{2,4}}}^{(2,4)} = \begin{bmatrix} E[r_1 r_1^*] + E[\tilde{r}_1 \tilde{r}_1^*] & E[r_1 r_2^*] + E[\tilde{r}_2 \tilde{r}_1^*] & E[r_1 r_3^*] + E[\tilde{r}_3 \tilde{r}_1^*] \\ E[r_2 r_1^*] + E[\tilde{r}_1 \tilde{r}_2^*] & E[r_2 r_2^*] + E[\tilde{r}_2 \tilde{r}_2^*] & E[r_2 r_3^*] + E[\tilde{r}_3 \tilde{r}_2^*] \\ E[r_3 r_1^*] + E[\tilde{r}_1 \tilde{r}_3^*] & E[r_3 r_2^*] + E[\tilde{r}_2 \tilde{r}_3^*] & E[r_3 r_3^*] + E[\tilde{r}_3 \tilde{r}_3^*] \end{bmatrix}^{-1} \\ \times \begin{bmatrix} E[r_1 r_4^*] + E[\tilde{r}_4 \tilde{r}_1^*] \\ E[r_2 r_4^*] + E[\tilde{r}_4 \tilde{r}_2^*] \\ E[r_3 r_4^*] + E[\tilde{r}_4 \tilde{r}_3^*] \end{bmatrix}. \quad (3.182)$$

It can be done the same calculations for $j = 3$ and $j = 2$, and the following results can be found as

$$\mathbf{k}_{r_{ASHP_{2,3}}}^{(2,4)} = \begin{bmatrix} E[r_1 r_1^*] + E[\tilde{r}_1 \tilde{r}_1^*] & E[r_1 r_2^*] + E[\tilde{r}_2 \tilde{r}_1^*] \\ E[r_2 r_1^*] + E[\tilde{r}_1 \tilde{r}_2^*] & E[r_2 r_2^*] + E[\tilde{r}_2 \tilde{r}_2^*] \end{bmatrix}^{-1} \\ \times \begin{bmatrix} E[r_1 r_3^*] + E[\tilde{r}_3 \tilde{r}_1^*] \\ E[r_2 r_3^*] + E[\tilde{r}_3 \tilde{r}_2^*] \end{bmatrix}, \quad (3.183)$$

and

$$\mathbf{k}_{r_{ASHP_{2,2}}}^{(2,4)} = \begin{bmatrix} E[r_1 r_1^*] + E[\tilde{r}_1 \tilde{r}_1^*] \end{bmatrix}^{-1} \begin{bmatrix} E[r_1 r_2^*] + E[\tilde{r}_2 \tilde{r}_1^*] \end{bmatrix}, \quad (3.184)$$

respectively.

3.3.3.4 ASHP 2-D lattice models of the PSR $S_{k,l}^{(3,2)}$

In this section, all the possible type-1 and type-2 ASHP lattice models by utilizing auxiliary horizontal and vertical forward/backward PEFs of the PSR $S_{k,l}^{(3,2)}$ depicted in Figure 3.5 will be shown. In Section 3.2.2, this PSR has been already utilized for demonstrating how to obtain all possible QP lattice models. Now, all possible type-1 and type-2 ASHP lattice filter models generated from the auxiliary vertical forward/backward PEFs, and type-1 and type-2 ASHP lattice filter models produced from the auxiliary horizontal forward/backward PEFs will be explained in detail, respectively.

The generation of the whole type-1 ASHP lattice models by employing the corresponding auxiliary vertical forward/backward PEFs is illustrated in Figure 3.23, and 3.24. It is clearly seen that it is possible to obtain two pairs of type-1 and two pairs of type-2 ASHP lattice models from the vertical forward/backward PEFs. In this example, M is 4, and j is defined as $j = 2, 3$.

As shown in Figure 3.23, to obtain $e_{ASHP_{1,2}}^{(3,3)}(k, l)$, the vertical PEF $e_2^{(3,3)}(k, l)$ is linearly predicted from $e_3^{(3,3)}(k, l)$ and $e_4^{(3,3)}(k, l)$, and $e_{ASHP_{1,3}}^{(3,3)}(k, l)$ is generated by linear

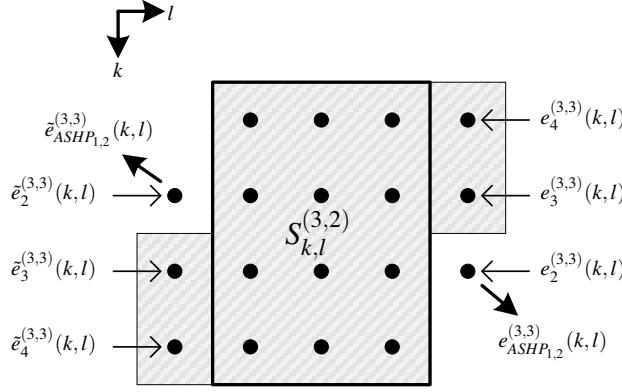


Figure 3.23: Type-1 ASHP lattice models of the PSR $S_{k,l}^{(3,2)}$: $e_{ASHP_{1,2}}^{(3,3)}(k,l)$ and $\tilde{e}_{ASHP_{1,2}}^{(3,3)}(k,l)$.

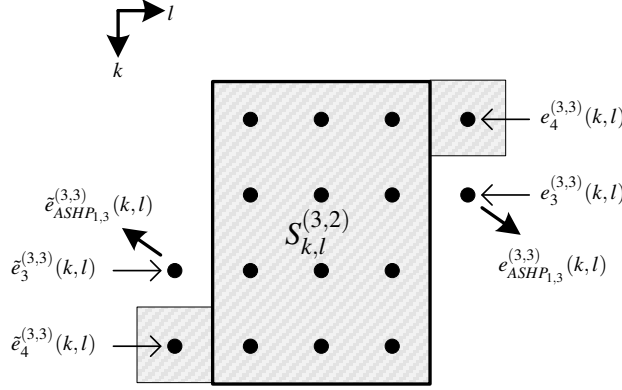


Figure 3.24: Type-1 ASHP lattice models of the PSR $S_{k,l}^{(3,2)}$: $e_{ASHP_{1,3}}^{(3,3)}(k,l)$ and $\tilde{e}_{ASHP_{1,3}}^{(3,3)}(k,l)$.

prediction of $e_3^{(3,3)}(k,l)$ from $e_4^{(3,3)}(k,l)$. The backward cases can be determined at the same way. The type-1 ASHP lattice models equations of these constructed type-1 ASHP lattice models of the PSR $S_{k,l}^{(3,2)}$ can be written for $j = 2, 3$ as

$$\begin{aligned} e_{ASHP_{1,2}}^{(3,3)}(k,l) &= e_2^{(3,3)}(k,l) - \mathbf{k}_{e_{ASHP_{1,2}}}^{(3,3)H} \Psi_{1,2} \mathbf{e}^{(3,3)}(k,l) \\ \tilde{e}_{ASHP_{1,2}}^{(3,3)}(k,l) &= \tilde{e}_2^{(3,3)}(k,l) - \mathbf{k}_{e_{ASHP_{1,2}}}^{(3,3)T} \Psi_{1,2} \tilde{\mathbf{e}}^{(3,3)}(k,l), \end{aligned} \quad (3.185)$$

and

$$\begin{aligned} e_{ASHP_{1,3}}^{(3,3)}(k,l) &= e_3^{(3,3)}(k,l) - \mathbf{k}_{e_{ASHP_{1,3}}}^{(3,3)H} \Psi_{1,2,3} \mathbf{e}^{(3,3)}(k,l) \\ \tilde{e}_{ASHP_{1,3}}^{(3,3)}(k,l) &= \tilde{e}_3^{(3,3)}(k,l) - \mathbf{k}_{e_{ASHP_{1,3}}}^{(3,3)T} \Psi_{1,2,3} \tilde{\mathbf{e}}^{(3,3)}(k,l), \end{aligned} \quad (3.186)$$

where

$$\mathbf{k}_{e_{ASHP_{1,2}}}^{(3,3)} = \begin{bmatrix} k_{e_{ASHP_{1,2}}}^{(3,3)}(1) \\ k_{e_{ASHP_{1,2}}}^{(3,3)}(2) \end{bmatrix}, \quad (3.187)$$

and

$$\mathbf{k}_{e_{ASHP_{1,3}}}^{(3,3)} = \begin{bmatrix} k_{e_{ASHP_{1,3}}}^{(3,3)} \end{bmatrix}. \quad (3.188)$$

From the definition of the $\Psi_{1,\dots,j}$, we can get for $j = 2, 3$,

$$\begin{aligned} \Psi_{1,2} &= \Psi_{ASHP_{1,2}} \\ &= \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \end{aligned} \quad (3.189)$$

$$\begin{aligned} \Psi_{1,2,3} &= \Psi_{ASHP_{1,3}} \\ &= \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}. \end{aligned} \quad (3.190)$$

By substituting (3.189), and (3.190) into (3.185), and (3.186), respectively, the type-1 ASHP model equations will be as follows

$$\begin{aligned} e_{ASHP_{1,2}}^{(3,3)}(k, l) &= e_2^{(3,3)}(k, l) - k_{e_{ASHP_{1,2}}}^{(3,3)*}(1)e_3^{(3,3)}(k, l) \\ &\quad - k_{e_{ASHP_{1,2}}}^{(3,3)*}(2)e_4^{(3,3)}(k, l) \\ \tilde{e}_{ASHP_{1,2}}^{(3,3)}(k, l) &= \tilde{e}_2^{(3,3)}(k, l) - k_{e_{ASHP_{1,2}}}^{(3,3)}(1)\tilde{e}_3^{(3,3)}(k, l) \\ &\quad - k_{e_{ASHP_{1,2}}}^{(3,3)}(2)\tilde{e}_4^{(3,3)}(k, l) \end{aligned} \quad (3.191)$$

$$\begin{aligned} e_{ASHP_{1,3}}^{(3,3)}(k, l) &= e_3^{(3,3)}(k, l) - k_{e_{ASHP_{1,3}}}^{(3,3)*}e_4^{(3,3)}(k, l) \\ \tilde{e}_{ASHP_{1,3}}^{(3,3)}(k, l) &= \tilde{e}_3^{(3,3)}(k, l) - k_{e_{ASHP_{1,3}}}^{(3,3)}(1)\tilde{e}_4^{(3,3)}(k, l) \end{aligned} \quad (3.192)$$

The power of the error propagation equations from the lattice model equations given in (3.191), and (3.192) can be written as

$$P_{e_{ASHP_{1,j}}}^{(3,3)} = E \left[\left| e_{ASHP_{1,j}}^{(3,3)}(k, l) \right|^2 \right] + E \left[\left| \tilde{e}_{ASHP_{1,j}}^{(3,3)}(k, l) \right|^2 \right], \quad (3.193)$$

for $j = 2, 3$.

As a result of the mean-square minimization of (3.193), the reflection coefficient vectors are calculated for $j = 2, 3$ as

$$\mathbf{k}_{e_{ASHP_{1,2}}}^{(3,3)} = \left(\Psi_{1,2} \Phi_{\mathbf{e}, \tilde{\mathbf{e}}}^{(3,3)} \Psi_{1,2}^T \right)^{-1} \Psi_{1,2} \Phi_{\mathbf{e}, \tilde{\mathbf{e}}}^{(3,3)} \Pi_2 \quad (3.194)$$

and

$$\mathbf{k}_{eASHP_{1,3}}^{(2,4)} = \left(\Psi_{1,2,3} \Phi_{\mathbf{e},\tilde{\mathbf{e}}}^{(3,3)} \Psi_{1,2,3}^T \right)^{-1} \Psi_{1,2,3} \Phi_{\mathbf{e},\tilde{\mathbf{e}}}^{(3,3)} \Pi_3, \quad (3.195)$$

where

$$\Phi_{\mathbf{e},\tilde{\mathbf{e}}}^{(3,3)} = \Sigma_{\mathbf{e},\mathbf{e}}^{(3,3)} + \Sigma_{\tilde{\mathbf{e}},\tilde{\mathbf{e}}}^{(3,3)*},$$

and

$$\begin{aligned} \Pi_2 &= \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix}^T, \\ \Pi_3 &= \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix}^T. \end{aligned}$$

In order to find the results of the lattice coefficient vectors for each type-1 ASHP lattice model, first we compute $\Sigma_{\mathbf{e},\mathbf{e}}$ and $\Sigma_{\tilde{\mathbf{e}},\tilde{\mathbf{e}}}^*$ as

$$\begin{aligned} \Sigma_{\mathbf{e},\mathbf{e}}^{(3,3)} &= E \left[\mathbf{e}^{(3,3)}(k,l) \mathbf{e}^{(3,3)H}(k,l) \right] \\ &= \begin{bmatrix} E[e_1 e_1^*] & E[e_1 e_2^*] & E[e_1 e_3^*] & E[e_1 e_4^*] \\ E[e_2 e_1^*] & E[e_2 e_2^*] & E[e_2 e_3^*] & E[e_2 e_4^*] \\ E[e_3 e_1^*] & E[e_3 e_2^*] & E[e_3 e_3^*] & E[e_3 e_4^*] \\ E[e_4 e_1^*] & E[e_4 e_2^*] & E[e_4 e_3^*] & E[e_4 e_4^*] \end{bmatrix}, \end{aligned} \quad (3.196)$$

$$\begin{aligned} \Sigma_{\tilde{\mathbf{e}},\tilde{\mathbf{e}}}^{(3,3)} &= E \left[\tilde{\mathbf{e}}^{(3,3)}(k,l) \tilde{\mathbf{e}}^{(3,3)H}(k,l) \right] \\ &= \begin{bmatrix} E[\tilde{e}_1 \tilde{e}_1^*] & E[\tilde{e}_1 \tilde{e}_2^*] & E[\tilde{e}_1 \tilde{e}_3^*] & E[\tilde{e}_1 \tilde{e}_4^*] \\ E[\tilde{e}_2 \tilde{e}_1^*] & E[\tilde{e}_2 \tilde{e}_2^*] & E[\tilde{e}_2 \tilde{e}_3^*] & E[\tilde{e}_2 \tilde{e}_4^*] \\ E[\tilde{e}_3 \tilde{e}_1^*] & E[\tilde{e}_3 \tilde{e}_2^*] & E[\tilde{e}_3 \tilde{e}_3^*] & E[\tilde{e}_3 \tilde{e}_4^*] \\ E[\tilde{e}_4 \tilde{e}_1^*] & E[\tilde{e}_4 \tilde{e}_2^*] & E[\tilde{e}_4 \tilde{e}_3^*] & E[\tilde{e}_4 \tilde{e}_4^*] \end{bmatrix}, \end{aligned} \quad (3.197)$$

where for the simplicity (k,l) and superscript representing stage order are omitted.

Now, for $j = 2$, let us calculate

$$\begin{aligned} & \left(\Psi_{1,2} \left[\Sigma_{\mathbf{e},\mathbf{e}}^{(3,3)} + \Sigma_{\tilde{\mathbf{e}},\tilde{\mathbf{e}}}^{(3,3)*} \right] \Psi_{1,2}^T \right) \\ &= \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \left[\Sigma_{\mathbf{e},\mathbf{e}}^{(3,3)} + \Sigma_{\tilde{\mathbf{e}},\tilde{\mathbf{e}}}^{(3,3)*} \right] \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} E[e_3 e_3^*] + E[\tilde{e}_3 \tilde{e}_3^*] & E[e_3 e_4^*] + E[\tilde{e}_4 \tilde{e}_3^*] \\ E[e_4 e_3^*] + E[\tilde{e}_3 \tilde{e}_4^*] & E[e_4 e_4^*] + E[\tilde{e}_4 \tilde{e}_4^*] \end{bmatrix}, \end{aligned} \quad (3.198)$$

and

$$\begin{aligned} \left(\Psi_{1,2} \left[\Sigma_{\mathbf{e},\mathbf{e}}^{(3,3)} + \Sigma_{\tilde{\mathbf{e}},\tilde{\mathbf{e}}}^{(3,3)*} \right] \Pi_2 \right) &= \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \left[\Sigma_{\mathbf{e},\mathbf{e}}^{(3,3)} + \Sigma_{\tilde{\mathbf{e}},\tilde{\mathbf{e}}}^{(3,3)*} \right] \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} E[e_3 e_2^*] + E[\tilde{e}_2 \tilde{e}_3^*] \\ E[e_4 e_2^*] + E[\tilde{e}_2 \tilde{e}_4^*] \end{bmatrix}. \end{aligned} \quad (3.199)$$

By substituting (3.198) and (3.199) into (3.194), we can rewrite the lattice coefficient vector for $j = 2$ as

$$\mathbf{k}_{e_{ASHP_{1,2}}}^{(3,3)} = \begin{bmatrix} E[e_3 e_3^*] + E[\tilde{e}_3 \tilde{e}_3^*] & E[e_3 e_4^*] + E[\tilde{e}_4 \tilde{e}_3^*] \\ E[e_4 e_3^*] + E[\tilde{e}_3 \tilde{e}_4^*] & E[e_4 e_4^*] + E[\tilde{e}_4 \tilde{e}_4^*] \end{bmatrix}^{-1} \times \begin{bmatrix} E[e_3 e_2^*] + E[\tilde{e}_2 \tilde{e}_3^*] \\ E[e_4 e_2^*] + E[\tilde{e}_2 \tilde{e}_4^*] \end{bmatrix}. \quad (3.200)$$

It is possible to do same calculations for $j = 3$, and the following result can be found as

$$\mathbf{k}_{e_{ASHP_{1,3}}}^{(3,3)} = \begin{bmatrix} E[e_4 e_4^*] + E[\tilde{e}_4 \tilde{e}_4^*] \end{bmatrix}^{-1} \times \begin{bmatrix} E[e_4 e_3^*] + E[\tilde{e}_3 \tilde{e}_4^*] \end{bmatrix}. \quad (3.201)$$

Similarly, the type-2 ASHP lattice models are also generated from the auxiliary vertical forward/backward PEFs as illustrated in Figure 3.25, and Figure 3.26. The type-2 ASHP lattice model, $\tilde{e}_{ASHP_{2,2}}^{(3,3)}(k, l)$ for $j = 2$, is generated by predicted $\tilde{e}_2^{(3,3)}(k, l)$ from $\tilde{e}_1^{(3,3)}(k, l)$ and type-2 ASHP model, $\tilde{e}_{ASHP_{2,3}}^{(3,3)}(k, l)$ is produced by linear prediction of $e_3^{(3,3)}(k, l)$ from $e_1^{(3,3)}(k, l)$ and $e_2^{(3,3)}(k, l)$. The type-2 ASHP lattice models equations for these ASHP lattice models of the PSR $S_{k,l}^{(3,2)}$ can be written for $j = 2, 3$ as

$$\begin{aligned} e_{ASHP_{2,j}}^{(3,3)}(k, l) &= e_j^{(3,3)}(k, l) - \mathbf{k}_{e_{ASHP_{2,j}}}^{(3,3)H} \boldsymbol{\Psi}_{ASHP_{2,j}} \mathbf{e}^{(3,3)}(k, l) \\ \tilde{e}_{ASHP_{2,j}}^{(3,3)}(k, l) &= \tilde{e}_j^{(3,3)}(k, l) - \mathbf{k}_{e_{ASHP_{2,j}}}^{(3,3)T} \boldsymbol{\Psi}_{ASHP_{2,j}} \tilde{\mathbf{e}}^{(3,3)}(k, l) \end{aligned} \quad (3.202)$$

where for $j = 2$,

$$\mathbf{k}_{e_{ASHP_{2,2}}}^{(3,3)} = \begin{bmatrix} k_{e_{ASHP_{2,2}}}^{(3,3)} \end{bmatrix}, \quad (3.203)$$

for $j = 3$,

$$\mathbf{k}_{e_{ASHP_{2,3}}}^{(3,3)} = \begin{bmatrix} k_{e_{ASHP_{2,3}}}^{(3,3)}(1) \\ k_{e_{ASHP_{2,3}}}^{(3,3)}(2) \end{bmatrix}. \quad (3.204)$$

For the type-2 ASHP case, we need the determination of the $\boldsymbol{\Psi}_{j,\dots,M=4} = \boldsymbol{\Psi}_{ASHP_{2,j}}$, we can write it for $j = 2, 3$ as

$$\begin{aligned} \boldsymbol{\Psi}_{2,3,4} &= \boldsymbol{\Psi}_{ASHP_{2,2}} \\ &= \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}, \end{aligned} \quad (3.205)$$

$$\begin{aligned} \boldsymbol{\Psi}_{3,4} &= \boldsymbol{\Psi}_{ASHP_{2,3}} \\ &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}. \end{aligned} \quad (3.206)$$

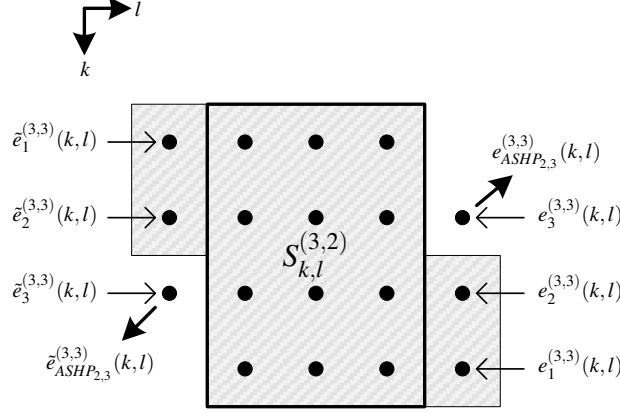


Figure 3.25: Type-2 ASHP lattice models of the PSR $S_{k,l}^{(3,2)}$: $e_{ASHP_{2,3}}^{(3,3)}(k,l)$ and $\tilde{e}_{ASHP_{2,3}}^{(3,3)}(k,l)$.

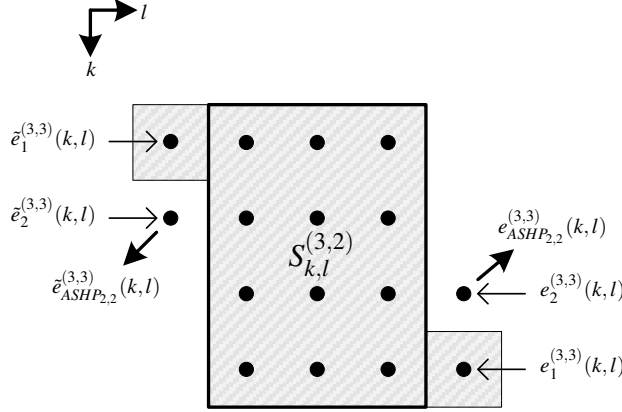


Figure 3.26: Type-2 ASHP lattice models of the PSR $S_{k,l}^{(3,2)}$: $e_{ASHP_{2,2}}^{(3,3)}(k,l)$ and $\tilde{e}_{ASHP_{2,2}}^{(3,3)}(k,l)$.

By substituting (3.205) and (3.206) into (3.202) for $j = 2, 3$, the type-2 ASHP lattice model equations will be as follows

$$\begin{aligned} e_{ASHP_{2,2}}^{(3,3)}(k,l) &= e_2^{(3,3)}(k,l) - k_{e_{ASHP_{2,2}}}^{(3,3)*} e_1^{(3,3)}(k,l) \\ \tilde{e}_{ASHP_{2,2}}^{(3,3)}(k,l) &= \tilde{e}_2^{(3,3)}(k,l) - k_{e_{ASHP_{2,2}}}^{(3,3)} \tilde{e}_1^{(3,3)}(k,l) \end{aligned} \quad (3.207)$$

$$\begin{aligned} e_{ASHP_{2,3}}^{(3,3)}(k,l) &= e_3^{(3,3)}(k,l) - k_{e_{ASHP_{2,3}}}^{(3,3)*} (1)r_1^{(3,3)}(k,l) \\ &\quad - k_{e_{ASHP_{2,3}}}^{(3,3)*} (2)r_2^{(3,3)}(k,l) \\ \tilde{e}_{ASHP_{2,3}}^{(3,3)}(k,l) &= \tilde{e}_3^{(3,3)}(k,l) - k_{e_{ASHP_{2,3}}}^{(3,3)} (1)\tilde{e}_1^{(3,3)}(k,l) \\ &\quad - k_{e_{ASHP_{2,3}}}^{(3,3)} (2)\tilde{e}_2^{(3,3)}(k,l). \end{aligned} \quad (3.208)$$

The power of the error propagation equations from the lattice model equations given in (3.207), and (3.208) can be written as

$$P_{e_{ASHP_{2,j}}}^{(3,3)} = E \left[\left| e_{ASHP_{2,j}}^{(3,3)}(k, l) \right|^2 \right] + E \left[\left| \tilde{e}_{ASHP_{2,j}}^{(3,3)}(k, l) \right|^2 \right], \quad (3.209)$$

for $j = 2, 3$.

As a result of the mean-square minimization of (3.209), the reflection coefficient vectors are computed for $j = 2, 3$ as

$$\mathbf{k}_{e_{ASHP_{2,j}}}^{(3,3)} = \left(\Psi_{ASHP_{2,j}} \Phi_{\mathbf{e}, \tilde{\mathbf{e}}}^{(3,3)} \Psi_{ASHP_{2,j}}^T \right)^{-1} \Psi_{ASHP_{2,j}} \Phi_{\mathbf{e}, \tilde{\mathbf{e}}}^{(3,3)} \Pi_j, \quad (3.210)$$

and the minimum mean-square error equation, from stage to stage, can be written as

$$P_{e_{ASHP_{2,j}}}^{(3,3)} = P_{e_j}^{(3,3)} - \mathbf{k}_{e_{ASHP_{2,j}}}^{(3,3)H} \Psi_{ASHP_{2,j}} \Phi_{\mathbf{e}, \tilde{\mathbf{e}}}^{(3,3)} \Psi_{ASHP_{2,j}}^T \mathbf{k}_{e_{ASHP_{2,j}}}^{(3,3)}, \quad (3.211)$$

where

$$\begin{aligned} \Phi_{\mathbf{e}, \tilde{\mathbf{e}}}^{(3,3)} &= \Sigma_{\mathbf{e}, \mathbf{e}}^{(3,3)} + \Sigma_{\tilde{\mathbf{e}}, \tilde{\mathbf{e}}}^{(3,3)*}, \\ P_{e_j}^{(3,3)} &= E \left[\left| e_j^{(3,3)}(k, l) \right|^2 \right] + E \left[\left| \tilde{e}_j^{(3,3)}(k, l) \right|^2 \right], \end{aligned} \quad (3.212)$$

and

$$\begin{aligned} \Pi_2 &= \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix}^T, \\ \Pi_3 &= \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix}^T. \end{aligned}$$

The calculation results of $\Sigma_{\mathbf{e}, \mathbf{e}}$ and $\Sigma_{\tilde{\mathbf{e}}, \tilde{\mathbf{e}}}$ are already given above in (3.196) and (3.197). Now, for $j = 3$, let us calculate

$$\begin{aligned} & \left(\Psi_{3,4} \left[\Sigma_{\mathbf{e}, \mathbf{e}}^{(3,3)} + \Sigma_{\tilde{\mathbf{e}}, \tilde{\mathbf{e}}}^{(3,3)*} \right] \Psi_{3,4}^T \right) \\ &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \left[\Sigma_{\mathbf{e}, \mathbf{e}}^{(3,3)} + \Sigma_{\tilde{\mathbf{e}}, \tilde{\mathbf{e}}}^{(3,3)*} \right] \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} E[e_1 e_1^*] + E[\tilde{e}_1 \tilde{e}_1^*] & E[e_1 e_2^*] + E[\tilde{e}_2 \tilde{e}_1^*] \\ E[e_2 e_1^*] + E[\tilde{e}_1 \tilde{e}_2^*] & E[e_2 e_2^*] + E[\tilde{e}_2 \tilde{e}_2^*] \end{bmatrix}. \end{aligned} \quad (3.213)$$

and

$$\begin{aligned} \left(\Psi_{3,4} \left[\Sigma_{\mathbf{e}, \mathbf{e}}^{(3,3)} + \Sigma_{\tilde{\mathbf{e}}, \tilde{\mathbf{e}}}^{(3,3)*} \right] \Pi_3 \right) &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \left[\Sigma_{\mathbf{e}, \mathbf{e}}^{(3,3)} + \Sigma_{\tilde{\mathbf{e}}, \tilde{\mathbf{e}}}^{(3,3)*} \right] \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} E[e_1 e_3^*] + E[\tilde{e}_3 \tilde{e}_1^*] \\ E[e_2 e_3^*] + E[\tilde{e}_3 \tilde{e}_2^*] \end{bmatrix}. \end{aligned} \quad (3.214)$$

By substituting (3.213) and (3.214) into (3.210) for $j = 3$, we can rewrite the lattice coefficient vector as

$$\mathbf{k}_{e_{ASHP_{2,3}}}^{(3,3)} = \begin{bmatrix} E[e_1 e_1^*] + E[\tilde{e}_1 \tilde{e}_1^*] & E[e_1 e_2^*] + E[\tilde{e}_2 \tilde{e}_1^*] \\ E[e_2 e_1^*] + E[\tilde{e}_1 \tilde{e}_2^*] & E[e_2 e_2^*] + E[\tilde{e}_2 \tilde{e}_2^*] \end{bmatrix}^{-1} \times \begin{bmatrix} E[e_1 e_3^*] + E[\tilde{e}_3 \tilde{e}_1^*] \\ E[e_2 e_3^*] + E[\tilde{e}_3 \tilde{e}_2^*] \end{bmatrix}. \quad (3.215)$$

If we do the same calculations for $j = 2$, the following results can be found as

$$\mathbf{k}_{e_{ASHP_{2,2}}}^{(3,3)} = [E[e_1 e_1^*] + E[\tilde{e}_1 \tilde{e}_1^*]]^{-1} [E[e_1 e_2^*] + E[\tilde{e}_2 \tilde{e}_1^*]]. \quad (3.216)$$

At this point we have populated all possible type-1 and type-2 ASHP lattice models of the PSR $S_{k,l}^{(3,2)}$ by employing the auxiliary vertical forward/backward PEFs. Since the number of the pair of horizontal PEFs is 3, it is still possible to define one pair type-1 and one pair type-2 ASHP lattice models obtained by linearly predicted from the auxiliary horizontal forward/backward PEFs. The type-1 and type-2 ASHP lattice models are represented in Figure 3.27 and Figure 3.28, respectively.

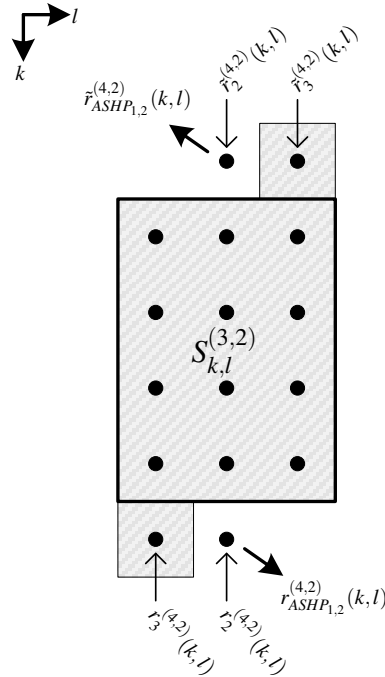


Figure 3.27: Type-1 ASHP lattice models of the PSR $S_{k,l}^{(3,2)}$: $r_{ASHP_{1,2}}^{(4,2)}(k,l)$ and $\tilde{r}_{ASHP_{1,2}}^{(4,2)}(k,l)$.

As seen in Figure 3.27, type-1 ASHP lattice model $r_{ASHP_{1,2}}^{(4,2)}(k,l)$ is obtained by predicting $r_2^{(4,2)}(k,l)$ from $r_3^{(4,2)}(k,l)$ and type-2 lattice model $r_{ASHP_{2,2}}^{(4,2)}(k,l)$ is generated

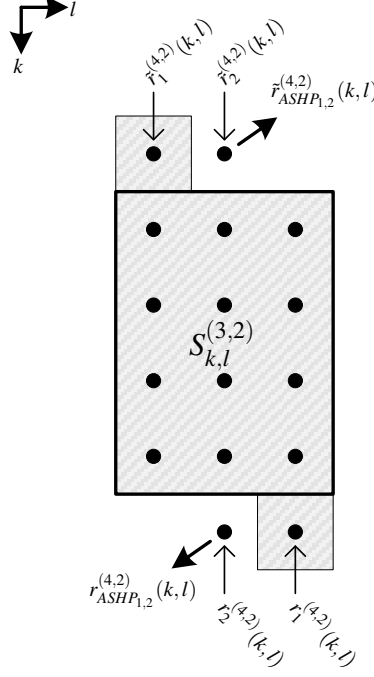


Figure 3.28: Type-2 ASHP lattice models of the PSR $S_{k,l}^{(3,2)}$: $r_{ASHP_{2,2}}^{(4,2)}(k,l)$ and $\tilde{r}_{ASHP_{2,2}}^{(4,2)}(k,l)$.

by linear prediction of $r_2^{(4,2)}(k,l)$ from $r_1^{(4,2)}(k,l)$. The type-1 ASHP lattice model equations of these constructed type-1 ASHP lattice model of the PSR $S_{k,l}^{(3,2)}$ can be written in an open form as

$$\begin{aligned} r_{ASHP_{1,2}}^{(4,2)}(k,l) &= r_2^{(4,2)}(k,l) - k_{r_{ASHP_{1,2}}}^{(4,2)*} r_3^{(4,2)}(k,l) \\ \tilde{r}_{ASHP_{1,2}}^{(4,2)}(k,l) &= \tilde{r}_2^{(4,2)}(k,l) - k_{r_{ASHP_{1,2}}}^{(4,2)} \tilde{r}_3^{(4,2)}(k,l) \end{aligned} \quad (3.217)$$

As a result of the mean-square minimization of $P_{r_{ASHP_{1,2}}}^{(4,2)} = E \left[\left| r_{ASHP_{1,2}}^{(4,2)}(k,l) \right|^2 \right] + E \left[\left| \tilde{r}_{ASHP_{1,2}}^{(4,2)}(k,l) \right|^2 \right]$, the reflection coefficient vector is calculated as

$$\mathbf{k}_{r_{ASHP_{1,2}}}^{(4,2)} = \left(\Psi_{1,2} \Phi_{\mathbf{r}, \tilde{\mathbf{r}}}^{(4,2)} \Psi_{1,2}^T \right)^{-1} \Psi_{1,2} \Phi_{\mathbf{r}, \tilde{\mathbf{r}}}^{(4,2)} \Pi_2 \quad (3.218)$$

where

$$\Phi_{\mathbf{r}, \tilde{\mathbf{r}}}^{(4,2)} = \Sigma_{\mathbf{r}, \mathbf{r}}^{(4,2)} + \Sigma_{\tilde{\mathbf{r}}, \tilde{\mathbf{r}}}^{(4,2)*},$$

$$\Psi_{1,2} = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix},$$

and

$$\Pi_2 = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}^T.$$

The computation of $\mathbf{\Sigma}_{\mathbf{r},\mathbf{r}}$ and $\mathbf{\Sigma}_{\tilde{\mathbf{r}},\tilde{\mathbf{r}}}^*$ are defined as

$$\begin{aligned}\mathbf{\Sigma}_{\mathbf{r},\mathbf{r}}^{(4,2)} &= E \left[\mathbf{r}^{(4,2)}(k,l) \mathbf{r}^{(4,2)H}(k,l) \right] \\ &= \begin{bmatrix} E[r_1 r_1^*] & E[r_1 r_2^*] & E[r_1 r_3^*] \\ E[r_2 r_1^*] & E[r_2 r_2^*] & E[r_2 r_3^*] \\ E[r_3 r_1^*] & E[r_3 r_2^*] & E[r_3 r_3^*] \end{bmatrix},\end{aligned}\quad (3.219)$$

$$\begin{aligned}\mathbf{\Sigma}_{\tilde{\mathbf{r}},\tilde{\mathbf{r}}}^{(4,2)} &= E \left[\tilde{\mathbf{r}}^{(4,2)}(k,l) \tilde{\mathbf{r}}^{(4,2)H}(k,l) \right] \\ &= \begin{bmatrix} E[\tilde{r}_1 \tilde{r}_1^*] & E[\tilde{r}_1 \tilde{r}_2^*] & E[\tilde{r}_1 \tilde{r}_3^*] \\ E[\tilde{r}_2 \tilde{r}_1^*] & E[\tilde{r}_2 \tilde{r}_2^*] & E[\tilde{r}_2 \tilde{r}_3^*] \\ E[\tilde{r}_3 \tilde{r}_1^*] & E[\tilde{r}_3 \tilde{r}_2^*] & E[\tilde{r}_3 \tilde{r}_3^*] \end{bmatrix},\end{aligned}\quad (3.220)$$

where for the simplicity (k,l) and superscript representing stage order are omitted.

Now, we can write that

$$\begin{aligned}\left(\mathbf{\Psi}_{1,2} \left[\mathbf{\Sigma}_{\mathbf{r},\mathbf{r}}^{(4,2)} + \mathbf{\Sigma}_{\tilde{\mathbf{r}},\tilde{\mathbf{r}}}^{(4,2)*} \right] \mathbf{\Psi}_{1,2}^T \right) &= \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \left[\mathbf{\Sigma}_{\mathbf{r},\mathbf{r}}^{(4,2)} + \mathbf{\Sigma}_{\tilde{\mathbf{r}},\tilde{\mathbf{r}}}^{(4,2)*} \right] \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} E[r_3 r_3^*] + E[\tilde{r}_3 \tilde{r}_3^*] \end{bmatrix},\end{aligned}\quad (3.221)$$

and

$$\begin{aligned}\left(\mathbf{\Psi}_{1,2} \left[\mathbf{\Sigma}_{\mathbf{r},\mathbf{r}}^{(4,2)} + \mathbf{\Sigma}_{\tilde{\mathbf{r}},\tilde{\mathbf{r}}}^{(4,2)*} \right] \mathbf{\Pi}_2 \right) &= \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \left[\mathbf{\Sigma}_{\mathbf{r},\mathbf{r}}^{(4,2)} + \mathbf{\Sigma}_{\tilde{\mathbf{r}},\tilde{\mathbf{r}}}^{(4,2)*} \right] \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} E[r_3 r_2^*] + E[\tilde{r}_3 \tilde{r}_3^*] \end{bmatrix}.\end{aligned}\quad (3.222)$$

By substituting (3.221) and (3.222) into (3.218), we can rewrite the lattice coefficient vector as

$$\mathbf{k}_{r_{ASHP_{1,2}}}^{(4,2)} = \begin{bmatrix} E[r_3 r_3^*] + E[\tilde{r}_3 \tilde{r}_3^*] \end{bmatrix}^{-1} \begin{bmatrix} E[r_3 r_2^*] + E[\tilde{r}_2 \tilde{r}_3^*] \end{bmatrix}. \quad (3.223)$$

Similarly, that what is calculated for type-1 ASHP lattice model can be done for the type-2 ASHP lattice model as well. The Figure 3.28 shows the construction of the type-2 ASHP lattice model from the auxiliary horizontal forward/backward PEFs. The type-2 ASHP lattice model equations for these ASHP lattice model of the PSR $S_{k,l}^{(3,2)}$ can be written in an open form as

$$\begin{aligned}r_{ASHP_{2,2}}^{(4,2)}(k,l) &= r_2^{(4,2)}(k,l) - k_{r_{ASHP_{2,2}}}^{(4,2)*} r_1^{(4,2)}(k,l) \\ \tilde{r}_{ASHP_{2,2}}^{(4,2)}(k,l) &= \tilde{r}_2^{(4,2)}(k,l) - k_{r_{ASHP_{2,2}}}^{(4,2)} \tilde{r}_1^{(4,2)}(k,l)\end{aligned}\quad (3.224)$$

As a result of the mean-square minimization of $P_{r_{ASHP_{2,2}}}^{(4,2)} = E \left[\left| r_{ASHP_{2,2}}^{(4,2)}(k, l) \right|^2 \right] + E \left[\left| \tilde{r}_{ASHP_{2,2}}^{(4,2)}(k, l) \right|^2 \right]$, the reflection coefficient vector is computed as

$$\mathbf{k}_{r_{ASHP_{2,2}}}^{(4,2)} = \left(\Psi_{ASHP_{2,2}} \Phi_{\mathbf{r}, \tilde{\mathbf{r}}}^{(4,2)} \Psi_{ASHP_{2,2}}^T \right)^{-1} \Psi_{ASHP_{2,2}} \Phi_{\mathbf{r}, \tilde{\mathbf{r}}}^{(4,2)} \mathbf{\Pi}_2, \quad (3.225)$$

where $\Psi_{ASHP_{2,2}} = [1 \ 0 \ 0]$ and $\mathbf{\Pi}_2 = [0 \ 1 \ 0]^T$.

The calculation results of $\Sigma_{\mathbf{e}, \mathbf{e}}$ and $\Sigma_{\tilde{\mathbf{e}}, \tilde{\mathbf{e}}}$ are already given above in (3.219) and (3.220).

Now, we can calculate

$$\begin{aligned} \left(\Psi_{ASHP_{2,2}} \left[\Sigma_{\mathbf{r}, \mathbf{r}}^{(4,2)} + \Sigma_{\tilde{\mathbf{r}}, \tilde{\mathbf{r}}}^{(4,2)*} \right] \Psi_{ASHP_{2,2}}^T \right) &= [1 \ 0 \ 0] \left[\Sigma_{\mathbf{r}, \mathbf{r}}^{(4,2)} + \Sigma_{\tilde{\mathbf{r}}, \tilde{\mathbf{r}}}^{(4,2)*} \right] \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \\ &= [E[r_1 r_1^*] + E[\tilde{r}_1 \tilde{r}_1^*]]. \end{aligned} \quad (3.226)$$

and

$$\begin{aligned} \left(\Psi_{ASHP_{2,2}} \left[\Sigma_{\mathbf{r}, \mathbf{r}}^{(4,2)} + \Sigma_{\tilde{\mathbf{r}}, \tilde{\mathbf{r}}}^{(4,2)*} \right] \mathbf{\Pi}_2 \right) &= [1 \ 0 \ 0] \left[\Sigma_{\mathbf{r}, \mathbf{r}}^{(4,2)} + \Sigma_{\tilde{\mathbf{r}}, \tilde{\mathbf{r}}}^{(4,2)*} \right] \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \\ &= [E[r_1 r_2^*] + E[\tilde{r}_2 \tilde{r}_1^*]]. \end{aligned} \quad (3.227)$$

By substituting (3.226) and (3.227) into (3.225), we can rewrite the lattice coefficient vector as

$$\mathbf{k}_{r_{ASHP_{2,2}}}^{(4,2)} = [E[r_1 r_1^*] + E[\tilde{r}_1 \tilde{r}_1^*]]^{-1} [E[r_1 r_2^*] + E[\tilde{r}_2 \tilde{r}_1^*]]. \quad (3.228)$$

3.4 NonCausal Half-Plane Models

In addition to developing computationally recursive either QP or ASHP models as explained above sections, it is also possible to define $2(M-2)$ noncausal half-plane model filters by using the auxiliary vertical forward/backward PEFs for the PSR as shown in Figure 3.29. Similarly, $2(N-2)$ noncausal half-plane model filters can be obtained by employing the auxiliary horizontal forward/backward PEFs.

In some applications noncausal half-plane models is preferable for the 2-D spectrum estimation with superior resolution [58]. Our proposed method is also capable of producing the necessary 2-D noncausal half-plane models without any additional computational cost as shown in Figure 3.29.

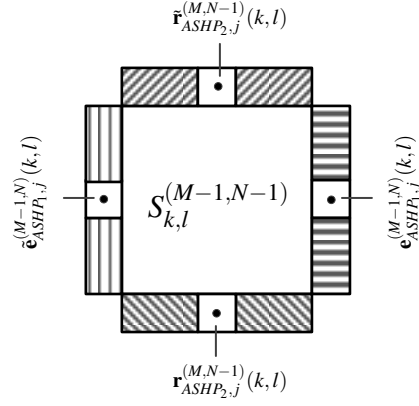


Figure 3.29: Generalized ASHP Type-1 and Type-2 model 2-D filters for the PSR $S_{k,l}^{(M-1,N-1)}$.

As seen in the Figure 3.29, for the PSR $S_{k,l}^{(M-1,N-1)}$, it is possible to generate $2(M+N-4)$ noncausal (NC) half-plane lattice models from all auxiliary vertical and horizontal forward/backward PEFs. The noncausal half-plane lattice models $e_{NCj}^{(M-1,N)}$ and $\tilde{e}_{NCj}^{(M-1,N)}$ for $j = 2, \dots, M-1$ are obtained as a linear combinations of the other auxiliary PEFs in the same column in the shaded area Figure 3.29. Similarly, the noncausal half-plane lattice models $r_{NCi}^{(M,N-1)}$ and $\tilde{r}_{NCi}^{(M,N-1)}$ for $j = 2, \dots, N-1$ are generated as a linear combinations of the other auxiliary PEFs in the same row in shaded area in Figure 3.29.

3.4.1 General case of noncausal half-plane models using vertical PEFs

The Figure 3.30 demonstrates the construction of the noncausal half-plane lattice model from the auxiliary vertical forward/backward PEFs.

The noncausal half-plane lattice model equations from the auxiliary vertical forward/backward PEFs can be written as follows,

$$\begin{aligned} e_{NCj}^{(M-1,N)}(k,l) &= e_j^{(M-1,N)}(k,l) - k_{e_{NCj}}^{(M-1,N)H} \Psi_j \mathbf{e}^{(M-1,N)}(k,l) \\ \tilde{e}_{NCj}^{(M-1,N)}(k,l) &= \tilde{e}_j^{(M-1,N)}(k,l) - k_{e_{NCj}}^{(M-1,N)T} \Psi_j \tilde{\mathbf{e}}^{(M-1,N)}(k,l) \end{aligned} \quad (3.229)$$

where $\mathbf{k}_{e_{NCj}}^{(M-1,N)} = \begin{bmatrix} k_{e_{NCj}}^{(M-1,N)}(1) & \dots & k_{e_{NCj}}^{(M-1,N)}(M) \end{bmatrix}^T$, for $j = 2, \dots, M-1$.

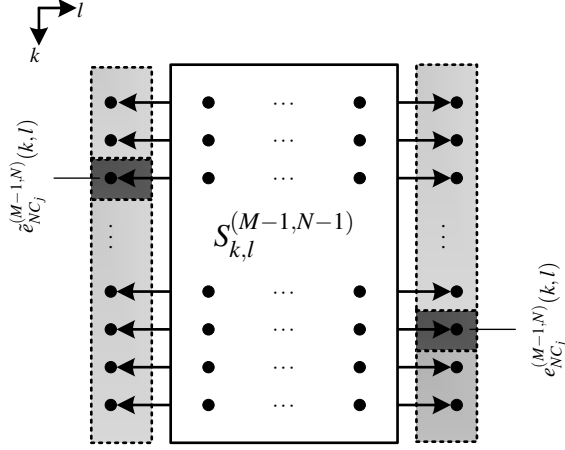


Figure 3.30: NC lattice filter model constructions by using auxiliary vertical PEFs of the PSR $S_{k,l}^{(M-1,N-1)}$ for $j = 2, \dots, M-1$.

As it has already explained in Sections 3.2 and 3.3, in order to find the lattice coefficient vector of the lattice equations, we can write the minimum mean-square equation as

$$\begin{aligned} P_{e_{NC_j}}^{(M-1,N)} &= \text{tr} \left[E \left[\mathbf{e}_{NC_j}^{(M-1,N)}(k,l) \mathbf{e}_{NC_j}^{(M-1,N)H}(k,l) \right] + E \left[\tilde{\mathbf{e}}_{NC_j}^{(M-1,N)}(k,l) \tilde{\mathbf{e}}_{NC_j}^{(M-1,N)H}(k,l) \right] \right] \\ &= E \left[\left| \mathbf{e}_{NC_j}^{(M-1,N)}(k,l) \right|^2 \right] + E \left[\left| \tilde{\mathbf{e}}_{NC_j}^{(M-1,N)}(k,l) \right|^2 \right] \end{aligned} \quad (3.230)$$

where tr means trace operator.

By minimizing the minimum mean-square equation (3.230) with respect to the lattice coefficient vector $\mathbf{k}_{e_{NC_j}}^{(M-1,N)}$ as shown below,

$$\frac{\partial}{\partial \mathbf{k}_{e_{NC_j}}^{(M-1,N)}} P_{e_{NC_j}}^{(M-1,N)} = 0 \quad (3.231)$$

and we can get the lattice coefficient vector as

$$\mathbf{k}_{e_{NC_j}}^{(M-1,N)} = [\Psi_j \Phi_{\mathbf{e}, \tilde{\mathbf{e}}} \Psi_j]^{-1} \Psi_j \Phi_{\mathbf{e}, \tilde{\mathbf{e}}} \Pi_j. \quad (3.232)$$

By using the equation 3.232 in the minimum mean-square equations, then we can write the common mean-square equation from stage to stage as follows

$$P_{NC_j}^{(M-1,N)} = P_{e_j}^{(M-1,N)} - \mathbf{k}_{e_{NC_j}}^{(M-1,N)H} \Psi_j \Phi_{\mathbf{e}, \tilde{\mathbf{e}}} \Psi_j \mathbf{k}_{e_{NC_j}}^{(M-1,N)} \quad (3.233)$$

where $P_{e_j} = E \left[\left| e_j^{(M-1,N)} \right|^2 \right] + E \left[\left| \tilde{e}_j^{(M-1,N)} \right|^2 \right]$.

As we defined for ASHP case, it is also possible to define a single matrix including all NC half-plane lattice filter models generated from the auxiliary vertical forward/backward PEFs as follows

$$\mathbf{K}_{e_{NC}}^{(M-1,N)} = \begin{bmatrix} \mathbf{k}_{e_{NC_2}}^{(M-1,N)} & \mathbf{k}_{e_{NC_3}}^{(M-1,N)} & \dots & \mathbf{k}_{e_{NC_{M-2}}}^{(M-1,N)} & \mathbf{k}_{e_{NC_{M-1}}}^{(M-1,N)} \end{bmatrix}_{M \times (M-2)}, \quad (3.234)$$

where

$$\mathbf{k}_{e_{NC_j}}^{(M-1,N)} = \left[\mathbf{k}_{e_{NC_j}}^{(M-1,N)}(1) \quad \dots \quad \mathbf{k}_{e_{NC_j}}^{(M-1,N)}(M-1) \right]^T, \quad (3.235)$$

for $j = 2, \dots, M-1$. In case of j is equal to the row number, a "0" will be inserted as a lattice coefficient value. In fact, this cell in the matrix where "0" is inserted represents the vertical PEF index where NC half-plane is initiated.

By putting all lattice vectors in the matrix as open form, then we can write the following matrix:

$$\mathbf{K}_{e_{NC}}^{(M-1,N)} = \begin{bmatrix} k_{e_{NC_2}}^{(M-1,N)}(1) & k_{e_{NC_3}}^{(M-1,N)}(1) & \dots & k_{e_{NC_{M-2}}}^{(M-1,N)}(1) & k_{e_{NC_{M-1}}}^{(M-1,N)}(1) \\ 0 & k_{e_{NC_3}}^{(M-1,N)}(2) & \dots & k_{e_{NC_{M-2}}}^{(M-1,N)}(2) & k_{e_{NC_{M-1}}}^{(M-1,N)}(2) \\ k_{e_{NC_2}}^{(M-1,N)}(2) & 0 & \dots & k_{e_{NC_{M-2}}}^{(M-1,N)}(3) & k_{e_{NC_{M-1}}}^{(M-1,N)}(3) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ k_{e_{NC_2}}^{(M-1,N)}(M-2) & k_{e_{NC_3}}^{(M-1,N)}(M-2) & \dots & k_{e_{NC_{M-2}}}^{(M-1,N)}(M-2) & 0 \\ k_{e_{NC_2}}^{(M-1,N)}(M-1) & k_{e_{NC_3}}^{(M-1,N)}(M-1) & \dots & k_{e_{NC_{M-2}}}^{(M-1,N)}(M-1) & k_{e_{NC_{M-1}}}^{(M-1,N)}(M-1) \end{bmatrix}. \quad (3.236)$$

We can write all NC half-plane lattice models produced from the vertical PEFs as

$$\mathbf{e}_{NC}^{(M-1,N)}(k, l) = \left[e_{NC_2}^{(M-1,N)}(k, l) \quad e_{NC_3}^{(M-1,N)}(k, l) \quad \dots \quad e_{NC_{M-1}}^{(M-1,N)}(k, l) \right]^T \quad (3.237)$$

and

$$\tilde{\mathbf{e}}_{NC}^{(M-1,N)}(k, l) = \left[\tilde{e}_{NC_2}^{(M-1,N)}(k, l) \quad \tilde{e}_{NC_3}^{(M-1,N)}(k, l) \quad \dots \quad \tilde{e}_{NC_{M-1}}^{(M-1,N)}(k, l) \right]^T. \quad (3.238)$$

By using already defined constant matrix $\Psi_{1,M}$, we can define one pair of lattice filter equations including all possible NC half-plane lattice filter equations as

$$\begin{aligned} \mathbf{e}_{NC}^{(M-1,N)}(k, l) &= \Psi_{1,M} \mathbf{e}^{(M-1,N)}(k, l) - \mathbf{K}_{e_{NC}}^{(M-1,N)H} \mathbf{e}^{(M-1,N)}(k, l) \\ \tilde{\mathbf{e}}_{NC}^{(M-1,N)}(k, l) &= \Psi_{1,M} \tilde{\mathbf{e}}^{(M-1,N)}(k, l) - \mathbf{K}_{e_{NC}}^{(M-1,N)T} \tilde{\mathbf{e}}^{(M-1,N)}(k, l). \end{aligned} \quad (3.239)$$

The sum of the mean-square values of 3.239 can be written as

$$P_{e_{NC}}^{(M-1,N)} = \text{tr} E \left[\mathbf{e}_{NC}^{(M-1,N)}(k,l) \mathbf{e}_{NC}^{(M-1,N)H}(k,l) \right] + \text{tr} E \left[\tilde{\mathbf{e}}_{NC}^{(M-1,N)}(k,l) \tilde{\mathbf{e}}_{NC}^{(M-1,N)H}(k,l) \right]. \quad (3.240)$$

By minimizing (3.240) with respect to the lattice coefficient matrix $\mathbf{K}_{e_{NC}}^{(M-1,N)}$ as follows

$$\frac{\partial}{\partial \mathbf{K}_{e_{NC}}^{(M-1,N)}} P_{e_{NC}}^{(M-1,N)} = 0,$$

we get the following result

$$\mathbf{K}_{e_{NC}}^{(M-1,N)} = \left[\Phi_{\mathbf{e}, \tilde{\mathbf{e}}}^{(M-1,N)} \right]^{-1} \Phi_{\mathbf{e}, \tilde{\mathbf{e}}}^{(M-1,N)} \Psi_{1,M}^T. \quad (3.241)$$

3.4.2 General case of noncausal half-plane models using horizontal PEFs

The Figure 3.30 demonstrates the construction of the noncausal half-plane lattice model from the auxiliary horizontal forward/backward PEFs.

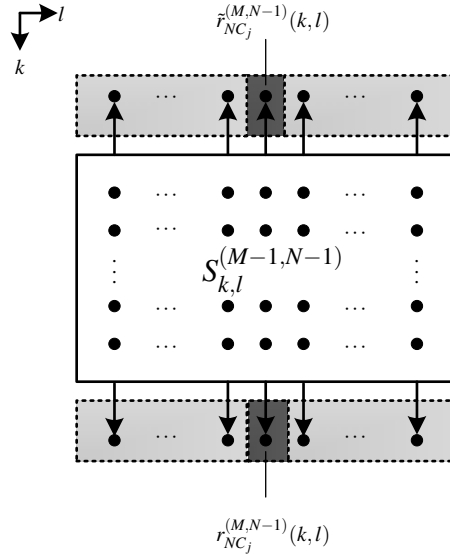


Figure 3.31: NC lattice filter model constructions by using auxiliary horizontal PEFs of the PSR $S_{k,l}^{(M-1,N-1)}$ for $j = 2, \dots, N-1$.

The noncausal half-plane lattice model equations from the auxiliary horizontal forward/backward PEFs can be written as follows,

$$\begin{aligned} r_{NC_j}^{(M,N-1)}(k,l) &= r_j^{(M,N-1)}(k,l) - k_{r_{NC_j}}^{(M,N-1)H} \Psi_j \mathbf{r}^{(M,N-1)}(k,l) \\ \tilde{r}_{NC_j}^{(M,N-2)}(k,l) &= \tilde{r}_j^{(M,N-1)}(k,l) - k_{r_{NC_j}}^{(M,N-1)T} \Psi_j \tilde{\mathbf{r}}^{(M,N-1)}(k,l) \end{aligned} \quad (3.242)$$

where $\mathbf{k}_{r_{NCj}}^{(M,N-1)} = \left[k_{r_{NCj}}^{(M,N-1)}(1) \ \cdots \ k_{r_{NCj}}^{(M,N-1)}(M) \right]^T$, for $j = 2, \dots, N-1$.

Similar to the Section 3.4.1, we can write the minimum mean-square equation for the noncausal half-plane lattice filter models obtained from the auxiliary horizontal forward/backward PEFs as follows

$$P_{r_{NCj}}^{(M,N-1)} = E \left[\left| \mathbf{r}_{r_{NCj}}^{(M,N-1)}(k,l) \right|^2 \right] + E \left[\left| \tilde{\mathbf{r}}_{r_{NCj}}^{(M,N-1)}(k,l) \right|^2 \right] \quad (3.243)$$

By minimizing the minimum mean-square equation (3.243) with respect to the lattice coefficient vector $\mathbf{k}_{r_{NCj}}^{(M,N-1)}$ as,

$$\frac{\partial}{\partial \mathbf{k}_{r_{NCj}}^{(M,N-1)}} P_{r_{NCj}}^{(M,N-1)} = 0 \quad (3.244)$$

and we can get the lattice coefficient vector as

$$\mathbf{k}_{r_{NCj}}^{(M,N-1)} = \left[\Psi_j \Phi_{\mathbf{r}, \tilde{\mathbf{r}}} \Psi_j \right]^{-1} \Psi_j \Phi_{\mathbf{r}, \tilde{\mathbf{r}}} \Pi_j. \quad (3.245)$$

By using the result of (3.245), the mean-square equation from stage to stage can be written as follows

$$P_{r_{NCj}}^{(M,N-1)} = P_{r_j}^{(M,N-1)} - \mathbf{k}_{r_{NCj}}^{(M,N-1)H} \Psi_j \Phi_{\mathbf{r}, \tilde{\mathbf{r}}} \Psi_j \mathbf{k}_{r_{NCj}}^{(M,N-1)} \quad (3.246)$$

where $P_{r_j} = E \left[\left| r_j^{(M,N-1)} \right|^2 \right] + E \left[\left| \tilde{r}_j^{(M,N-1)} \right|^2 \right]$.

Similarly, it can also be defined a single matrix including all NC half-plane lattice filter models generated from the auxiliary horizontal forward/backward PEFs as follows

$$\mathbf{K}_{r_{NC}}^{(M,N-1)} = \begin{bmatrix} \mathbf{k}_{r_{NC2}}^{(M,N-1)} & \mathbf{k}_{r_{NC3}}^{(M,N-1)} & \cdots & \mathbf{k}_{r_{NCM-2}}^{(M,N-1)} & \mathbf{k}_{r_{NCM-1}}^{(M,N-1)} \end{bmatrix}_{N \times (N-2)}, \quad (3.247)$$

where

$$\mathbf{k}_{r_{NCj}}^{(M,N-1)} = \left[\mathbf{k}_{r_{NCj}}^{(M,N-1)}(1) \ \cdots \ \mathbf{k}_{r_{NCj}}^{(M,N-1)}(M-1) \right]^T, \quad (3.248)$$

for $j = 2, \dots, N-1$. In case of j is equal to the row number, a 0 will be inserted as a lattice coefficient value. In fact, this cell in the matrix where 0 is inserted represents the horizontal PEF index where NC half-plane is initiated.

By putting all lattice vectors in the matrix as open form, then we can write the following matrix:

$$\mathbf{K}_{r_{NC}}^{(M,N-1)} = \begin{bmatrix} k_{r_{NC_2}}^{(M,N-1)}(1) & k_{r_{NC_3}}^{(M,N-1)}(1) & \cdots & k_{r_{NC_{M-2}}}^{(M,N-1)}(1) & k_{r_{NC_{M-1}}}^{(M,N-1)}(1) \\ 0 & k_{r_{NC_3}}^{(M,N-1)}(2) & \cdots & k_{r_{NC_{M-2}}}^{(M,N-1)}(2) & k_{r_{NC_{M-1}}}^{(M,N-1)}(2) \\ k_{r_{NC_2}}^{(M,N-1)}(2) & 0 & \cdots & k_{r_{NC_{M-2}}}^{(M,N-1)}(3) & k_{r_{NC_{M-1}}}^{(M,N-1)}(3) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ k_{r_{NC_2}}^{(M,N-1)}(M-2) & k_{r_{NC_3}}^{(M,N-1)}(M-2) & \cdots & k_{r_{NC_{M-2}}}^{(M,N-1)}(M-2) & 0 \\ k_{r_{NC_2}}^{(M,N-1)}(M-1) & k_{r_{NC_3}}^{(M,N-1)}(M-1) & \cdots & k_{r_{NC_{M-2}}}^{(M,N-1)}(M-1) & k_{r_{NC_{M-1}}}^{(M,N-1)}(M-1) \end{bmatrix}. \quad (3.249)$$

We can write all NC half-plane lattice models generated from the horizontal PEFs as

$$\mathbf{r}_{NC}^{(M,N-1)}(k,l) = \begin{bmatrix} r_{NC_2}^{(M,N-1)}(k,l) & r_{NC_3}^{(M,N-1)}(k,l) & \cdots & r_{NC_{M-1}}^{(M,N-1)}(k,l) \end{bmatrix}^T \quad (3.250)$$

and

$$\tilde{\mathbf{r}}_{NC}^{(M,N-1)}(k,l) = \begin{bmatrix} \tilde{r}_{NC_2}^{(M,N-1)}(k,l) & \tilde{r}_{NC_3}^{(M,N-1)}(k,l) & \cdots & \tilde{r}_{NC_{M-1}}^{(M,N-1)}(k,l) \end{bmatrix}^T. \quad (3.251)$$

Similar to the NC half-plane lattice model obtained from vertical PEFs, we can also define one pair of lattice filter equations including all possible NC half-plane lattice filter equations as

$$\begin{aligned} \mathbf{r}_{NC}^{(M,N-1)}(k,l) &= \Psi_{1,M} \mathbf{r}^{(M,N-1)}(k,l) - \mathbf{K}_{r_{NC}}^{(M,N-1)H} \mathbf{r}^{(M,N-1)}(k,l) \\ \tilde{\mathbf{r}}_{NC}^{(M,N-1)}(k,l) &= \Psi_{1,M} \tilde{\mathbf{r}}^{(M,N-1)}(k,l) - \mathbf{K}_{r_{NC}}^{(M,N-1)T} \tilde{\mathbf{r}}^{(M,N-1)}(k,l). \end{aligned} \quad (3.252)$$

The sum of the mean-square values of 3.252 can be written as

$$\begin{aligned} P_{r_{NC}}^{(M,N-1)} &= tr E \left[\mathbf{r}_{NC}^{(M,N-1)}(k,l) \mathbf{r}_{NC}^{(M,N-1)H}(k,l) \right] \\ &\quad + tr E \left[\tilde{\mathbf{r}}_{NC}^{(M,N-1)}(k,l) \tilde{\mathbf{r}}_{NC}^{(M,N-1)H}(k,l) \right] \end{aligned} \quad (3.253)$$

where tr means trace operator.

By minimizing (3.253) with respect to the lattice coefficient matrix $\mathbf{K}_{r_{NC}}^{(M,N-1)}$ as follows

$$\frac{\partial}{\partial \mathbf{K}_{r_{NC}}^{(M,N-1)}} P_{r_{NC}}^{(M,N-1)} = 0,$$

we get the following result

$$\mathbf{K}_{r_{NC}}^{(M,N-1)} = \left[\Phi_{\mathbf{r},\tilde{\mathbf{r}}}^{(M,N-1)} \right]^{-1} \Phi_{\mathbf{r},\tilde{\mathbf{r}}}^{(M,N-1)} \Psi_{1,M}^T. \quad (3.254)$$

3.4.3 Application examples of the NC half-plane 2-D lattice models

In this section, some examples of the NC half-plane lattice models obtained from vertical and horizontal forward/backward PEFs will be introduced. For each case, all NC half-plane lattice models will be elaborated.

According to our NC half-plane definition presented above, it is not possible to define any NC half-plane lattice model from the PSRs $S_{k,l}^{(1,0)}$ and $S_{k,l}^{(0,1)}$. In order to define any NC half-plane lattice model, the order of the PSR should be higher than (1,0) and (0,1).

3.4.3.1 NC half-plane 2-D lattice models of the PSR $S_{k,l}^{(2,0)}$

It is clearly seen that, only one pair of NC half-plane lattice model using auxiliary vertical forward/backward PEFs can be generated. It is possible to construct NC half-plane models, namely, $e_{NC_2}^{(2,1)}(k,l)$ and $\tilde{e}_{NC_2}^{(2,1)}(k,l)$ by predicting $e_2^{(2,1)}(k,l)$ and $\tilde{e}_2^{(2,1)}(k,l)$ from the rest of the vertical forward PEFs $e_1^{(2,1)}(k,l)$ and $e_3^{(2,1)}(k,l)$, and from the rest of the vertical backward PEFs $\tilde{e}_1^{(2,1)}(k,l)$ and $\tilde{e}_3^{(2,1)}(k,l)$, respectively.

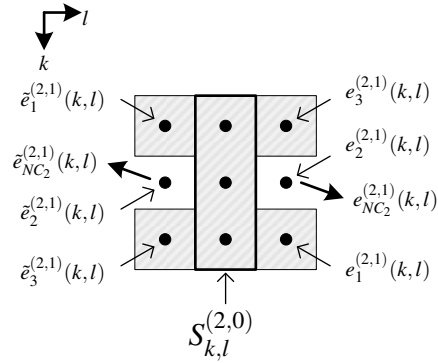


Figure 3.32: NC half-plane lattice model from the corresponding auxiliary vertical PEFs of the PSR $S_{k,l}^{(2,0)}$.

Figure 3.32 shows the illustration of the NC half-plane lattice model construction of the PSR $S_{k,l}^{(2,0)}$. The lattice model equations for NC half-plane lattice models can be written as,

$$\begin{aligned} e_{NC_2}^{(2,1)}(k,l) &= e_2^{(2,1)}(k,l) - \mathbf{k}_{e_{NC_2}}^{(2,1)H} \Psi_2 \mathbf{e}^{(2,1)}(k,l) \\ \tilde{e}_{NC_2}^{(2,1)}(k,l) &= \tilde{e}_2^{(2,1)}(k,l) - \mathbf{k}_{e_{NC_2}}^{(2,1)T} \Psi_2 \tilde{\mathbf{e}}^{(2,1)}(k,l). \end{aligned} \quad (3.255)$$

From the definition of the constant matrix Ψ , we can get $\Psi_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ from removing the second row of the identity matrix $\mathbf{I}_{3 \times 3}$. Hence, we can write the lattice equation (3.255) in the open form as

$$\begin{aligned} e_{NC_2}^{(2,1)}(k, l) &= e_2^{(2,1)}(k, l) - k_{e_{NC_2}}^{(2,1)*}(1) e_1^{(2,1)}(k, l) \\ &\quad - k_{e_{NC_2}}^{(2,1)*}(2) e_3^{(2,1)}(k, l) \\ \tilde{e}_{NC_2}^{(2,1)}(k, l) &= \tilde{e}_2^{(2,1)}(k, l) - k_{e_{NC_2}}^{(2,1)}(1) \tilde{e}_1^{(2,1)}(k, l) \\ &\quad - k_{e_{NC_2}}^{(2,1)}(2) \tilde{e}_3^{(2,1)}(k, l). \end{aligned} \quad (3.256)$$

The sum of the mean-square value of the lattice form equations (3.256) can be written as

$$P_{e_{NC_2}}^{(2,1)} = E \left[\left| e_{NC_2}^{(2,1)}(k, l) \right|^2 \right] + E \left[\left| \tilde{e}_{NC_2}^{(2,1)}(k, l) \right|^2 \right]. \quad (3.257)$$

As a result of the mean-square minimization of (3.257), the reflection coefficient vector is calculated as

$$\mathbf{k}_{e_{NC_2}}^{(2,1)} = \left(\Psi_2 \Phi_{\mathbf{e}, \tilde{\mathbf{e}}}^{(2,1)} \Psi_2^T \right)^{-1} \Psi_2 \Phi_{\mathbf{e}, \tilde{\mathbf{e}}}^{(2,1)} \Pi_2. \quad (3.258)$$

We can restructure the power of the error propagation equation from stage to stage by using (3.258) as

$$P_{e_{NC_2}}^{(2,1)} = P_{e_2}^{(2,1)} - \mathbf{k}_{e_{NC_2}}^{(2,1)H} \Psi_2 \Phi_{\mathbf{e}, \tilde{\mathbf{e}}}^{(2,1)} \Psi_2 \mathbf{k}_{e_{NC_2}}^{(2,1)}, \quad (3.259)$$

where

$$P_{e_2}^{(2,1)} = E \left[\left| e_2^{(2,1)}(k, l) \right|^2 \right] + E \left[\left| \tilde{e}_2^{(2,1)}(k, l) \right|^2 \right].$$

3.4.3.2 NC half-plane 2-D lattice models of the PSR $S_{k,l}^{(3,2)}$

In this application, it will be demonstrated all the possible NC half-plane lattice models by utilizing auxiliary horizontal and vertical forward/backward PEFs of the PSR $S_{k,l}^{(3,2)}$ shown in Figure 3.5. In Sections 3.2.2 and 3.3.3.4, this PSR has been already utilized for showing how to obtain all possible QP lattice models and all possible type-1 and type-2 ASHP lattice models, respectively. Now, first, all possible NC half-plane lattice filter models generated from the auxiliary vertical forward/backward PEFs,

and then NC half-plane lattice filter models produced from the auxiliary horizontal forward/backward PEFs will be elaborated.

The construction of the whole NC half-plane lattice models by employing the corresponding auxiliary vertical forward/backward PEFs is illustrated in Figure 3.33, and 3.34. It is clearly seen that it is possible to obtain two pairs of NC half-plane lattice models from the vertical forward/backward PEFs. Here, M is 4, and j is defined as $j = 2, 3$.

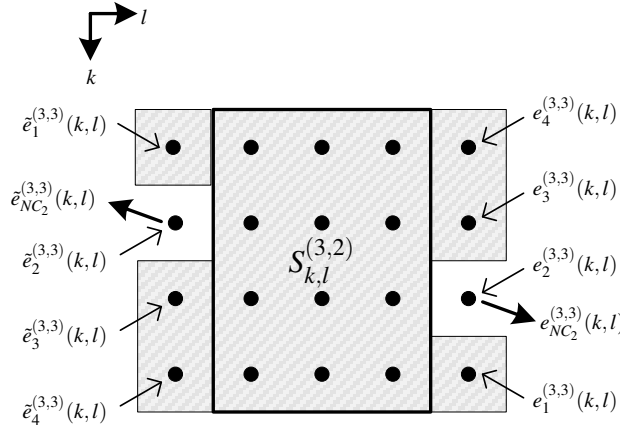


Figure 3.33: NC half-plane lattice models of the PSR $S_{k,l}^{(3,2)}$: $e_{NC_2}^{(3,3)}(k,l)$ and $\tilde{e}_{NC_2}^{(3,3)}(k,l)$.

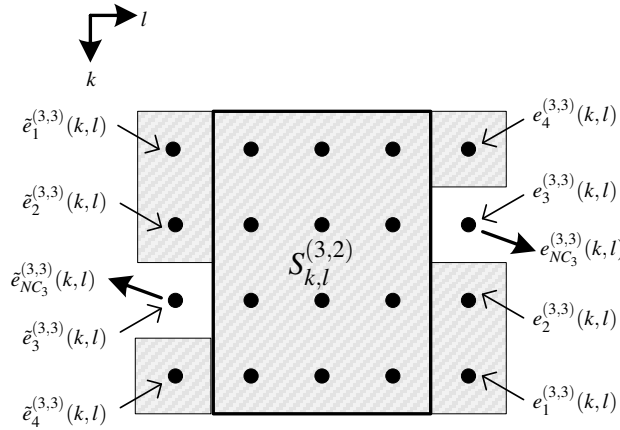


Figure 3.34: NC half-plane lattice models of the PSR $S_{k,l}^{(3,2)}$: $e_{NC_3}^{(3,3)}(k,l)$ and $\tilde{e}_{NC_3}^{(3,3)}(k,l)$.

As shown in Figure 3.33 and Figure 3.34, to construct $e_{NC_2}^{(3,3)}(k,l)$, the vertical PEF $e_2^{(3,3)}(k,l)$ is linearly predicted from $e_1^{(3,3)}(k,l)$, $e_3^{(3,3)}(k,l)$, and $e_4^{(3,3)}(k,l)$ and $e_{NC_3}^{(3,3)}(k,l)$ is generated by linear prediction of $e_3^{(3,3)}(k,l)$ from $e_1^{(3,3)}(k,l)$, $e_2^{(3,3)}(k,l)$,

and $e_4^{(3,3)}(k, l)$, respectively. The backward cases can be determined at the same way. The NC half-plane lattice models equations of these NC half-plane lattice models of the PSR $S_{k,l}^{(3,2)}$ can be written for $j = 2, 3$ as

$$\begin{aligned} e_{NC_2}^{(3,3)}(k, l) &= e_2^{(3,3)}(k, l) - \mathbf{k}_{e_{NC_2}}^{(3,3)H} \Psi_2 \mathbf{e}^{(3,3)}(k, l) \\ \tilde{e}_{NC_2}^{(3,3)}(k, l) &= \tilde{e}_2^{(3,3)}(k, l) - \mathbf{k}_{e_{NC_2}}^{(3,3)T} \Psi_2 \tilde{\mathbf{e}}^{(3,3)}(k, l), \end{aligned} \quad (3.260)$$

and

$$\begin{aligned} e_{NC_3}^{(3,3)}(k, l) &= e_3^{(3,3)}(k, l) - \mathbf{k}_{e_{NC_3}}^{(3,3)H} \Psi_3 \mathbf{e}^{(3,3)}(k, l) \\ \tilde{e}_{NC_3}^{(3,3)}(k, l) &= \tilde{e}_3^{(3,3)}(k, l) - \mathbf{k}_{e_{NC_3}}^{(3,3)T} \Psi_3 \tilde{\mathbf{e}}^{(3,3)}(k, l), \end{aligned} \quad (3.261)$$

where

$$\mathbf{k}_{e_{NC_2}}^{(3,3)} = \begin{bmatrix} k_{e_{NC_2}}^{(3,3)}(1) \\ k_{e_{NC_2}}^{(3,3)}(2) \\ k_{e_{NC_2}}^{(3,3)}(3) \end{bmatrix}, \quad (3.262)$$

and

$$\mathbf{k}_{e_{NC_3}}^{(3,3)} = \begin{bmatrix} k_{e_{NC_3}}^{(3,3)}(1) \\ k_{e_{NC_3}}^{(3,3)}(2) \\ k_{e_{NC_3}}^{(3,3)}(3) \end{bmatrix}. \quad (3.263)$$

By substituting constant matrix $\Psi_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ into (3.260), and (3.261), respectively, the NC half-plane model equations will be as follows

$$\begin{aligned} e_{NC_2}^{(3,3)}(k, l) &= e_2^{(3,3)}(k, l) - k_{e_{NC_2}}^{(3,3)*}(1) e_1^{(3,3)}(k, l) \\ &\quad - k_{e_{NC_2}}^{(3,3)*}(2) e_3^{(3,3)}(k, l) - k_{e_{NC_2}}^{(3,3)*}(3) e_4^{(3,3)}(k, l) \\ \tilde{e}_{NC_2}^{(3,3)}(k, l) &= \tilde{e}_2^{(3,3)}(k, l) - k_{e_{NC_2}}^{(3,3)}(1) \tilde{e}_1^{(3,3)}(k, l) \\ &\quad - k_{e_{NC_2}}^{(3,3)}(2) \tilde{e}_3^{(3,3)}(k, l) - k_{e_{NC_2}}^{(3,3)}(3) \tilde{e}_4^{(3,3)}(k, l), \end{aligned} \quad (3.264)$$

$$\begin{aligned} e_{NC_3}^{(3,3)}(k, l) &= e_3^{(3,3)}(k, l) - k_{e_{NC_3}}^{(3,3)*}(1) e_1^{(3,3)}(k, l) \\ &\quad - k_{e_{NC_3}}^{(3,3)*}(2) e_2^{(3,3)}(k, l) - k_{e_{NC_3}}^{(3,3)*}(3) e_4^{(3,3)}(k, l) \\ \tilde{e}_{NC_3}^{(3,3)}(k, l) &= \tilde{e}_3^{(3,3)}(k, l) - k_{e_{NC_3}}^{(3,3)}(1) \tilde{e}_1^{(3,3)}(k, l) \\ &\quad - k_{e_{NC_3}}^{(3,3)}(2) \tilde{e}_2^{(3,3)}(k, l) - k_{e_{NC_3}}^{(3,3)}(3) \tilde{e}_4^{(3,3)}(k, l). \end{aligned} \quad (3.265)$$

The sum of the mean-square value of the lattice model equations given in (3.264), and (3.265) can be written as

$$P_{e_{NC_j}}^{(3,3)} = E \left[\left| e_{NC_j}^{(3,3)}(k, l) \right|^2 \right] + E \left[\left| \tilde{e}_{NC_j}^{(3,3)}(k, l) \right|^2 \right], \quad (3.266)$$

for $j = 2, 3$.

As a result of the mean-square minimization of (3.266), the reflection coefficient vectors are calculated for $j = 2, 3$ as

$$\mathbf{k}_{e_{NC_2}}^{(3,3)} = \left(\Psi_2 \Phi_{\mathbf{e}, \tilde{\mathbf{e}}}^{(3,3)} \Psi_2^T \right)^{-1} \Psi_2 \Phi_{\mathbf{e}, \tilde{\mathbf{e}}}^{(3,3)} \Pi_2 \quad (3.267)$$

and

$$\mathbf{k}_{e_{NC_3}}^{(3,3)} = \left(\Psi_3 \Phi_{\mathbf{e}, \tilde{\mathbf{e}}}^{(3,3)} \Psi_3^T \right)^{-1} \Psi_3 \Phi_{\mathbf{e}, \tilde{\mathbf{e}}}^{(3,3)} \Pi_3, \quad (3.268)$$

where

$$\Phi_{\mathbf{e}, \tilde{\mathbf{e}}}^{(3,3)} = \Sigma_{\mathbf{e}, \mathbf{e}}^{(3,3)} + \Sigma_{\tilde{\mathbf{e}}, \tilde{\mathbf{e}}}^{(3,3)*},$$

and

$$\begin{aligned} \Pi_2 &= \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix}^T, \\ \Pi_3 &= \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix}^T. \end{aligned}$$

Now, for $j = 2$, let us calculate

$$\begin{aligned} & \left(\Psi_2 \left[\Sigma_{\mathbf{e}, \mathbf{e}}^{(3,3)} + \Sigma_{\tilde{\mathbf{e}}, \tilde{\mathbf{e}}}^{(3,3)*} \right] \Psi_2^T \right) \\ &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \left[\Sigma_{\mathbf{e}, \mathbf{e}}^{(3,3)} + \Sigma_{\tilde{\mathbf{e}}, \tilde{\mathbf{e}}}^{(3,3)*} \right] \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} E[e_1 e_1^*] + E[\tilde{e}_1 \tilde{e}_1^*] & E[e_1 e_3^*] + E[\tilde{e}_3 \tilde{e}_1^*] & E[e_1 e_4^*] + E[\tilde{e}_4 \tilde{e}_1^*] \\ E[e_3 e_1^*] + E[\tilde{e}_1 \tilde{e}_3^*] & E[e_3 e_3^*] + E[\tilde{e}_3 \tilde{e}_3^*] & E[e_3 e_4^*] + E[\tilde{e}_4 \tilde{e}_3^*] \\ E[e_4 e_1^*] + E[\tilde{e}_1 \tilde{e}_4^*] & E[e_4 e_3^*] + E[\tilde{e}_3 \tilde{e}_4^*] & E[e_4 e_4^*] + E[\tilde{e}_4 \tilde{e}_4^*] \end{bmatrix} \quad (3.269) \end{aligned}$$

and

$$\begin{aligned} \left(\Psi_2 \left[\Sigma_{\mathbf{e}, \mathbf{e}}^{(3,3)} + \Sigma_{\tilde{\mathbf{e}}, \tilde{\mathbf{e}}}^{(3,3)*} \right] \Pi_2 \right) &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \left[\Sigma_{\mathbf{e}, \mathbf{e}}^{(3,3)} + \Sigma_{\tilde{\mathbf{e}}, \tilde{\mathbf{e}}}^{(3,3)*} \right] \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} E[e_1 e_2^*] + E[\tilde{e}_2 \tilde{e}_1^*] \\ E[e_3 e_2^*] + E[\tilde{e}_2 \tilde{e}_3^*] \\ E[e_4 e_2^*] + E[\tilde{e}_2 \tilde{e}_4^*] \end{bmatrix}. \quad (3.270) \end{aligned}$$

By substituting (3.269) and (3.270) into (3.267), we can rewrite the lattice coefficient vector for $j = 2$ as

$$\mathbf{k}_{e_{NC_2}}^{(3,3)} = \begin{bmatrix} E[e_1 e_1^*] + E[\tilde{e}_1 \tilde{e}_1^*] & E[e_1 e_3^*] + E[\tilde{e}_3 \tilde{e}_1^*] & E[e_1 e_4^*] + E[\tilde{e}_4 \tilde{e}_1^*] \\ E[e_3 e_1^*] + E[\tilde{e}_1 \tilde{e}_3^*] & E[e_3 e_3^*] + E[\tilde{e}_3 \tilde{e}_3^*] & E[e_3 e_4^*] + E[\tilde{e}_4 \tilde{e}_3^*] \\ E[e_4 e_1^*] + E[\tilde{e}_1 \tilde{e}_4^*] & E[e_4 e_3^*] + E[\tilde{e}_3 \tilde{e}_4^*] & E[e_4 e_4^*] + E[\tilde{e}_4 \tilde{e}_4^*] \end{bmatrix}^{-1} \\ \times \begin{bmatrix} E[e_1 e_2^*] + E[\tilde{e}_2 \tilde{e}_1^*] \\ E[e_3 e_2^*] + E[\tilde{e}_2 \tilde{e}_3^*] \\ E[e_4 e_2^*] + E[\tilde{e}_2 \tilde{e}_4^*] \end{bmatrix}. \quad (3.271)$$

It is possible to do same calculations for $j = 3$, and the following result can be found as

$$\mathbf{k}_{r_{NC_3}}^{(3,3)} = \begin{bmatrix} E[e_1 e_1^*] + E[\tilde{e}_1 \tilde{e}_1^*] & E[e_1 e_2^*] + E[\tilde{e}_2 \tilde{e}_1^*] & E[e_1 e_4^*] + E[\tilde{e}_4 \tilde{e}_1^*] \\ E[e_3 e_1^*] + E[\tilde{e}_1 \tilde{e}_3^*] & E[e_3 e_2^*] + E[\tilde{e}_2 \tilde{e}_3^*] & E[e_3 e_4^*] + E[\tilde{e}_4 \tilde{e}_3^*] \\ E[e_4 e_1^*] + E[\tilde{e}_1 \tilde{e}_4^*] & E[e_4 e_2^*] + E[\tilde{e}_2 \tilde{e}_4^*] & E[e_4 e_4^*] + E[\tilde{e}_4 \tilde{e}_4^*] \end{bmatrix}^{-1} \\ \times \begin{bmatrix} E[e_1 e_3^*] + E[\tilde{e}_3 \tilde{e}_1^*] \\ E[e_2 e_3^*] + E[\tilde{e}_3 \tilde{e}_2^*] \\ E[e_4 e_3^*] + E[\tilde{e}_3 \tilde{e}_4^*] \end{bmatrix}. \quad (3.272)$$

Now we have illustrated to obtain all possible NC half-plane lattice models of the PSR $S_{k,l}^{(3,2)}$ by employing the auxiliary vertical forward/backward PEFs. Since the N is 3, it is still possible to define one pair of NC half-plane lattice models, namely $r_{NC_2}^{(4,2)}(k, l)$ and $\tilde{r}_{NC_2}^{(4,2)}(k, l)$, obtained by linearly predicting $r_2^{(4,2)}(k, l)$ and $\tilde{r}_2^{(4,2)}(k, l)$ from the rest of auxiliary horizontal forward and backward PEFs, respectively. The NC half-plane lattice models are shown in Figure 3.35.

As seen in Figure 3.35, NC half-plane lattice model $r_{NC_2}^{(4,2)}(k, l)$ is generated by predicting $r_2^{(4,2)}(k, l)$ from $r_1^{(4,2)}(k, l)$ and $r_3^{(4,2)}(k, l)$, and $\tilde{r}_{NC_2}^{(4,2)}(k, l)$ is produced by linear prediction of $\tilde{r}_2^{(4,2)}(k, l)$ from $\tilde{r}_1^{(4,2)}(k, l)$ and $\tilde{r}_3^{(4,2)}(k, l)$. The NC half-plane lattice model equations of these constructed lattice models of the PSR $S_{k,l}^{(3,2)}$ can be written in an open form as

$$\begin{aligned} r_{NC_2}^{(4,2)}(k, l) &= r_2^{(4,2)}(k, l) - k_{r_{NC_2}}^{(4,2)*}(1) r_1^{(4,2)}(k, l) \\ &\quad - k_{r_{NC_2}}^{(4,2)*}(2) r_3^{(4,2)}(k, l) \\ \tilde{r}_{NC_2}^{(4,2)}(k, l) &= \tilde{r}_2^{(4,2)}(k, l) - k_{r_{NC_2}}^{(4,2)} \tilde{r}_1^{(4,2)}(k, l) \\ &\quad - k_{r_{NC_2}}^{(4,2)}(2) \tilde{r}_3^{(4,2)}(k, l). \end{aligned} \quad (3.273)$$

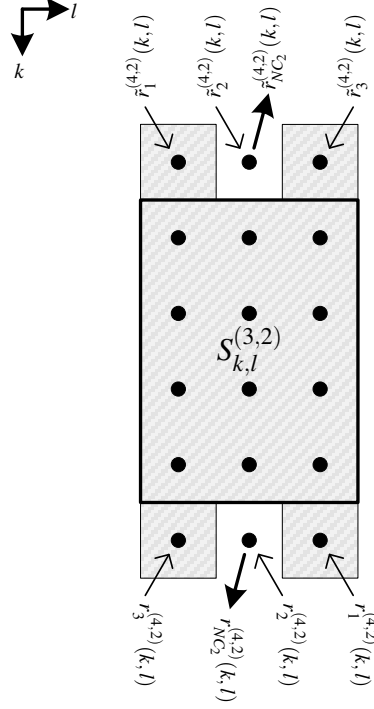


Figure 3.35: NC half-plane lattice models of the PSR $S_{k,l}^{(3,2)}$: $r_{NC_2}^{(4,2)}(k,l)$ and $\tilde{r}_{NC_2}^{(4,2)}(k,l)$.

As a result of the mean-square minimization of $P_{r_{NC_2}}^{(4,2)} = E \left[\left| r_{NC_2}^{(4,2)}(k,l) \right|^2 \right] + E \left[\left| \tilde{r}_{NC_2}^{(4,2)}(k,l) \right|^2 \right]$, the reflection coefficient vector is calculated as

$$\mathbf{k}_{r_{NC_2}}^{(4,2)} = \left(\Psi_2 \Phi_{\mathbf{r}, \tilde{\mathbf{r}}}^{(4,2)} \Psi_2^T \right)^{-1} \Psi_2 \Phi_{\mathbf{r}, \tilde{\mathbf{r}}}^{(4,2)} \Pi_2 \quad (3.274)$$

where

$$\Phi_{\mathbf{r}, \tilde{\mathbf{r}}}^{(4,2)} = \Sigma_{\mathbf{r}, \mathbf{r}}^{(4,2)} + \Sigma_{\tilde{\mathbf{r}}, \tilde{\mathbf{r}}}^{(4,2)*},$$

and

$$\Psi_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}^T,$$

and

$$\Pi_2 = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}^T.$$

The computation of $\Sigma_{\mathbf{r}, \mathbf{r}}$ and $\Sigma_{\tilde{\mathbf{r}}, \tilde{\mathbf{r}}}^*$ are defined as

$$\begin{aligned} \Sigma_{\mathbf{r}, \mathbf{r}}^{(4,2)} &= E \left[\mathbf{4}^{(4,2)}(k,l) \mathbf{r}^{(4,2)H}(k,l) \right] \\ &= \begin{bmatrix} E[r_1 r_1^*] & E[r_1 r_2^*] & E[r_1 r_3^*] \\ E[r_2 r_1^*] & E[r_2 r_2^*] & E[r_2 r_3^*] \\ E[r_3 r_1^*] & E[r_3 r_2^*] & E[r_3 r_3^*] \end{bmatrix}, \end{aligned} \quad (3.275)$$

$$\begin{aligned}
\mathbf{\Sigma}_{\tilde{\mathbf{r}}, \tilde{\mathbf{r}}}^{(4,2)} &= E \left[\tilde{\mathbf{r}}^{(4,2)}(k, l) \tilde{\mathbf{r}}^{(4,2)H}(k, l) \right] \\
&= \begin{bmatrix} E[\tilde{r}_1 \tilde{r}_1^*] & E[\tilde{r}_1 \tilde{r}_2^*] & E[\tilde{r}_1 \tilde{r}_3^*] \\ E[\tilde{r}_2 \tilde{r}_1^*] & E[\tilde{r}_2 \tilde{r}_2^*] & E[\tilde{r}_2 \tilde{r}_3^*] \\ E[\tilde{r}_3 \tilde{r}_1^*] & E[\tilde{r}_3 \tilde{r}_2^*] & E[\tilde{r}_3 \tilde{r}_3^*] \end{bmatrix}, \tag{3.276}
\end{aligned}$$

where (k, l) and superscript representing stage order are omitted for the simplicity.

Now, we can write that

$$\begin{aligned}
\left(\mathbf{\Psi}_2 \left[\mathbf{\Sigma}_{\mathbf{r}, \mathbf{r}}^{(4,2)} + \mathbf{\Sigma}_{\tilde{\mathbf{r}}, \tilde{\mathbf{r}}}^{(4,2)*} \right] \mathbf{\Psi}_2^T \right) &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \left[\mathbf{\Sigma}_{\mathbf{r}, \mathbf{r}}^{(4,2)} + \mathbf{\Sigma}_{\tilde{\mathbf{r}}, \tilde{\mathbf{r}}}^{(4,2)*} \right] \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} E[r_1 r_1^*] + E[\tilde{r}_1 \tilde{r}_1^*] \\ E[r_3 r_3^*] + E[\tilde{r}_3 \tilde{r}_3^*] \end{bmatrix}, \tag{3.277}
\end{aligned}$$

and

$$\begin{aligned}
\left(\mathbf{\Psi}_2 \left[\mathbf{\Sigma}_{\mathbf{r}, \mathbf{r}}^{(4,2)} + \mathbf{\Sigma}_{\tilde{\mathbf{r}}, \tilde{\mathbf{r}}}^{(4,2)*} \right] \mathbf{\Pi}_2 \right) &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \left[\mathbf{\Sigma}_{\mathbf{r}, \mathbf{r}}^{(4,2)} + \mathbf{\Sigma}_{\tilde{\mathbf{r}}, \tilde{\mathbf{r}}}^{(4,2)*} \right] \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} E[r_1 r_2^*] + E[\tilde{r}_2 \tilde{r}_1^*] \\ E[r_3 r_2^*] + E[\tilde{r}_2 \tilde{r}_3^*] \end{bmatrix}. \tag{3.278}
\end{aligned}$$

By substituting (3.277) and (3.278) into (3.274), we can rewrite the lattice coefficient vector as

$$\mathbf{k}_{r_{NC2}}^{(4,2)} = \begin{bmatrix} E[r_1 r_1^*] + E[\tilde{r}_1 \tilde{r}_1^*] \\ E[r_3 r_3^*] + E[\tilde{r}_3 \tilde{r}_3^*] \end{bmatrix}^{-1} \begin{bmatrix} E[r_1 r_2^*] + E[\tilde{r}_2 \tilde{r}_1^*] \\ E[r_3 r_2^*] + E[\tilde{r}_2 \tilde{r}_3^*] \end{bmatrix}. \tag{3.279}$$

3.5 Unification of All Prediction Models

In the previous sections, we presented how to obtain all causal and noncausal 2-D lattice prediction models by employing the vertical and horizontal auxiliary PEFs. It is possible to write a single general lattice error propagation equation for all of the proposed models. The unified prediction for the general modeling equation for the auxiliary vertical forward/backward PEFs can be given by,

$$\begin{aligned}
\begin{bmatrix} e_x^{(M-1,N)}(k, l) \\ \tilde{e}_x^{(M-1,N)}(k, l) \end{bmatrix} &= \begin{bmatrix} \mathbf{\Pi}_x - \mathbf{k}_{e_x}^{(M-1,N)H} \mathbf{\Psi}_x & -\mathbf{O}_M^T \\ -\mathbf{O}_M^T & \mathbf{\Pi}_x - \mathbf{k}_{e_x}^{(M-1,N)T} \mathbf{\Psi}_x \end{bmatrix} \\
&\times \begin{bmatrix} \mathbf{e}^{(M-1,N)}(k, l) \\ \tilde{\mathbf{e}}^{(M-1,N)}(k-1, l) \end{bmatrix} \tag{3.280}
\end{aligned}$$

where the choice of constant matrices $\mathbf{\Pi}_x$ and $\mathbf{\Psi}_x$ for the 2-D causal/noncausal prediction models is shown in Table 3.1. $\mathbf{k}_{e_x}^{(M-1,N)}$ is the reflection coefficient vector. The minimum mean-square values of prediction model PEF (3.280) can be written as,

Table 3.1: The choice of constant matrices Π_j ve Ψ_j for the 2-D causal/noncausal prediction models using the vertical auxiliary PEFs for a PSR $S_{k,l}^{(M,N)}$. For horizontal auxiliary PEFs M will be replaced with N .

x	QP1	QP2	ASHP1,i	ASHP2,i	NC,i
Π_x	Π_1	Π_M	Π_i	Π_i	Π_i
Ψ_x	Ψ_1	Ψ_M	$\Psi_{1,\dots,i}$	$\Psi_{1,\dots,M}$	Ψ_i
			<i>for $i = 2, \dots, M-1$</i>		

$$P_{e_x} = E \left[\left| e_x^{(M-1,N)}(k,l) \right|^2 \right] + E \left[\left| \tilde{e}_x^{(M-1,N)}(k,l) \right|^2 \right]. \quad (3.281)$$

The minimization of (3.281) with respect to the reflection coefficient vector $\mathbf{k}_{e_x}^{(M-1,N)}$ will yield

$$\mathbf{k}_{e_x}^{(M-1,N)} = \left(\Psi_x \Sigma_{\mathbf{e}, \tilde{\mathbf{e}}}^{(M-1,N)} \Psi_x^T \right)^{-1} \Psi_x \Sigma_{\mathbf{e}, \tilde{\mathbf{e}}}^{(M-1,N)} \Pi_x^T \quad (3.282)$$

and the minimum mean-square value equation can be written from stage to stage as follows

$$P_{e_x}^{(M-1,N)} = \Pi_x \Sigma_{\mathbf{e}, \tilde{\mathbf{e}}}^{(M-1,N)} \Pi_x^T - \mathbf{k}_{e_x}^{(M-1,N)H} \Psi_x \Sigma_{\mathbf{e}, \tilde{\mathbf{e}}}^{(M-1,N)} \Psi_x^T \mathbf{k}_{e_x}^{(M-1,N)}. \quad (3.283)$$

Similarly, by using the auxiliary horizontal forward/backward PEFs $r_j^{(M-1,N)}$ and $\tilde{r}_j^{(M-1,N)}$, the general modeling equation can be given as

$$\begin{bmatrix} r_x^{(M-1,N)}(k,l) \\ \tilde{r}_x^{(M-1,N)}(k,l) \end{bmatrix} = \begin{bmatrix} \Pi_x - \mathbf{k}_{r_x}^{(M-1,N)H} \Psi_x & -\mathbf{O}_M^T \\ -\mathbf{O}_M^T & \Pi_x - \mathbf{k}_{r_x}^{(M-1,N)T} \Psi_x \end{bmatrix} \times \begin{bmatrix} \mathbf{r}^{(M-1,N)}(k,l) \\ \tilde{\mathbf{r}}^{(M-1,N)}(k-1,l) \end{bmatrix}, \quad (3.284)$$

where $\mathbf{k}_{r_x}^{(M-1,N)}$ is the reflection coefficient vector.

The minimum mean-square value of prediction model PEF (3.284) can be written as,

$$P_{r_x} = E \left[\left| r_x^{(M-1,N)}(k,l) \right|^2 \right] + E \left[\left| \tilde{r}_x^{(M-1,N)}(k,l) \right|^2 \right]. \quad (3.285)$$

The minimization of (3.285) with respect to the reflection coefficient vector $\mathbf{k}_{r_x}^{(M-1,N)}$ will give the following result

$$\mathbf{k}_{r_x}^{(M-1,N)} = \left(\Psi_x \Sigma_{\mathbf{r}, \tilde{\mathbf{r}}}^{(M-1,N)} \Psi_x^T \right)^{-1} \Psi_x \Sigma_{\mathbf{r}, \tilde{\mathbf{r}}}^{(M-1,N)} \Pi_x^T \quad (3.286)$$

and the minimum mean-square value equation can be written from stage to stage as follows

$$P_{r_x}^{(M-1,N)} = \Pi_x \Sigma_{\mathbf{r}, \tilde{\mathbf{r}}}^{(M-1,N)} \Pi_x^T - \mathbf{k}_{r_x}^{(M-1,N)H} \Psi_x \Sigma_{\mathbf{r}, \tilde{\mathbf{r}}}^{(M-1,N)} \Psi_x^T \mathbf{k}_{r_x}^{(M-1,N)}. \quad (3.287)$$

3.6 Stability of All Prediction Models

Stability is not guaranteed in this procedure, there is no "stability-by-inspection" [11] property for these proposed 2-D lattice filters. However, the synthesis models for the auxiliary prediction error filters and the QP model filters as shown in Figure 2.21 and Figure 3.1, respectively, are found to be almost always stable, as shown in *Example 1* of the Chapter V. This is due to the fact that the 2-D lattice parameters factors are calculated by Burg's method [60] which definitely also leads the maximum entropy. On the other hand, a few of the ASHP AR lattice filters in Figure 3.7 may have stable inverse models.

The stability of a first-quadrant QP filter will be assured by using the stability check rules determined by DeCarlo-Strintzis [61]. If the 2-D transfer function of the first-quadrant QP is

$$H(z_1, z_2) = \frac{1}{A(z_1, z_2)}, \quad (3.288)$$

then it is stable if only if

$$1) \quad A(z_1, 1) \neq 0 \quad \text{for} \quad |z_1| \geq 1 \quad (3.289)$$

$$2) \quad A(1, z_2) \neq 0 \quad \text{for} \quad |z_2| \geq 1 \quad (3.290)$$

$$3) \quad A(e^{j\omega_1}, e^{j\omega_2}) \neq 0 \quad \text{for all} \quad \omega_1, \omega_2. \quad (3.291)$$

Similarly, ASHP filters are also tested by the rules determined in [62] and [63]. For the type-1 ASHP filters obtained from the horizontal PEFs, the filter is stable in *bounded-input bounded-output* (BIBO) sense, if the transfer function satisfies

$$1) \quad A(z_1, \infty) \neq 0 \quad \text{for} \quad |z_1| \geq 1 \quad (3.292)$$

$$2) \quad A(e^{j\omega_1}, z_2) \neq 0 \quad \text{for} \quad |z_2| \geq 1, \text{ and for all } \omega_1. \quad (3.293)$$

3.7 Complexity Comparisons of the Proposed Method

We compared the proposed method with the alternative structures [11, 14, 16–19, 29, 32, 35, 36, 39, 40, 42, 55, 57, 59, 64–67], both in terms of conceptual background and in terms of complexity. Our initial motivation is to present a general 2-D lattice modeling technique which is capable of simultaneously providing all possible causal and noncausal models to obtain superior spectral estimation accuracy. Hence, this new formulation for constructing orthogonal backward prediction error vectors is quite different from the other attempts in [39], [40], and [66].

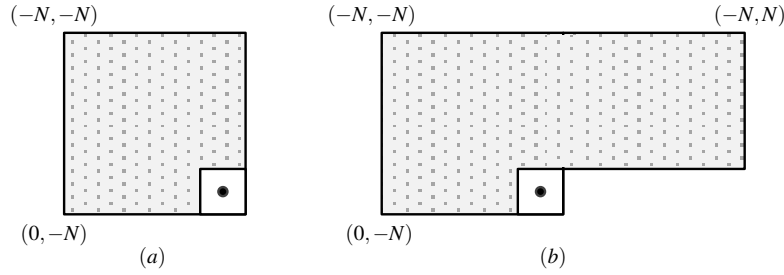


Figure 3.36: The models used for complexity comparisons. (a) QP model with $N^2 + 2N$ points in the PSR, $S_{k,l}^{(N,N)}$, (b) ASHP model with $2(N^2 + N)$ points in the PSR, $S_{k,l}^{(N,2N)}$.

The first-quadrant QP model shown in Figure 3.36(a), the complexity of the proposed approach in this thesis is given in Table 3.2. For comparison, it is also given the complexity of the orthogonal 2-D lattice filter [39], the four-field 2-D lattice filter [40] and the multichannel 2-D lattice filter [14, 65]. As shown in Table 3.2, our 2-D lattice filter requires the fewest number of lattice parameters for > 2 . Furthermore, the number of lattice sections for our 2D lattice filter is on the order of only 2.

Moreover, we consider an ASHP model shown in Figure 3.36(b). For this model, it is presented the complexities for the orthogonal 2-D lattice filter [39, 57], the optimum 2-D lattice filter [42] and the proposed 2-D lattice filter in Table 3.2.

From the Table 3.2 which shows the complexities of the existing 2-D lattice filters and the proposed one, it can deduced that the new introduced 2-D lattice filter is superior in terms of the number of lattice parameter as well as the number of lattice sections. It is interesting to note that our method computes all possible QP and ASHP models

Table 3.2: Complexity Comparisons for the QP Model 2-D Lattice Filters with $N^2 + 2N$ Points in the PSR, $S_{k,l}^{(N,N)}$ and for the ASHP Model 2-D Lattice Filter with $2(N^2 + N)$ points in the PSR, $S_{k,l}^{(N,2N)}$.

QP MODELS	Number of Lattice Sections	Number of Lattice Parameters
Orthogonal [39]	$N^3 + 2N^2 - N$	$2N^3 + 4N^2 + 2N$
Four-Field [40]	$2N^2$	$4N^3 + 2N$
Multichannel [14, 65]	$N^3 + N^2 + N - 1$	$2N^3 + 2N^2 + 2N - 2$
Proposed method	$2N^2 + 3N - 2$	$\frac{4}{3}N^3 + \frac{5}{2}N^2 + \frac{19}{6}N$
ASHP MODELS		
Orthogonal [39, 57]	$4N^3 + 4N^2 + N$	$8N^3 + 8N^2 + 2N$
Optimum [42]	$2N^2 + 4N - 4$	$\frac{4}{3}(4N^3 + 5N^2 + N)$
Proposed method	$4N^2 + 3N - 1$	$\frac{1}{3}(7N^3 + 9N^2 + 11N)$

simultaneously. However, all other approaches [11, 14, 16–19, 29, 32, 35, 36, 39, 40, 42, 55, 57, 59, 64–67] computes only a single 2-D prediction model.

4. RELATIONSHIP OF THE LATTICE MODEL PARAMETERS TO THE TRANSFER FUNCTION

In the above sections, we introduced a new 2-D lattice filter modeling and presented how to obtain QP, ASHP, and NC half-plane lattice prediction models. Now, an algorithm will be developed for the construction of the prediction error transfer functions from the 2-D lattice parameters calculated by using the proposed 2-D lattice structure. For a $(p-1, q-1)$ th PSR shown in Figure 2.10, we can write the forward/backward vertical and horizontal auxiliary PEFs in terms of the 2-D z -transform for $i = 1, \dots, p$ and $j = 1, \dots, q$

$$E_i^{(p-1,q)}(z_1, z_2) = A_i^{(p-1,q)}(z_1, z_2)Y(z_1, z_2) \quad (4.1)$$

$$\tilde{E}_i^{(p-1,q)}(z_1, z_2) = \tilde{A}_i^{(p-1,q)}(z_1, z_2)Y(z_1, z_2) \quad (4.2)$$

$$R_j^{(p,q-1)}(z_1, z_2) = B_j^{(p,q-1)}(z_1, z_2)Y(z_1, z_2) \quad (4.3)$$

$$\tilde{R}_j^{(p,q-1)}(z_1, z_2) = \tilde{B}_j^{(p,q-1)}(z_1, z_2)Y(z_1, z_2) \quad (4.4)$$

where $Y(z_1, z_2)$ is the z -transform of the input data. The transfer functions of the forward PEFs, $A_i^{(p-1,q)}(z_1, z_2)$ and $B_j^{(p,q-1)}(z_1, z_2)$ can be represented in terms of the coefficient matrices $\mathbf{A}_i^{(p-1,q)}$ and $\mathbf{B}_j^{(p,q-1)}$, with the order of $p \times (q+1)$ and $(p+1) \times q$, respectively. Similarly, the transfer functions of the backward PEFs, $\tilde{A}_i^{(p-1,q)}(z_1, z_2)$ and $\tilde{B}_j^{(p,q-1)}(z_1, z_2)$ can be written in terms of the coefficient matrices $\tilde{\mathbf{A}}_i^{(p-1,q)}$ and $\tilde{\mathbf{B}}_j^{(p,q-1)}$, with the same order as defined for the forward PEFs. The transfer functions of the forward PEFs can be given as,

$$\begin{aligned} A_i^{(p-1,q)}(z_1, z_2) &= \mathbf{\Delta}^{(p-1)^T}(z_1) \mathbf{A}_i^{(p-1,q)} \mathbf{\Delta}^{(q)}(z_2) \\ B_j^{(p,q-1)}(z_1, z_2) &= \mathbf{\Delta}^{(p)^T}(z_1) \mathbf{B}_j^{(p,q-1)} \mathbf{\Delta}^{(q-1)}(z_2) \end{aligned} \quad (4.5)$$

where the vectors $\mathbf{\Delta}^{(m)^T}(z_x) = [1 \quad z_x^{-1} \quad \dots \quad z_x^{-m}]^T$, for $x = 1, 2$ defines the position of the sample points used in the PSR. Owing to the prediction of the neighboring points of the PSR $S_{k,l}^{(p-1,q-1)}$, $a_i^{(p-1,q)}(i, 1) = b_j^{(p,q-1)}(1, j) = 1$ and all other entries of the first column and the first row of $\mathbf{A}_i^{(p-1,q)}$ and $\mathbf{B}_j^{(p,q-1)}$ are zero, respectively. For example,

if we write a transfer function for $i = 1$, it will be that

$$A_1^{(p-1,q)}(z_1, z_2) = \begin{bmatrix} 1 & z_1^{-1} & \cdots & z_1^{-(p-1)} \end{bmatrix} \mathbf{A}_1^{(p-1,q)} \begin{bmatrix} 1 & z_2^{-1} & \cdots & z_2^{-q} \end{bmatrix}^T. \quad (4.6)$$

where the coefficient matrix $A_1^{(p-1,q)}(z_1, z_2)$ is

$$A_1^{(p-1,q)} = \begin{bmatrix} 1 & a_1^{(p-1,q)}(1,2) & \cdots & a_1^{(p-1,q)}(1,q) \\ 0 & a_1^{(p-1,q)}(2,2) & \cdots & a_1^{(p-1,q)}(2,q) \\ 0 & a_1^{(p-1,q)}(3,2) & \cdots & a_1^{(p-1,q)}(3,q) \\ \vdots & \vdots & \ddots & \vdots \\ 0 & a_1^{(p-1,q)}(p-1,2) & \cdots & a_1^{(p-1,q)}(p-1,q) \end{bmatrix}. \quad (4.7)$$

The backward vertical and horizontal PEFs can be expressed in terms of the forward ones, and it can be written as

$$\begin{aligned} \tilde{A}_i^{(p-1,q)}(z_1, z_2) &= z_1^{-(p-1)} z_2^{-q} A_i^{(p-1,q)*} \left(\frac{1}{z_1^*}, \frac{1}{z_2^*} \right) \\ \tilde{B}_j^{(p,q-1)}(z_1, z_2) &= z_1^{-p} z_2^{-(q-1)} B_j^{(p,q-1)*} \left(\frac{1}{z_1^*}, \frac{1}{z_2^*} \right). \end{aligned} \quad (4.8)$$

Thus, the backward coefficient matrices $\tilde{\mathbf{A}}_i^{(p-1,q)}$ and $\tilde{\mathbf{B}}_j^{(p,q-1)}$ for the backward PEFs can be obtained by permuting the forward coefficient matrices $\mathbf{A}_i^{(p-1,q)}$ and $\mathbf{B}_j^{(p,q-1)}$, respectively. Indeed,

$$\begin{aligned} \tilde{\mathbf{A}}_i^{(p-1,q)} &= \mathbf{J}_p \mathbf{A}_i^{(p-1,q)*} \mathbf{J}_{q+1} \\ \tilde{\mathbf{B}}_j^{(p,q-1)} &= \mathbf{J}_{p+1} \mathbf{B}_j^{(p,q-1)*} \mathbf{J}_q \end{aligned} \quad (4.9)$$

where $\mathbf{J}_n$ is the $n \times n$ exchange matrix. (4.9) indicates the row and column reversal of the matrix.

Let us show how to get (4.9). First we need to find $A_i^{(p-1,q)*} \left(\frac{1}{z_1^*}, \frac{1}{z_2^*} \right)$. For that, from the equation (4.5), we can determine

$$A_i^{(p-1,q)*}(z_1, z_2) = \mathbf{\Delta}^{(p-1)T} \left(\frac{1}{z_1} \right) \mathbf{A}_i^{(p-1,q)*} \mathbf{\Delta}^{(q)} \left(\frac{1}{z_2} \right). \quad (4.10)$$

By substituting (4.10) into (4.8), we get

$$\begin{aligned} \tilde{A}_i^{(p-1,q)}(z_1, z_2) &= z_1^{-(p-1)} z_2^{-q} A_i^{(p-1,q)*} \left(\frac{1}{z_1^*}, \frac{1}{z_2^*} \right) \\ &= z_1^{-(p-1)} \mathbf{\Delta}^{(p-1)T} \left(\frac{1}{z_1} \right) \mathbf{A}_i^{(p-1,q)*} z_2^{-q} \mathbf{\Delta}^{(q)} \left(\frac{1}{z_2} \right). \end{aligned} \quad (4.11)$$

And with the following definitions,

$$\begin{aligned} z_1^{-(p-1)} \mathbf{\Delta}^{(p-1)T} \left(\frac{1}{z_1} \right) &= \mathbf{\Delta}^{(p-1)T}(z_1) \mathbf{J}_p \\ z_2^{-q} \mathbf{\Delta}^{(q)} \left(\frac{1}{z_2} \right) &= \mathbf{J}_{q+1} \mathbf{\Delta}^{(q)}(z_2) \end{aligned}$$

Table 4.1: Augmentation and Shifting Operations for the Coefficient Matrix $\mathbf{C}_i^{(M,N)}$.

	Augmentation	Shifting
horizontal	$\hat{\mathbf{C}}_{i,h}^{(M,N)} = \begin{bmatrix} \mathbf{C}_i^{(M,N)} & \\ & \mathbf{0}_{N+1}^T \end{bmatrix}$	$\mathbf{C}_{i,z_1}^{(M,N)} = \begin{bmatrix} \mathbf{0}_{N+1}^T \\ \mathbf{C}_i^{(M,N)} \end{bmatrix}$
vertical	$\hat{\mathbf{C}}_{i,v}^{(M,N)} = \begin{bmatrix} \mathbf{C}_i^{(M,N)} & \mathbf{0}_{M+1} \end{bmatrix}$	$\mathbf{C}_{i,z_2}^{(M,N)} = \begin{bmatrix} \mathbf{0}_{M+1} & \mathbf{C}_i^{(M,N)} \end{bmatrix}$

we can write (4.11) as

$$\tilde{A}_i^{(p-1,q)}(z_1, z_2) = \mathbf{\Delta}^{(p-1)T}(z_1) \mathbf{J}_p \mathbf{A}_i^{(p-1,q)*} \mathbf{J}_{q+1} \mathbf{\Delta}^{(q)}(z_2). \quad (4.12)$$

It is clearly seen from (4.12),

$$\tilde{\mathbf{A}}_i^{(p-1,q)} = \mathbf{J}_p \mathbf{A}_i^{(p-1,q)*} \mathbf{J}_{q+1}. \quad (4.13)$$

It is known that in order to calculate the coefficient vector of the $(N+1)$ th order 1-D lattice filter, $\mathbf{a}^{(N+1)}$, the well-known Levinson algorithm [60] requires the augmentation of the (N) th order forward coefficient vector, $\hat{\mathbf{a}}^{(N)} = \begin{bmatrix} \mathbf{a}^{(N)T} & 0 \end{bmatrix}^T$ and the delayed backward coefficient vector, $\tilde{\mathbf{a}}_z^{(N)} = \begin{bmatrix} 0 & \tilde{\mathbf{a}}^{(N)T} \end{bmatrix}^T$. But, for the determination of the 2-D coefficient matrices defined in (4.5) and (4.9), we must perform these "augmentation" and "shifting" operations for both vertical and horizontal directions. Therefore, we can define four different augmented and shifted matrices as depicted in Table 4.1, for the (M, N) th order coefficient matrix, $\mathbf{C}_i^{(M,N)}$. The horizontally augmented matrices $\hat{\mathbf{C}}_{i,h}^{(M,N)}$ and $\hat{\mathbf{C}}_{i,v}^{(M,N)}$ are obtained by adding one row and column of zeros, respectively. On the other hand, the horizontally and vertically shifted coefficient matrices $\mathbf{C}_{i,z_1}^{(M,N)}$ and $\mathbf{C}_{i,z_2}^{(M,N)}$ are obtained by shifting the whole matrix one row and column as demonstrated in Table 4.1, respectively.

4.1 2-D Levinson Algorithm by using Matrices

The 2-D coefficient matrices of the transfer functions of the auxiliary PEFs can be computed from the 2-D lattice coefficients using the following procedure:

1. Set $(p, q) = (0, 0)$. Define the initial zeroth order stage coefficient matrices as $\mathbf{A}_1^{(0,0)} = \tilde{\mathbf{A}}_1^{(0,0)} = [1]$ and $\mathbf{B}_1^{(0,0)} = \tilde{\mathbf{B}}_1^{(0,0)} = [1]$.

2. At the first order stage, given the lattice parameters $k_{e_1}^{(0,1)}$, for $q = 1$, and $k_{r_1}^{(1,0)}$, for $p = 1$, calculate the coefficient matrices

$$\begin{aligned}
\mathbf{A}_1^{(0,1)} &= \mathbf{A}_{1,h}^{(0,0)} - k_{e_1}^{(0,1)*} \mathbf{A}_{l,z_1}^{(0,0)} \\
&= \begin{bmatrix} 1 & \vdots & 0 \end{bmatrix} - k_{e_1}^{(0,1)*} \begin{bmatrix} 0 & \vdots & 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & -k_{e_1}^{(0,1)*} \end{bmatrix}
\end{aligned} \tag{4.14}$$

and

$$\begin{aligned}
\mathbf{B}_1^{(1,0)} &= \mathbf{B}_{1,v}^{(0,0)} - k_{r_1}^{(1,0)*} \mathbf{B}_{l,z_2}^{(0,0)} \\
&= \begin{bmatrix} 1 \\ \vdots \\ 0 \end{bmatrix} - k_{r_1}^{(1,0)*} \begin{bmatrix} 0 \\ \vdots \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 \\ -k_{r_1}^{(1,0)*} \end{bmatrix}
\end{aligned} \tag{4.15}$$

Determine $\tilde{\mathbf{A}}_1^{(0,1)}$ and $\tilde{\mathbf{B}}_1^{(1,0)}$ using (4.9).

3. Incrementing p by unity, from the last calculated $\mathbf{A}_i^{(p-1,q)}$, $\mathbf{B}_j^{(p,q-1)}$, and the reflection coefficient vectors, compute the horizontally augmented stage coefficient matrices, using (2.22), (2.32) and Table 4.1 as for $i = 1, \dots, p$ and $j = 1, \dots, q$

$$\begin{aligned}
\mathbf{B}_j^{(p+1,q-1)} &= \hat{\mathbf{B}}_{j,h}^{(p,q-1)} - \sum_{l=1}^q k_{r_j}^{(p+1,q-1)*} (l) \tilde{\mathbf{B}}_{l,z_1}^{(p,q-1)} \\
\mathbf{A}_i^{(p,q)} &= \hat{\mathbf{A}}_{i,h}^{(p-1,q)} - \sum_{l=1}^q k_{e_i}^{(p,q)*} (l) \tilde{\mathbf{B}}_{l,z_2}^{(p,q-1)} \\
\mathbf{A}_{p+1}^{(p,q)} &= \mathbf{A}_{p,z_1}^{(p-1,q)} - \sum_{l=1}^q k_{e_{p+1}}^{(p,q)*} (l) \mathbf{B}_{l,z_2}^{(p,q-1)}.
\end{aligned} \tag{4.16}$$

Determine $\tilde{\mathbf{A}}_i^{(p,q)}$ and $\tilde{\mathbf{B}}_j^{(p+1,q-1)}$ by column and row reversal as in (4.9).

4. Incrementing q by unity, vertically augmented stage coefficient matrices, using (2.47) and (2.55) are given as for $i = 1, \dots, p$ and $j = 1, \dots, q$

$$\begin{aligned}
\mathbf{A}_i^{(p-1,q+1)} &= \hat{\mathbf{A}}_{i,v}^{(p-1,q)} - \sum_{l=1}^p k_{e_i}^{(p-1,q+1)*} (l) \tilde{\mathbf{A}}_{l,z_2}^{(p-1,q)} \\
\mathbf{B}_j^{(p,q)} &= \hat{\mathbf{B}}_{i,v}^{(p,q-1)} - \sum_{l=1}^p k_{r_i}^{(p,q)*} (l) \tilde{\mathbf{A}}_{l,z_1}^{(p-1,q)} \\
\mathbf{B}_{q+1}^{(p,q)} &= \mathbf{B}_{q,z_2}^{(p,q-1)} - \sum_{l=1}^p k_{r_{q+1}}^{(p,q)*} (l) \mathbf{A}_{l,z_1}^{(p-1,q)}.
\end{aligned} \tag{4.17}$$

Determine $\tilde{\mathbf{A}}_i^{(p-1,q+1)}$ and $\tilde{\mathbf{B}}_j^{(p,q)}$ using (4.9).

5. The foregoing process is continued stage by stage depending on the selected augmentation direction by going to **Step 3** or **Step 4**.

After the calculation of the coefficient matrices of the transfer functions for the auxiliary PEF filters, similarly the transfer functions for the QP, ASHP, and NC prediction lattice models can be obtained by using (3.281) and (3.285). In the following sections, how to determine the transfer functions for the QP, ASHP, and NC half-plane prediction lattice models will be elaborated.

4.1.1 Coefficient matrices at the initial stage

Now, we will verify our proposed algorithm by giving an example. We have already defined the coefficient matrices for the initial case given in (4.14) and (4.15). Let us find the same result by using the error propagation equations. The vertical forward PEFs equation is

$$e_1^{(0,1)}(k, l) = e_1^{(0,0)}(k, l) - k_{e_1}^{(0,1)*} e_1^{(0,0)}(k, l-1) \quad (4.18)$$

where $e_1^{(0,0)}(k, l) = y(k, l)$ at the initial stage. By getting z -transform, we have

$$\begin{aligned} E_1^{(0,1)}(z_1, z_2) &= Y(z_1, z_2) - k_{e_1}^{(0,1)*} z_2^{-1} Y(z_1, z_2) \\ &= \underbrace{\left[1 - k_{e_1}^{(0,1)*} z_2^{-1} \right]}_{A_1^{(0,1)}(z_1, z_2)} Y(z_1, z_2). \end{aligned} \quad (4.19)$$

From (4.19), $A_1^{(0,1)}(z_1, z_2)$ is written as

$$\begin{aligned} A_1^{(0,1)}(z_1, z_2) &= \left[1 - k_{e_1}^{(0,1)*} z_2^{-1} \right] \\ &= \underbrace{\begin{bmatrix} 1 & -k_{e_1}^{(0,1)*} \end{bmatrix}}_{\mathbf{A}_1^{(0,1)}} \underbrace{\begin{bmatrix} 1 \\ z_2^{-1} \end{bmatrix}}_{\mathbf{\Delta}^{(1)}(z_2)} = \mathbf{A}_1^{(0,1)} \mathbf{\Delta}^{(1)}(z_2) \end{aligned} \quad (4.20)$$

where $\mathbf{\Delta}^{(0)}(z_1) = [1]$.

Coefficient matrix for the vertical forward PEFs is

$$A_1^{(0,1)} = \begin{bmatrix} 1 & -k_{e_1}^{(0,1)*} \end{bmatrix}. \quad (4.21)$$

At the same way, we can find

$$\begin{aligned}
\tilde{E}_1^{(0,1)}(z_1, z_2) &= z_2^{-1}Y(z_1, z_2) - k_{e_1}^{(0,1)}Y(z_1, z_2) \\
&= \left[z_2^{-1} - k_{e_1}^{(0,1)*} \right] Y(z_1, z_2) \\
&= \underbrace{\begin{bmatrix} -k_{e_1}^{(0,1)} & 1 \end{bmatrix}}_{\tilde{\mathbf{A}}_1^{(0,1)}} \underbrace{\begin{bmatrix} 1 \\ z_2^{-1} \end{bmatrix}}_{\mathbf{\Delta}^{(1)}(z_2)} Y(z_1, z_2)
\end{aligned} \tag{4.22}$$

From (4.22), it is clearly seen that

$$\tilde{A}_1(z_1, z_2) = \tilde{\mathbf{A}}_1^{(0,1)} \mathbf{\Delta}^{(1)}(z_2). \tag{4.23}$$

Coefficient matrix for the vertical backward PEFs is

$$\tilde{A}_1^{(0,1)} = \begin{bmatrix} -k_{e_1}^{(0,1)} & 1 \end{bmatrix}. \tag{4.24}$$

By using (4.9), we can also determine $\tilde{\mathbf{A}}_1^{(0,1)}$ from (4.14) as

$$\begin{aligned}
\tilde{\mathbf{A}}_1^{(0,1)} &= \mathbf{J}_1 \mathbf{A}_1^{(0,1)*} \mathbf{J}_2 \\
&= [1] \begin{bmatrix} 1 & -k_{e_1}^{(0,1)} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \\
&= \begin{bmatrix} -k_{e_1}^{(0,1)} & 1 \end{bmatrix}.
\end{aligned} \tag{4.25}$$

As it is seen that, indeed, the forward coefficient matrix (4.14) is equal to (4.21) and the backward coefficient matrix (4.25) is also equal to (4.24).

Similarly, we can calculate the coefficient matrices for the horizontal forward and backward PEFs. For the horizontal forward PEFs, the z -transform can be defined as

$$\begin{aligned}
R_1^{(1,0)}(z_1, z_2) &= Y(z_1, z_2) - k_{r_1}^{(1,0)*} z_1^{-1} Y(z_1, z_2) \\
&= \underbrace{\left[1 - k_{r_1}^{(1,0)*} z_1^{-1} \right]}_{\mathbf{B}_1^{(1,0)}(z_1, z_2)} Y(z_1, z_2) \\
&= \underbrace{\begin{bmatrix} 1 & z_1^{-1} \end{bmatrix}}_{\mathbf{\Delta}^{(1)T}(z_1)} \underbrace{\begin{bmatrix} 1 \\ -k_{r_1}^{(1,0)*} \end{bmatrix}}_{\mathbf{B}_1^{(1,0)}} Y(z_1, z_2) \\
&= \underbrace{\mathbf{\Delta}^{(1)T}(z_1) \mathbf{B}_1^{(1,0)}}_{\mathbf{B}_1^{(1,0)}(z_1, z_2)} Y(z_1, z_2)
\end{aligned} \tag{4.26}$$

$$\tag{4.27}$$

where $\mathbf{\Delta}^{(0)}(z_2) = [1]$.

Coefficient matrix for the horizontal forward PEFs is

$$B_1^{(1,0)} = \begin{bmatrix} 1 \\ -k_{r_1}^{(1,0)*} \end{bmatrix}. \quad (4.28)$$

For the horizontal backward PEFs, the z -transform can be written as

$$\begin{aligned} \tilde{R}_1^{(1,0)}(z_1, z_2) &= z_1^{-1}Y(z_1, z_2) - k_{r_1}^{(1,0)}Y(z_1, z_2) \\ &= \underbrace{\begin{bmatrix} z_1^{-1} - k_{r_1}^{(1,0)} \end{bmatrix}}_{\tilde{B}_1^{(1,0)}(z_1, z_2)} Y(z_1, z_2) \\ &= \underbrace{\begin{bmatrix} 1 & z_1^{-1} \end{bmatrix}}_{\Delta^{(1)T}(z_1)} \underbrace{\begin{bmatrix} -k_{r_1}^{(1,0)} \\ 1 \end{bmatrix}}_{\tilde{\mathbf{B}}_1^{(1,0)}} Y(z_1, z_2). \end{aligned} \quad (4.29)$$

Coefficient matrix for the horizontal backward PEFs is

$$\tilde{B}_1^{(1,0)} = \begin{bmatrix} -k_{r_1}^{(1,0)} \\ 1 \end{bmatrix}. \quad (4.30)$$

By using (4.9), we can determine $\tilde{\mathbf{B}}_1^{(1,0)}$ from (4.15) as

$$\begin{aligned} \tilde{\mathbf{B}}_1^{(1,0)} &= \mathbf{J}_2 \mathbf{B}_1^{(1,0)*} \mathbf{J}_1 \\ &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ -k_{r_1}^{(1,0)} \end{bmatrix} [1] \\ &= \begin{bmatrix} -k_{r_1}^{(1,0)} \\ 1 \end{bmatrix}. \end{aligned} \quad (4.31)$$

As a consequence, the coefficient matrices we calculated from the z -transform of the error propagation equations are equal to the ones computed by using the proposed algorithm.

Now we can go one step further and calculate the coefficient matrices after applying a horizontal augmentation to the initial stage. This augmentation case will be studied in the next section.

4.1.2 Coefficient matrices of the horizontal augmentation after the initial stage

In the above section, we found the coefficient matrices for the initial case. Herein, we will find the coefficient matrices of all auxiliary PEFs in case of horizontal augmentation. The coefficient matrices will be calculated by using our proposed algorithm. In **Step 3**, the equations are given for updated horizontal and vertical, and the newly initiated vertical PEFs.

We know that in case of horizontal augmentation the number of vertical forward/backward PEFs increases one unity. In other words, at the initial stage, we had one pair vertical forward/backward PEFs, after the horizontal augmentation, the number of vertical forward/backward PEFs increases and we have two pairs of them. But, there will be only one pair of horizontal forward/backward PEFs.

For the horizontal augmentation, by increasing p unitary from the initial stage, we will have the following error propagation equation for the auxiliary horizontal forward PEFs,

$$r_1^{(2,0)}(k, l) = r_1^{(1,0)}(k, l) - k_{r_1}^{(2,0)*} \tilde{r}_1^{(1,0)}(k-1, l) \quad (4.32)$$

and by getting z -transform of (4.32), we can write

$$R_1^{(2,0)}(z_1, z_2) = R_1^{(1,0)}(z_1, z_2) - k_{r_1}^{(2,0)*} z_1^{-1} \tilde{R}_1^{(1,0)}(z_1, z_2) \quad (4.33)$$

If we substitute (4.26) and (4.29) into (4.33), then the equation will be

$$\begin{aligned} R_1^{(2,0)}(z_1, z_2) &= \left(1 - k_{r_1}^{(1,0)*} z_1^{-1}\right) Y(z_1, z_2) - k_{r_1}^{(2,0)*} z_1^{-1} \left(z_1^{-1} - k_{r_1}^{(1,0)}\right) Y(z_1, z_2) \\ &= \left[\left(1 - k_{r_1}^{(1,0)*} z_1^{-1}\right) - k_{r_1}^{(2,0)*} z_1^{-1} \left(z_1^{-1} - k_{r_1}^{(1,0)}\right)\right] Y(z_1, z_2) \\ &= \underbrace{\left[1 + \left(k_{r_1}^{(1,0)} k_{r_1}^{(2,0)*} - k_{r_1}^{(1,0)*}\right) z_1^{-1} - k_{r_1}^{(2,0)*} z_1^{-2}\right]}_{B_1^{(2,0)}(z_1, z_2)} Y(z_1, z_2). \end{aligned} \quad (4.34)$$

Hence, we can write that

$$B_1^{(2,0)}(z_1, z_2) = \underbrace{\begin{bmatrix} 1 & z_1^{-1} & z_1^{-2} \end{bmatrix}}_{\Delta^{(2)}(z_1)} \underbrace{\begin{bmatrix} 1 \\ k_{r_1}^{(1,0)} k_{r_1}^{(2,0)*} - k_{r_1}^{(1,0)*} \\ -k_{r_1}^{(2,0)*} \end{bmatrix}}_{\mathbf{B}_1^{(2,0)}} \underbrace{\begin{bmatrix} 1 \end{bmatrix}}_{\Delta^{(0)}(z_2)} \quad (4.35)$$

The coefficient matrix for the horizontal forward PEF, $\mathbf{B}_1^{(2,0)}$ is

$$\mathbf{B}_1^{(2,0)} = \begin{bmatrix} 1 \\ k_{r_1}^{(1,0)} k_{r_1}^{(2,0)*} - k_{r_1}^{(1,0)*} \\ -k_{r_1}^{(2,0)*} \end{bmatrix}. \quad (4.36)$$

After the horizontal augmentation, the error propagation equation for the auxiliary horizontal backward PEFs will be as follows

$$\tilde{r}_1^{(2,0)}(k, l) = \tilde{r}_1^{(1,0)}(k-1, l) - k_{r_1}^{(2,0)} r_1^{(1,0)}(k-1, l) \quad (4.37)$$

and the z -transform of (4.37) is

$$\tilde{R}_1^{(2,0)}(z_1, z_2) = z_1^{-1} \tilde{R}_1^{(1,0)}(z_1, z_2) - k_{r_1}^{(2,0)} R_1^{(1,0)}(z_1, z_2) \quad (4.38)$$

If we substitute (4.29) and (4.26) into (4.38), then we can write

$$\begin{aligned} \tilde{R}_1^{(2,0)}(z_1, z_2) &= z_1^{-1} \left(z_1^{-1} - k_{r_1}^{(1,0)} \right) Y(z_1, z_2) - k_{r_1}^{(2,0)} \left(1 - k_{r_1}^{(1,0)*} z_1^{-1} \right) Y(z_1, z_2) \\ &= \underbrace{\left[z_1^{-1} \left(z_1^{-1} - k_{r_1}^{(1,0)} \right) - k_{r_1}^{(2,0)} \left(1 - k_{r_1}^{(1,0)*} z_1^{-1} \right) \right]}_{\tilde{B}_1^{(2,0)}(z_1, z_2)} Y(z_1, z_2). \end{aligned} \quad (4.39)$$

Thus, we can write that

$$\tilde{B}_1^{(2,0)}(z_1, z_2) = \underbrace{\begin{bmatrix} 1 & z_1^{-1} & z_1^{-2} \end{bmatrix}}_{\mathbf{\Delta}^{(2)}(z_1)} \underbrace{\begin{bmatrix} -k_{r_1}^{(2,0)} \\ k_{r_1}^{(1,0)*} k_{r_1}^{(2,0)} - k_{r_1}^{(1,0)} \\ 1 \end{bmatrix}}_{\tilde{\mathbf{B}}_1^{(2,0)}} \underbrace{[1]}_{\mathbf{\Delta}^{(0)}(z_2)}. \quad (4.40)$$

The coefficient matrix for the horizontal backward PEF, $\tilde{\mathbf{B}}_1^{(2,0)}$ is

$$\tilde{\mathbf{B}}_1^{(2,0)} = \begin{bmatrix} -k_{r_1}^{(2,0)} \\ k_{r_1}^{(1,0)*} k_{r_1}^{(2,0)} - k_{r_1}^{(1,0)} \\ 1 \end{bmatrix}. \quad (4.41)$$

Similarly, by using z -transforms of the updated and newly initiated vertical forward/backward PEFs, the coefficient matrices can be found easily. For the updated vertical forward/backward PEFs, the lattice equations are given as

$$e_1^{(1,1)}(k, l) = e_1^{(0,1)}(k, l) - k_{e_1}^{(1,1)*} \tilde{r}_1^{(1,0)}(k, l-1) \quad (4.42)$$

$$\tilde{e}_1^{(1,1)}(k, l) = \tilde{e}_1^{(0,1)}(k-1, l) - k_{e_1}^{(1,1)} r_1^{(1,0)}(k, l) \quad (4.43)$$

By getting z -transforms of (4.42) and (4.43), we can write

$$E_1^{(1,1)}(z_1, z_2) = E_1^{(1,1)}(z_1, z_2) - k_{e_1}^{(1,1)*} z_2^{-1} \tilde{R}_1^{(1,0)}(z_1, z_2) \quad (4.44)$$

$$\tilde{E}_1^{(1,1)}(z_1, z_2) = z_1^{-1} \tilde{E}_1^{(1,1)}(z_1, z_2) - k_{e_1}^{(1,1)} R_1^{(1,0)}(z_1, z_2) \quad (4.45)$$

If we substitute (4.19) and (4.29) into (4.44), then the equation will be

$$\begin{aligned} E_1^{(1,1)}(z_1, z_2) &= \left[\left(1 - k_{e_1}^{(0,1)*} z_2^{-1} \right) - k_{e_1}^{(1,1)*} z_2^{-1} \left(z_1^{-1} - k_{r_1}^{(1,0)} \right) \right] Y(z_1, z_2) \\ &= \underbrace{\left[1 + \left(k_{e_1}^{(1,1)*} k_{r_1}^{(1,0)} - k_{e_1}^{(1,0)*} \right) z_2^{-1} - k_{e_1}^{(1,1)*} z_1^{-1} z_2^{-1} \right]}_{A_1^{(1,1)}(z_1, z_2)} Y(z_1, z_2) \end{aligned} \quad (4.46)$$

From (4.46), we can write that

$$A_1^{(1,1)}(z_1, z_2) = \underbrace{\begin{bmatrix} 1 & z_1^{-1} \end{bmatrix}}_{\Delta^{(1)}(z_1)} \underbrace{\begin{bmatrix} 1 & k_{e_1}^{(1,1)*} k_{r_1}^{(1,0)} - k_{e_1}^{(1,0)*} \\ 0 & k_{e_1}^{(1,1)*} \end{bmatrix}}_{\mathbf{A}_1^{(1,1)}} \underbrace{\begin{bmatrix} 1 \\ z_2^{-1} \end{bmatrix}}_{\Delta^{(1)}(z_2)}. \quad (4.47)$$

The coefficient matrix for the first vertical forward PEFs $\mathbf{A}_1^{(1,1)}$ is

$$\mathbf{A}_1^{(1,1)} = \begin{bmatrix} 1 & k_{e_1}^{(1,1)*} k_{r_1}^{(1,0)} - k_{e_1}^{(1,0)*} \\ 0 & k_{e_1}^{(1,1)*} \end{bmatrix}. \quad (4.48)$$

For finding coefficient matrix of the vertical backward PEFs, (4.22) and (4.26) are substituted into (4.45), and the equation will be written as

$$\begin{aligned} \tilde{E}_1^{(1,1)}(z_1, z_2) &= \left[z_1^{-1} \left(z_2^{-1} - k_{e_1}^{(0,1)} \right) - k_{e_1}^{(1,1)} \left(1 - k_{r_1}^{(1,0)*} z_1^{-1} \right) \right] Y(z_1, z_2) \\ &= \underbrace{\left[z_1^{-1} z_2^{-1} + \left(k_{e_1}^{(1,1)} k_{r_1}^{(1,0)*} - k_{e_1}^{(0,1)} \right) z_1^{-1} - k_{e_1}^{(1,1)} \right]}_{\tilde{A}_1^{(1,1)}(z_1, z_2)} Y(z_1, z_2). \end{aligned} \quad (4.49)$$

And from (4.49), we can write that

$$\tilde{A}_1^{(1,1)}(z_1, z_2) = \underbrace{\begin{bmatrix} 1 & z_1^{-1} \end{bmatrix}}_{\Delta^{(1)}(z_1)} \underbrace{\begin{bmatrix} -k_{e_1}^{(1,1)} & 0 \\ k_{e_1}^{(1,1)} k_{r_1}^{(1,0)*} - k_{e_1}^{(0,1)} & 1 \end{bmatrix}}_{\tilde{\mathbf{A}}_1^{(1,1)}} \underbrace{\begin{bmatrix} 1 \\ z_2^{-1} \end{bmatrix}}_{\Delta^{(1)}(z_2)}. \quad (4.50)$$

The coefficient matrix for the first vertical backward PEFs $\tilde{\mathbf{A}}_1^{(1,1)}$ is

$$\tilde{\mathbf{A}}_1^{(1,1)} = \begin{bmatrix} -k_{e_1}^{(1,1)} & 0 \\ k_{e_1}^{(1,1)} k_{r_1}^{(1,0)*} - k_{e_1}^{(0,1)} & 1 \end{bmatrix}. \quad (4.51)$$

Now, it is turn to find the coefficient matrices for the newly initiated vertical forward/backward PEFs. For these newly initiated ones which are the second vertical forward/backward PEFs, the error propagation equations are given as

$$e_2^{(1,1)}(k, l) = e_1^{(0,1)}(k-1, l) - k_{e_2}^{(1,1)*} r_1^{(1,0)}(k, l-1) \quad (4.52)$$

$$\tilde{e}_2^{(1,1)}(k, l) = \tilde{e}_1^{(0,1)}(k, l) - k_{e_2}^{(1,1)} \tilde{r}_1^{(1,0)}(k, l). \quad (4.53)$$

By getting z -transforms of (4.52) and (4.53), we can write

$$E_2^{(1,1)}(z_1, z_2) = z_1^{-1} E_1^{(1,1)}(z_1, z_2) - k_{e_2}^{(1,1)*} z_2^{-1} R_1^{(1,0)}(z_1, z_2) \quad (4.54)$$

$$\tilde{E}_2^{(1,1)}(z_1, z_2) = \tilde{E}_1^{(1,1)}(z_1, z_2) - k_{e_2}^{(1,1)} \tilde{R}_1^{(1,0)}(z_1, z_2). \quad (4.55)$$

If we substitute (4.19) and (4.26) into (4.54), then the equation will be

$$\begin{aligned} E_2^{(1,1)}(z_1, z_2) &= \left[z_1^{-1} \left(1 - k_{e_1}^{(0,1)*} z_2^{-1} \right) - k_{e_2}^{(1,1)*} z_2^{-1} \left(1 - k_{r_1}^{(1,0)*} z_1^{-1} \right) \right] Y(z_1, z_2) \\ &= \underbrace{\left[z_1^{-1} - k_{e_2}^{(1,1)*} z_2^{-1} + \left(k_{e_2}^{(1,1)*} k_{r_1}^{(1,0)*} - k_{e_1}^{(0,1)*} \right) z_1^{-1} z_2^{-1} \right]}_{A_2^{(1,1)}(z_1, z_2)} Y(z_1, z_2). \end{aligned} \quad (4.56)$$

From (4.56), we can write that

$$A_2^{(1,1)}(z_1, z_2) = \underbrace{\begin{bmatrix} 1 & z_1^{-1} \end{bmatrix}}_{\Delta^{(1)}(z_1)} \underbrace{\begin{bmatrix} 0 & -k_{e_2}^{(1,1)*} \\ 1 & k_{e_2}^{(1,1)*} k_{r_1}^{(1,0)*} - k_{e_1}^{(0,1)*} \end{bmatrix}}_{\mathbf{A}_2^{(1,1)}} \underbrace{\begin{bmatrix} 1 \\ z_2^{-1} \end{bmatrix}}_{\Delta^{(1)}(z_2)}. \quad (4.57)$$

The coefficient matrix for the second vertical forward PEFs $\mathbf{A}_2^{(1,1)}$ is found as

$$\mathbf{A}_2^{(1,1)} = \begin{bmatrix} 0 & -k_{e_2}^{(1,1)*} \\ 1 & k_{e_2}^{(1,1)*} k_{r_1}^{(1,0)*} - k_{e_1}^{(0,1)*} \end{bmatrix}. \quad (4.58)$$

For the coefficient matrix of the second vertical backward PEFs, (4.22) and (4.29) are substituted into (4.55), and the equation will be written as

$$\begin{aligned} \tilde{E}_2^{(1,1)}(z_1, z_2) &= \left[\left(z_2^{-1} - k_{e_1}^{(0,1)} \right) - k_{e_1}^{(1,1)} \left(z_1^{-1} - k_{r_1}^{(1,0)} \right) \right] Y(z_1, z_2) \\ &= \underbrace{\left[\left(k_{e_2}^{(1,1)} k_{r_1}^{(1,0)} - k_{e_1}^{(0,1)} \right) - k_{e_2}^{(1,1)} z_1^{-1} + z_2^{-1} \right]}_{\tilde{A}_2^{(1,1)}(z_1, z_2)} Y(z_1, z_2). \end{aligned} \quad (4.59)$$

And from (4.59), we can write that

$$\tilde{A}_2^{(1,1)}(z_1, z_2) = \underbrace{\begin{bmatrix} 1 & z_1^{-1} \end{bmatrix}}_{\Delta^{(1)}(z_1)} \underbrace{\begin{bmatrix} k_{e_2}^{(1,1)} k_{r_1}^{(1,0)*} - k_{e_1}^{(0,1)} & 1 \\ -k_{e_2}^{(1,1)} & 0 \end{bmatrix}}_{\tilde{\mathbf{A}}_2^{(1,1)}} \underbrace{\begin{bmatrix} 1 \\ z_2^{-1} \end{bmatrix}}_{\Delta^{(1)}(z_2)}. \quad (4.60)$$

The coefficient matrix for the first vertical backward PEFs $\tilde{\mathbf{A}}_2^{(1,1)}$ is

$$\tilde{\mathbf{A}}_2^{(1,1)} = \begin{bmatrix} k_{e_2}^{(1,1)} k_{r_1}^{(1,0)} - k_{e_1}^{(0,1)} & 1 \\ -k_{e_2}^{(1,1)} & 0 \end{bmatrix}. \quad (4.61)$$

Until now, we have found all coefficient matrices of the updated horizontal and vertical and the newly initiated vertical forward/backward PEFs by calculating directly from the error propagation equations. From now on, we will show how to obtain same values by using our proposed algorithm.

After finding the coefficient matrices at the initial stage, the values of p and q become equal to 1. Hence, we can define the i and j as $i = 1$ and $j = 1$, as defined in **Step 3**. If we used the equations given in (4.16), we can find the coefficient matrix of the horizontal forward and backward PEFs as

$$\begin{aligned} \mathbf{B}_j^{(p+1,q-1)} &= \hat{\mathbf{B}}_{j,h}^{(p,q-1)} - \sum_{l=1}^q k_{r_j}^{(p+1,q-1)*}(l) \tilde{\mathbf{B}}_{l,z_1}^{(p,q-1)} \\ \mathbf{B}_1^{(1+1,1-1)} &= \hat{\mathbf{B}}_{1,h}^{(1,1-1)} - \sum_{l=1}^1 k_{r_1}^{(1+1,1-1)*}(l) \tilde{\mathbf{B}}_{l,z_1}^{(1,1-1)} \\ \mathbf{B}_1^{(2,0)} &= \hat{\mathbf{B}}_{1,h}^{(1,0)} - \sum_{l=1}^1 k_{r_1}^{(2,0)*}(l) \tilde{\mathbf{B}}_{l,z_1}^{(1,0)} \\ &= \begin{bmatrix} \mathbf{B}_1^{(1,0)} \\ \mathbf{O}_1^T \end{bmatrix} - k_{r_1}^{(2,0)*}(1) \begin{bmatrix} \mathbf{O}_1^T \\ \tilde{\mathbf{B}}_1^{(1,0)} \end{bmatrix} \\ &= \begin{bmatrix} 1 \\ -k_{r_1}^{(1,0)*} \\ 0 \end{bmatrix} - k_{r_1}^{(2,0)*}(1) \begin{bmatrix} 0 \\ -k_{r_1}^{(1,0)} \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 \\ k_{r_1}^{(2,0)*} k_{r_1}^{(1,0)} - k_{r_1}^{(1,0)*} \\ -k_{r_1}^{(2,0)*} \end{bmatrix}. \end{aligned} \quad (4.62)$$

By using (4.9), we can find $\tilde{\mathbf{B}}_j(2,0)$ as

$$\begin{aligned} \tilde{\mathbf{B}}_1^{(2,0)} &= \mathbf{J}_3 \mathbf{B}_1^{(2,0)*} \mathbf{J}_1 \\ &= \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ k_{r_1}^{(2,0)} k_{r_1}^{(1,0)*} - k_{r_1}^{(1,0)} \\ -k_{r_1}^{(2,0)} \end{bmatrix} [1] \\ &= \begin{bmatrix} -k_{r_1}^{(2,0)} \\ k_{r_1}^{(2,0)} k_{r_1}^{(1,0)*} - k_{r_1}^{(1,0)} \\ 1 \end{bmatrix}. \end{aligned} \quad (4.63)$$

If we compare the coefficient matrices (4.62) and (4.63) with the ones (4.36) and (4.41), respectively, we can see that they are identical.

Similarly, we can calculate the coefficient matrix of the updated vertical forward and backward PEFs as,

$$\begin{aligned}
\mathbf{A}_1^{(1,1)} &= \hat{\mathbf{A}}_{1,h}^{(0,1)} - \sum_{l=1}^1 k_{e_1}^{(1,1)*}(l) \tilde{\mathbf{B}}_{l,z_2}^{(1,0)} \\
&= \begin{bmatrix} \mathbf{A}_1^{(0,1)} \\ \mathbf{O}_1^T \end{bmatrix} - k_{e_1}^{(1,1)*}(1) \begin{bmatrix} \mathbf{O}_2 \\ \tilde{\mathbf{B}}_1^{(1,0)} \end{bmatrix} \\
&= \begin{bmatrix} 1 & -k_{e_1}^{(0,1)*} \\ 0 & 0 \end{bmatrix} - k_{e_1}^{(1,1)*}(1) \begin{bmatrix} 0 & -k_{r_1}^{(1,0)} \\ 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & k_{e_1}^{(1,1)*} k_{r_1}^{(1,0)} - k_{e_1}^{(0,1)*} \\ 0 & -k_{e_1}^{(1,1)*} \end{bmatrix}, \tag{4.64}
\end{aligned}$$

and

$$\begin{aligned}
\tilde{\mathbf{A}}_1^{(1,1)} &= \mathbf{J}_2 \mathbf{A}_1^{(1,1)*} \mathbf{J}_2 \\
&= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & k_{e_1}^{(1,1)} k_{r_1}^{(1,0)*} - k_{e_1}^{(0,1)} \\ 0 & -k_{e_1}^{(1,1)} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \\
&= \begin{bmatrix} -k_{e_1}^{(1,1)} & 0 \\ k_{e_1}^{(1,1)} k_{r_1}^{(1,0)*} - k_{e_1}^{(0,1)} & 1 \end{bmatrix}, \tag{4.65}
\end{aligned}$$

respectively. It is seen that the coefficient matrices (4.64) and (4.65) for the updated vertical forward and backward PEFs are equal to the ones (4.48) and (4.51), respectively, calculated from the error propagation equations.

We can do the same calculations for the newly initiated vertical forward and the backward PEFs. To find the coefficient matrix of new updated vertical forward PEF, we use the last equation given in the **Step 3**.

$$\begin{aligned}
\mathbf{A}_2^{(1,1)} &= \hat{\mathbf{A}}_{1,z_1}^{(0,1)} - \sum_{l=1}^1 k_{e_2}^{(1,1)*}(l) \mathbf{B}_{l,z_2}^{(1,0)} \\
&= \begin{bmatrix} \mathbf{O}_2^T \\ \mathbf{A}_1^{(0,1)} \end{bmatrix} - k_{e_2}^{(1,1)*}(1) \begin{bmatrix} \mathbf{O}_2 \\ \mathbf{B}_1^{(1,0)} \end{bmatrix} \\
&= \begin{bmatrix} 0 & 0 \\ 1 & -k_{e_1}^{(0,1)*} \end{bmatrix} - k_{e_2}^{(1,1)*}(1) \begin{bmatrix} 0 & 1 \\ 0 & -k_{r_1}^{(1,0)*} \end{bmatrix} \\
&= \begin{bmatrix} 0 & -k_{e_2}^{(1,1)*} \\ 1 & k_{e_2}^{(1,1)*} k_{r_1}^{(1,0)*} - k_{e_1}^{(0,1)*} \end{bmatrix}, \tag{4.66}
\end{aligned}$$

and

$$\begin{aligned}
\tilde{\mathbf{A}}_2^{(1,1)} &= \mathbf{J}_2 \mathbf{A}_2^{(1,1)*} \mathbf{J}_2 \\
&= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -k_{e_2}^{(1,1)} \\ 1 & k_{e_2}^{(1,1)} k_{r_1}^{(1,0)} - k_{e_1}^{(0,1)} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \\
&= \begin{bmatrix} k_{e_2}^{(1,1)} k_{r_1}^{(1,0)} - k_{e_1}^{(0,1)} & 1 \\ -k_{e_2}^{(1,1)} & 0 \end{bmatrix}, \tag{4.67}
\end{aligned}$$

respectively. If we compare the results (4.66) and (4.67) with ones, (4.58) and (4.61), calculated for the newly initiated vertical forward and backward PEFs, we can see that they are the same.

Consequently, the coefficient matrices for each horizontal and vertical PEFs at the initial stage and the succeeding horizontal augmentation process have been calculated by using the z -transforms of the error propagation equations and the proposed algorithm. The results have been compared and it is clearly seen that our algorithm works correctly. In the following section, some more examples for the further augmentation process will be given.

4.1.3 Determination of coefficient matrices for the further augmentations

The coefficient matrices of the auxiliary PEFs of the initial stage, the PSR $S_{k,l}^{(0,0)}$, and the horizontally augmented case, the PSR $S_{k,l}^{(1,0)}$ have been determined in the above sections. After the horizontal augmentation, the coefficient matrices $\mathbf{B}_1^{(2,0)}$, $\tilde{\mathbf{B}}_1^{(2,0)}$ for the horizontal PEFs, $\mathbf{A}_1^{(1,1)}$, $\tilde{\mathbf{A}}_1^{(1,1)}$ for the first vertical PEFs, and $\mathbf{A}_2^{(1,1)}$, $\tilde{\mathbf{A}}_2^{(1,1)}$ for the second vertical PEFs which are the newly initiated ones have been obtained.

Now, first a vertical, then horizontal, and finally a vertical augmentation will be applied and all coefficient matrices of the whole horizontal and vertical PEFs of the PSR $S_{k,l}^{(1,1)}$, $S_{k,l}^{(2,1)}$, and $S_{k,l}^{(2,2)}$ will be calculated, respectively, by using the proposed algorithm in Section 4.1.

4.1.3.1 Coefficient matrices of the PSR $S_{k,l}^{(1,1)}$ after vertical augmentation

We know that the PSR $S_{k,l}^{(1,1)}$ have two pairs of vertical forward/backward PEFs and two pairs of horizontal forward/backward PEFs whose one pair of the horizontal forward/backward PEFs is the newly initiated because of vertical augmentation. By increasing q unitary from the last horizontal augmentation, we proceed a vertical

augmentation. That means the PSR $S_{k,l}^{(1,0)}$ is going to one step higher order on l -direction. Hence, from the equation (4.17) given in the **Step 4**, we can write the coefficient matrices $\mathbf{A}_1^{(1,2)}$ and $\mathbf{A}_2^{(1,2)}$ for the vertical PEFs as for $p = 2$, $q = 1$, $i = 1, 2$, and $j = 1$:

$$\begin{aligned}
\mathbf{A}_1^{(1,2)} &= \hat{\mathbf{A}}_{1,v}^{(1,1)} - \sum_{l=1}^2 k_{e_1}^{(1,2)*}(l) \tilde{\mathbf{A}}_{l,z_2}^{(1,1)} \\
&= \hat{\mathbf{A}}_{1,v}^{(1,1)} - k_{e_1}^{(1,2)*}(1) \tilde{\mathbf{A}}_{1,z_2}^{(1,1)} - k_{e_1}^{(1,2)*}(2) \tilde{\mathbf{A}}_{2,z_2}^{(1,1)} \\
&= \begin{bmatrix} \mathbf{A}_1^{(1,1)} & \vdots & \mathbf{O}_2 \end{bmatrix} - k_{e_1}^{(1,2)*}(1) \begin{bmatrix} \mathbf{O}_2 & \vdots & \tilde{\mathbf{A}}_1^{(1,1)} \end{bmatrix} - k_{e_1}^{(1,2)*}(2) \begin{bmatrix} \mathbf{O}_2 & \vdots & \tilde{\mathbf{A}}_2^{(1,1)} \end{bmatrix} \\
&= \begin{bmatrix} 1 & k_{e_1}^{(1,1)*} k_{r_1}^{(1,0)} - k_{e_1}^{(0,1)*} & 0 \\ 0 & -k_{e_1}^{(1,1)*} & 0 \end{bmatrix} - k_{e_1}^{(1,2)*}(1) \begin{bmatrix} 0 & -k_{e_1}^{(1,1)} & 0 \\ 0 & k_{e_1}^{(1,1)} k_{r_1}^{(1,0)*} - k_{e_1}^{(0,1)} & 1 \end{bmatrix} \\
&\quad - k_{e_1}^{(1,2)*}(2) \begin{bmatrix} 0 & k_{e_2}^{(1,1)} k_{r_1}^{(1,0)} - k_{e_1}^{(0,1)} & 1 \\ 0 & -k_{e_2}^{(1,1)} & 0 \end{bmatrix} \\
&= \begin{bmatrix} 1 & k_{e_1}^{(1,1)*} k_{r_1}^{(1,0)} - k_{e_1}^{(0,1)*} + k_{e_2}^{(1,2)*}(1) k_{e_1}^{(1,1)} - k_{e_2}^{(1,2)*}(2) (k_{e_2}^{(1,1)} k_{r_1}^{(1,0)} - k_{e_1}^{(0,1)}) \\ 0 & -k_{e_1}^{(1,1)*} - k_{e_2}^{(1,2)*}(1) (k_{e_1}^{(1,1)} k_{r_1}^{(1,0)*} - k_{e_1}^{(0,1)}) + k_{e_2}^{(1,2)*}(2) k_{e_2}^{(1,2)} \\ -k_{e_1}^{(1,2)*}(2) \\ -k_{e_1}^{(1,2)*}(1) \end{bmatrix}, \tag{4.68}
\end{aligned}$$

and

$$\begin{aligned}
\mathbf{A}_2^{(1,2)} &= \hat{\mathbf{A}}_{2,v}^{(1,1)} - \sum_{l=1}^2 k_{e_2}^{(1,2)*}(l) \tilde{\mathbf{A}}_{l,z_2}^{(1,1)} \\
&= \hat{\mathbf{A}}_{2,v}^{(1,1)} - k_{e_2}^{(1,2)*}(1) \tilde{\mathbf{A}}_{1,z_2}^{(1,1)} - k_{e_2}^{(1,2)*}(2) \tilde{\mathbf{A}}_{2,z_2}^{(1,1)} \\
&= \begin{bmatrix} \mathbf{A}_2^{(1,1)} & \vdots & \mathbf{O}_2 \end{bmatrix} - k_{e_2}^{(1,2)*}(1) \begin{bmatrix} \mathbf{O}_2 & \vdots & \tilde{\mathbf{A}}_1^{(1,1)} \end{bmatrix} - k_{e_2}^{(1,2)*}(2) \begin{bmatrix} \mathbf{O}_2 & \vdots & \tilde{\mathbf{A}}_2^{(1,1)} \end{bmatrix} \\
&= \begin{bmatrix} 0 & -k_{e_2}^{(1,1)*} & 0 \\ 1 & k_{e_2}^{(1,1)*} k_{r_1}^{(1,0)} - k_{e_1}^{(0,1)*} & 0 \end{bmatrix} - k_{e_2}^{(1,2)*}(1) \begin{bmatrix} 0 & -k_{e_1}^{(1,1)} & 0 \\ 0 & k_{e_1}^{(1,1)} k_{r_1}^{(1,0)*} - k_{e_1}^{(0,1)} & 1 \end{bmatrix} \\
&\quad - k_{e_2}^{(1,2)*}(2) \begin{bmatrix} 0 & k_{e_2}^{(1,1)} k_{r_1}^{(1,0)} - k_{e_1}^{(0,1)} & 1 \\ 0 & -k_{e_2}^{(1,1)} & 0 \end{bmatrix} \\
&= \begin{bmatrix} 0 & -k_{e_2}^{(1,1)*} + k_{e_2}^{(1,2)*}(1) k_{e_1}^{(1,1)} - k_{e_2}^{(1,2)*}(2) (k_{e_2}^{(1,1)} k_{r_1}^{(1,0)} - k_{e_1}^{(0,1)}) \\ 1 & -k_{e_2}^{(1,1)*} k_{r_1}^{(1,0)} - k_{e_1}^{(0,1)*} - k_{e_2}^{(1,2)*}(1) (k_{e_1}^{(1,1)} k_{r_1}^{(1,0)*} - k_{e_1}^{(0,1)}) + k_{e_2}^{(1,2)*}(2) k_{e_2}^{(1,2)} \\ -k_{e_2}^{(1,2)*}(2) \\ -k_{e_2}^{(1,2)*}(1) \end{bmatrix}. \tag{4.69}
\end{aligned}$$

$\tilde{\mathbf{A}}_1(1,2)$ and $\tilde{\mathbf{A}}_1(1,2)$ are determined using (4.9) as

$$\begin{aligned}
\tilde{\mathbf{A}}_1^{(1,2)} &= \mathbf{J}_2 \mathbf{A}_l^{(1,2)*} \mathbf{J}_3 \\
&= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \mathbf{A}_l^{(1,2)*} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \\
&= \begin{bmatrix} -k_{e_1}^{(1,2)}(1) \\ -k_{e_1}^{(1,2)}(2) \\ -k_{e_1}^{(1,1)} - k_{e_2}^{(1,2)}(1)(k_{e_1}^{(1,1)*} k_{r_1}^{(1,0)} - k_{e_1}^{(0,1)*}) + k_{e_2}^{(1,2)}(2)k_{e_2}^{(1,2)*} \\ k_{e_1}^{(1,1)} k_{r_1}^{(1,0)*} - k_{e_1}^{(0,1)} + k_{e_2}^{(1,2)}(1)k_{e_1}^{(1,1)*} - k_{e_2}^{(1,2)}(2)(k_{e_2}^{(1,1)*} k_{r_1}^{(1,0)*} - k_{e_1}^{(0,1)*}) \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \\
&\quad (4.70)
\end{aligned}$$

and

$$\begin{aligned}
\tilde{\mathbf{A}}_2^{(1,2)} &= \mathbf{J}_2 \mathbf{A}_2^{(1,2)*} \mathbf{J}_3 \\
&= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \mathbf{A}_2^{(1,2)*} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \\
&= \begin{bmatrix} -k_{e_2}^{(1,2)}(1) \\ -k_{e_2}^{(1,2)}(2) \\ -k_{e_2}^{(1,1)} k_{r_1}^{(1,0)*} - k_{e_1}^{(0,1)} - k_{e_2}^{(1,2)}(1)(k_{e_1}^{(1,1)*} k_{r_1}^{(1,0)} - k_{e_1}^{(0,1)*}) + k_{e_2}^{(1,2)}(2)k_{e_2}^{(1,1)*} \\ -k_{e_2}^{(1,1)} + k_{e_2}^{(1,2)}(1)k_{e_1}^{(1,1)*} - k_{e_2}^{(1,2)*}(2)(k_{e_2}^{(1,1)*} k_{r_1}^{(1,0)*} - k_{e_1}^{(0,1)*}) \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \\
&\quad (4.71)
\end{aligned}$$

respectively.

The coefficient matrices $\mathbf{B}_1^{(2,1)}$ and $\tilde{\mathbf{B}}_1^{(2,1)}$ of the updated horizontal PEFs are

$$\begin{aligned}
\mathbf{B}_1^{(2,1)} &= \hat{\mathbf{B}}_{1,v}^{(2,0)} - \sum_{l=1}^2 k_{r_1}^{(2,1)*}(l) \tilde{\mathbf{A}}_{l,z_1}^{(1,1)} \\
&= \begin{bmatrix} \mathbf{B}_1^{(2,0)} & \vdots & \mathbf{O}_3 \end{bmatrix} - k_{r_1}^{(2,1)*}(1) \begin{bmatrix} \mathbf{O}_2^T & \vdots & \tilde{\mathbf{A}}_1^{(1,1)} \end{bmatrix} - k_{r_1}^{(2,1)*}(2) \begin{bmatrix} \mathbf{O}_2^T & \vdots & \tilde{\mathbf{A}}_2^{(1,1)} \end{bmatrix} \\
&= \begin{bmatrix} 1 & \vdots & 0 \\ k_{r_1}^{(2,0)*} k_{r_1}^{(1,0)} - k_{r_1}^{(1,0)*} & \vdots & 0 \\ -k_{r_1}^{(2,0)*} & \vdots & 0 \end{bmatrix} - k_{r_1}^{(2,1)*}(1) \begin{bmatrix} 0 & 0 \\ -k_{e_1}^{(1,1)} & 0 \\ k_{e_1}^{(1,1)} k_{r_1}^{(1,0)*} - k_{e_1}^{(0,1)} & 1 \end{bmatrix} \\
&\quad - k_{r_1}^{(2,1)*}(2) \begin{bmatrix} 0 & 0 \\ k_{e_2}^{(1,1)} k_{r_1}^{(1,0)} - k_{e_1}^{(0,1)} & 1 \\ -k_{e_2}^{(1,1)} & 0 \end{bmatrix} \\
&= \begin{bmatrix} 1 \\ k_{r_1}^{(2,0)*} k_{r_1}^{(1,0)} - k_{r_1}^{(1,0)*} + k_{r_1}^{(2,1)*}(1) k_{e_1}^{(1,1)} - k_{r_1}^{(2,1)*}(2) (k_{e_2}^{(1,1)} k_{r_1}^{(1,0)} - k_{e_1}^{(0,1)}) \\ -k_{r_1}^{(2,0)*} - k_{r_1}^{(2,1)*}(1) (k_{e_1}^{(1,1)} k_{r_1}^{(1,0)*} - k_{e_1}^{(0,1)}) + k_{r_1}^{(2,1)*}(2) k_{e_2}^{(1,1)} \\ 0 \\ -k_{r_1}^{(2,1)*}(2) \\ -k_{r_1}^{(2,1)*}(1) \end{bmatrix}
\end{aligned} \tag{4.72}$$

and

$$\begin{aligned}
\tilde{\mathbf{B}}_1^{(2,1)} &= \mathbf{J}_3 \mathbf{B}_1^{(2,1)*} \mathbf{J}_2 \\
&= \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \mathbf{B}_1^{(2,1)*} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \\
&= \begin{bmatrix} -k_{r_1}^{(2,1)}(1) \\ -k_{r_1}^{(2,1)}(2) \\ 0 \\ -k_{r_1}^{(2,0)} - k_{r_1}^{(2,1)}(1) (k_{e_1}^{(1,1)*} k_{r_1}^{(1,0)} - k_{e_1}^{(0,1)*}) + k_{r_1}^{(2,1)}(2) k_{e_2}^{(1,1)*} \\ k_{r_1}^{(2,0)} k_{r_1}^{(1,0)*} - k_{r_1}^{(1,0)} + k_{r_1}^{(2,1)}(1) k_{e_1}^{(1,1)*} - k_{r_1}^{(2,1)}(2) (k_{e_2}^{(1,1)*} k_{r_1}^{(1,0)*} - k_{e_1}^{(0,1)*}) \\ 1 \end{bmatrix},
\end{aligned} \tag{4.73}$$

respectively.

For the newly initiated horizontal PEFs, the coefficient matrices $\mathbf{B}_2^{(2,1)}$ and $\tilde{\mathbf{B}}_2^{(2,1)}$ are calculated as follows

$$\begin{aligned}
\mathbf{B}_2^{(2,1)} &= \hat{\mathbf{B}}_{1,v}^{(2,0)} - \sum_{l=1}^2 k_{r_2}^{(2,1)*}(l) \mathbf{A}_{l,z_1}^{(1,1)} \\
&= \begin{bmatrix} \mathbf{O}_3 & \vdots & \mathbf{B}_1^{(2,0)} \end{bmatrix} - k_{r_1}^{(2,1)*}(1) \begin{bmatrix} \mathbf{O}_2^T & \vdots & \mathbf{A}_1^{(1,1)} \end{bmatrix} - k_{r_1}^{(2,1)*}(2) \begin{bmatrix} \mathbf{O}_2^T & \vdots & \mathbf{A}_2^{(1,1)} \end{bmatrix} \\
&= \begin{bmatrix} 0 & \vdots & 1 \\ 0 & \vdots & k_{r_1}^{(2,0)*} k_{r_1}^{(1,0)} - k_{r_1}^{(1,0)*} \\ 0 & \vdots & -k_{r_1}^{(2,0)*} \end{bmatrix} - k_{r_2}^{(2,1)*}(1) \begin{bmatrix} 0 & 0 \\ 1 & k_{e_1}^{(1,1)*} k_{r_1}^{(1,0)} - k_{e_1}^{(0,1)*} \\ 0 & -k_{e_1}^{(1,1)*} \end{bmatrix} \\
&\quad - k_{r_1}^{(2,1)*}(2) \begin{bmatrix} 0 & 0 \\ 0 & -k_{e_2}^{(1,1)*} \\ 1 & k_{e_2}^{(1,1)*} k_{r_1}^{(1,0)*} - k_{e_1}^{(0,1)*} \end{bmatrix} \\
&= \begin{bmatrix} 0 & 1 \\ -k_{r_1}^{(2,1)*}(1) & k_{r_1}^{(2,1)*}(1)(k_{e_1}^{(1,1)*} k_{r_1}^{(1,0)} - k_{e_1}^{(0,1)*}) + k_{r_1}^{(2,1)*}(1)k_{e_2}^{(1,1)*} \\ -k_{r_1}^{(2,1)*}(2) & k_{r_2}^{(2,1)*}(1)(k_{e_1}^{(1,1)*} k_{r_1}^{(1,0)*} - k_{e_1}^{(0,1)*}) + k_{r_1}^{(2,1)*}(2)(k_{e_2}^{(1,1)*} k_{r_1}^{(1,0)*} - k_{e_1}^{(0,1)*}) \end{bmatrix}, \tag{4.74}
\end{aligned}$$

and

$$\begin{aligned}
\tilde{\mathbf{B}}_2^{(2,1)} &= \mathbf{J}_3 \mathbf{B}_2^{(2,1)*} \mathbf{J}_2 \\
&= \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \mathbf{B}_2^{(2,1)*} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \\
&= \begin{bmatrix} k_{r_2}^{(2,1)}(1)(k_{e_1}^{(1,1)} - k_{r_2}^{(2,1)}(2)(k_{e_2}^{(1,1)*} k_{r_1}^{(1,0)} - k_{e_1}^{(0,1)}) & -k_{r_1}^{(2,1)}(2) \\ k_{r_1}^{(2,1)}(1)(k_{e_1}^{(1,1)} k_{r_1}^{(1,0)*} - k_{e_1}^{(0,1)}) + k_{r_1}^{(2,1)}(1)k_{e_2}^{(1,1)} & -k_{r_1}^{(2,1)}(1) \\ 1 & 0 \end{bmatrix}, \tag{4.75}
\end{aligned}$$

respectively.

4.1.3.2 Coefficient matrices of the PSR $S_{k,l}^{(2,1)}$ after horizontal augmentation

If a horizontal augmentation is applied to the PSR $S_{k,l}^{(1,1)}$, the new PSR $S_{k,l}^{(2,1)}$ is obtained. This new PSR $S_{k,l}^{(2,1)}$ has *three* pairs of vertical PEFs whose one pair of them are newly initiated, and *two* pairs of horizontal PEFs. Hereafter, for simplicity, the values which are the result of the multiplication/addition of the coefficient values in the coefficient matrices will not be written in open form, they will be represented with x or $*$ or any arbitrary symbol.

First, let us write the coefficient matrices of the horizontal PEFs. $\mathbf{B}_1^{(3,1)}$ and $\tilde{\mathbf{B}}_1^{(3,1)}$ are

$$\begin{aligned}\mathbf{B}_1^{(3,1)} &= \hat{\mathbf{B}}_{1,h}^{(2,1)} - k_{r_1}^{(3,1)*}(1)\tilde{\mathbf{B}}_{1,z_1}^{(2,1)} - k_{r_1}^{(3,1)*}(2)\tilde{\mathbf{B}}_{2,z_1}^{(2,1)} \\ &= \begin{bmatrix} 1 & 0 \\ x & x \\ x & x \\ -k_{r_1}^{(3,1)*}(2) & -k_{r_1}^{(3,1)*}(1) \end{bmatrix},\end{aligned}\quad (4.76)$$

and

$$\begin{aligned}\tilde{\mathbf{B}}_1^{(3,1)} &= \mathbf{J}_4 \mathbf{B}_1^{(3,1)} \mathbf{J}_2 \\ &= \begin{bmatrix} -k_{r_1}^{(3,1)}(1) & -k_{r_1}^{(3,1)}(2) \\ x & x \\ x & x \\ 0 & 1 \end{bmatrix}.\end{aligned}\quad (4.77)$$

The coefficient matrices of the second pair of horizontal PEFs $\mathbf{B}_2^{(3,1)}$ and $\tilde{\mathbf{B}}_2^{(3,1)}$ are found as

$$\begin{aligned}\mathbf{B}_2^{(3,1)} &= \hat{\mathbf{B}}_{2,h}^{(2,1)} - k_{r_2}^{(3,1)*}(1)\tilde{\mathbf{B}}_{1,z_1}^{(2,1)} - k_{r_2}^{(3,1)*}(2)\tilde{\mathbf{B}}_{2,z_1}^{(2,1)} \\ &= \begin{bmatrix} 0 & 1 \\ x & x \\ x & x \\ -k_{r_2}^{(3,1)*}(2) & -k_{r_2}^{(3,1)*}(1) \end{bmatrix},\end{aligned}\quad (4.78)$$

and

$$\begin{aligned}\tilde{\mathbf{B}}_2^{(3,1)} &= \mathbf{J}_4 \mathbf{B}_2^{(3,1)} \mathbf{J}_2 \\ &= \begin{bmatrix} -k_{r_2}^{(3,1)}(1) & -k_{r_2}^{(3,1)}(2) \\ x & x \\ x & x \\ 1 & 0 \end{bmatrix}.\end{aligned}\quad (4.79)$$

The coefficient matrices of the updated vertical PEFs $\mathbf{A}_1^{(2,2)}$, $\mathbf{A}_2^{(2,2)}$, $\tilde{\mathbf{A}}_1^{(2,2)}$, and $\tilde{\mathbf{A}}_2^{(2,2)}$ are given as

$$\begin{aligned}\mathbf{A}_1^{(2,2)} &= \hat{\mathbf{A}}_{1,h}^{(1,2)} - k_{e_1}^{(2,2)*}(1)\tilde{\mathbf{B}}_{1,z_2}^{(2,1)} - k_{e_1}^{(2,2)*}(2)\tilde{\mathbf{B}}_{2,z_2}^{(2,1)} \\ &= \begin{bmatrix} 1 & x & x \\ 0 & x & x \\ 0 & -k_{e_1}^{(2,2)*}(2) & -k_{e_1}^{(2,2)*}(1) \end{bmatrix},\end{aligned}\quad (4.80)$$

$$\begin{aligned}\mathbf{A}_2^{(2,2)} &= \hat{\mathbf{A}}_{2,h}^{(1,2)} - k_{e_2}^{(2,2)*}(1)\tilde{\mathbf{B}}_{1,z_2}^{(2,1)} - k_{e_2}^{(2,2)*}(2)\tilde{\mathbf{B}}_{2,z_2}^{(2,1)} \\ &= \begin{bmatrix} 0 & x & x \\ 1 & x & x \\ 0 & -k_{e_2}^{(2,2)*}(2) & -k_{e_2}^{(2,2)*}(1) \end{bmatrix},\end{aligned}\quad (4.81)$$

$$\begin{aligned}
\tilde{\mathbf{A}}_1^{(2,2)} &= \mathbf{J}_3 \mathbf{A}_1^{(2,2)*} \mathbf{J}_3 \\
&= \begin{bmatrix} -k_{e_1}^{(2,2)}(1) & -k_{e_1}^{(2,2)}(2) & 0 \\ x & x & 0 \\ x & x & 1 \end{bmatrix}, \tag{4.82}
\end{aligned}$$

and

$$\begin{aligned}
\tilde{\mathbf{A}}_2^{(2,2)} &= \mathbf{J}_3 \mathbf{A}_2^{(2,2)*} \mathbf{J}_3 \\
&= \begin{bmatrix} -k_{e_2}^{(2,2)}(1) & -k_{e_2}^{(2,2)}(2) & 0 \\ x & x & 1 \\ x & x & 0 \end{bmatrix}, \tag{4.83}
\end{aligned}$$

respectively.

The coefficient matrices for the newly initiated vertical PEFs $\mathbf{A}_3^{(2,2)}$, and $\tilde{\mathbf{A}}_3^{(2,2)}$ are defined as

$$\begin{aligned}
\mathbf{A}_3^{(2,2)} &= \mathbf{A}_{2,z_1}^{(1,2)} - k_{e_3}^{(2,2)*}(1)\mathbf{B}_{1,z_2}^{(2,1)} - k_{e_3}^{(2,2)*}(2)\mathbf{B}_{2,z_2}^{(2,1)} \\
&= \begin{bmatrix} 0 & -k_{e_3}^{(2,2)*}(1) & -k_{e_3}^{(2,2)*}(2) \\ 0 & x & x \\ 1 & x & x \end{bmatrix}, \tag{4.84}
\end{aligned}$$

$$\begin{aligned}
\tilde{\mathbf{A}}_3^{(2,2)} &= \mathbf{J}_3 \mathbf{A}_3^{(2,2)*} \mathbf{J}_3 \\
&= \begin{bmatrix} x & x & 1 \\ x & x & 0 \\ -k_{e_3}^{(2,2)}(2) & -k_{e_3}^{(2,2)}(1) & 0 \end{bmatrix}, \tag{4.85}
\end{aligned}$$

respectively.

4.1.3.3 Coefficient matrices of the PSR $S_{k,l}^{(2,2)}$ after vertical augmentation

Now, we apply a vertical augmentation to the PSR $S_{k,l}^{(2,1)}$, then the new PSR will be $S_{k,l}^{(2,2)}$. This new PSR $S_{k,l}^{(2,2)}$ will have *three* pairs of vertical PEFs, and *three* pairs of horizontal PEFs of which one pair are the newly initiated.

The coefficient matrices of the updated vertical PEFs $\mathbf{A}_1^{(2,2)}$, $\mathbf{A}_2^{(2,2)}$, $\mathbf{A}_3^{(2,2)}$, $\tilde{\mathbf{A}}_1^{(2,2)}$, $\tilde{\mathbf{A}}_2^{(2,2)}$, and $\tilde{\mathbf{A}}_3^{(2,2)}$ can be calculated as

$$\begin{aligned}
\mathbf{A}_1^{(2,3)} &= \hat{\mathbf{A}}_{1,v}^{(2,2)} - k_{e_1}^{(2,3)*}(1)\tilde{\mathbf{A}}_{1,z_2}^{(2,2)} - k_{e_1}^{(2,3)*}(2)\tilde{\mathbf{A}}_{2,z_2}^{(2,2)} \\
&\quad - k_{e_1}^{(2,3)*}(3)\tilde{\mathbf{A}}_{3,z_2}^{(2,2)} \\
&= \begin{bmatrix} 1 & x & x & -k_{e_1}^{(2,3)*}(3) \\ 0 & x & x & -k_{e_1}^{(2,3)*}(2) \\ 0 & x & x & -k_{e_1}^{(2,3)*}(1) \end{bmatrix}, \tag{4.86}
\end{aligned}$$

$$\begin{aligned}
\mathbf{A}_2^{(2,3)} &= \hat{\mathbf{A}}_{2,v}^{(2,2)} - k_{e_2}^{(2,3)*}(1)\tilde{\mathbf{A}}_{1,z_2}^{(2,2)} - k_{e_2}^{(2,3)*}(2)\tilde{\mathbf{A}}_{2,z_2}^{(2,2)} \\
&\quad - k_{e_2}^{(2,3)*}(3)\tilde{\mathbf{A}}_{3,z_2}^{(2,2)} \\
&= \begin{bmatrix} 0 & x & x & -k_{e_2}^{(2,3)*}(3) \\ 1 & x & x & -k_{e_2}^{(2,3)*}(2) \\ 0 & x & x & -k_{e_2}^{(2,3)*}(1) \end{bmatrix}, \tag{4.87}
\end{aligned}$$

$$\begin{aligned}
\mathbf{A}_3^{(2,3)} &= \hat{\mathbf{A}}_{3,v}^{(2,2)} - k_{e_3}^{(2,3)*}(1)\tilde{\mathbf{A}}_{1,z_2}^{(2,2)} - k_{e_3}^{(2,3)*}(2)\tilde{\mathbf{A}}_{2,z_2}^{(2,2)} \\
&\quad - k_{e_3}^{(2,3)*}(3)\tilde{\mathbf{A}}_{3,z_2}^{(2,2)} \\
&= \begin{bmatrix} 0 & x & x & -k_{e_2}^{(2,3)*}(3) \\ 0 & x & x & -k_{e_2}^{(2,3)*}(2) \\ 1 & x & x & -k_{e_2}^{(2,3)*}(1) \end{bmatrix}, \tag{4.88}
\end{aligned}$$

$$\begin{aligned}
\tilde{\mathbf{A}}_1^{(2,3)} &= \mathbf{J}_3 \mathbf{A}_1^{(2,3)*} \mathbf{J}_4 \\
&= \begin{bmatrix} -k_{e_1}^{(2,3)}(1) & x & x & 0 \\ -k_{e_1}^{(2,3)}(2) & x & x & 0 \\ -k_{e_1}^{(2,3)}(2) & x & x & 1 \end{bmatrix}, \tag{4.89}
\end{aligned}$$

$$\begin{aligned}
\tilde{\mathbf{A}}_2^{(2,3)} &= \mathbf{J}_3 \mathbf{A}_2^{(2,3)*} \mathbf{J}_4 \\
&= \begin{bmatrix} -k_{e_2}^{(2,3)}(1) & x & x & 0 \\ -k_{e_2}^{(2,3)}(2) & x & x & 1 \\ -k_{e_2}^{(2,3)}(2) & x & x & 0 \end{bmatrix}, \tag{4.90}
\end{aligned}$$

and

$$\begin{aligned}
\tilde{\mathbf{A}}_3^{(2,3)} &= \mathbf{J}_3 \mathbf{A}_3^{(2,3)*} \mathbf{J}_4 \\
&= \begin{bmatrix} -k_{e_3}^{(2,3)}(1) & x & x & 1 \\ -k_{e_3}^{(2,3)}(2) & x & x & 0 \\ -k_{e_3}^{(2,3)}(2) & x & x & 0 \end{bmatrix}, \tag{4.91}
\end{aligned}$$

respectively.

Now, let us write the coefficient matrices of the updated horizontal PEFs $\mathbf{B}_1^{(3,2)}$, $\mathbf{B}_2^{(3,2)}$, $\tilde{\mathbf{B}}_1^{(3,2)}$, and $\tilde{\mathbf{B}}_2^{(3,2)}$ as

$$\begin{aligned}
\mathbf{B}_1^{(3,2)} &= \hat{\mathbf{B}}_{1,v}^{(3,1)} - k_{r_1}^{(3,2)*}(1)\tilde{\mathbf{A}}_{1,z_1}^{(2,2)} - k_{r_1}^{(3,2)*}(2)\tilde{\mathbf{A}}_{2,z_1}^{(2,2)} \\
&\quad - k_{r_1}^{(3,2)*}(3)\tilde{\mathbf{A}}_{3,z_1}^{(2,2)} \\
&= \begin{bmatrix} 1 & 0 & 0 \\ x & x & -k_{r_1}^{(3,2)*}(3) \\ x & x & -k_{r_1}^{(3,2)*}(2) \\ x & x & -k_{r_1}^{(3,2)*}(1) \end{bmatrix}, \tag{4.92}
\end{aligned}$$

$$\begin{aligned}
\mathbf{B}_2^{(3,2)} &= \hat{\mathbf{B}}_{2,v}^{(3,1)} - k_{r_2}^{(3,2)*}(1)\tilde{\mathbf{A}}_{1,z_1}^{(2,2)} - k_{r_2}^{(3,2)*}(2)\tilde{\mathbf{A}}_{2,z_1}^{(2,2)} \\
&\quad - k_{r_2}^{(3,2)*}(3)\tilde{\mathbf{A}}_{3,z_1}^{(2,2)} \\
&= \begin{bmatrix} 0 & 1 & 0 \\ x & x & -k_{r_2}^{(3,2)*}(3) \\ x & x & -k_{r_2}^{(3,2)*}(2) \\ x & x & -k_{r_2}^{(3,2)*}(1) \end{bmatrix}, \tag{4.93}
\end{aligned}$$

$$\begin{aligned}
\tilde{\mathbf{B}}_1^{(3,2)} &= \mathbf{J}_4 \mathbf{B}_1^{(3,2)*} \mathbf{J}_3 \\
&= \begin{bmatrix} -k_{r_1}^{(3,2)}(1) & x & x \\ -k_{r_1}^{(3,2)}(2) & x & x \\ -k_{r_1}^{(3,2)}(3) & x & x \\ 0 & 0 & 1 \end{bmatrix}, \tag{4.94}
\end{aligned}$$

and

$$\begin{aligned}
\tilde{\mathbf{B}}_2^{(3,2)} &= \mathbf{J}_4 \mathbf{B}_2^{(3,2)*} \mathbf{J}_3 \\
&= \begin{bmatrix} -k_{r_2}^{(3,2)}(1) & x & x \\ -k_{r_2}^{(3,2)}(2) & x & x \\ -k_{r_2}^{(3,2)}(3) & x & x \\ 0 & 1 & 0 \end{bmatrix}, \tag{4.95}
\end{aligned}$$

respectively.

The coefficient matrices for the newly initiated horizontal PEFs $\mathbf{B}_3^{(3,2)}$, and $\tilde{\mathbf{B}}_3^{(3,2)}$ are defined as

$$\begin{aligned}
\mathbf{B}_3^{(3,2)} &= \mathbf{B}_{3,z_2}^{(3,1)} - k_{r_3}^{(3,2)*}(1)\mathbf{A}_{1,z_1}^{(2,2)} - k_{r_3}^{(3,2)*}(2)\mathbf{A}_{2,z_1}^{(2,2)} - k_{r_3}^{(3,2)*}(3)\mathbf{A}_{3,z_1}^{(2,2)} \\
&= \begin{bmatrix} 0 & 0 & 1 \\ -k_{r_3}^{(3,2)*}(1) & x & x \\ -k_{r_3}^{(3,2)*}(2) & x & x \\ -k_{r_3}^{(3,2)*}(3) & x & x \end{bmatrix}, \tag{4.96}
\end{aligned}$$

and

$$\begin{aligned}
\tilde{\mathbf{B}}_3^{(3,2)} &= \mathbf{J}_4 \mathbf{B}_3^{(3,2)*} \mathbf{J}_3 \\
&= \begin{bmatrix} x & x & -k_{r_3}^{(3,2)}(1) \\ x & x & -k_{r_3}^{(3,2)}(2) \\ x & x & -k_{r_3}^{(3,2)}(1) \\ 1 & 0 & 0 \end{bmatrix}, \tag{4.97}
\end{aligned}$$

respectively.

By now, we introduced how to calculate coefficient matrices of the horizontal and vertical PEFs. It is clearly seen that first column or first row of the coefficient matrices

calculated for the forward PEFs contain 0s except the point presented with 1 where PEF is initiated. In addition the last row or the last column contains the computed lattice coefficient vectors depending on the augmentation type.

4.2 Transfer Functions of the Proposed Lattice Prediction Models

In the above sections, we explained how to find coefficient matrices by using the calculated lattice reflection vectors in case of horizontal and vertical directions, and at that point, we defined the coefficient matrices of all horizontal and vertical PEFs. Now, we will present how to obtain the coefficient matrices for the lattice prediction models as QP, ASHP, and NC half-plane.

4.2.1 Transfer functions of the QP prediction error fields

In Section 3.2, we presented how to define a QP lattice model and to calculate the QP lattice filter coefficient vector. Now, we will define a coefficient matrix equation by using the coefficient matrices of the related PEFs.

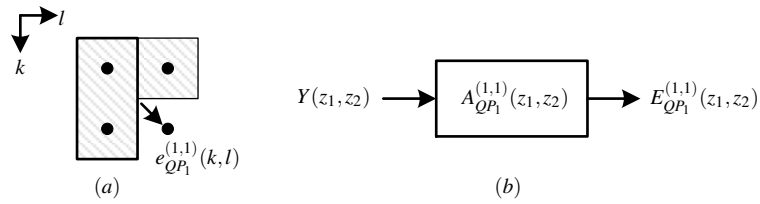


Figure 4.1: The first-quadrant QP lattice model and its z -transform demonstration of the PSR $S_{k,l}^{(1,0)}$ using the vertical PEFs.

For example, let us assume that we have a given PSR $S_{k,l}^{(1,0)}$ and its z -transform demonstration shown in Figure 4.1. From the Figure 4.1(b), we can write that

$$E_{QP_1}^{(1,1)}(z_1, z_2) = A_{QP_1}^{(1,1)}(z_1, z_2)Y(z_1, z_2). \quad (4.98)$$

The transfer function $A_{QP_1}^{(1,1)}(z_1, z_2)$ can be defined as

$$A_{QP_1}^{(1,1)}(z_1, z_2) = \begin{bmatrix} 1 & z_1^{-1} \end{bmatrix} \underbrace{\begin{bmatrix} 1 & x \\ x & x \end{bmatrix}}_{\mathbf{A}_{Q_1}^{(1,1)}} \begin{bmatrix} 1 \\ z_2^{-1} \end{bmatrix}. \quad (4.99)$$

We need to determine how to find the coefficient matrix $\mathbf{A}_{Q_1}^{(1,1)}$. We know that the QP PEFs equation is

$$e_{QP_1}^{(1,1)}(k, l) = e_1^{(1,1)}(k, l) - k_{e_{QP_1}}^{(1,1)*} e_2^{(1,1)}(k, l). \quad (4.100)$$

If we write the equation (4.100) in terms of coefficient matrices, then we can write it as

$$\begin{aligned}\mathbf{A}_{QP_1}^{(1,1)} &= \underbrace{\begin{bmatrix} 1 & x_1 \\ 0 & x_2 \end{bmatrix}}_{\mathbf{A}_1^{(1,1)}} - k_{e_{QP_1}}^{(1,1)*} \underbrace{\begin{bmatrix} 0 & y_1 \\ 1 & y_2 \end{bmatrix}}_{\mathbf{A}_2^{(1,1)}} \\ &= \begin{bmatrix} 1 & x_1 - y_1 k_{e_{QP_1}}^{(1,1)*} \\ -k_{e_{QP_1}}^{(1,1)*} & x_2 - y_2 k_{e_{QP_1}}^{(1,1)*} \end{bmatrix}.\end{aligned}\quad (4.101)$$

The second vertical QP PEFs coefficient matrix can be given as

$$\begin{aligned}\mathbf{A}_{QP_2}^{(1,1)} &= \underbrace{\begin{bmatrix} 1 & y_1 \\ 0 & y_2 \end{bmatrix}}_{\mathbf{A}_2^{(1,1)}} - k_{e_{QP_2}}^{(1,1)*} \underbrace{\begin{bmatrix} 0 & x_1 \\ 1 & x_2 \end{bmatrix}}_{\mathbf{A}_1^{(1,1)}} \\ &= \begin{bmatrix} -k_{e_{QP_1}}^{(1,1)*} & y_1 - x_1 k_{e_{QP_1}}^{(1,1)*} \\ 1 & y_2 - x_2 k_{e_{QP_1}}^{(1,1)*} \end{bmatrix}.\end{aligned}\quad (4.102)$$

The backward coefficient matrices $\tilde{\mathbf{A}}_{QP_1}^{(1,1)}$ and $\tilde{\mathbf{A}}_{QP_2}^{(1,1)}$ are found as

$$\tilde{\mathbf{A}}_{QP_1}^{(1,1)} = \mathbf{J}_2 \mathbf{A}_{QP_1}^{(1,1)*} \mathbf{J}_2 \quad (4.103)$$

$$\tilde{\mathbf{A}}_{QP_2}^{(1,1)} = \mathbf{J}_2 \mathbf{A}_{QP_2}^{(1,1)*} \mathbf{J}_2. \quad (4.104)$$

After this example, a general equation can be prepared for the QP PEFs produced from the auxiliary vertical PEFs. We have already introduced in Section 3.2, for the PSR $S_{k,l}^{(p-1,q-1)}$, the first-quadrant QP model forward error propagation equation can be written to predict the first one, $e_1^{(p-1,q)}(k,l)$, from the rest of the vertical PEFs as

$$e_{QP_1}^{(p-1,q)}(k,l) = e_1^{(p-1,q)}(k,l) - \mathbf{k}_{e_{QP_1}}^{(p-1,q)H} \boldsymbol{\Psi}_1 \mathbf{e}^{(p-1,q)}(k,l), \quad (4.105)$$

and we can write the second-quadrant forward error propagation equation as

$$e_{QP_2}^{(p-1,q)}(k,l) = e_p^{(p-1,q)}(k,l) - \mathbf{k}_{e_{QP_2}}^{(p-1,q)H} \boldsymbol{\Psi}_p \mathbf{e}^{(p-1,q)}(k,l), \quad (4.106)$$

where

$$\begin{aligned}\mathbf{e}^{(p-1,q)}(k,l) &= \begin{bmatrix} e_1^{(p-1,q)}(k,l) & \dots & e_p^{(p-1,q)}(k,l) \end{bmatrix}^T, \\ \mathbf{k}_{e_{QP_1}}^{(p-1,q)} &= \begin{bmatrix} k_{e_{QP_1}}^{(p-1,q)} & \dots & k_{e_{QP_1}}^{(p-1,q)} \end{bmatrix}^T, \\ \mathbf{k}_{e_{QP_2}}^{(p-1,q)} &= \begin{bmatrix} k_{e_{QP_2}}^{(p-1,q)} & \dots & k_{e_{QP_2}}^{(p-1,q)} \end{bmatrix}^T.\end{aligned}$$

We can define the equations (4.105) and (4.106) in terms of the coefficient matrices of the related PEFs as

$$\mathbf{A}_{QP_1}^{(p-1,q)} = \mathbf{A}_1^{(p-1,q)} - \sum_{i=1}^{p-1} k_{e_{QP_1}}^{(p-1,q)*}(i) \mathbf{A}_{1+i}^{(p-1,q)}, \quad (4.107)$$

$$\mathbf{A}_{QP_2}^{(p-1,q)} = \mathbf{A}_p^{(p-1,q)} - \sum_{i=1}^{p-1} k_{e_{QP_2}}^{(p-1,q)*}(i) \mathbf{A}_i^{(p-1,q)}, \quad (4.108)$$

respectively. The backward coefficient matrices of (4.107) and (4.108) are given as

$$\tilde{\mathbf{A}}_{QP_1}^{(p-1,q)} = \mathbf{J}_p \mathbf{A}_{QP_1}^{(p-1,q)*} \mathbf{J}_{q+1}, \quad (4.109)$$

$$\tilde{\mathbf{A}}_{QP_2}^{(p-1,q)} = \mathbf{J}_p \mathbf{A}_{QP_2}^{(p-1,q)*} \mathbf{J}_{q+1}. \quad (4.110)$$

Similarly, we can write the forward propagation equations of the first- and second-quadrant QP PEFs as follows

$$r_{QP_1}^{(p,q-1)}(k,l) = r_1^{(p,q-1)}(k,l) - \mathbf{k}_{r_{QP_1}}^{(p,q-1)H} \boldsymbol{\Psi}_1 \mathbf{r}^{(p,q-1)}(k,l), \quad (4.111)$$

$$r_{QP_2}^{(p,q-1)}(k,l) = r_p^{(p,q-1)}(k,l) - \mathbf{k}_{r_{QP_2}}^{(p,q-1)H} \boldsymbol{\Psi}_q \mathbf{r}^{(p,q-1)}(k,l), \quad (4.112)$$

where

$$\begin{aligned} \mathbf{r}^{(p,q-1)}(k,l) &= \begin{bmatrix} r_1^{(p,q-1)}(k,l) & \dots & r_p^{(p,q-1)}(k,l) \end{bmatrix}^T, \\ \mathbf{k}_{r_{QP_1}}^{(p,q-1)} &= \begin{bmatrix} k_{r_{QP_1}}^{(p,q-1)} & \dots & k_{r_{QP_1}}^{(p,q-1)} \end{bmatrix}^T, \\ \mathbf{k}_{r_{QP_2}}^{(p,q-1)} &= \begin{bmatrix} k_{r_{QP_2}}^{(p,q-1)} & \dots & k_{r_{QP_2}}^{(p,q-1)} \end{bmatrix}^T. \end{aligned}$$

So far, we introduced the equations to find the coefficient matrices of the QP models obtained using vertical and horizontal PEFs. Now we apply them on the PSR $S_{k,l}^{(3,1)}$ as shown in Figure 4.2.

The error propagation equations for forward vertical PEFs $e_{QP_1}^{(3,2)}$ and $e_{QP_2}^{(3,2)}$ can be written as

$$e_{QP_1}^{(3,2)}(k,l) = e_1^{(3,2)}(k,l) - \sum_{i=1}^3 \mathbf{k}_{e_{QP_1}}^{(3,2)*}(i) e_{i+1}^{(3,2)}(k,l), \quad (4.113)$$

$$e_{QP_2}^{(3,2)}(k,l) = e_4^{(3,2)}(k,l) - \sum_{i=1}^3 \mathbf{k}_{e_{QP_2}}^{(3,2)*}(i) e_i^{(3,2)}(k,l). \quad (4.114)$$

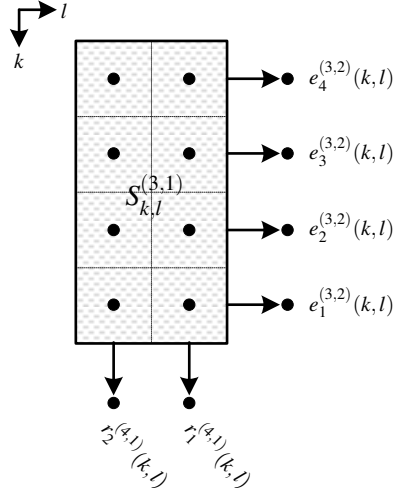


Figure 4.2: The forward vertical and horizontal auxiliary PEFs of the PSR $S_{k,l}^{(3,1)}$.

The coefficient matrices can be defined from the equations (4.113) and (4.114) as follows

$$\begin{aligned} \mathbf{A}_{QP_1}^{(3,2)}(k,l) &= \mathbf{A}_1^{(3,2)}(k,l) - \sum_{i=1}^3 \mathbf{k}_{e_{QP_1}}^{(3,2)*}(i) \mathbf{A}_{i+1}^{(3,2)}(k,l) \\ &= \begin{bmatrix} 1 & x & x \\ -k_{e_{QP_1}}^{(3,2)*}(1) & x & x \\ -k_{e_{QP_1}}^{(3,2)*}(2) & x & x \\ -k_{e_{QP_1}}^{(3,2)*}(3) & x & x \end{bmatrix}, \end{aligned} \quad (4.115)$$

and

$$\begin{aligned} \mathbf{A}_{QP_2}^{(3,2)} &= \mathbf{A}_4^{(3,2)} - \sum_{i=1}^3 \mathbf{k}_{e_{QP_2}}^{(3,2)*}(i) \mathbf{A}_i^{(3,2)} \\ &= \begin{bmatrix} -k_{e_{QP_2}}^{(3,2)*}(1) & x & x \\ -k_{e_{QP_2}}^{(3,2)*}(2) & x & x \\ -k_{e_{QP_2}}^{(3,2)*}(3) & x & x \\ 1 & x & x \end{bmatrix}. \end{aligned} \quad (4.116)$$

The backward of the coefficient matrices are defined as

$$\tilde{\mathbf{A}}_{QP_1}^{(3,2)} = \mathbf{J}_4 \mathbf{A}_{QP_1}^{(3,2)*} \mathbf{J}_3, \quad (4.117)$$

$$\tilde{\mathbf{A}}_{QP_2}^{(3,2)} = \mathbf{J}_4 \mathbf{A}_{QP_2}^{(3,2)*} \mathbf{J}_3. \quad (4.118)$$

In order to find coefficient matrices of the error propagation equations for forward horizontal PEFs $r_{QP_1}^{(4,1)}$ and $r_{QP_2}^{(4,1)}$, we can write the following equations

$$r_{QP_1}^{(4,1)}(k,l) = r_1^{(4,1)}(k,l) - \mathbf{k}_{r_{QP_1}}^{(4,1)*} r_2^{(4,1)}(k,l), \quad (4.119)$$

$$r_{QP_2}^{(4,1)}(k,l) = r_2^{(4,1)}(k,l) - \mathbf{k}_{r_{QP_2}}^{(4,1)*} r_2^{(4,1)}(k,l). \quad (4.120)$$

And we can write the coefficient matrices from the equations (4.119) and (4.120) as follows

$$\begin{aligned}
\mathbf{B}_{QP_1}^{(4,1)} &= \mathbf{B}_1^{(4,1)} - \mathbf{k}_{rQP_1}^{(4,1)*} \mathbf{B}_2^{(4,1)} \\
&= \underbrace{\begin{bmatrix} 1 & 0 \\ x_1 & x_2 \\ x_3 & x_4 \\ x_5 & x_6 \\ x_7 & x_8 \end{bmatrix}}_{\mathbf{B}_1^{(4,1)}} - \mathbf{k}_{rQP_1}^{(4,1)*} \underbrace{\begin{bmatrix} 0 & 1 \\ y_1 & y_2 \\ y_3 & y_4 \\ y_5 & y_6 \\ y_7 & y_8 \end{bmatrix}}_{\mathbf{B}_2^{(4,1)}} \\
&= \begin{bmatrix} 1 & -k_{rQP_1}^{(4,1)*} \\ x_1 - k_{rQP_1}^{(4,1)*} y_1 & x_2 - k_{rQP_1}^{(4,1)*} y_2 \\ x_3 - k_{rQP_1}^{(4,1)*} y_3 & x_4 - k_{rQP_1}^{(4,1)*} y_4 \\ x_5 - k_{rQP_1}^{(4,1)*} y_5 & x_6 - k_{rQP_1}^{(4,1)*} y_6 \\ x_7 - k_{rQP_1}^{(4,1)*} y_7 & x_8 - k_{rQP_1}^{(4,1)*} y_8 \end{bmatrix}, \tag{4.121}
\end{aligned}$$

$$\begin{aligned}
\mathbf{B}_{QP_2}^{(4,1)} &= \mathbf{B}_2^{(4,1)} - \mathbf{k}_{rQP_2}^{(4,1)*} \mathbf{B}_1^{(4,1)} \\
&= \underbrace{\begin{bmatrix} 0 & 1 \\ y_1 & y_2 \\ y_3 & y_4 \\ y_5 & y_6 \\ y_7 & y_8 \end{bmatrix}}_{\mathbf{B}_2^{(4,1)}} - \mathbf{k}_{rQP_2}^{(4,1)*} \underbrace{\begin{bmatrix} 1 & 0 \\ x_1 & x_2 \\ x_3 & x_4 \\ x_5 & x_6 \\ x_7 & x_8 \end{bmatrix}}_{\mathbf{B}_1^{(4,1)}} \\
&= \begin{bmatrix} -k_{rQP_2}^{(4,1)*} & 1 \\ y_1 - k_{rQP_2}^{(4,1)*} x_1 & y_2 - k_{rQP_2}^{(4,1)*} x_2 \\ y_3 - k_{rQP_2}^{(4,1)*} x_3 & y_4 - k_{rQP_2}^{(4,1)*} x_4 \\ y_5 - k_{rQP_2}^{(4,1)*} x_5 & y_6 - k_{rQP_2}^{(4,1)*} x_6 \\ y_7 - k_{rQP_2}^{(4,1)*} x_7 & y_8 - k_{rQP_2}^{(4,1)*} x_8 \end{bmatrix}. \tag{4.122}
\end{aligned}$$

The backward of the coefficient matrices of $\mathbf{B}_{QP_1}^{(4,1)}$ and $\mathbf{B}_{QP_2}^{(4,1)}$ are defined as

$$\tilde{\mathbf{B}}_{QP_1}^{(4,1)} = \mathbf{J}_5 \mathbf{B}_{QP_1}^{(4,1)*} \mathbf{J}_2, \tag{4.123}$$

$$\tilde{\mathbf{B}}_{QP_2}^{(4,1)} = \mathbf{J}_5 \mathbf{B}_{QP_2}^{(4,1)*} \mathbf{J}_2. \tag{4.124}$$

4.2.2 Coefficient Matrices of ASHP Prediction Error Fields

Herein, we will explain how to obtain coefficient matrices for type-1 and type-2 ASHP models from the horizontal and vertical PEFs. In addition, some examples will be given. First, the type-1 and type-2 ASHP models using the horizontal PEFs will be

taken into account, and then the coefficient matrices of the type-1 and type-2 ASHP models using the vertical PEFs will be determined.

4.2.2.1 Coefficient matrices of type-1 ASHP models using horizontal PEFs

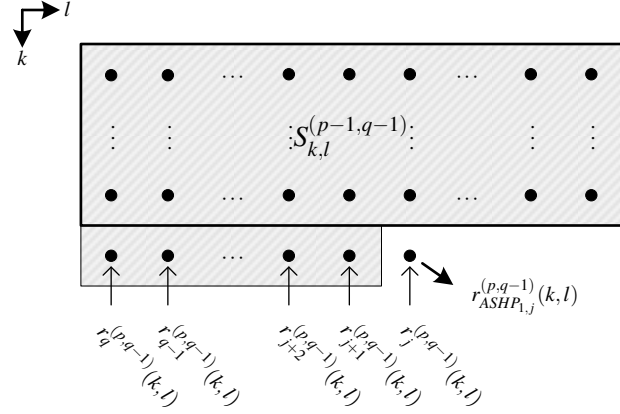


Figure 4.3: The forward horizontal auxiliary PEFs of the PSR $S_{k,l}^{(p-1,q-1)}$ for the type-1 ASHP Model.

For the type-1 ASHP prediction error fields predicted at the j th horizontal PEFs of the PSR $S_{k,l}^{(p-1,q-1)}$ shown in Figure 4.3, the forward error propagation equation can be written as, for $j = 2, 3, \dots, q-1$

$$r_{ASHP_{1,j}}^{(p,q-1)}(k,l) = r_j^{(p,q-1)}(k,l) - \sum_{i=1}^{q-j} \mathbf{k}_{r_{ASHP_{1,j}}}^{(p,q-1)*}(i) r_{j+i}^{(p,q-1)}(k,l). \quad (4.125)$$

The coefficient matrix of type-1 ASHP can be determined from (4.125) as

$$\begin{aligned}
\mathbf{B}_{ASHP_{1,j}}^{(p,q-1)} &= \mathbf{B}_j^{(p,q-1)} - \sum_{i=1}^{q-j} \mathbf{k}_{r_{ASHP_{1,j}}}^{(p,q-1)*}(i) \mathbf{B}_{j+i}^{(p,q-1)} \\
&= \underbrace{\begin{bmatrix} \overbrace{0 \cdots 0}^{j-1} & \overbrace{1}^j & \overbrace{0 \cdots 0}^{q-j} \\ x & \cdots & x \\ \vdots & \cdots & \vdots \\ x & \cdots & x \end{bmatrix}}_{\mathbf{B}_j^{(p,q-1)}} \\
&\quad - \sum_{i=1}^{q-j} \mathbf{k}_{r_{ASHP_{1,j}}}^{(p,q-1)*}(i) \underbrace{\begin{bmatrix} \overbrace{0 \cdots 0}^{j+i} & \overbrace{1}^{j+i} & \overbrace{0 \cdots 0}^{q-j-i} \\ x & \cdots & x \\ \vdots & \cdots & \vdots \\ x & \cdots & x \end{bmatrix}}_{\mathbf{B}_{j+i}^{(p,q-1)}} \\
&= \begin{bmatrix} \overbrace{0 \cdots 0}^{j-1} & \overbrace{1}^j & -k_{ASHP_{1,j}}^{(p,q-1)*}(1) & \cdots & -k_{ASHP_{1,j}}^{(p,q-1)*}(q-j) \\ * & \cdots & * & \cdots & * \\ \vdots & \cdots & \vdots & \cdots & \vdots \\ * & \cdots & * & \cdots & * \end{bmatrix}_{(p+1) \times q} \quad (4.126)
\end{aligned}$$

The backward coefficient matrix $\tilde{\mathbf{B}}_{ASHP_{1,j}}^{(p,q-1)}$ can be calculated as

$$\tilde{\mathbf{B}}_{ASHP_{1,j}}^{(p,q-1)} = \mathbf{J}_{p+1} \mathbf{B}_{ASHP_{1,j}}^{(p,q-1)*} \mathbf{J}_q. \quad (4.127)$$

In Section 3.3.3.2, we introduced how to find type-1 ASHP lattice model using horizontal PEFs of the PSR $S_{k,l}^{(0,2)}$ as shown in Figure 3.15. If we want to find the coefficient matrix of this type-1 ASHP lattice model, we can write the lattice error propagation equations in terms of the coefficient matrices of the corresponding horizontal PEFs. From the equation (3.131), we can define the forward error propagation equation as

$$r_{ASHP_{1,2}}^{(1,2)}(k,l) = r_2^{(1,2)}(k,l) - \mathbf{k}_{r_{ASHP_{1,2}}}^{(1,2)*} r_3^{(1,2)}(k,l). \quad (4.128)$$

From (4.128), the following coefficient matrix equation can be written as

$$\begin{aligned}
\mathbf{B}_{ASHP_{1,2}}^{(1,2)} &= \mathbf{B}_2^{(1,2)} - \mathbf{k}_{r_{ASHP_{1,2}}}^{(1,2)*} \mathbf{B}_3^{(1,2)} \\
&= \begin{bmatrix} 0 & 1 & 0 \\ x_1 & x_2 & x_3 \end{bmatrix} - \mathbf{k}_{r_{ASHP_{1,2}}}^{(1,2)*} \begin{bmatrix} 0 & 0 & 1 \\ y_1 & y_2 & y_3 \end{bmatrix} \\
&= \begin{bmatrix} 0 & 1 & -k_{ASHP_{1,2}}^{(1,2)*} \\ x_1 - y_1 k_{ASHP_{1,2}}^{(1,2)*} & x_2 - y_2 k_{ASHP_{1,2}}^{(1,2)*} & x_3 - y_3 k_{ASHP_{1,2}}^{(1,2)*} \end{bmatrix}_{2 \times 3} \quad (4.129)
\end{aligned}$$

The backward coefficient matrix is found as

$$\tilde{\mathbf{B}}_{ASHP_{1,2}}^{(1,2)} = \mathbf{J}_2 \mathbf{B}_{ASHP_{1,2}}^{(1,2)*} \mathbf{J}_3. \quad (4.130)$$

4.2.2.2 Coefficient matrices of type-2 ASHP models using horizontal PEFs

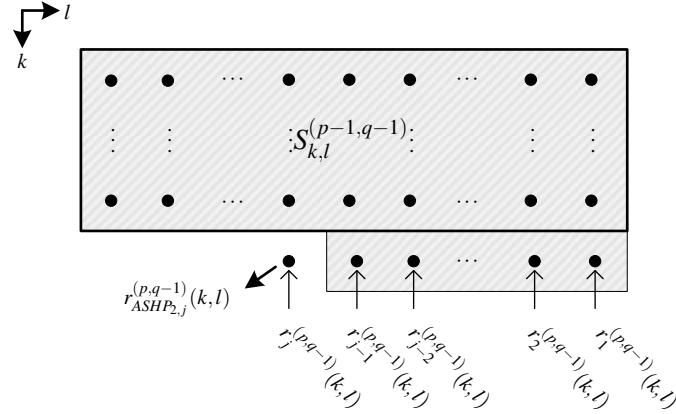


Figure 4.4: The forward horizontal auxiliary PEFs of the PSR $S_{k,l}^{(p-1,q-1)}$ for the type-2 ASHP Model.

So as to find the coefficient matrix for the type-2 ASHP prediction error fields predicted at the j th horizontal PEFs of the PSR $S_{k,l}^{(p-1,q-1)}$ shown in Figure 4.4, the forward error propagation equation can be written as, for $j = 2, \dots, q-1$

$$r_{ASHP_{2,j}}^{(p,q-1)}(k,l) = r_j^{(p,q-1)}(k,l) - \sum_{i=1}^{q-j} \mathbf{k}_{r_{ASHP_{2,j}}}^{(p,q-1)*}(i) r_i^{(p,q-1)}(k,l). \quad (4.131)$$

The coefficient matrix of type-2 ASHP can be determined from (4.131) as

$$\begin{aligned}
\mathbf{B}_{ASHP_{2,j}}^{(p,q-1)} &= \mathbf{B}_j^{(p,q-1)} - \sum_{i=1}^{j-1} \mathbf{k}_{r_{ASHP_{2,j}}}^{(p,q-1)*}(i) \mathbf{B}_i^{(p,q-1)} \\
&= \underbrace{\begin{bmatrix} 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ x & \cdots & x & x & x & \cdots & x \\ \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ x & \cdots & x & x & x & \cdots & x \end{bmatrix}}_{\mathbf{B}_j^{(p,q-1)}} \\
&\quad - \sum_{i=1}^{q-j} \mathbf{k}_{r_{ASHP_{1,j}}}^{(p,q-1)*}(i) \underbrace{\begin{bmatrix} 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ x & \cdots & x & x & x & \cdots & x \\ \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ x & \cdots & x & x & x & \cdots & x \end{bmatrix}}_{\mathbf{B}_i^{(p,q-1)}} \\
&= \begin{bmatrix} -k_{ASHP_{2,j}}^{(p,q-1)*}(1) & \cdots & -k_{ASHP_{1,j}}^{(p,q-1)*}(j-1) & 1 & 0 & \cdots & 0 \\ * & \cdots & * & * & * & \cdots & * \\ \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ * & \cdots & * & * & * & \cdots & * \end{bmatrix}_{(p+1) \times q} \quad (4.132)
\end{aligned}$$

The backward coefficient matrix $\tilde{\mathbf{B}}_{ASHP_{2,j}}^{(p,q-1)}$ can be determined as

$$\tilde{\mathbf{B}}_{ASHP_{2,j}}^{(p,q-1)} = \mathbf{J}_{p+1} \mathbf{B}_{ASHP_{2,j}}^{(p,q-1)*} \mathbf{J}_q. \quad (4.133)$$

In Section 3.3.3.2, the construction of the type-2 ASHP lattice model using horizontal PEFs of the PSR $S_{k,l}^{(0,2)}$ as shown in Figure 3.16, is also introduced. In order to find coefficient matrix of this type-2 ASHP lattice model, from the equation (3.139), we can write the forward error propagation equation as

$$r_{ASHP_{2,2}}^{(1,2)}(k,l) = r_2^{(1,2)}(k,l) - \mathbf{k}_{r_{ASHP_{2,2}}}^{(1,2)*} r_1^{(1,2)}(k,l). \quad (4.134)$$

From (4.134), the following coefficient matrix equation can be written as

$$\begin{aligned}
\mathbf{B}_{ASHP_{2,2}}^{(1,2)} &= \mathbf{B}_2^{(1,2)} - \mathbf{k}_{r_{ASHP_{2,2}}}^{(1,2)*} \mathbf{B}_1^{(1,2)} \\
&= \begin{bmatrix} 0 & 1 & 0 \\ x_1 & x_2 & x_3 \end{bmatrix} - \mathbf{k}_{r_{ASHP_{1,2}}}^{(1,2)*} \begin{bmatrix} 1 & 0 & 0 \\ t_1 & t_2 & t_3 \end{bmatrix} \\
&= \begin{bmatrix} -k_{ASHP_{2,2}}^{(1,2)*} & 1 & 0 \\ x_1 - t_1 k_{ASHP_{2,2}}^{(1,2)*} & x_2 - t_2 k_{ASHP_{1,2}}^{(2,2)*} & x_3 - t_3 k_{ASHP_{1,2}}^{(2,2)*} \end{bmatrix}_{2 \times 3}. \quad (4.135)
\end{aligned}$$

The backward coefficient matrix can be calculated as

$$\tilde{\mathbf{B}}_{ASHP_{2,2}}^{(1,2)} = \mathbf{J}_2 \mathbf{B}_{ASHP_{2,2}}^{(1,2)*} \mathbf{J}_3. \quad (4.136)$$

4.2.2.3 Coefficient matrices of type-1 ASHP models using vertical PEFs

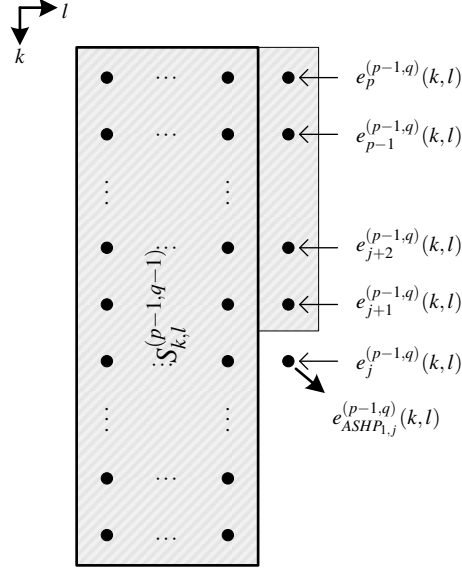


Figure 4.5: The forward vertical auxiliary PEFs of the PSR $S_{k,l}^{(p-1,q-1)}$ for the type-1 ASHP Model.

Similar to the type-1 ASHP model obtained by using horizontal PEFs, we can also determine the coefficient matrix for the type-1 ASHP prediction error fields predicted at the j th vertical PEFs of the PSR $S_{k,l}^{(p-1,q-1)}$ shown in Figure 4.5. The forward error propagation equation can be written as, for $j = 2, 3, \dots, p-1$

$$e_{ASHP_{1,j}}^{(p-1,q)}(k,l) = e_j^{(p-1,q)}(k,l) - \sum_{i=1}^{p-j} \mathbf{k}_{e_{ASHP_{1,j}}}^{(p-1,q)*}(i) e_{j+i}^{(p-1,q)}(k,l). \quad (4.137)$$

The coefficient matrix of type-1 ASHP can be determined from (4.137) as

$$\begin{aligned}
\mathbf{A}_{ASHP_{1,j}}^{(p-1,q)} &= \mathbf{A}_j^{(p-1,q)} - \sum_{i=1}^{p-j} \mathbf{k}_{e_{ASHP_{1,j}}}^{(p-1,q)*}(i) \mathbf{A}_{j+i}^{(p-1,q)} \\
&= \underbrace{\begin{bmatrix} 0 & \cdots & x & x & \cdots & x \\ \vdots & \cdots & \vdots & \vdots & \cdots & \vdots \\ 0 & \cdots & x & x & \cdots & x \\ 1 & \cdots & x & x & \cdots & x \\ 0 & \cdots & x & x & \cdots & x \\ \vdots & \cdots & \vdots & \vdots & \cdots & \vdots \\ 0 & \cdots & x & x & \cdots & x \end{bmatrix}}_{\mathbf{A}_j^{(p-1,q)}} - \sum_{i=1}^{p-j} \mathbf{k}_{e_{ASHP_{1,j}}}^{(p-1,q)*}(i) \underbrace{\begin{bmatrix} 0 & \cdots & x & x & \cdots & x \\ \vdots & \cdots & \vdots & \vdots & \cdots & \vdots \\ 0 & \cdots & x & x & \cdots & x \\ 1 & \cdots & x & x & \cdots & x \\ 0 & \cdots & x & x & \cdots & x \\ \vdots & \cdots & \vdots & \vdots & \cdots & \vdots \\ 0 & \cdots & x & x & \cdots & x \end{bmatrix}}_{\mathbf{A}_{j+i}^{(p-1,q)}} \\
&= \begin{bmatrix} 0 & \cdots & * & * & \cdots & * \\ \vdots & \cdots & \vdots & \vdots & \cdots & \vdots \\ 0 & \cdots & * & * & \cdots & * \\ 1 & \cdots & * & * & \cdots & * \\ -k_{ASHP_{1,j}}^{(p-1,q)*}(1) & \cdots & * & * & \cdots & * \\ \vdots & \cdots & \vdots & \vdots & \cdots & \vdots \\ -k_{ASHP_{1,j}}^{(p-1,q)*}(p-j) & \cdots & * & * & \cdots & * \end{bmatrix}_{p \times (q+1)}. \tag{4.138}
\end{aligned}$$

The backward coefficient matrix $\tilde{\mathbf{A}}_{ASHP_{1,j}}^{(p-1,q)}$ can be calculated as

$$\tilde{\mathbf{A}}_{ASHP_{1,j}}^{(p-1,q)} = \mathbf{J}_p \mathbf{A}_{ASHP_{1,j}}^{(p-1,q)*} \mathbf{J}_{q+1}. \tag{4.139}$$

As an example we can find the coefficient matrix of the PSR $S_{k,l}^{(2,0)}$ which is presented in Section 3.3.3.1. The Figure 3.12 shows the type-1 ASHP lattice model construction from the related vertical PEFs. The forward error propagation equation is given as

$$e_{ASHP_{1,2}}^{(2,1)}(k,l) = e_2^{(2,1)}(k,l) - \mathbf{k}_{e_{ASHP_{1,2}}}^{(2,1)*} e_3^{(1,2)}(k,l). \tag{4.140}$$

We can write the forward error propagation equation (4.140) in terms of the coefficient matrices of the corresponding vertical PEFs as

$$\begin{aligned}
\mathbf{A}_{ASHP_{1,2}}^{(2,1)} &= \mathbf{A}_2^{(2,1)} - \mathbf{k}_{e_{ASHP_{1,2}}}^{(2,1)*} \mathbf{A}_3^{(2,1)} \\
&= \begin{bmatrix} 0 & x_1 \\ 1 & x_2 \\ 0 & x_3 \end{bmatrix} - \mathbf{k}_{e_{ASHP_{1,2}}}^{(2,1)*} \begin{bmatrix} 0 & y_1 \\ 0 & y_2 \\ 1 & y_3 \end{bmatrix} \\
&= \begin{bmatrix} 0 & x_1 - y_1 k_{ASHP_{1,2}}^{(2,1)*} \\ 1 & x_2 - y_2 k_{ASHP_{1,2}}^{(2,1)*} \\ k_{ASHP_{1,2}}^{(2,1)*} & x_3 - y_3 k_{ASHP_{1,2}}^{(1,2)*} \end{bmatrix}_{3 \times 2}. \tag{4.141}
\end{aligned}$$

The backward coefficient matrix is found as

$$\tilde{\mathbf{A}}_{ASHP_{1,2}}^{(2,1)} = \mathbf{J}_3 \mathbf{A}_{ASHP_{1,2}}^{(2,1)*} \mathbf{J}_2. \quad (4.142)$$

4.2.2.4 Coefficient matrices of type-2 ASHP models using vertical PEFs

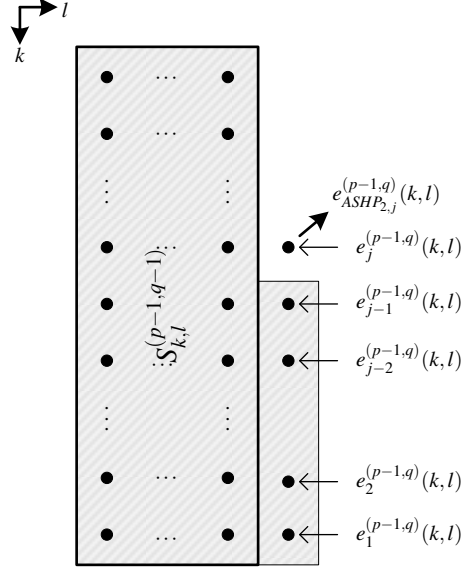


Figure 4.6: The forward horizontal auxiliary PEFs of the PSR $S_{k,l}^{(p-1,q-1)}$ for the type-2 ASHP Model.

For the type-2 ASHP prediction error fields predicted at the j th vertical PEFs of the PSR $S_{k,l}^{(p-1,q-1)}$ shown in Figure 4.6, the forward error propagation equation can be written as, for $j = 2, \dots, p-1$

$$e_{ASHP_{2,j}}^{(p-1,q)}(k,l) = e_j^{(p-1,q)}(k,l) - \sum_{i=1}^{p-j} \mathbf{k}_{e_{ASHP_{2,j}}}^{(p-1,q)*}(i) e_i^{(p-1,q)}(k,l). \quad (4.143)$$

The coefficient matrix of type-2 ASHP can be determined from (4.143) as

$$\begin{aligned} \mathbf{A}_{ASHP_{2,j}}^{(p-1,q)} &= \mathbf{A}_j^{(p-1,q)} - \sum_{i=1}^{j-1} \mathbf{k}_{e_{ASHP_{2,j}}}^{(p-1,q)*}(i) \mathbf{A}_i^{(p-1,q)} \\ &= \begin{bmatrix} -k_{ASHP_{2,j}}^{(p-1,q)*}(1) & \dots & * & * & \dots & * \\ -k_{ASHP_{2,j}}^{(p-1,q)*}(2) & \dots & * & * & \dots & * \\ \vdots & \dots & \vdots & \vdots & \dots & \vdots \\ -k_{ASHP_{2,j}}^{(p-1,q)*}(j-1) & \dots & * & * & \dots & * \\ 1 & \dots & * & * & \dots & * \\ 0 & \dots & * & * & \dots & * \\ \vdots & \dots & \vdots & \vdots & \dots & \vdots \\ 0 & \dots & * & * & \dots & * \end{bmatrix}_{p \times (q+1)}. \end{aligned} \quad (4.144)$$

The backward coefficient matrix $\tilde{\mathbf{A}}_{ASHP_{2,j}}^{(p-1,q)}$ can be determined as

$$\tilde{\mathbf{A}}_{ASHP_{2,j}}^{(p-1,q)} = \mathbf{J}_p \mathbf{A}_{ASHP_{2,j}}^{(p-1,q)*} \mathbf{J}_{q+1}. \quad (4.145)$$

Herein, we can also find the coefficient matrix of the PSR $S_{k,l}^{(2,0)}$ for type-2 ASHP lattice model which is presented in Section 3.3.3.1. The Figure 3.14 shows the type-2 ASHP lattice model construction from the related vertical PEFs. In the equation (3.126), the forward error propagation equation is given as

$$e_{ASHP_{2,2}}^{(2,1)}(k,l) = e_2^{(2,1)}(k,l) - \mathbf{k}_{e_{ASHP_{2,2}}}^{(2,1)*} e_1^{(1,2)}(k,l). \quad (4.146)$$

The forward error propagation equation (4.146) can be written in terms of the coefficient matrices of the corresponding vertical PEFs as

$$\begin{aligned} \mathbf{A}_{ASHP_{2,2}}^{(2,1)} &= \mathbf{A}_2^{(2,1)} - \mathbf{k}_{e_{ASHP_{2,2}}}^{(2,1)*} \mathbf{A}_1^{(2,1)} \\ &= \begin{bmatrix} 0 & x_1 \\ 1 & x_2 \\ 0 & x_3 \end{bmatrix} - \mathbf{k}_{e_{ASHP_{1,2}}}^{(2,1)*} \begin{bmatrix} 0 & t_1 \\ 0 & t_2 \\ 1 & t_3 \end{bmatrix} \\ &= \begin{bmatrix} k_{ASHP_{2,2}}^{(2,1)*} & x_1 - t_1 k_{ASHP_{2,2}}^{(2,1)*} \\ 1 & x_2 - t_2 k_{ASHP_{2,2}}^{(2,1)*} \\ 0 & x_3 - t_3 k_{ASHP_{2,2}}^{(1,2)*} \end{bmatrix}_{3 \times 2}. \end{aligned} \quad (4.147)$$

The backward coefficient matrix can be found as

$$\tilde{\mathbf{A}}_{ASHP_{2,2}}^{(2,1)} = \mathbf{J}_3 \mathbf{A}_{ASHP_{2,2}}^{(2,1)*} \mathbf{J}_2. \quad (4.148)$$

4.2.3 Coefficient matrices of NC half-plane prediction error fields

We know that it is possible to have NC half-plane lattice filters by using vertical or horizontal forward/backward PEFs presented in Section 3.4.1 and 3.4.2, respectively. In the following sections, we will introduce how to determine the coefficient matrices for NC half-plane models.

4.2.3.1 Coefficient matrices of NC half-plane models using vertical PEFs

Similar to the other proposed QP and ASHP lattice models, it is also possible to determine the coefficient matrix for the NC half-plane prediction error fields predicted at the j th vertical PEFs of the PSR $S_{k,l}^{(p-1,q-1)}$ shown in Figure 4.7. By using (3.230),

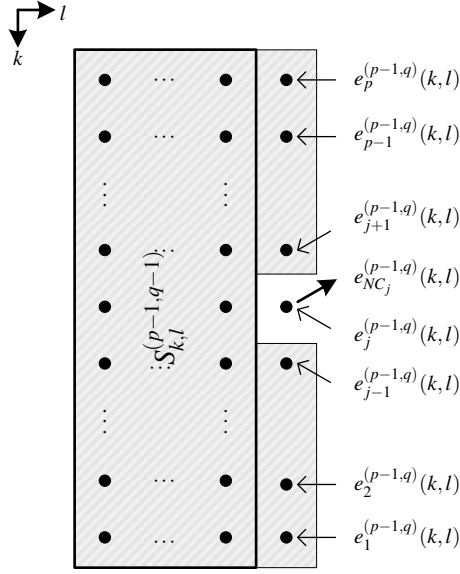


Figure 4.7: The forward vertical auxiliary PEFs of the PSR $S_{k,l}^{(p-1, q-1)}$ for the NC half-plane lattice model.

the forward error propagation equation can be written as, for $j = 2, 3, \dots, p-1$

$$\begin{aligned}
 e_{NC_j}^{(p-1, q)}(k, l) &= e_j^{(p-1, q)}(k, l) - \sum_{i=1}^{j-1} \mathbf{k}_{e_{NC_j}}^{(p-1, q)*}(i) e_i^{(p-1, q)}(k, l) \\
 &\quad - \sum_{i=j+1}^p \mathbf{k}_{e_{NC_j}}^{(p-1, q)*}(i) e_i^{(p-1, q)}(k, l)
 \end{aligned} \tag{4.149}$$

The coefficient matrix of NC half-plane lattice filter can be determined from (4.149)

as

$$\begin{aligned}
 \mathbf{A}_{NC_j}^{(p-1, q)} &= \mathbf{A}_j^{(p-1, q)} - \sum_{i=1}^{j-1} \mathbf{k}_{e_{NC_j}}^{(p-1, q)*}(i) \mathbf{A}_i^{(p-1, q)} - \sum_{i=j+1}^p \mathbf{k}_{e_{NC_j}}^{(p-1, q)*}(i) \mathbf{A}_i^{(p-1, q)} \\
 &= \begin{bmatrix} -k_{e_{NC_2}}^{(p-1, q)*}(1) & \dots & * & * & \dots & * \\ \vdots & \dots & \vdots & \vdots & \dots & \vdots \\ -k_{e_{NC_2}}^{(p-1, q)*}(2) & \dots & * & * & \dots & * \\ 1 & \dots & * & * & \dots & * \\ -k_{e_{NC_2}}^{(p-1, q)*}(j) & \dots & * & * & \dots & * \\ \vdots & \dots & \vdots & \vdots & \dots & \vdots \\ -k_{e_{NC_2}}^{(p-1, q)*}(p-1) & \dots & * & * & \dots & * \end{bmatrix}_{p \times (q+1)}.
 \end{aligned} \tag{4.150}$$

The backward coefficient matrix $\tilde{\mathbf{A}}_{NC_j}^{(p-1, q)}$ can be calculated as

$$\tilde{\mathbf{A}}_{NC_j}^{(p-1, q)} = \mathbf{J}_p \mathbf{A}_{NC_j}^{(p-1, q)*} \mathbf{J}_{q+1}. \tag{4.151}$$

We can find the coefficient matrix of the PSR $S_{k,l}^{(2,0)}$ which is presented in Section 3.4.3.1. The Figure 3.22 shows the NC half-plane lattice model construction from the related vertical PEFs. In the equation (3.257), the forward error propagation equation is given as

$$e_{NC_2}^{(2,1)}(k,l) = e_2^{(2,1)}(k,l) - \mathbf{k}_{e_{NC_2}}^{(2,1)*}(1)e_1^{(2,1)}(k,l) - \mathbf{k}_{e_{NC_2}}^{(2,1)*}(2)e_3^{(2,1)}(k,l). \quad (4.152)$$

We can write the forward error propagation equation (4.152) in terms of the coefficient matrices of the related vertical PEFs as

$$\begin{aligned} \mathbf{A}_{NC_2}^{(2,1)} &= \mathbf{A}_2^{(2,1)} - \mathbf{k}_{e_{NC_2}}^{(2,1)*} \mathbf{A}_1^{(2,1)} - \mathbf{k}_{e_{NC_2}}^{(2,1)*} \mathbf{A}_3^{(2,1)} \\ &= \begin{bmatrix} 0 & x_1 \\ 1 & x_2 \\ 0 & x_3 \end{bmatrix} - \mathbf{k}_{e_{NC_2}}^{(2,1)*}(1) \begin{bmatrix} 1 & y_1 \\ 0 & y_2 \\ 0 & y_3 \end{bmatrix} - \mathbf{k}_{e_{NC_2}}^{(2,1)*}(2) \begin{bmatrix} 0 & t_1 \\ 0 & t_2 \\ 1 & t_3 \end{bmatrix} \\ &= \begin{bmatrix} k_{e_{NC_2}}^{(2,1)*}(1) & x_1 - y_1 k_{e_{NC_2}}^{(2,1)*}(1) - t_1 k_{e_{NC_2}}^{(2,1)*}(2) \\ 1 & x_2 - y_2 k_{e_{NC_2}}^{(2,1)*}(1) - t_2 k_{e_{NC_2}}^{(2,1)*}(2) \\ k_{e_{NC_2}}^{(2,1)*}(2) & x_3 - y_3 k_{e_{NC_2}}^{(2,1)*}(1) - t_3 k_{e_{NC_2}}^{(2,1)*}(2) \end{bmatrix}_{3 \times 2}. \end{aligned} \quad (4.153)$$

The backward coefficient matrix is found as

$$\tilde{\mathbf{B}}_{NC_2}^{(1,2)} = \mathbf{J}_2 \mathbf{B}_{NC_2}^{(1,2)*} \mathbf{J}_3. \quad (4.154)$$

4.2.3.2 Coefficient matrices of NC half-plane models using horizontal PEFs

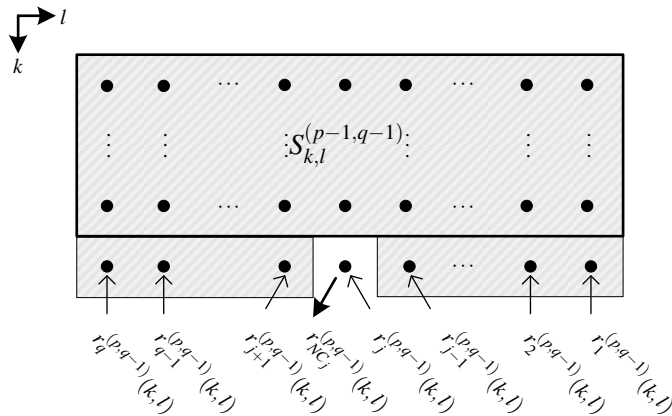


Figure 4.8: The forward horizontal auxiliary PEFs of the PSR $S_{k,l}^{(p-1,q-1)}$ for the NC half-plane lattice model.

In order to determine the coefficient matrix for the NC half-plane prediction error fields predicted at the j th horizontal PEFs of the PSR $S_{k,l}^{(p-1,q-1)}$ shown in Figure 4.8,

by using (3.243), the forward error propagation equation can be written as, for $j = 2, 3, \dots, q-1$

$$\begin{aligned} r_{NC_j}^{(p,q-1)}(k,l) &= r_j^{(p,q-1)}(k,l) - \sum_{i=1}^{j-1} \mathbf{k}_{r_{NC_j}}^{(p,q-1)*}(i) r_i^{(p,q-1)}(k,l) \\ &\quad - \sum_{i=j+1}^q \mathbf{k}_{r_{NC_j}}^{(p,q-1)*}(i) r_i^{(p,q-1)}(k,l) \end{aligned} \quad (4.155)$$

The coefficient matrix of NC half-plane lattice filter can be determined from (4.155) as

$$\begin{aligned} \mathbf{B}_{NC_j}^{(p,q-1)} &= \mathbf{B}_j^{(p,q-1)} - \sum_{i=1}^{j-1} \mathbf{k}_{r_{NC_j}}^{(p,q-1)*}(i) \mathbf{B}_i^{(p,q-1)} - \sum_{i=j+1}^q \mathbf{k}_{r_{NC_j}}^{(p,q-1)*}(i) \mathbf{B}_i^{(p,q-1)} \\ &= \begin{bmatrix} -\mathbf{k}_{r_{NC_j}}^{(p,q-1)*}(1) & \dots & -\mathbf{k}_{r_{NC_j}}^{(p,q-1)*}(j-1) & 1 & -\mathbf{k}_{r_{NC_j}}^{(p,q-1)*}(j+1) \\ * & \dots & * & * & * \\ \vdots & \dots & \vdots & \vdots & \vdots \\ * & \dots & * & * & * \\ \dots & -\mathbf{k}_{r_{NC_j}}^{(p,q-1)*}(q) & & & \\ \dots & * & & & \\ \dots & \vdots & & & \\ \dots & * & & & \end{bmatrix}_{p \times (q+1)}. \end{aligned} \quad (4.156)$$

The backward coefficient matrix $\tilde{\mathbf{A}}_{NC_j}^{(p-1,q)}$ can be calculated as

$$\tilde{\mathbf{B}}_{NC_j}^{(p,q-1)} = \mathbf{J}_{p+1} \mathbf{B}_{NC_j}^{(p,q-1)*} \mathbf{J}_q. \quad (4.157)$$

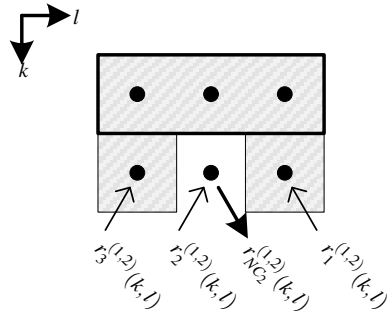


Figure 4.9: NC half-plane lattice model from the corresponding auxiliary horizontal PEFs of the PSR $S_{k,l}^{(0,2)}$.

Now, as an example, we can find the coefficient matrix of the PSR $S_{k,l}^{(0,2)}$ for the NC half-plane lattice model generated from the horizontal PEFs as shown in Figure 4.9.

From the equation (3.253), the forward error propagation equation can be written as

$$r_{NC_2}^{(1,2)}(k,l) = r_2^{(1,2)}(k,l) - \mathbf{k}_{r_{NC_2}}^{(1,2)*}(1)r_1^{(1,2)}(k,l) - \mathbf{k}_{r_{NC_2}}^{(1,2)*}(2)r_3^{(1,2)}(k,l). \quad (4.158)$$

We can write the forward error propagation equation (4.152) in terms of the coefficient matrices of the related vertical PEFs as

$$\begin{aligned} \mathbf{B}_{NC_2}^{(1,2)} &= \mathbf{B}_2^{(1,2)} - \mathbf{k}_{r_{NC_2}}^{(1,2)*} \mathbf{B}_1^{(1,2)} - \mathbf{k}_{r_{NC_2}}^{(1,2)*} \mathbf{B}_3^{(1,2)} \\ &= \begin{bmatrix} 0 & 1 & 0 \\ x_1 & x_2 & x_3 \end{bmatrix} - \mathbf{k}_{r_{NC_2}}^{(1,2)*}(1) \begin{bmatrix} 1 & 0 & 0 \\ y_1 & y_2 & y_3 \end{bmatrix} - \mathbf{k}_{r_{NC_2}}^{(1,2)*}(2) \begin{bmatrix} 0 & 0 & 1 \\ t_1 & t_2 & t_3 \end{bmatrix} \\ &= \begin{bmatrix} k_{r_{NC_2}}^{(1,2)*}(1) & 1 & 0 \\ x_1 - y_1 k_{r_{NC_2}}^{(1,2)*}(1) - t_1 k_{r_{NC_2}}^{(1,2)*}(2) & x_2 - y_2 k_{r_{NC_2}}^{(1,2)*}(1) - t_2 k_{r_{NC_2}}^{(1,2)*}(2) & x_3 - y_3 k_{r_{NC_2}}^{(1,2)*}(1) - t_3 k_{r_{NC_2}}^{(1,2)*}(2) \end{bmatrix}_{2 \times 3}. \end{aligned} \quad (4.159)$$

The backward coefficient matrix is found as

$$\tilde{\mathbf{B}}_{NC_2}^{(1,2)} = \mathbf{J}_2 \mathbf{B}_{NC_2}^{(1,2)*} \mathbf{J}_3. \quad (4.160)$$

5. NUMERICAL EXAMPLES

Computer simulations using the proposed 2-D lattice filter structures introduced in the previous sections are performed to verify the validity and effectiveness of the theory of the new proposed method. Numerical examples will consist of three parts.

In the first part consisting of Example 1, 2, 3, and 4, the input data field will be a 2-D AR random fields which is produced by passing the Gaussian noise generated by a Matlab function. The AR data field obtained by using the Gaussian noise is used as an input to the proposed lattice filter method described in the previous sections.

In the second part consisting of Example 5, the 2-D data will be obtained by sampling some sinusoids buried in white Gaussian noise with a variance σ^2 to be defined in the examples will be used for the proposed method. Our aim is to estimate the power spectral density of the random field we generated. Spectrum estimation approaches used in the examples are explained in Appendix 5. Generally, we will estimate the locations of the sinusoids.

In the third part consisting of Example 6, and 7, the 2-D complex data will be generated as the sum of 2-D complex sinusoids corrupted by additive complex Gaussian noise and the proposed method will be applied on this data to find the location of the peaks.

In the fourth path consisting of Example 8, the data will be generated using ARMA 2-D model and will be modeled using the proposed 2-D AR lattice models. Three simulations will be considered.

For each example, first initial order stage will be computed. Then, the vertical and horizontal augmentation process, or vice versa, will be repeated until the desired stage order are acquired. After each augmentation step, the lattice reflection coefficients will be calculated and the coefficient matrix of each vertical and horizontal PEFs will be determined. The magnitude and contour plots of the spectrum of them will be demonstrated.

In some examples, it will be shown that the start of the augmentation method makes no difference at the desired stage order. That means, to obtain the desired stage order, the free augmentation sequence can be used.

After calculating all lattice reflection coefficients and determining all coefficient matrices, it will be shown how to establish the QP, ASHP and NC half-plane lattice filters. The coefficient matrices, the magnitude and contour plots of all lattice filter models will be shown in detail.

In the second and third part examples, the spectra of the harmonic mean estimate of the all QP and NC half-plane lattice filters will be calculated and the results will be illustrated. In order to improve the spectrum geometry and estimation accuracy, we combined all 8 QP and $2(M+N-4)$ NC half-plane 2-D lattice filter models calculated concurrently from a rectangular PSR $S_{k,l}^{(M,N)}$ by computing their harmonic mean [58].

In some examples, Itakura's LPC distance measure [68] which is used to find the similarity of given two sets of AR model parameters will be calculated (Appendix A.4). The distances for the filters are computed numerically by sampling the spectrum with $\pi/60$ spacing.

In the first part of examples, the AR processes used in the study [34] and [41] are exercised. Thus, we will be able to compare our results with those studies.

5.1 Example 1: AR Data Field with the (4,4)th order

In this example, we will show that the augmentation sequence is not important to reach to the desired PSR when modeling a 2-D AR data field. In order to arrive to the same stage order, two different augmentation sequences will be applied as Choice I and Choice II. For all cases, the coefficient matrices will be estimated for the 2-D filters to be generated.

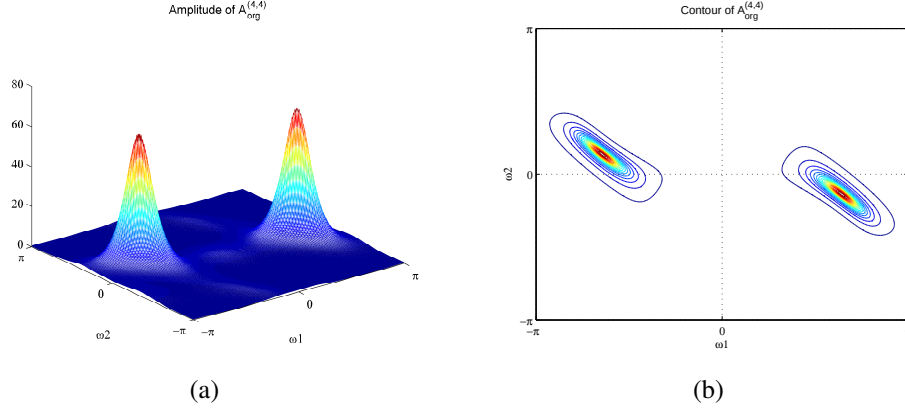


Figure 5.1: (a) Amplitude; (b) Contour plots of the original spectrum of $A_{org}^{(4,4)}$ in Example 1.

AR data field with the (4,4)th order stage with the size $85 - by - 85$ is generated from the following equation with Gaussian white noise $N(\mu = 0, \sigma^2 = 1)$.

$$\begin{aligned}
 y(k, l) = & w(k, l) - 0.03 * y(k, l - 1) + 0.015 * y(k, l - 2) + 0.011 * y(k, l - 3) \\
 & - 0.033 * y(k, l - 4) + 0.47 * y(k - 1, l) - 0.195 * y(k - 1, l - 1) \\
 & - 0.25 * y(k - 1, l - 2) + 0.11 * y(k - 1, l - 3) + 0.085 * y(k - 1, l - 4) \\
 & + 0.3 * y(k - 2, l - 1) - 0.003 * y(k - 2, l - 2) - 0.022 * y(k - 2, l - 3) \\
 & + 0.0001 * y(k - 2, l - 4) - 0.0067 * y(k - 4, l - 1) + 0.002 * y(k - 4, l - 2) \\
 & - 0.0001 * y(k - 4, l - 4)
 \end{aligned} \tag{5.1}$$

where $w(k, l)$ is a Gaussian white noise $N(\mu = 0, \sigma^2 = 1)$. This AR data field is stable as shown in Figure 5.2.

The coefficient matrix of the original AR data field is

$$A_{org}^{(4,4)} = \begin{bmatrix} 1 & 0.03 & -0.015 & -0.011 & 0.033 \\ -0.47 & 0.195 & 0.25 & -0.11 & -0.085 \\ 0 & -0.3 & 0.003 & 0.022 & -0.0001 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0.0067 & -0.002 & 0 & 0.0001 \end{bmatrix}. \tag{5.2}$$

The magnitude and contour plots of the original spectrum are shown in Figure 5.1.

The results of the stability test is shown in Figure 5.2.

By using the proposed method, it is possible to have $A_{QP_1}^{(4,4)}$ from the auxiliary vertical PEFs of the PSR $S_{k,l}^{(4,3)}$ and $B_{QP_1}^{(4,4)}$ from the auxiliary horizontal PEFs of the PSR $S_{k,l}^{(3,4)}$. By applying convenient augmentation steps, it is pretty easy to reach the desired PSR. For example, if we have the PSR $S_{k,l}^{(3,3)}$, we can have the PSR $S_{k,l}^{(4,3)}$ by applying the horizontal augmentation, or the PSR $S_{k,l}^{(3,4)}$ by applying the vertical augmentation.

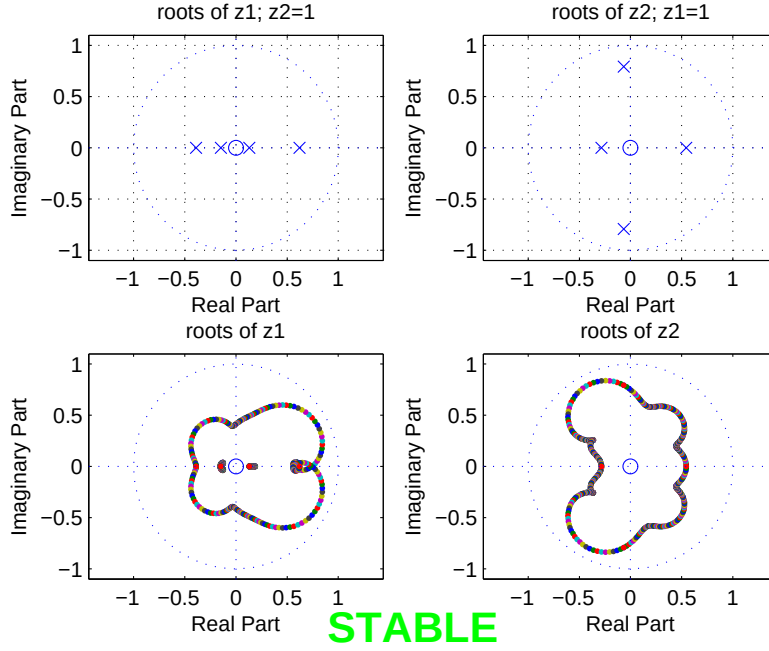


Figure 5.2: The stability test of the original filter in Example 1.

Table 5.1: Reflection Coefficient Values and the Coefficient Matrices for the initial first order stage.

Reflection Coefficients	Coefficient Matrix
$k_{e_1}^{(0,1)} = \begin{bmatrix} -0.1547 \end{bmatrix}$	$A_1^{(0,1)} = \begin{bmatrix} 1 & 0.1547 \end{bmatrix}$
$k_{r_1}^{(1,0)} = \begin{bmatrix} 0.5319 \end{bmatrix}$	$B_1^{(1,0)} = \begin{bmatrix} 1 \\ -0.5319 \end{bmatrix}$

As a first step, the reflection coefficient values and the coefficient matrices are given in Table 5.1. In the table, A_j and B_j represent the coefficient matrices calculated from vertical and horizontal prediction error fields, $e_j(k, l)$ and $r_j(k, l)$, respectively.

The amplitudes and equal magnitude contours of $A_1^{(0,1)}$ and $B_1^{(1,0)}$ will not be shown here since they are 1-D coefficient vectors.

After the initial first order stage, it is possible to go further by applying horizontal or vertical augmentation. Here, we will show the result for both cases. Hence, as a first choice we will apply a horizontal, and then vertical augmentation and we will repeat it until we have the PSR $S_{k,l}^{(4,3)}$. In order to see the improvements, after each augmentation the first-quadrant QP lattice filter coefficient matrix obtained from the auxiliary vertical PEFs will be calculated and the amplitude and equal magnitude contour of this QP model will be demonstrated in this example, as well. In addition, at the square PSR cases, the other first-quadrant QP model from the horizontal PEFs will be generated.

Table 5.2: Lattice Reflection Coefficients and the Coefficient Matrices for the PSR $S_{k,l}^{(1,0)}$ in Example 1, by Choice I.

Reflection Coefficients	Coefficient Matrices
$k_{e_1}^{(1,1)} = [-0.0086]$	$A_1^{(1,1)} = \begin{bmatrix} 1 & 0.1501 \\ 0 & 0.0086 \end{bmatrix}$
$k_{e_{new}}^{(1,1)} = [-0.1574]$	$A_2^{(1,1)} = \begin{bmatrix} 0 & 0.1574 \\ 1 & 0.0709 \end{bmatrix}$
$k_{r_1}^{(2,0)} = [-0.0717]$	$B_1^{(2,0)} = \begin{bmatrix} 1 \\ -0.5701 \\ 0.0717 \end{bmatrix}$

In the second choice, we will apply first vertical, and then horizontal augmentation until the desired stage order.

Moreover, at the desired order QP lattice model, Itakura's LPC distance measure [68] will be calculated (Appendix A.4).

5.1.1 Augmentation application sequences: Choice I

5.1.1.1 Horizontal augmentation applied: Obtained PSR $S_{k,l}^{(1,0)}$

Lattice reflection coefficients and the coefficient matrices are shown in Table 5.2.

The amplitudes and equal magnitude contours of $A_1^{(1,1)}$ and $A_2^{(1,1)}$ are demonstrated in Figure 5.3. $B_1^{(2,0)}$ will not be shown because it is 1-D coefficient vector.

The coefficient matrix of the first-quadrant QP model, $A_{QP1}^{(1,1)}$ is

$$A_{QP1}^{(1,1)} = \begin{bmatrix} 1 & 0.0679 \\ -0.5221 & -0.0284 \end{bmatrix} \quad (5.3)$$

and the amplitude and equal magnitude contours of $A_{QP1}^{(1,1)}$ are shown in Figure 5.4.

The results of the stability test of $A_{QP1}^{(1,1)}$ are shown in Figure 5.5.

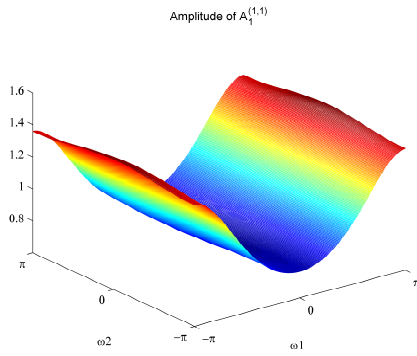
5.1.1.2 Vertical augmentation applied: Obtained PSR $S_{k,l}^{(1,1)}$

Lattice reflection coefficients and the coefficient matrices are found in Table 5.3.

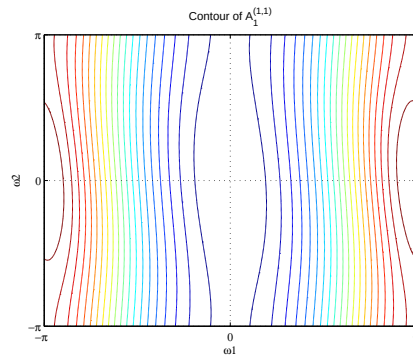
The amplitudes and equal magnitude contours of $A_1^{(1,2)}$, $A_2^{(1,2)}$, and $B_1^{(2,1)}$, $B_2^{(2,1)}$ are demonstrated in Figure 5.6.

The coefficient matrix of the first-quadrant QP model, $A_{QP1}^{(1,2)}$ is

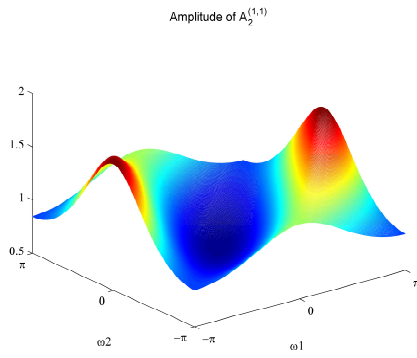
$$A_{QP1}^{(1,2)} = \begin{bmatrix} 1 & 0.0823 & 0.0266 \\ -0.4597 & 0.0266 & 0.3090 \end{bmatrix} \quad (5.4)$$



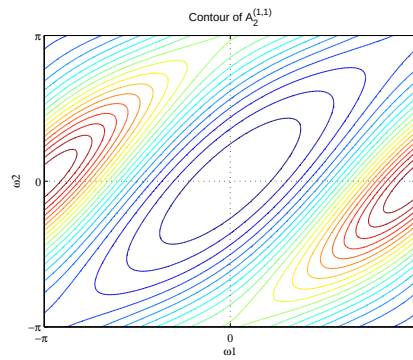
(a)



(b)

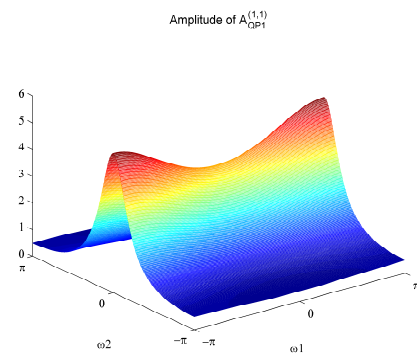


(c)

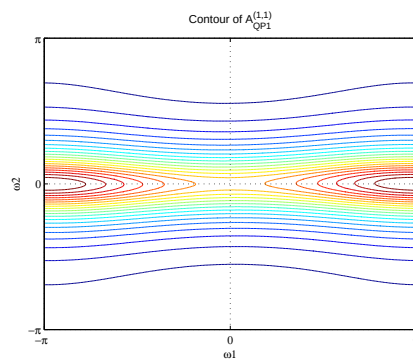


(d)

Figure 5.3: (a,c) Amplitude; (b,d) Contour plots of the spectrum of $A_1^{(1,1)}$ and $A_2^{(1,1)}$, respectively, in Example 1, by Choice I.



(a)



(b)

Figure 5.4: (a) Amplitude; (b) Contour plots of the spectrum of $A_{QP1}^{(1,1)}$ in Example 1, by Choice I.

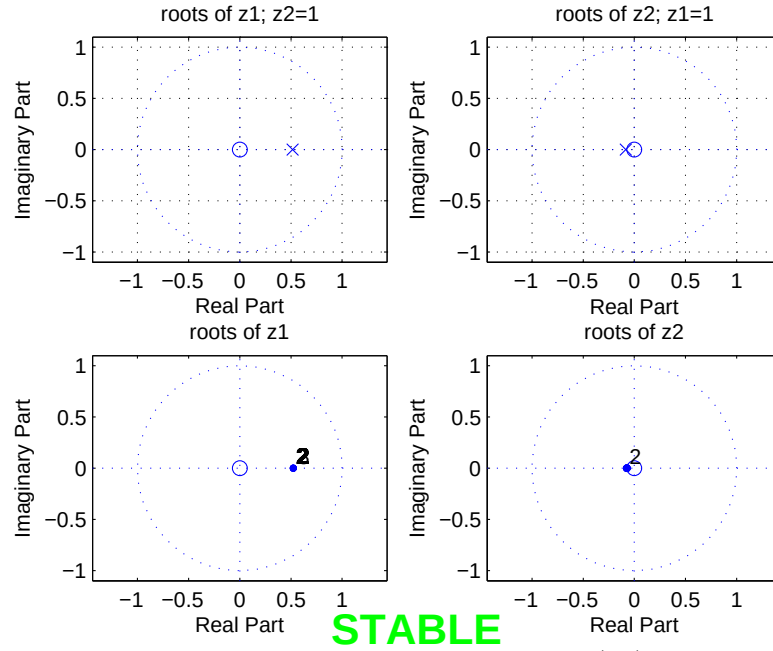


Figure 5.5: The stability test of the first-quadrant QP, $A_{QP1}^{(1,1)}$ in Example 1, by Choice I.

Table 5.3: Lattice Reflection Coefficients and the Coefficient Matrices for the PSR $S_{k,l}^{(1,1)}$ in Example 1, by Choice I.

Reflection Coefficients	Coefficient Matrices
$\mathbf{k}_e^{(1,2)} = \begin{bmatrix} -0.4236 & -0.2494 \\ 0.0189 & 0.0990 \end{bmatrix}$	$A_1^{(1,2)} = \begin{bmatrix} 1 & 0.1524 & -0.0189 \\ 0 & 0.0692 & 0.4236 \end{bmatrix}$ $A_2^{(1,2)} = \begin{bmatrix} 0 & 0.1526 & -0.0990 \\ 1 & 0.0928 & 0.2494 \end{bmatrix}$
$k_{r_1}^{(2,1)} = \begin{bmatrix} 0.3456 \\ -0.1947 \end{bmatrix}$	$B_1^{(2,1)} = \begin{bmatrix} 1 & 0 \\ -0.5592 & 0.1947 \\ 0.0505 & -0.3456 \end{bmatrix}$
$k_{r_{new}}^{(2,1)} = \begin{bmatrix} -0.1119 \\ -0.0096 \end{bmatrix}$	$B_2^{(2,1)} = \begin{bmatrix} 0 & 1 \\ 0.1119 & -0.5518 \\ 0.0096 & 0.0733 \end{bmatrix}$

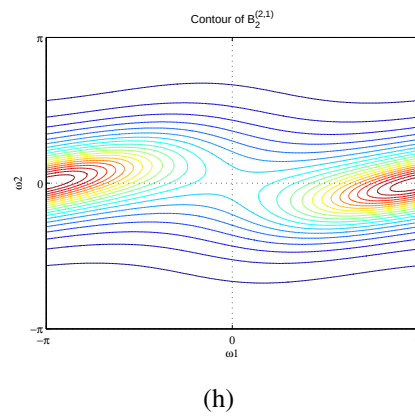
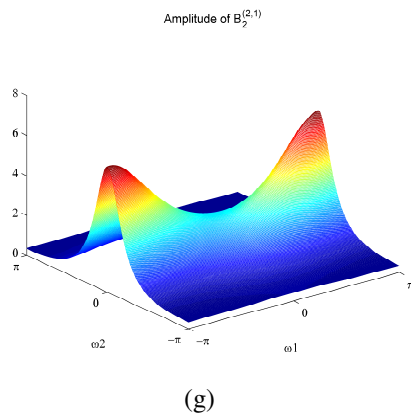
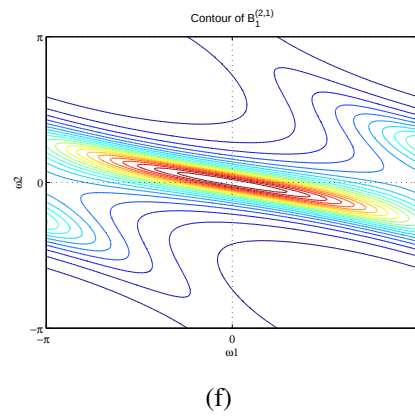
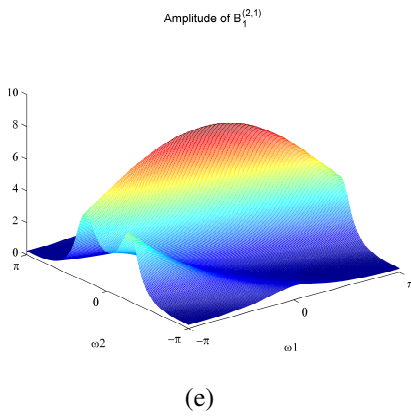
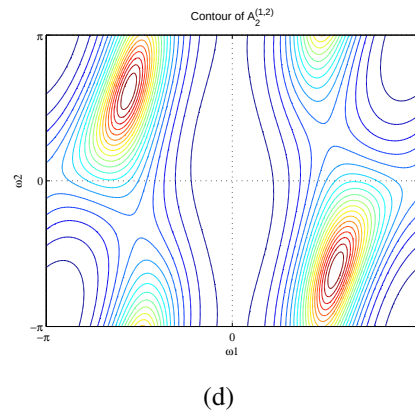
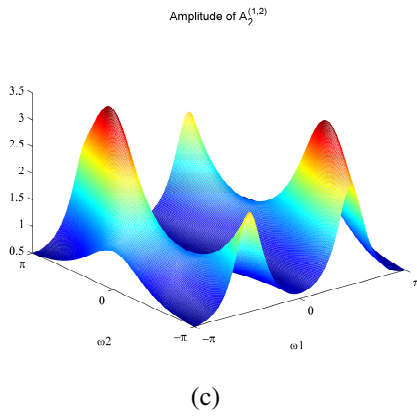
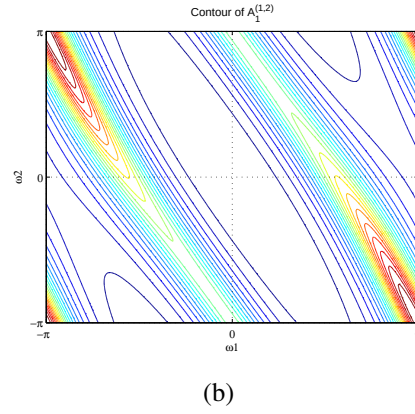
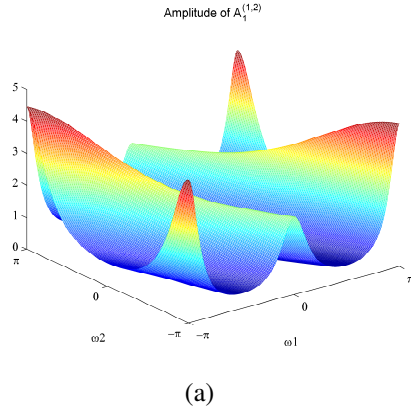


Figure 5.6: (a,c,e,g) Amplitude; (b,d,f,h) Contour plots of the spectrum of $A_1^{(1,2)}$, $A_2^{(1,2)}$, $B_1^{(2,1)}$, and $B_2^{(2,1)}$, respectively, in Example 1, by Choice I.

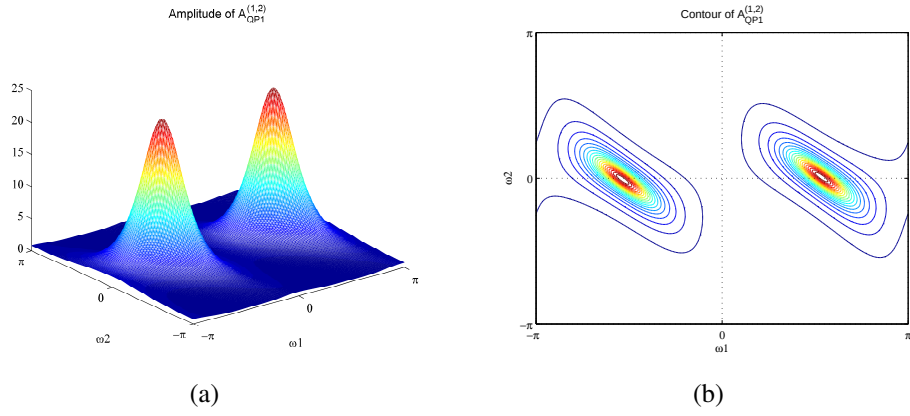


Figure 5.7: (a) Amplitude; (b) Contour plots of the spectrum of $A_{QP1}^{(1,2)}$ in Example 1, by Choice I.

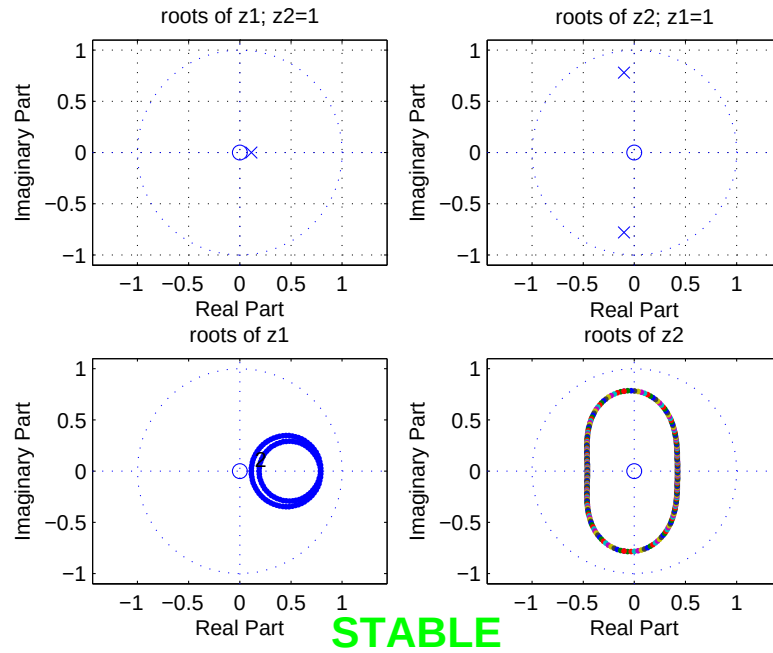


Figure 5.8: The stability test of the first-quadrant QP, $A_{QP1}^{(1,2)}$ in Example 1, by Choice I.

and the amplitudes and equal magnitude contours of $A_{QP1}^{(1,2)}$ are shown in Figure 5.7.

The results of the stability test of $A_{QP1}^{(1,2)}$ is shown in Figure 5.8.

Similarly, the coefficient matrix of the first-quadrant QP model from the horizontal PEFs, $B_{QP1}^{(2,1)}$ can also be obtained as

$$B_{QP1}^{(2,1)} = \begin{bmatrix} 1 & 0.0435 \\ -0.5544 & 0.1707 \\ 0.0509 & -0.3424 \end{bmatrix} \quad (5.5)$$

and the amplitude and equal magnitude contours of $B_{QP1}^{(2,1)}$ are shown in Figure 5.9.

The results of the stability test of $B_{QP1}^{(2,1)}$ are shown in Figure 5.10.

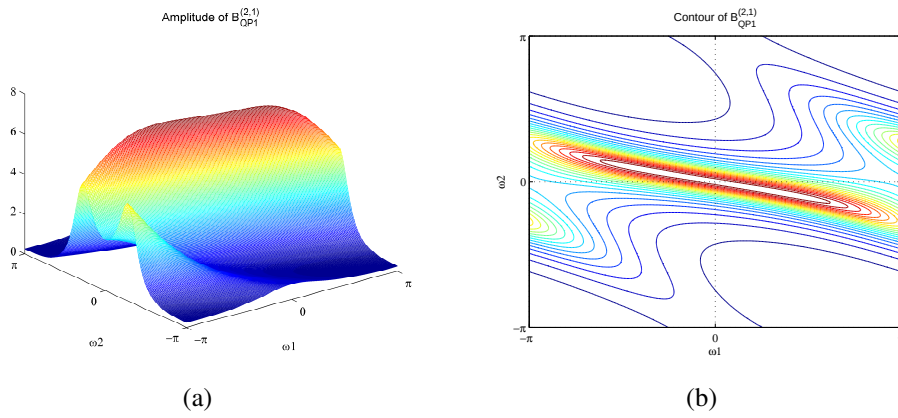


Figure 5.9: (a) Amplitude; (b) Contour plots of the spectrum of $B_{QP1}^{(2,1)}$ in Example 1, by Choice I.

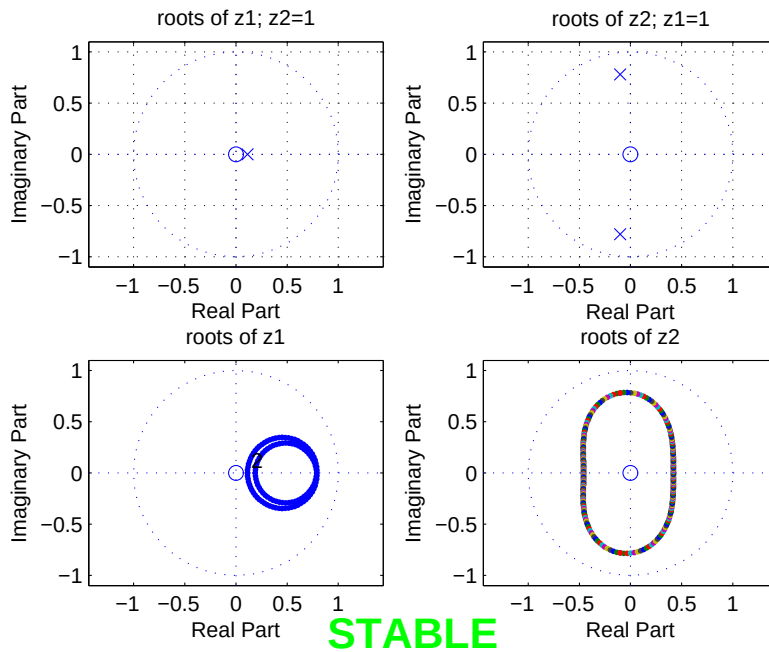


Figure 5.10: The stability test of the first-quadrant QP, $B_{QP1}^{(2,1)}$ in Example 1, by Choice I.

Table 5.4: Lattice Reflection Coefficients and the Coefficient Matrices for the PSR $S_{k,l}^{(2,1)}$ in Example 1, by Choice I.

Reflection Coefficients	Coefficient Matrices
$\mathbf{k}_e^{(2,2)} = \begin{bmatrix} -0.2047 & -0.4136 \\ 0.2656 & -0.0685 \end{bmatrix}$	$A_1^{(2,2)} = \begin{bmatrix} 1 & 0.0622 & -0.0111 \\ 0 & 0.2556 & 0.2794 \\ 0 & -0.2656 & 0.2047 \end{bmatrix}$
	$A_2^{(2,2)} = \begin{bmatrix} 0 & 0.0146 & -0.0775 \\ 1 & 0.1355 & 0.0257 \\ 0 & 0.0685 & 0.4136 \end{bmatrix}$
$k_{e_{new}}^{(2,2)} = \begin{bmatrix} -0.0043 \\ 0.0204 \end{bmatrix}$	$A_3^{(2,2)} = \begin{bmatrix} 0 & 0.0043 & -0.0204 \\ 0 & 0.1479 & -0.0870 \\ 1 & 0.0628 & 0.2464 \end{bmatrix}$
$\mathbf{k}_r^{(3,1)} = \begin{bmatrix} -0.0060 & -0.0187 \\ -0.0187 & 0.0376 \end{bmatrix}$	$B_1^{(3,1)} = \begin{bmatrix} 1 & 0 \\ -0.5599 & 0.1952 \\ 0.0413 & -0.3468 \\ 0.0187 & 0.0060 \end{bmatrix}$
	$B_2^{(3,1)} = \begin{bmatrix} 0 & 1 \\ 0.1027 & -0.5512 \\ 0.0339 & 0.0586 \\ -0.0376 & 0.0187 \end{bmatrix}$

5.1.1.3 Horizontal augmentation applied: Obtained PSR $S_{k,l}^{(2,1)}$

Lattice reflection coefficients and the coefficient matrices are given in Table 5.4.

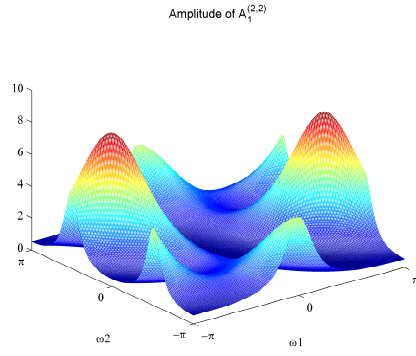
The amplitudes and equal magnitude contours of $A_1^{(2,2)}, \dots, A_3^{(2,2)}$ from the vertical PEFs, and $B_1^{(3,1)}$, and $B_2^{(3,1)}$ from the horizontal PEFs are demonstrated in Figure 5.11 and Figure 5.12, respectively.

The coefficient matrix of the first-quadrant QP model, $A_{QP1}^{(2,2)}$ is

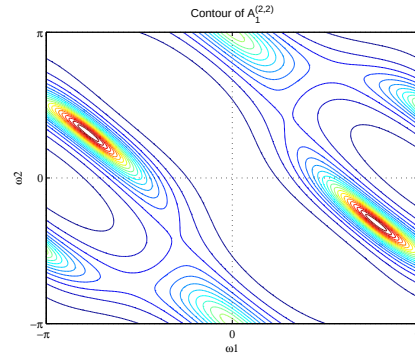
$$A_{QP1}^{(2,2)} = \begin{bmatrix} 1 & 0.0553 & 0.0257 \\ -0.4806 & 0.1938 & 0.2651 \\ 0.0225 & -0.2964 & 0.0114 \end{bmatrix} \quad (5.6)$$

and the amplitude and equal magnitude contours of $A_{QP1}^{(2,2)}$ are shown in Figure 5.13.

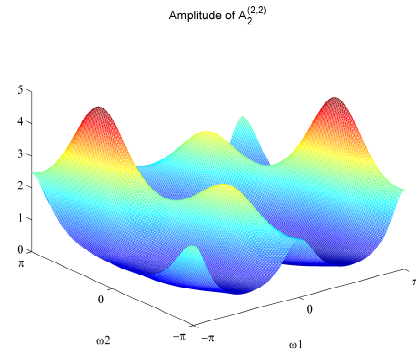
The results of the stability test of $A_{QP1}^{(2,2)}$ are shown in Figure 5.14.



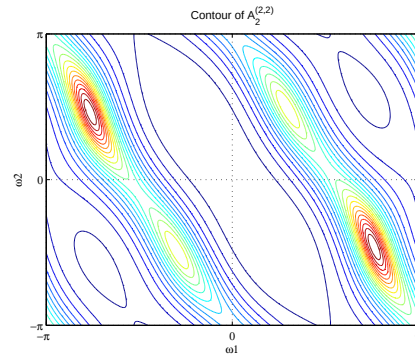
(a)



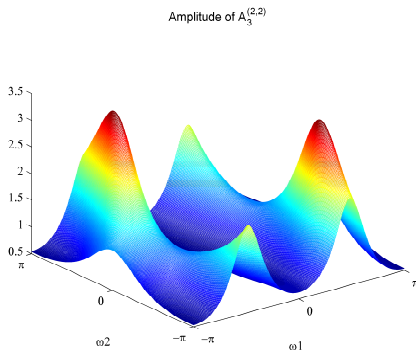
(b)



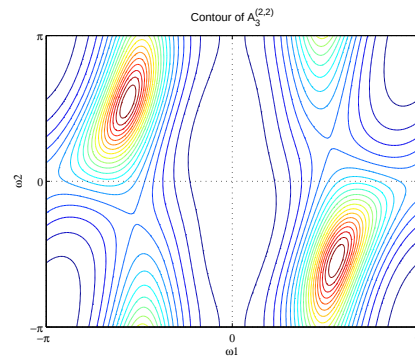
(c)



(d)

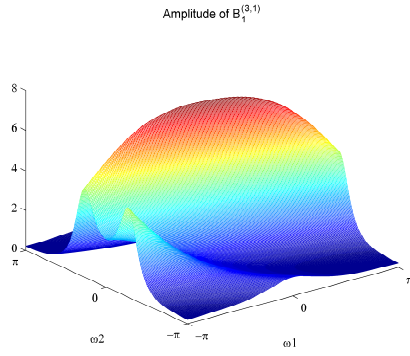


(e)

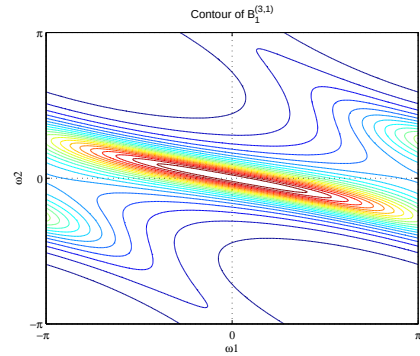


(f)

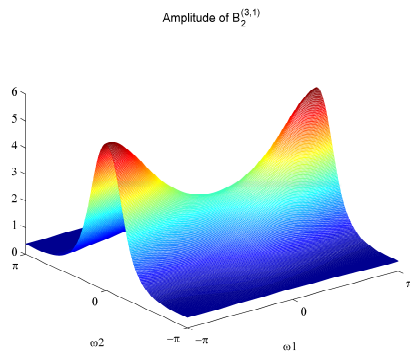
Figure 5.11: (a,c,e) Amplitude; (b,d,f) Contour plots of the spectrum of $A_1^{(2,2)}$, $A_2^{(2,2)}$, and $A_3^{(2,2)}$, respectively, in Example 1, by Choice I.



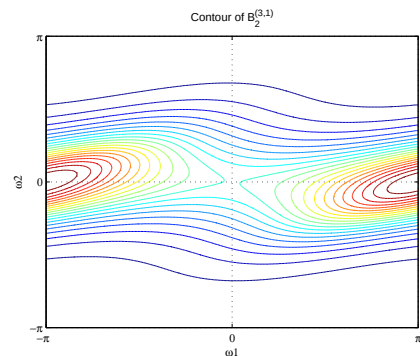
(a)



(b)

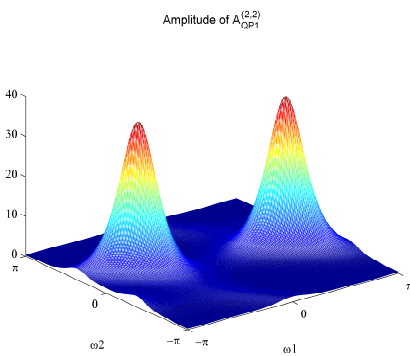


(c)

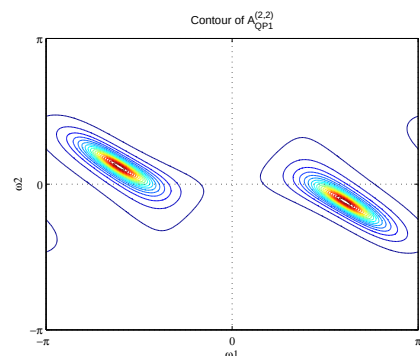


(d)

Figure 5.12: (a,c) Amplitude; (b,d) Contour plots of the spectrum of $B_1^{(3,1)}$, and $B_2^{(3,1)}$, respectively, in Example 1, by Choice I.



(a)



(b)

Figure 5.13: (a) Amplitude; (b) Contour plots of the spectrum of $A_{QP1}^{(2,2)}$ in Example 1, by Choice I.

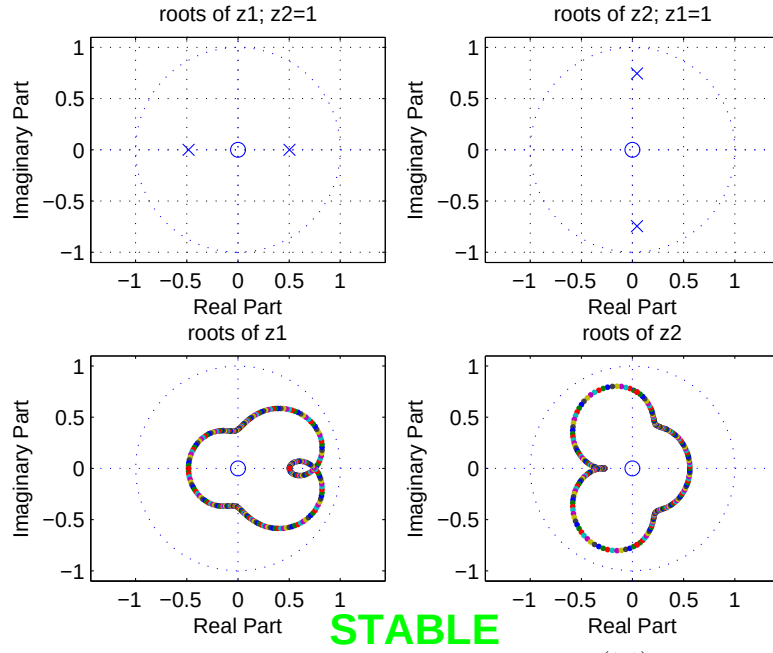


Figure 5.14: The stability test of the first-quadrant QP, $A_{QP1}^{(2,2)}$ in Example 1, by Choice I.

5.1.1.4 Vertical augmentation applied: Obtained PSR $S_{k,l}^{(2,2)}$

Lattice reflection coefficients and the coefficient matrices are given in Table 5.5.

Table 5.5: Lattice Reflection Coefficients and the Coefficient Matrices for the PSR $S_{k,l}^{(2,2)}$ in Example 1, by Choice I.

Reflection Coefficients				Coefficient Matrices			
$\mathbf{k}_e^{(2,3)} = \begin{bmatrix} 0.0386 & 0.1384 & 0.0579 \\ 0.1083 & -0.0094 & 0.0440 \\ 0.0092 & 0.0441 & 0.0096 \end{bmatrix}$				$A_1^{(2,3)} = \begin{bmatrix} 1 & 0.0072 & -0.0091 & -0.0092 \\ 0 & 0.2428 & 0.2535 & -0.1083 \\ 0 & -0.2566 & 0.2006 & -0.0386 \\ 0 & -0.0206 & -0.0442 & -0.0441 \end{bmatrix}$			
				$A_2^{(2,3)} = \begin{bmatrix} 1 & 0.1009 & -0.0149 & 0.0094 \\ 0 & 0.0702 & 0.4050 & -0.1384 \\ 0 & -0.0282 & -0.0089 & -0.0096 \\ 0 & 0.1314 & -0.1092 & -0.0440 \end{bmatrix}$			
$\mathbf{k}_r^{(3,2)} = \begin{bmatrix} 0.0012 & -0.0076 \\ -0.0291 & 0.3408 \\ -0.2691 & -0.1950 \end{bmatrix}$				$B_1^{(3,2)} = \begin{bmatrix} 1 & 0 & 0 \\ -0.4818 & 0.2225 & 0.2691 \\ 0.0183 & -0.3034 & 0.0291 \\ 0.0110 & 0.0075 & -0.0012 \end{bmatrix}$			
				$B_2^{(3,2)} = \begin{bmatrix} 0 & 1 & 0 \\ 0.0114 & -0.5585 & 0.1950 \\ 0.0103 & 0.0433 & -0.3408 \\ -0.0152 & 0.0150 & 0.0076 \end{bmatrix}$			
$k_{r_{new}}^{(3,2)} = \begin{bmatrix} 0.0721 \\ -0.0226 \\ 0.0014 \end{bmatrix}$				$B_3^{(3,2)} = \begin{bmatrix} 0 & 0 & 1 \\ -0.0721 & 0.0986 & -0.5521 \\ 0.0226 & 0.0184 & 0.0392 \\ -0.0014 & -0.0170 & 0.0130 \end{bmatrix}$			

Table 5.6: Lattice Reflection Coefficients of the PSR $S_{k,l}^{(3,2)}$ in Example 1, by Choice I.

Reflection Coefficients			
$\mathbf{k}_e^{(3,3)} =$	$\begin{bmatrix} 0.0086 & 0.0391 & 0.1391 \\ -0.0875 & -0.2025 & -0.4073 \\ 0.1184 & 0.2563 & -0.0699 \end{bmatrix}$		
$k_{e_{new}}^{(3,3)} =$	$\begin{bmatrix} 0.0386 \\ -0.0123 \\ 0.0133 \end{bmatrix}$		
$\mathbf{k}_r^{(4,2)} =$	$\begin{bmatrix} 0.0041 & -0.0206 & 0.0029 \\ -0.0183 & 0.0031 & 0.0053 \\ 0.0019 & 0.0052 & -0.0209 \end{bmatrix}$		

The amplitudes and equal magnitude contours of $A_1^{(2,3)}, \dots, A_3^{(2,3)}$ obtained from the vertical PEFs, and $B_1^{(3,2)}, \dots, B_3^{(3,2)}$ generated from the horizontal PEFs are demonstrated in Figure 5.15 and Figure 5.16, respectively.

The coefficient matrix of the first-quadrant QP model, $A_{QP1}^{(2,3)}$ is

$$A_{QP1}^{(2,3)} = \begin{bmatrix} 1 & 0.0164 & 0.0117 & 0.0116 \\ -0.4781 & 0.1981 & 0.2577 & -0.1140 \\ 0.0269 & -0.2875 & 0.0135 & 0.0260 \end{bmatrix} \quad (5.7)$$

and the amplitude and equal magnitude contours of $A_{QP1}^{(2,3)}$ are shown in Figure 5.17.

The results of the stability test of $A_{QP1}^{(2,3)}$ are shown in Figure 5.18.

Similarly, a first-quadrant QP model from the horizontal PEFs, $B_{QP1}^{(3,2)}$ is also determined and its coefficient matrix can be found as

$$B_{QP1}^{(3,2)} = \begin{bmatrix} 1 & 0.0549 & 0.0264 \\ -0.4831 & 0.1944 & 0.2652 \\ 0.0195 & -0.3006 & 0.0115 \\ 0.0101 & 0.0078 & -0.0004 \end{bmatrix} \quad (5.8)$$

and the amplitude and equal magnitude contours of $B_{QP1}^{(3,2)}$ are shown in Figure 5.19.

The results of the stability test of $B_{QP1}^{(3,2)}$ are shown in Figure 5.20.

5.1.1.5 Horizontal augmentation applied: Obtained PSR $S_{k,l}^{(3,2)}$

Lattice reflection coefficients and the coefficient matrices are given in Table 5.6 and Table 5.7, respectively.

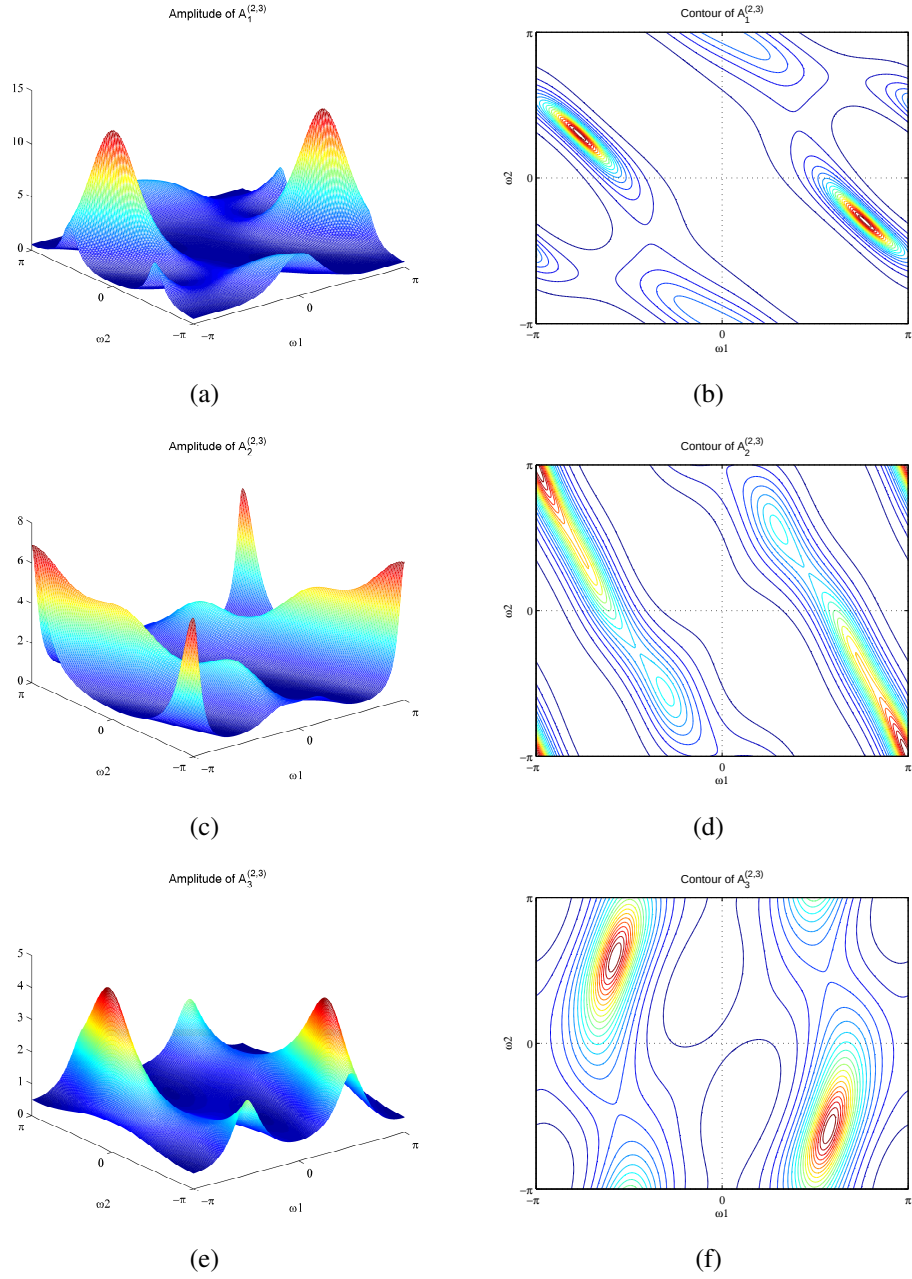
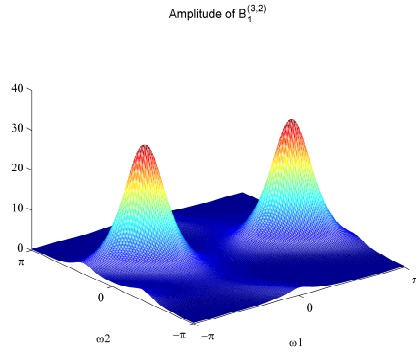
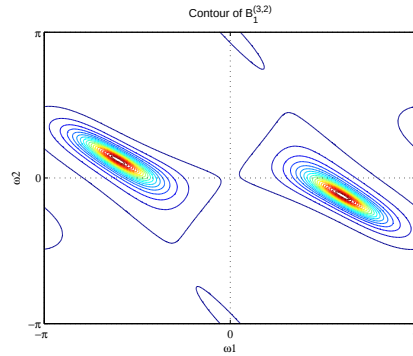


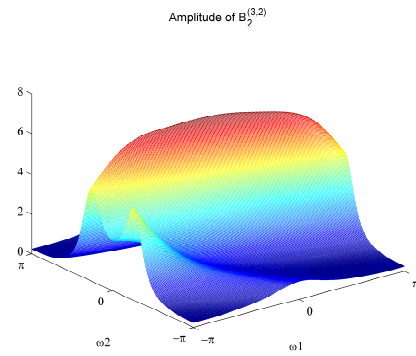
Figure 5.15: (a, c, e) Amplitude; (b, d, f) Contour plots of the spectrum of $A_1^{(2,3)}$, $A_2^{(2,3)}$, and $A_3^{(2,3)}$, respectively, in Example 1, by Choice I.



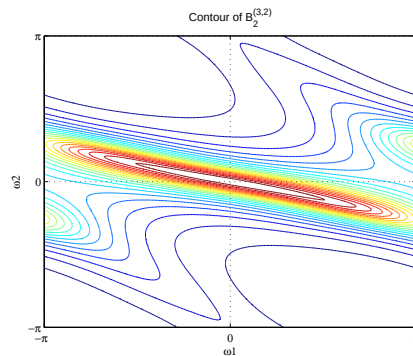
(a)



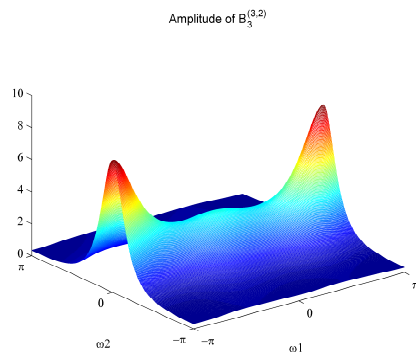
(b)



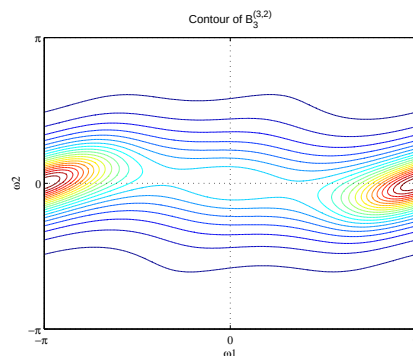
(c)



(d)



(e)



(f)

Figure 5.16: (a,c,e) Amplitude; (b,d,f) Contour plots of the spectrum of $B_1^{(3,2)}$, $B_2^{(3,2)}$, and $B_3^{(3,2)}$, respectively, in Example 1, by Choice I.

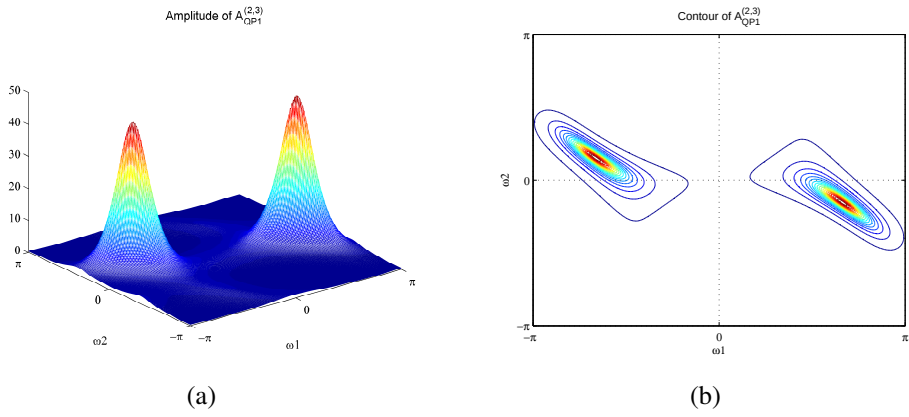


Figure 5.17: (a) Amplitude; (b) Contour plots of the spectrum of $A_{QP1}^{(2,3)}$ in Example 1, by Choice I.

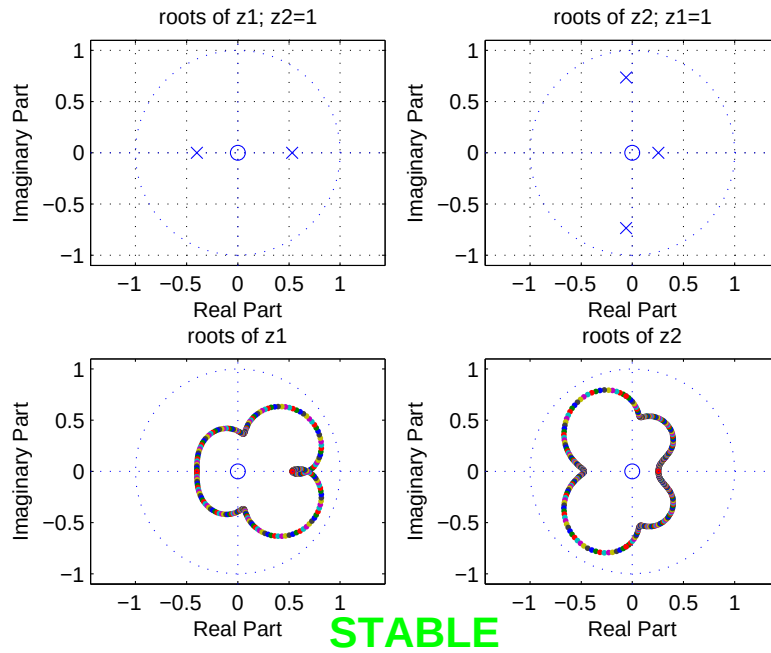


Figure 5.18: The stability test of the first-quadrant QP, $A_{QP1}^{(2,3)}$ in Example 1, by Choice I.

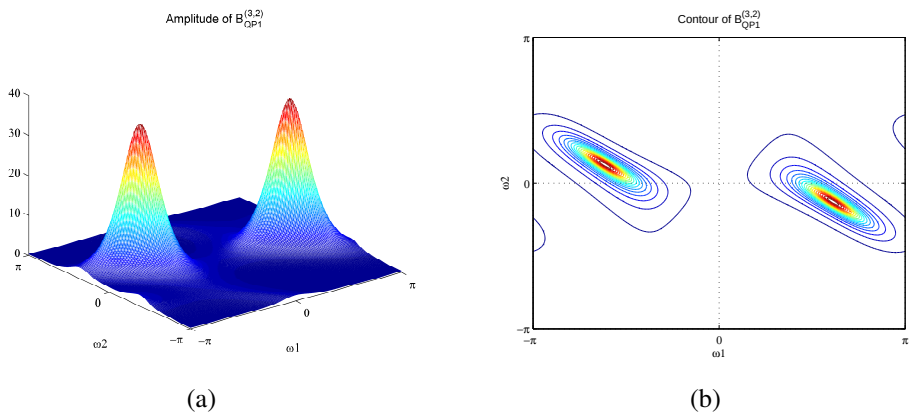


Figure 5.19: (a) Amplitude; (b) Contour plots of the spectrum of $B_{QP1}^{(3,2)}$ in Example 1, by Choice I.

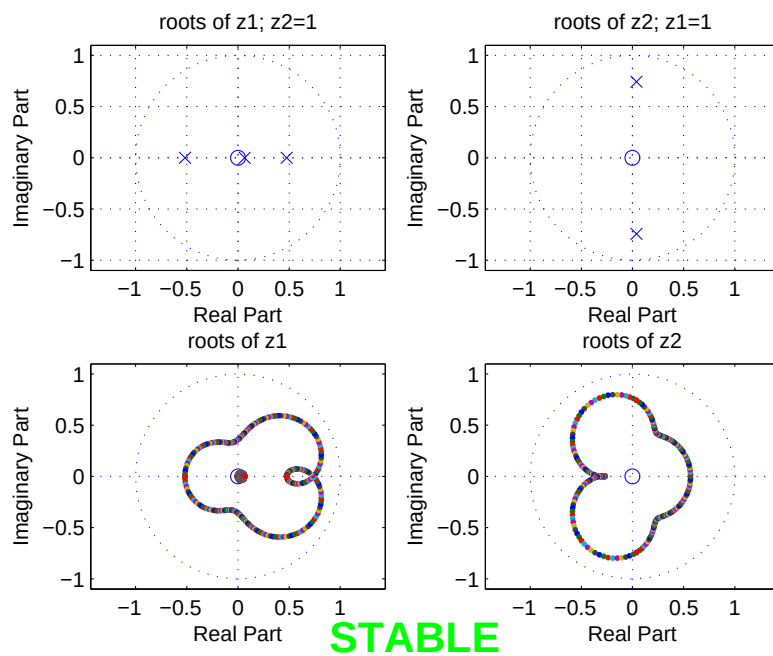


Figure 5.20: The stability test of the first-quadrant QP, $B_{QP1}^{(3,2)}$ in Example 1, by Choice I.

Table 5.7: Coefficient Matrices of the Horizontal and Vertical PEFs of the PSR $S_{k,l}^{(3,2)}$ in Example 1, by Choice I.

Coefficient Matrices of Vertical PEFs		of Horizontal PEFs	
$A_1^{(3,3)} =$	$\begin{bmatrix} 1 & 0.0064 & -0.0059 & -0.0105 \\ 0 & 0.2081 & 0.2578 & -0.1102 \\ 0 & -0.1765 & 0.1382 & -0.0249 \\ 0 & -0.1184 & 0.0875 & -0.0086 \end{bmatrix}$	$B_1^{(4,2)} =$	$\begin{bmatrix} 1 & 0 & 0 \\ -0.4817 & 0.2227 & 0.2687 \\ 0.0119 & -0.3014 & 0.0292 \\ 0.0145 & -0.0038 & 0.0011 \\ -0.0019 & 0.0183 & -0.0041 \end{bmatrix}$
$A_2^{(3,3)} =$	$\begin{bmatrix} 0 & -0.0224 & -0.0371 & -0.0473 \\ 1 & 0.0207 & 0.0010 & 0.0050 \\ 0 & 0.2407 & 0.2579 & -0.0988 \\ 0 & -0.2563 & 0.2025 & -0.0391 \end{bmatrix}$	$B_2^{(4,2)} =$	$\begin{bmatrix} 0 & 1 & 0 \\ 0.0112 & -0.5583 & 0.1953 \\ 0.0118 & 0.0368 & -0.3406 \\ -0.0074 & 0.0208 & -0.0020 \\ -0.0052 & -0.0031 & 0.0206 \end{bmatrix}$
$A_3^{(3,3)} =$	$\begin{bmatrix} 0 & -0.0240 & -0.0050 & -0.0174 \\ 0 & -0.0087 & -0.0480 & -0.0408 \\ 1 & 0.1005 & -0.0094 & 0.0087 \\ 0 & 0.0699 & 0.4073 & -0.1391 \end{bmatrix}$	$B_3^{(4,2)} =$	$\begin{bmatrix} 0 & 0 & 1 \\ -0.0719 & 0.0981 & -0.5521 \\ 0.0252 & 0.0194 & 0.0396 \\ -0.0148 & -0.0126 & 0.0128 \\ 0.0209 & -0.0053 & -0.0029 \end{bmatrix}$
$A_4^{(3,3)} =$	$\begin{bmatrix} 0 & -0.0386 & 0.0123 & -0.0133 \\ 0 & -0.0084 & -0.0257 & -0.0103 \\ 0 & 0.1305 & -0.0971 & -0.0498 \\ 1 & 0.0964 & 0.2422 & -0.0580 \end{bmatrix}$		

Table 5.8: Lattice Reflection Coefficients of the PSR $S_{k,l}^{(3,3)}$ in Example 1, by Choice I.

Reflection Coefficients				
$\mathbf{k}_e^{(3,4)} =$	0.0274	0.0730	0.1265	0.0078
	0.0578	0.0871	-0.0386	-0.0184
	0.0886	-0.0347	-0.0150	0.0171
	-0.0483	-0.0042	0.0291	0.0331
$\mathbf{k}_r^{(4,3)} =$	0.0065	0.0011	-0.0200	
	-0.0087	-0.0015	0.0035	
	-0.0268	-0.0329	0.3377	
	0.1149	-0.2698	-0.1949	
$k_{r_{new}}^{(4,3)} =$	0.0489			
	0.0004			
	-0.0002			
	0.0015			

The amplitudes and equal magnitude contours of $A_1^{(3,3)}, \dots, A_4^{(3,3)}$ from the vertical PEFs, and $B_1^{(4,2)}, \dots, B_3^{(4,2)}$ from the horizontal PEFs are demonstrated in Figure 5.21 and Figure 5.22, respectively.

The coefficient matrix of the first-quadrant QP model, $A_{QP1}^{(3,3)}$ is

$$A_{QP1}^{(3,3)} = \begin{bmatrix} 1 & 0.0161 & 0.0119 & 0.0117 \\ -0.4799 & 0.1979 & 0.2559 & -0.1137 \\ 0.0224 & -0.2882 & 0.0131 & 0.0221 \\ 0.0115 & 0.0073 & 0.0022 & 0.0063 \end{bmatrix} \quad (5.9)$$

and the amplitude and equal magnitude contours of $A_{QP1}^{(3,3)}$ are shown in Figure 5.23.

The results of the stability test of $A_{QP1}^{(3,3)}$ are shown in Figure 5.24.

5.1.1.6 Vertical augmentation applied: Obtained PSR $S_{k,l}^{(3,3)}$

Lattice reflection coefficients and the coefficient matrices are given in Table 5.8 and Table 5.9, respectively.

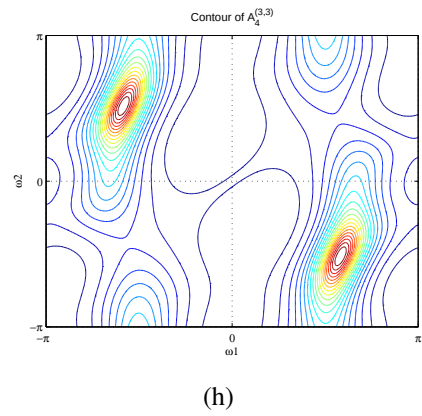
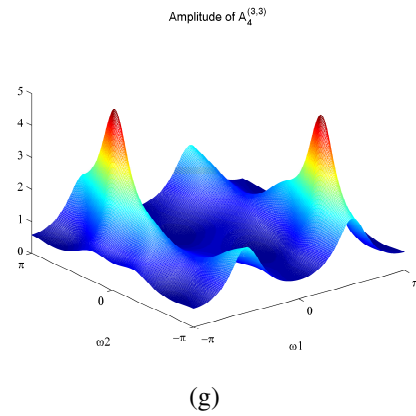
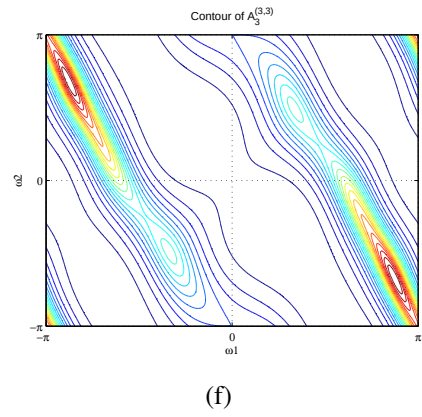
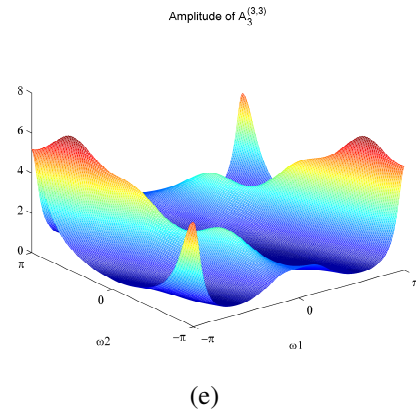
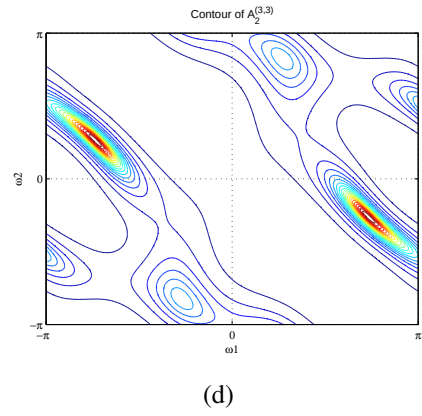
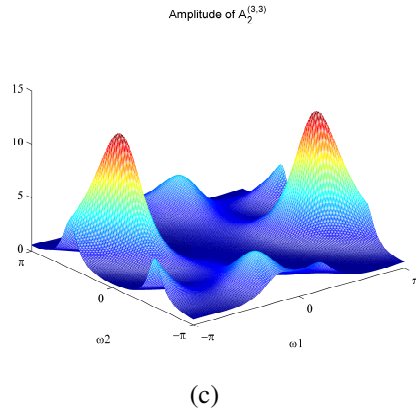
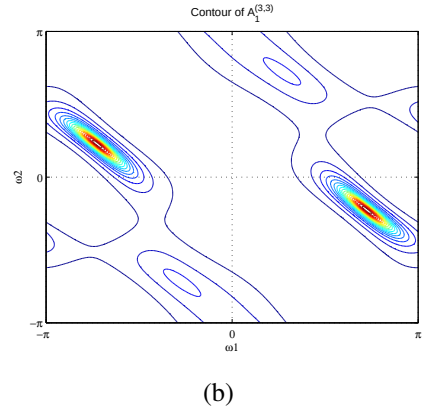
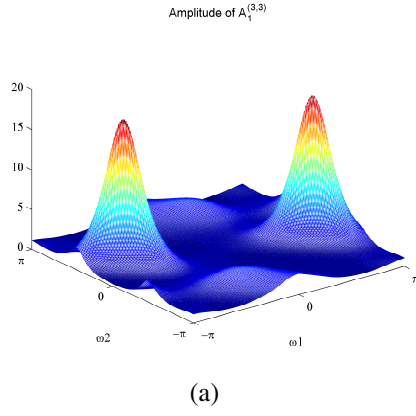
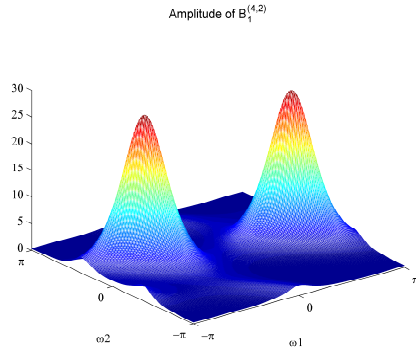
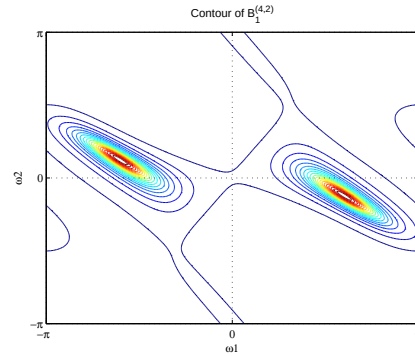


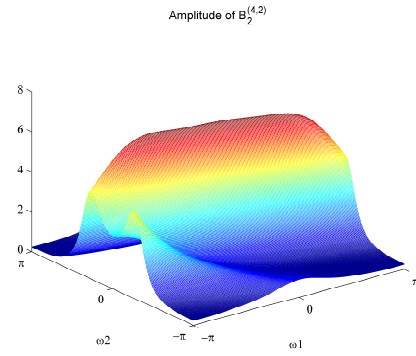
Figure 5.21: (a,c,e,g) Amplitude; (b,d,f,h) Contour plots of the spectrum of $A_1^{(3,3)}, \dots, A_4^{(3,3)}$, respectively, in Example 1, by Choice I.



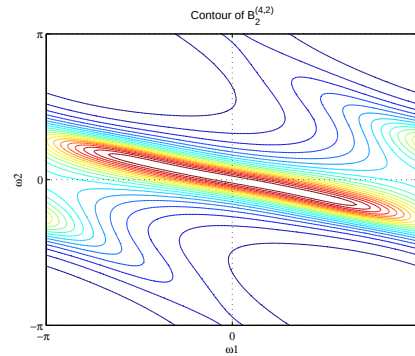
(a)



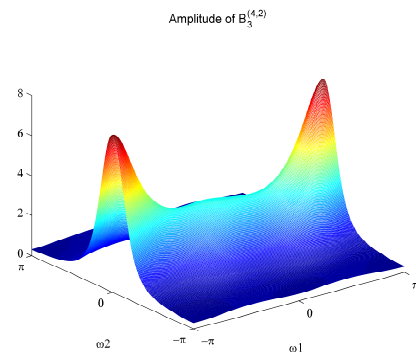
(b)



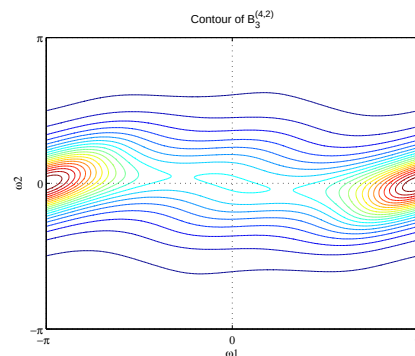
(c)



(d)



(e)



(f)

Figure 5.22: (a,c,e) Amplitude; (b,d,f) Contour plots of the spectrum of $B_1^{(4,2)}$, $B_2^{(4,2)}$, and $B_3^{(4,2)}$, respectively, in Example 1, by Choice I.

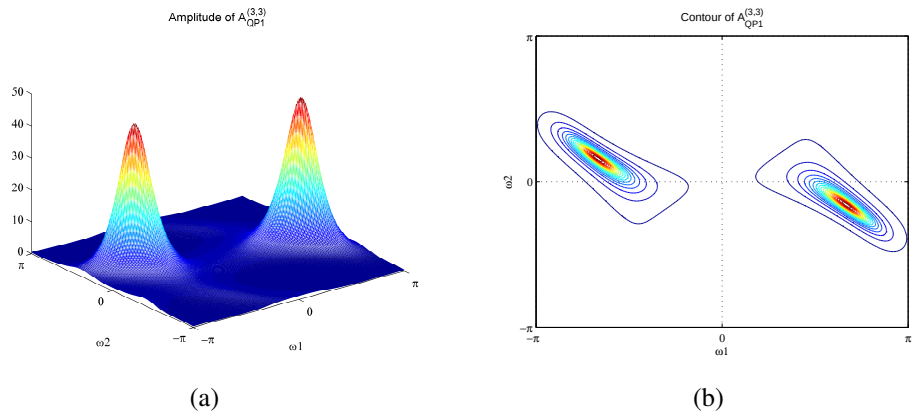


Figure 5.23: (a) Amplitude; (b) Contour plots of the spectrum of $A_{QP1}^{(3,3)}$ in Example 1, by Choice I.

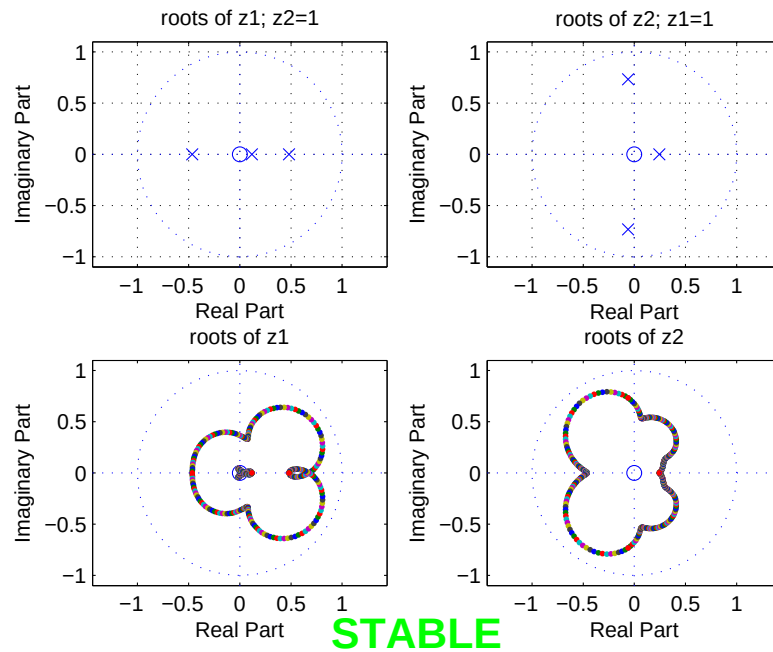


Figure 5.24: The stability test of the first-quadrant QP, $A_{QP1}^{(3,3)}$ in Example 1, by Choice I.

Table 5.9: Coefficient Matrices of the horizontal and vertical PEFs of the PSR $S_{k,l}^{(3,3)}$ in Example 1, by Choice I.

Coefficient Matrices of Vertical PEFs					of Horizontal PEFs				
$A_1^{(3,4)} =$	$\begin{bmatrix} 1 & 0.0184 & -0.0444 & 0.0061 & 0.0483 \\ 0 & 0.2113 & 0.2352 & -0.1219 & -0.0886 \\ 0 & -0.1706 & 0.1341 & -0.0314 & -0.0578 \\ 0 & -0.1145 & 0.0908 & -0.0073 & -0.0274 \end{bmatrix}$	$B_1^{(4,3)} =$	$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -0.4790 & 0.2070 & 0.2581 & -0.1149 & 0 \\ 0.0172 & -0.2891 & 0.0201 & 0.0268 & 0 \\ 0.0153 & -0.0038 & 0.0006 & 0.0087 & 0 \\ -0.0012 & 0.0164 & -0.0005 & -0.0065 & 0 \end{bmatrix}$	$B_1^{(4,3)} =$					
$A_2^{(3,4)} =$	$\begin{bmatrix} 0 & -0.0234 & -0.0460 & -0.0135 & 0.0042 \\ 1 & 0.0312 & -0.0322 & 0.0009 & 0.0347 \\ 0 & 0.2468 & 0.2372 & -0.1161 & -0.0871 \\ 0 & -0.2520 & 0.2060 & -0.0386 & -0.0730 \end{bmatrix}$	$B_2^{(4,3)} =$	$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ -0.0090 & -0.4793 & 0.2233 & 0.2698 & 0 \\ -0.0015 & 0.0105 & -0.3015 & 0.0329 & 0 \\ 0.0114 & 0.0120 & -0.0047 & 0.0015 & 0 \\ -0.0094 & 0.00001 & 0.0093 & -0.0011 & 0 \end{bmatrix}$	$B_2^{(4,3)} =$					
$A_3^{(3,4)} =$	$\begin{bmatrix} 0 & -0.0248 & -0.0092 & -0.0141 & -0.0291 \\ 0 & -0.0078 & -0.0529 & -0.0114 & 0.0150 \\ 1 & 0.1143 & -0.0420 & -0.0167 & 0.0386 \\ 0 & 0.0695 & 0.4062 & -0.1400 & -0.1265 \end{bmatrix}$	$B_3^{(4,3)} =$	$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ -0.0362 & 0.0088 & -0.5583 & 0.1949 & 0 \\ 0.0124 & 0.0055 & 0.0267 & -0.3377 & 0 \\ -0.0052 & 0.0038 & 0.0182 & -0.0035 & 0 \\ 0.0241 & -0.0013 & -0.0021 & 0.0200 & 0 \end{bmatrix}$	$B_3^{(4,3)} =$					
$A_4^{(3,4)} =$	$\begin{bmatrix} 0 & -0.0350 & 0.0003 & -0.0214 & -0.0331 \\ 0 & -0.0086 & -0.0186 & -0.0105 & -0.0171 \\ 0 & 0.1325 & -0.0975 & -0.0507 & 0.0184 \\ 1 & 0.0964 & 0.2413 & -0.0567 & -0.0078 \end{bmatrix}$	$B_4^{(4,3)} =$	$\begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ -0.0489 & -0.0721 & 0.0984 & -0.5515 & 0 \\ -0.0004 & 0.0150 & 0.0068 & 0.0450 & 0 \\ 0.0002 & -0.0064 & -0.0193 & 0.0141 & 0 \\ -0.0015 & 0.0266 & -0.0100 & -0.0024 & 0 \end{bmatrix}$	$B_4^{(4,3)} =$					

The amplitudes and equal magnitude contours of $A_1^{(3,4)}, \dots, A_4^{(3,4)}$ calculated from vertical PEFs, and $B_1^{(4,3)}, \dots, B_4^{(4,3)}$ computed from horizontal PEFs are demonstrated in Figure 5.25 and Figure 5.26, respectively.

The coefficient matrix of the first-quadrant QP model, $A_{QP1}^{(3,4)}$ is

$$A_{QP1}^{(3,4)} = \begin{bmatrix} 1 & 0.0286 & -0.0228 & 0.0119 & 0.0454 \\ -0.4731 & 0.1963 & 0.2493 & -0.1227 & -0.1050 \\ 0.0170 & -0.2838 & 0.0199 & 0.0226 & -0.0157 \\ 0.0124 & 0.0072 & 0.0032 & 0.0079 & 0.0048 \end{bmatrix} \quad (5.10)$$

and the amplitude and equal magnitude contours of $A_{QP1}^{(3,4)}$ are shown in Figure 5.27.

The results of the stability test of $A_{QP1}^{(3,4)}$ are shown in Figure 5.28.

Similarly, the first-quadrant QP model, $B_{QP1}^{(4,3)}$ can be produced from the horizontal PEFs and its coefficient matrix can be determined as

$$B_{QP1}^{(4,3)} = \begin{bmatrix} 1 & 0.0154 & 0.0123 & 0.0148 \\ -0.4803 & 0.1987 & 0.2561 & -0.1165 \\ 0.0173 & -0.2887 & 0.0159 & 0.0238 \\ 0.0151 & -0.0037 & 0.0005 & 0.0089 \\ -0.0010 & 0.0168 & -0.0005 & -0.0063 \end{bmatrix} \quad (5.11)$$

and the amplitude and equal magnitude contours of $B_{QP1}^{(4,3)}$ are shown in Figure 5.29.

The results of the stability test of $B_{QP1}^{(4,3)}$ are shown in Figure 5.30.

For now, we applied the horizontal and vertical augmentation consecutively, and we reach to the PSR $S_{k,l}^{(3,3)}$. At that point, if we want to generate the (4,3)th order coefficient matrix from the vertical PEFs, we should apply a horizontal augmentation. If we want to get the (3,4)th order coefficient matrix from the horizontal PEFs, a vertical augmentation should be applied to this PSR. In the following subsection both cases will be taken into account.

5.1.1.7 Horizontal augmentation applied: Obtained PSR $S_{k,l}^{(4,3)}$ from $S_{k,l}^{(3,3)}$

Lattice reflection coefficients and the coefficient matrices are given in Table 5.10 and Table 5.11-5.12, respectively.

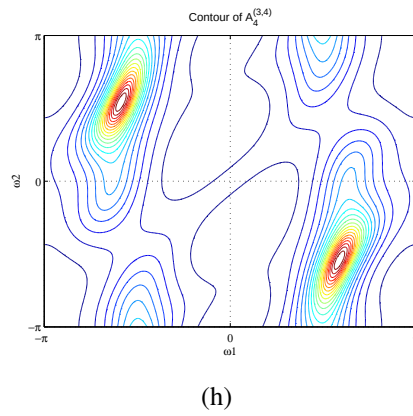
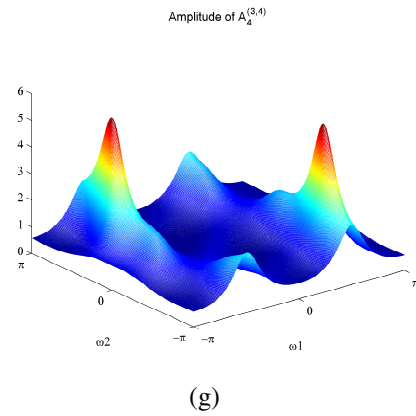
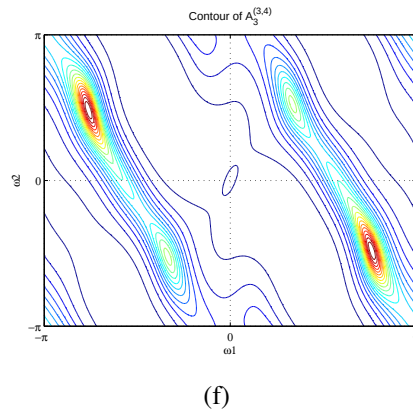
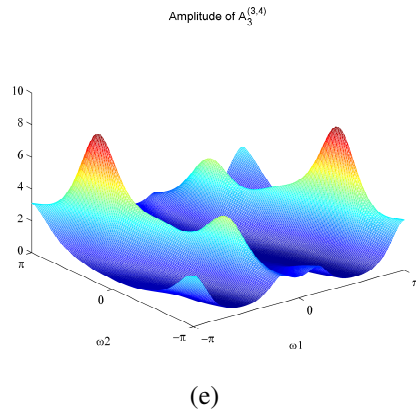
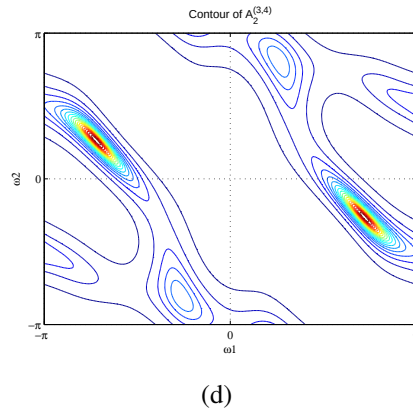
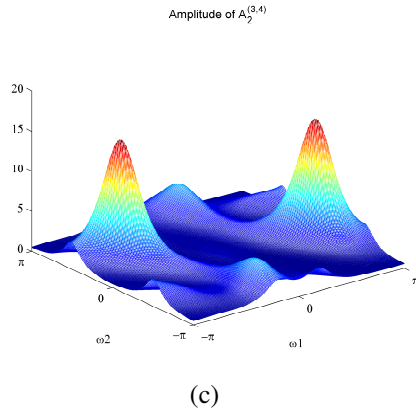
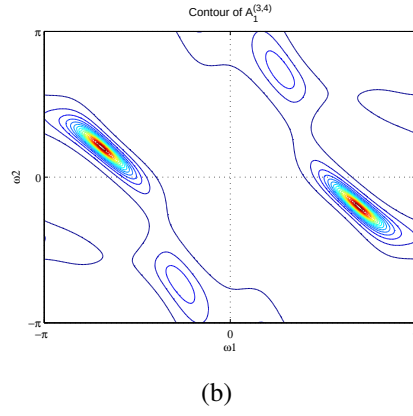
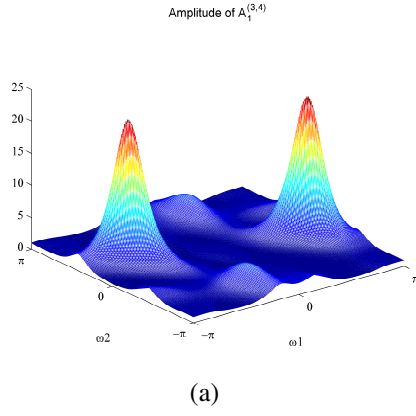


Figure 5.25: (a,c,e,g) Amplitude; (b,d,f,h) Contour plots of the spectrum of $A_1^{(3,4)}, \dots, A_4^{(3,4)}$, respectively, in Example 1, by Choice I.

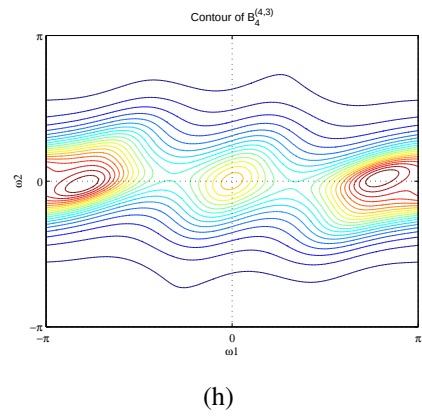
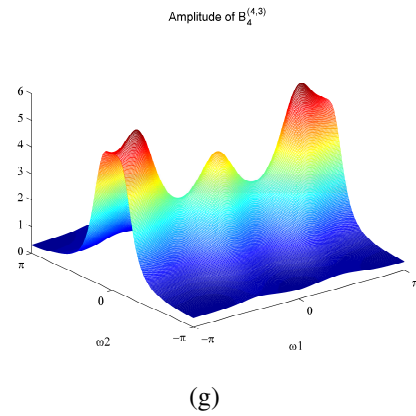
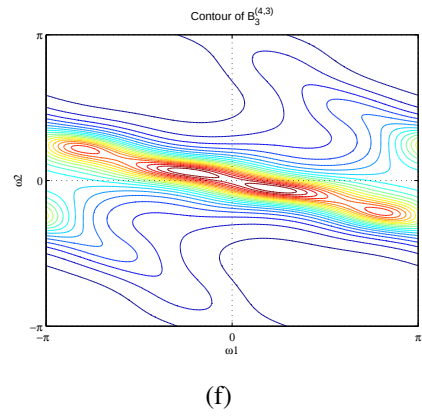
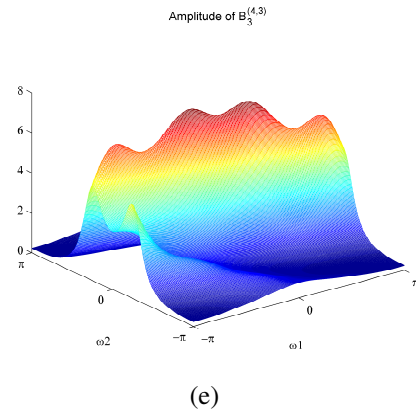
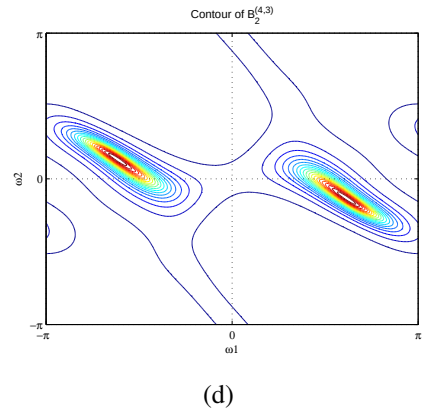
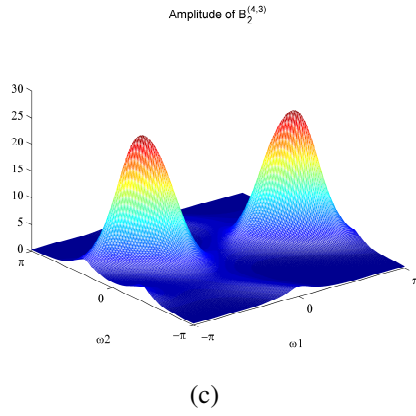
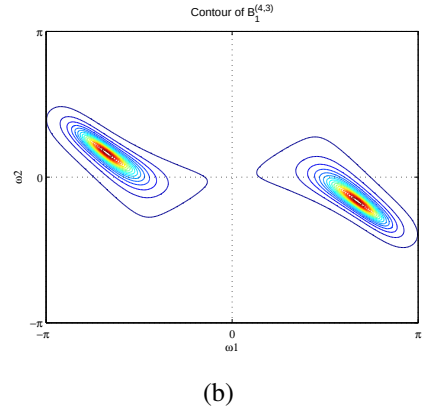
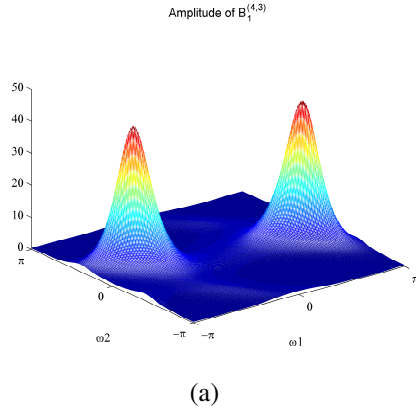


Figure 5.26: (a,c,e,g) Amplitude; (b,d,f,h) Contour plots of the spectrum of $B_1^{(4,3)}, \dots, B_4^{(4,3)}$, respectively, in Example 1, by Choice I.

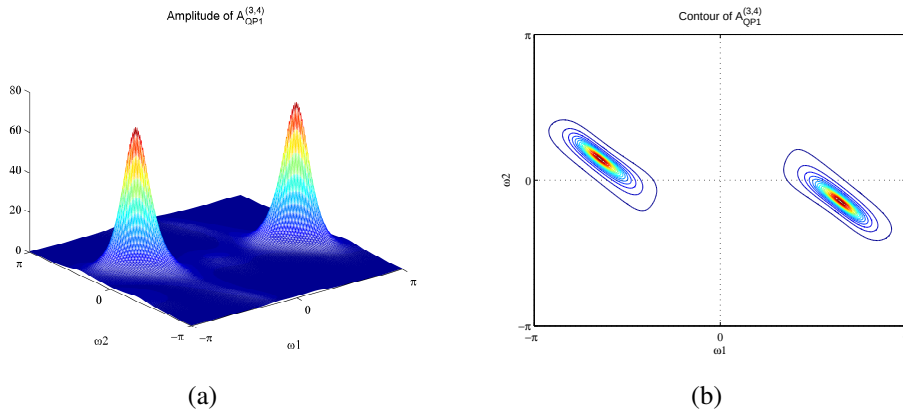


Figure 5.27: (a) Amplitude; (b) Contour plots of the spectrum of $A_{QP1}^{(3,4)}$ in Example 1, by Choice I.

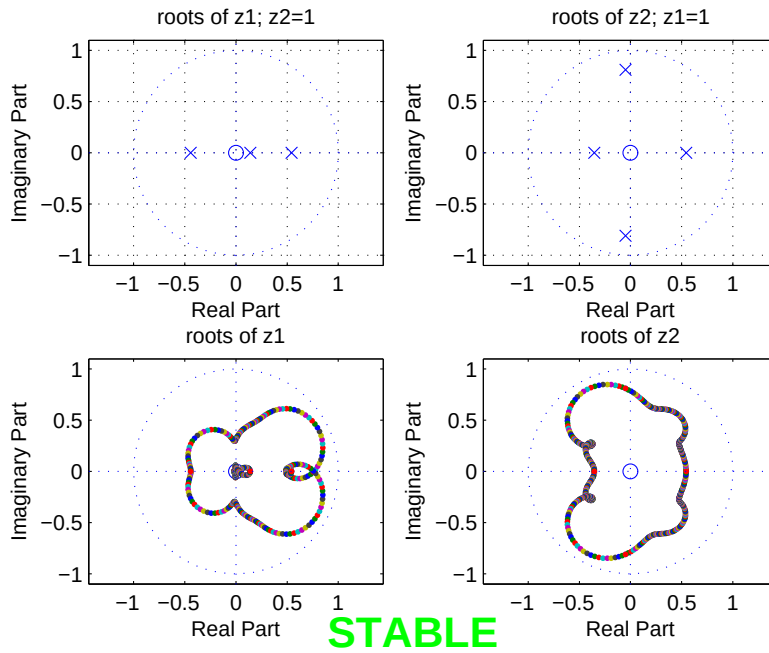


Figure 5.28: The stability test of the first-quadrant QP, $A_{QP1}^{(3,4)}$ in Example 1, by Choice I.

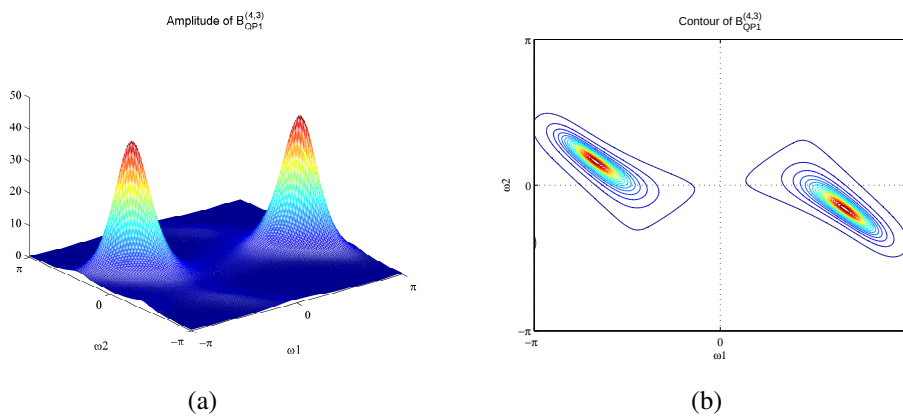


Figure 5.29: (a) Amplitude; (b) Contour plots of the spectrum of $B_{QP1}^{(4,3)}$ in Example 1, by Choice I.

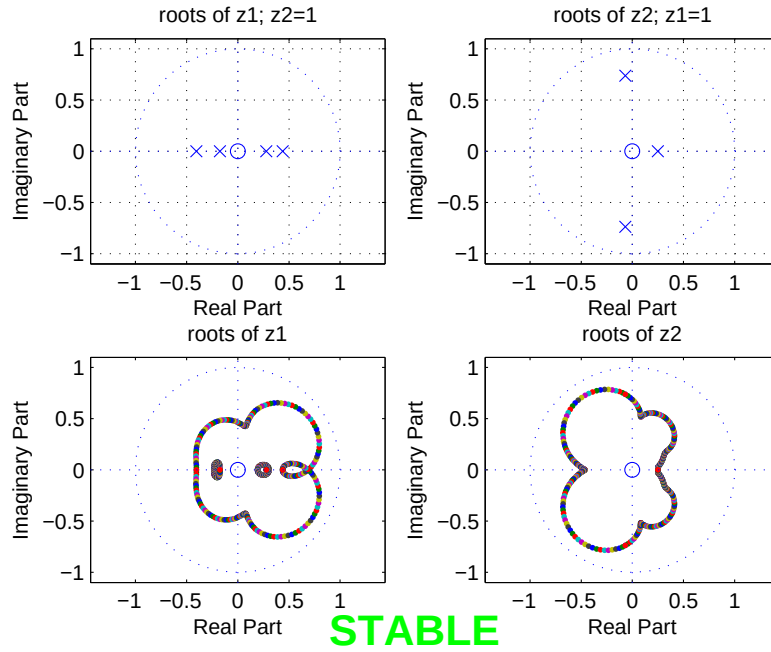


Figure 5.30: The stability test of the first-quadrant QP, $B_{QP1}^{(4,3)}$ in Example 1, by Choice I.

Table 5.10: Lattice Reflection Coefficients of the PSR $S_{k,l}^{(4,3)}$ in Example 1, by Choice I.

Reflection Coefficients				
$\mathbf{k}_e^{(4,4)} =$	0.0162	0.0276	0.0740	0.1282
	0.0083	0.0095	0.0385	0.1409
	-0.0343	-0.0899	-0.2061	-0.4053
	0.0332	0.1140	0.2521	-0.0670
$k_{e_{new}}^{(4,4)} =$	0.0210			
	-0.0222			
	0.0058			
	-0.0037			
$\mathbf{k}_r^{(5,3)} =$	0.0024	0.0162	-0.0227	0.0219
	0.0154	-0.0172	0.0153	0.0019
	-0.0183	0.0132	-0.0049	0.0074
	0.0166	0.0010	0.0058	0.0107

Table 5.11: Coefficient Matrices of the horizontal and vertical PEFs of the PSR $S_{k,l}^{(4,3)}$ in Example 1.

Coefficient Matrices of Vertical PEFs					of Horizontal PEFs				
$A_1^{(4,4)} =$	$\begin{bmatrix} 1 & 0.0193 & -0.0442 & 0.0049 & 0.0493 \\ 0 & 0.2106 & 0.2365 & -0.1216 & -0.0889 \\ 0 & -0.1844 & 0.1369 & -0.0272 & -0.0576 \\ 0 & -0.0899 & 0.0624 & -0.0040 & -0.0192 \\ 0 & -0.0332 & 0.0343 & -0.0083 & -0.0162 \end{bmatrix}$	$B_1^{(5,3)} =$	$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -0.4786 & 0.2070 & 0.2576 & -0.1143 \\ 0.0168 & -0.2884 & 0.0201 & 0.0268 \\ 0.0078 & 0.0011 & 0.0011 & 0.0087 \\ 0.0077 & 0.0005 & 0.0077 & -0.0050 \\ -0.0166 & 0.0183 & -0.0154 & -0.0024 \end{bmatrix}$	$B_2^{(5,3)} =$	$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ -0.0092 & -0.4791 & 0.2231 & 0.2693 \\ -0.0016 & 0.0102 & -0.3012 & 0.0325 \\ -0.0069 & 0.0062 & 0.00005 & 0.0011 \\ -0.0049 & 0.0070 & -0.0023 & 0.0070 \\ -0.0010 & -0.0132 & 0.0172 & -0.0162 \end{bmatrix}$	$B_3^{(5,3)} =$	$\begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ -0.0362 & 0.0087 & -0.5581 & 0.1951 \\ 0.0125 & 0.0058 & 0.0265 & -0.3372 \\ -0.0070 & 0.0089 & 0.0114 & -0.0031 \\ 0.0215 & -0.0021 & 0.0104 & 0.0094 \\ -0.0058 & 0.0049 & -0.0153 & 0.0227 \end{bmatrix}$	$B_4^{(5,3)} =$	$\begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ -0.0489 & -0.0720 & 0.0977 & -0.5516 \\ -0.0007 & 0.0151 & 0.0069 & 0.0447 \\ 0.0016 & -0.0066 & -0.0132 & 0.0136 \\ 0.0049 & 0.0236 & -0.0129 & 0.0090 \\ -0.0107 & -0.0074 & -0.0019 & -0.0219 \end{bmatrix}$
$A_2^{(4,4)} =$	$\begin{bmatrix} 0 & -0.0211 & -0.0451 & -0.0171 & 0.0066 \\ 1 & 0.0290 & -0.0284 & 0.0020 & 0.0339 \\ 0 & 0.2103 & 0.2412 & -0.1094 & -0.0864 \\ 0 & -0.1711 & 0.1354 & -0.0307 & -0.0574 \\ 0 & -0.1140 & 0.0899 & -0.0095 & -0.0276 \end{bmatrix}$								
$A_3^{(4,4)} =$	$\begin{bmatrix} 0 & -0.0196 & -0.0074 & -0.0223 & -0.0233 \\ 0 & -0.0128 & -0.0441 & -0.0092 & 0.0132 \\ 1 & 0.0301 & -0.0281 & 0.0016 & 0.0401 \\ 0 & 0.2468 & 0.2386 & -0.1169 & -0.0858 \\ 0 & -0.2521 & 0.2061 & -0.0385 & -0.0740 \end{bmatrix}$								
$A_4^{(4,4)} =$	$\begin{bmatrix} 0 & -0.0260 & -0.0024 & -0.0223 & -0.0219 \\ 0 & -0.0104 & -0.0120 & -0.0106 & -0.0195 \\ 0 & -0.0094 & -0.0463 & -0.0118 & 0.0214 \\ 1 & 0.1151 & -0.0430 & -0.0170 & 0.0369 \\ 0 & 0.0670 & 0.4053 & -0.1409 & -0.1282 \end{bmatrix}$								

Table 5.12: Coefficient Matrices of the last vertical PEFs of the PSR $S_{k,l}^{(4,3)}$ in Example 1, by Choice I.

Coefficient Matrix	
$A_5^{(4,4)} =$	$\begin{bmatrix} 0 & -0.0210 & 0.0222 & -0.0058 & 0.0037 \\ 0 & -0.0251 & -0.0149 & -0.0183 & -0.0278 \\ 0 & -0.0090 & -0.0123 & -0.0178 & -0.0148 \\ 0 & 0.1320 & -0.0972 & -0.0510 & 0.0183 \\ 1 & 0.0961 & 0.2410 & -0.0566 & -0.0078 \end{bmatrix}$

The amplitude and equal magnitude contours of $A_1^{(4,4)}, \dots, A_5^{(4,4)}$, and $B_1^{(5,3)}, \dots, B_4^{(5,3)}$ are demonstrated in Figure 5.31-5.32 and Figure 5.33, respectively.

The coefficient matrix of the first-quadrant QP model, $A_{QP1}^{(4,4)}$ is

$$A_{QP1}^{(4,4)} = \begin{bmatrix} 1 & 0.0286 & -0.0229 & 0.0123 & 0.0456 \\ -0.4732 & 0.1965 & 0.2493 & -0.1228 & -0.1052 \\ 0.0105 & -0.2838 & 0.0217 & 0.0244 & -0.0160 \\ 0.0163 & -0.0043 & 0.00006 & 0.0090 & 0.0076 \\ 0.0007 & 0.0193 & 0.0007 & -0.0065 & -0.0060 \end{bmatrix} \quad (5.12)$$

and the amplitude and equal magnitude contours of $A_{QP1}^{(4,4)}$ are shown in Figure 5.34.

The results of the stability test of $A_{QP1}^{(4,4)}$ are shown in Figure 5.35.

Now, we reached to the desired order PSR and we generated a first-quadrant QP filter from the vertical PEFs, $A_{QP1}^{(4,4)}$ as shown in Figure 5.34. As seen in the Figure 5.35, this QP filter is stable.

We know that it is also possible to obtain a first-quadrant QP filter from the horizontal PEFs as $B_{QP1}^{(5,3)}$. The coefficient matrix is

$$B_{QP1}^{(5,3)} = \begin{bmatrix} 1 & 0.0132 & 0.0121 & 0.0157 \\ -0.4799 & 0.1997 & 0.2553 & -0.1171 \\ 0.0169 & -0.2880 & 0.0166 & 0.0239 \\ 0.0076 & 0.0012 & 0.0009 & 0.0091 \\ 0.0080 & 0.0009 & 0.0076 & -0.0047 \\ -0.0169 & 0.0181 & -0.0154 & -0.0027 \end{bmatrix} \quad (5.13)$$

and the amplitude and equal magnitude contours of $B_{QP1}^{(5,3)}$ are illustrated in Figure 5.36.

As seen in the Figure 5.37, this QP filter, $B_{QP1}^{(5,3)}$ is also stable.

Not only QP lattice filter, but the type-1 or type-2 ASHP and NC half-plane lattice filters can also be generated from the horizontal and vertical PEFs of this PSR $S_{k,l}^{(4,3)}$. Here, one example of each filter models will be produced from the vertical PEFs.

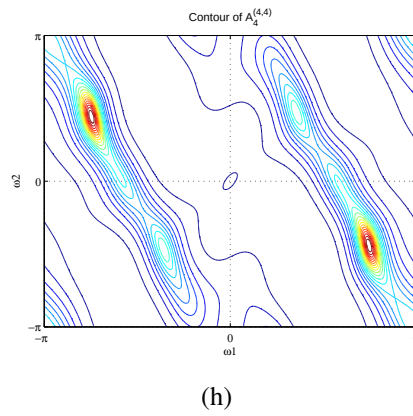
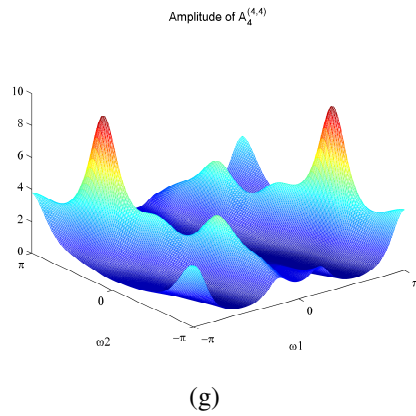
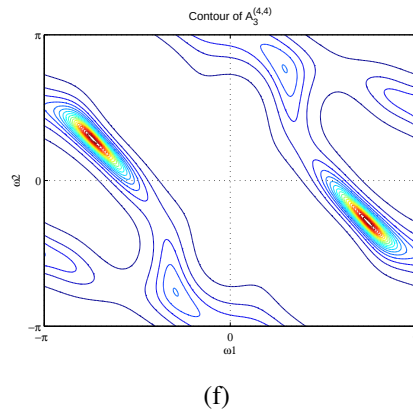
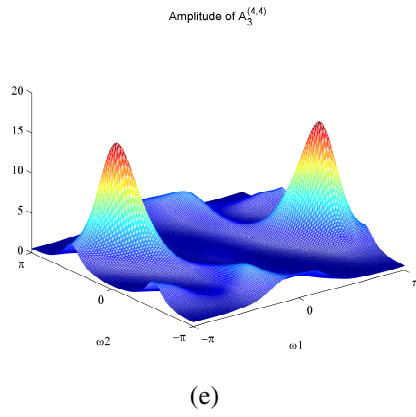
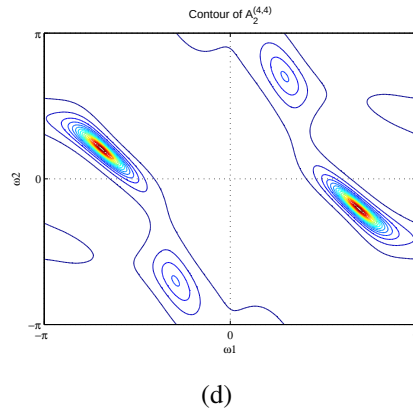
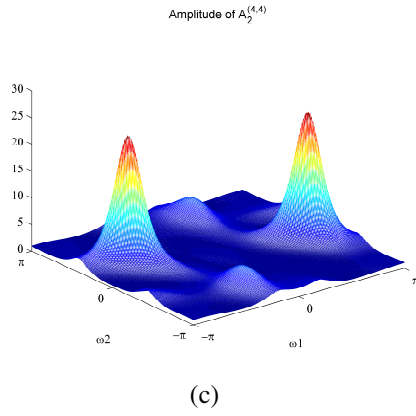
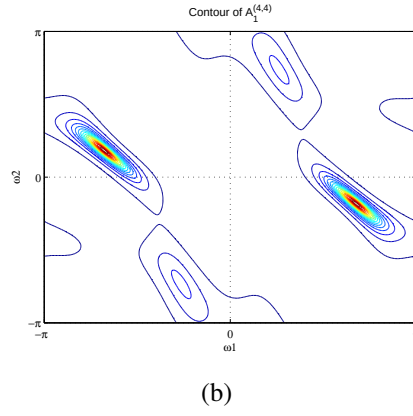
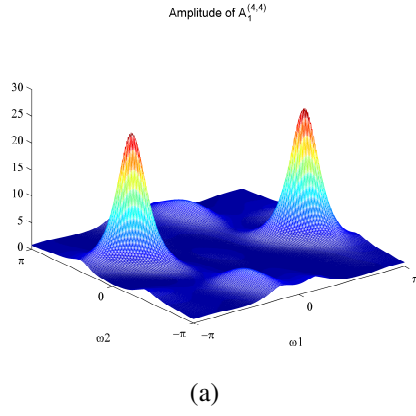


Figure 5.31: (a,c,e,g) Amplitude; (b,d,f,h) Contour plots of the spectrum of $A_1^{(4,4)}, \dots, A_4^{(4,4)}$, respectively, in Example 1, by Choice I.

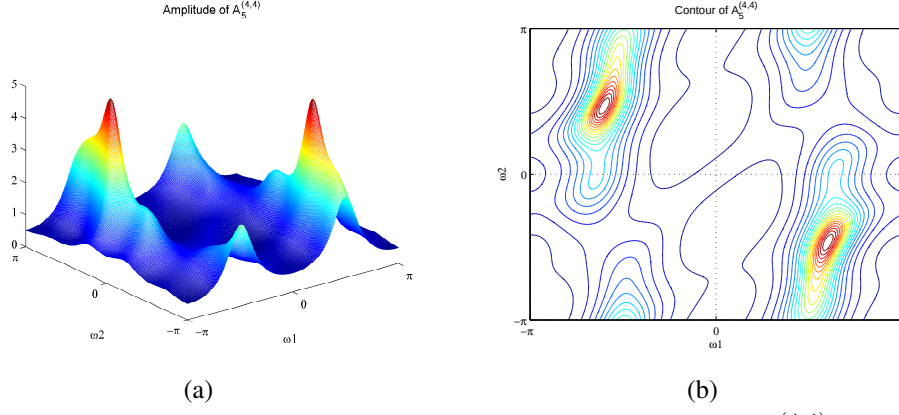


Figure 5.32: (a) Amplitude; (b) Contour plots of the spectrum of $A_5^{(4,4)}$, respectively, in Example 1, by Choice I.

The coefficient matrix of the type-1 ASHP lattice filter from the second vertical PEFs, $A_{ASHP_{1,2}}^{(4,4)}$ can be found as

$$A_{ASHP_{1,2}}^{(4,4)} = \begin{bmatrix} 0 & -0.0126 & -0.0414 & -0.0070 & 0.0173 \\ 1 & 0.0346 & -0.0078 & 0.0060 & 0.0269 \\ -0.4744 & 0.1958 & 0.2535 & -0.1106 & -0.1052 \\ 0.0191 & -0.2844 & 0.0203 & 0.0238 & -0.0157 \\ 0.0115 & 0.0080 & 0.0026 & 0.0054 & 0.0050 \end{bmatrix} \quad (5.14)$$

and the amplitude and equal magnitude contours of $A_{ASHP_{1,2}}^{(4,4)}$ are illustrated in Figure 5.38.

If we generate the type-2 ASHP from the same vertical PEFs, the coefficient matrix, $A_{ASHP_{2,2}}^{(4,4)}$ can be produced as

$$A_{ASHP_{2,2}}^{(4,4)} = \begin{bmatrix} -0.4648 & -0.0301 & -0.0246 & -0.0193 & -0.0163 \\ 1 & -0.0688 & -0.1383 & 0.0585 & 0.0752 \\ 0 & 0.2960 & 0.1775 & -0.0968 & -0.0596 \\ 0 & -0.1293 & 0.1064 & -0.0289 & -0.0484 \\ 0 & -0.0986 & 0.0739 & -0.0057 & -0.0201 \end{bmatrix} \quad (5.15)$$

and the amplitude and equal magnitude contours of $A_{ASHP_{2,2}}^{(4,4)}$ are illustrated in Figure 5.39.

The type-1 ASHP filter $A_{ASHP_{1,2}}^{(4,4)}$ and the type-2 ASHP filter $A_{ASHP_{2,2}}^{(4,4)}$ were found unstable.

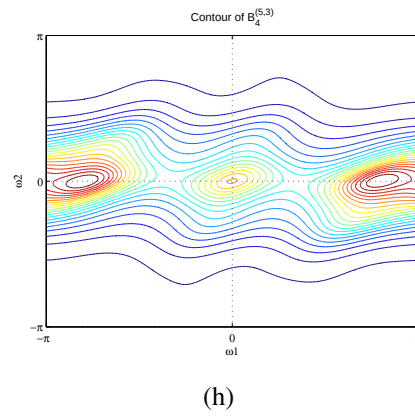
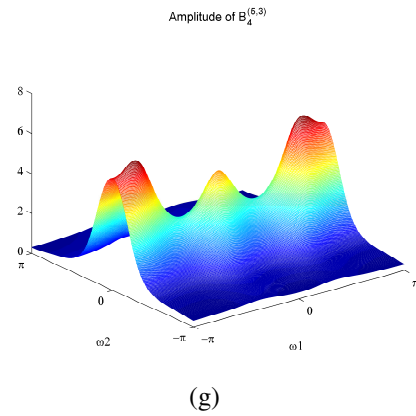
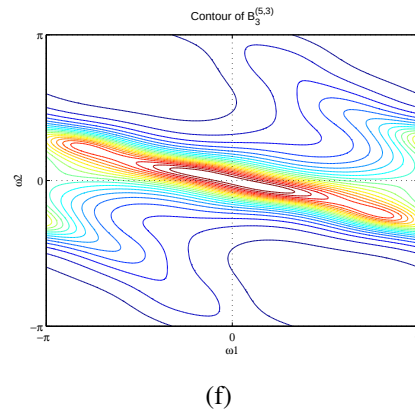
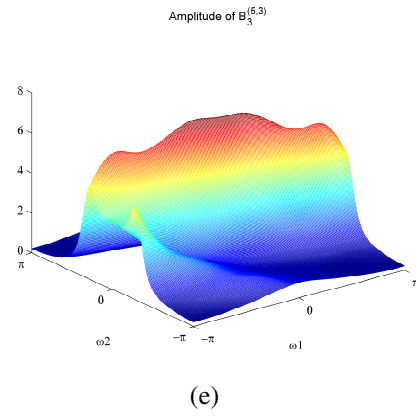
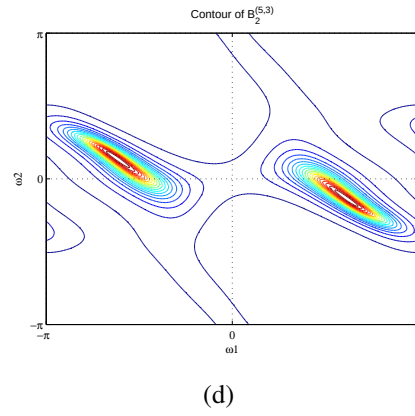
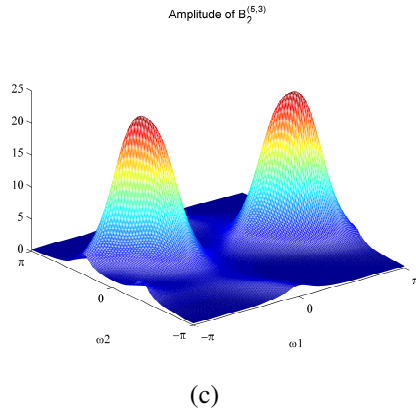
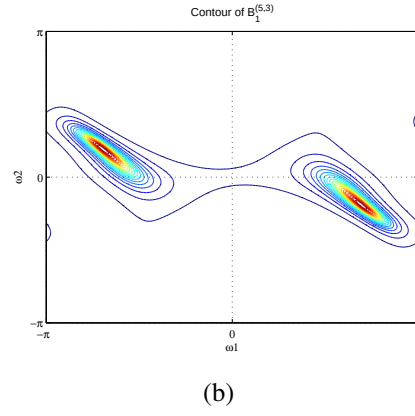
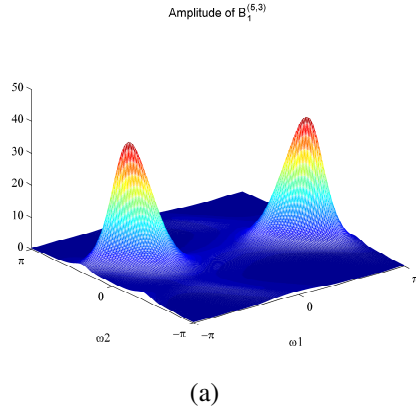


Figure 5.33: (a,c,e,g) Amplitude; (b,d,f,h) Contour plots of the spectrum of $B_1^{(5,3)}, \dots, B_4^{(5,3)}$, respectively, in Example 1, by Choice I.

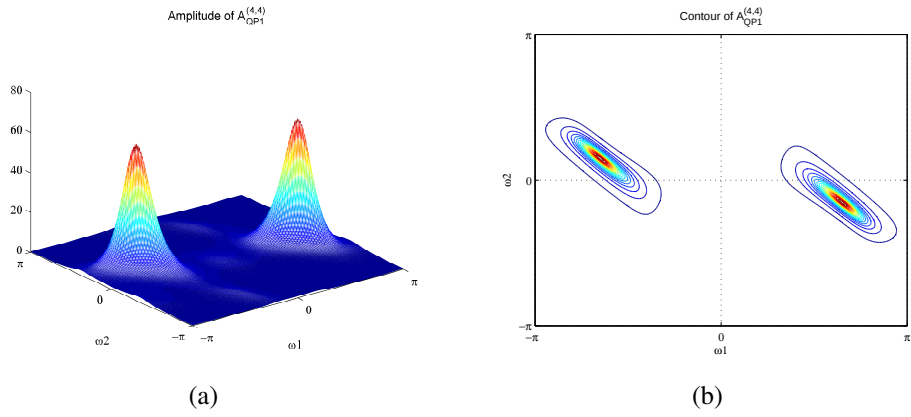


Figure 5.34: (a) Amplitude; (b) Contour plots of the spectrum of $A_{QP1}^{(4,4)}$ in Example 1, by Choice I.

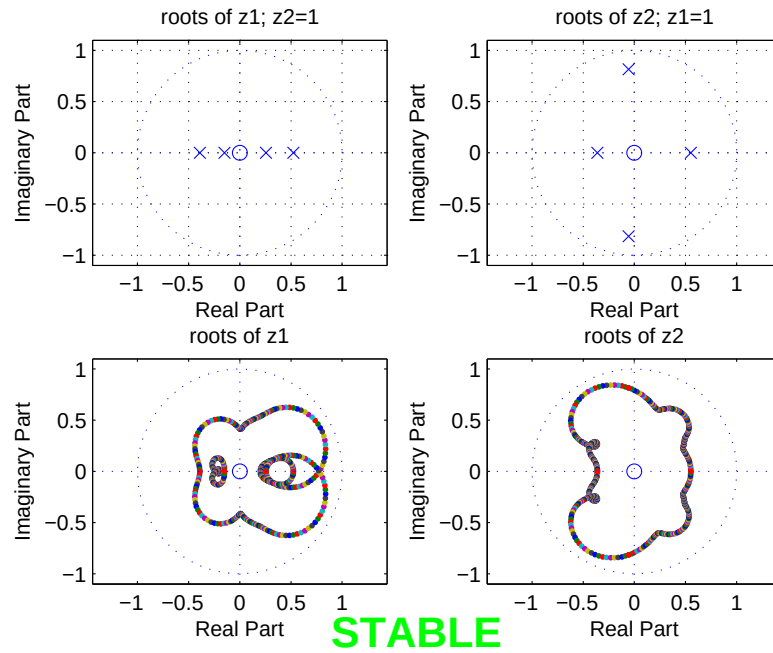


Figure 5.35: The stability test of the first-quadrant QP, $A_{QP1}^{(4,4)}$ in Example 1, by Choice I.

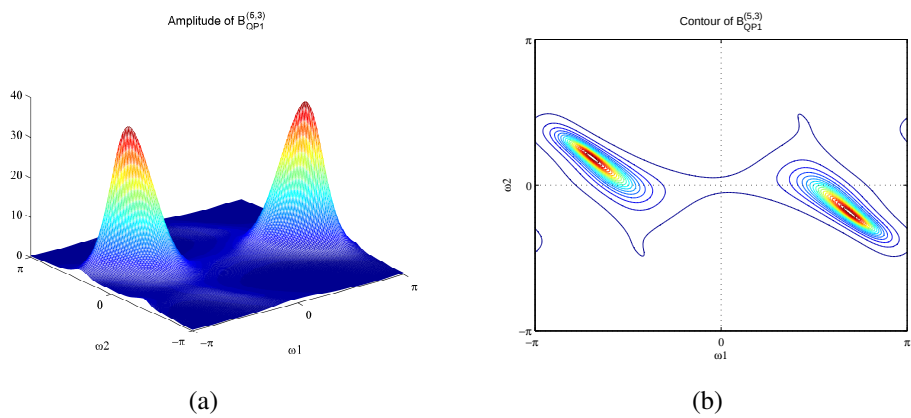


Figure 5.36: (a) Amplitude; (b) Contour plots of the spectrum of $B_{QP1}^{(5,3)}$ in Example 1, by Choice I.

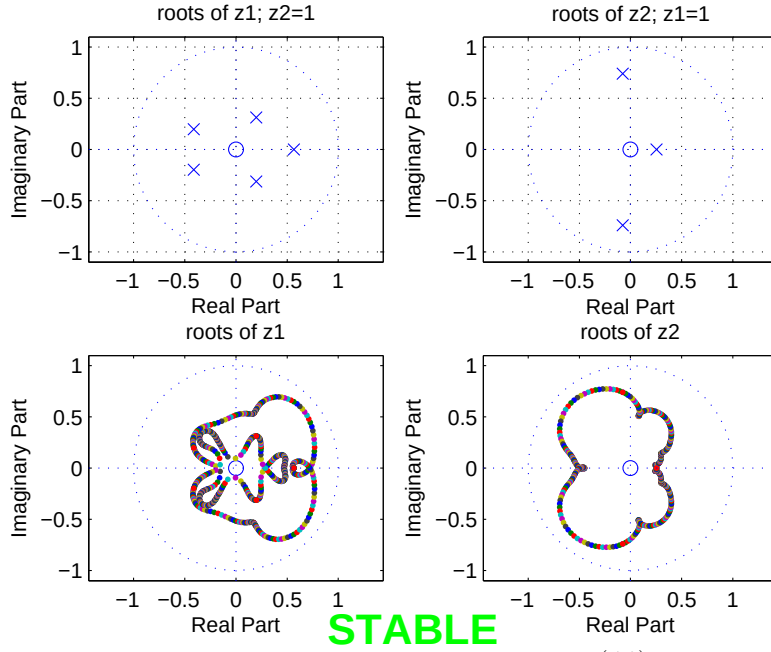


Figure 5.37: The stability test of the first-quadrant QP, $B_{QP1}^{(5,3)}$ in Example 1, by Choice I.

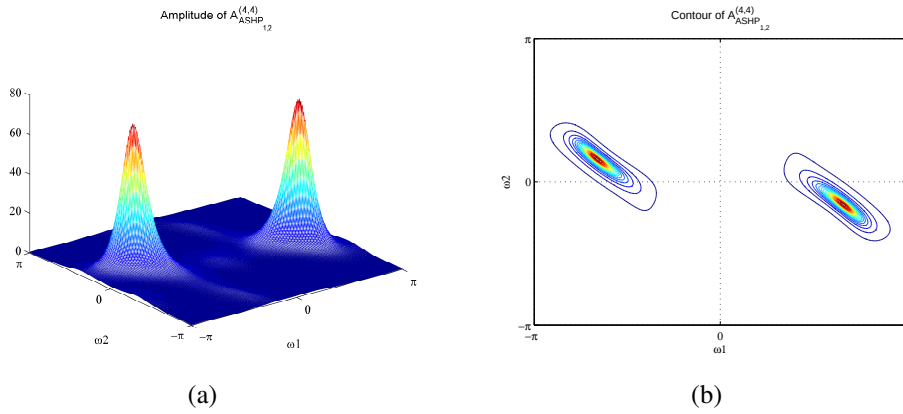


Figure 5.38: (a) Amplitude; (b) Contour plots of the spectrum of $A_{ASHP_{1,2}}^{(4,4)}$ in Example 1, by Choice I.

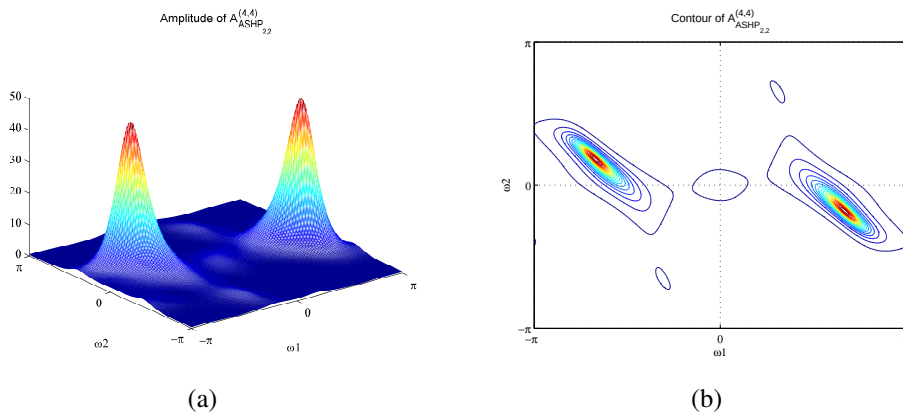


Figure 5.39: (a) Amplitude; (b) Contour plots of the spectrum of $A_{ASHP_{2,2}}^{(4,4)}$ in Example 1, by Choice I.

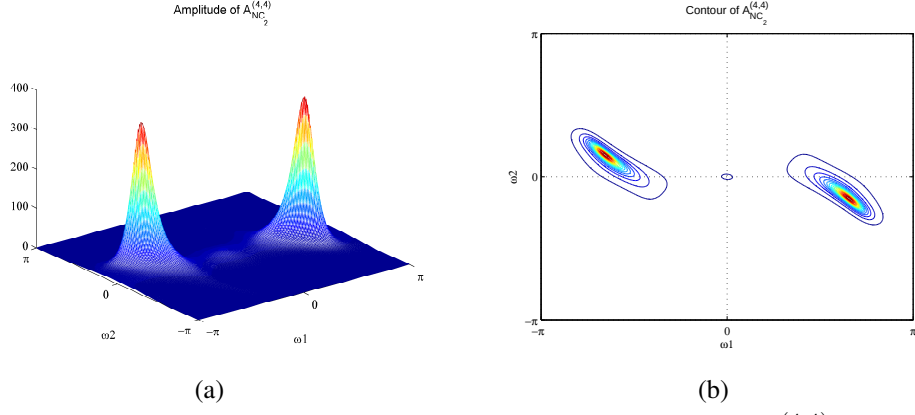


Figure 5.40: (a) Amplitude; (b) Contour plots of the spectrum of $A_{NC_2}^{(4,4)}$ in Example 1, by Choice I.

Table 5.13: Lattice Reflection Coefficients of the PSR $S_{k,l}^{(3,4)}$ in Example 1, by Choice I.

Reflection Coefficients				
$\mathbf{k}_e^{(3,5)} =$	0.0171	0.0128	-0.0190	0.0029
	0.0079	-0.0187	0.0066	-0.0037
	-0.0247	0.0081	0.0069	-0.0106
	0.0081	-0.0056	-0.0098	0.0073
$\mathbf{k}_r^{(4,4)} =$	0.0063	0.0068	0.0001	-0.0227
	-0.0072	-0.0083	-0.0018	0.0025
	0.0130	-0.0279	-0.0368	0.3363
	0.0750	0.1156	-0.2686	-0.1928
$k_{r_{new}}^{(4,4)} =$	-0.0244			
	0.0271			
	0.0169			
	-0.0139			

Similarly, an NC half-plane filter can be generated from the same vertical PEFs. The coefficient matrix, $A_{NC_2}^{(4,4)}$ can be calculated as

$$A_{NC_2}^{(4,4)} = \begin{bmatrix} -0.3859 & -0.0214 & -0.0250 & -0.0105 & -0.0035 \\ 1 & -0.0476 & -0.1026 & 0.0523 & 0.0626 \\ -0.3918 & 0.2695 & 0.1988 & -0.0999 & -0.0798 \\ 0.0093 & -0.2308 & 0.0166 & 0.0160 & -0.0158 \\ 0.0091 & -0.0009 & 0.0019 & 0.0069 & 0.0064 \end{bmatrix} \quad (5.16)$$

and the amplitude and equal magnitude contours of $A_{NC_2}^{(4,4)}$ are illustrated in Figure 5.40.

5.1.1.8 Vertical augmentation applied: Obtained PSR $S_{k,l}^{(3,4)}$ from $S_{k,l}^{(3,3)}$

Lattice reflection coefficients and the coefficient matrices are given in Table 5.13 and Table 5.14-5.15, respectively.

Table 5.14: Coefficient Matrices of the horizontal and vertical PEFs of the PSR $S_{k,l}^{(4,3)}$ in Example 1, by Choice I.

Coefficient Matrices of Vertical PEFs					of Horizontal PEFs									
$A_1^{(3,5)} =$					$B_1^{(4,4)} =$					$B_2^{(4,4)} =$				
1	0.0164	-0.0469	0.0110	0.0532	-0.0081	1	0	0	0	0	0	0	0	0
0	0.2138	0.2367	-0.1263	-0.0859	0.0247	-0.4771	0.2129	0.2356	-0.1241	-0.4769	0.2079	0.2593	-0.1156	-0.0750
0	-0.1689	0.1359	-0.0364	-0.0618	-0.0079	0.0150	-0.2858	0.0288	0.0182	0.0151	-0.2903	0.0240	0.0279	-0.0130
0	-0.1158	0.0907	-0.0064	-0.0279	-0.0171	0.0172	-0.0021	0.0010	0.0084	0.0138	-0.0059	0.0011	0.0083	0.0072
0						0.0014	0.0181	-0.0004	-0.0038	0.0019	0.0090	0.0019	-0.0068	-0.0063
$A_2^{(3,5)} =$					$B_3^{(4,4)} =$					$B_4^{(4,4)} =$				
0	-0.0234	-0.0458	-0.0127	0.0009	0.0056	0	0	1	0	0	0	0	0	0
1	0.0301	-0.0342	0.0035	0.0413	-0.0081	-0.0121	-0.4769	0.2079	0.2593	-0.0121	-0.4769	0.2079	0.2593	-0.1156
0	0.2484	0.2389	-0.1194	-0.0892	0.0187	-0.0029	0.0151	-0.2903	0.0240	-0.0029	0.0151	-0.2903	0.0240	0.0279
0	-0.2525	0.2057	-0.0388	-0.0736	-0.0128	-0.0081	0.0138	-0.0059	0.0011	-0.0081	0.0138	-0.0059	0.0011	0.0083
0						-0.0067	0.0019	0.0090	0.0019	-0.0067	0.0019	0.0090	0.0019	-0.0068
$A_3^{(3,5)} =$					$B_4^{(4,4)} =$					$B_5^{(4,4)} =$				
0	-0.0241	-0.0086	-0.0142	-0.0291	0.0098	0	0	1	0	0	0	0	0	0
0	-0.0084	-0.0531	-0.0111	0.0107	-0.0069	-0.0431	-0.0117	-0.4782	0.2229	-0.0431	-0.0117	-0.4782	0.2229	0.2686
1	0.1121	-0.0443	-0.0119	0.0424	-0.0066	0.0186	-0.0089	-0.0006	-0.2974	0.0186	-0.0089	-0.0006	-0.2974	0.0368
0	0.0703	0.4063	-0.1405	-0.1262	0.0190	-0.0092	0.0005	0.0111	-0.0061	-0.0092	0.0005	0.0111	-0.0061	0.0018
0						0.0142	-0.0076	-0.0024	0.0097	0.0142	-0.0076	-0.0024	0.0097	-0.0001
$A_4^{(3,5)} =$					$B_6^{(4,4)} =$					$B_7^{(4,4)} =$				
0	-0.0365	-0.0008	-0.0184	-0.0337	-0.0073	0	0	0	1	0	0	0	0	0
0	-0.0084	-0.0188	-0.0098	-0.0154	0.0106	-0.0083	-0.0360	0.0098	-0.5583	-0.0083	-0.0360	0.0098	-0.5583	0.1928
0	0.1332	-0.0972	-0.0519	0.0179	0.0037	-0.0110	0.0104	0.0046	0.0276	-0.0110	0.0104	0.0046	0.0276	-0.3363
1	0.0962	0.2412	-0.0569	-0.0080	-0.0029	-0.0103	-0.0074	0.0002	0.0198	-0.0103	-0.0074	0.0002	0.0198	-0.0025
0						0.0029	0.0274	-0.0078	-0.0003	0.0029	0.0274	-0.0078	-0.0003	0.0227

Table 5.15: Coefficient Matrices of the last horizontal PEFs of the PSR $S_{k,l}^{(3,4)}$ in Example 1, by Choice I.

Coefficient Matrix					
$B_5^{(4,4)} =$	0	0	0	0	1
	0.0244	-0.0479	-0.0718	0.0988	-0.5504
	-0.0271	0.0039	0.0223	0.0039	0.0414
	-0.0169	-0.0108	-0.0102	-0.0174	0.0147
	0.0139	0.0027	0.0197	-0.0076	0.0009

The amplitudes and equal magnitude contours of $A_1^{(3,5)}, \dots, A_4^{(3,5)}$, and $B_1^{(4,4)}, \dots, B_5^{(4,4)}$ are demonstrated in Figure 5.41 and Figure 5.42-5.43, respectively.

The coefficient matrix of the first-quadrant QP model, $B_{QP1}^{(4,4)}$ is

$$B_{QP1}^{(4,4)} = \begin{bmatrix} 1 & 0.0284 & -0.0225 & 0.0146 & 0.0457 \\ -0.4755 & 0.1969 & 0.2491 & -0.1255 & -0.1066 \\ 0.0131 & -0.2848 & 0.0217 & 0.0261 & -0.0161 \\ 0.0162 & -0.0023 & 0.0001 & 0.0080 & 0.0080 \\ 0.0016 & 0.0188 & 0.0006 & -0.0043 & -0.0061 \end{bmatrix} \quad (5.17)$$

and the amplitude and equal magnitude contours of $B_{QP1}^{(4,4)}$ are shown in Figure 5.44.

The results of the stability test of $B_{QP1}^{(4,4)}$ are shown in Figure 5.45.

Now, we reached to the desired order PSR $S_{k,l}^{(3,4)}$ and the first-quadrant QP filter from the horizontal PEFs, $B_{QP1}^{(4,4)}$ was generated as shown in Figure 5.44. As seen in the Figure 5.45, this QP filter is stable.

According to the proposed method, it is also possible to obtain a first-quadrant QP filter from the vertical PEFs as $A_{QP1}^{(3,5)}$ whose coefficient matrix is

$$A_{QP1}^{(3,5)} = \begin{bmatrix} 1 & 0.0266 & -0.0254 & 0.0165 & 0.0519 & -0.0107 \\ -0.4735 & 0.1993 & 0.2518 & -0.1282 & -0.1055 & 0.0286 \\ 0.0157 & -0.2830 & 0.0209 & 0.0193 & -0.0186 & -0.0168 \\ 0.0132 & 0.0012 & 0.0028 & 0.0090 & 0.0049 & -0.0108 \end{bmatrix} \quad (5.18)$$

and the amplitude and equal magnitude contours of $A_{QP1}^{(3,5)}$ are illustrated in Figure 5.46.

As seen in the Figure 5.47, this QP filter, $A_{QP1}^{(3,5)}$ is also stable.

Now, one example of each filter models, type-1, type-2 ASHP, and NC half-plane, will be generated from the horizontal PEFs of this PSR $S_{k,l}^{(3,4)}$. The coefficient matrix of the type-1 ASHP lattice filter from the second horizontal PEFs, $B_{ASHP1,2}^{(4,4)}$ can be obtained

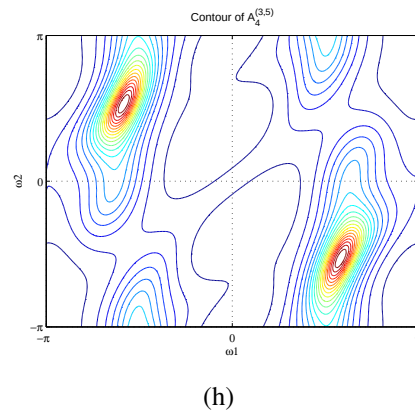
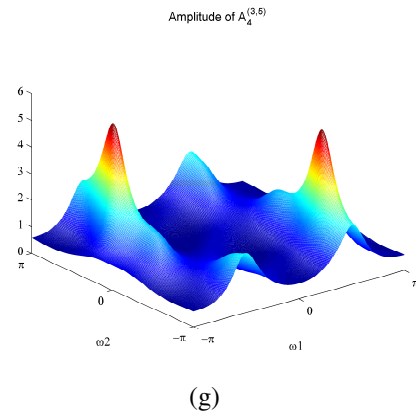
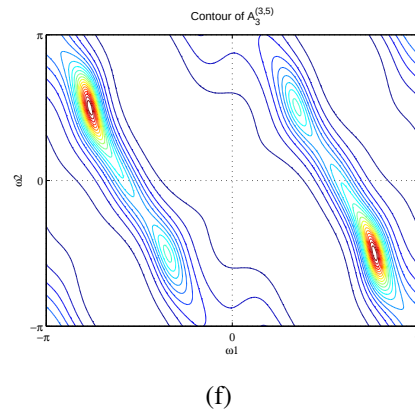
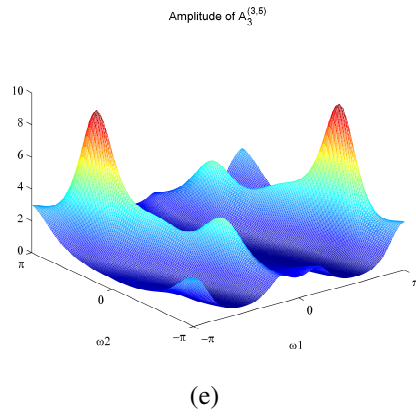
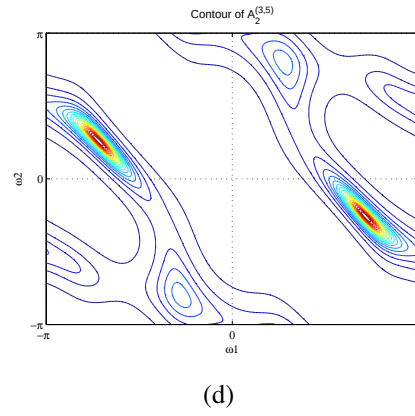
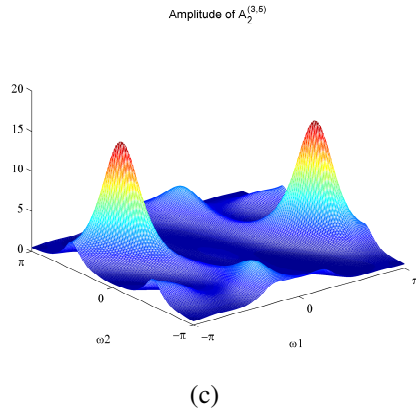
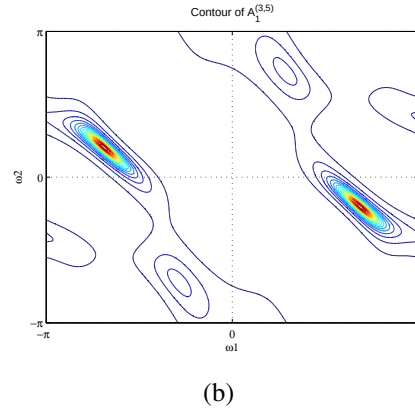
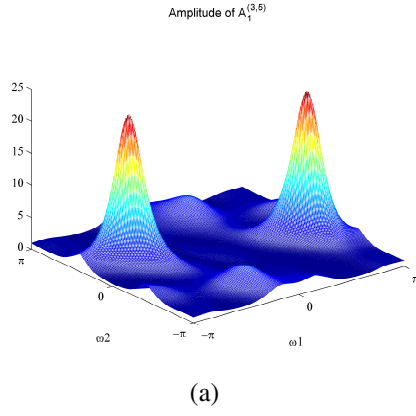
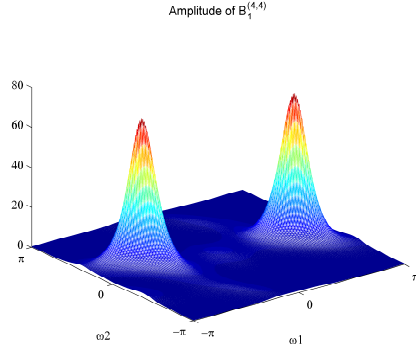
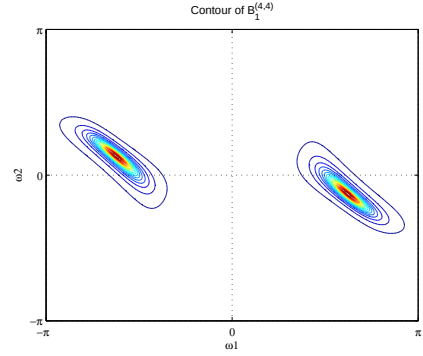


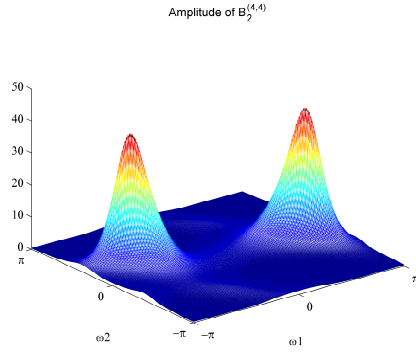
Figure 5.41: (a, c, e, g) Amplitude; (b, d, f, h) Contour plots of the spectrum of $A_1^{(3,5)}, \dots, A_4^{(3,5)}$, respectively, in Example 1, by Choice I.



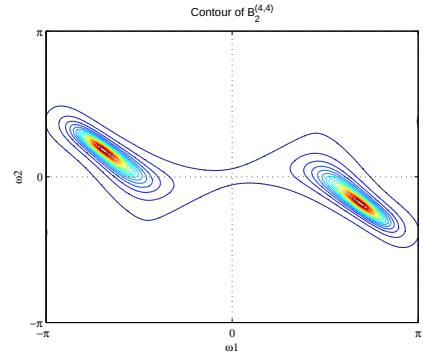
(a)



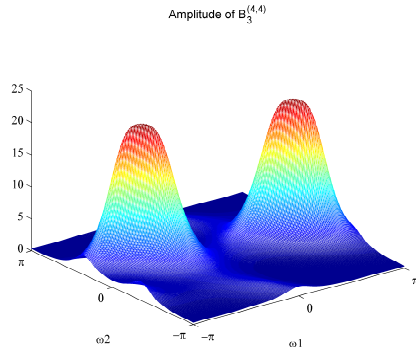
(b)



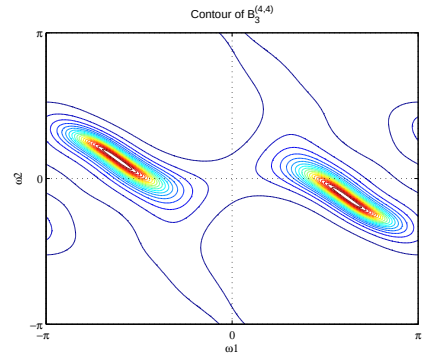
(c)



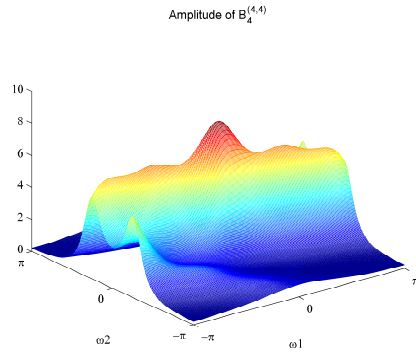
(d)



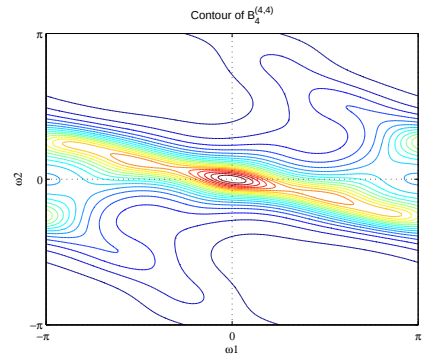
(e)



(f)

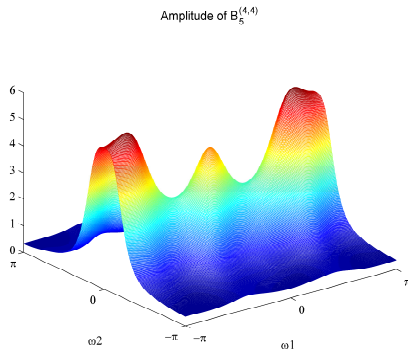


(g)

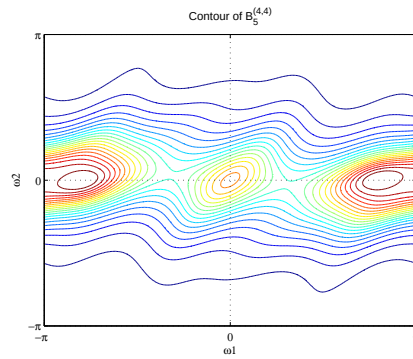


(h)

Figure 5.42: (a, c, e, g) Amplitude; (b, d, f, h) Contour plots of the spectrum of $B_1^{(5,3)}, \dots, B_4^{(5,3)}$, respectively, in Example 1, by Choice I.

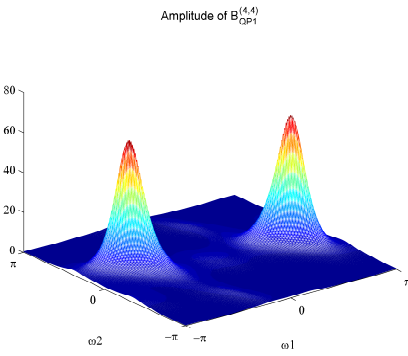


(a)

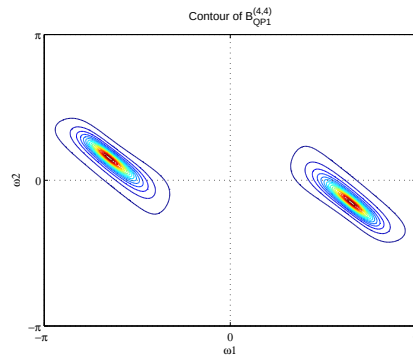


(b)

Figure 5.43: (a) Amplitude; (b) Contour plots of the spectrum of $B_5^{(4,4)}$, respectively, in Example 1, by Choice I.



(a)



(b)

Figure 5.44: (a) Amplitude; (b) Contour plots of the spectrum of $B_{QP1}^{(4,4)}$ in Example 1, by Choice I.

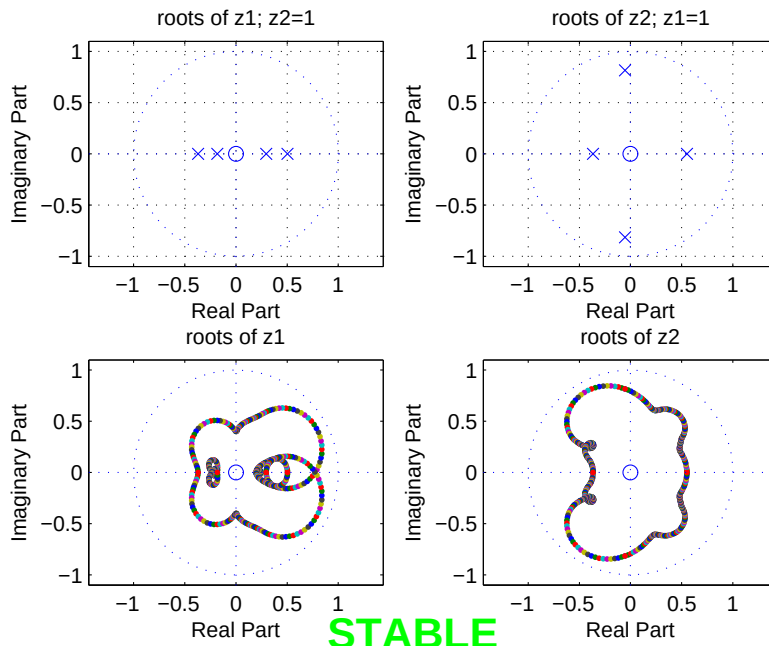


Figure 5.45: The stability test of the first-quadrant QP, $B_{QP1}^{(4,4)}$ in Example 1, by Choice I.

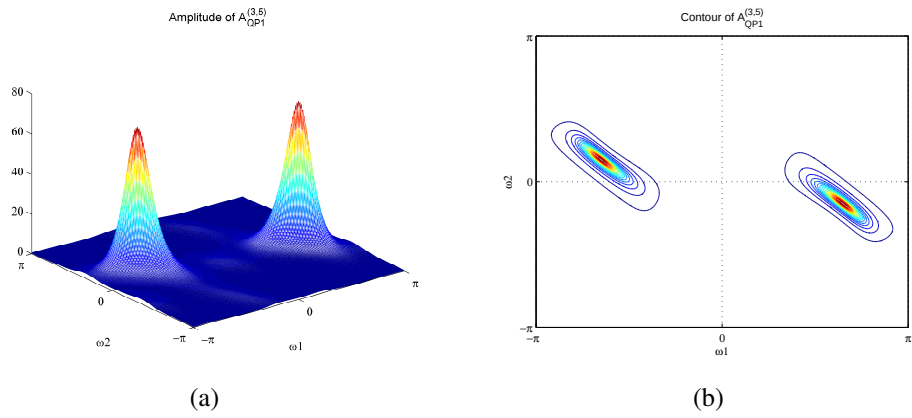


Figure 5.46: (a) Amplitude; (b) Contour plots of the spectrum of $A_{QP1}^{(3,5)}$ in Example 1, by Choice I.

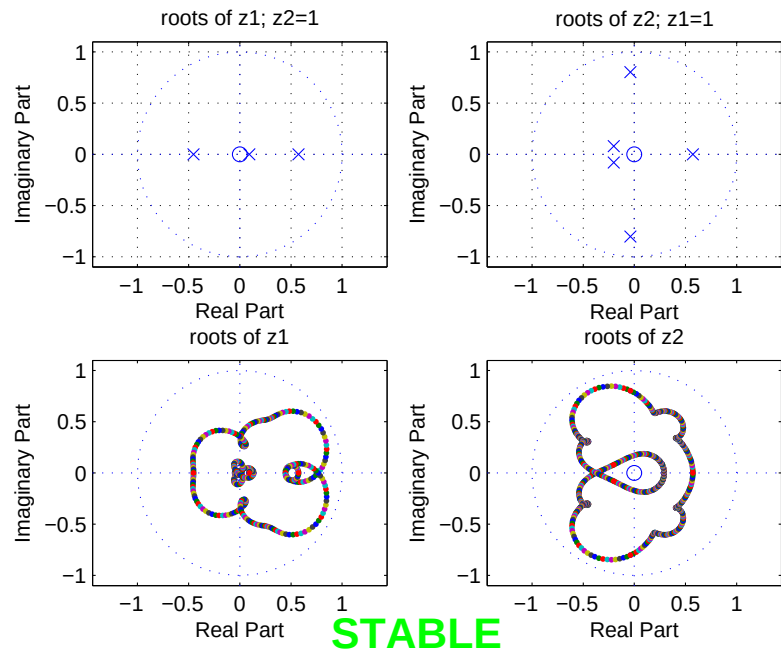


Figure 5.47: The stability test of the first-quadrant QP, $A_{QP1}^{(3,5)}$ in Example 1, by Choice I.

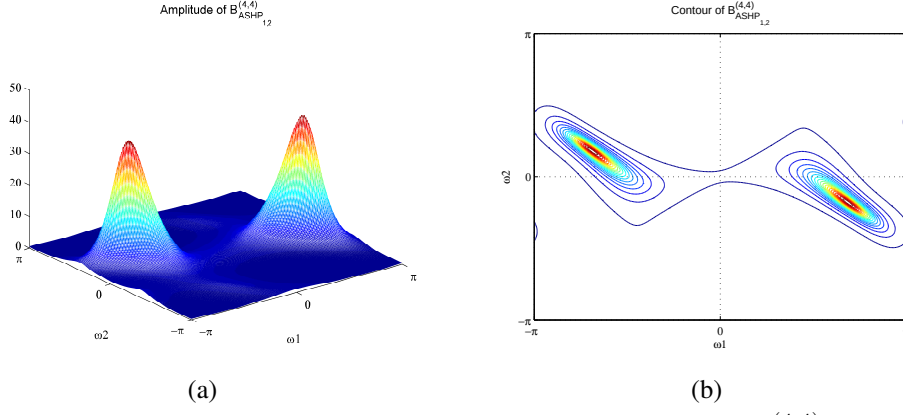


Figure 5.48: (a) Amplitude; (b) Contour plots of the spectrum of $B_{ASHP_{1,2}}^{(4,4)}$ in Example 1, by Choice I.

as

$$B_{ASHP_{1,2}}^{(4,4)} = \begin{bmatrix} 0 & 1 & 0.0152 & 0.0132 & 0.0142 \\ -0.0125 & -0.4783 & 0.1997 & 0.2567 & -0.1167 \\ -0.0031 & 0.0152 & -0.2899 & 0.0199 & 0.0247 \\ -0.0087 & 0.0135 & -0.0059 & 0.0010 & 0.0085 \\ -0.0062 & 0.0022 & 0.0091 & 0.0019 & -0.0065 \end{bmatrix} \quad (5.19)$$

and the amplitude and equal magnitude contours of $B_{ASHP_{1,2}}^{(4,4)}$ are illustrated in Figure 5.48.

If we generate the type-2 ASHP from the same vertical PEFs, the coefficient matrix, $B_{ASHP_{2,2}}^{(4,4)}$ can be found as

$$B_{ASHP_{2,2}}^{(4,4)} = \begin{bmatrix} 0.0280 & 1 & 0 & 0 & 0 \\ -0.0254 & -0.4710 & 0.2145 & 0.2558 & -0.1176 \\ -0.0024 & 0.0071 & -0.2895 & 0.0245 & 0.0276 \\ -0.0077 & 0.0137 & -0.0059 & 0.0013 & 0.0085 \\ -0.0066 & 0.0024 & 0.0089 & 0.0018 & -0.0070 \end{bmatrix} \quad (5.20)$$

and the amplitude and equal magnitude contours of $B_{ASHP_{2,2}}^{(4,4)}$ are illustrated in Figure 5.49.

The type-1 ASHP filter $B_{ASHP_{1,2}}^{(4,4)}$ and the type-2 ASHP filter $B_{ASHP_{2,2}}^{(4,4)}$ were found unstable.

Similarly, an NC half-plane filter can be generated from the same horizontal PEF. The coefficient matrix, $B_{NC_2}^{(4,4)}$ can be calculated as

$$B_{NC_2}^{(4,4)} = \begin{bmatrix} 0.0287 & 1 & 0.0146 & 0.0136 & 0.0155 \\ -0.0261 & -0.4722 & 0.2067 & 0.2529 & -0.1197 \\ -0.0027 & 0.0070 & -0.2891 & 0.0206 & 0.0242 \\ -0.0082 & 0.0134 & -0.0059 & 0.0013 & 0.0088 \\ -0.0061 & 0.0028 & 0.0091 & 0.0018 & -0.0067 \end{bmatrix} \quad (5.21)$$

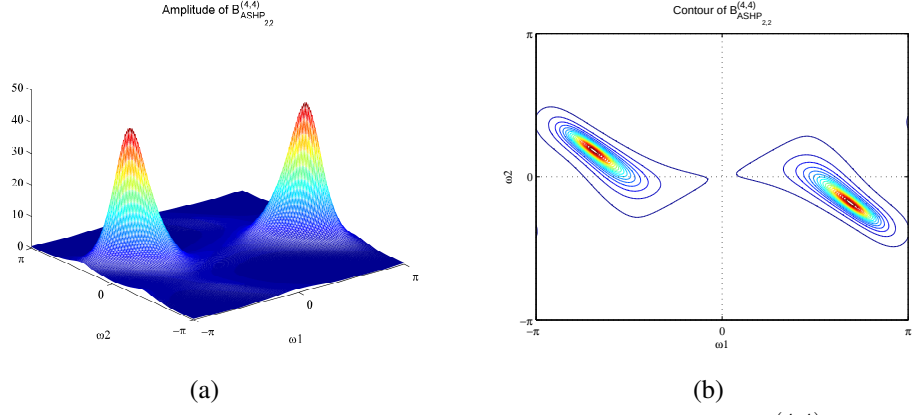


Figure 5.49: (a) Amplitude; (b) Contour plots of the spectrum of $B_{ASHP_{2,2}}^{(4,4)}$ in Example 1, by Choice I.

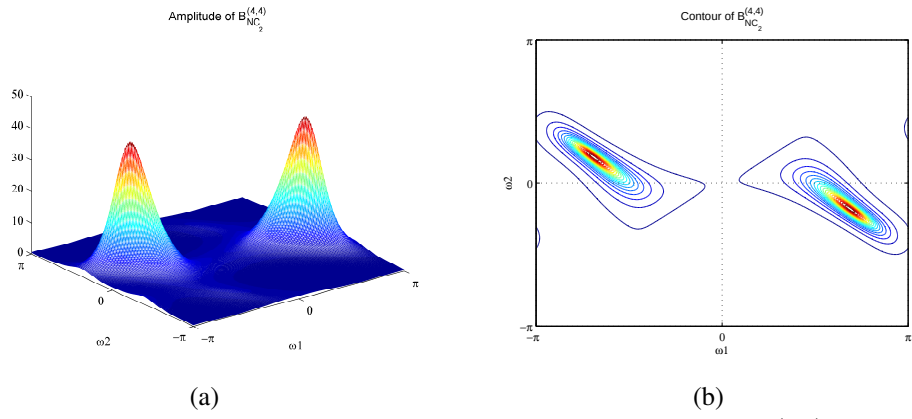


Figure 5.50: (a) Amplitude; (b) Contour plots of the spectrum of $B_{NC_2}^{(4,4)}$ in Example 1, by Choice I.

and the amplitude and equal magnitude contours of $B_{NC_2}^{(4,4)}$ are illustrated in Figure 5.50.

5.1.2 Augmentation application sequences: Choice II

As a second selection, first a vertical then a horizontal augmentation will be applied after the initial first order stage. The sequences will be repeated until the desired PSR $S_{k,l}^{(0,1)}$ is reached.

5.1.2.1 Vertical augmentation applied: Obtained PSR $S_{k,l}^{(0,1)}$

Lattice reflection coefficients and the coefficient matrices are shown in Table 5.16.

The amplitudes and equal magnitude contours of $B_1^{(1,1)}$ and $B_2^{(1,1)}$ are illustrated in Figure 5.51. $A_1^{(0,2)}$ will not be shown because it is 1-D coefficient vector.

Table 5.16: Lattice Reflection Coefficients and the Coefficient Matrices for the PSR $S_{k,l}^{(0,1)}$ in Example 1, by Choice II.

Reflection Coefficients	Coefficient Matrices
$k_{e_1}^{(0,2)} = [-0.1987]$	$A_1^{(1,1)} = \begin{bmatrix} 1 & 0.1854 & 0.1987 \end{bmatrix}$
$k_{r_1}^{(1,1)} = [-0.0063]$	$B_1^{(1,1)} = \begin{bmatrix} 1 & 0 \\ -0.5310 & 0.0063 \end{bmatrix}$
$k_{r_{new}}^{(1,1)} = [-0.1161]$	$B_2^{(1,1)} = \begin{bmatrix} 0 & 1 \\ 0.1161 & -0.5140 \end{bmatrix}$

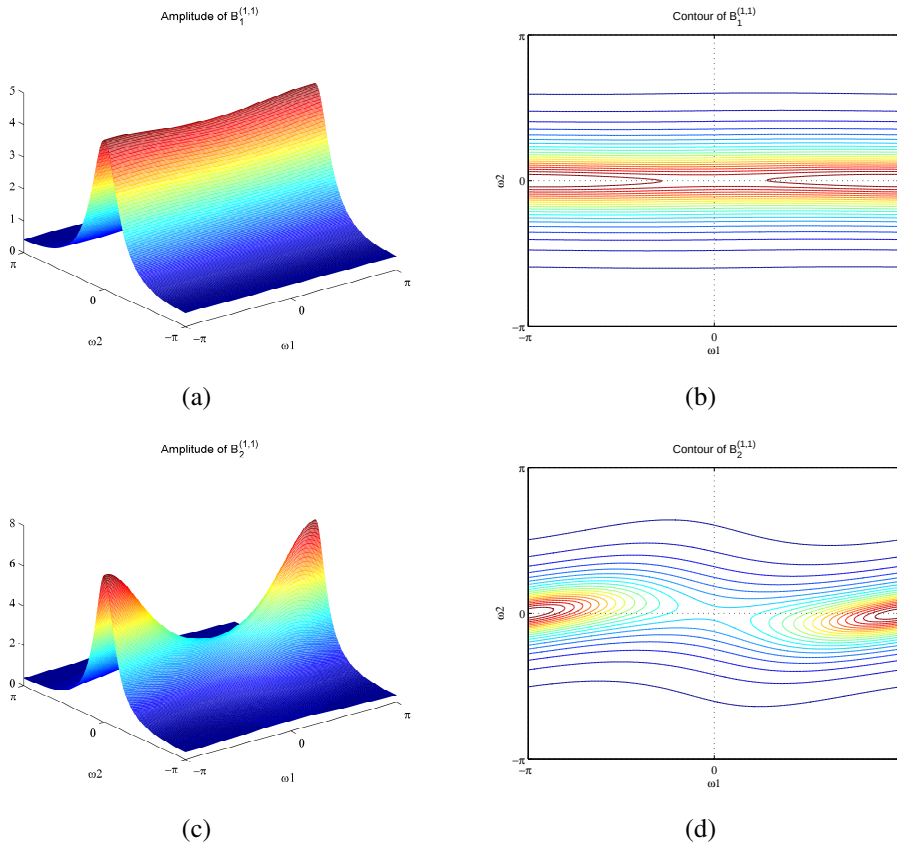


Figure 5.51: (a, c) Amplitude; (b, d) Contour plots of the spectrum of $B_1^{(1,1)}$ and $B_2^{(1,1)}$, respectively, in Example 1, by Choice II.

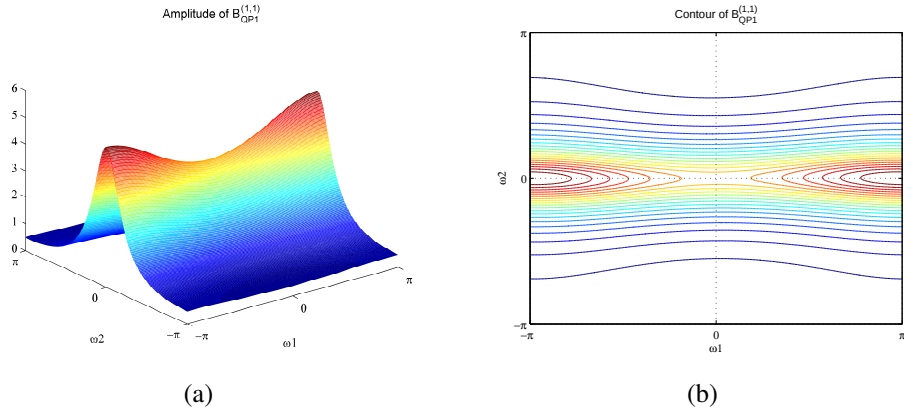


Figure 5.52: (a) Amplitude; (b) Contour plots of the spectrum of $B_{QP1}^{(1,1)}$ in Example 1, by Choice II.

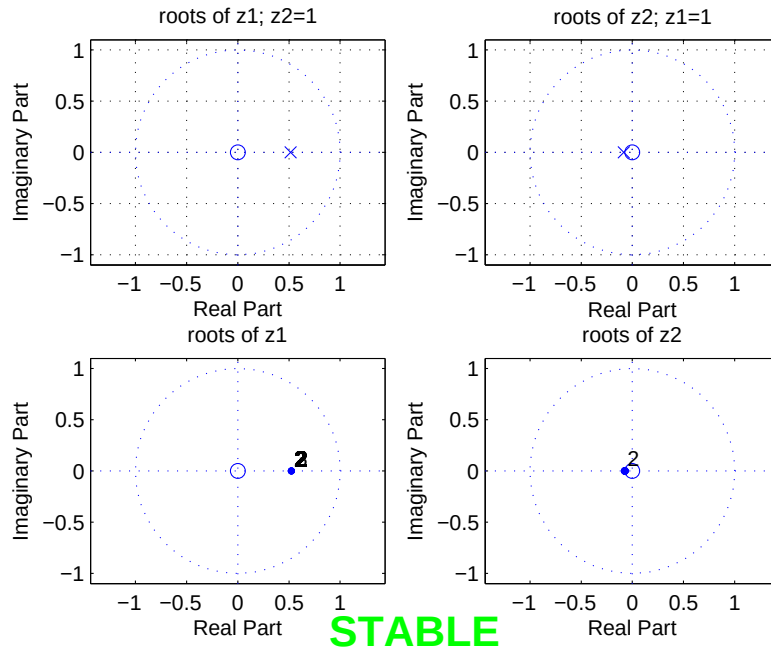


Figure 5.53: The stability test of the first-quadrant QP, $B_{QP1}^{(1,1)}$ in Example 1, by Choice II.

The coefficient matrix of the first-quadrant QP model, $B_{QP1}^{(1,1)}$ is

$$B_{QP1}^{(1,1)} = \begin{bmatrix} 1 & 0.0681 \\ -0.5211 & -0.0287 \end{bmatrix} \quad (5.22)$$

and the amplitude and equal magnitude contours of $B_{QP1}^{(1,1)}$ are shown in Figure 5.52.

The results of the stability test of $B_{QP1}^{(1,1)}$ are shown in Figure 5.53.

Table 5.17: Lattice Reflection Coefficients and the Coefficient Matrices for the PSR $S_{k,l}^{(1,1)}$ in Example 1, by Choice II.

Reflection Coefficients	Coefficient Matrices
$\mathbf{k}_e^{(1,2)} = \begin{bmatrix} -0.4235 \\ -0.0689 \end{bmatrix}$	$A_1^{(1,2)} = \begin{bmatrix} 1 & 0.1527 & -0.0182 \\ 0 & 0.0689 & 0.4235 \end{bmatrix}$
$k_{e_{new}}^{(1,2)} = \begin{bmatrix} -0.1536 \\ 0.0992 \end{bmatrix}$	$A_2^{(1,2)} = \begin{bmatrix} 0 & 0.1536 & -0.0992 \\ 1 & 0.0923 & 0.2506 \end{bmatrix}$
$\mathbf{k}_r^{(2,1)} = \begin{bmatrix} 0.3456 & -0.0730 \\ -0.0502 & -0.0094 \end{bmatrix}$	$B_1^{(2,1)} = \begin{bmatrix} 1 & 0 \\ -0.5589 & 0.1957 \\ 0.0502 & -0.3456 \end{bmatrix}$
	$B_2^{(2,1)} = \begin{bmatrix} 0 & 1 \\ 0.1117 & -0.5516 \\ 0.0094 & 0.0730 \end{bmatrix}$

5.1.2.2 Horizontal augmentation applied: Obtained PSR $S_{k,l}^{(1,1)}$

Lattice reflection coefficients and the coefficient matrices are found in Table 5.17.

The amplitudes and equal magnitude contours of $A_1^{(1,2)}$, $A_2^{(1,2)}$, and $B_1^{(2,1)}$, $B_2^{(2,1)}$ are demonstrated in Figure 5.54.

The coefficient matrix of the first-quadrant QP model, $B_{QP1}^{(2,1)}$ is

$$B_{QP1}^{(2,1)} = \begin{bmatrix} 1 & 0.0435 \\ -0.5541 & 0.01717 \\ 0.0506 & -0.3425 \end{bmatrix} \quad (5.23)$$

and the amplitude and equal magnitude contours of $B_{QP1}^{(2,1)}$ are shown in Figure 5.55.

The results of the stability test of $B_{QP1}^{(2,1)}$ are shown in Figure 5.56.

For making comparison with the choice I result, the coefficient matrix of the first-quadrant QP model, $A_{QP1}^{(1,2)}$ is also given as

$$A_{QP1}^{(1,2)} = \begin{bmatrix} 1 & 0.0821 & 0.0274 \\ -0.4597 & 0.0264 & 0.3083 \end{bmatrix} \quad (5.24)$$

and the amplitude and equal magnitude contours of $A_{QP1}^{(1,2)}$ are shown in Figure 5.57.

The results of the stability test of $A_{QP1}^{(1,2)}$ are shown in Figure 5.58.

The coefficient matrix (5.24) is closely the same as the coefficient matrix (5.4) generated by using choice I.

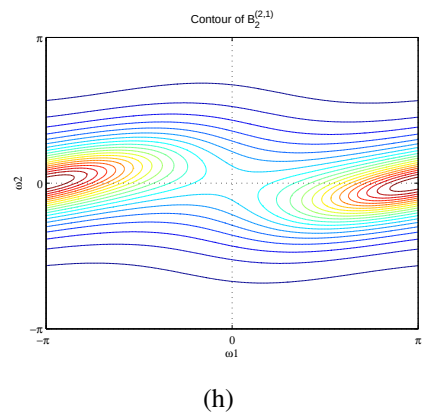
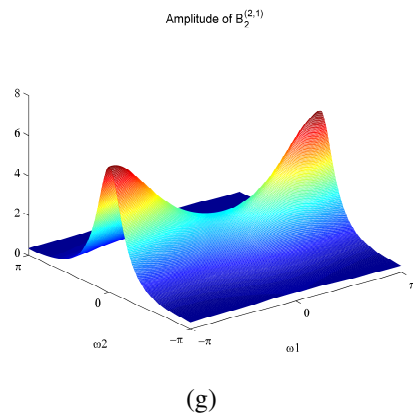
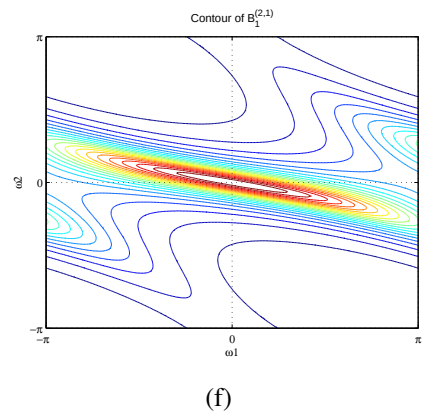
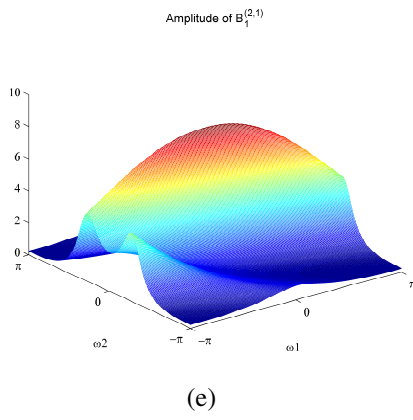
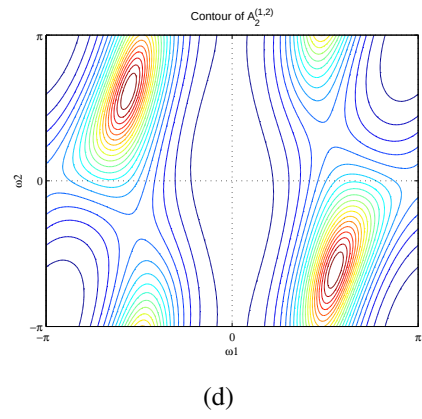
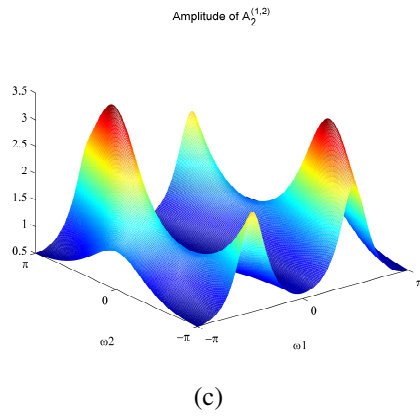
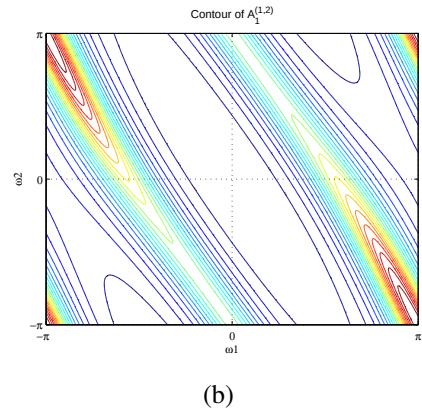
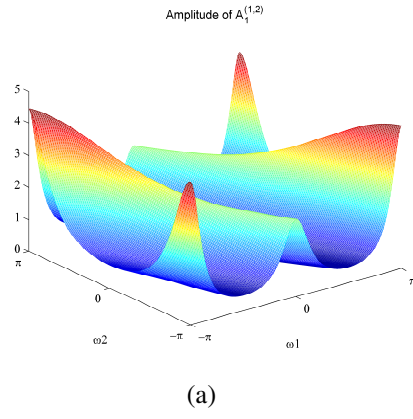


Figure 5.54: (a,c,e,g) Amplitude; (b,d,f,h) Contour plots of the spectrum of $A_1^{(1,2)}$, $A_2^{(1,2)}$, $B_1^{(2,1)}$, and $B_2^{(2,1)}$, respectively, in Example 1, by Choice II.

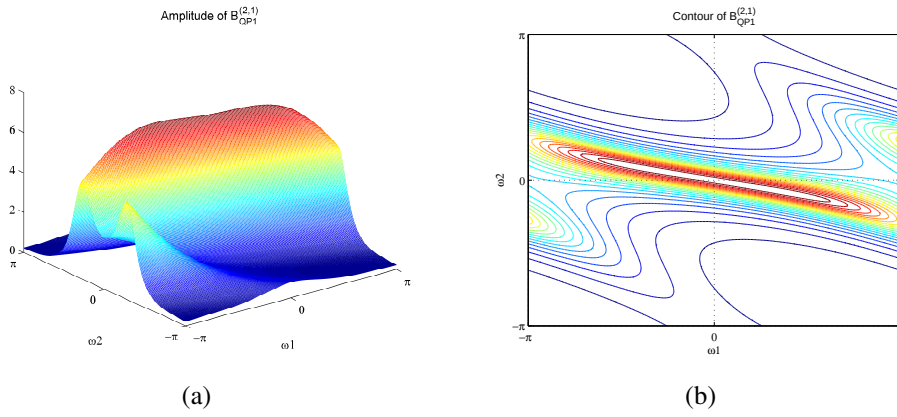


Figure 5.55: (a) Amplitude; (b) Contour plots of the spectrum of $B_{QP1}^{(2,1)}$ in Example 1, by Choice II.

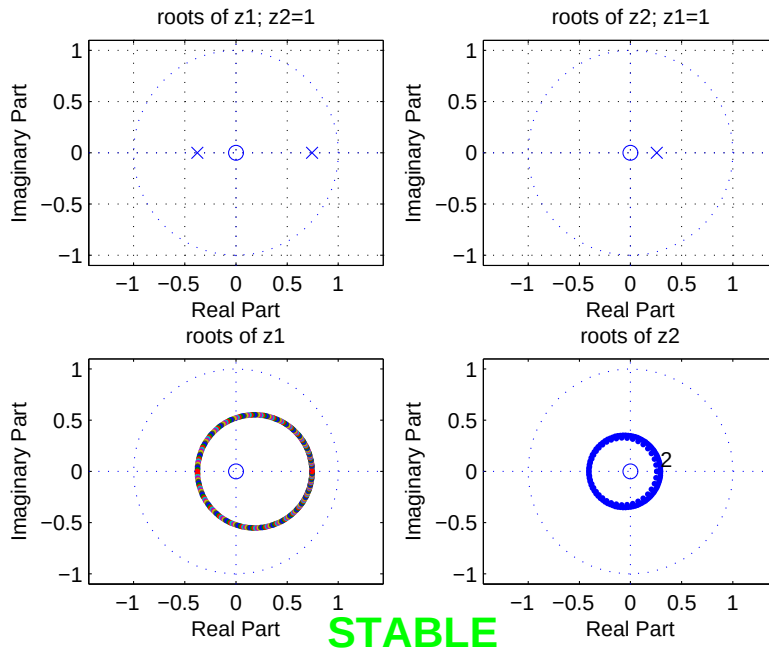


Figure 5.56: The stability test of the first-quadrant QP, $B_{QP1}^{(2,1)}$ in Example 1, by Choice II.

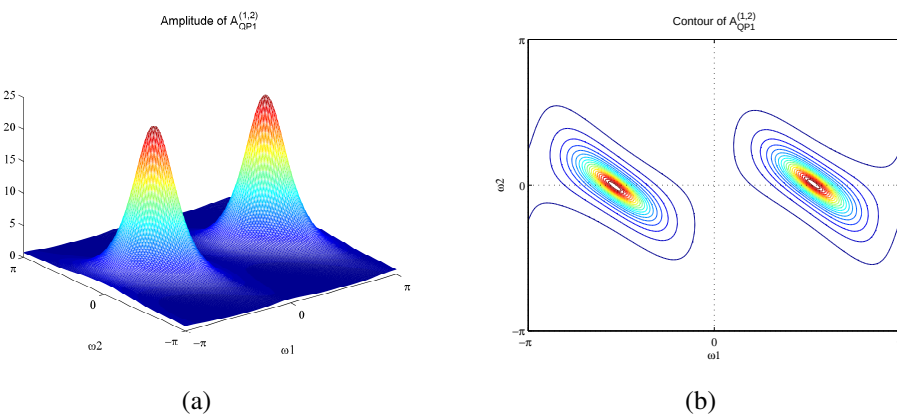


Figure 5.57: (a) Amplitude; (b) Contour plots of the spectrum of $A_{QP1}^{(1,2)}$ in Example 1, by Choice II.

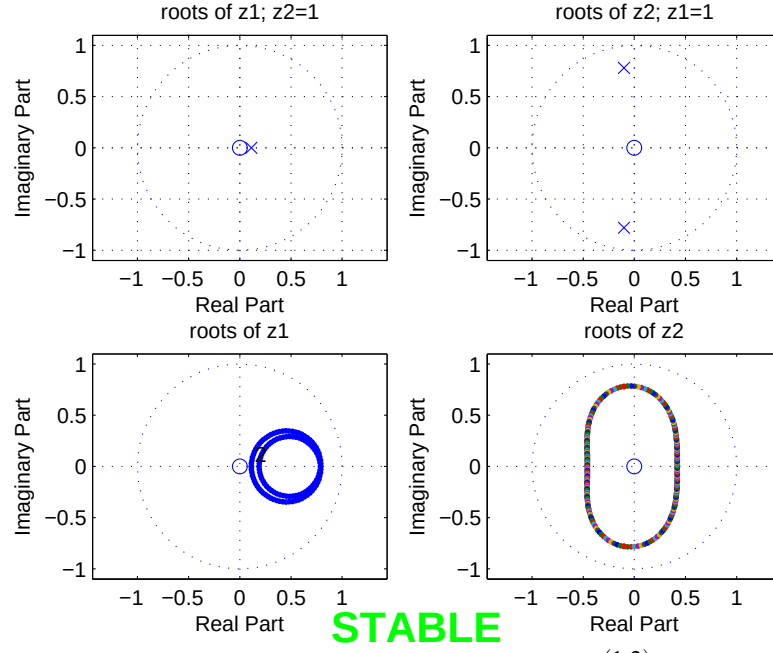


Figure 5.58: The stability test of the first-quadrant QP, $A_{QP1}^{(1,2)}$ in Example 1, by Choice II.

5.1.2.3 Vertical augmentation applied: Obtained PSR $S_{k,l}^{(1,2)}$

Lattice reflection coefficients and the coefficient matrices are given in Table 5.18.

The amplitudes and equal magnitude contours of $A_1^{(1,3)}$ and $A_2^{(1,3)}$, and $B_1^{(2,2)}, \dots, B_2^{(2,2)}$ are demonstrated in Figure 5.59 and Figure 5.60, respectively.

The coefficient matrix of the first-quadrant QP model, $B_{QP1}^{(2,2)}$ is

$$B_{QP1}^{(2,2)} = \begin{bmatrix} 1 & 0.0566 & 0.0275 \\ -0.4803 & 0.1938 & 0.2648 \\ 0.0233 & -0.2967 & 0.0116 \end{bmatrix} \quad (5.25)$$

and the amplitude and equal magnitude contours of $B_{QP1}^{(2,2)}$ are shown in Figure 5.61.

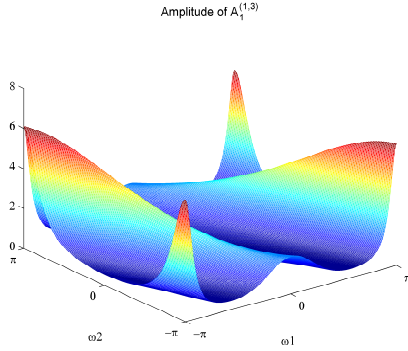
The results of the stability test of $B_{QP1}^{(2,2)}$ are shown in Figure 5.62.

5.1.2.4 Horizontal augmentation applied: Obtained PSR $S_{k,l}^{(2,2)}$

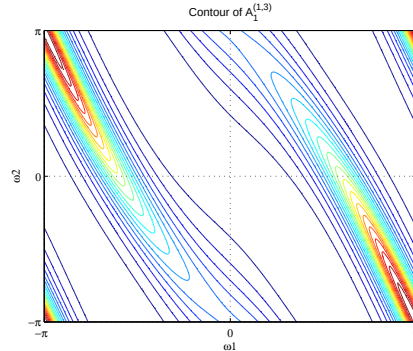
Lattice reflection coefficients and the coefficient matrices are given in Table 5.19.

Table 5.18: Lattice Reflection Coefficients and the Coefficient Matrices for the PSR $S_{k,l}^{(1,2)}$ in Example 1, by Choice II.

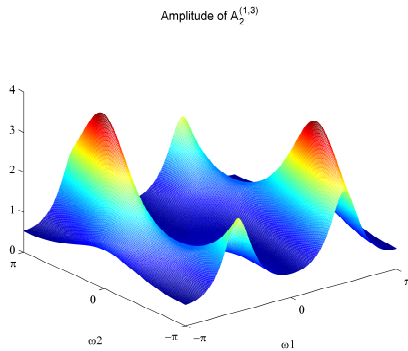
Reflection Coefficients	Coefficient Matrices
$\mathbf{k}_e^{(1,3)} = \begin{bmatrix} 0.1517 & 0.0599 \\ 0.0019 & 0.0405 \end{bmatrix}$	$A_1^{(1,3)} = \begin{bmatrix} 1 & 0.0880 & -0.0288 & -0.0019 \\ 0 & 0.0718 & 0.4000 & -0.1517 \\ 0 & 0.1181 & -0.1071 & -0.0405 \\ 1 & 0.0974 & 0.2353 & -0.0599 \end{bmatrix}$ $A_2^{(1,3)} = \begin{bmatrix} 1 & 0 & 0 \\ -0.4790 & 0.2225 & 0.2690 \\ 0.0230 & -0.2998 & 0.0295 \\ 0 & 1 & 0 \\ 0.0172 & -0.5570 & 0.1943 \\ -0.0037 & 0.0512 & -0.3382 \\ 0 & 0 & 1 \\ -0.0800 & 0.1027 & -0.5523 \\ 0.0210 & 0.0058 & 0.0444 \end{bmatrix}$
$\mathbf{k}_r^{(2,2)} = \begin{bmatrix} -0.0295 & 0.3382 \\ -0.2690 & -0.1943 \end{bmatrix}$	$B_1^{(2,2)} = \begin{bmatrix} 1 & 0 & 0 \\ -0.4790 & 0.2225 & 0.2690 \\ 0.0230 & -0.2998 & 0.0295 \end{bmatrix}$
$k_{r_{new}}^{(2,2)} = \begin{bmatrix} 0.0800 \\ -0.0210 \end{bmatrix}$	$B_2^{(2,2)} = \begin{bmatrix} 0 & 1 & 0 \\ 0.0172 & -0.5570 & 0.1943 \\ -0.0037 & 0.0512 & -0.3382 \end{bmatrix}$
	$B_3^{(2,2)} = \begin{bmatrix} 0 & 0 & 1 \\ -0.0800 & 0.1027 & -0.5523 \\ 0.0210 & 0.0058 & 0.0444 \end{bmatrix}$



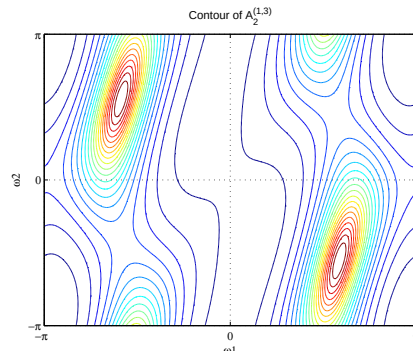
(a)



(b)

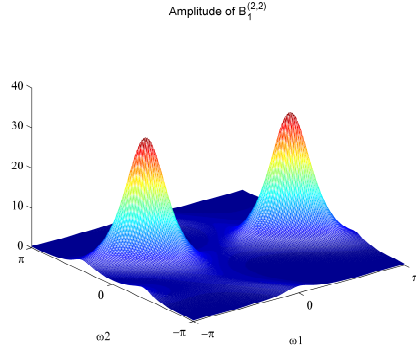


(c)

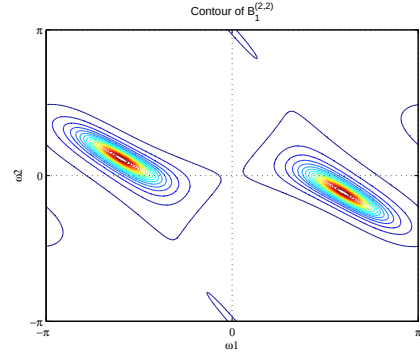


(d)

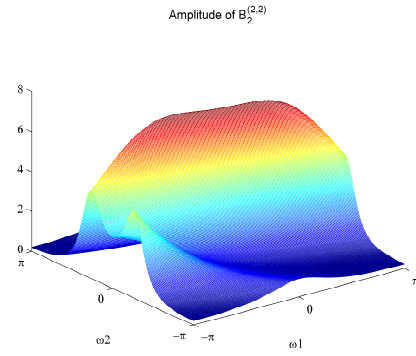
Figure 5.59: (a,c) Amplitude; (b,d) Contour plots of the spectrum of $A_1^{(1,3)}$, and $A_2^{(1,3)}$, respectively, in Example 1, by Choice II.



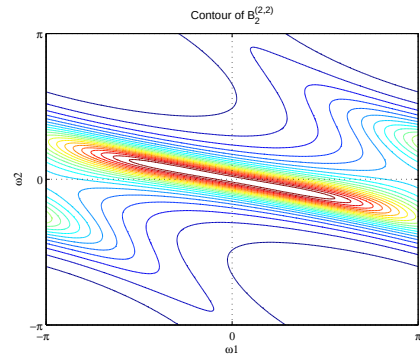
(a)



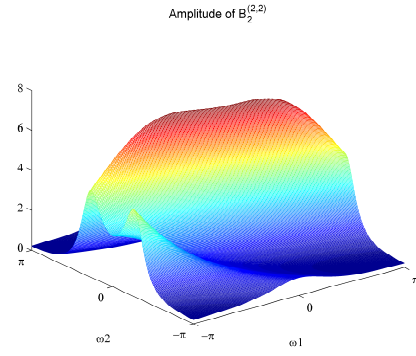
(b)



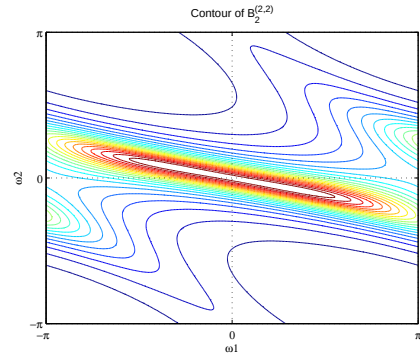
(c)



(d)

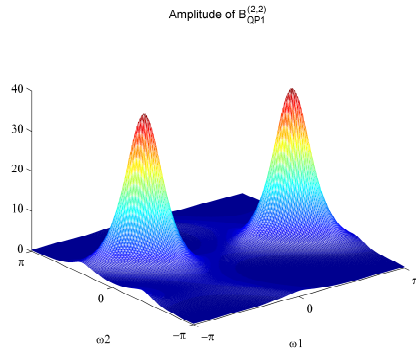


(e)

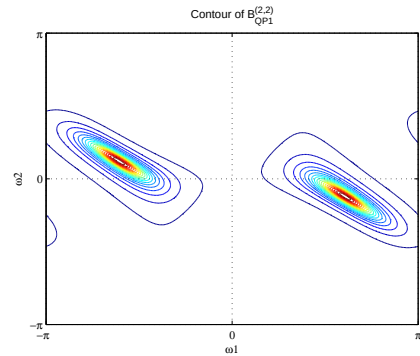


(f)

Figure 5.60: (a,c,e) Amplitude; (b,d,f) Contour plots of the spectrum of $B_1^{(2,2)}, \dots, B_2^{(2,2)}$, respectively, in Example 1, by Choice II.



(a)



(b)

Figure 5.61: (a) Amplitude; (b) Contour plots of the spectrum of $B_{QP1}^{(2,2)}$ in Example 1, by Choice II.

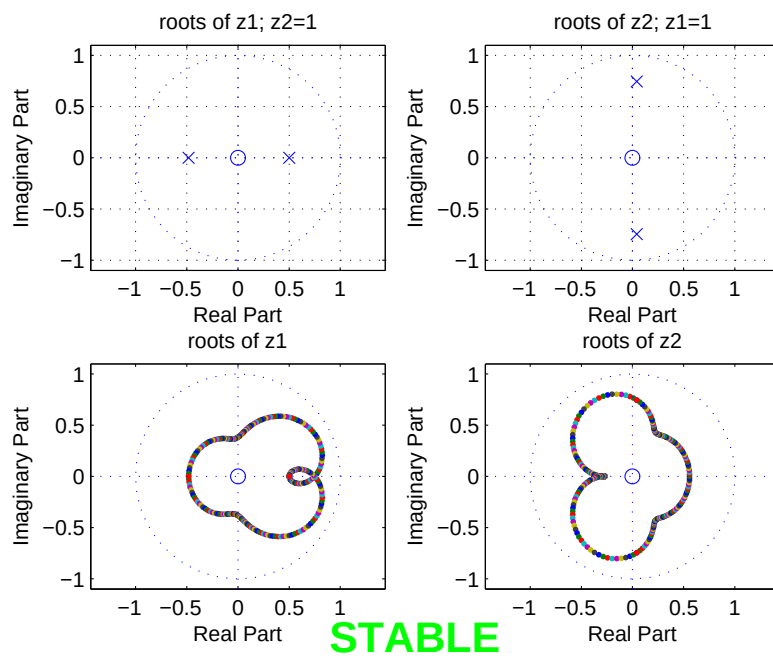


Figure 5.62: The stability test of the first-quadrant QP, $B_{QP1}^{(2,2)}$ in Example 1, by Choice II.

Table 5.19: Lattice Reflection Coefficients and the Coefficient Matrices for the PSR $S_{k,l}^{(2,2)}$ in Example 1, by Choice II.

Reflection Coefficients		Coefficient Matrices	
$\mathbf{k}_e^{(2,3)} =$	$\begin{bmatrix} 0.0383 & 0.1381 \\ -0.2016 & -0.4065 \\ 0.2562 & -0.0696 \end{bmatrix}$	$A_1^{(2,3)} =$	$\begin{bmatrix} 1 & 0.0073 & -0.0085 & -0.0090 \\ 0 & 0.2422 & 0.2528 & -0.1094 \\ 0 & -0.2562 & 0.2016 & -0.0383 \\ 0 & -0.0203 & -0.0445 & -0.0437 \end{bmatrix}$
			$A_2^{(2,3)} =$
			$A_3^{(2,3)} =$
$k_{e_{new}}^{(2,3)} =$	$\begin{bmatrix} 0.0281 \\ 0.0088 \\ 0.0096 \end{bmatrix}$	$A_1^{(2,3)} =$	$\begin{bmatrix} 1 & 0.0073 & -0.0085 & -0.0090 \\ 0 & 0.2422 & 0.2528 & -0.1094 \\ 0 & -0.2562 & 0.2016 & -0.0383 \\ 0 & -0.0203 & -0.0445 & -0.0437 \end{bmatrix}$
			$A_2^{(2,3)} =$
			$A_3^{(2,3)} =$
$\mathbf{k}_r^{(3,2)} =$	$\begin{bmatrix} 0.0009 & -0.0077 & -0.0126 \\ -0.0070 & -0.0149 & 0.0185 \\ -0.0113 & 0.0164 & 0.0016 \end{bmatrix}$	$B_1^{(3,2)} =$	$\begin{bmatrix} 1 & 0 & 0 \\ -0.4809 & 0.2232 & 0.2692 \\ 0.0178 & -0.3027 & 0.0292 \\ 0.0113 & 0.0070 & -0.0009 \\ 0 & 1 & 0 \\ 0.0117 & -0.5587 & 0.1940 \\ 0.0103 & 0.0429 & -0.3404 \\ -0.0164 & 0.0149 & 0.0077 \\ 0 & 0 & 1 \\ -0.0735 & 0.0980 & -0.5520 \\ 0.0217 & 0.0188 & 0.0382 \\ -0.0016 & -0.0185 & 0.0126 \end{bmatrix}$
			$B_2^{(3,2)} =$
			$B_3^{(3,2)} =$

The amplitudes and equal magnitude contours of $A_1^{(2,3)}, \dots, A_3^{(2,3)}$, and $B_1^{(3,2)}, \dots, B_3^{(3,2)}$ are demonstrated in Figure 5.63 and Figure 5.64, respectively.

The coefficient matrix of the first-quadrant QP model, $B_{QP1}^{(3,2)}$ is

$$B_{QP1}^{(3,2)} = \begin{bmatrix} 1 & 0.0164 & 0.0117 & 0.0116 \\ -0.4781 & 0.1981 & 0.2577 & -0.1140 \\ 0.0269 & -0.2875 & 0.0135 & 0.0260 \end{bmatrix} \quad (5.26)$$

and the amplitude and equal magnitude contours of $B_{QP1}^{(3,2)}$ are shown in Figure 5.65.

The results of the stability test of $B_{QP1}^{(3,2)}$ are shown in Figure 5.66.

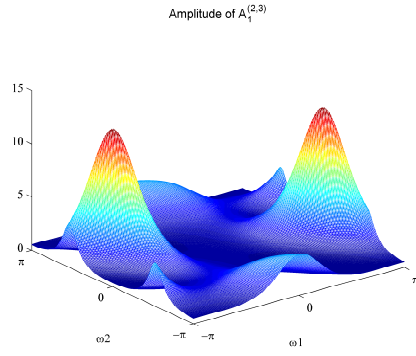
Again, to compare the result of the first-quadrant QP filter from the vertical PEFs given in the equation (5.7) calculated by using the first choice, here, we generate the same QP filter $A_{QP1}^{(2,3)}$. The coefficient matrix is

$$A_{QP1}^{(2,3)} = \begin{bmatrix} 1 & 0.0162 & 0.0125 & 0.0117 \\ -0.4781 & 0.1975 & 0.2570 & -0.1142 \\ 0.0269 & -0.2869 & 0.0138 & 0.0261 \end{bmatrix} \quad (5.27)$$

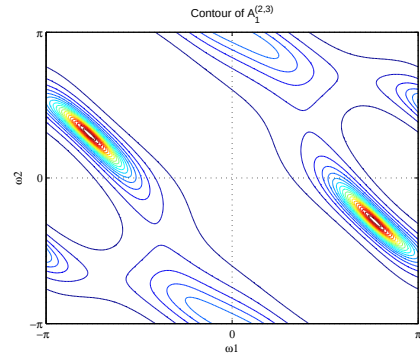
and the amplitude and equal magnitude contours of $A_{QP1}^{(2,3)}$ are shown in Figure 5.67.

The results of the stability test of $A_{QP1}^{(2,3)}$ are shown in Figure 5.68.

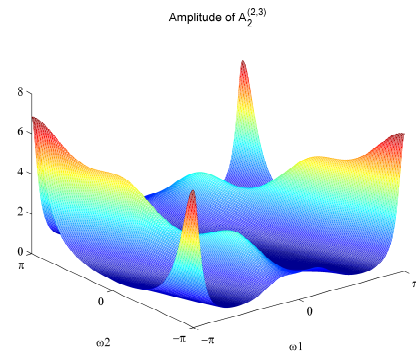
As seen that regardless of the augmentation directions, at the desired stage order the coefficient matrices of the generated filters with the same order are closely the same.



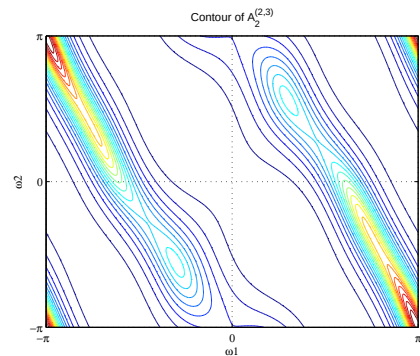
(a)



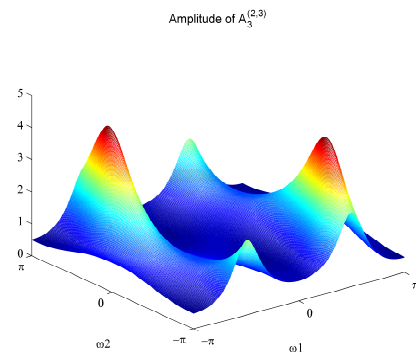
(b)



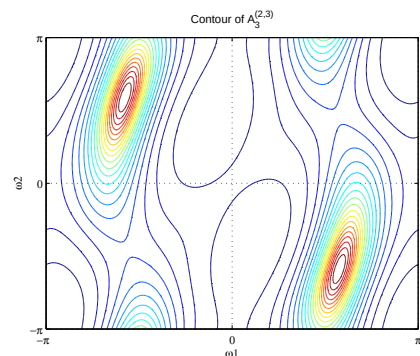
(c)



(d)

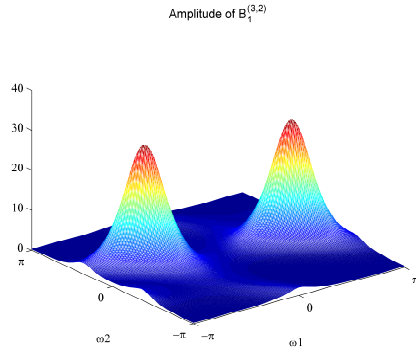


(e)

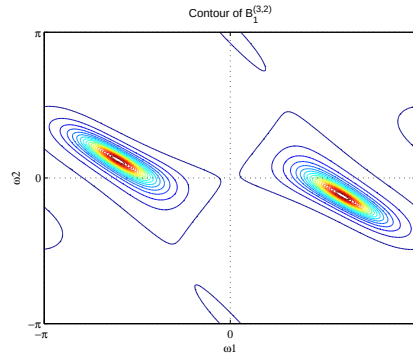


(f)

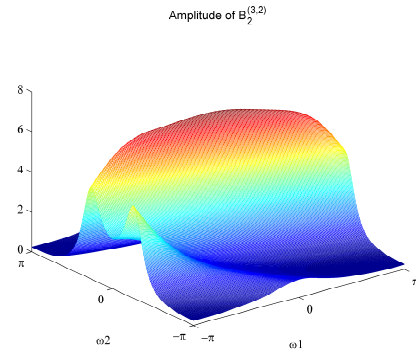
Figure 5.63: (a,c,e) Amplitude; (b,d,f) Contour plots of the spectrum of $A_1^{(2,3)}$, $A_2^{(2,3)}$, and $A_3^{(2,3)}$, respectively, in Example 1, by Choice II.



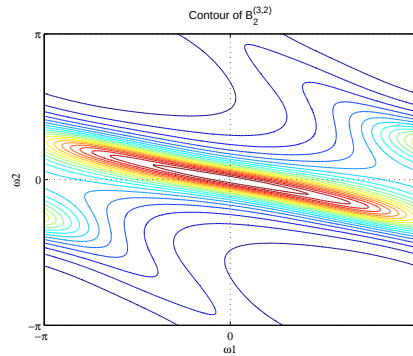
(a)



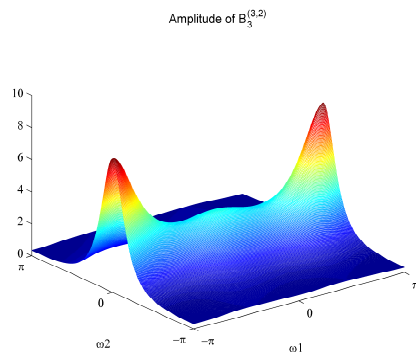
(b)



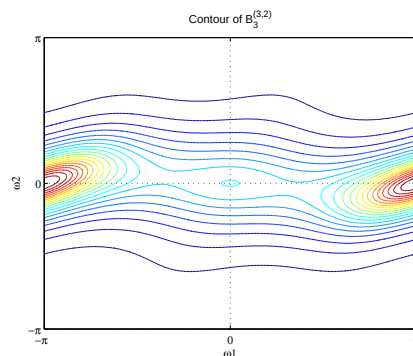
(c)



(d)



(e)



(f)

Figure 5.64: (a,c,e) Amplitude; (b,d,f) Contour plots of the spectrum of $B_1^{(3,2)}$, $B_2^{(3,2)}$, and $B_3^{(3,2)}$, respectively, in Example 1, by Choice II.

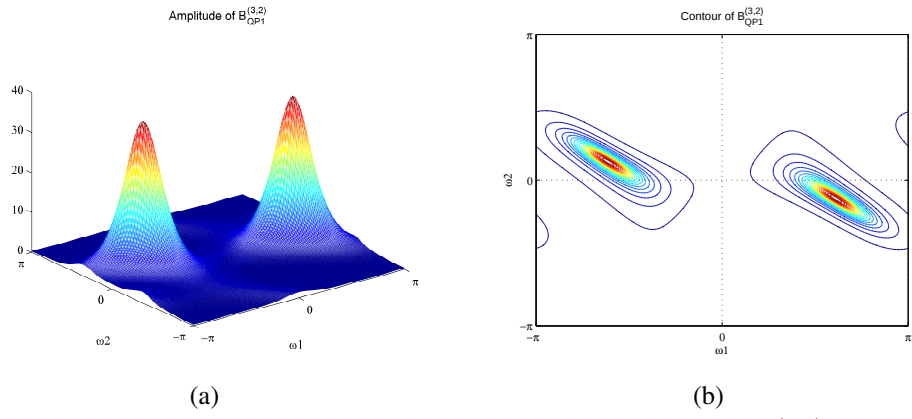


Figure 5.65: (a) Amplitude; (b) Contour plots of the spectrum of $B_{QP1}^{(3,2)}$ in Example 1, by Choice II.

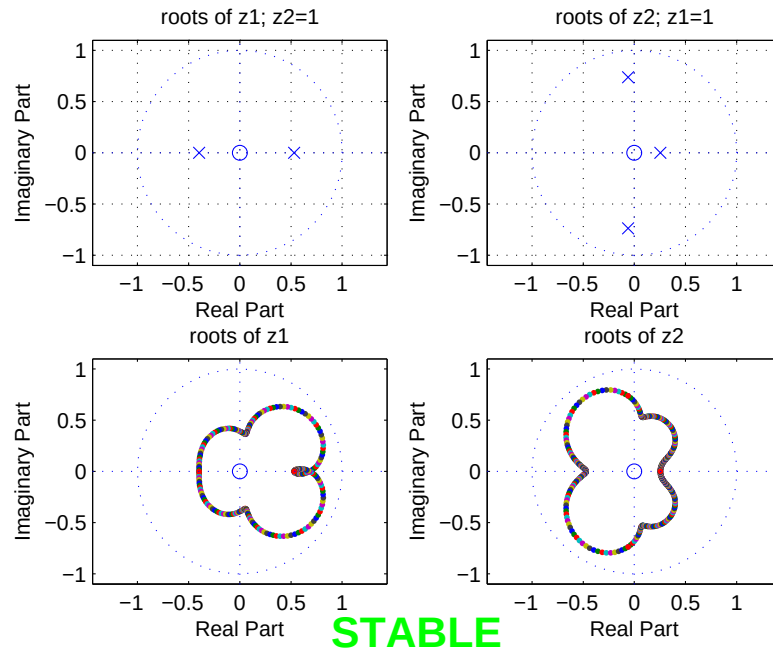


Figure 5.66: The stability test of the first-quadrant QP, $B_{QP1}^{(3,2)}$ in Example 1, by Choice II.

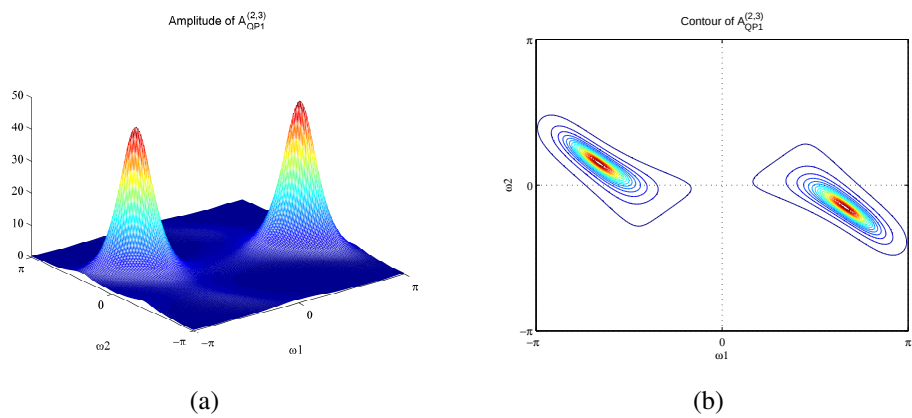


Figure 5.67: (a) Amplitude; (b) Contour plots of the spectrum of $A_{QP1}^{(2,3)}$ in Example 1, by Choice II.

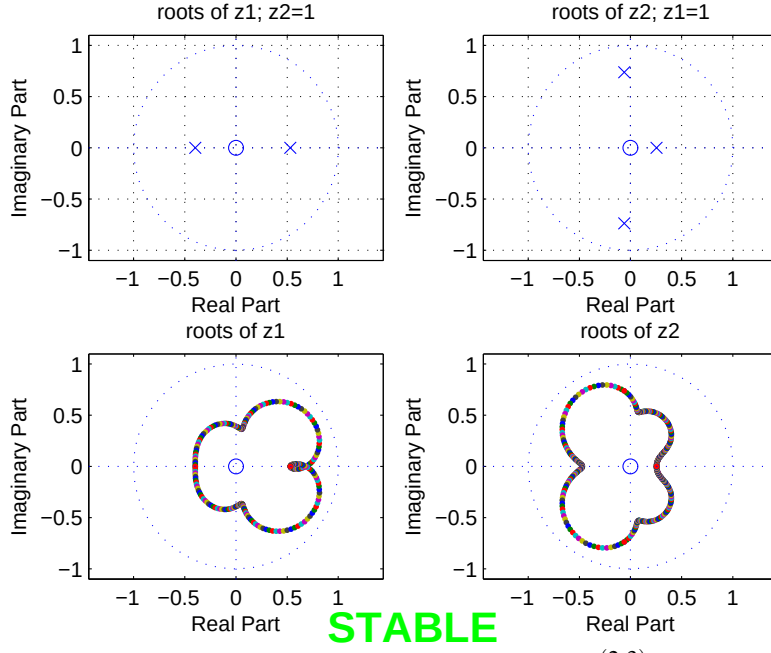


Figure 5.68: The stability test of the first-quadrant QP, $A_{QP1}^{(2,3)}$ in Example 1, by Choice II.

Table 5.20: Lattice Reflection Coefficients of the PSR $S_{k,l}^{(2,3)}$ in Example 1, by Choice II.

Reflection Coefficients			
$\mathbf{k}_e^{(4,2)} =$	$\begin{bmatrix} 0.0724 & 0.1268 & 0.0056 \\ 0.0896 & -0.0362 & -0.0112 \\ -0.0478 & 0.0003 & -0.0339 \end{bmatrix}$		
$\mathbf{k}_r^{(3,3)} =$	$\begin{bmatrix} -0.0062 & -0.0012 & -0.0074 \\ -0.0260 & -0.0311 & 0.3383 \\ 0.1147 & -0.2707 & -0.1949 \end{bmatrix}$		
$k_{r_{new}}^{(3,3)} =$	$\begin{bmatrix} 0.0488 \\ -0.0101 \\ 0.0052 \end{bmatrix}$		

5.1.2.5 Vertical augmentation applied: Obtained PSR $S_{k,l}^{(2,3)}$

Lattice reflection coefficients and the coefficient matrices are given in Table 5.20 and Table 5.21, respectively.

Table 5.21: Coefficient Matrices of the horizontal and vertical PEFs of the PSR $S_{k,l}^{(3,2)}$ in Example 1, by Choice II.

Coefficient Matrices of Vertical PEFs					of Horizontal PEFs				
$A_1^{(2,4)} =$	$\begin{bmatrix} 1 & 0.0196 & -0.0479 & 0.0080 & 0.0478 \\ 0 & 0.2473 & 0.2306 & -0.1296 & -0.0896 \\ 0 & -0.2521 & 0.2058 & -0.0384 & -0.0724 \end{bmatrix}$	$B_1^{(3,3)} =$	$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -0.4781 & 0.2072 & 0.2583 & -0.1147 & 0 \\ 0.0225 & -0.2890 & 0.0181 & 0.0260 & 0 \\ 0.0112 & 0.0068 & 0.0018 & 0.0062 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}$						
$A_2^{(2,4)} =$	$\begin{bmatrix} 0 & -0.0205 & -0.0553 & -0.0087 & -0.0003 \\ 1 & 0.1150 & -0.0473 & -0.0194 & 0.0362 \\ 0 & 0.0692 & 0.4060 & -0.1397 & -0.1268 \end{bmatrix}$	$B_2^{(3,3)} =$	$\begin{bmatrix} -0.0084 & -0.4800 & 0.2221 & 0.2707 & 0 \\ -0.0016 & 0.0132 & -0.3012 & 0.0311 & 0 \\ -0.0204 & 0.0111 & -0.0005 & 0.0012 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$						
$A_3^{(2,4)} =$	$\begin{bmatrix} 0 & -0.0275 & -0.0136 & -0.0106 \\ 0 & 0.1344 & -0.1073 & -0.0492 \\ 1 & 0.0965 & 0.2431 & -0.0575 \end{bmatrix}$	$B_3^{(3,3)} =$	$\begin{bmatrix} -0.0384 & 0.0094 & -0.5586 & 0.1949 & 0 \\ 0.0096 & 0.0043 & 0.0316 & -0.3383 & 0 \\ 0.0113 & -0.0053 & 0.0140 & 0.0074 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -0.0488 & -0.0739 & 0.0980 & -0.5519 & 0 \\ 0.0101 & 0.0102 & 0.0069 & 0.0438 & 0 \\ -0.0052 & 0.0111 & -0.0255 & 0.0134 & 0 \end{bmatrix}$						

The amplitudes and equal magnitude contours of $A_1^{(2,4)}, \dots, A_3^{(2,4)}$, and $B_1^{(3,3)}, \dots, B_4^{(3,3)}$ are demonstrated in Figure 5.69 and Figure 5.70, respectively.

The coefficient matrix of the first-quadrant QP model, $B_{QP1}^{(3,3)}$ is

$$B_{QP1}^{(3,3)} = \begin{bmatrix} 1 & 0.0150 & 0.0132 & 0.0144 \\ -0.4794 & 0.1990 & 0.2557 & -0.1160 \\ 0.0227 & -0.2886 & 0.0141 & 0.0226 \\ 0.0110 & 0.0070 & 0.0016 & 0.0065 \end{bmatrix} \quad (5.28)$$

and the amplitude and equal magnitude contours of $B_{QP1}^{(3,3)}$ are shown in Figure 5.71.

The results of the stability test of $B_{QP1}^{(3,3)}$ are shown in Figure 5.72.

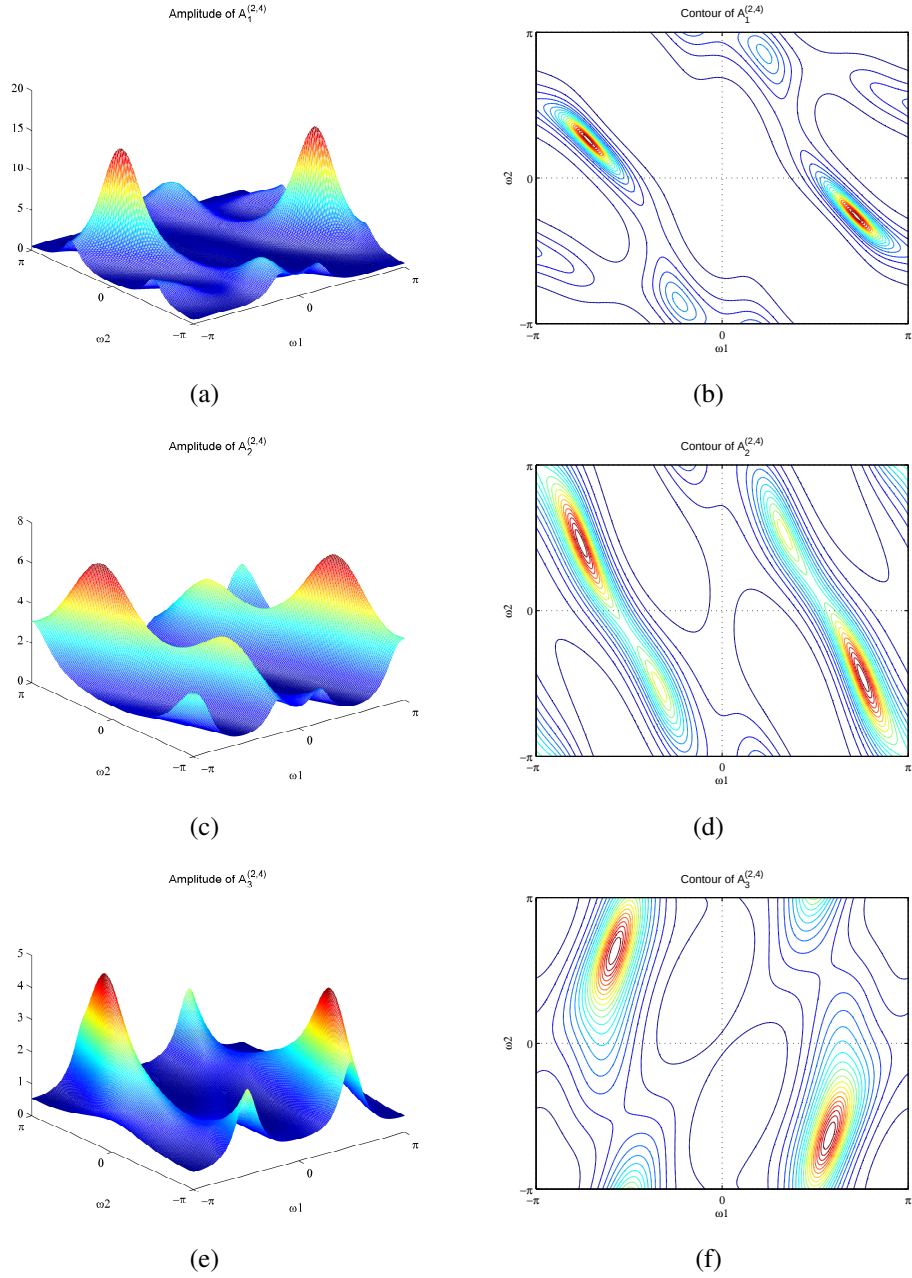


Figure 5.69: (a, c, e) Amplitude; (b, d, f) Contour plots of the spectrum of $A_1^{(2,4)}, \dots, A_3^{(2,4)}$, respectively, in Example 1, by Choice II.

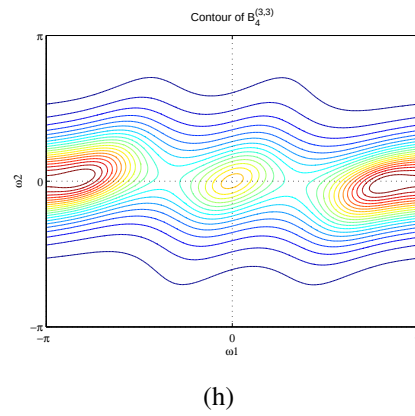
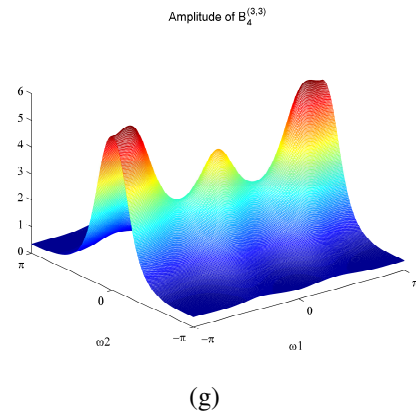
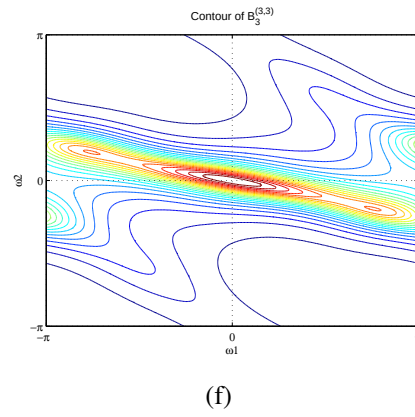
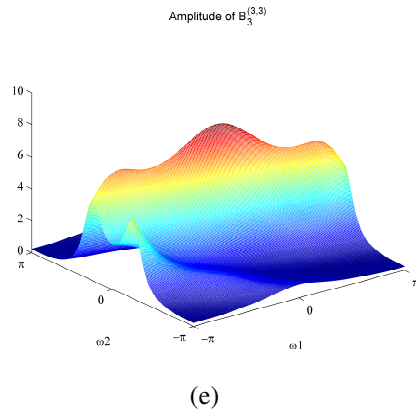
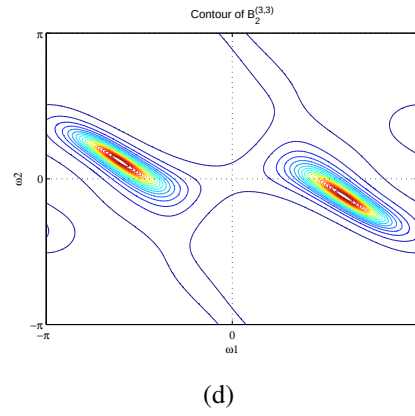
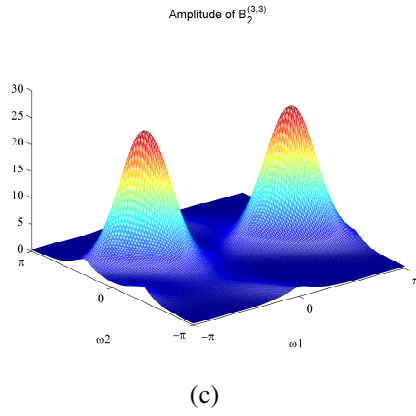
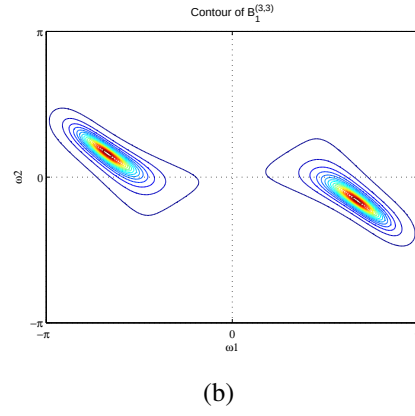
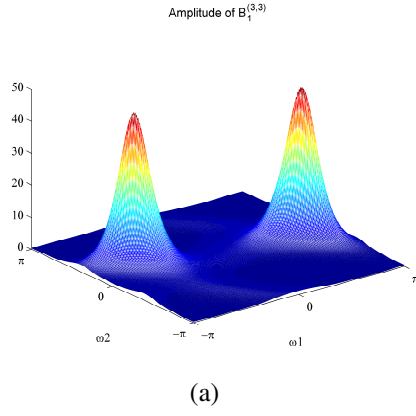


Figure 5.70: (a,c,e,g) Amplitude; (b,d,f,h) Contour plots of the spectrum of $B_1^{(4,2)}, \dots, B_4^{(3,3)}$, respectively, in Example 1, by Choice II.

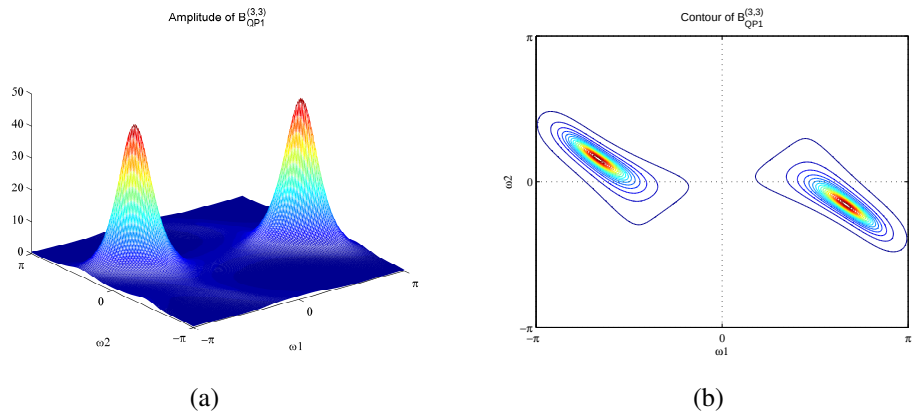


Figure 5.71: (a) Amplitude; (b) Contour plots of the spectrum of $B_{QP1}^{(3,3)}$ in Example 1, by Choice II.

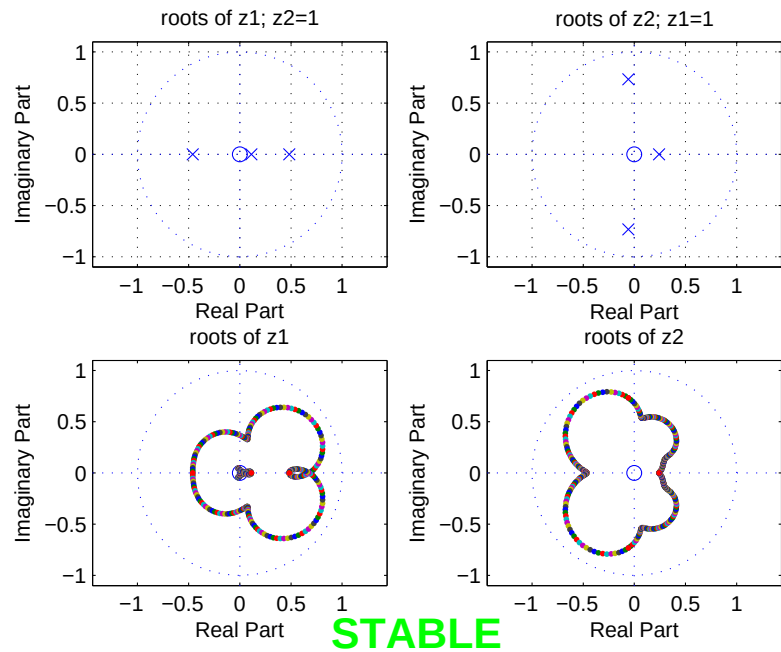


Figure 5.72: The stability test of the first-quadrant QP, $B_{QP1}^{(3,3)}$ in Example 1, by Choice II.

Table 5.22: Lattice Reflection Coefficients of the PSR $S_{k,l}^{(3,3)}$ in Example 1, by Choice II.

Reflection Coefficients				
$\mathbf{k}_e^{(3,4)} =$	$\begin{bmatrix} 0.0275 & 0.0733 & 0.1264 \\ 0.0098 & 0.0396 & 0.1409 \\ -0.0899 & -0.2051 & -0.4057 \\ 0.1134 & 0.2517 & -0.0675 \end{bmatrix}$			
$k_{e_{new}}^{(3,4)} =$	$\begin{bmatrix} 0.0363 \\ 0.0001 \\ 0.0223 \\ 0.0332 \end{bmatrix}$			
$\mathbf{k}_r^{(4,3)} =$	$\begin{bmatrix} 0.0063 & 0.0015 & -0.0199 & 0.0026 \\ 0.0006 & -0.0089 & -0.0012 & 0.0089 \\ -0.0167 & -0.0002 & 0.0004 & -0.0285 \\ 0.0016 & 0.0082 & -0.0266 & 0.0015 \end{bmatrix}$			

5.1.2.6 Horizontal augmentation applied: Obtained PSR $S_{k,l}^{(3,3)}$

Lattice reflection coefficients and the coefficient matrices are given in Table 5.22 and Table 5.23, respectively.

Table 5.23: Coefficient Matrices of the horizontal and vertical PEFs of the PSR $S_{k,l}^{(3,3)}$ in Example 1, by Choice II.

Coefficient Matrices of Vertical PEFs					of Horizontal PEFs									
$A_1^{(3,4)} =$	$\begin{bmatrix} 1 & 0.0186 & -0.0438 & 0.0059 & 0.0493 \\ 0 & 0.2109 & 0.2351 & -0.1226 & -0.0905 \\ 0 & -0.1715 & 0.1352 & -0.0301 & -0.0571 \\ 0 & -0.1134 & 0.0899 & -0.0098 & -0.0275 \end{bmatrix}$					$B_1^{(4,3)} =$	1	0	0	0	0	0	0	
$A_2^{(3,4)} =$	$\begin{bmatrix} 0 & -0.0228 & -0.0462 & -0.0135 & 0.0033 \\ 1 & 0.0314 & -0.0319 & 0.0004 & 0.0341 \\ 0 & 0.2457 & 0.2391 & -0.1154 & -0.0869 \\ 0 & -0.2517 & 0.2051 & -0.0396 & -0.0733 \end{bmatrix}$					$B_2^{(4,3)} =$	0	0.0085	-0.4797	0.2221	0.2705	0.0310	0.0022	
$A_3^{(3,4)} =$	$\begin{bmatrix} 0 & -0.0245 & -0.0098 & -0.0144 & -0.0283 \\ 0 & -0.0075 & -0.0538 & -0.0121 & 0.0131 \\ 1 & 0.1146 & -0.0409 & -0.0172 & 0.0372 \\ 0 & 0.0675 & 0.4057 & -0.1409 & -0.1264 \end{bmatrix}$					$B_3^{(4,3)} =$	0	0	0.0087	-0.5581	0.1950	-0.3376	-0.0034	
$A_4^{(3,4)} =$	$\begin{bmatrix} 0 & -0.0363 & -0.0001 & -0.0223 & -0.0332 \\ 0 & -0.0076 & -0.0189 & -0.0108 & -0.0158 \\ 0 & 0.1331 & -0.0972 & -0.0507 & 0.0163 \\ 1 & 0.0960 & 0.2426 & -0.0570 & -0.0064 \end{bmatrix}$					$B_4^{(4,3)} =$	0	0	0	0.0012	0.0199	0	1	
							-0.0486	-0.0734	0.0977	-0.5514	0.0440	0.0137	-0.0026	
							0.00005	0.0137	0.0076	0.0440	0.0137	-0.0026	-0.0089	
							-0.0009	-0.0076	-0.0214	0.0137	-0.0026	-0.0089	-0.0026	
							-0.0015	0.0285	-0.0089	-0.0026	-0.0089	-0.0026	-0.0026	

The amplitudes and equal magnitude contours of $A_1^{(3,4)}, \dots, A_4^{(3,4)}$, and $B_1^{(4,3)}, \dots, B_4^{(4,3)}$ are shown in Figure 5.73 and Figure 5.74, respectively.

The coefficient matrix of the first-quadrant QP model, $B_{QP1}^{(4,3)}$ is

$$B_{QP1}^{(4,3)} = \begin{bmatrix} 1 & 0.0154 & 0.0123 & 0.0148 \\ -0.4793 & 0.1990 & 0.2562 & -0.1162 \\ 0.0167 & -0.2880 & 0.0158 & 0.0230 \\ 0.0156 & -0.0044 & 0.0008 & 0.0088 \\ -0.0015 & 0.0172 & -0.0005 & -0.0061 \end{bmatrix} \quad (5.29)$$

and the amplitude and equal magnitude contours of $B_{QP1}^{(4,3)}$ are shown in Figure 5.75.

The results of the stability test of $B_{QP1}^{(4,3)}$ are shown in Figure 5.76.

The coefficient matrix of the first-quadrant QP model, $A_{QP1}^{(3,4)}$ is

$$A_{QP1}^{(3,4)} = \begin{bmatrix} 1 & 0.0285 & -0.0221 & 0.0118 & 0.0469 \\ -0.4731 & 0.1958 & 0.2491 & -0.1227 & -0.1066 \\ 0.0170 & -0.2842 & 0.0202 & 0.0236 & -0.0151 \\ 0.0124 & 0.0081 & 0.0028 & 0.0058 & 0.0049 \end{bmatrix} \quad (5.30)$$

and the amplitude and equal magnitude contours of $A_{QP1}^{(3,4)}$ are shown in Figure 5.77.

The results of the stability test of $A_{QP1}^{(3,4)}$ are shown in Figure 5.78.

For now, we applied the vertical and horizontal augmentation consecutively, and reached to the PSR $S_{k,l}^{(3,3)}$. As the first choice, we show the results of the augmentations applied on the both direction. To obtain the (4,3)th order coefficient matrix from the vertical PEFs, a horizontal augmentation, or to obtain the (3,4)th order coefficient matrix from the horizontal PEFs, a vertical augmentation should be applied to this PSR. In the following subsection both cases will be elaborated.

5.1.2.7 Vertical augmentation applied: Obtained PSR $S_{k,l}^{3,4}$ from $S_{k,l}^{(3,3)}$

Lattice reflection coefficients and the coefficient matrices are given in Table 5.24 and Table 5.25-5.26, respectively.

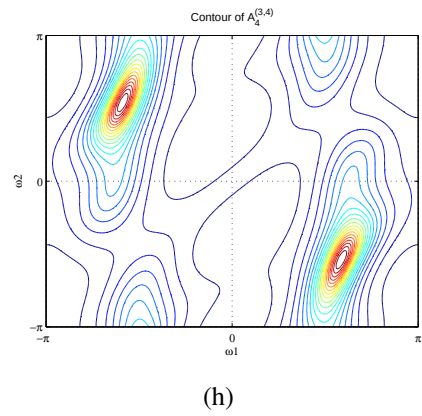
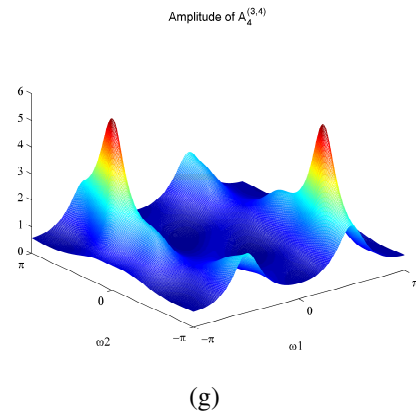
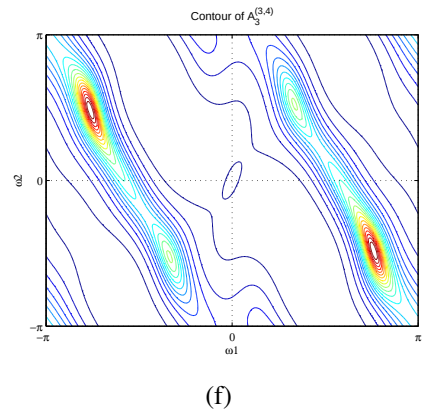
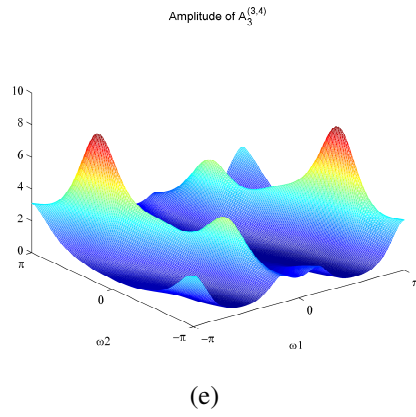
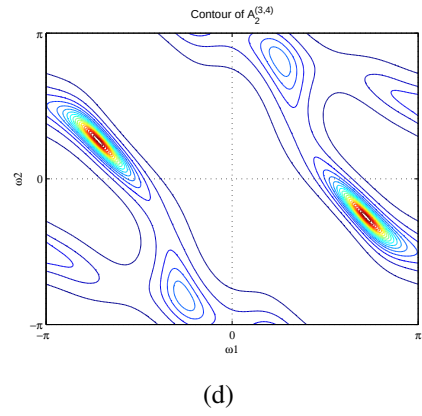
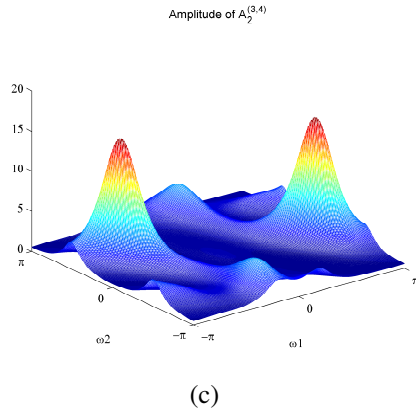
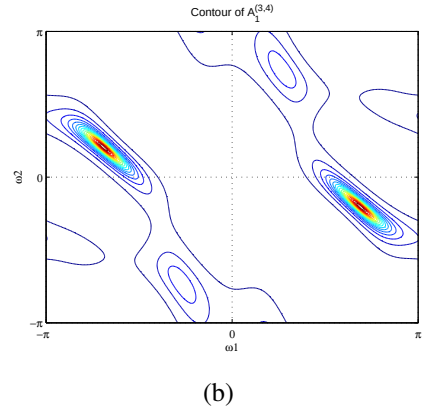
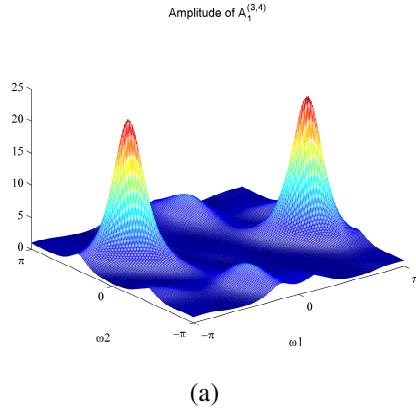


Figure 5.73: (a,c,e,g) Amplitude; (b,d,f,h) Contour plots of the spectrum of $A_1^{(3,4)}, \dots, A_4^{(3,4)}$, respectively, in Example 1, by Choice II.

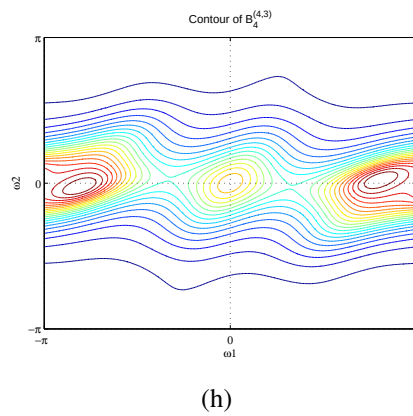
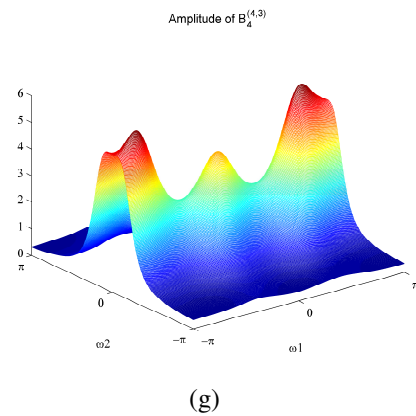
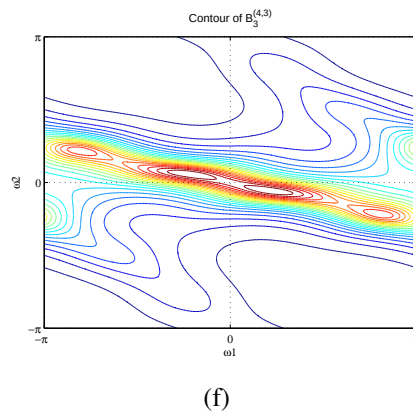
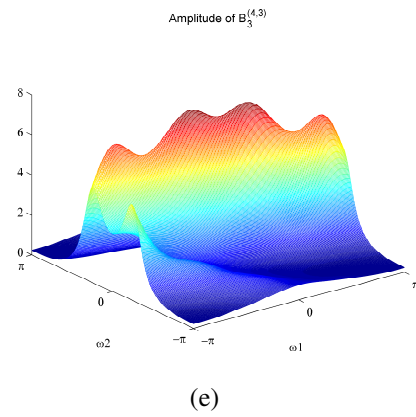
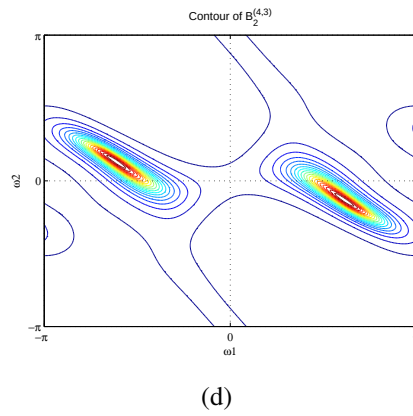
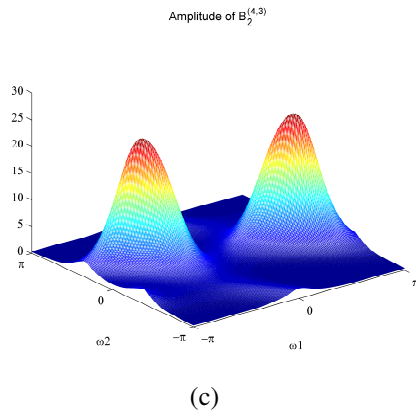
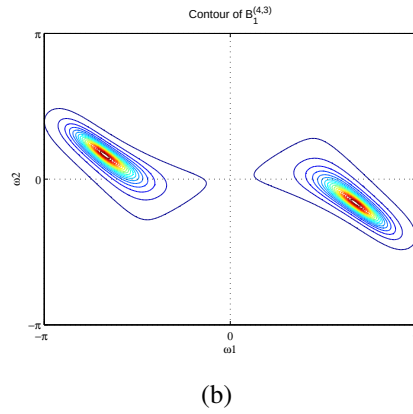
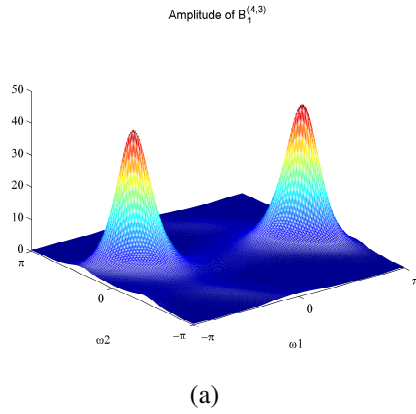


Figure 5.74: (a,c,e,g) Amplitude; (b,d,f,h) Contour plots of the spectrum of $B_1^{(4,3)}, \dots, B_4^{(4,3)}$, respectively, in Example 1, by Choice II.

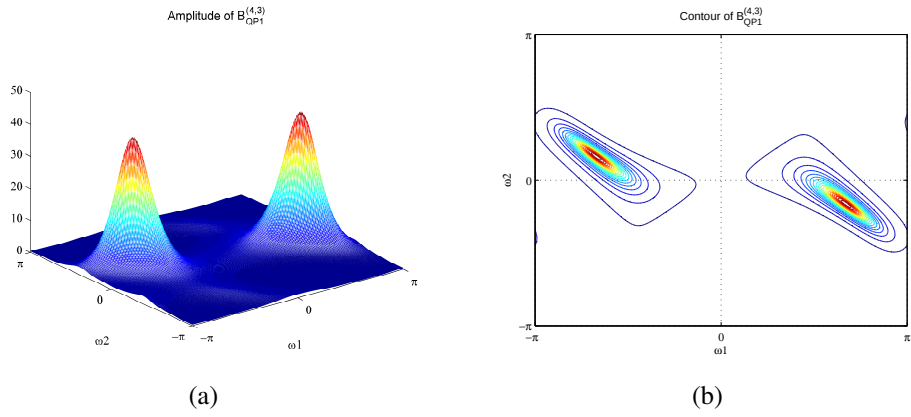


Figure 5.75: (a) Amplitude; (b) Contour plots of the spectrum of $B_{QP1}^{(4,3)}$ in Example 1, by Choice II.

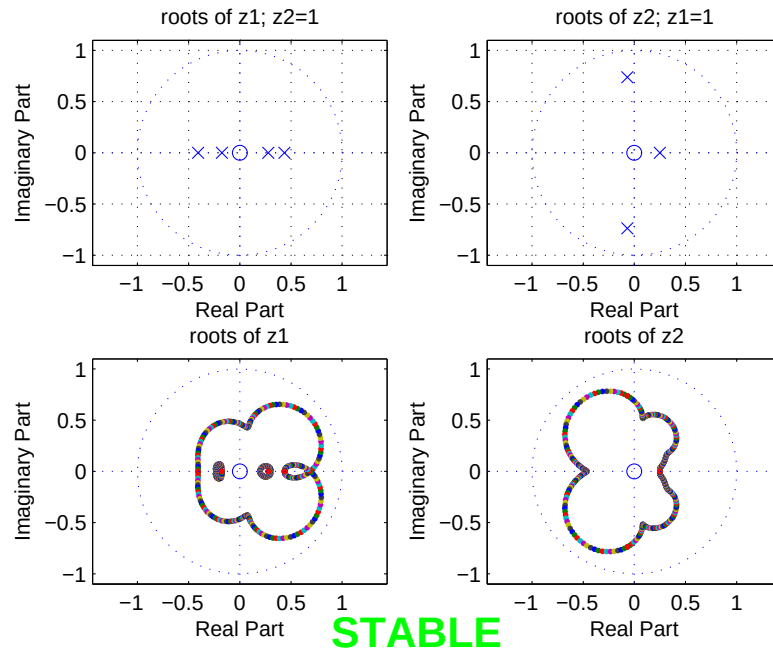


Figure 5.76: The stability test of the first-quadrant QP, $B_{QP1}^{(4,3)}$ in Example 1, by Choice II.

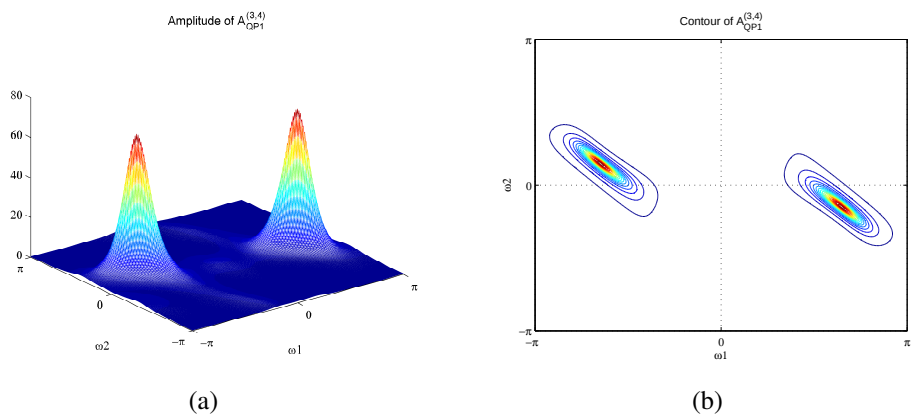


Figure 5.77: (a) Amplitude; (b) Contour plots of the spectrum of $A_{QP1}^{(3,4)}$ in Example 1, by Choice II.

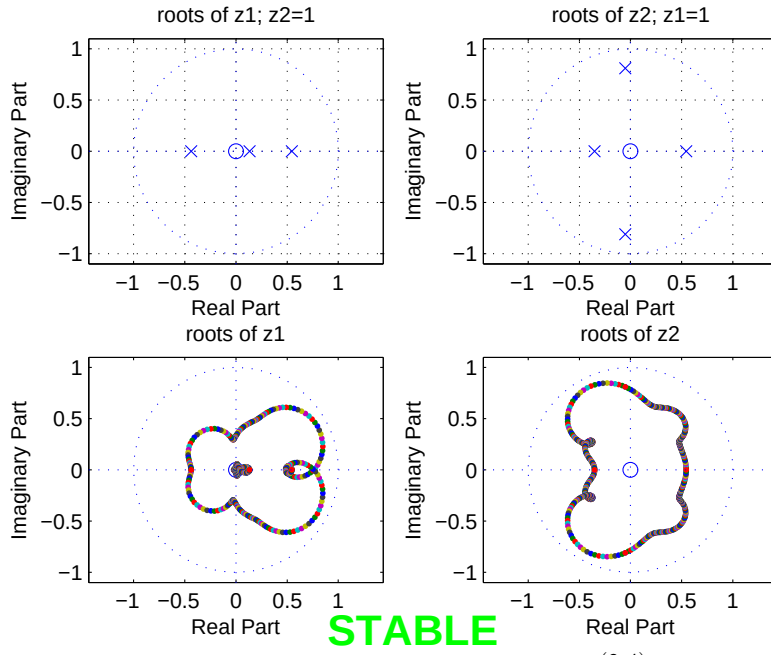


Figure 5.78: The stability test of the first-quadrant QP, $A_{QP1}^{(3,4)}$ in Example 1, by Choice II.

Table 5.24: Lattice Reflection Coefficients of the PSR $S_{k,l}^{(3,4)}$ in Example 1, by Choice II.

Reflection Coefficients				
$\mathbf{k}_e^{(3,5)} =$	0.0171	0.0128	-0.0190	0.0029
	0.0079	-0.0187	0.0066	-0.0037
	-0.0247	0.0081	0.0069	-0.0106
	0.0081	-0.0056	-0.0098	0.0073
$\mathbf{k}_r^{(4,4)} =$	0.0063	0.0068	0.0002	-0.0227
	-0.0072	-0.0083	-0.0018	0.0025
	0.0130	-0.0280	-0.0368	0.3363
	0.0750	0.1156	-0.2686	-0.1928
$k_{r_{new}}^{(4,4)} =$	-0.0244			
	0.0271			
	0.0169			
	-0.0138			

Table 5.25: Coefficient Matrices of the horizontal and vertical PEFs of the PSR $S_{k,l}^{(4,3)}$ in Example 1, by Choice II.

Coefficient Matrices of Vertical PEFs					of Horizontal PEFs									
$A_1^{(3,5)} =$					$B_1^{(4,4)} =$					$B_2^{(4,4)} =$				
1	0.0166	-0.0463	0.0109	0.0541	-0.0081	1	0	0	0	-0.4762	0.2133	0.2356	-0.1237	0
0	0.2133	0.2366	-0.1270	-0.0878	0.0247	0.0146	-0.2850	0.0287	0.0174	0.0146	-0.2850	0.0287	0.0174	-0.0130
0	-0.1698	0.1371	-0.0351	-0.0611	-0.0079	0.0178	-0.0028	0.0015	0.0082	0.0178	-0.0028	0.0015	0.0082	0.0072
0	-0.1147	0.0898	-0.0090	-0.0280	-0.0171	0.0009	0.0185	-0.0005	-0.0035	0.0009	0.0185	-0.0005	-0.0035	-0.0063
$A_2^{(3,5)} =$					$B_2^{(4,4)} =$					$B_3^{(4,4)} =$				
0	-0.0229	-0.0459	-0.0128	0.00004	0.0056	0	1	0	0	-0.0117	-0.4774	0.2065	0.2600	-0.1156
1	0.0303	-0.0339	0.0021	0.0407	-0.0081	-0.0030	0.0150	-0.2895	0.0220	-0.0030	0.0150	-0.2895	0.0220	0.0280
0	0.2473	0.2407	-0.1187	-0.0890	0.0187	-0.0101	0.0135	-0.0057	0.0017	-0.0101	0.0135	-0.0057	0.0017	0.0083
0	-0.2522	0.2047	-0.0398	-0.0740	-0.0128	-0.0055	0.0022	0.0085	0.0017	-0.0055	0.0022	0.0085	0.0017	-0.0068
$A_3^{(3,5)} =$					$B_3^{(4,4)} =$					$B_4^{(4,4)} =$				
0	-0.0238	-0.0093	-0.0145	-0.0283	0.0098	0	0	1	0	-0.0444	-0.0118	-0.4777	0.2228	0.2686
0	-0.0081	-0.0540	-0.0117	0.0088	-0.0069	0.0170	-0.0100	-0.0011	-0.2971	0.0170	-0.0100	-0.0011	-0.2971	0.0368
1	0.1125	-0.0432	-0.0123	0.0409	-0.0066	-0.0091	-0.0003	0.0085	-0.0057	-0.0091	-0.0003	0.0085	-0.0057	0.0018
0	0.0683	0.4058	-0.1414	-0.1261	0.0190	0.0166	-0.0069	0.0007	0.0092	0.0166	-0.0069	0.0007	0.0092	-0.0001
$A_4^{(3,5)} =$					$B_4^{(4,4)} =$					$B_4^{(4,4)} =$				
0	-0.0378	-0.0013	-0.0192	-0.0337	-0.0073	0	0	0	1	0	0	0	1	0
0	-0.0075	-0.0190	-0.0101	-0.0142	0.0106	-0.0078	-0.0372	0.0096	-0.5576	-0.0078	-0.0372	0.0096	-0.5576	0.1928
0	0.1337	-0.0969	-0.0520	0.0158	0.0037	-0.0104	0.0093	0.0051	0.0266	-0.0104	0.0093	0.0051	0.0266	-0.3363
1	0.0958	0.2425	-0.0571	-0.0066	-0.0029	-0.0105	-0.0084	-0.0015	0.0194	-0.0105	-0.0084	-0.0015	0.0194	-0.0025
						0.0027	0.0292	-0.0065	-0.0008	0.0027	0.0292	-0.0065	-0.0008	0.0227

Table 5.26: Coefficient Matrices of the last horizontal PEFs of the PSR $S_{k,l}^{(3,4)}$ in Example 1, by Choice II.

Coefficient Matrix					
$B_5^{(4,4)} =$	0	0	0	0	1
	0.0244	-0.0477	-0.0731	0.0982	-0.5503
	-0.0271	0.0044	0.0210	0.0047	0.0405
	-0.0169	-0.0119	-0.0115	-0.0194	0.0142
	0.0138	0.0027	0.0217	-0.0065	0.0007

The amplitudes and equal magnitude contours of $A_1^{(3,5)}, \dots, A_4^{(3,5)}$, and $B_1^{(4,4)}, \dots, B_5^{(4,4)}$ are demonstrated in Figure 5.79 and Figure 5.80-5.81, respectively.

The coefficient matrix of the first-quadrant QP model, $B_{QP1}^{(4,4)}$ is

$$B_{QP1}^{(4,4)} = \begin{bmatrix} 1 & 0.0284 & -0.0225 & 0.0147 & 0.0457 \\ -0.4746 & 0.1973 & 0.2490 & -0.1251 & -0.1066 \\ 0.0127 & -0.2841 & 0.0217 & 0.0253 & -0.0161 \\ 0.0167 & -0.0031 & 0.0005 & 0.0078 & 0.0080 \\ 0.0011 & 0.0192 & 0.0006 & -0.0040 & -0.0061 \end{bmatrix} \quad (5.31)$$

and the amplitude and equal magnitude contours of $B_{QP1}^{(4,4)}$ are shown in Figure 5.82.

The results of the stability test of $B_{QP1}^{(4,4)}$ are shown in Figure 5.83.

At that point, we reached to the desired order PSR $S_{k,l}^{(3,4)}$ and the first-quadrant QP filter from the horizontal PEFs, $B_{QP1}^{(4,4)}$ has been generated as shown in Figure 5.44. As seen in the Figure 5.45, this QP filter is stable.

According to the proposed method, it is also possible to obtain a first-quadrant QP filter from the vertical PEFs as $A_{QP1}^{(3,5)}$ whose coefficient matrix is

$$A_{QP1}^{(3,5)} = \begin{bmatrix} 1 & 0.0265 & -0.0247 & 0.0164 & 0.0532 & -0.0107 \\ -0.4734 & 0.1988 & 0.2515 & -0.1283 & -0.1071 & 0.0286 \\ 0.0157 & -0.2834 & 0.0212 & 0.0202 & -0.0181 & -0.0168 \\ 0.0132 & 0.0071 & 0.0024 & 0.0069 & 0.0050 & -0.0108 \end{bmatrix} \quad (5.32)$$

and the amplitude and equal magnitude contours of $A_{QP1}^{(3,5)}$ are illustrated in Figure 5.84.

As seen in the Figure 5.85, this QP filter, $A_{QP1}^{(3,5)}$ is also stable.

Now, one example of each filter models, type-1, type-2 ASHP, and NC half-plane, will be generated from the horizontal PEFs of this PSR $S_{k,l}^{(3,4)}$.

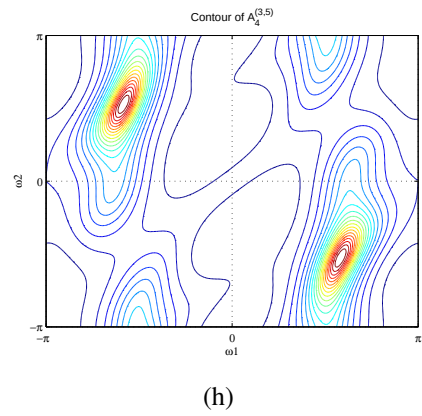
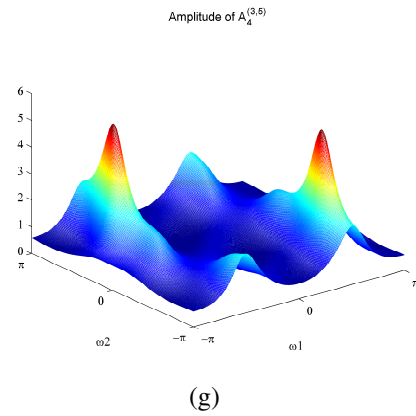
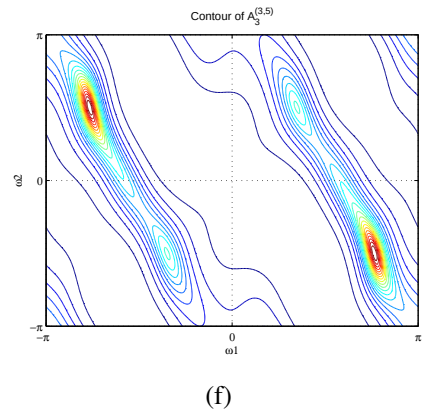
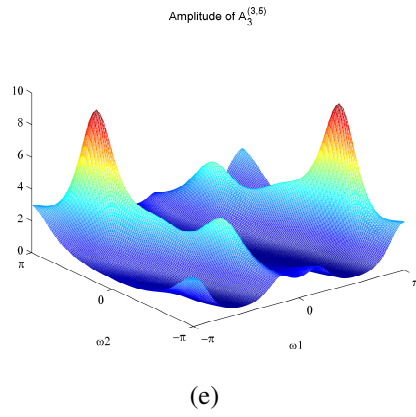
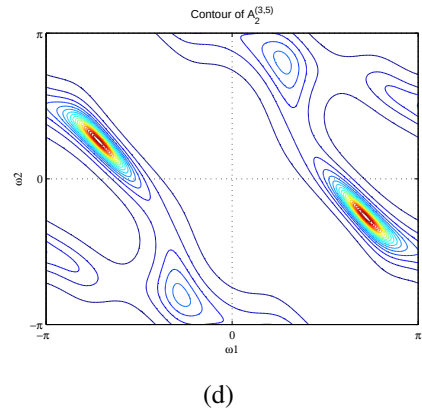
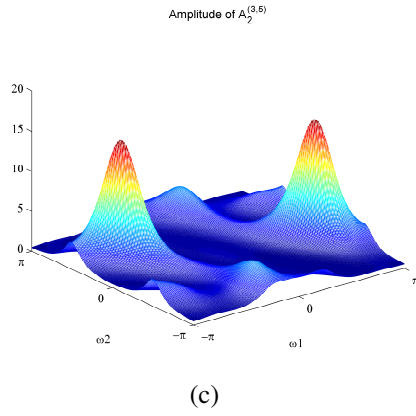
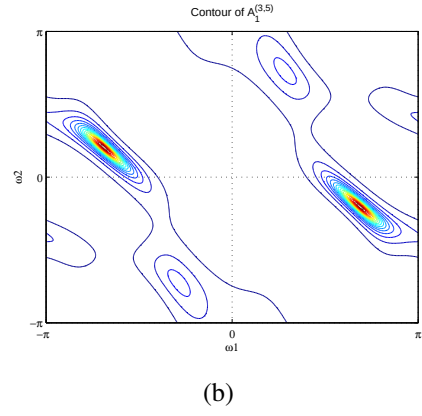
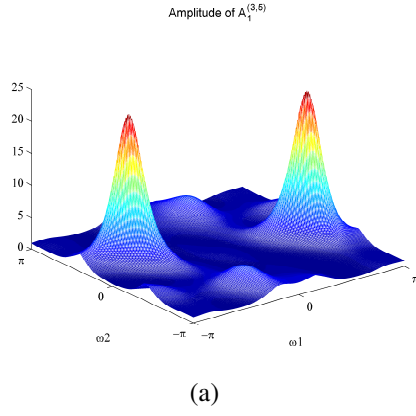


Figure 5.79: (a,c,e,g) Amplitude; (b,d,f,h) Contour plots of the spectrum of $A_1^{(3,5)}, \dots, A_4^{(3,5)}$, respectively, in Example 1, by Choice II.

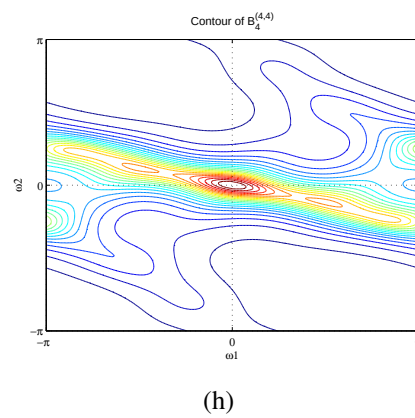
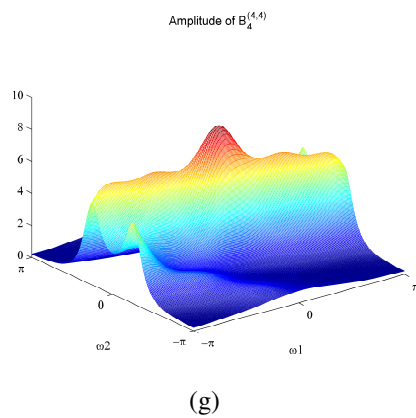
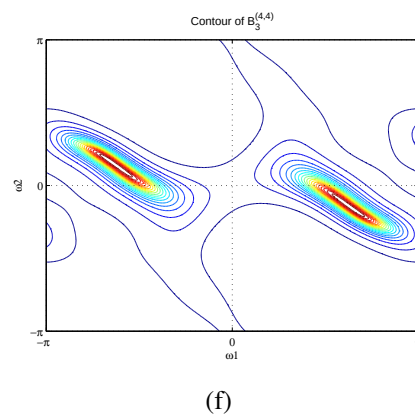
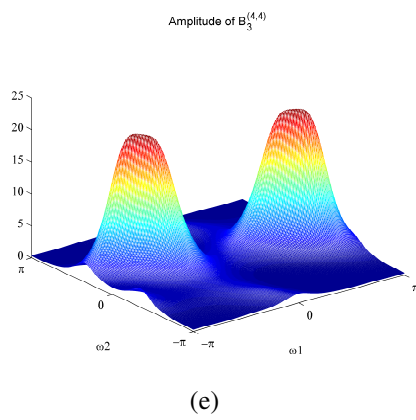
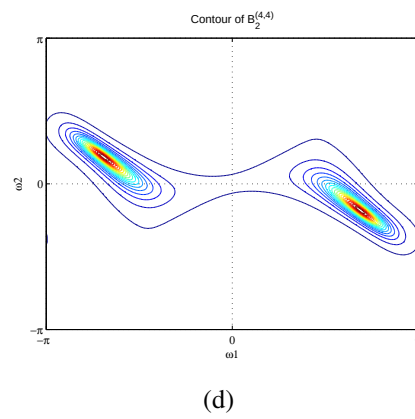
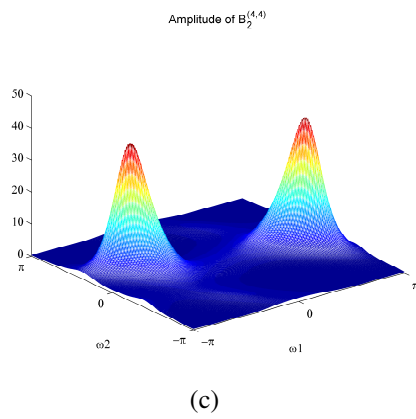
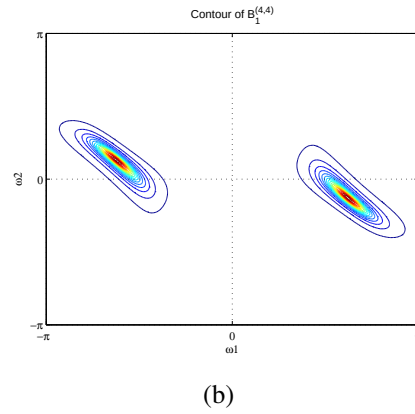
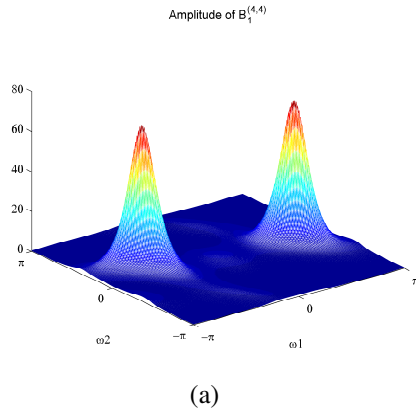


Figure 5.80: (a, c, e, g) Amplitude; (b, d, f, h) Contour plots of the spectrum of $B_1^{(5,3)}, \dots, B_4^{(5,3)}$, respectively, in Example 1, by Choice II.

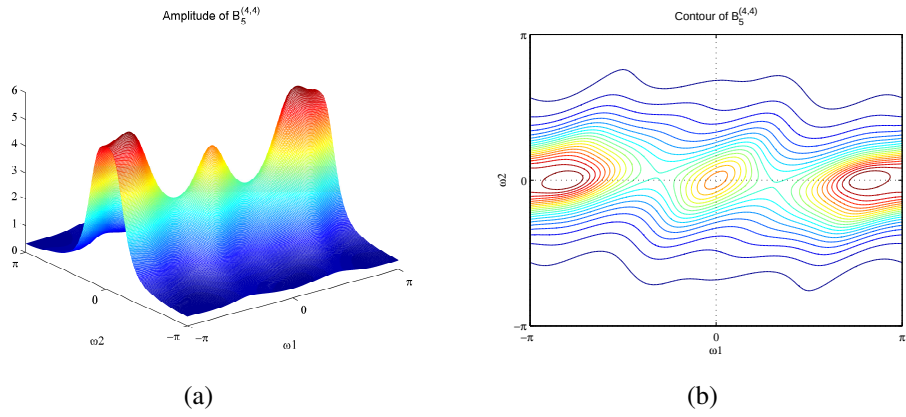


Figure 5.81: (a) Amplitude; (b) Contour plots of the spectrum of $B_5^{(4,4)}$, respectively, in Example 1, by Choice II.

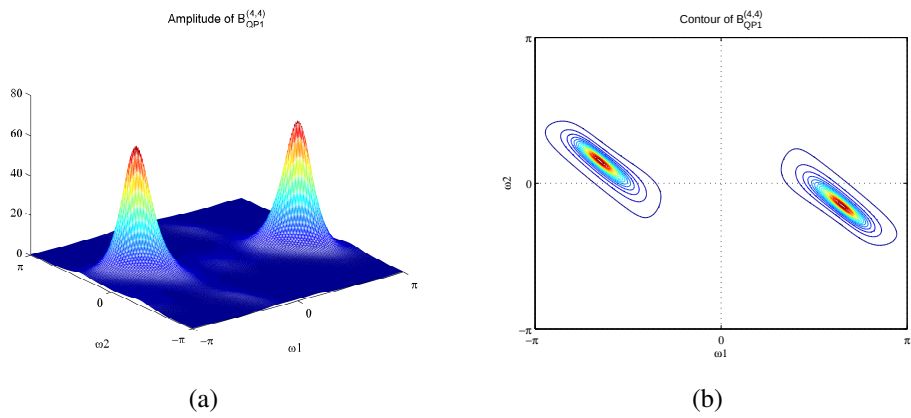


Figure 5.82: (a) Amplitude; (b) Contour plots of the spectrum of $B_{QP1}^{(4,4)}$ in Example 1, by Choice II.

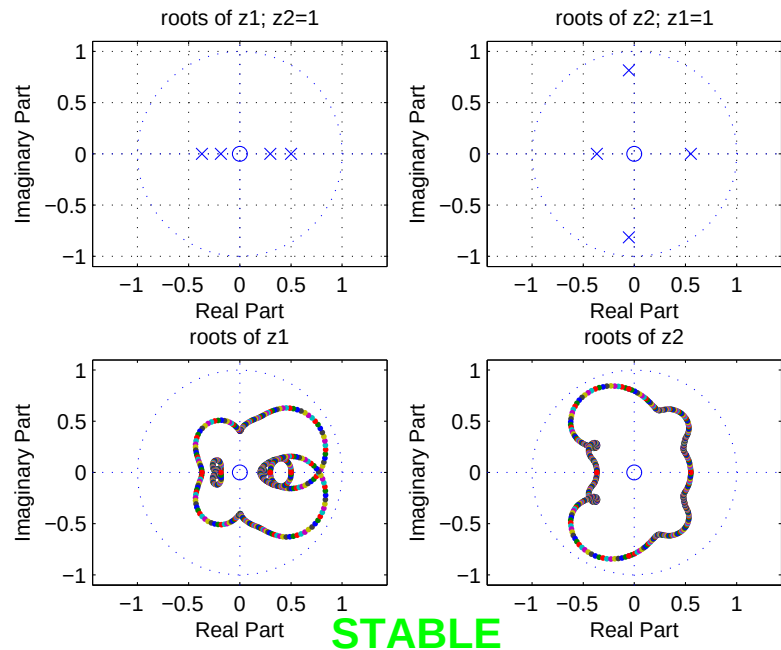


Figure 5.83: The stability test of the first-quadrant QP, $B_{QP1}^{(4,4)}$ in Example 1, by Choice II.

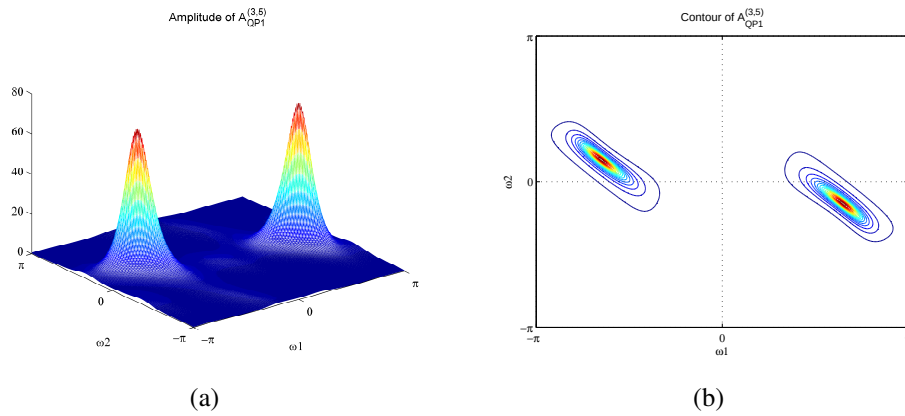


Figure 5.84: (a) Amplitude; (b) Contour plots of the spectrum of $A_{QP1}^{(3,5)}$ in Example 1, by Choice II.

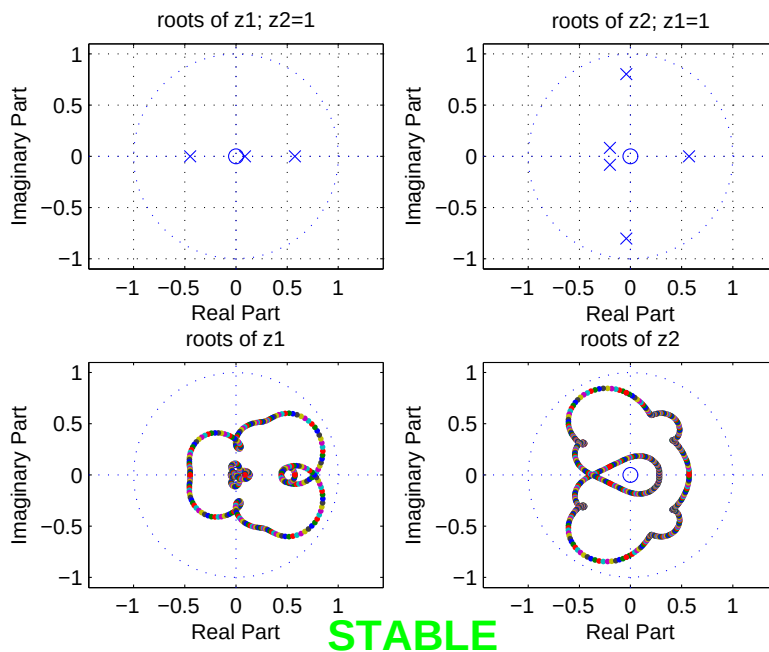


Figure 5.85: The stability test of the first-quadrant QP, $A_{QP1}^{(3,5)}$ in Example 1, by Choice II.

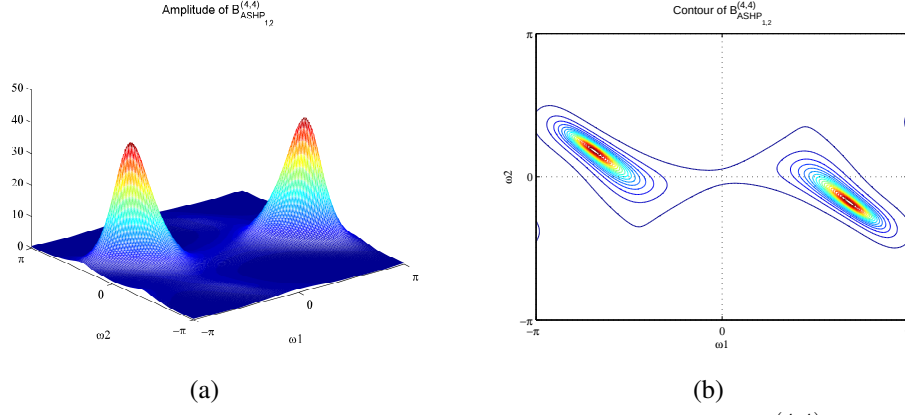


Figure 5.86: (a) Amplitude; (b) Contour plots of the spectrum of $B_{ASHP_{1,2}}^{(4,4)}$ in Example 1, by Choice II.

The coefficient matrix of the type-1 ASHP lattice filter from the second horizontal PEFs, $B_{ASHP_{1,2}}^{(4,4)}$ can be obtained as

$$B_{ASHP_{1,2}}^{(4,4)} = \begin{bmatrix} 0 & 1 & 0.0152 & 0.0132 & 0.0142 \\ -0.0122 & -0.4787 & 0.1983 & 0.2574 & -0.1168 \\ -0.0032 & 0.0151 & -0.2892 & 0.0179 & 0.0246 \\ -0.0106 & 0.0132 & -0.0057 & 0.0016 & 0.0085 \\ -0.0050 & 0.0025 & 0.0088 & 0.0017 & -0.0065 \end{bmatrix} \quad (5.33)$$

and the amplitude and equal magnitude contours of $B_{ASHP_{1,2}}^{(4,4)}$ are illustrated in Figure 5.86.

If we generate the type-2 ASHP from the same vertical PEFs, the coefficient matrix, $B_{ASHP_{2,2}}^{(4,4)}$ can be found as

$$B_{ASHP_{2,2}}^{(4,4)} = \begin{bmatrix} 0.0280 & 1 & 0 & 0 & 0 \\ -0.0251 & -0.4714 & 0.2131 & 0.2565 & -0.1177 \\ -0.0026 & 0.0071 & -0.2887 & 0.0225 & 0.0276 \\ -0.0096 & 0.0135 & -0.0056 & 0.0019 & 0.0085 \\ -0.0055 & 0.0027 & 0.0085 & 0.0016 & -0.0070 \end{bmatrix} \quad (5.34)$$

and the amplitude and equal magnitude contours of $B_{ASHP_{2,2}}^{(4,4)}$ are illustrated in Figure 5.87.

The type-1 ASHP filter $B_{ASHP_{1,2}}^{(4,4)}$ and the type-2 ASHP filter $B_{ASHP_{2,2}}^{(4,4)}$ were found unstable.

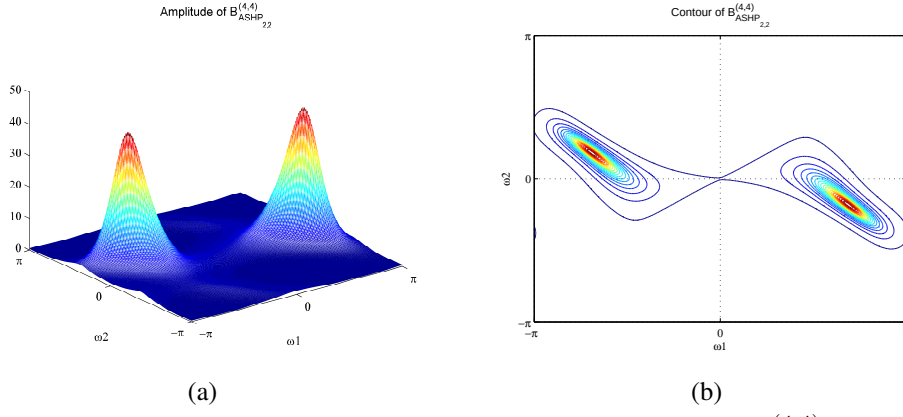


Figure 5.87: (a) Amplitude; (b) Contour plots of the spectrum of $B_{ASHP_{2,2}}^{(4,4)}$ in Example 1, by Choice II.

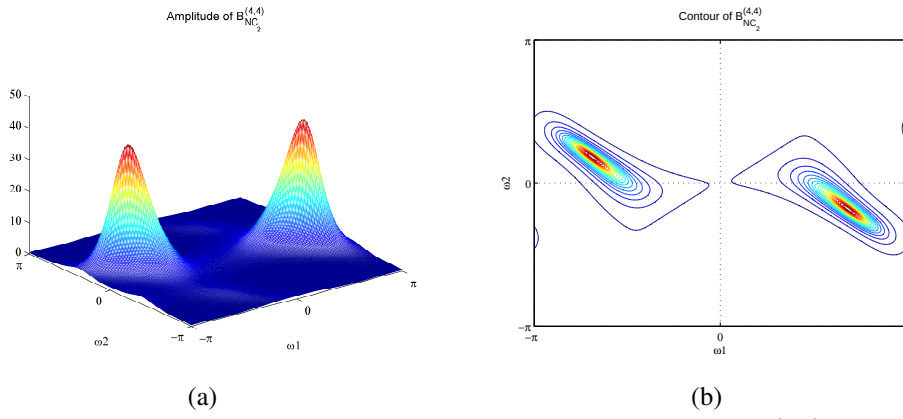


Figure 5.88: (a) Amplitude; (b) Contour plots of the spectrum of $B_{NC_2}^{(4,4)}$ in Example 1, by Choice II.

Similarly, an NC half-plane filter can be generated from the same horizontal PEF. The coefficient matrix, $B_{NC_2}^{(4,4)}$ can be calculated as

$$B_{NC_2}^{(4,4)} = \begin{bmatrix} 0.0287 & 1 & 0.0146 & 0.0136 & 0.0155 \\ -0.0258 & -0.4727 & 0.2053 & 0.2536 & -0.1197 \\ -0.0029 & 0.0069 & -0.2883 & 0.0186 & 0.0242 \\ -0.0102 & 0.0131 & -0.0057 & 0.0018 & 0.0088 \\ -0.0050 & 0.0031 & 0.0088 & 0.0016 & -0.0067 \end{bmatrix} \quad (5.35)$$

and the amplitude and equal magnitude contours of $B_{NC_2}^{(4,4)}$ are illustrated in Figure 5.88.

Table 5.27: Lattice Reflection Coefficients of the PSR $S_{k,l}^{(4,3)}$ in Example 1, by Choice II.

Reflection Coefficients				
$\mathbf{k}_e^{(4,4)} =$	$\begin{bmatrix} 0.0162 & 0.0276 & 0.0740 & 0.1282 \\ 0.0083 & 0.0095 & 0.0385 & 0.1410 \\ -0.0343 & -0.0899 & -0.2061 & -0.4053 \\ 0.0332 & 0.1140 & 0.2521 & -0.0670 \end{bmatrix}$			
$k_{e_{new}}^{(4,4)} =$	$\begin{bmatrix} 0.0210 \\ -0.0222 \\ 0.0058 \\ -0.0037 \end{bmatrix}$			
$\mathbf{k}_r^{(5,3)} =$	$\begin{bmatrix} 0.0024 & 0.0162 & -0.0227 & 0.0219 \\ 0.0154 & -0.0172 & 0.0153 & 0.0019 \\ -0.0183 & 0.0133 & -0.0049 & 0.0074 \\ 0.0166 & 0.0010 & 0.0058 & 0.0107 \end{bmatrix}$			

5.1.2.8 Horizontal augmentation applied: Obtained PSR $S_{k,l}^{(4,3)}$ from $S_{k,l}^{(3,3)}$

Lattice reflection coefficients and the coefficient matrices are given in Table 5.27 and Table 5.28-5.29, respectively.

Table 5.28: Coefficient Matrices of the horizontal and vertical PEFs of the PSR $S_{k,l}^{(4,3)}$ in Example 1, by Choice II.

Coefficient Matrices of Vertical PEFs					of Horizontal PEFs				
$A_1^{(4,4)} =$	$\begin{bmatrix} 1 & 0.0195 & -0.0435 & 0.0047 & 0.0504 \\ 0 & 0.2102 & 0.2364 & -0.1222 & -0.0908 \\ 0 & -0.1852 & 0.1380 & -0.0258 & -0.0570 \\ 0 & -0.0888 & 0.0615 & -0.0065 & -0.0194 \\ 0 & -0.0332 & 0.0343 & -0.0083 & -0.0162 \end{bmatrix}$	$B_1^{(5,3)} =$	$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -0.4776 & 0.2075 & 0.2576 & -0.1139 & 0 \\ 0.0162 & -0.2877 & 0.0200 & 0.0261 & 0 \\ 0.0085 & 0.0004 & 0.0011 & 0.0088 & 0 \\ 0.0072 & 0.0008 & 0.0077 & -0.0049 & 0 \\ -0.0166 & 0.0183 & -0.0154 & -0.0024 & 0 \end{bmatrix}$						
$A_2^{(4,4)} =$	$\begin{bmatrix} 0 & -0.0206 & -0.0451 & -0.0173 & 0.0060 \\ 1 & 0.0293 & -0.0281 & 0.0007 & 0.0334 \\ 0 & 0.2094 & 0.2429 & -0.1087 & -0.0864 \\ 0 & -0.1707 & 0.1345 & -0.0316 & -0.0579 \\ 0 & -0.1140 & 0.0899 & -0.0095 & -0.0276 \end{bmatrix}$	$B_2^{(5,3)} =$	$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ -0.0087 & -0.4796 & 0.2218 & 0.2700 & 0 \\ -0.0019 & 0.0101 & -0.3005 & 0.0306 & 0 \\ -0.0087 & 0.0059 & 0.0003 & 0.0017 & 0 \\ -0.0037 & 0.0071 & -0.0028 & 0.0066 & 0 \\ -0.0010 & -0.0133 & 0.0172 & -0.0162 & 0 \end{bmatrix}$						
$A_3^{(4,4)} =$	$\begin{bmatrix} 0 & -0.0196 & -0.0076 & -0.0229 & -0.0220 \\ 0 & -0.0124 & -0.0451 & -0.0096 & 0.0116 \\ 1 & 0.0309 & -0.0273 & 0.0012 & 0.0384 \\ 0 & 0.2448 & 0.2384 & -0.1175 & -0.0863 \\ 0 & -0.2521 & 0.2061 & -0.0385 & -0.0740 \end{bmatrix}$	$B_3^{(5,3)} =$	$\begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ -0.0379 & 0.0086 & -0.5579 & 0.1952 & 0 \\ 0.0115 & 0.0048 & 0.0259 & -0.3370 & 0 \\ -0.0072 & 0.0081 & 0.0089 & -0.0030 & 0 \\ 0.0240 & -0.0012 & 0.0137 & 0.0093 & 0 \\ -0.0058 & 0.0049 & -0.0153 & 0.0227 & 0 \end{bmatrix}$						
$A_4^{(4,4)} =$	$\begin{bmatrix} 0 & -0.0274 & -0.0014 & -0.0227 & -0.0211 \\ 0 & -0.0095 & -0.0135 & -0.0112 & -0.0183 \\ 0 & -0.0085 & -0.0463 & -0.0125 & 0.0191 \\ 1 & 0.1147 & -0.0415 & -0.0173 & 0.0374 \\ 0 & 0.0670 & 0.4053 & -0.1410 & -0.1282 \end{bmatrix}$	$B_4^{(5,3)} =$	$\begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ -0.0486 & -0.0733 & 0.0971 & -0.5516 & 0 \\ -0.0002 & 0.0138 & 0.0078 & 0.0438 & 0 \\ 0.0004 & -0.0078 & -0.0153 & 0.0132 & 0 \\ 0.0049 & 0.0255 & -0.0118 & 0.0087 & 0 \\ -0.0107 & -0.0074 & -0.0019 & -0.0219 & 0 \end{bmatrix}$						

Table 5.29: Coefficient Matrices of the last vertical PEFs of the PSR $S_{k,l}^{(4,3)}$ in Example 1, by Choice II.

Coefficient Matrix					
$A_5^{(4,4)} =$	0	-0.0210	0.0222	-0.0058	0.0037
	0	-0.0265	-0.0155	-0.0192	-0.0279
	0	-0.0081	-0.0126	-0.0181	-0.0136
	0	0.1325	-0.0969	-0.0510	0.0162
	1	0.0957	0.2423	-0.0568	-0.0065

The amplitudes and equal magnitude contours of $A_1^{(4,4)}, \dots, A_5^{(4,4)}$, and $B_1^{(5,3)}, \dots, B_4^{(5,3)}$ are demonstrated in Figure 5.89-5.90 and Figure 5.91, respectively.

The coefficient matrix of the first-quadrant QP model, $A_{QP1}^{(4,4)}$ is

$$A_{QP1}^{(4,4)} = \begin{bmatrix} 1 & 0.0285 & -0.0223 & 0.0123 & 0.0470 \\ -0.4732 & 0.1960 & 0.2490 & -0.1229 & -0.1068 \\ 0.0105 & -0.2841 & 0.0220 & 0.0254 & -0.0154 \\ 0.0163 & -0.0034 & -0.0003 & 0.0069 & 0.0077 \\ 0.0007 & 0.0193 & 0.0007 & -0.0065 & -0.0060 \end{bmatrix} \quad (5.36)$$

and the amplitude and equal magnitude contours of $A_{QP1}^{(4,4)}$ are shown in Figure 5.92.

The results of the stability test of $A_{QP1}^{(4,4)}$ are shown in Figure 5.93.

Now, we reached to the desired order PSR and we generated a first-quadrant QP filter from the vertical PEFs, $A_{QP1}^{(4,4)}$ as shown in Figure 5.34. As seen in the Figure 5.93, this QP filter is stable.

It is also possible to obtain a first-quadrant QP filter from the horizontal PEFs as $B_{QP1}^{(5,3)}$.

The coefficient matrix is

$$B_{QP1}^{(5,3)} = \begin{bmatrix} 1 & 0.0132 & 0.0121 & 0.0157 \\ -0.4789 & 0.2001 & 0.2553 & -0.1167 \\ 0.0163 & -0.2873 & 0.0165 & 0.0231 \\ 0.0083 & 0.0004 & 0.0013 & 0.0090 \\ 0.0075 & 0.0013 & 0.0076 & -0.0046 \\ -0.0169 & 0.0181 & -0.0154 & -0.0027 \end{bmatrix} \quad (5.37)$$

and the amplitude and equal magnitude contours of $B_{QP1}^{(5,3)}$ are illustrated in Figure 5.94.

As seen in the Figure 5.95, this QP filter, $B_{QP1}^{(5,3)}$ is also stable.

The type-1 or type-2 ASHP and NC half-plane lattice filters can also be generated from the horizontal and vertical PEFs of this PSR $S_{k,l}^{(4,3)}$. Here, one example of each filter models will be produced from the vertical PEFs.

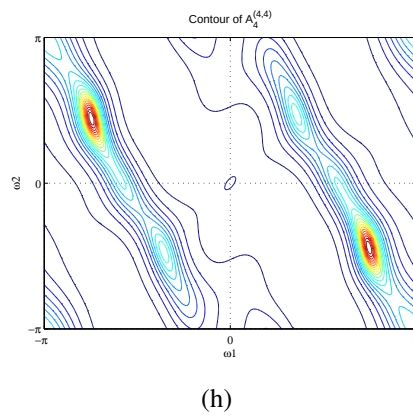
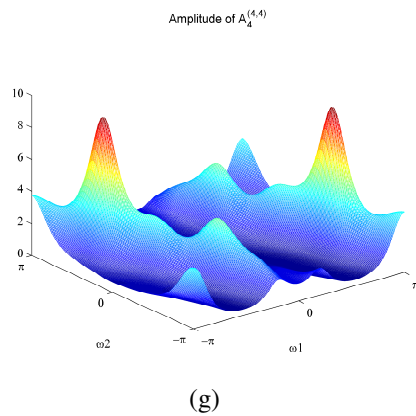
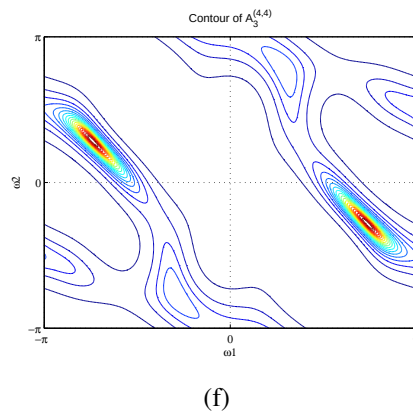
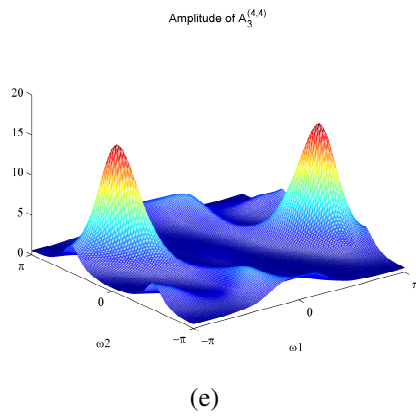
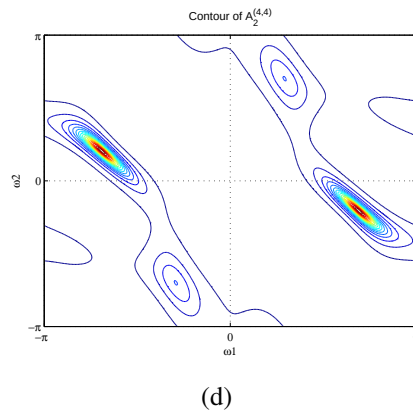
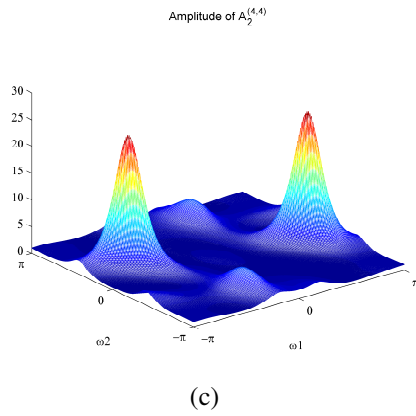
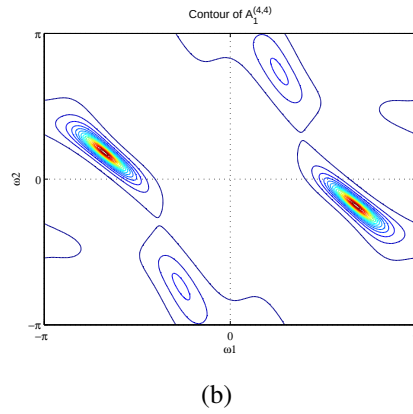
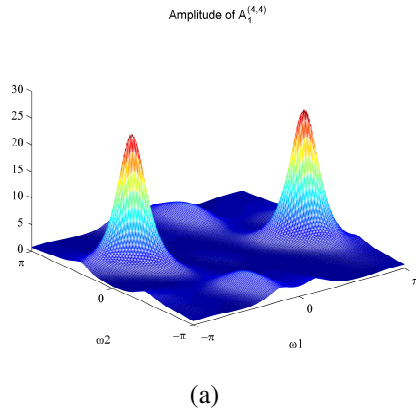


Figure 5.89: (a,c,e,g) Amplitude; (b,d,f,h) Contour plots of the spectrum of $A_1^{(4,4)}, \dots, A_4^{(4,4)}$, respectively, in Example 1, by Choice II.

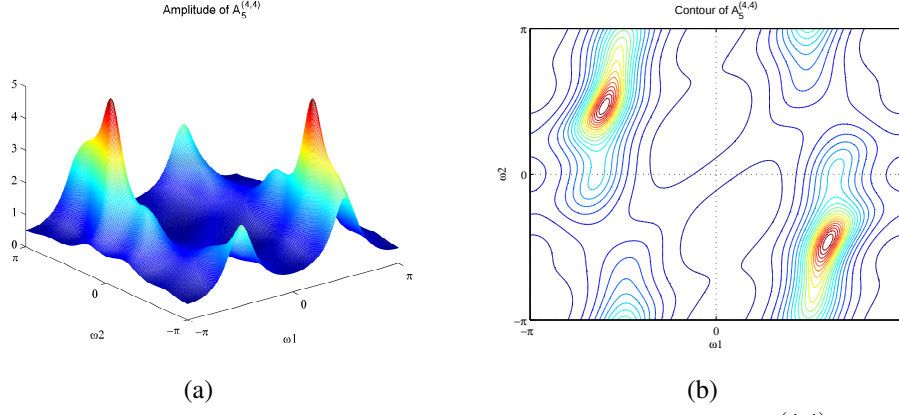


Figure 5.90: (a) Amplitude; (b) Contour plots of the spectrum of $A_5^{(4,4)}$, respectively, in Example 1, by Choice II.

The coefficient matrix of the type-1 ASHP lattice filter from the second vertical PEFs, $A_{ASHP_{1,2}}^{(4,4)}$ can be found as

$$A_{ASHP_{1,2}}^{(4,4)} = \begin{bmatrix} 0 & -0.0122 & -0.0413 & -0.0069 & 0.0161 \\ 1 & 0.0347 & -0.0071 & 0.0049 & 0.0272 \\ -0.4744 & 0.1945 & 0.2548 & -0.1097 & -0.1044 \\ 0.0191 & -0.2831 & 0.0196 & 0.0232 & -0.0161 \\ 0.0115 & 0.0080 & 0.0026 & 0.0054 & 0.0050 \end{bmatrix} \quad (5.38)$$

and the amplitude and equal magnitude contours of $A_{ASHP_{1,2}}^{(4,4)}$ are illustrated in Figure 5.96.

If we generate the type-2 ASHP from the same vertical PEFs, the coefficient matrix, $A_{ASHP_{2,2}}^{(4,4)}$ can be produced as

$$A_{ASHP_{2,2}}^{(4,4)} = \begin{bmatrix} -0.4648 & -0.0296 & -0.0249 & -0.0195 & -0.0174 \\ 1 & -0.0684 & -0.1380 & 0.0576 & 0.0756 \\ 0 & 0.2955 & 0.1787 & -0.0967 & -0.0599 \\ 0 & -0.1295 & 0.1059 & -0.0286 & -0.0489 \\ 0 & -0.0986 & 0.0739 & -0.0057 & -0.0201 \end{bmatrix} \quad (5.39)$$

and the amplitude and equal magnitude contours of $A_{ASHP_{2,2}}^{(4,4)}$ are illustrated in Figure 5.97.

The type-1 ASHP filter $A_{ASHP_{1,2}}^{(4,4)}$ and the type-2 ASHP filter $A_{ASHP_{2,2}}^{(4,4)}$ were found unstable.

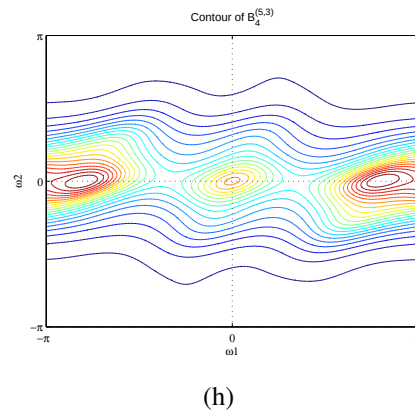
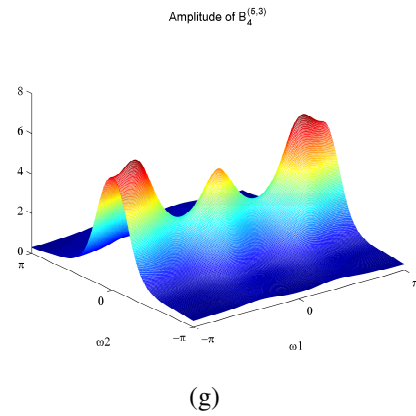
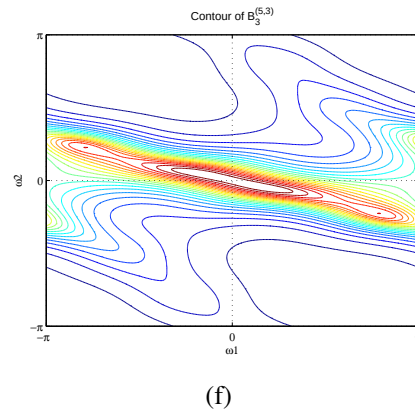
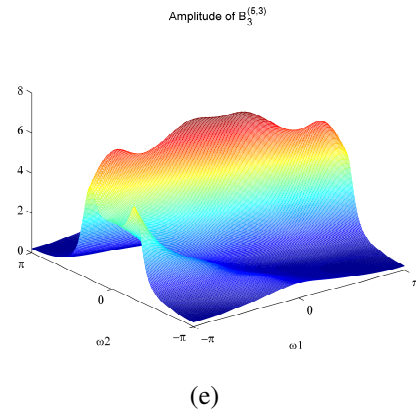
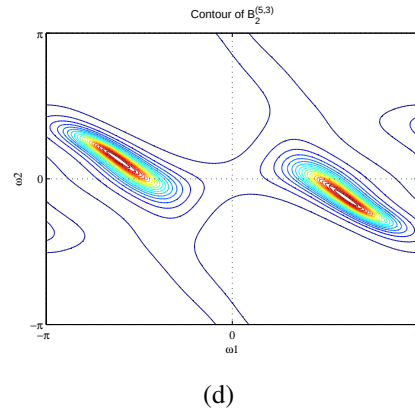
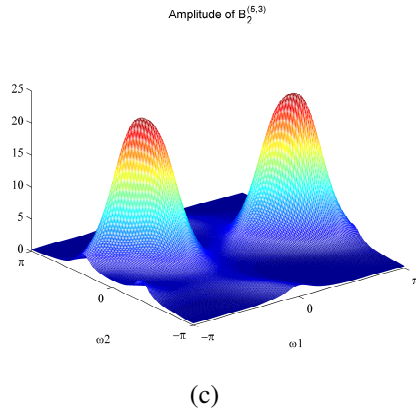
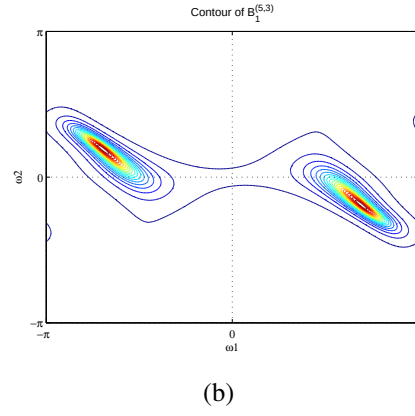
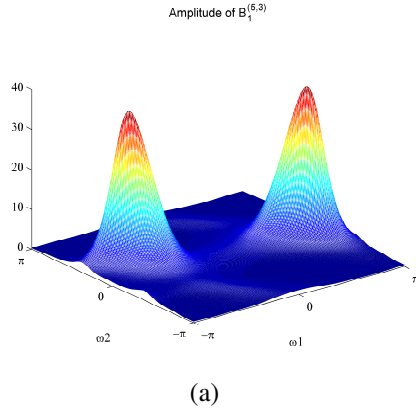


Figure 5.91: (a,c,e,g) Amplitude; (b,d,f,h) Contour plots of the spectrum of $B_1^{(5,3)}, \dots, B_4^{(5,3)}$, respectively, in Example 1, by Choice II.

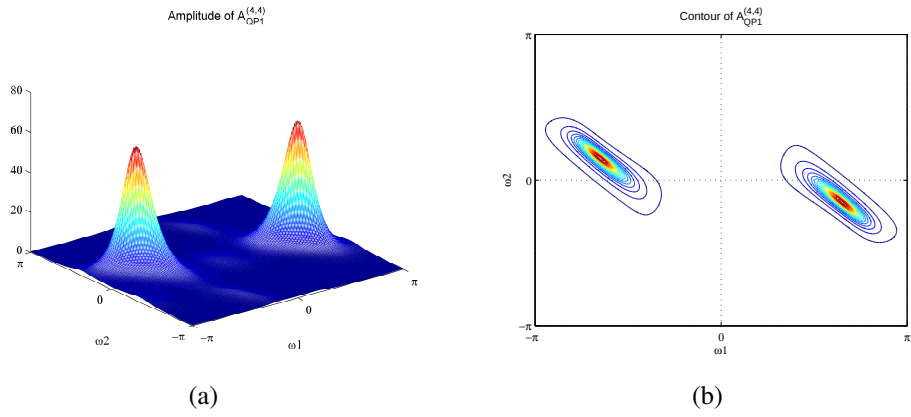


Figure 5.92: (a) Amplitude; (b) Contour plots of the spectrum of $A_{QP1}^{(4,4)}$ in Example 1, by Choice II.

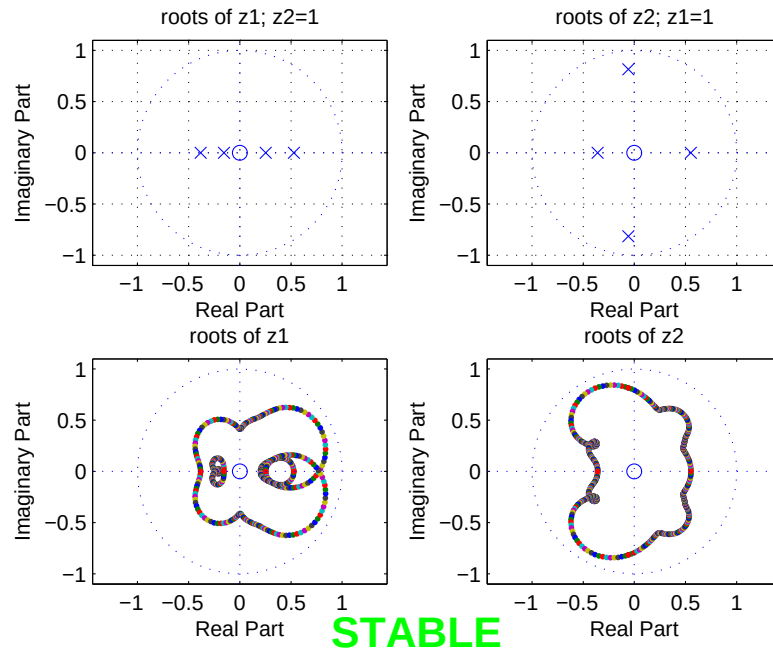


Figure 5.93: The stability test of the first-quadrant QP, $A_{QP1}^{(4,4)}$ in Example 1, by Choice II.

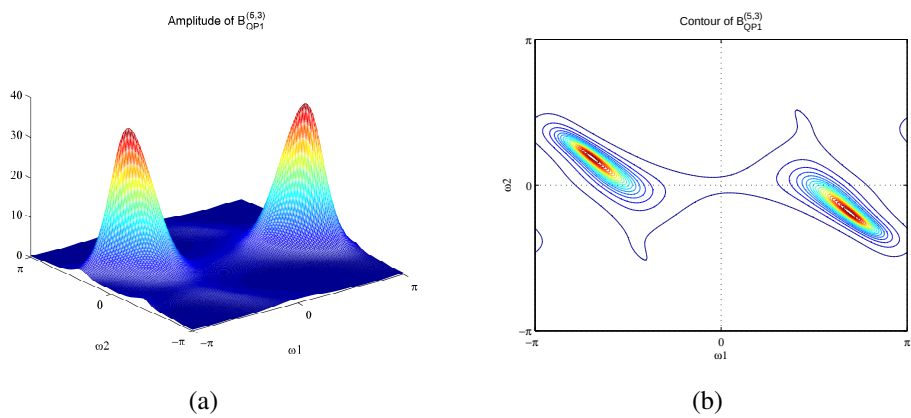


Figure 5.94: (a) Amplitude; (b) Contour plots of the spectrum of $B_{QP1}^{(5,3)}$ in Example 1, by Choice II.

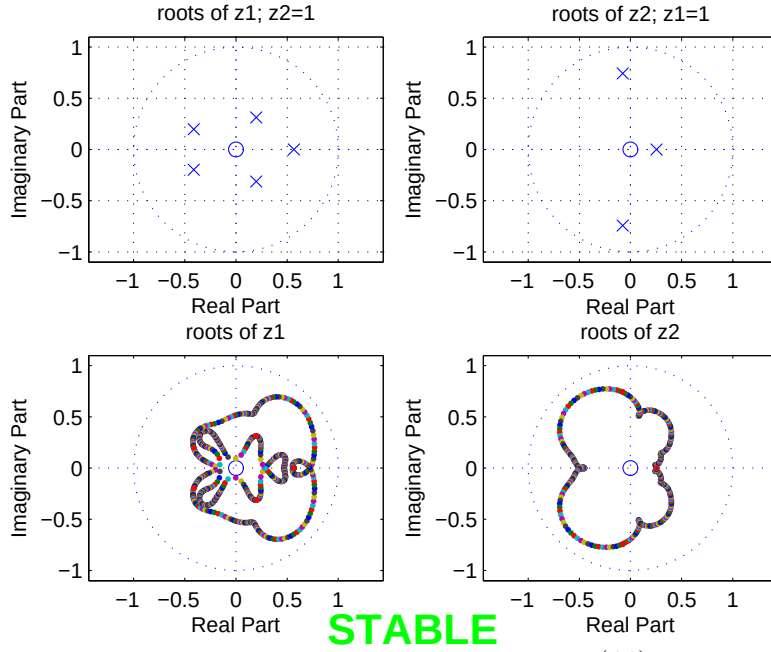


Figure 5.95: The stability test of the first-quadrant QP, $B_{QP1}^{(5,3)}$ in Example 1, by Choice II.

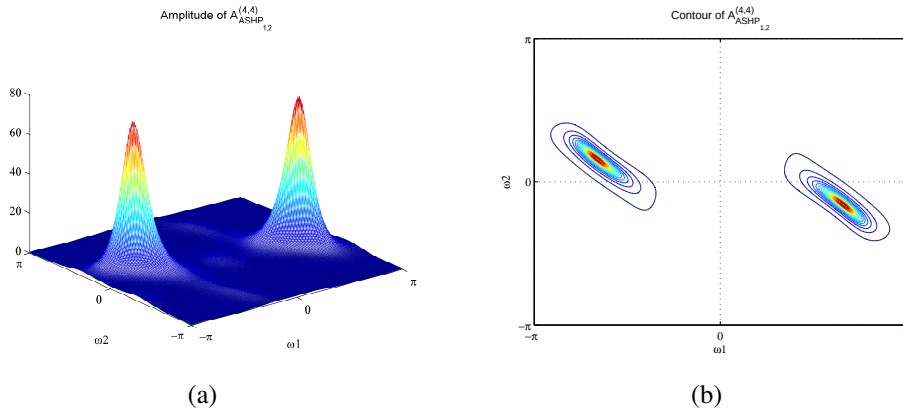


Figure 5.96: (a) Amplitude; (b) Contour plots of the spectrum of $A_{ASHP,1,2}^{(4,4)}$ in Example 1, by Choice II.

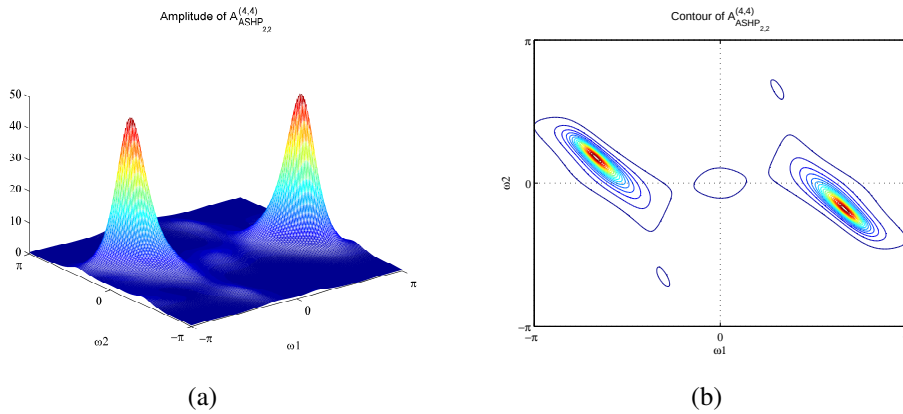


Figure 5.97: (a) Amplitude; (b) Contour plots of the spectrum of $A_{ASHP,2,2}^{(4,4)}$ in Example 1, by Choice II.

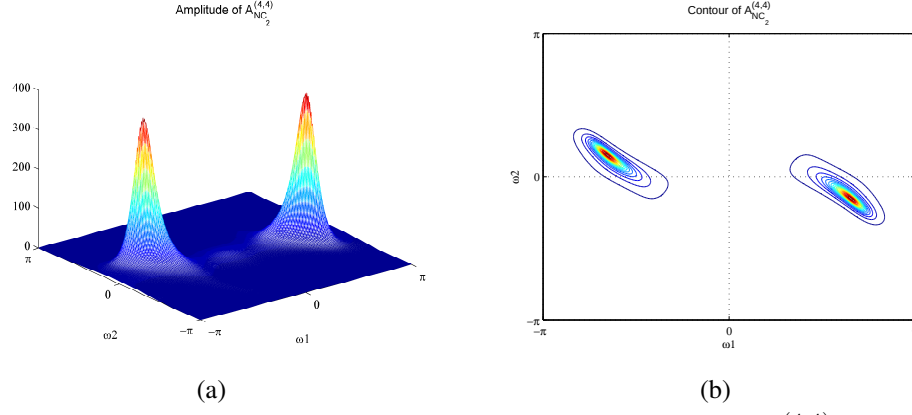


Figure 5.98: (a) Amplitude; (b) Contour plots of the spectrum of $A_{NC_2}^{(4,4)}$ in Example 1, by Choice II.

Similarly, an NC half-plane filter can be generated from the same vertical PEFs. The coefficient matrix, $A_{NC_2}^{(4,4)}$ can be calculated as

$$A_{NC_2}^{(4,4)} = \begin{bmatrix} -0.3859 & -0.0210 & -0.0251 & -0.0104 & -0.0050 \\ 1 & -0.0473 & -0.1019 & 0.0514 & 0.0635 \\ -0.3918 & 0.2686 & 0.1998 & -0.0995 & -0.0794 \\ 0.0093 & -0.2301 & 0.0161 & 0.0163 & -0.0161 \\ 0.0091 & -0.0009 & 0.0019 & 0.0069 & 0.0064 \end{bmatrix} \quad (5.40)$$

and the amplitude and equal magnitude contours of $A_{NC_2}^{(4,4)}$ are illustrated in Figure 5.98.

5.1.3 Discussion of example 1

In this example, we applied augmentation order for both cases and we have seen that regardless of the augmentation sequences, at the desired stage order PSR, the lattice filters generated by using the proposed method are very close to each other.

The Itakura's LPC distance measure of each filters obtained by using the augmentation sequence Choice I and Choice II are calculated and the results are demonstrated in the Table 5.30 and Table 5.31, respectively. If we compare both tables, we can see that the distances calculated are very close to each other for the same filter models.

We also compared the Itakura's LPC distance measures of the results found in the study [34] with the results of the proposed method as shown in Table 5.32. As it is seen the proposed method gives the best result at the (3,3)th order stage for the first-quadrant QP lattice filter.

Example 1 shows us that augmentation direction is not important to reach to the certain desired stage order PSR. Augmentation direction can be chosen freely to scan the 2-D

Table 5.30: Comparison by Itakura's LPC distance measures of the 2-D lattice filters from the Choice I augmentation direction.

Filter Type	Distance	Filter Type	Distance
$A_{QP1}^{(1,1)}$	0.1200	$B_{QP1}^{(5,3)}$	0.0051
$A_{QP1}^{(1,2)}$	0.0580	$A_{ASHP_{1,2}}^{(4,4)}$	0.0023
$B_{QP1}^{(2,1)}$	0.0627	$A_{ASHP_{2,2}}^{(4,4)}$	0.0036
$A_{QP1}^{(2,2)}$	0.0106	$A_{NC_2}^{(4,4)}$	-0.0822
$A_{QP1}^{(2,3)}$	0.0044	$B_{QP1}^{(4,4)}$	0.0014
$B_{QP1}^{(3,2)}$	0.0107	$A_{QP1}^{(3,5)}$	0.0017
$A_{QP1}^{(3,3)}$	0.0044	$B_{ASHP_{1,2}}^{(4,4)}$	0.0049
$A_{QP1}^{(3,4)}$	0.0012	$A_{ASHP_{2,2}}^{(4,4)}$	0.0043
$B_{QP1}^{(4,3)}$	0.0045	$B_{NC_2}^{(4,4)}$	0.0043
$A_{QP1}^{(4,4)}$	0.0013		

Table 5.31: Comparison by Itakura's LPC distance measures of the 2-D lattice filters from the Choice II augmentation direction.

Filter Type	Distance	Filter Type	Distance
$B_{QP1}^{(1,1)}$	0.1200	$B_{QP1}^{(5,3)}$	0.0051
$A_{QP1}^{(1,2)}$	0.0580	$A_{ASHP_{1,2}}^{(4,4)}$	0.0022
$B_{QP1}^{(2,1)}$	0.0627	$A_{ASHP_{2,2}}^{(4,4)}$	0.0036
$B_{QP1}^{(2,2)}$	0.0107	$A_{NC_2}^{(4,4)}$	-0.0823
$A_{QP1}^{(2,3)}$	0.0044	$B_{QP1}^{(4,4)}$	0.0014
$B_{QP1}^{(3,2)}$	0.0107	$A_{QP1}^{(3,5)}$	0.0018
$B_{QP1}^{(3,3)}$	0.0045	$B_{ASHP_{1,2}}^{(4,4)}$	0.0049
$A_{QP1}^{(3,4)}$	0.0013	$A_{ASHP_{2,2}}^{(4,4)}$	0.0043
$B_{QP1}^{(4,3)}$	0.0045	$B_{NC_2}^{(4,4)}$	0.0043
$A_{QP1}^{(4,4)}$	0.0013		

Table 5.32: Comparison by Itakura's LPC distance measures in Example 1.

Lattice Algorithm	Order Stage	Itakura's LPC Distance
Three-Parameter Lattice Filter [17]	(3, 3)	0.1053
Extended Lattice Filter Perpendicular [34]	(3, 3)	0.0087
Extended Lattice Filter Diagonal [34]	(3, 3)	0.0162
Proposed Lattice Filter	(3, 3)	0.0044
Proposed Lattice Filter	(4, 4)	0.0013

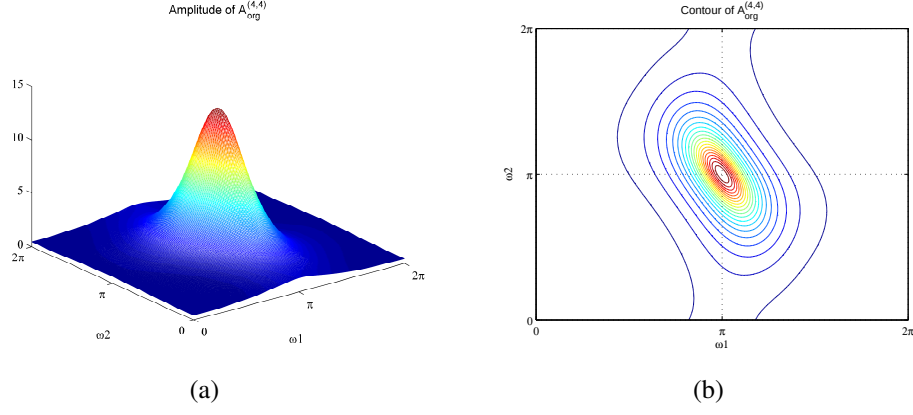


Figure 5.99: (a) Amplitude; (b) Contour plots of the original spectrum of $A_{org}^{(4,4)}$ in Example 2.

AR data field. In addition, with this example, it has been demonstrated that causal and noncausal 2D lattice filters can be simultaneously generated by using the appropriate forward/backward vertical and horizontal auxiliary PEFs from a given PSR.

5.2 Example 2: AR Data Field with the (4,4)th order

Here, we will use the proposed method to model a 2-D AR data field and generate the QP, ASHP and NC half-plane lattice filters. In addition, we will find their spectrum estimations. AR data field with the (4,4)th order stage with the size 85×85 is generated with Gaussian white noise $N(\mu = 0, \sigma^2 = 1)$ by using the following coefficient matrix defined as

$$A_{org}^{(4,4)} = \begin{bmatrix} 1 & 0.4 & -0.02 & 0 & -0.0001 \\ 0.2 & -0.1 & 0.005 & 0 & 0.001 \\ 0.0000006 & -0.0001 & 0.0011 & 0 & -0.0044 \\ 0 & 0 & -0.0005 & 0.0001 & 0 \\ -0.001 & 0 & 0 & 0.0005 & -0.0005 \end{bmatrix}. \quad (5.41)$$

The magnitude and contour plots of the original spectrum are shown in Figure 5.99.

The results of the stability test are shown in Figure 5.100.

By applying our proposed method on this AR field, we obtain the coefficient matrices of the first-quadrant QP filter, type-1 ASHP, type-2 ASHP, and NC half-plane, namely, $A_{QP_1}^{(3,3)}$, $A_{ASHP_{1,2}}^{(3,3)}$, $A_{ASHP_{2,2}}^{(3,3)}$ and $A_{NC_2}^{(3,3)}$ from the auxiliary vertical PEFs of the PSR $S_{k,l}^{(3,2)}$. The coefficient matrices of the filters obtained for the (3,3)th order stage are shown in Table 5.33 and their amplitudes and equal magnitude contours are illustrated in Figure 5.101.

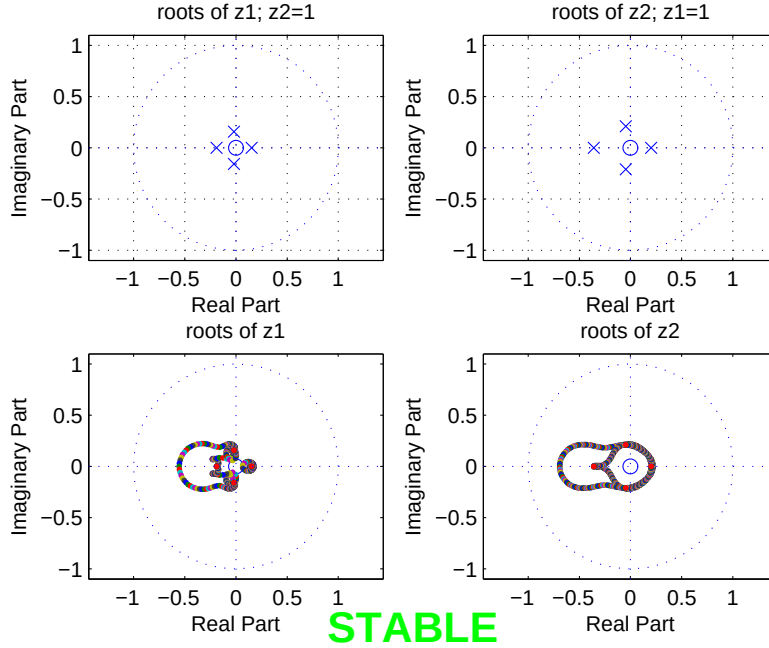


Figure 5.100: The stability test of the original filter in Example 2.

The stability test results of $A_{QP1}^{(3,3)}$ are shown in Figure 5.102.

Similarly, we can go one step further and obtain the same filters from the vertical PEFs of the PSR $S_{k,l}^{(4,3)}$. The coefficient matrices of the filters obtained, $A_{QP1}^{(4,4)}$, $A_{ASHP1,2}^{(4,4)}$, $A_{ASHP2,2}^{(4,4)}$ and $A_{NC2}^{(4,4)}$ for the (4,4)th order stage are given in Table 5.34 and their amplitudes and equal magnitude contours are demonstrated in Figure 5.103.

The stability test results of $A_{QP1}^{(4,4)}$ are shown in Figure 5.104.

Itakura's LPC distance measures of the (3,3) and (4,4)th order stage filters obtained by using the proposed method are given in Table 5.35.

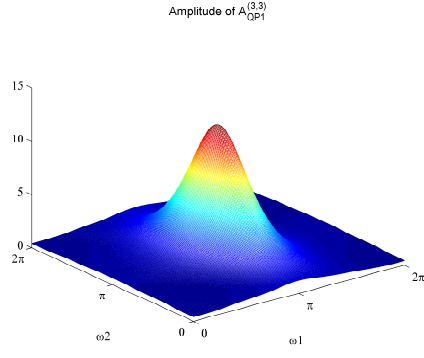
A visual comparison of the similarity between magnitude and contour plots of the original and estimated spectra is very close and the (3,3)th order stage QP filter obtained by proposed method gives the best Itakura's LPC distance measure value. According to the Table 5.35, the (4,4)th order stage ASHP filters give the best measurement values.

Table 5.33: Coefficient Matrices of the filters from the vertical PEFs of the PSR $S_{k,l}^{(3,2)}$ in Example 2.

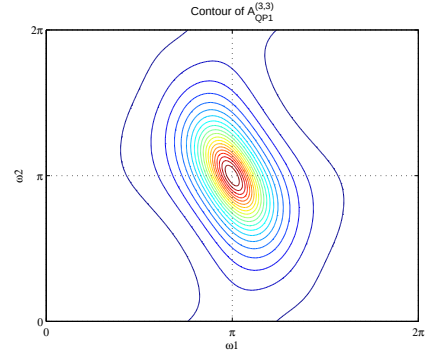
Coefficient Matrices					
$A_{QP_1}^{(3,3)} =$	1	0.4078	-0.0232	0.0032	
	0.2088	-0.0896	0.0200	0.0113	
	0.0002	-0.0110	0.0295	0.0115	
	0.0020	0.0082	0.0107	0.0015	
$A_{ASHP_{1,2}}^{(3,3)} =$	0	-0.0101	-0.0167	0.0018	
	1	0.4057	-0.0247	0.0073	
	0.2079	-0.0896	0.0201	0.0113	
	-0.0001	-0.0133	0.0267	0.0103	
$A_{ASHP_{2,2}}^{(3,3)} =$	0.2088	0.0759	-0.0192	0.0059	
	1	0.3713	-0.0161	0.0092	
	0	-0.1692	0.0297	0.0120	
	0	0.0242	0.0235	0.0084	
$A_{NC_2}^{(3,3)} =$	0.2001	0.0720	-0.0206	0.0024	
	1	0.3708	-0.0196	0.0093	
	0.1993	-0.0881	0.0252	0.0132	
	0.0003	-0.0111	0.0277	0.0102	

Table 5.34: Coefficient Matrices of the filters from the vertical PEFs of the PSR $S_{k,l}^{(4,3)}$ in Example 2.

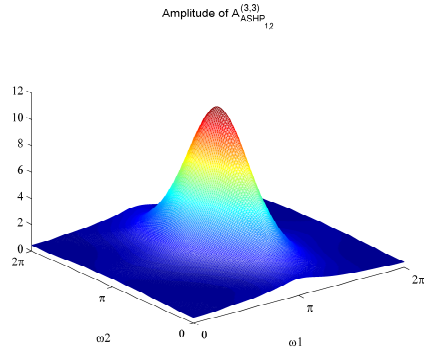
Coefficient Matrices					
$A_{QP_1}^{(4,4)} =$	1	0.4078	-0.0237	0.0064	0.0104
	0.2085	-0.0893	0.0202	0.0160	0.0070
	-0.00048	-0.0109	0.0297	0.0047	-0.0089
	0.0046	0.0085	0.0079	0.0110	0.0156
	0.0168	-0.0013	-0.0042	0.0092	-0.0028
$A_{ASHP_{1,2}}^{(4,4)} =$	0	-0.0097	-0.0163	0.0009	0.0001
	1	0.4058	-0.0246	0.0087	0.0067
	0.2084	-0.0895	0.0200	0.0165	0.0081
	-0.0012	-0.0138	0.0292	0.0039	-0.0083
	0.0004	0.0075	0.0107	0.0081	0.0170
$A_{ASHP_{2,2}}^{(4,4)} =$	0.2086	0.0761	-0.0193	0.0090	0.0091
	1	0.3714	-0.0164	0.0109	0.0067
	0	-0.1693	0.0297	0.0154	0.0050
	0	0.0236	0.0250	0.0033	-0.0057
	0	0.0003	0.0050	0.0097	0.0184
$A_{NC_2}^{(4,4)} =$	0.1998	0.0722	-0.0204	0.0021	0.0022
	1	0.3711	-0.0195	0.0115	0.0078
	0.1996	-0.0879	0.0251	0.0167	0.0060
	-0.0002	-0.0115	0.0295	0.0059	-0.0049
	0.0037	0.0069	0.0094	0.0096	0.0158



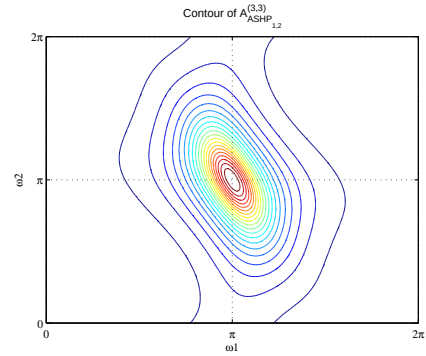
(a)



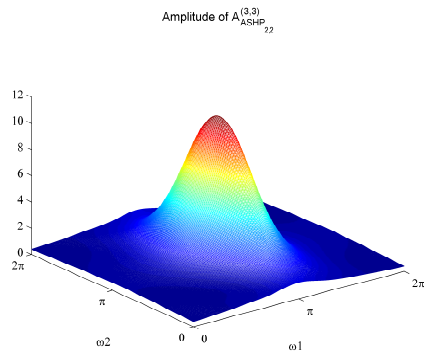
(b)



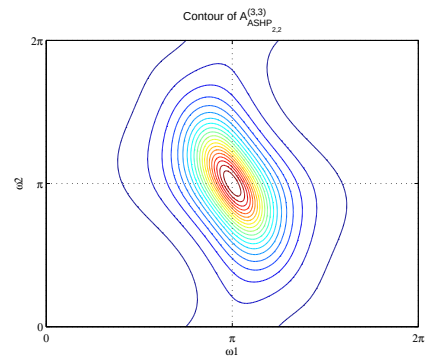
(c)



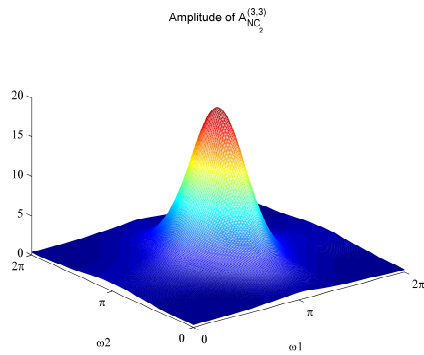
(d)



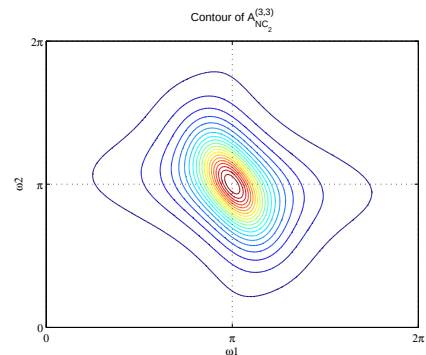
(e)



(f)



(g)



(h)

Figure 5.101: (a, c, e, g) Amplitude; (b, d, f, h) Contour plots of the spectrum of $A_{QP1}^{(3,3)}$, $A_{ASHP1,2}^{(3,3)}$, $A_{ASHP2,2}^{(3,3)}$, and $A_{NC2}^{(3,3)}$, respectively, in Example 2.

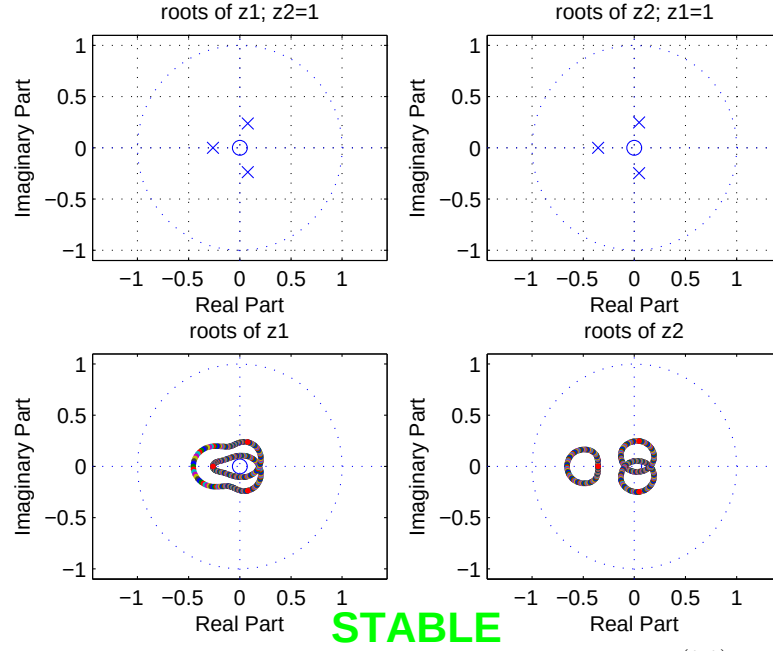


Figure 5.102: The stability test of the first-quadrant QP, $A_{QP1}^{(3,3)}$ in Example 2.

Table 5.35: Comparison by Itakura's LPC distance measures of the 2-D lattice filters in Example 2.

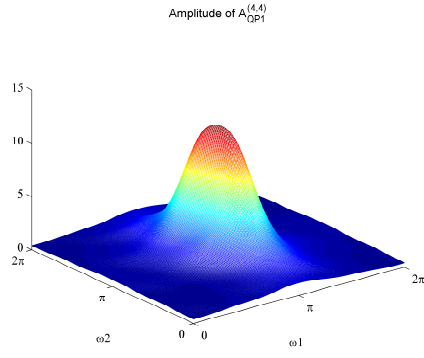
Filter Type	Distance	Filter Type	Distance
$A_{QP1}^{(3,3)}$	0.00048	$A_{QP1}^{(4,4)}$	0.0011
$A_{ASHP1,2}^{(3,3)}$	0.00064	$A_{ASHP1,2}^{(4,4)}$	0.00092
$A_{ASHP2,2}^{(3,3)}$	0.00070	$A_{ASHP2,2}^{(4,4)}$	0.00091
$A_{NC2}^{(3,3)}$	-0.0178	$A_{NC2}^{(4,4)}$	-0.0175

5.3 Example 3: AR Data Field with the (4,4)th order

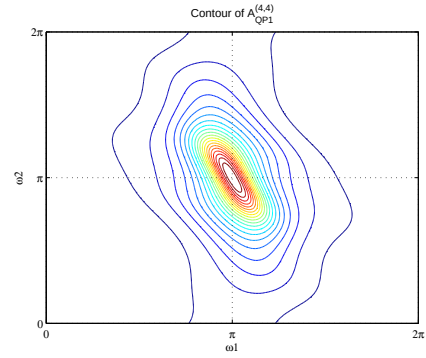
In this example, we will model a 2-D AR data field by using the proposed method and generate the QP, ASHP and NC half-plane lattice filters. In addition, we will find their spectrum estimations. AR data field with the (4,4)th order stage with the size 85 – by – 85 is generated with Gaussian white noise $N(\mu = 0, \sigma^2 = 1)$ by using the following coefficient matrix defined as

$$A_{org}^{(4,4)} = \begin{bmatrix} 1 & -0.4400 & 0.0074 & 0.1600 & -0.4000 \\ -0.4400 & 0.2200 & -0.0095 & -0.0700 & 0.1700 \\ -0.1200 & 0.0690 & 0.0140 & 0.0200 & -0.0480 \\ -0.0520 & 0.0200 & -0.0030 & 0.0015 & 0.0180 \\ -0.0060 & 0.0027 & 0.000012 & -0.0032 & 0.0070 \end{bmatrix}. \quad (5.42)$$

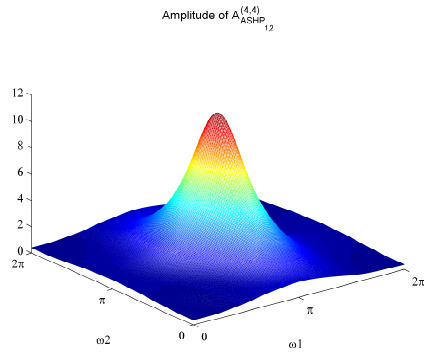
The magnitude and contour plots of the original spectrum are shown in Figure 5.105.



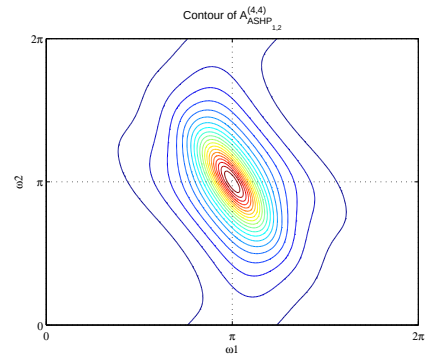
(a)



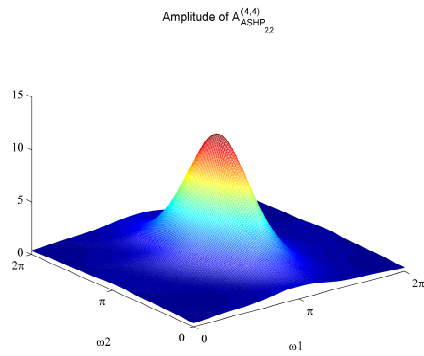
(b)



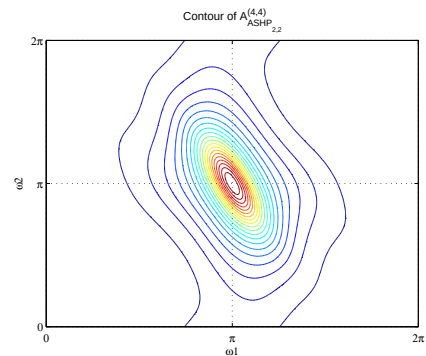
(c)



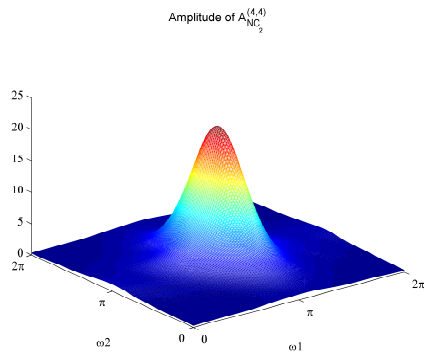
(d)



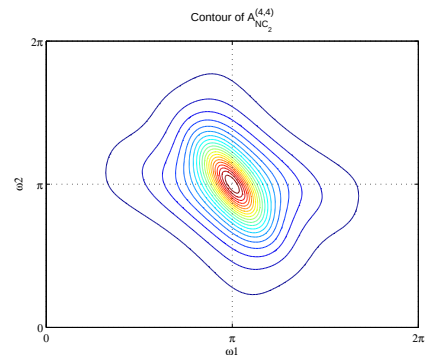
(e)



(f)



(g)



(h)

Figure 5.103: (a, c, e, g) Amplitude; (b, d, f, h) Contour plots of the spectrum of $A_{QP1}^{(4,4)}$, $A_{ASHP1,2}^{(4,4)}$, $A_{ASHP2,2}^{(4,4)}$, and $A_{NC2}^{(4,4)}$, respectively, in Example 2.

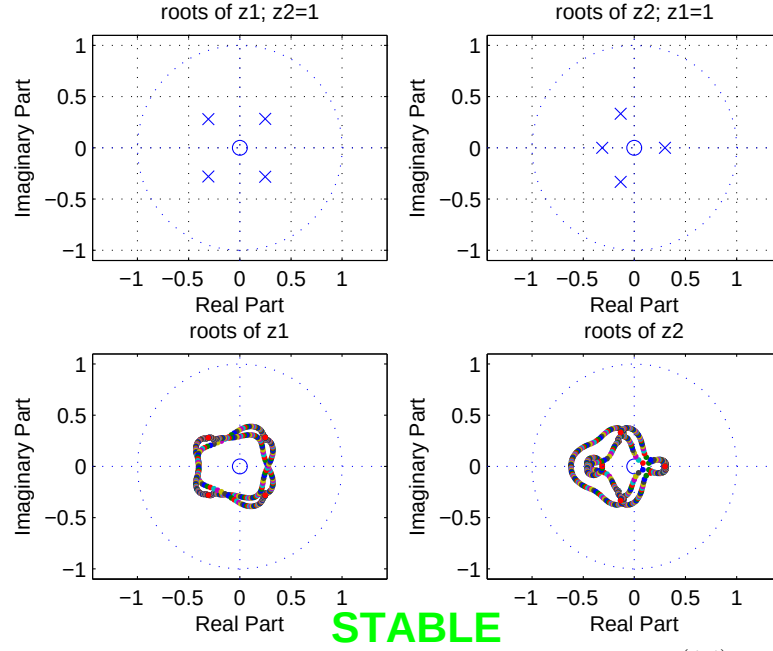


Figure 5.104: The stability test of the first-quadrant QP, $A_{QP1}^{(4,4)}$ in Example 2.

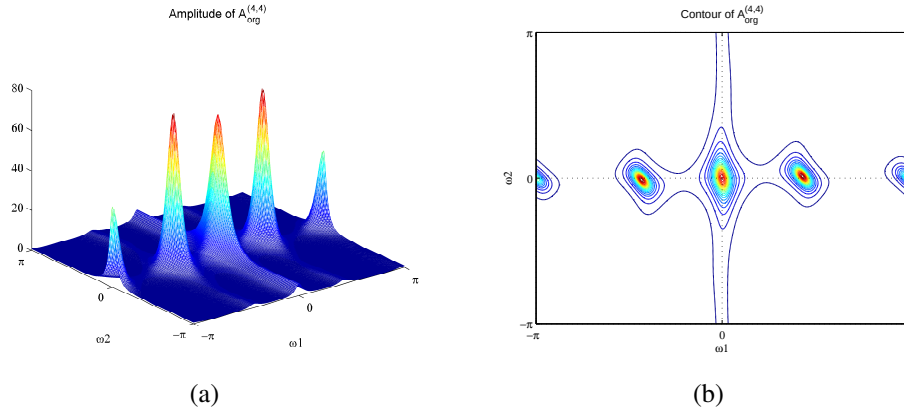


Figure 5.105: (a) Amplitude; (b) Contour plots of the original spectrum of $A_{org}^{(4,4)}$ in Example 3.

The results of the stability test is shown in Figure 5.106.

By applying our proposed method on this AR field, we obtain the coefficient matrices of first-quadrant QP filter, type-1 ASHP, type-2 ASHP, and NC half-plane, namely, $A_{QP1}^{(4,4)}$, $A_{ASHP1,2}^{(4,4)}$, $A_{ASHP2,2}^{(4,4)}$ and $A_{NC2}^{(4,4)}$, respectively, from the auxiliary vertical PEFs of the PSR $S_{k,l}^{(4,3)}$. The coefficient matrices of the filters obtained for the (4,4)th order stage are shown in Table 5.36 and their amplitudes and equal magnitude contours are illustrated in Figure 5.107.

The stability test result of $A_{QP1}^{(4,4)}$ is shown in Figure 5.108.

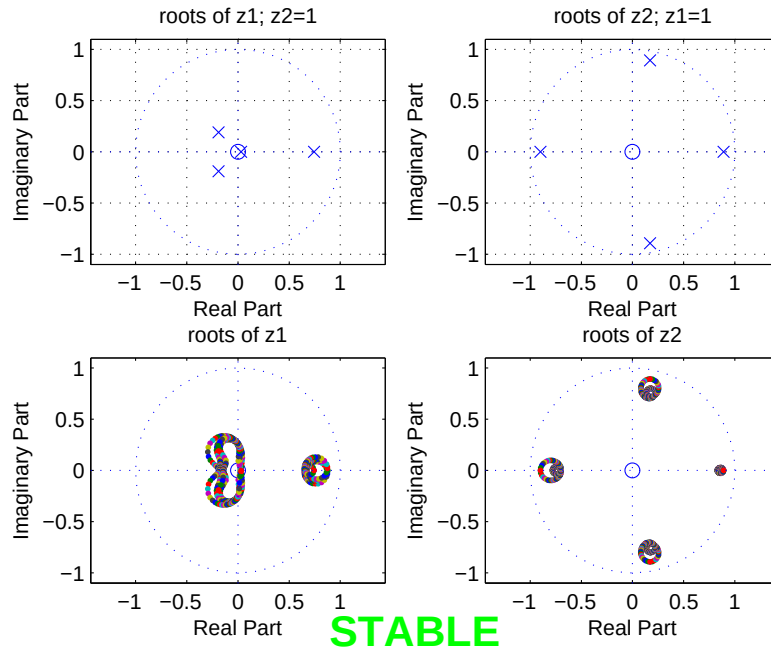


Figure 5.106: The stability test of the original filter in Example 3.

Itakura's LPC distance measures of the (4,4)th order stage filters obtained by using the proposed method are given in Table 5.37.

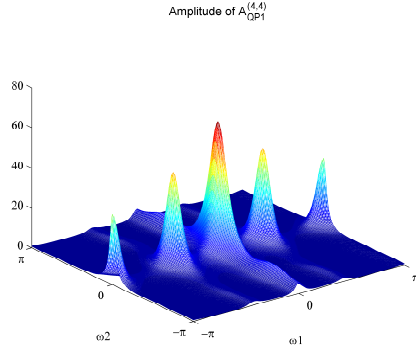
A visual comparison of the similarity between magnitude and contour plots of the original and estimated spectra is very close and the first-quadrant QP filter gives the lowest Itakura's LPC distance measure value.

Table 5.36: Coefficient Matrices of the filters from the vertical PEFs of the PSR $S_{k,l}^{(4,3)}$ in Example 3.

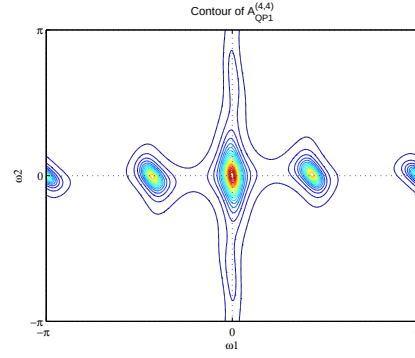
Coefficient Matrices					
$A_{QP_1}^{(4,4)} =$	1	-0.4591	0.0110	0.1597	-0.4098
	-0.4343	0.2304	-0.0266	-0.0757	0.1911
	-0.0966	0.0563	0.0223	0.0222	-0.0467
	-0.0453	0.0087	0.0033	0.0162	-0.0220
	-0.0153	0.0167	-0.0137	0.0010	0.0281
$A_{ASHP_{1,2}}^{(4,4)} =$	0	-0.0216	0.0066	-0.0024	0.0159
	1	-0.4488	0.0070	0.1590	-0.4167
	-0.4373	0.2346	-0.0267	-0.0746	0.1896
	-0.0950	0.0576	0.0183	0.0231	-0.0449
	-0.0540	0.0167	-0.0034	0.0177	-0.0115
$A_{ASHP_{2,2}}^{(4,4)} =$	-0.5046	0.2179	-0.0114	-0.0746	0.2179
	1	-0.4710	0.0174	0.1633	-0.4173
	0	0.0212	-0.0301	-0.0212	0.0493
	0	0.0276	0.0072	0.0217	-0.0481
	0	0.0021	0.0059	0.0439	-0.0798
$A_{NC_2}^{(4,4)} =$	-0.3648	0.1493	0.0016	-0.0602	0.1629
	1	-0.4618	0.0156	0.1614	-0.4204
	-0.3328	0.1769	-0.0306	-0.0709	0.1766
	-0.0634	0.0453	0.0142	0.0136	-0.0298
	-0.0398	0.0080	0.0021	0.0145	-0.0199

Table 5.37: Comparison by Itakura's LPC distance measures of the 2-D lattice filters in Example 3.

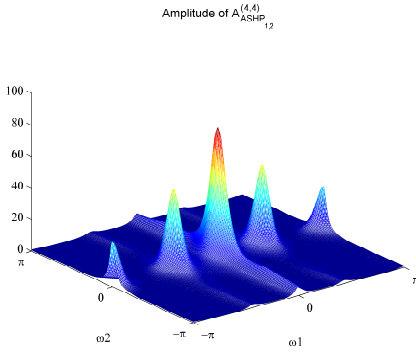
Filter Type	Distance
$A_{QP_1}^{(4,4)}$	0.0021
$A_{ASHP_{1,2}}^{(4,4)}$	0.0023
$A_{ASHP_{2,2}}^{(4,4)}$	0.0149
$A_{NC_2}^{(4,4)}$	-0.0722



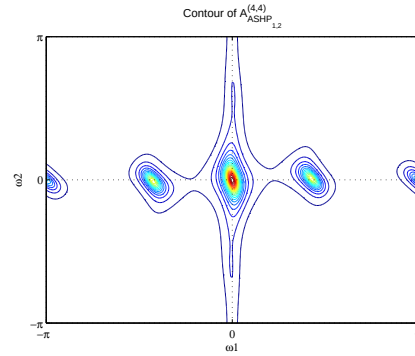
(a)



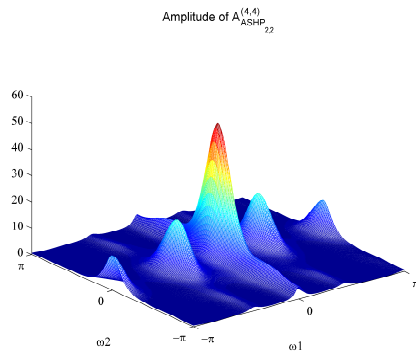
(b)



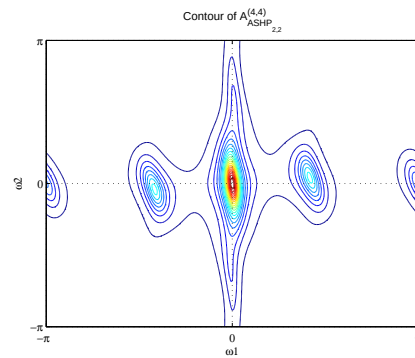
(c)



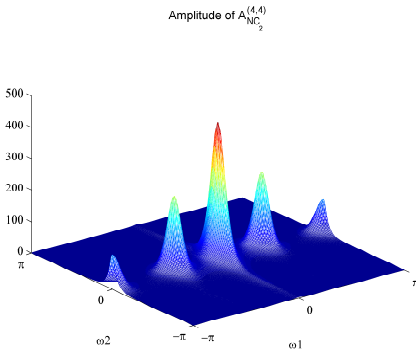
(d)



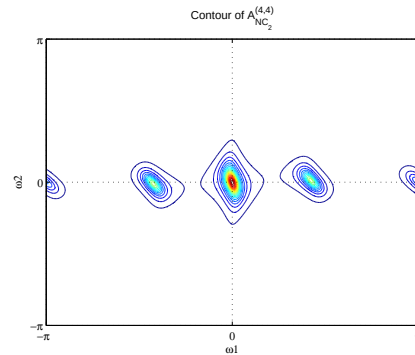
(e)



(f)



(g)



(h)

Figure 5.107: (a, c, e, g) Amplitude; (b, d, f, h) Contour plots of the spectrum of $A_{QP1}^{(4,4)}$, $A_{ASHP1,2}^{(4,4)}$, $A_{ASHP2,2}^{(4,4)}$, and $A_{NC2}^{(4,4)}$, respectively, in Example 3.

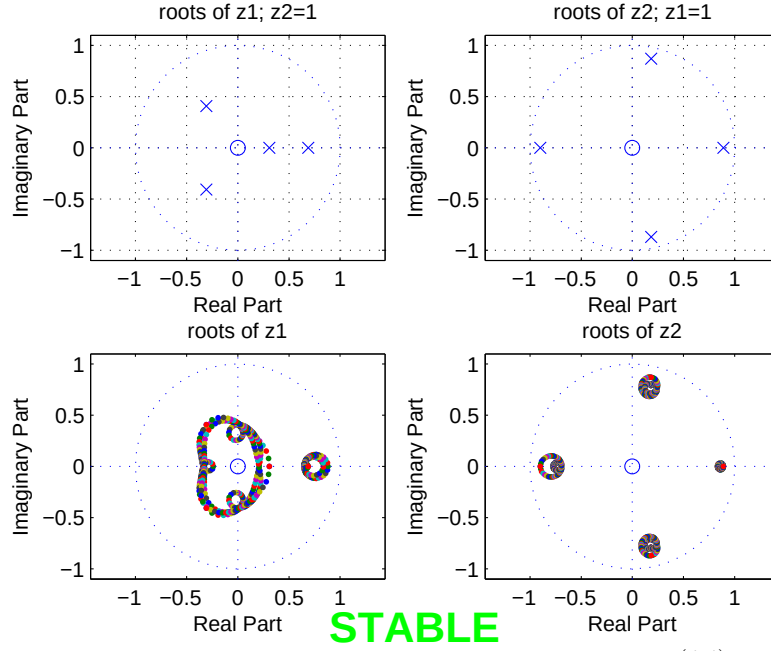


Figure 5.108: The stability test of the first-quadrant QP, $A_{QP1}^{(4,4)}$ in Example 3.

5.4 Example 4: AR Data Field with the (3,3)th order

In this example, we used the AR data field with the (3,3)th order stage with the size $75 - by - 75$ is generated with Gaussian white noise $N(\mu = 0, \sigma^2 = 1)$ by using the following coefficient matrix defined as

$$A_{org}^{(4,4)} = \begin{bmatrix} 1 & -0.1 & 0.1 & -0.1 \\ -0.1 & 0.1 & -0.1 & 0.1 \\ 0.1 & -0.1 & 0.1 & -0.1 \\ -0.1 & 0.1 & -0.1 & 0.1 \end{bmatrix}. \quad (5.43)$$

We will model this 2-D AR data field by using the proposed method and generate the QP, ASHP and NC half-plane lattice filters. We will calculate their spectrum estimations. We will compare the results with the other methods outcome by calculating the Itakura's distance measure.

The magnitude and contour plots of the original spectrum are shown in Figure 5.109.

The results of the stability test are shown in Figure 5.110.

This AR data field coefficient matrix has already been used in the study [41]. We will just apply the proposed method and compare the results with the ones found in the study [41]. Here, we will generate the all possible filters from the vertical PEFs of the

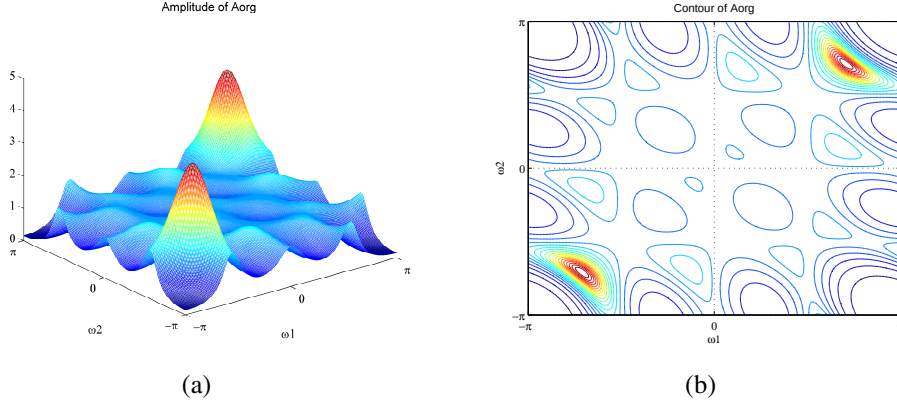


Figure 5.109: (a) Amplitude; (b) Contour plots of the original spectrum of $A_{org}^{(3,3)}$ in Example 4.

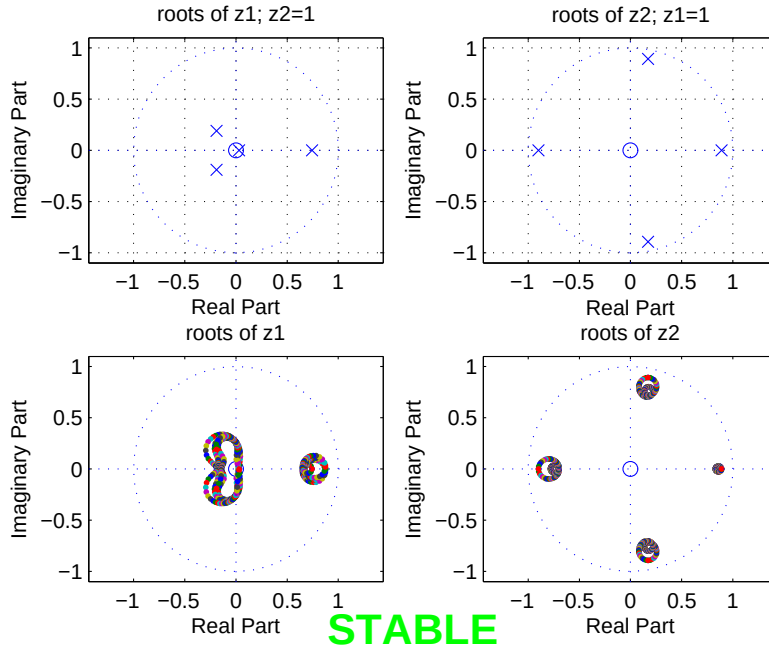


Figure 5.110: The stability test of the original filter in Example 4.

PSR $S_{k,l}^{(3,2)}$ and in addition, we will produce the all possible filters from the horizontal PEFs of the PSR $S_{k,l}^{(2,3)}$, as well. All Itakura's LPC distance measures of the obtained filters will be calculated. As a last one, we will compare the results found in the study [41] with our results.

First, by applying our proposed method, we obtain the coefficient matrices of the first-quadrant QP, type-1 ASHP, type-2 ASHP, and NC half-plane, namely, $A_{QP_1}^{(3,3)}$, $A_{ASHP_{1,2}}^{(3,3)}$, $A_{ASHP_{2,2}}^{(3,3)}$, and $A_{NC_2}^{(3,3)}$, respectively, from the auxiliary vertical PEFs of the PSR $S_{k,l}^{(3,2)}$. The coefficient matrices of the filters obtained for the (3,3)th order stage are shown in Table 5.38 and their amplitudes and equal magnitude contours are illustrated in Figure 5.111.

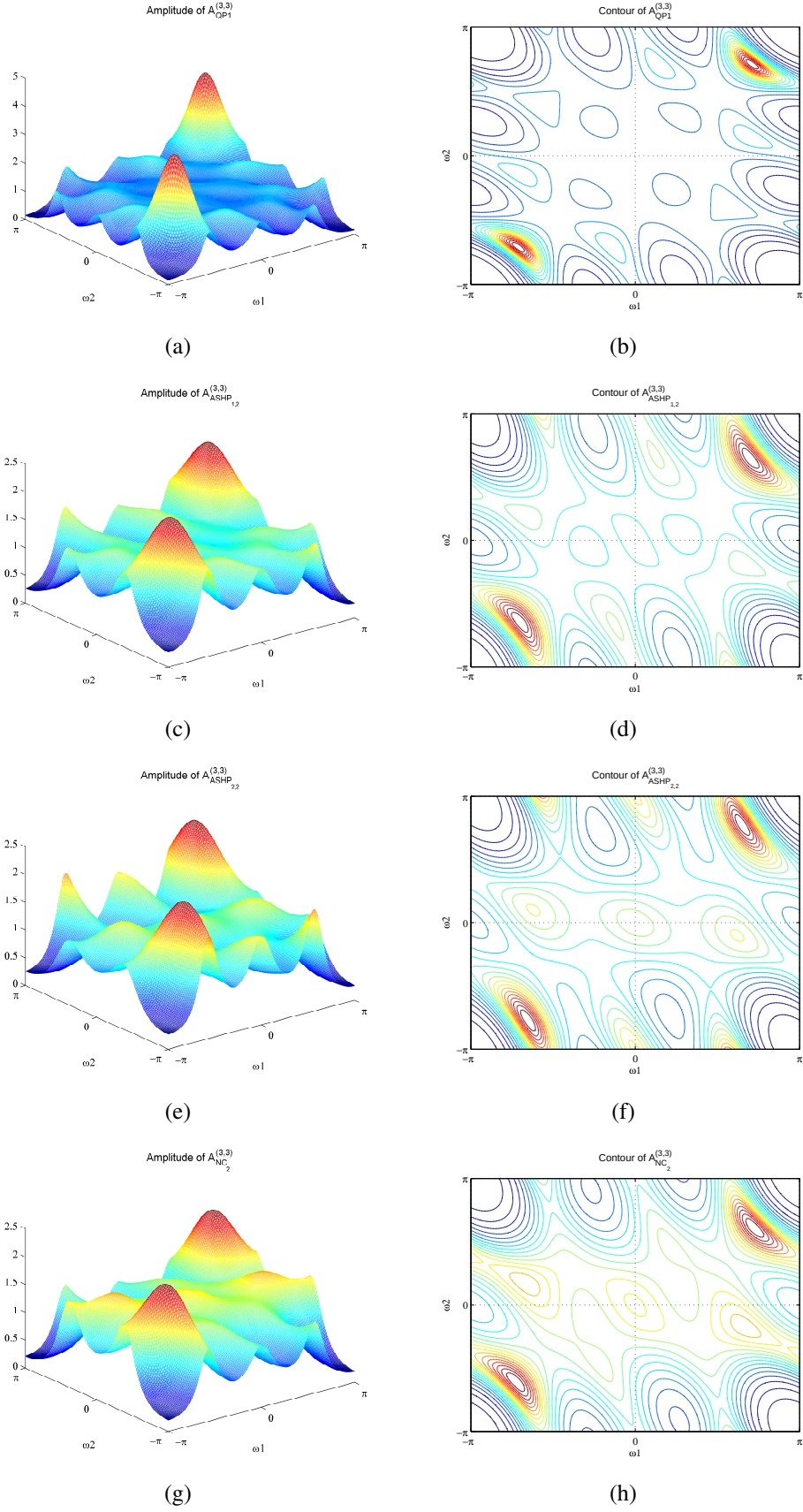


Figure 5.111: (a, c, e, g) Amplitude; (b, d, f, h) Contour plots of the spectrum of $A_{QP1}^{(3,3)}$, $A_{ASHP_{1,2}}^{(3,3)}$, $A_{ASHP_{2,2}}^{(3,3)}$, and $A_{NC_2}^{(3,3)}$, respectively, obtained from the PSTR $S_{k,l}^{(3,2)}$ in Example 4.

Table 5.38: Coefficient Matrices of the filters from the vertical PEFs of the PSR $S_{k,l}^{(3,2)}$ in Example 4.

Coefficient Matrices				
$A_{QP_1}^{(3,3)} =$	1	-0.0928	0.0973	-0.0850
	-0.1005	0.1074	-0.0969	0.1062
	-0.0923	-0.1126	0.0891	-0.0954
	-0.0862	0.1059	-0.1026	0.0923
$A_{ASHP_{1,2}}^{(3,3)} =$	0	0.0028	-0.0148	0.0009
	1	-0.0709	0.0847	-0.0811
	-0.0748	0.0888	-0.0868	0.1043
	0.0678	-0.0964	0.0775	-0.0895
$A_{ASHP_{2,2}}^{(3,3)} =$	-0.0904	0.0125	-0.0234	0.0094
	1	-0.0835	0.0940	-0.0911
	0	0.0960	-0.0885	0.1063
	0	-0.0960	0.0753	-0.0840
$A_{NC_2}^{(4,4)} =$	-0.1034	0.0124	-0.0247	0.0097
	1	-0.0813	0.0938	-0.0912
	-0.0836	0.0995	-0.0951	0.1131
	0.0760	-0.1063	0.0873	-0.0981

Table 5.39: Comparison by Itakura's LPC distance measures of the 2-D lattice filters obtained from the vertical PEFs of the PSR $S_{k,l}^{(3,2)}$ in Example 4.

Filter Type	Distance
$A_{QP_1}^{(3,3)}$	0.00020
$A_{ASHP_{1,2}}^{(3,3)}$	0.0138
$A_{ASHP_{2,2}}^{(3,3)}$	0.0149
$A_{NC_2}^{(3,3)}$	0.0100

The stability test results of $A_{QP_1}^{(3,3)}$ are shown in Figure 5.112.

Itakura's LPC distance measures of the (3,3)th order stage filters obtained from the vertical PEFs of the PSR $S_{k,l}^{(3,2)}$ are given in Table 5.39.

Now, it is turn to obtain the coefficient matrices of the first-quadrant QP, type-1 ASHP, type-2 ASHP, and NC half-plane, namely, $B_{QP_1}^{(3,3)}$, $B_{ASHP_{1,2}}^{(3,3)}$, $B_{ASHP_{2,2}}^{(3,3)}$, and $B_{NC_2}^{(3,3)}$, respectively, from the auxiliary horizontal PEFs of the PSR $S_{k,l}^{(2,3)}$. The coefficient matrices of the filters obtained for the (3,3)th order stage are shown in Table 5.40 and their amplitudes and equal magnitude contours are illustrated in Figure 5.113.

The stability test results of $B_{QP_1}^{(3,3)}$ are shown in Figure 5.114.

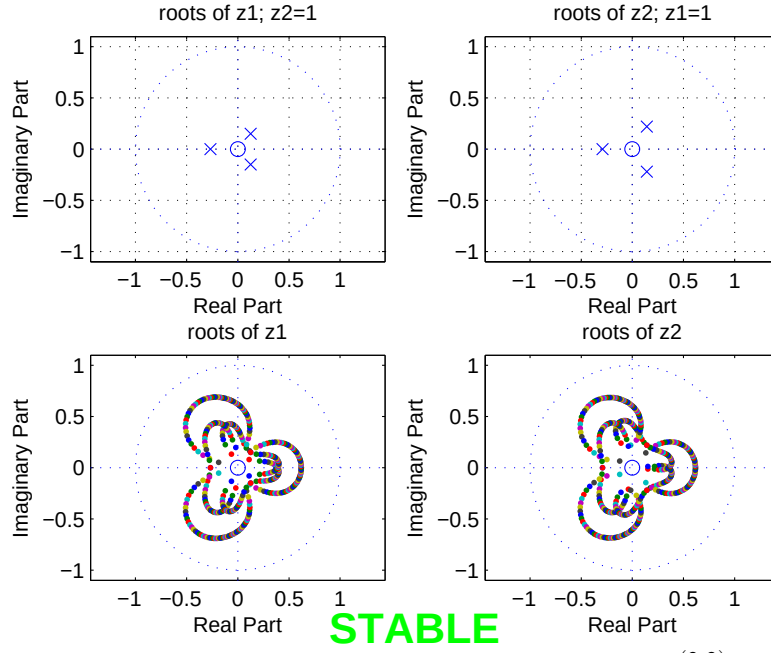
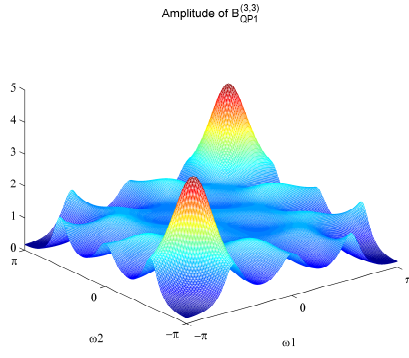


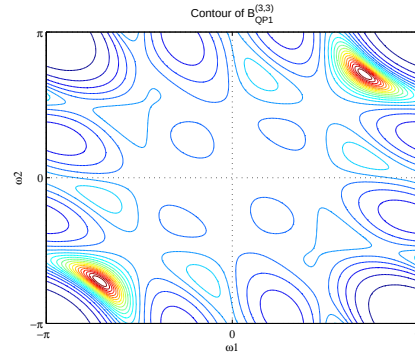
Figure 5.112: The stability test of the first-quadrant QP, $A_{QP1}^{(3,3)}$ in Example 3.

Table 5.40: Coefficient Matrices of the filters from the horizontal PEFs of the PSR $S_{k,l}^{(2,3)}$ in Example 4.

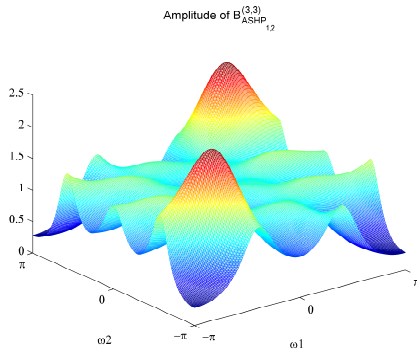
Coefficient Matrices				
$B_{QP1}^{(3,3)} =$	1	-0.0908	0.0988	-0.0837
	-0.1018	0.1072	-0.0971	0.1063
	0.0938	-0.1120	0.0893	-0.0955
	-0.0869	0.1055	-0.1025	0.0923
$B_{ASHP_{1,2}}^{(3,3)} =$	0	1	-0.0662	0.0748
	0.0026	-0.0809	0.0886	-0.0796
	0.0087	0.0806	-0.1002	0.0795
	-0.0093	-0.0815	0.0976	-0.0980
$B_{ASHP_{2,2}}^{(3,3)} =$	-0.0802	1	0	0
	0.0097	-0.0905	0.0933	-0.0780
	0.0005	0.0922	-0.1040	0.0749
	-0.0023	-0.0901	0.1012	-0.0920
$B_{NC_2}^{(4,4)} =$	-0.0934	1	-0.0749	0.0820
	0.0120	-0.0903	0.0969	-0.0889
	-0.00017	0.0904	-0.1076	0.0877
	-0.0011	-0.0907	0.1064	-0.1057



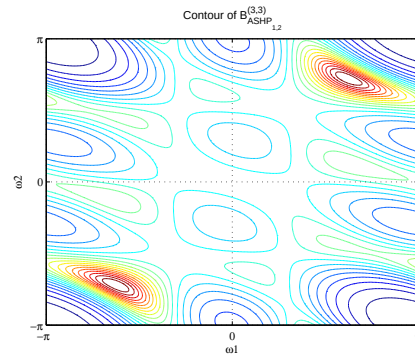
(a)



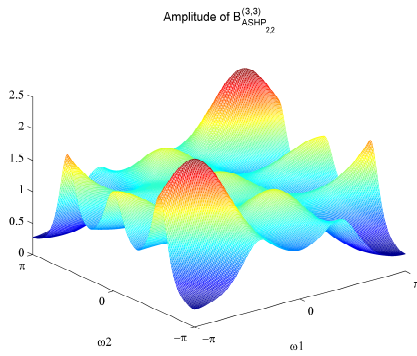
(b)



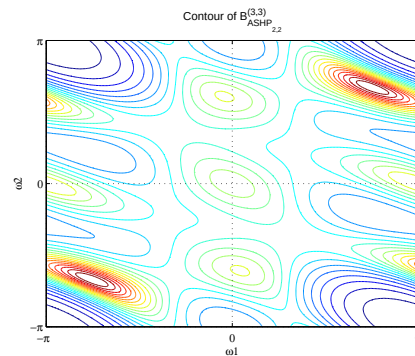
(c)



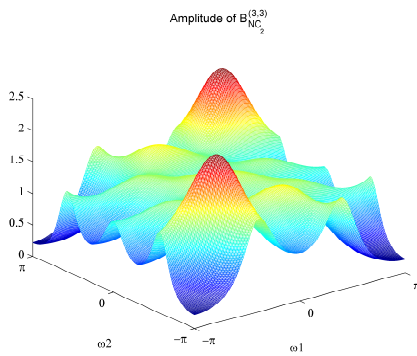
(d)



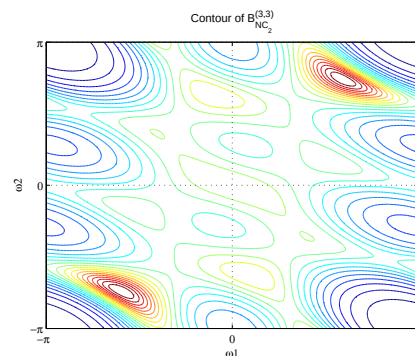
(e)



(f)



(g)



(h)

Figure 5.113: (a, c, e, g) Amplitude; (b, d, f, h) Contour plots of the spectrum of $B_{QP1}^{(3,3)}$, $B_{ASHP_{1,2}}^{(3,3)}$, $B_{ASHP_{2,2}}^{(3,3)}$, and $B_{NC_2}^{(3,3)}$, respectively, obtained from the PSR $S_{k,l}^{(2,3)}$ in Example 4.

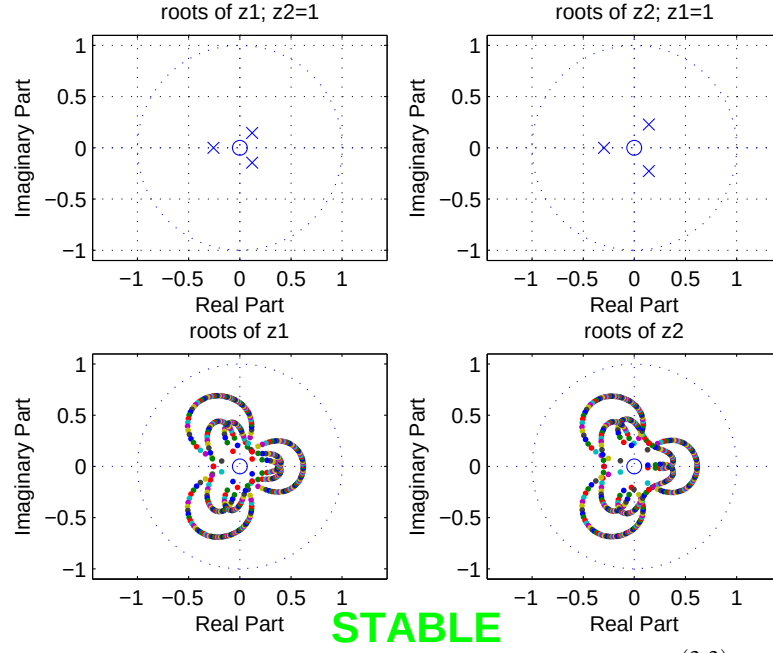


Figure 5.114: The stability test of the first-quadrant QP, $B_{QP1}^{(3,3)}$ in Example 4.

Table 5.41: Comparison by Itakura's LPC distance measures of the 2-D lattice filters obtained from the horizontal PEFs of the PSR $S_{k,l}^{(2,3)}$ in Example 4.

Filter Type	Distance
$B_{QP1}^{(3,3)}$	0.00022
$B_{ASHP1,2}^{(3,3)}$	0.0138
$B_{ASHP2,2}^{(3,3)}$	0.0149
$B_{NC2}^{(3,3)}$	0.0099

Itakura's LPC distance measures of the (3,3)th order stage filters obtained from the horizontal PEFs of the PSR $S_{k,l}^{(2,3)}$ are given in Table 5.41.

If we compare the obtained filters from vertical and horizontal PEFs and the values of Itakura's LPC distance measures, we can see that the first-quadrant QP filter is the best for this example. A visual comparison of the similarity between magnitude and contour plots of the original and estimated spectra of the first-quadrant QP filters are very close to original one and they have the lowest Itakura's LPC distance measure value. It can be seen that the contours of the magnitude plot of the spectrum of the proposed model clearly demonstrate of the locations and the shapes of all peaks and the other characteristics of the original spectrum. In addition, the estimated QP filters are also stable.

Table 5.42: Comparison by Itakura's LPC distance measures of (3,3)th order stage filters in Example 4.

Algorithm	Itakura's LPC Distance
Three-Parameter Lattice Filter [17]	0.0193
Six-Parameter Lattice Filter [69]	0.0404
Improved Lattice Filter [35]	0.0088
Solution of the Normal Equation [4]	0.0011
Two-Dimensional Schur Algorithm [41]	0.00088
Proposed Lattice Filter ($A_{QP1}^{(3,3)}$)	0.00020
Proposed Lattice Filter ($B_{QP1}^{(3,3)}$)	0.00022

Moreover, if we can compare the result of the first-quadrant QP filters obtained by using the proposed method with the ones given in the study [41], it is clearly seen that our results are the superior. The results found in the study [41] and calculated by using the proposed method are demonstrated in Table 5.42.

5.5 Example 5: Finding Real Sinusoidal Peaks

In this example, the 2-D data are obtained by sampling two sinusoids buried in white Gaussian noise with variance $\sigma^2 = 1$. The AR coefficients of this 2-D data field consisting sinusoids in noise are predicted with the proposed lattice filter and spectra are obtained as explained in Appendix 5. First classical method which is FFT based transform, second the power spectral density of the first-quadrant QP filter, and last the harmonic mean estimate of all QP and non-causal half-plane filters are computed and demonstrated.

Two sinusoids in different quadrants buried in noise are generated with the following equation

$$y(k, l) = a_1 \cos(-0.3\omega_1 k + 0.3\omega_2 l) + a_2 \cos(0.1\omega_1 k + 0.2\omega_2 l) + w(k, l) \quad (5.44)$$

where $w(k, l)$ is Gaussian white noise with zero mean and $\sigma_w^2 = 1$. Signal-to-noise ratio will be computed with the following equation

$$SNR = \frac{\sum_{i=1}^2 a_i^2}{\sigma_w^2} \quad (5.45)$$

where σ_w^2 is the noise power.

In order to show the usefulness our proposed method, first we will use it on a 65 – by – 65 populated 2-D data. Secondly, we will use the short data of size 13×13 to find the sinusoidal peaks, and lastly we will use the data of size 7×7 to show the effect of SNR.

5.5.1 Data field of size 65×65 consisting of two 2-D sinusoids in noise

In this simulation example, the amplitudes of all signals are equal to $a_1 = a_2 = 0.75$, an SNR is 0.5115 dB.

We know that classical method gives good results with the large dimensional data. Here, we will show the proposed method also gives good result.

The amplitude of the classical FFT based estimation and equal magnitude contours of the generated data by using the equation (5.44) are illustrated in Figure 5.115.

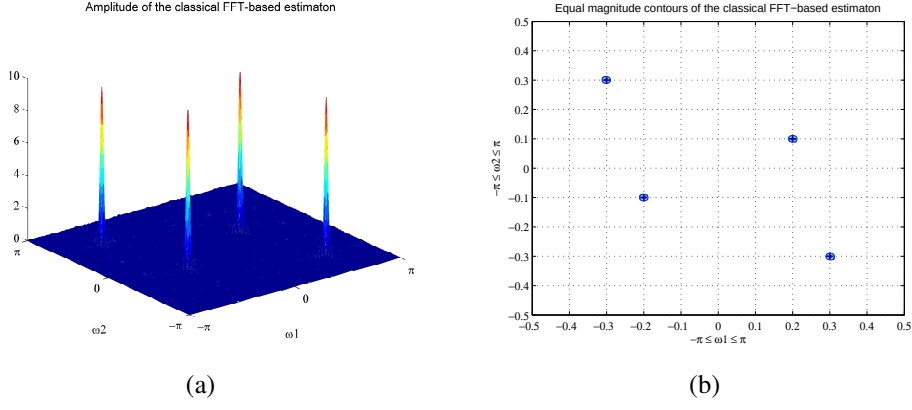


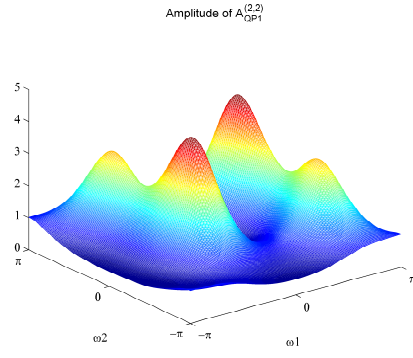
Figure 5.115: (a) Amplitude; (b) Contour plots of the original spectrum of two sinusoids buried with Gaussian noise $N(\mu = 0, \sigma^2 = 1)$ in Example 5.

Table 5.43: Coefficient Matrices of the first-quadrant QP filters for two sinusoids buried with white noise in Example 5.

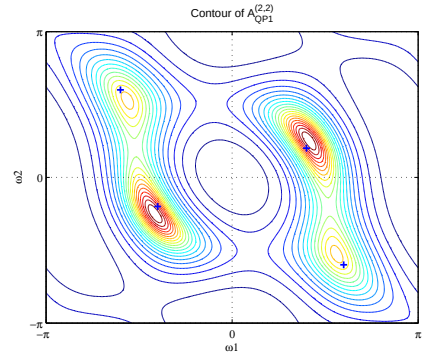
PSR	Coefficient Matrices						
$S_{k,l}^{(2,1)}$	$A_{QP_1}^{(2,2)} =$	1	-0.0354	0.2340			
		-0.0715	-0.1174	0.1884			
		0.0831	0.1370	0.0145			
$S_{k,l}^{(3,2)}$	$A_{QP_1}^{(3,3)} =$	1	-0.0179	0.1523	0.0014		
		-0.0491	-0.0375	0.1232	0.0895		
		0.0757	0.0898	0.0296	0.0014		
		-0.0693	0.1846	0.0541	0.1460		
$S_{k,l}^{(3,3)}$	$A_{QP_1}^{(3,4)} =$	1	-0.0181	0.1281	-0.0074	-0.0366	
		-0.0499	-0.0390	0.1031	0.0853	-0.1196	
		0.0661	0.0711	0.0251	0.0081	0.0021	
		-0.0735	0.1659	0.0480	-0.1231	-0.0649	
$S_{k,l}^{(4,3)}$	$A_{QP_1}^{(4,4)} =$	1	-0.0146	0.1220	-0.0115	0.0402	
		-0.0388	-0.0286	0.0904	0.0701	-0.1065	
		0.0709	0.0658	0.0204	0.00026	0.0021	
		-0.0659	0.1529	0.0387	-0.1153	-0.0586	
		0.0491	-0.0021	0.0260	-0.0442	-0.0890	

We applied the proposed method on this data field and in order to demonstrate the improvement, we calculated the first-quadrant QP filters and harmonic mean estimates of the PSR $S_{k,l}^{(2,1)}$, $S_{k,l}^{(3,2)}$, $S_{k,l}^{(3,3)}$, and $S_{k,l}^{(4,3)}$. Table 5.43 shows the coefficient matrices of the first-quadrant QP filters of the PSRs. The amplitudes and equal magnitude contours are illustrated in Figure 5.116. All of these QP filters are stable.

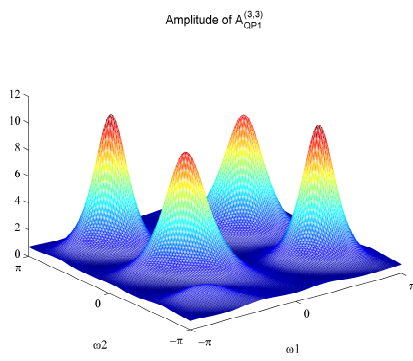
The harmonic mean estimates for the PSR $S_{k,l}^{(2,1)}$, $S_{k,l}^{(3,2)}$, $S_{k,l}^{(3,3)}$, and $S_{k,l}^{(4,3)}$ are shown in Figure 5.117. It is seen that very good results are obtained in the higher orders. Indeed, if we increase the order, for example PSR $S_{k,l}^{(6,4)}$ and $S_{k,l}^{(9,4)}$, we can see the accuracy is getting better as shown in Figure 5.118.



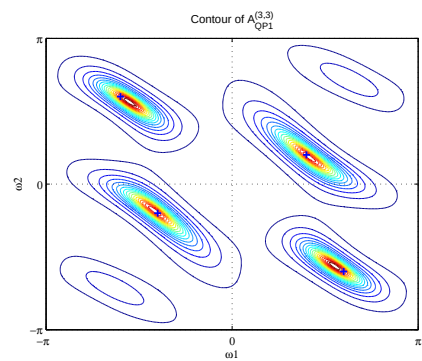
(a)



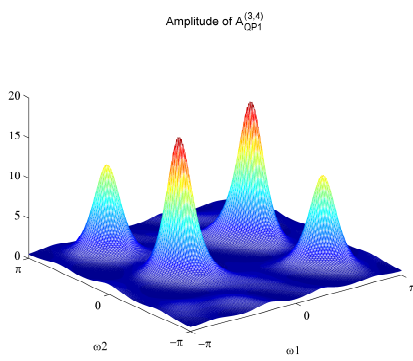
(b)



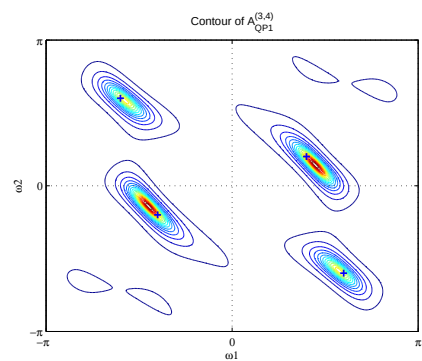
(c)



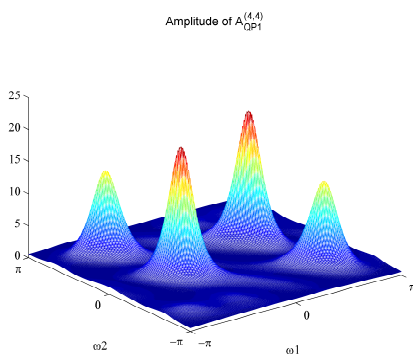
(d)



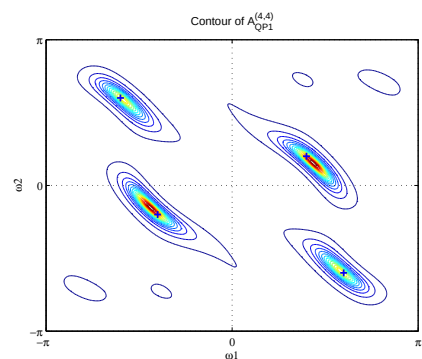
(e)



(f)



(g)



(h)

Figure 5.116: (a, c, e, g) Amplitude; (b, d, f, h) Contour plots of the spectrum of $A_{QP1}^{(2,2)}$, $A_{QP1}^{(3,3)}$, $A_{QP1}^{(3,4)}$, and $A_{QP1}^{(4,4)}$, respectively, in Example 5.

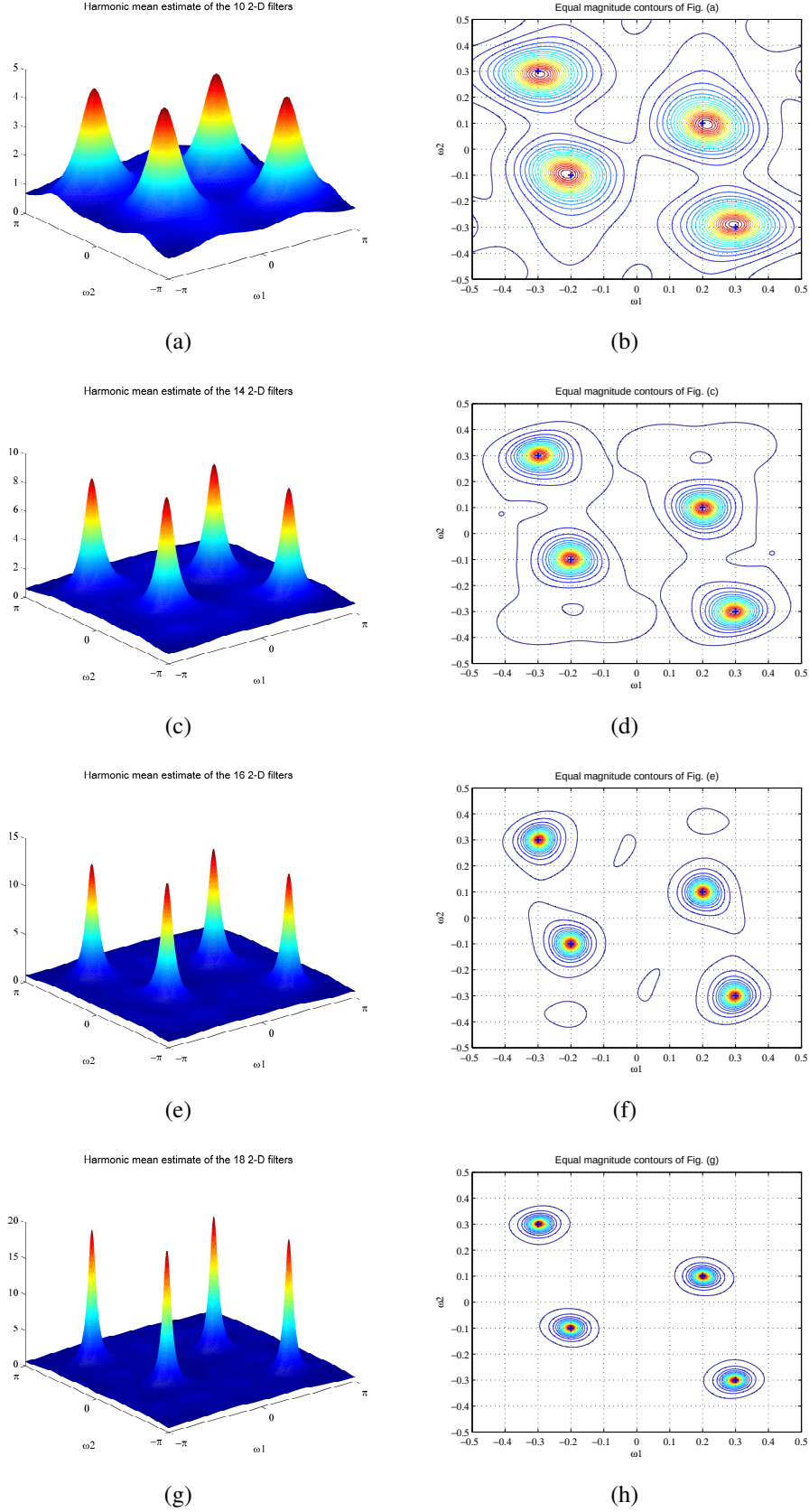


Figure 5.117: Spectral estimates of a data of size 65×65 consisting of two 2-D sinusoids in noise with frequencies $(f_1, f_2) = (-0.3, 0.3), (0.2, 0.1)$: (a, b) , (c, d) , (e, f) , and (g, h) Harmonic mean estimates for the PSR $S_{(k,l)}^{(2,1)}$, $S_{k,l}^{(3,2)}$, $S_{(k,l)}^{(3,3)}$, and $S_{k,l}^{(4,3)}$, respectively, in Example 5.

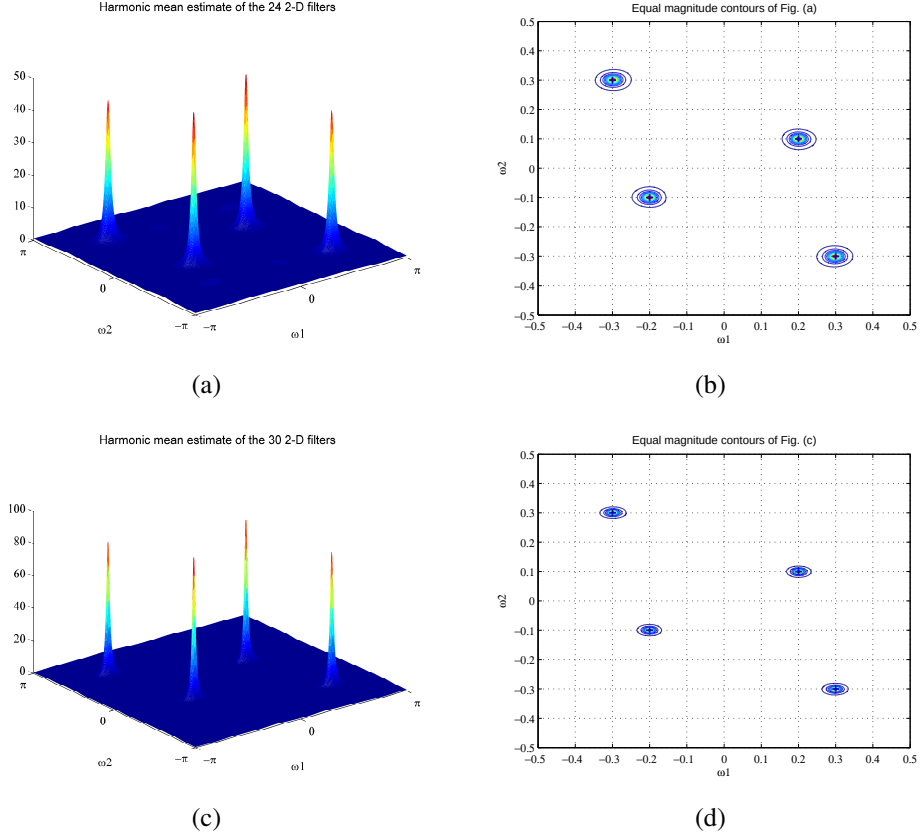


Figure 5.118: (a, c) Amplitude; (b, d) Contour plots of the harmonic mean estimates of the PSR $S_{k,l}^{(6,4)}$, and $S_{k,l}^{(9,4)}$, respectively, in Example 5.

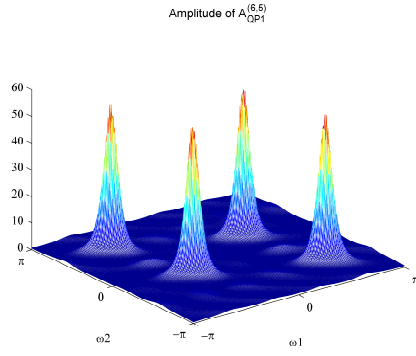
In addition, it is observed that some ASHP filters obtained by using the proposed method give better results than QP filters. Amplitudes and equal magnitude contours of one of QP and ASHP filters are shown in Figure 5.119.

5.5.2 Data field of size 13×13 consisting of two 2-D sinusoids in noise

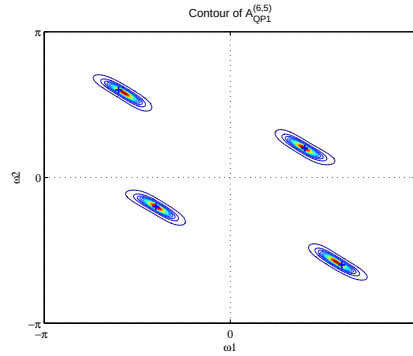
In this simulation example, the amplitudes of all signals are equal to $a_1 = a_2 = 0.75$, an SNR is 0.5115 dB . Here, we will show that the proposed method gives a good result in short data of size 13×13 , as well.

The amplitude of the classical FFT based estimation and equal magnitude contours of the generated data by using the equation (5.44) are illustrated in Figure 5.120.

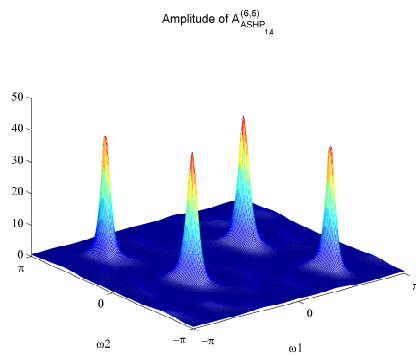
The harmonic mean estimates of the rectangular PSR $S_{k,l}^{(2,1)}$, $S_{k,l}^{(3,2)}$, $S_{k,l}^{(4,3)}$, and $S_{k,l}^{(6,4)}$ are calculated by applying the proposed method. The spectral estimates of them are shown in Figure 5.121. It is seen that very good result is obtained in the higher order although the data size is small. In the short data size cases, the QP, ASHP, and NC half-plane filters obtained in the higher order by using the proposed method may not reflect the



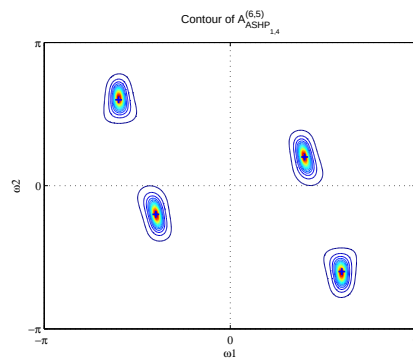
(a)



(b)

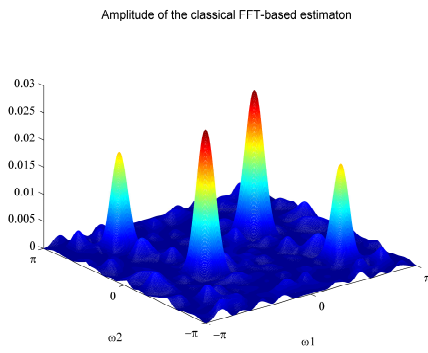


(c)

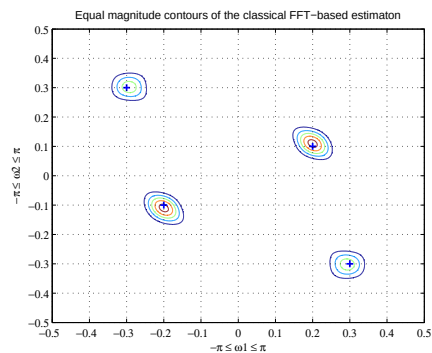


(d)

Figure 5.119: (a,c) Amplitude; (b,d) Contour plots of the QP $A_{QP1}^{(6,5)}$ and the ASHP filters $A_{ASHP_{1,4}}^{(6,5)}$ of the PSR $S_{k,l}^{(6,4)}$, respectively, in Example 5.



(a)



(b)

Figure 5.120: (a) Amplitude; (b) Contour plots of the original spectrum of two sinusoids buried with Gaussian noise $N(\mu = 0, \sigma^2 = 1)$ in Example 5.

correct amplitude and equal magnitude contours when comparing them visually. But they still have the information in and the harmonic mean estimations of them give the accurate results to us.

In order to give a notion, amplitudes and equal magnitude contours of the first-quadrant QP filters of the rectangular PSR $S_{k,l}^{(2,1)}$, $S_{k,l}^{(3,2)}$, $S_{k,l}^{(4,3)}$, and the type-1 ASHP filter at 5th vertical PEF of the rectangular PSR $S_{k,l}^{(6,4)}$ are shown in Figure 5.122.

5.5.3 Data field of size 7×7 consisting of two 2-D sinusoids in noise

In this simulation example, we will increase the SNR by changing the amplitudes of the sinusoids given in the equation (5.44). In case of low data size, the increase of SNR yields a better result. We will generate different data fields of size 7×7 by changing the amplitudes of the sinusoids given in the equation (5.44).

The amplitude of the classical FFT based estimation and its equal magnitude contours of each generated data having the different SNR 0.5115 dB, 3.0103 dB, 6.0206 dB by using the equation (5.44) are illustrated in Figure 5.123, 5.124, and 5.125. The amplitudes of the sinusoids are defined as $a_1 = a_2 = 0.75$, $a_1 = a_2 = 1$, and $a_1 = a_2 = \sqrt{2}$ for the SNR 0.5115 dB, 3.0103 dB, and 6.0206 dB, respectively.

The harmonic mean estimates of the rectangular PSR $S_{k,l}^{(2,1)}$ are calculated by applying the proposed method. The spectral estimates of them are shown in Figure 5.126. As seen that even the SNR near zero, the proposed method gives a good result with respect to the classical FFT based method.

5.5.4 Discussion of example 5

In the first simulation of Example 5, we showed the proposed method found the sinusoidal peaks in the data of big size correctly as Fourier transform did. It is known that Fourier transform can be used to locate the sinusoidal peaks in the data of big size. Figure 5.115 shows the amplitude and equal magnitude contours of the classical 2-D FFT-based estimates and one can easily see the peaks clearly.

We applied the proposed method and generated QP, ASHP and NC model 2-D filters. QP filter started to locate the peaks clearly at (3,3)th order as given in Figure 5.116. In the higher orders, the lobes was getting narrower, and we observed that the ASHP 2-D

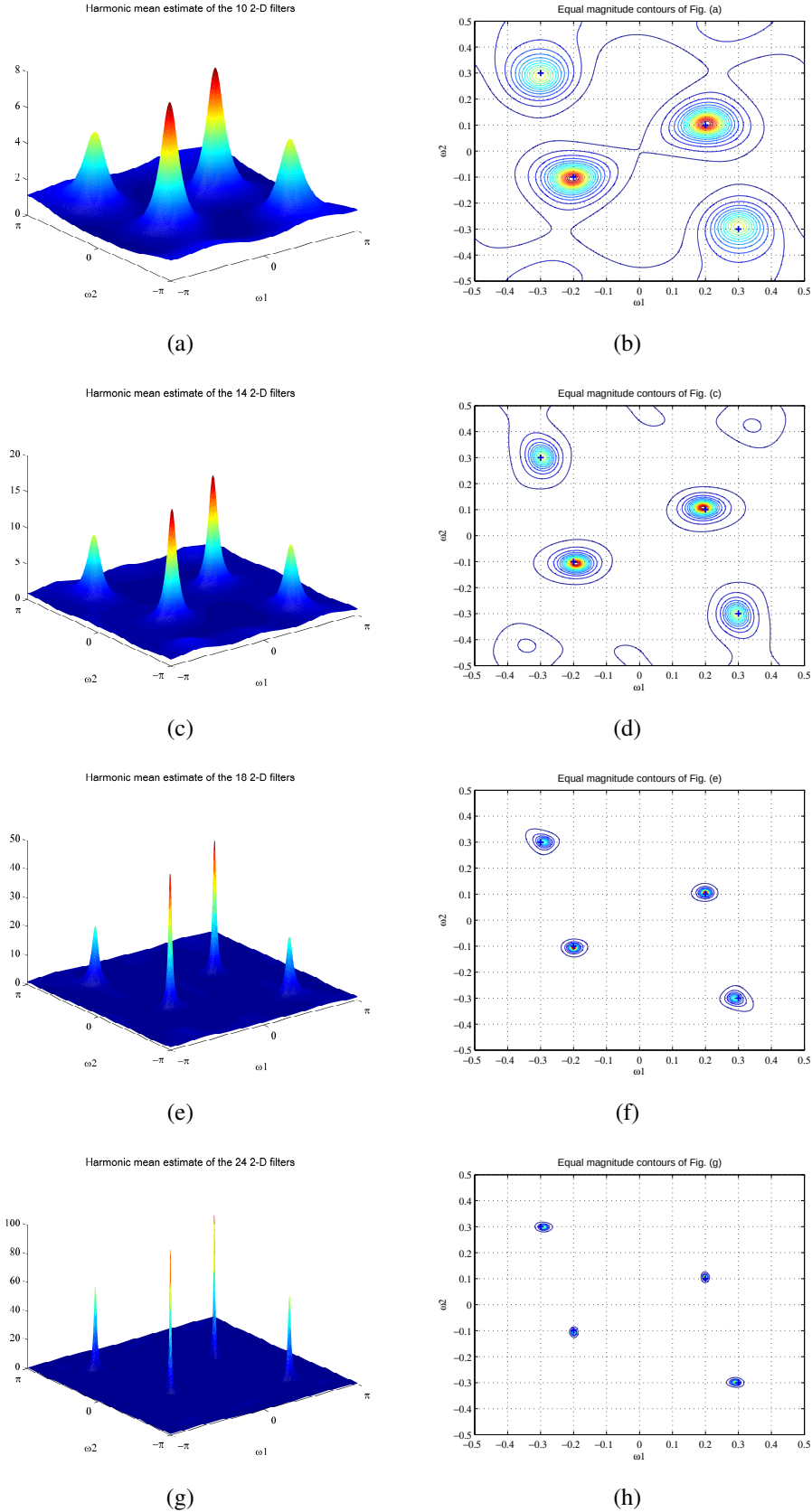


Figure 5.121: Spectral estimates of a data of size 13×13 consisting of two 2-D sinusoids in noise with frequencies $(f_1, f_2) = (-0.3, 0.3), (0.2, 0.1)$: (a,b), (c,d), (e,f), and (g,h) Harmonic mean estimates for the rectangular PSR $S_{k,l}^{(2,1)}$, $S_{k,l}^{(3,2)}$, $S_{k,l}^{(4,3)}$, and $S_{k,l}^{(6,4)}$, respectively, in Example 5.

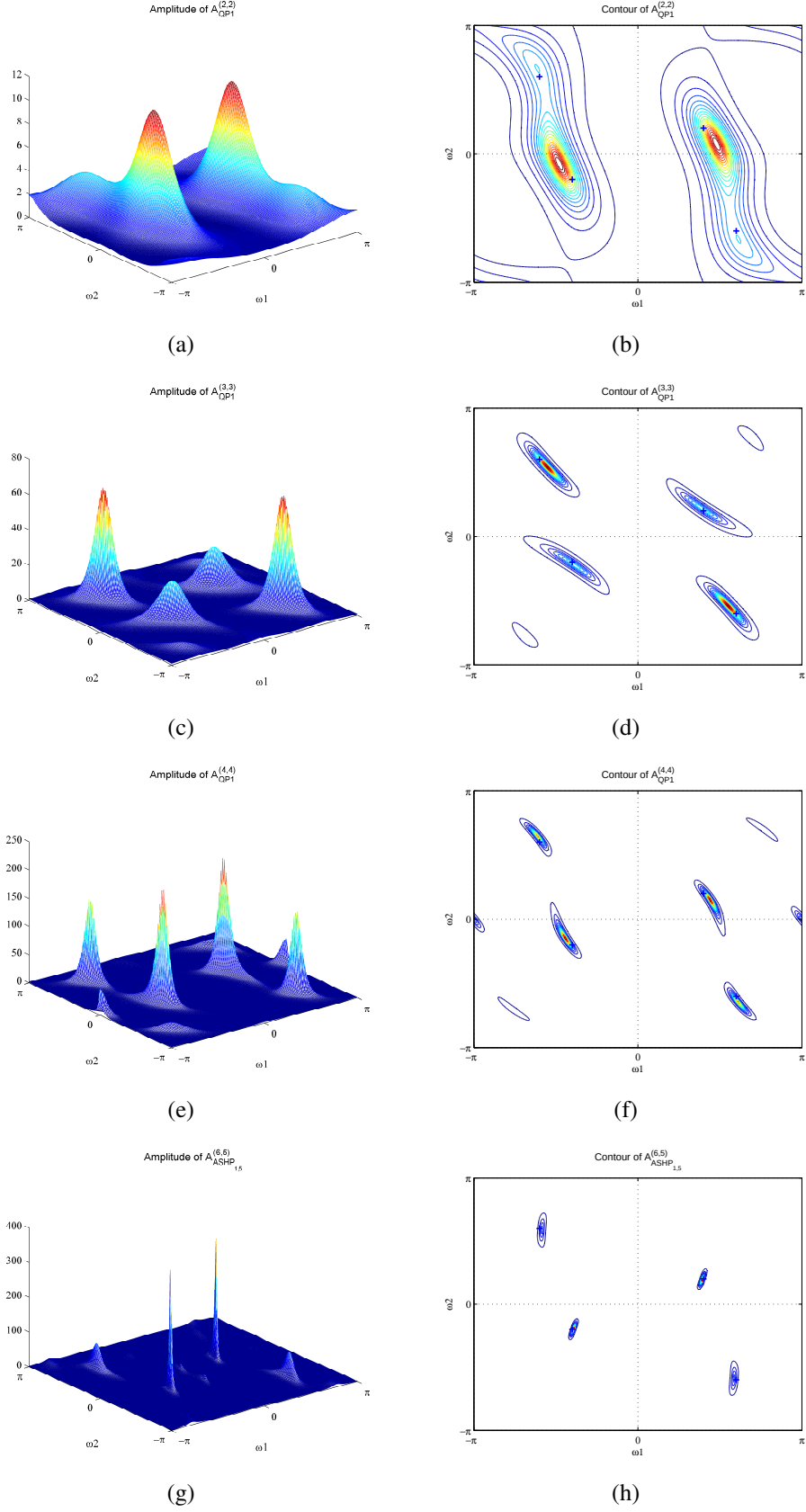


Figure 5.122: (a, c, e, g) Amplitude; (b, d, f, h) Contour plots of the QP filters $A_{QP1}^{(2,2)}$, $A_{QP1}^{(3,3)}$, $A_{QP1}^{(4,4)}$, and type-1 ASHP filter $A_{ASHP_{1,5}}^{(6,5)}$ of the rectangular PSR $S_{k,l}^{(2,1)}$, $S_{k,l}^{(3,2)}$, $S_{k,l}^{(4,3)}$, and $S_{k,l}^{(6,4)}$, respectively, in Example 5.

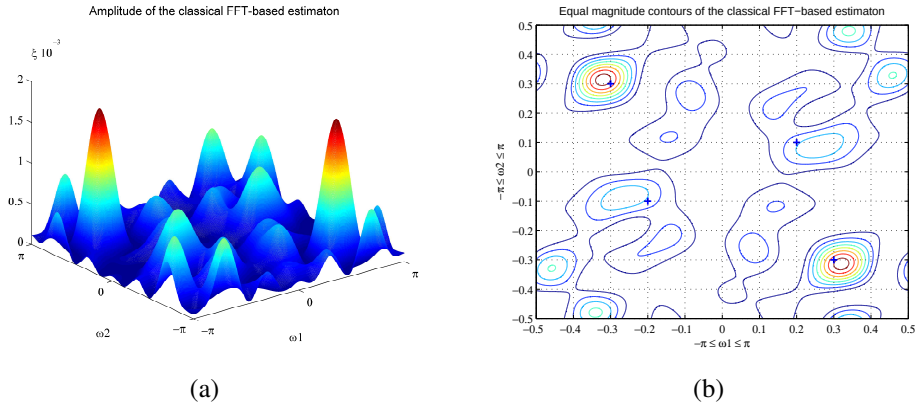


Figure 5.123: (a) Amplitude; (b) Contour plots of the original spectrum of two sinusoids buried with Gaussian noise $N(\mu = 0, \sigma^2 = 1)$ with $SNR = 0.5115$ dB, in Example 5.

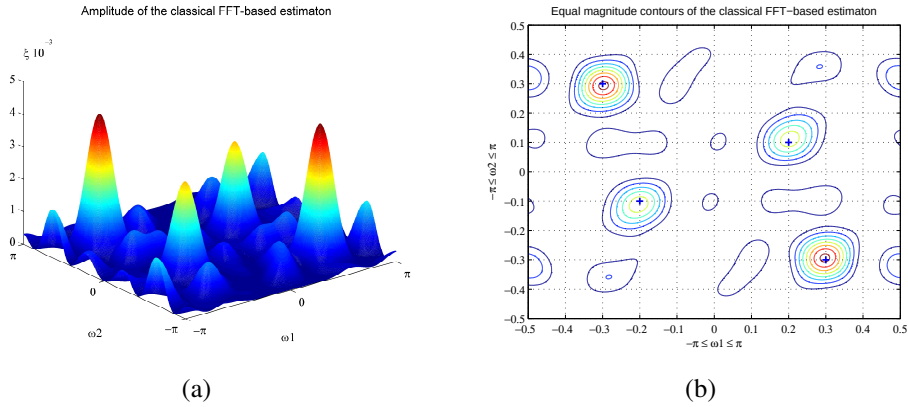


Figure 5.124: (a) Amplitude; (b) Contour plots of the original spectrum of two sinusoids buried with Gaussian noise $N(\mu = 0, \sigma^2 = 1)$ with $SNR = 3.0103$ dB, in Example 5.

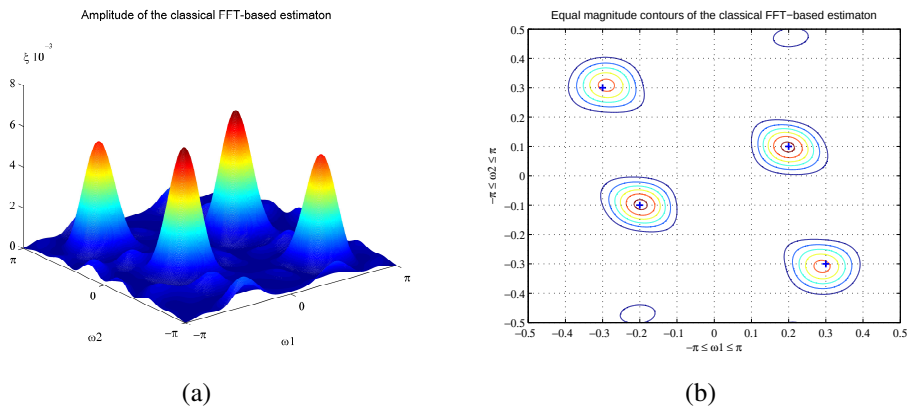


Figure 5.125: (a) Amplitude; (b) Contour plots of the original spectrum of two sinusoids buried with Gaussian noise $N(\mu = 0, \sigma^2 = 1)$ with $SNR = 6.0206$ dB, in Example 5.

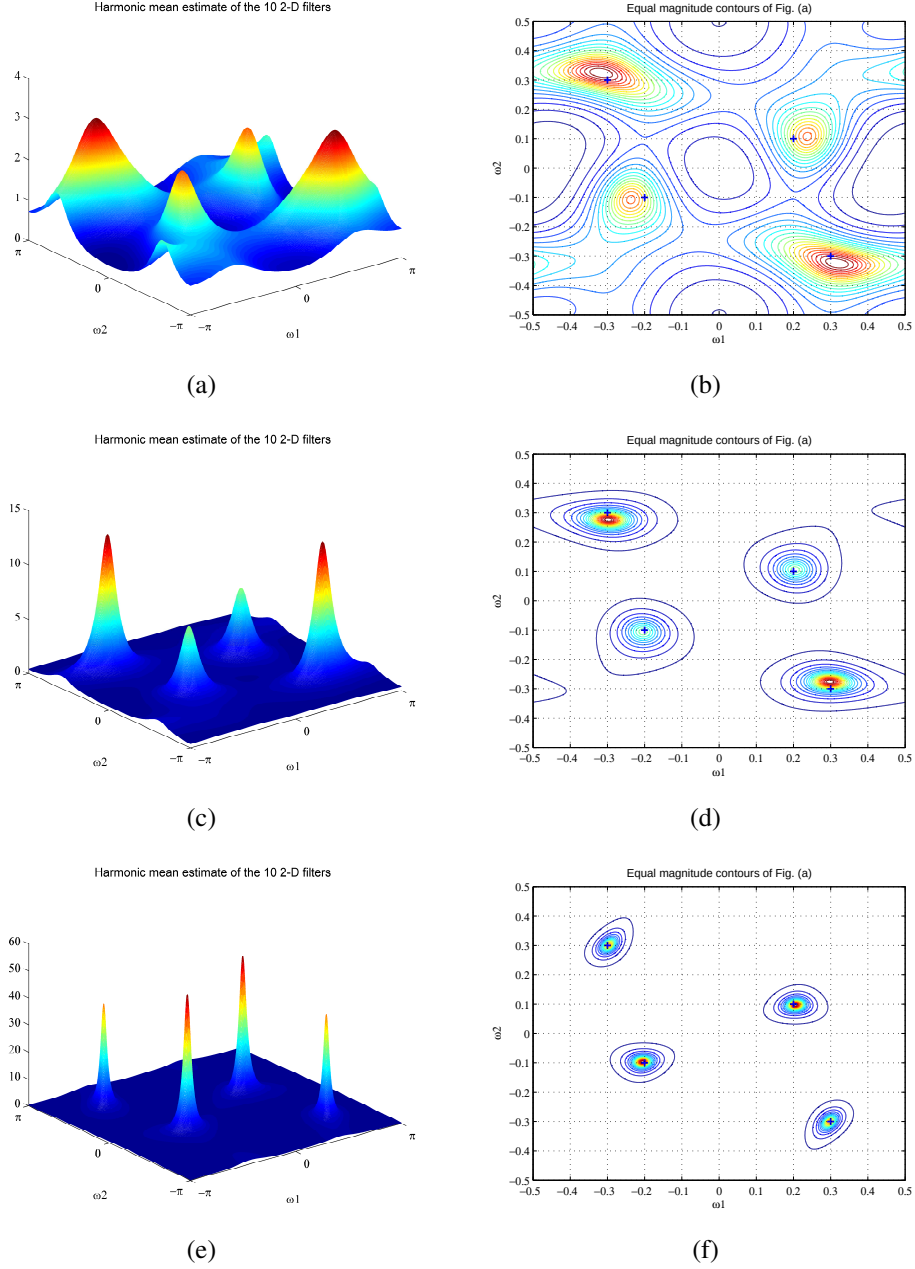


Figure 5.126: Spectral estimates of three different data with the SNR 0.5115 dB, 3.0103 dB, and 6.0206 dB, of size 7×7 consisting of two 2-D sinusoids in noise with frequencies $(f_1, f_2) = (-0.3, 0.3), (0.2, 0.1)$: (a, b), (c, d), and (e, f) Harmonic mean estimates for the rectangular PSR $S_{k,l}^{(2,1)}$, respectively in Example 5.

lattice filter gave better results as demonstrated in Figure 5.119. However, geometric shape of the spectrum was not satisfied. In order to improve the spectrum geometry and estimation accuracy, we combined all possible obtained QP and NC half-plane 2-D filter models by computing their harmonic mean [58]. In Figures 5.117(a) – 5.117(h), we demonstrated the harmonic mean estimates of the PSR $S_{k,l}^{(2,1)}$, $S_{k,l}^{(3,2)}$, $S_{k,l}^{(3,3)}$, and $S_{k,l}^{(4,3)}$, respectively. It was seen that the resolution capacities of the resulting 2-D filters were getting better as long as the order increased. Indeed, at the higher order, we got the result which was very close to Fourier transform. Figure 5.118(a,b)-5.118(c,d) demonstrates the harmonic mean estimates for the PSR $S_{k,l}^{(6,4)}$ and $S_{k,l}^{(9,4)}$. The result of the PSR $S_{k,l}^{(9,4)}$ was the best.

In the second simulation of Example 5, we used the data of size 13×13 with same SNR as the first simulation. Figure 5.120 shows the amplitude and equal magnitude contours of the classical 2-D FFT-based estimates of the 13×13 size data generated. Here, 4 lobes are seen clearly but lobes a bit wider than the ones we demonstrated in the first simulation and there are more side lobes as well. We applied the proposed method and computed the harmonic mean estimates of different order lattice stages shown in Figure 5.121. As the first simulation, at the higher order lattice filters, we got the best results.

In the third simulation of Example 5, we used the data of size 7×7 and we generated data field with the different SNR values. Figures 5.123, 5.124, and 5.125 show the amplitude and equal magnitude contours of the classical FFT-based estimates of the generated data with the SNR 0.5115 dB, 3.0103 dB, and 6.0206 dB, respectively. As seen that the higher SNR uses the better 2-D FFT-based estimates get.

Although the lobes are not seen clearly in 2-D FFT estimates of the data generated with the SNR 0.5115 dB, the proposed method is capable of locating the peaks as shown in Figures 5.126(a)-5.126(b).

FFT-based estimates of the generated data with the SNR 3.0103 dB have more side lobes. Harmonic mean estimates computed for this data give more accurate result without side lobes as illustrated in Figures 5.126(c)-5.126(d).

Moreover, when applying the proposed method and calculating the harmonic mean estimates of the last generated data with the SNR 6.0206 dB, we found better result than the FFT-based estimates as demonstrated in Figures 5.126(a)-5.126(b).

5.6 Example 6: Finding Complex Sinusoidal Peaks

In this example the 2-D data field has been generated as the sum of 2-D complex sinusoids corrupted by additive complex Gaussian noise. It is considered the data file of size 11×11 consisting of two sinusoids with frequencies $(f_1, f_2) = (0.1, 0.25), (0.16, 0.25)$. The complex sinusoids all have equal amplitude $a_1 = a_2 = 0.8$ and the SNR is 1.0721 dB. We will locate these sinusoids by using the proposed method.

Two 2-D complex sinusoids at the first-quadrant buried in 2-D complex Gaussian noise are generated with the following equation

$$y(k, l) = a_1 \exp(-j * (l - 1) f_1(1) * \omega_1) \exp(-j(k - 1) f_2(1) \omega_2) \\ + a_1 \exp(-j(l - 1) f_1(2) \omega_1) \exp(-j(k - 1) f_2(2) \omega_2) + w(k, l) \quad (5.46)$$

where $w(k, l)$ is 2-D complex Gaussian white noise with zero mean and $\sigma_w^2 = 1$.

The amplitude of the classical FFT based estimation and equal magnitude contours of the generated data are illustrated in Figures 5.127(a) and 5.127(b), respectively, and the spectra of the harmonic mean estimates of three different order lattice stages are shown in Figures 5.127(c)-5.127(h).

In Figures 5.127(a) and 5.127(b), the amplitude and equal magnitude contours of the classical 2-D FFT-based estimates, respectively. Only one lobe is clearly seen. The image has poor resolution. This is an expected result with Fourier transform for limited data case. Figures 5.127(c)-5.127(h) show the spectra of the harmonic mean estimates of three different order stages. The lowest order model with 6 observation points in the PSR $S_{k,l}^{(1,2)}$ produces only 2 NC half-plane 2-D lattice filters along with the usual 8 QP filters. At this order, the 2-D lattice filters fail to estimate the locations of the sinusoids as illustrated in Figures 5.127(c) and 5.127(d). When the order of the 2-D auxiliary lattice filters is increased, we can obtain more NC half-plane models for the computation of the harmonic mean spectral estimation. In addition, the order increase will enhance the resolution capacities of lattice filters as seen in the Example 5. Indeed,

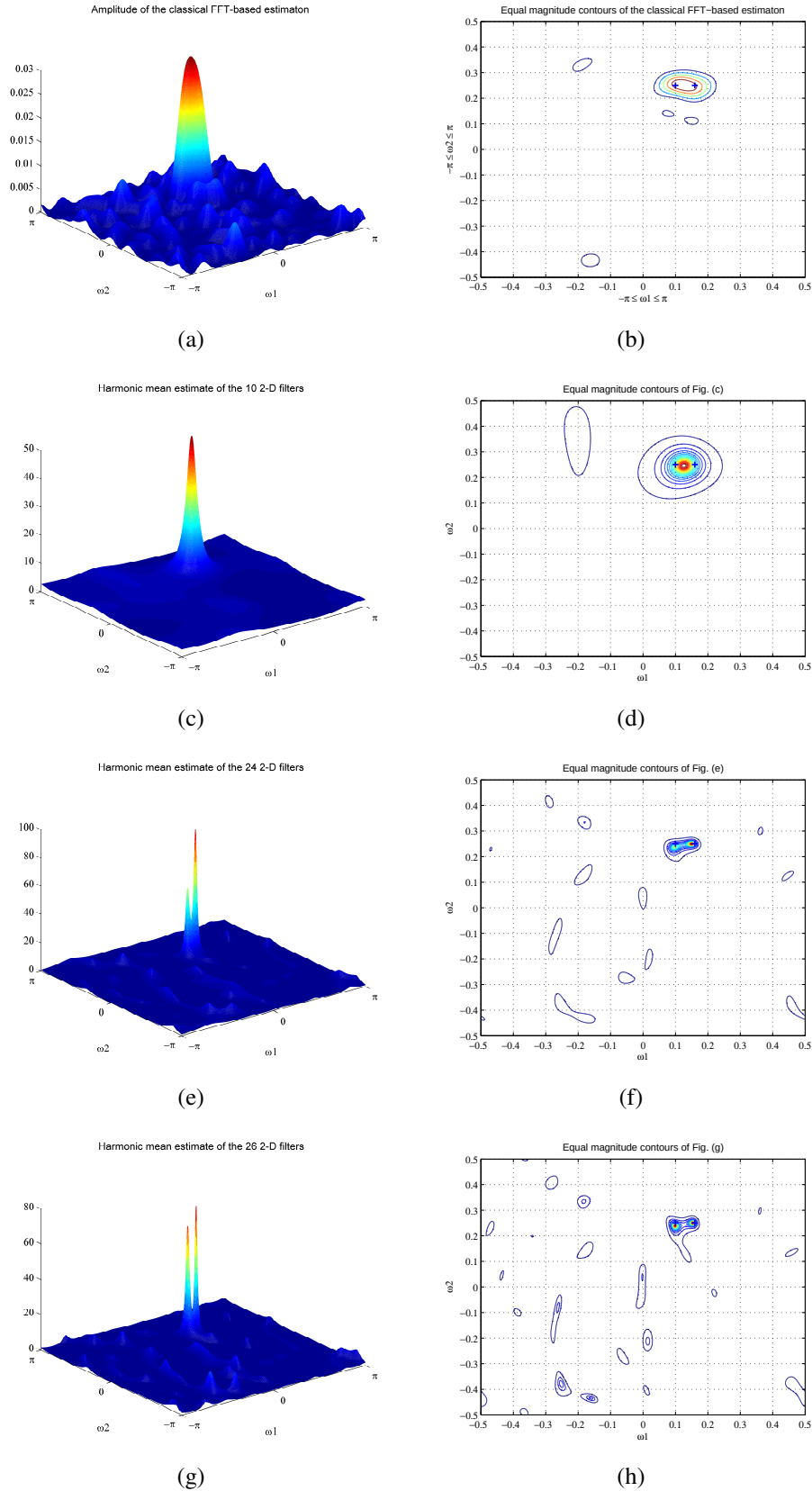


Figure 5.127: Spectral estimates of a data of size 11×11 consisting of two 2-D complex sinusoids in noise with frequencies $(f_1, f_2) = (0.1, 0.25), (0.16, 0.25)$: (a, b) Classical FFT-based estimate; (c, d), (e, f), and (g, h) Harmonic mean estimates for the rectangular PSR $S_{k,l}^{(1,2)}$, $S_{k,l}^{(4,6)}$, and $S_{k,l}^{(4,7)}$, respectively, in Example 6.

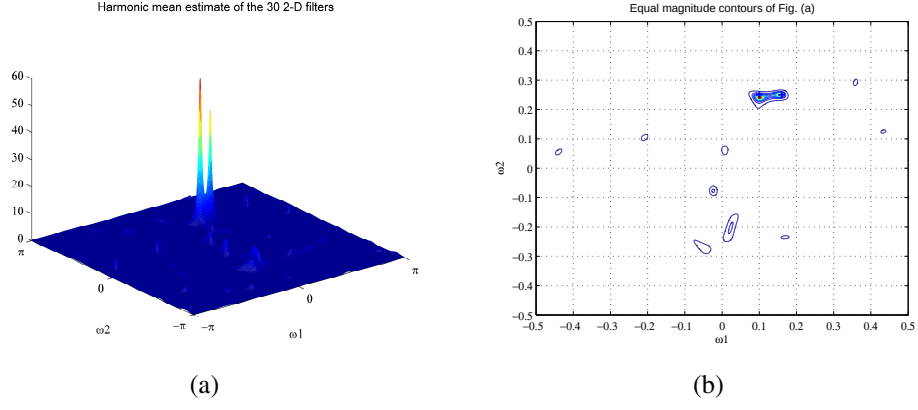


Figure 5.128: Spectral estimates of a data of size 11×11 consisting of two 2-D complex sinusoids in noise with frequencies $(f_1, f_2) = (0.1, 0.25), (0.16, 0.25)$: (a, b) Harmonic mean estimates for the rectangular PSR $S_{k,l}^{(6,7)}$, in Example 6.

the higher order 2-D lattice filters with the PSR $S_{k,l}^{(4,6)}$ having 35 points can be used to construct 16 NC model 2-D filters. The spectra of the harmonic mean estimate for $S_{k,l}^{(4,6)}$ is shown in Figures 5.127(e) and 5.127(f). It is seen that the one lobe is starting to bifurcate and the frequency locations of the sinusoids are coming into focus. We know that if we carry on increasing the number of lattice stages, the resolution capacities of the resulting 2-D lattice filters will be improved in the parallel manner. For this example, the spectrum of the harmonic mean estimate of the PSR $S_{k,l}^{(4,7)}$ gives a good result and two complex sinusoids are clearly recovered as shown in Figures 5.127(g) and 5.127(h). However, the spectrum of the harmonic mean estimate for the PSR $S_{k,l}^{(6,7)}$ leads the best result and two complex sinusoids are also clearly seen and the side lobes diminished as shown in Figures 5.128(a) and 5.128(b).

5.7 Example 7: Finding Complex Sinusoidal Peaks

In this example, the 2-D data field has been generated as the sum of 2-D complex sinusoids corrupted by additive complex Gaussian noise as it has been done in Example 6. It is considered the data file of size 13×13 consisting of eight sinusoids with frequencies $(f_1, f_2) = (0.22, 0.26), (0.28, 0.20), (-0.25, 0.10), (0.27, 0.16), (0.19, -0.39), (0.14, -0.32), (0.09, -0.22),$ and $(0.03, -0.16)$. The complex sinusoids all have equal amplitude $a_i = 0.8$ for $i = 1, \dots, 8$ and the SNR is 7.00927 dB . These sinusoids will be located by using the proposed method.

Eight 2-D complex sinusoids at the first-quadrant buried in 2-D complex Gaussian noise are generated with the following equation

$$y(k, l) = \sum_{i=1}^8 a_i \exp(-j(l-1)f_1(i)\omega_1) \exp(-j(k-1)f_2(i)\omega_2) + w(k, l) \quad (5.47)$$

where $w(k, l)$ is 2-D complex Gaussian white noise with zero mean and $\sigma_w^2 = 1$.

The amplitude of the classical FFT based estimation and equal magnitude contours of the generated data are illustrated in Figures 5.129(a) and 5.129(b), respectively, and the spectra of the harmonic mean estimate of three different order lattice stages are shown in Figures 5.129(c)-5.129(h).

The amplitude and equal magnitude contours of the classical 2-D FFT-based estimates are shown in Figures 5.129(a) and 5.129(b), respectively. The FFT spectrum has poor resolution and has only nine lobes. Figures 5.129(c)-5.129(h) and 5.129(a)-5.129(f) show the spectra of the harmonic mean estimates of six different order stages. At the lowest order model with 8 observation points in the PSR $S_{k,l}^{(3,1)}$, the 2-D lattice filters fail to estimate the locations of the sinusoids as illustrated in Figures 5.129(c) and 5.129(d). When the order of the 2-D lattice filters is increased, the lattice filters start to estimate the locations of the sinusoids. As it is seen in Figures 5.129(e) and 5.129(f), the spectral peaks in the second-quadrant are getting estimated but rest of the spectral peaks in the first- and fourth-quadrant are not resolved yet. Just one unit increase of the number of lattice stage, we get better solution as seen in Figures 5.129(g) and 5.129(h). More or less, eight spectral peaks are resolved. If we carry on increasing the number of lattice stages, the resolution capacities of the resulting 2-D lattice filters will be

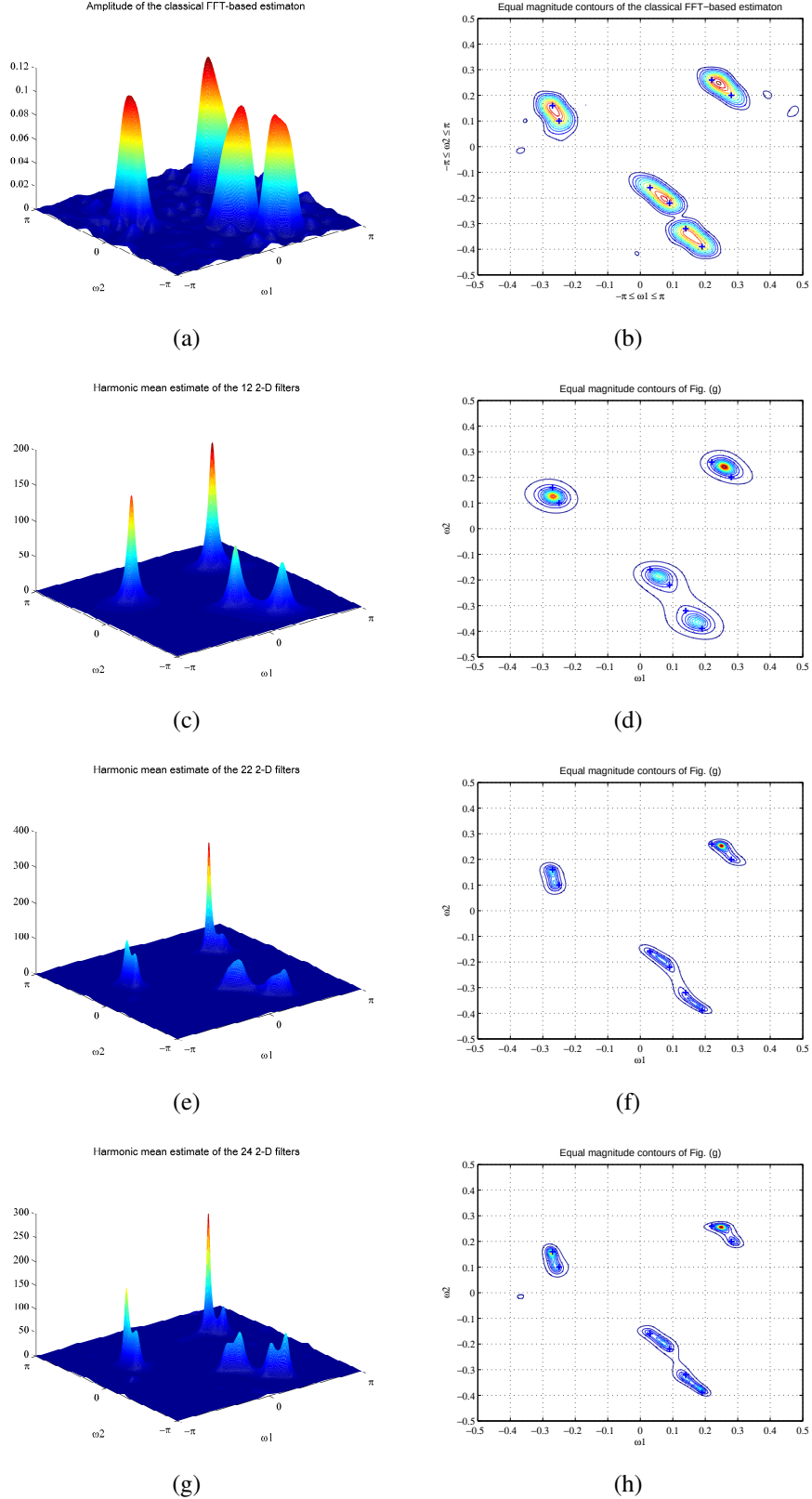
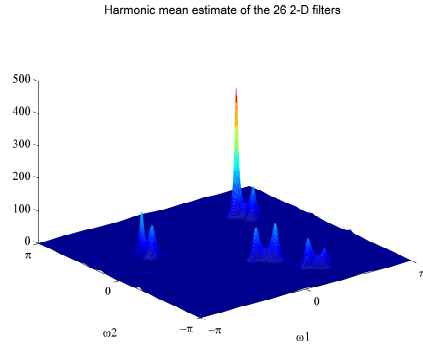


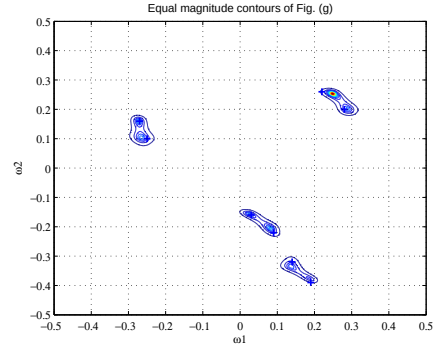
Figure 5.129: Spectral estimates of a data of size 13×13 consisting of eight 2-D complex sinusoids in noise with frequencies $(f_1, f_2) = (0.22, 0.26), (0.28, 0.20), (-0.25, 0.10), (0.27, 0.16), (0.19, -0.39), (0.14, -0.32), (0.09, -0.22),$ and $(0.03, -0.16)$: (a, b) Classical FFT-based estimate; (c, d), (e, f), and (g, h) Harmonic mean estimates for the rectangular PSR $S_{k,l}^{(3,1)}$, $S_{k,l}^{(6,3)}$, and $S_{k,l}^{(6,4)}$, respectively, in Example 7.

improved in the parallel manner as seen in Figures 5.130(a)-5.130(f). The spectra of the harmonic mean estimate for the PSR $S_{k,l}^{(9,4)}$ yields the best result and eight complex sinusoids are clearly recovered as shown in Figures 5.130(e) and 5.130(f).

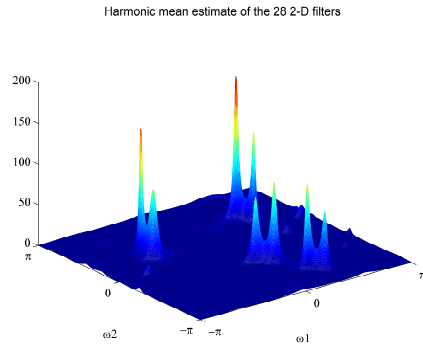
Consequently, although FFT spectra has poor resolution, the frequency spectrum of the harmonic mean of our 2-D lattice filters estimates all eight spectral peaks at their actual locations with high accuracy as displayed in Figures 5.130(e) and 5.130(f). The proposed auxiliary forward/backward prediction error filters have 50 observation points in the PSR $S_{k,l}^{(9,4)}$ and we have employed 8 QP and 22 noncausal half-plane 2-D lattice filters for the harmonic mean estimate.



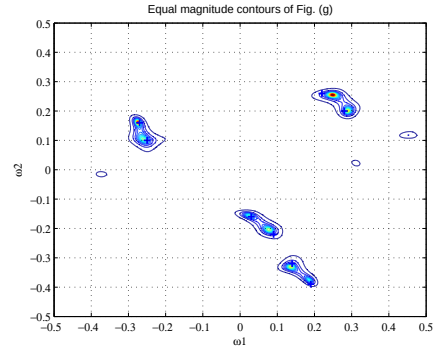
(a)



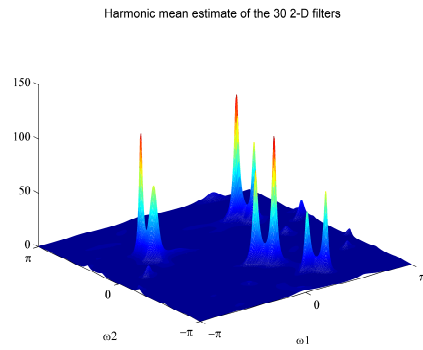
(b)



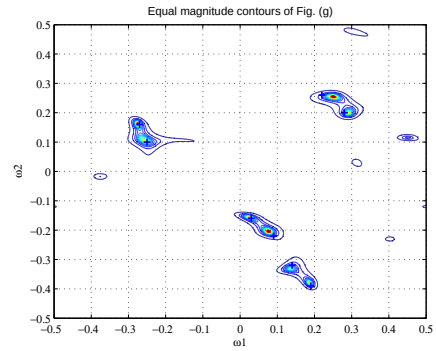
(c)



(d)



(e)



(f)

Figure 5.130: Continued of Figure 5.129: (a,b) , (c,d) , and (e,f) Harmonic mean estimates for the rectangular PSR $S_{k,l}^{(8,3)}$, $S_{k,l}^{(8,4)}$, and $S_{k,l}^{(9,4)}$ in Example 7.

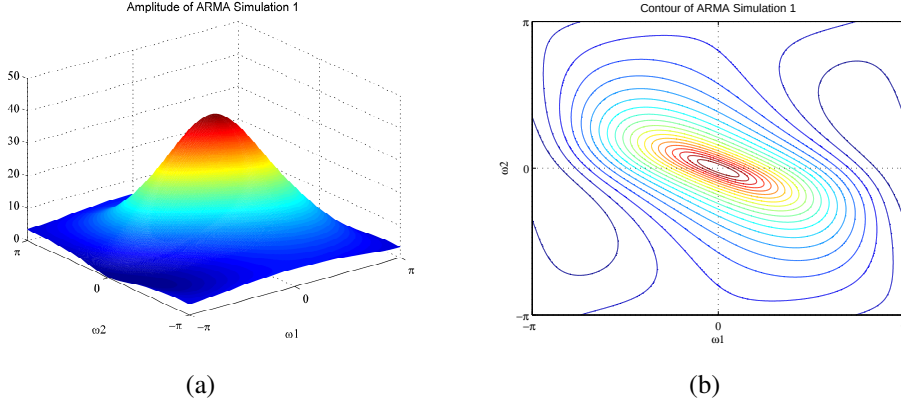


Figure 5.131: (a) Amplitude; (b) Contour plots of the original spectrum of the first ARMA simulation in Example 8.

5.8 Example 8: ARMA 2-D Models Simulations

In this example, three different ARMA 2-D models will be considered. In order to model 2-D ARMA data field by using the proposed 2-D AR lattice structures, the 2-D ARMA data will be generated and the proposed method will be applied on it. The ARMA spectrum of each generated 2-D data will be compared with the ones obtained by using the proposed method. The ARMA model is defined as,

$$H(z_1, z_2) = \frac{A(z_1, z_2)}{B(z_1, z_2)}. \quad (5.48)$$

5.8.1 ARMA 2-D models simulation 1

In this simulation a stable ARMA model generated by using the following coefficient matrices of $A(z_1, z_2)$ and $B(z_1, z_2)$ defined as

$$A^{(1,1)} = \begin{bmatrix} 1 & 0.6 \\ -0.25 & 0.3 \end{bmatrix} \quad (5.49)$$

and

$$B^{(1,1)} = \begin{bmatrix} 1 & -0.1 \\ -0.5 & -0.25 \end{bmatrix}, \quad (5.50)$$

respectively.

The amplitudes and equal magnitude contours are given in Figures 5.131(a)-5.131(b).

After applied the proposed method for the rectangular PSR $S_{k,l}^{(1,0)}$, the amplitudes and equal magnitude contours of the obtained vertical QP filters are shown in Figures 5.132(a)-5.132(b).

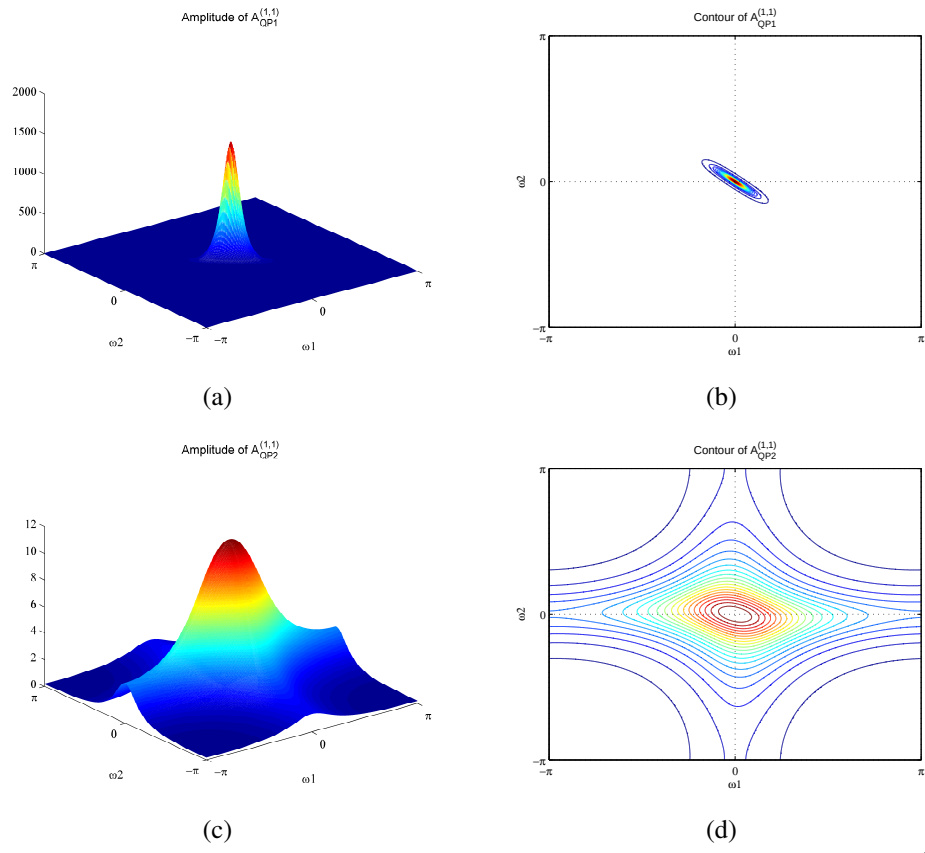


Figure 5.132: (a, c) Amplitude; (b, d) Contour plots of the spectrum of the $A_{QP1}^{(1,1)}$ and $A_{QP2}^{(1,1)}$, respectively, in the first simulation of Example 8.

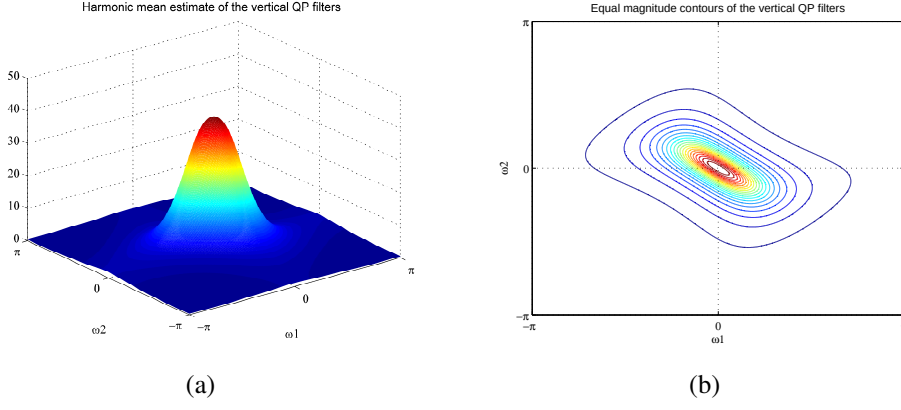


Figure 5.133: (a) Amplitude; (b) Contour plots of the Harmonic mean estimates of the QP filters $A_{QP1}^{(1,1)}$ and $A_{QP2}^{(1,1)}$ obtained from the vertical PEFs, in the first simulation of Example 8.

The coefficient matrices of the vertical QP filters, $A_{QP1}^{(1,1)}$ and $A_{QP2}^{(1,1)}$ are computed as

$$A_{QP1}^{(1,1)} = \begin{bmatrix} 1 & -0.3205 \\ -0.4279 & -0.2259 \end{bmatrix}, \quad (5.51)$$

and

$$A_{QP2}^{(1,1)} = \begin{bmatrix} -0.5734 & 0.2790 \\ 1 & -0.4226 \end{bmatrix}, \quad (5.52)$$

respectively.

The appearances of the QP filters obtained from the vertical PEFs are not exactly the similar to the original one but if we calculate the harmonic mean of these first- and second-quadrant QP filters, we obtain very sufficient results. The location of the peak and its contours are very similar to the original one as demonstrated in Figures 5.133(a)-5.133(b).

5.8.2 ARMA 2-D models simulation 2

In this simulation another stable ARMA model generated by using the following coefficient matrices of $A(z_1, z_2)$ and $B(z_1, z_2)$ defined as

$$A^{(2,2)} = \begin{bmatrix} 1 & 0.15 & 0.09 \\ 0.1 & -0.1 & 0.05 \\ -0.05 & 0.13 & 0.075 \end{bmatrix} \quad (5.53)$$

and

$$B^{(2,2)} = \begin{bmatrix} 1 & 0.2 & 0.23 \\ 0.15 & 0.18 & 0.16 \\ 0.17 & 0.24 & 0.21 \end{bmatrix}, \quad (5.54)$$

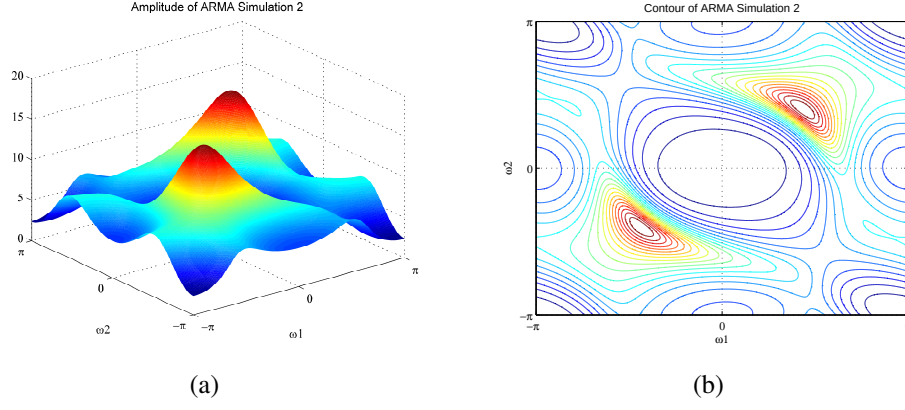


Figure 5.134: (a) Amplitude; (b) Contour plots of the original spectrum of the second ARMA simulation in Example 8.

respectively.

The amplitudes and equal magnitude contours are given in Figures 5.134(a)-5.134(b).

After applied the proposed method for the rectangular PSR $S_{k,l}^{(4,2)}$, the amplitudes and equal magnitude contours of the obtained first-quadrant QP filters obtained from vertical and horizontal PEFs are shown in Figures 5.135(a)-5.135(b).

The coefficient matrices of the QP filters, $A_{QP1}^{(4,3)}$ and $B_{QP1}^{(5,2)}$ are computed as

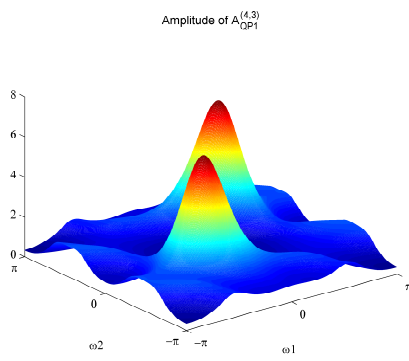
$$A_{QP1}^{(4,3)} = \begin{bmatrix} 1 & 0.0539 & 0.1167 & -0.0030 \\ 0.0654 & 0.2819 & 0.0725 & 0.0062 \\ 0.2127 & 0.0759 & 0.1469 & -0.0757 \\ 0.0043 & 0.0019 & -0.0485 & 0.0112 \\ 0.0179 & -0.0380 & 0.0078 & -0.0373 \end{bmatrix}, \quad (5.55)$$

and

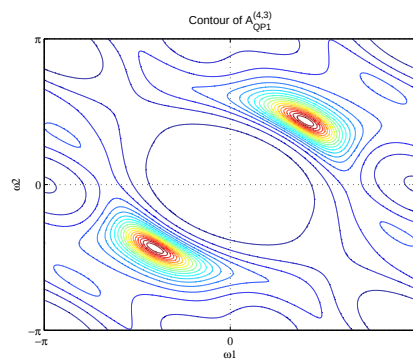
$$B_{QP1}^{(5,2)} = \begin{bmatrix} 1 & 0.0597 & 0.1170 \\ 0.0678 & 0.2818 & 0.0895 \\ 0.2146 & 0.0880 & 0.1511 \\ 0.0033 & 0.0029 & -0.0367 \\ 0.0198 & -0.0314 & 0.0128 \\ -0.0093 & -0.0047 & 0.0097 \end{bmatrix}, \quad (5.56)$$

respectively.

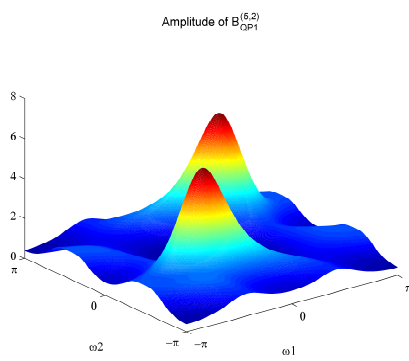
The appearances of the first-quadrant filters are very close to the original one. The location of the peaks and their contours are quite similar.



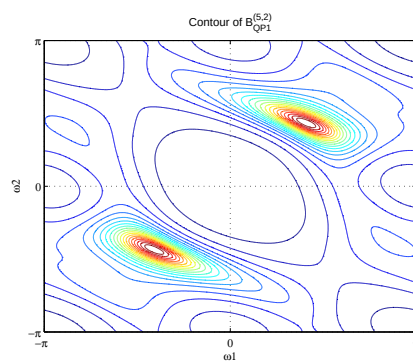
(a)



(b)



(c)



(d)

Figure 5.135: (a, c) Amplitude; (b, d) Contour plots of the spectrum of the $A_{QP1}^{(4,3)}$ and $B_{QP1}^{(5,2)}$, respectively, in the second simulation of Example 8.

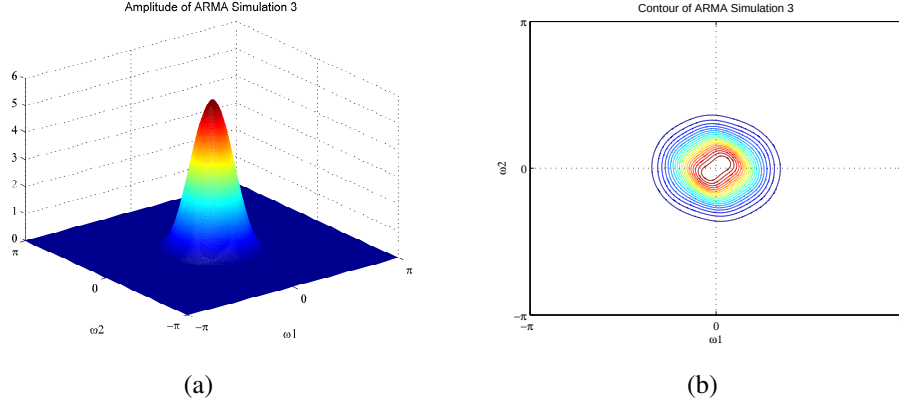


Figure 5.136: (a) Amplitude; (b) Contour plots of the original spectrum of the third ARMA simulation in Example 8.

5.8.3 ARMA 2-D models simulation 3

In this simulation, we tried to model a stable circular symmetric filter generated by using the following coefficient matrices of $A(z_1, z_2)$ and $B(z_1, z_2)$ defined as

$$A^{(2,2)} = \begin{bmatrix} 1 & -1.494467 & 0.986784 & -0.330467 & 0.041826 \\ -1.239595 & 1.475852 & -0.703064 & 0.136813 & 0 \\ 0.652381 & -0.537711 & 0.131311 & 0 & 0 \\ -0.172892 & 0.075504 & 0 & 0 & 0 \\ 0.018507 & 0 & 0 & 0 & 0 \end{bmatrix} \quad (5.57)$$

and

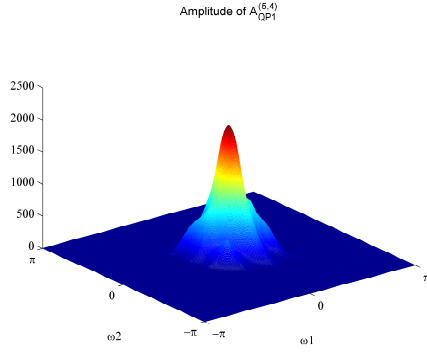
$$B^{(2,2)} = \begin{bmatrix} 1 & 2.851480 & 4.708566 & 0.020212 & -0.886100 \\ 3.963252 & 3.987310 & -4.625246 & 4.268581 & 0 \\ 5.992568 & -6.055965 & 12.42206 & 0 & 0 \\ 2.721233 & 4.374989 & 0 & 0 & 0 \\ -0.995091 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad (5.58)$$

respectively.

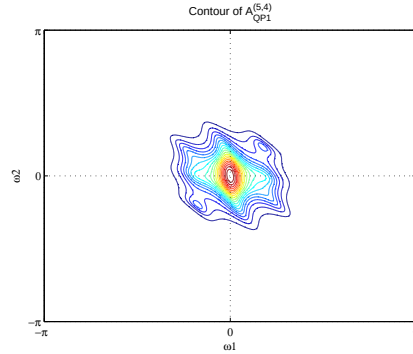
The amplitudes and equal magnitude contours are given in Figures 5.136(a)-5.134(b).

After applied the proposed method for the rectangular PSR $S_{k,l}^{(5,3)}$, the amplitudes and equal magnitude contours of the QP filters obtained from vertical and horizontal PEFs are shown in Figures 5.137(a)-5.137(h).

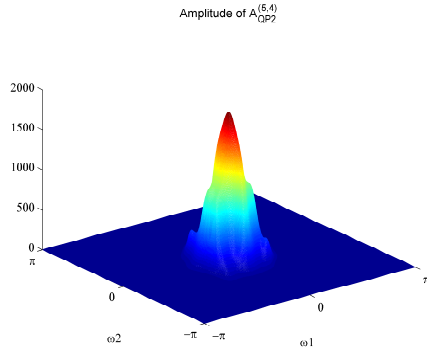
The appearances of the QP filters are not close to the original one as seen in Figures 5.137(a)-5.137(h). However, if we calculate the harmonic mean of these first- and second-quadrant QP filters, the results are getting good. The location of the peak



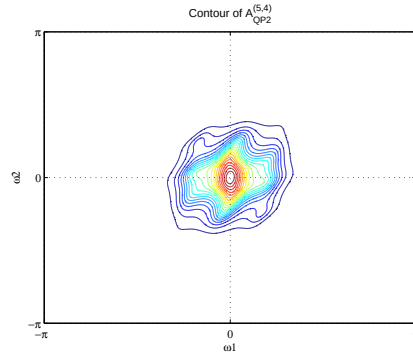
(a)



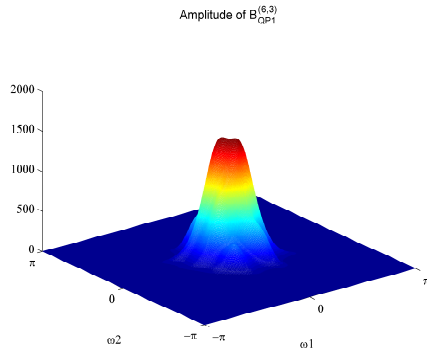
(b)



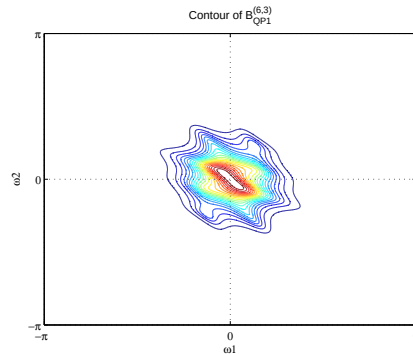
(c)



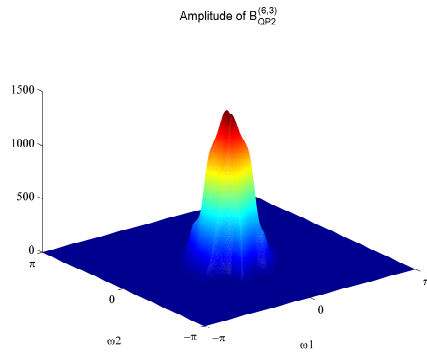
(d)



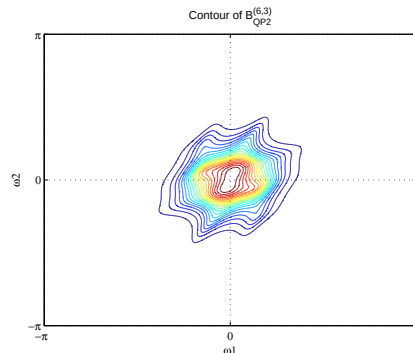
(e)



(f)



(g)



(h)

Figure 5.137: (a, c, e, g) Amplitude; (b, d, f, h) Contour plots of the spectrum of the QP filters $A_{QP1}^{(5,4)}$, $A_{QP2}^{(5,4)}$, $B_{QP1}^{(6,3)}$, and $B_{QP2}^{(6,3)}$, respectively, in the third simulation of Example 8.

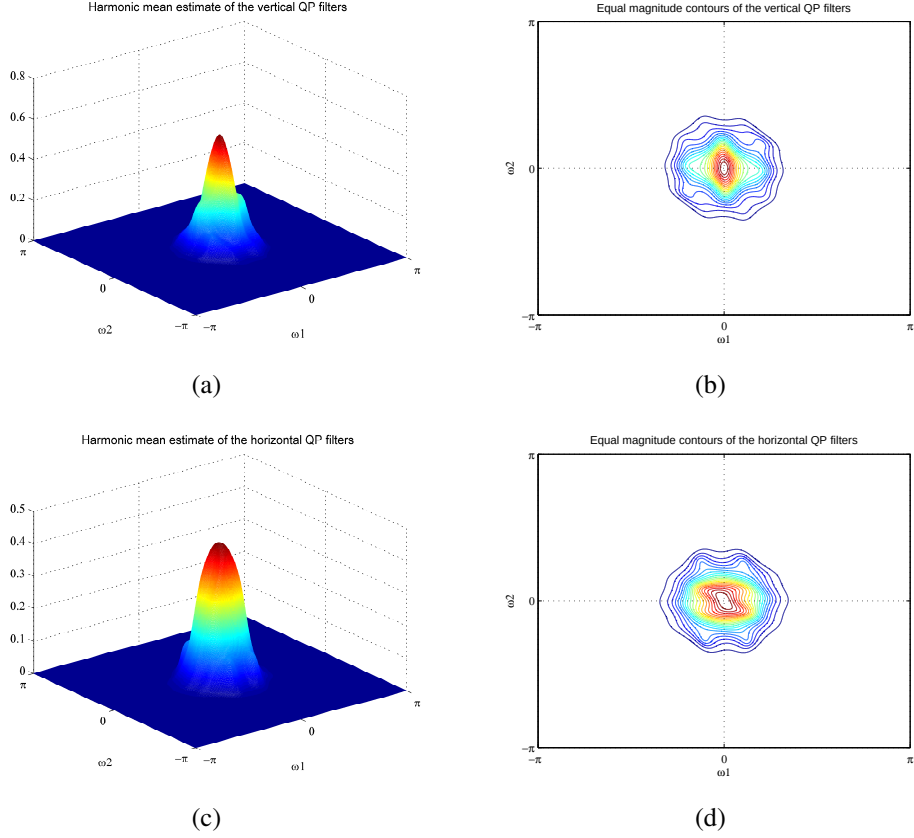


Figure 5.138: (a,c) Amplitude; (b,d) Contour plots of the Harmonic mean estimates of the QP filters obtained from the vertical and horizontal PEFs, respectively, in the third simulation of Example 8.

and its contours are quite similar to the original one as demonstrated in Figures 5.138(a)-5.138(d).

5.8.4 Discussion of example 8

Here, we produced the 2-D data using an ARMA 2-D model and applied the proposed method. In other words, we tried to model this generated 2-D data by using a 2-D AR lattice model.

It is seen that at some stage orders, we are able to obtain very similar results. Moreover, by getting harmonic mean of the obtained filters we are also able to get quite similar results. These results lead that it is possible to model a 2-D ARMA model by using a 2-D AR model. It can be done either to get only any obtained 2-D filter, for example to get one of QP filter, or any combination of the obtained 2-D filters, for example to use first- and second-quadrant QP filters by cascading.

It is also possible to design 2-D IIR filter by using the proposed method. To model 2-D ARMA by 2-D AR and 2-D IIR filter design issues will be elaborated in a different study since there is need to do some further investigations on this subject.

5.9 Discussion of Examples

Examples from 1 to 4 show that coefficient matrix can be very closely estimated and high resolution characteristic of the model in the frequency domain are obtained. In addition, the all QP model filters are found stable. Example 1 also shows augmentation direction is not important to reach the desired stage order PSR.

In the Example 5, the proposed method is applied to find the sinusoids buried in white noise. Low SNR and small data size give the good results

In the Example 6 and 7, the proposed method is applied to find the peaks in the 2-D data field generated as the sum of 2-D complex sinusoids corrupted by additive complex Gaussian noise. The results show that for short data records and low SNR, the new method which depends on harmonic mean estimate has a better performance than the classical FFT-base estimate.

In Example 8, we have seen that we are able to obtain very similar spectrum results if we applied the proposed method on a 2-D ARMA model. The results lead that it can be possible to model a 2-D ARMA filter by using the proposed 2-D AR structure by using either one of QP filter or any combination of the obtained filters.

6. GENERAL ORTHOGONAL 2-D BASES VECTORS AND ENTROPY RELATIONS OF 2-D LATTICE FILTERS

In the first part of this chapter, the orthogonality feature of the proposed 2-D lattice filter structure will be analyzed and then in the second part of the chapter, the information loss between the input vector and the backward prediction error vector constituted from the 2-D lattice filter is investigated by extending the method given in [34].

6.1 General Orthogonal 2-D Bases Vectors

In this section, we will show that the information content of the PSR in Figure 2.10 can be equivalently represented in terms of the orthogonal 2-D bases realization vectors by using the backward auxiliary vertical and horizontal PEFs. In Section 2.3, it is explained that the PSR $S_{k,l}^{(p,q)}$ can be obtained by augmenting the PSR $S_{k,l}^{(p-1,q-1)}$ first horizontally and then vertically as shown in Figure 2.7. Therefore, the rectangular shaped PSR $S_{k,l}^{(p-1,q-1)}$, for $p < q$ can be decomposed into shifted vertical and horizontal 1-D subspaces depending upon the ordering arrangements as illustrated in Figure 6.1. It can be expressed as

$$S_{k,l}^{(p,q)} = S_{k,l}^{(0,0)} + \sum_{i=1}^p \left(S_{k-i,l}^{(0,i-1)} + S_{k,l-i}^{(i,0)} \right) + \sum_{j=p+1}^q S_{k,l-i}^{(p,0)}. \quad (6.1)$$

The proposed general 2-D AR lattice structure generates the mutually orthogonal realization subspaces spanned by vertical and horizontal auxiliary backward PEFs,

$$\tilde{E}_{k,l}^{(m,n)} = \text{Span} \left\{ \tilde{e}_j^{(m,n)}(k,l); 1 \leq j \leq n \right\}, \quad (6.2)$$

and

$$\tilde{R}_{k,l}^{(m,n)} = \text{Span} \left\{ \tilde{r}_i^{(m,n)}(k,l); 1 \leq i \leq m \right\}, \quad (6.3)$$

respectively. This yields

$$S_{k,l}^{(p,q)} = \tilde{E}_{k-1,l-1}^{(0,0)} \oplus \sum_{i=1}^p \left(\tilde{R}_{k-1,l-1}^{(i,i-1)} \oplus \tilde{E}_{k-1,l-1}^{(i,i)} \right) \oplus \sum_{i=p+1}^q \tilde{E}_{k-1,l-1}^{(p-1,i)} \quad (6.4)$$

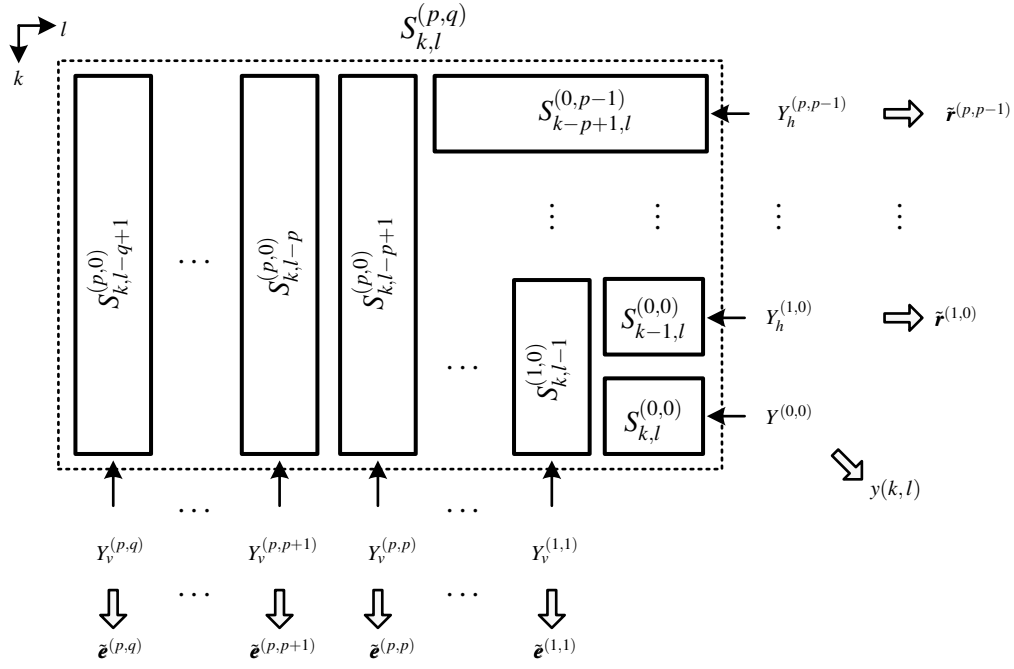


Figure 6.1: Decomposition of the rectangular PSR, $S_{k,l}^{(p,q)}$. Ordering arrangements of the PSR, $S_{k,l}^{(p,q)}$, for $p < q$, and corresponding mutually orthogonal backward horizontal and vertical auxiliary PEFs.

where $\oplus$ is used to indicate a direct sum, which means that each vector in $S_{k,l}^{(p,q)}$ may be written uniquely as the sum of vectors in the subspaces $\tilde{E}_{k,l}^{(m,n)}$ and $\tilde{R}_{k,l}^{(m,n)}$. The initial subspaces are defined as

$$\tilde{E}_{k-1,l-1}^{(0,0)} = \tilde{R}_{k-1,l-1}^{(0,0)} = S_{k,l}^{(0,0)}. \quad (6.5)$$

Let us consider to order the observation points of the PSR $S_{k,l}^{(p,q)}$ in Fig. 6.1 for $p < q$, by defining the input vector in terms of vertical and horizontal building observation blocks

$$\mathbf{Y}_{k,l}^{(p,q)} = \begin{bmatrix} Y^{(0,0)} & : & Y_h^{(1,0)} & : & \mathbf{Y}_v^{(1,1)T} & : & \dots & : & \mathbf{Y}_h^{(p,p-1)T} & : & \mathbf{Y}_v^{(p,p)T} & : \\ \mathbf{Y}_v^{(p,p+1)T} & : & \dots & : & \mathbf{Y}_v^{(p,q)T} \end{bmatrix}^T \quad (6.6)$$

where

$$\mathbf{Y}_h^{(s,t)} = [y(k-s, l-t) \quad \dots \quad y(k-s, l)]^T$$

and

$$\mathbf{Y}_v^{(s,t)} = [y(k-s, l-t) \quad \dots \quad y(k, l-t)]^T.$$

In order to transform the input vector defined in (6.6) into an auxiliary backward prediction error vector

$$\boldsymbol{\beta}^{(p,q)}(k,l) = \begin{bmatrix} \tilde{\mathbf{e}}^{(0,0)}(k,l) & \tilde{\mathbf{r}}^{(1,0)}(k,l) & \tilde{\mathbf{e}}^{(1,1)^T}(k,l) & \dots & \\ \tilde{\mathbf{r}}^{(p,p-1)^T}(k,l) & \tilde{\mathbf{e}}^{(p,p)^T}(k,l) & \tilde{\mathbf{e}}^{(p,p+1)^T}(k,l) & & \\ \vdots & \dots & \tilde{\mathbf{e}}^{(p,q)^T}(k,l) & & \end{bmatrix}^T, \quad (6.7)$$

we premultiply the input vector by a lower block-triangular matrix $\mathbf{L}^{(p,q)}$ with 1s along the main diagonal,

$$\boldsymbol{\beta}^{(p,q)}(k,l) = \mathbf{L}^{(p,q)} \mathbf{Y}_{k,l}^{(p,q)}. \quad (6.8)$$

where

$$\mathbf{L}^{(p,q)} = \begin{bmatrix} \mathbf{I}_1 & \mathbf{O}_1 & \mathbf{O}_2^T & \dots & \mathbf{O}_{q-1}^T & \mathbf{O}_q^T & \mathbf{O}_{q+1}^T \\ \mathbf{G}_{00}^{(1,0)} & \mathbf{I}_1 & \mathbf{O}_2^T & \dots & \mathbf{O}_{q-1}^T & \mathbf{O}_q^T & \mathbf{O}_{q+1}^T \\ \mathbf{G}_{00}^{(1,1)} & \mathbf{G}_{10}^{(1,1)} & \mathbf{I}_2 & \dots & \mathbf{O}_{2 \times (q-1)} & \mathbf{O}_{2 \times q} & \mathbf{O}_{2 \times (q+1)} \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots \\ \mathbf{G}_{00}^{(p,q-1)} & \mathbf{G}_{10}^{(p,q-1)} & \mathbf{G}_{11}^{(p,q-1)} & \dots & \mathbf{G}_{(p-1) \times (q-1)}^{(p,q-1)} & \mathbf{I}_p & \mathbf{O}_{p \times (q+1)} \\ \mathbf{G}_{00}^{(p,q)} & \mathbf{G}_{10}^{(p,q)} & \mathbf{G}_{11}^{(p,q)} & \dots & \mathbf{G}_{(p-1) \times (q-1)}^{(p,q)} & \mathbf{G}_{p \times (q-1)}^{(p,q)} & \mathbf{I}_{p+1} \end{bmatrix}. \quad (6.9)$$

In (6.9), the $\mathbf{G}_{ij}^{(p,q)}$ consist of the elements of the coefficient matrices calculated for each backward PEF and $\mathbf{O}$ is zero matrices. Since all diagonal block matrices are equal to the identity matrices with the appropriate order, *i.e.*, $\mathbf{G}_{pq}^{(p,q)} = \mathbf{I}_{p+1}$, the determinant of $\mathbf{L}^{(p,q)}$ equals to the unity. Hence, the matrix $\mathbf{L}^{(p,q)}$ is nonsingular. Moreover, there is a one-to-one correspondence between the input vector $\mathbf{Y}_{k,l}^{(p,q)}$ and the backward prediction error vector $\boldsymbol{\beta}^{(p,q)}(k,l)$. Therefore, one may be obtained from the other without loss of information if the lower triangular matrix $\mathbf{L}^{(p,q)}$ is derived from the lattice model parameters explained in Section 3.

The most interesting structural property of the proposed 2-D general lattice realization is that the auxiliary backward vertical and horizontal prediction error vectors are orthogonal to each other. In order to show this remarkable property, we form the matrix product $\mathfrak{R}_Y^{(p,q)} \mathbf{L}^{(p,q)H}$ where $\mathfrak{R}_Y^{(p,q)} = E \left[\mathbf{Y}_{k,l}^{(p,q)} \mathbf{Y}_{k,l}^{(p,q)H} \right]$ is the block-symmetric correlation matrix of the input vector defined in (6.6). $\mathfrak{R}_Y^{(p,q)} \mathbf{L}^{(p,q)H}$ will be a lower triangular matrix since the normal equation is satisfied by each prediction error filter. We then multiply $\mathfrak{R}_Y^{(p,q)} \mathbf{L}^{(p,q)H}$ from the left by $\mathbf{L}^{(p,q)}$ to get $\mathbf{L}^{(p,q)} \mathfrak{R}_Y^{(p,q)} \mathbf{L}^{(p,q)H}$.

Since $\mathbf{L}^{(p,q)}$ is by definition lower block triangular, and $\mathbf{L}^{(p,q)} \mathfrak{R}_Y^{(p,q)} \mathbf{L}^{(p,q)H} = \mathfrak{R}_\beta^{(p,q)}$ is hermitian, where $\mathfrak{R}_\beta^{(p,q)}$ is the correlation matrix of the backward prediction error vector in (6.7)

$$\begin{aligned}\mathfrak{R}_\beta^{(p,q)} &= E \left[\boldsymbol{\beta}^{(p,q)}(k,l) \boldsymbol{\beta}^{(p,q)H}(k,l) \right] \\ &= \text{Block diag.} \left[\begin{array}{ccccc} \Sigma_{\tilde{\mathbf{e}}}^{(0,0)} & \Sigma_{\tilde{\mathbf{r}}}^{(1,0)} & \dots & \Sigma_{\tilde{\mathbf{r}}}^{(p,q-1)} & \Sigma_{\tilde{\mathbf{e}}}^{(p,q)} \end{array} \right] \quad (6.10)\end{aligned}$$

where $\Sigma_{\tilde{\mathbf{e}}}^{(m,n)} = E \left[\tilde{\mathbf{e}}^{(m,n)} \tilde{\mathbf{e}}^{(m,n)H} \right]$ and $\Sigma_{\tilde{\mathbf{r}}}^{(m,n)} = E \left[\tilde{\mathbf{r}}^{(m,n)} \tilde{\mathbf{r}}^{(m,n)H} \right]$ are the minimized backward vertical and horizontal prediction error vector covariance matrices.

Now, since $\mathbf{L}^{(p,q)}$ is clearly nonsingular, one can express the inverse of the input correlation matrix as follows

$$\mathfrak{R}_Y^{(p,q)^{-1}} = \mathbf{L}^{(p,q)H} \mathfrak{R}_\beta^{(p,q)^{-1}} \mathbf{L}^{(p,q)}. \quad (6.11)$$

Using (6.11), it is possible to find the inverse of a large size, i.e pq -by- pq , input correlation matrix $\mathfrak{R}_Y^{(p,q)}$ by only taking the inverses of the backward covariance matrices with small sizes as shown in (6.10). Indeed, for $p < q$, their sizes start from 1-by-1 and go up to p -by- p . This is an expected result of the proposed 2-D lattice orthogonalization scheme.

Of course, it is also possible to decompose the rectangular shaped PSR $S_{k,l}^{(p,q)}$, for $q < p$ into shifted vertical and horizontal 1-D subspaces depending on the ordering arrangements as depicted in Figure 6.2. It can be defined as

$$S_{k,l}^{(p,q)} = S_{k,l}^{(0,0)} + \sum_{i=1}^q \left(S_{k-i,l}^{(0,i-1)} + S_{k,l-i}^{(i,0)} \right) + \sum_{j=q+1}^p S_{k-i,l}^{(0,q)}. \quad (6.12)$$

The proposed general 2-D AR lattice structure generates the mutually orthogonal realization subspaces spanned by vertical and horizontal auxiliary backward PEFs for $q < p$, and it can be written as

$$S_{k,l}^{(p,q)} = \tilde{E}_{k-1,l-1}^{(0,0)} \oplus \sum_{i=1}^q \left(\tilde{R}_{k-1,l-1}^{(i,i-1)} \oplus \tilde{E}_{k-1,l-1}^{(i,i)} \right) \oplus \sum_{i=q+1}^p \tilde{R}_{k-1,l-1}^{(i,q-1)}. \quad (6.13)$$

The observation points of the PSR $S_{k,l}^{(p,q)}$ in Figure 6.2 for $q < p$, can be ordered by defining the input vector in terms of vertical and horizontal building observation blocks

$$\mathbf{Y}_{k,l}^{(p,q)} = \begin{bmatrix} Y^{(0,0)} & Y_h^{(1,0)} & \mathbf{Y}_v^{(1,1)T} & \dots & \mathbf{Y}_v^{(q,q)T} & \mathbf{Y}_h^{(q+1,q)T} & \\ & \mathbf{Y}_h^{(q+2,q)T} & \dots & \mathbf{Y}_h^{(p,q)T} \end{bmatrix}^T. \quad (6.14)$$

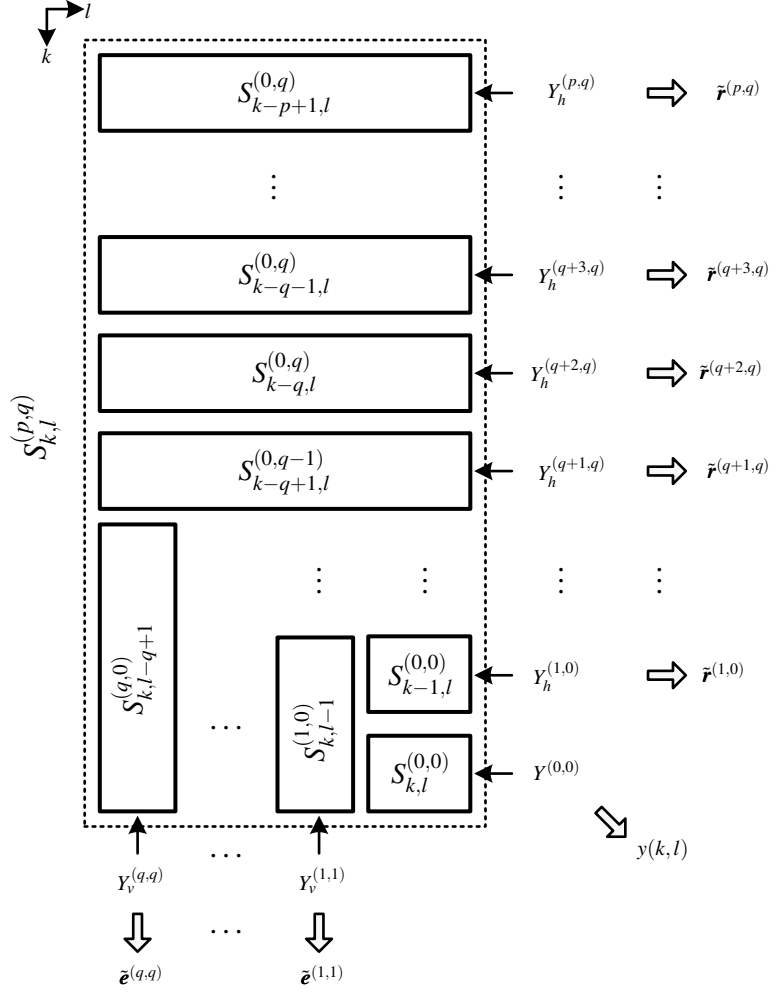


Figure 6.2: Decomposition of the rectangular PSR, $S_{k,l}^{(p,q)}$. Ordering arrangements of the PSR, $S_{k,l}^{(p,q)}$, for $q < p$, and corresponding mutually orthogonal backward horizontal and vertical auxiliary PEFs.

In order to transform the input vector defined in (6.14) into an auxiliary backward prediction error vector

$$\begin{aligned} \boldsymbol{\beta}^{(p,q)}(k,l) = & \begin{bmatrix} \tilde{e}^{(0,0)}(k,l) & : & \tilde{r}^{(0,1)}(k,l) & : & \tilde{\mathbf{e}}^{(1,1)^T}(k,l) & : & \dots & : \\ \tilde{\mathbf{e}}^{(q,q)^T}(k,l) & : & \tilde{\mathbf{r}}^{(q+1,q)^T}(k,l) & : & \tilde{\mathbf{r}}^{(q+2,q)^T}(k,l) & : & \dots & : \\ \vdots & \dots & : & \tilde{\mathbf{r}}^{(p,q)^T}(k,l) \end{bmatrix}^T. \end{aligned} \quad (6.15)$$

In addition, it is also possible to decompose the rectangular shaped PSR $S_{k,l}^{(p,q)}$, for $q < p$ into shifted horizontal and vertical 1-D subspaces as

$$S_{k,l}^{(p,q)} = S_{k,l}^{(0,0)} + \sum_{i=1}^q \left(S_{k,l-i}^{(i-1,0)} + S_{k-i,l}^{(0,i)} \right) + \sum_{j=q+1}^p S_{k-i,l}^{(0,q)}. \quad (6.16)$$

In this case, the mutually orthogonal realization subspaces spanned by vertical and horizontal auxiliary backward PEFs for $q < p$, can be written as

$$S_{k,l}^{(p,q)} = \tilde{R}_{k-1,l-1}^{(0,0)} \oplus \sum_{i=1}^q \left(\tilde{E}_{k-1,l-1}^{(i-1,i)} \oplus \tilde{R}_{k-1,l-1}^{(i,i)} \right) \oplus \sum_{i=q+1}^p \tilde{R}_{k-1,l-1}^{(i,q-1)}. \quad (6.17)$$

6.2 Entropy Relations of 2-D Lattice Filters

In this section, the information loss between the input vector and the backward prediction error vector in 2-D lattice filtering will be analyzed. The entropy calculation method given in [34] will be adapted to the proposed 2-D lattice filter method.

It is known that, in 1-D lattice filtering case, the input vector $\mathbf{y}(n)$, consisting of elements $y(n), y(n-1), \dots, y(n-p)$ and the corresponding backward prediction error vector $\mathbf{b}(n)$, consisting of elements $b^{(0)}(n), b^{(1)}(n), \dots, b^{(p)}(n)$ are probabilistically equivalent. Both contain exactly the same amount of information and one can be regained from the knowledge of the other [4]. Extension of this probabilistic relationship between the input and the backward prediction error vectors has already been presented in [34] for the 2-D case where the input and the backward prediction error vectors are lexicographic ordered vectors. Here, this extension will be adapted for the proposed 2-D lattice filter method and it will be shown how to calculate the entropies of these two random vectors. Contrary to the 2-D methods introduced in [34], it will be shown that the information loss is less even for the higher ordered filters for the proposed method.

It is known that the increase in the number of error fields decreases the information loss [34]. In our proposed 2-D lattice method, number of error fields increases after each augmentation process. Hence, the proposed filter models approximate the input entropy more closely.

The amount of information loss is calculated for Gaussian distributed data. To compute the entropy of the 2-D lattice filter, lexicographic ordered input vector $\mathbf{Y}_{k,l}^{(p,q)}$ can be defined as (6.6) according to the augmentation sequence. If the augmentation processes are applied as shown in Figure 6.1, *i.e.*, after initial first order stage, first horizontal, then vertical augmentation and repeat this sequence until the desired stage order, we

can write the input data as for $p = q$

$$\mathbf{Y}_{k,l}^{(p,q)} = \begin{bmatrix} Y^{(0,0)} \\ \vdots \\ Y_h^{(1,0)} \\ \vdots \\ \mathbf{Y}_v^{(1,1)T} \\ \vdots \\ \mathbf{Y}_h^{(p-1,q-2)T} \\ \vdots \\ \mathbf{Y}_v^{(p-1,q-1)T} \\ \vdots \\ \mathbf{Y}_h^{(p,q-1)T} \\ \vdots \\ \mathbf{Y}_v^{(p,q)T} \end{bmatrix} \quad (6.18)$$

where

$$\mathbf{Y}_h^{(s,t)} = [y(k-s, l-t) \quad \cdots \quad y(k-s, l)]^T$$

and

$$\mathbf{Y}_v^{(s,t)} = [y(k-s, l-t) \quad \cdots \quad y(k, l-t)]^T.$$

and if the augmentation processes are applied as first vertically, then horizontally and repeat this sequence until the desired stage order, we can write the input data as for $p = q$

$$\mathbf{Y}_{k,l}^{(p,q)} = \begin{bmatrix} Y^{(0,0)} \\ \vdots \\ Y_v^{(0,1)} \\ \vdots \\ \mathbf{Y}_h^{(1,1)T} \\ \vdots \\ \mathbf{Y}_v^{(p-2,q-1)T} \\ \vdots \\ \mathbf{Y}_h^{(p-1,q-1)T} \\ \vdots \\ \mathbf{Y}_v^{(p-1,q)T} \\ \vdots \\ \mathbf{Y}_h^{(p,q)T} \end{bmatrix}. \quad (6.19)$$

A lexicographic ordered backward prediction error vector $\boldsymbol{\beta}^{(p,q)}(k, l)$ can also be defined as input vector. But, for the proposed 2-D lattice filter, the backward error predictions to be taken into account will be determined according to the augmentation direction. For example, after the initial stage order, if we apply first horizontal and then vertical augmentation and repeat this sequence, the backward error vector $\boldsymbol{\beta}^{(p,q)}(k, l)$

will be defined as for $p = q$

$$\boldsymbol{\beta}^{(p,q)}(k,l) = \begin{bmatrix} \tilde{\mathbf{e}}^{(0,0)}(k,l) \\ \tilde{\mathbf{r}}_1^{(1,0)}(k,l) \\ \tilde{\mathbf{e}}^{(1,1)}(k,l) \\ \tilde{\mathbf{r}}^{(2,1)}(k,l) \\ \tilde{\mathbf{e}}^{(2,2)}(k,l) \\ \vdots \\ \tilde{\mathbf{r}}^{(p-1,p-2)}(k,l) \\ \tilde{\mathbf{e}}^{(p-1,p-1)}(k,l) \\ \tilde{\mathbf{r}}^{(p,p-1)}(k,l) \\ \tilde{\mathbf{e}}^{(p,p)}(k,l) \end{bmatrix}. \quad (6.20)$$

In this backward error vector, each partition of the vector $\boldsymbol{\beta}^{(p,q)}(k,l)$ consists of the backward PEFs obtained at the corresponding stage order.

As explained in the study [34], the entropy, the information content, of a random input vector $\mathbf{Y}_{k,l}^{(p,q)}$ is defined by [70]

$$H_Y = - \int_{-\infty}^{\infty} p_Y(\mathbf{Y}) \ln[p_Y(\mathbf{Y})] dY \quad (6.21)$$

where $p_Y(\mathbf{Y})$ is the joint probability density function of the input vector $\mathbf{Y}_{k,l}^{(p,q)}$. The integral in the equation (6.21) is a multiple integral and the variables of the integration is the individual elements of the input vector $\mathbf{Y}_{k,l}^{(p,q)}$. For the special case where the real valued random vector $\mathbf{Y}_{k,l}^{(p,q)}$ is Gaussian distributed, with zero mean and with correlation matrix $\mathfrak{R}_Y^{(p,q)} = E[\mathbf{Y}_{k,l}^{(p,q)} \mathbf{Y}_{k,l}^{(p,q)T}]$, its joint probability density function can be written as

$$p_Y(\mathbf{Y}) = \left\{ (2\pi)^{L_Y} \det[\mathfrak{R}_Y^{(p,q)}] \right\}^{-\frac{1}{2}} \exp \left(-\frac{1}{2} \mathbf{Y}^{(p,q)T} \mathfrak{R}_Y^{(p,q)-1} \mathbf{Y}_{k,l}^{(p,q)} \right) \quad (6.22)$$

where L_Y denotes the number of elements of the vector $\mathbf{Y}_{k,l}^{(p,q)}$ which is $(p+1) \times (q+1)$. By substituting (6.22) into (6.21) and simplifying, the following result is obtained for the entropy H_Y of the input vector as

$$H_Y = \frac{1}{2} \ln \left(\det[\mathfrak{R}_Y^{(p,q)}] \right) + \frac{1}{2} L_Y \ln(2\pi e). \quad (6.23)$$

Similarly the entropy H_B of the backward prediction error vector can be given as

$$H_B = \frac{1}{2} \ln \left(\det[\mathfrak{R}_\beta^{(p,q)}] \right) + \frac{1}{2} L_\beta \ln(2\pi e). \quad (6.24)$$

where L_β is the number of elements of the vector $\boldsymbol{\beta}^{(p,q)}(k,l)$.

For this proposed 2-D lattice filter method, the support data can vary as a rectangular function of the filter order and concurrently the number of the backward prediction error vector increases after applying any vertical or horizontal augmentation.

So far, only the entropy calculations for the proposed method are studied. Now, it will be explained the calculation of entropy by giving some examples. In the examples, (k, l) of the PEFs will be omitted for convenience and it will be explained as if it is applied on a real data. That means, the left upper corner is the beginning point of the data.

6.2.1 Entropy calculation examples for the proposed 2-D lattice filter

As explained in Section 2.3.2, at the *zero*-th order state, the PSR $S_{k,l}^{(0,0)}$ is empty set and the backward PEFs are all equal to the given 2-D observation data:

$$\tilde{e}_1^{(0,0)}(k, l) = \tilde{r}_1^{(0,0)}(k, l) = y(k, l) \quad (6.25)$$

Hence, it is clearly seen that the *entropies* of the input data and the backward prediction error at the *zero*-th order stage are exactly equal. The entropy of the input data is calculated as

$$H_Y = \frac{1}{2} \ln \left(\det \left(\mathfrak{R}_Y^{(0,0)} \right) \right) + \frac{1}{2} L_Y \ln (2\pi e) \quad (6.26)$$

where $\mathfrak{R}_Y^{(0,0)} = E [y(k, l)y(k, l)^*]$ and $L_Y = 1$.

Similarly the entropy of the backward PEF is calculated as

$$H_\beta = \frac{1}{2} \ln \left(\det \left(\mathfrak{R}_\beta^{(0,0)} \right) \right) + \frac{1}{2} L_\beta \ln (2\pi e) \quad (6.27)$$

where $\mathfrak{R}_\beta^{(0,0)} = E \left[\tilde{e}_1^{(0,0)}(k, l) \tilde{e}_1^{(0,0)*}(k, l) \right]$ and $L_\beta = 1$. Since $\mathfrak{R}_Y^{(0,0)} = \mathfrak{R}_\beta^{(0,0)}$ and $L_Y = L_\beta$, the calculated entropies H_Y and H_β are equal to each other.

In our methods, we have vertical and horizontal backward PEFs and therefore, it is possible to calculate entropy by either vertical or horizontal backward PEFs. Here, we will calculate the entropy separately for each direction but just only one will be taken into account according to the augmentation direction. For example, at the initial stage order, we have two backward PEFs, $\tilde{e}_1^{(0,1)}$ and $\tilde{r}_1^{(1,0)}$, we will show the entropy calculation for each vertical and horizontal backward PEFs. But for the next stage order entropy calculations, just only one of them will be considered. In other words, the

backward PEFs at the applied augmentation direction will be taken into the backward prediction error vector.

At the initial stage order, only the samples $y(k, l)$ and $y(k, l - 1)$ are spanned for making a prediction of vertical error fields. That means the input data $\mathbf{Y}_{k,l}^{(0,1)}$ is

$$\mathbf{Y}_{k,l}^{(0,1)} = \begin{bmatrix} y(k, l) \\ y(k, l - 1) \end{bmatrix} \quad (6.28)$$

and $L_Y = 2$. $\mathfrak{R}_Y^{(0,1)}$ is computed as

$$\begin{aligned} \mathfrak{R}_Y^{(0,1)} &= E \left[\mathbf{Y}_{k,l}^{(0,1)} \mathbf{Y}_{k,l}^{(0,1)H} \right] \\ &= \begin{bmatrix} \mathfrak{R}_Y(0, 0) & \mathfrak{R}_Y(0, 1) \\ \mathfrak{R}_Y(0, 1) & \mathfrak{R}_Y(0, 0) \end{bmatrix}. \end{aligned} \quad (6.29)$$

Now, the backward prediction error vector consists of $\tilde{e}_1^{(0,0)}$ from the *zero*-th order and $\tilde{e}_1^{(0,1)}$ from the initial first order stage. Hereafter, (k, l) will be omitted for PEFs. The backward prediction error vector $\boldsymbol{\beta}^{(0,1)}$ is

$$\boldsymbol{\beta}^{(0,1)} = \begin{bmatrix} \tilde{e}_1^{(0,0)} \\ \tilde{e}_1^{(0,1)} \end{bmatrix} \quad (6.30)$$

and $L_B = 2$. The auto-covariance matrix $\mathfrak{R}_\beta^{(0,1)}$ is calculated as

$$\begin{aligned} \mathfrak{R}_\beta^{(0,1)} &= E \left[\boldsymbol{\beta}^{(0,1)} \boldsymbol{\beta}^{(0,1)H} \right] \\ &= \begin{bmatrix} E \left[\tilde{e}_1^{(0,0)} \tilde{e}_1^{(0,0)*} \right] & E \left[\tilde{e}_1^{(0,0)} \tilde{e}_1^{(0,1)*} \right] \\ E \left[\tilde{e}_1^{(0,1)} \tilde{e}_1^{(0,0)*} \right] & E \left[\tilde{e}_1^{(0,1)} \tilde{e}_1^{(0,1)*} \right] \end{bmatrix}. \end{aligned} \quad (6.31)$$

To constitute the matrix $\mathbf{L}^{(p,q)}$ we need the coefficient matrices of $\tilde{e}_1^{(0,0)}$ and $\tilde{e}_1^{(0,1)}$. The coefficient matrices of $\tilde{e}_1^{(0,0)}$ and $\tilde{e}_1^{(0,1)}$ are $\tilde{E}_1^{(0,0)} = [1]$ and $\tilde{E}_1^{(0,1)} = \begin{bmatrix} a_{00}^{(0,1)} & 1 \end{bmatrix}$, respectively. The matrix $\mathbf{L}^{(0,1)}$ can be built up as follows:

$$\mathbf{L}^{(0,1)} = \begin{bmatrix} 1 & 0 \\ a_{00}^{(0,1)} & 1 \end{bmatrix} \quad (6.32)$$

As seen that the determinant of the matrix $\mathbf{L}^{(p,q)}$ is equal to 1. Thus, we expect the input and backward prediction error entropy will be equal to each other.

Similarly, for the horizontal PEFs, only the samples $y(k, l)$ and $y(k - 1, l)$ are spanned. The input data $\mathbf{Y}_{k,l}^{(1,0)}$ is

$$\mathbf{Y}_{k,l}^{(1,0)} = \begin{bmatrix} y(k, l) \\ y(k - 1, l) \end{bmatrix} \quad (6.33)$$

and $L_Y = 2$.

$\mathfrak{R}_Y^{(1,0)}$ is computed as

$$\begin{aligned}\mathfrak{R}_Y^{(1,0)} &= E \left[\mathbf{Y}_{k,l}^{(1,0)} \mathbf{Y}_{k,l}^{(1,0)H} \right] \\ &= \begin{bmatrix} \mathfrak{R}_Y(0,0) & \mathfrak{R}_Y(1,0) \\ \mathfrak{R}_Y(1,0) & \mathfrak{R}_Y(0,0) \end{bmatrix}.\end{aligned}\quad (6.34)$$

The backward prediction error vector includes $\tilde{e}_1^{(0,0)}$ from the *zero*-th order and $\tilde{r}_1^{(1,0)}$ from the initial first order stage. The backward prediction error vector $\boldsymbol{\beta}^{(1,0)}$ is

$$\boldsymbol{\beta}^{(1,0)} = \begin{bmatrix} \tilde{e}_1^{(0,0)} \\ \tilde{r}_1^{(1,0)} \end{bmatrix} \quad (6.35)$$

and $L_\beta = 2$. The auto-covariance matrix $\mathfrak{R}_\beta^{(1,0)}$ is calculated as

$$\begin{aligned}\mathfrak{R}_\beta^{(1,0)} &= E \left[\boldsymbol{\beta}^{(1,0)} \boldsymbol{\beta}^{(1,0)H} \right] \\ &= \begin{bmatrix} E \left[\tilde{e}_1^{(0,0)} \tilde{e}_1^{(0,0)*} \right] & E \left[\tilde{e}_1^{(0,0)} \tilde{r}_1^{(1,0)*} \right] \\ E \left[\tilde{r}_1^{(1,0)} \tilde{e}_1^{(0,0)*} \right] & E \left[\tilde{r}_1^{(1,0)} \tilde{r}_1^{(1,0)*} \right] \end{bmatrix}.\end{aligned}\quad (6.36)$$

The coefficient matrix of $\tilde{r}_1^{(1,0)}$ is found as $\tilde{R}_1^{(1,0)} = \begin{bmatrix} b_{00}^{(1,0)} \\ 1 \end{bmatrix}$ and the matrix $\mathbf{L}^{(1,0)}$ can be formed as

$$\mathbf{L}^{(1,0)} = \begin{bmatrix} 1 & 0 \\ b_{00}^{(1,0)} & 1 \end{bmatrix}. \quad (6.37)$$

At that point, we have two backward prediction vectors, (6.30) and (6.35). From this point, according to the augmentation type, one of these backward prediction vectors will be passed to the next stage order. If a horizontal augmentation is applied, (6.35) will be taken into account to calculate the entropy at the new stage order. If a vertical augmentation is realized, then (6.30) will be used in the entropy calculations. These calculations will be shown in the following sub sections.

6.2.1.1 Entropy calculation for $S_{k,l}^{(1,0)}$

We know that the PSR will be obtained if a horizontal augmentation is applied on the PSR $S_{k,l}^{(0,0)}$. Naturally, number of samples spanned increased. Again we will show the entropy calculation for both vertical and horizontal backward PEFs, separately.

First, we calculate the entropy by adding the new vertical backward PEFs into the backward prediction error vector. In that case the input data will be defined as

$$\mathbf{Y}_{k,l}^{(1,1)} = \begin{bmatrix} y(k,l) \\ y(k-1,l) \\ y(k-1,l-1) \\ y(k,l-1) \end{bmatrix} \quad (6.38)$$

and $L_Y = 4$. The auto-covariance matrix $\mathfrak{R}_Y^{(1,1)}$ is computed as

$$\begin{aligned} \mathfrak{R}_Y^{(1,1)} &= E \left[\mathbf{Y}_{k,l}^{(1,1)} \mathbf{Y}_{k,l}^{(1,1)H} \right] \\ &= \begin{bmatrix} \mathfrak{R}_Y(0,0) & \mathfrak{R}_Y(1,0) & \mathfrak{R}_Y(1,1) & \mathfrak{R}_Y(0,1) \\ \mathfrak{R}_Y(1,0) & \mathfrak{R}_Y(0,0) & \mathfrak{R}_Y(0,1) & \mathfrak{R}_Y(1,1) \\ \mathfrak{R}_Y(1,1) & \mathfrak{R}_Y(0,1) & \mathfrak{R}_Y(0,0) & \mathfrak{R}_Y(1,0) \\ \mathfrak{R}_Y(0,1) & \mathfrak{R}_Y(1,1) & \mathfrak{R}_Y(1,0) & \mathfrak{R}_Y(0,0) \end{bmatrix}. \end{aligned} \quad (6.39)$$

Since a horizontal augmentation is applied, the backward prediction error field (6.35) obtained at the initial first order stage is updated by adding two new backward PEFs, $\tilde{e}_1^{(1,1)}$ and $\tilde{e}_2^{(1,1)}$. The new backward prediction error vector $\boldsymbol{\beta}^{(1,1)}$ will be

$$\boldsymbol{\beta}^{(1,1)} = \begin{bmatrix} \tilde{e}_1^{(0,0)} \\ \tilde{r}_1^{(1,0)} \\ \tilde{e}_1^{(1,1)} \\ \tilde{e}_2^{(1,1)} \end{bmatrix} \quad (6.40)$$

and $L_\beta = 4$. The auto-covariance matrix $\mathfrak{R}_\beta^{(1,1)}$ is

$$\begin{aligned} \mathfrak{R}_\beta^{(1,1)} &= E \left[\boldsymbol{\beta}^{(1,1)} \boldsymbol{\beta}^{(1,1)H} \right] \\ &= \begin{bmatrix} E \begin{bmatrix} \tilde{e}_1^{(0,0)} \tilde{e}_1^{(0,0)*} \\ \tilde{r}_1^{(1,0)} \tilde{e}_1^{(0,0)*} \\ \tilde{e}_1^{(1,1)} \tilde{e}_1^{(0,0)*} \\ \tilde{e}_2^{(1,1)} \tilde{e}_1^{(0,0)*} \end{bmatrix} & E \begin{bmatrix} \tilde{e}_1^{(0,0)} \tilde{r}_1^{(1,0)*} \\ \tilde{r}_1^{(1,0)} \tilde{r}_1^{(1,0)*} \\ \tilde{e}_1^{(1,1)} \tilde{r}_1^{(1,0)*} \\ \tilde{e}_2^{(1,1)} \tilde{r}_1^{(1,0)*} \end{bmatrix} & E \begin{bmatrix} \tilde{e}_1^{(0,0)} \tilde{e}_1^{(1,1)*} \\ \tilde{r}_1^{(1,0)} \tilde{e}_1^{(1,1)*} \\ \tilde{e}_1^{(1,1)} \tilde{e}_1^{(1,1)*} \\ \tilde{e}_2^{(1,1)} \tilde{e}_1^{(1,1)*} \end{bmatrix} & E \begin{bmatrix} \tilde{e}_1^{(0,0)} \tilde{e}_2^{(1,1)*} \\ \tilde{r}_1^{(1,0)} \tilde{e}_2^{(1,1)*} \\ \tilde{e}_1^{(1,1)} \tilde{e}_2^{(1,1)*} \\ \tilde{e}_2^{(1,1)} \tilde{e}_2^{(1,1)*} \end{bmatrix} \end{bmatrix}. \end{aligned} \quad (6.41)$$

The coefficient matrices of $\tilde{e}_1^{(1,1)}$ and $\tilde{e}_2^{(1,1)}$ can be defined as

$$\tilde{E}_1^{(1,1)} = \begin{bmatrix} a1_{00}^{(1,1)} & 0 \\ a1_{10}^{(1,1)} & 1 \end{bmatrix}, \quad (6.42)$$

and

$$\tilde{E}_2^{(1,1)} = \begin{bmatrix} a2_{00}^{(1,1)} & 1 \\ a2_{10}^{(1,1)} & 0 \end{bmatrix}, \quad (6.43)$$

respectively. The matrix $\mathbf{L}^{(1,1)}$ can be constituted as

$$\mathbf{L}^{(1,1)} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ b_{00}^{(1,0)} & 1 & 0 & 0 \\ a1_{00}^{(1,1)} & a1_{10}^{(1,1)} & 1 & 0 \\ a2_{00}^{(1,1)} & a2_{10}^{(1,1)} & 0 & 1 \end{bmatrix}. \quad (6.44)$$

Entropies of the input data (6.38) and the backward prediction error vector (6.40) at this stage order can be calculated by using (6.26) and (6.27), respectively.

Secondarily, we can also calculate the entropy by adding the horizontal backward PEF into the backward prediction error vector. In that case the input data will be written as

$$\mathbf{Y}_{k,l}^{(2,0)} = \begin{bmatrix} y(k,l) \\ y(k-1,l) \\ y(k-2,l) \end{bmatrix} \quad (6.45)$$

and $L_Y = 3$. The auto-covariance matrix $\mathfrak{R}_Y^{(2,0)}$ is computed as

$$\begin{aligned} \mathfrak{R}_Y^{(2,0)} &= E \left[\mathbf{Y}_{k,l}^{(2,0)} \mathbf{Y}_{k,l}^{(2,0)H} \right] \\ &= \begin{bmatrix} \mathfrak{R}_Y(0,0) & \mathfrak{R}_Y(1,0) & \mathfrak{R}_Y(2,0) \\ \mathfrak{R}_Y(1,0) & \mathfrak{R}_Y(0,0) & \mathfrak{R}_Y(1,0) \\ \mathfrak{R}_Y(2,0) & \mathfrak{R}_Y(1,0) & \mathfrak{R}_Y(0,0) \end{bmatrix}. \end{aligned} \quad (6.46)$$

After a horizontal augmentation is applied to the PSR $S_{k,l}^{(0,0)}$, the backward prediction error field (6.35) obtained at the initial first order stage will be updated by adding the horizontal backward PEF, $\tilde{r}_1^{(2,0)}$. The new backward prediction error vector $\boldsymbol{\beta}^{(2,0)}$ will be

$$\boldsymbol{\beta}^{(2,0)} = \begin{bmatrix} \tilde{e}_1^{(0,0)} \\ \tilde{r}_1^{(1,0)} \\ \tilde{r}_1^{(2,0)} \end{bmatrix} \quad (6.47)$$

and $L_\beta = 3$. The auto-covariance matrix $\mathfrak{R}_\beta^{(2,0)}$ is

$$\begin{aligned} \mathfrak{R}_\beta^{(2,0)} &= E \left[\boldsymbol{\beta}^{(2,0)} \boldsymbol{\beta}^{(2,0)H} \right] \\ &= \begin{bmatrix} E \left[\tilde{e}_1^{(0,0)} \tilde{e}_1^{(0,0)*} \right] & E \left[\tilde{e}_1^{(0,0)} \tilde{r}_1^{(1,0)*} \right] & E \left[\tilde{e}_1^{(0,0)} \tilde{r}_1^{(2,0)*} \right] \\ E \left[\tilde{r}_1^{(1,0)} \tilde{e}_1^{(0,0)*} \right] & E \left[\tilde{r}_1^{(1,0)} \tilde{r}_1^{(1,0)*} \right] & E \left[\tilde{r}_1^{(1,0)} \tilde{r}_1^{(2,0)*} \right] \\ E \left[\tilde{r}_1^{(2,0)} \tilde{e}_1^{(0,0)*} \right] & E \left[\tilde{r}_1^{(2,0)} \tilde{r}_1^{(1,0)*} \right] & E \left[\tilde{r}_1^{(2,0)} \tilde{r}_1^{(2,0)*} \right] \end{bmatrix}. \end{aligned} \quad (6.48)$$

The coefficient matrix of $\tilde{r}_1^{(2,0)}$ is given as

$$\tilde{R}_1^{(2,0)} = \begin{bmatrix} b1_{00}^{(2,0)} \\ b1_{10}^{(2,0)} \\ 1 \end{bmatrix}. \quad (6.49)$$

The matrix $\mathbf{L}^{(2,0)}$ can be formed as

$$\mathbf{L}^{(2,0)} = \begin{bmatrix} 1 & 0 & 0 \\ b_{00}^{(1,0)} & 1 & 0 \\ b1_{00}^{(2,0)} & b1_{10}^{(2,0)} & 1 \end{bmatrix}. \quad (6.50)$$

Entropies of the input data (6.45) and the backward prediction error vector (6.47) at this stage order can be calculated by using (6.26) and (6.27), respectively.

Hereby, we showed how to calculate the entropies of the input data and the backward prediction error vector from vertical and horizontal backward PEFs and we determined two new prediction error vectors to be used in the new stage order after the augmentation method.

6.2.1.2 Entropy calculation for $S_{k,l}^{(0,1)}$

The PSR $S_{k,l}^{(1,0)}$ is obtained if a vertical augmentation is applied on the PSR $S_{k,l}^{(0,0)}$. The entropy calculation for both vertical and horizontal backward PEFs will be determined, separately.

At the first step, we calculate the entropy by adding the new horizontal backward PEFs into the backward prediction error vector (6.30). In that case the input data will be defined as

$$\mathbf{Y}_{k,l}^{(1,1)} = \begin{bmatrix} y(k,l) \\ y(k,l-1) \\ y(k-1,l-1) \\ y(k-1,l) \end{bmatrix} \quad (6.51)$$

and $L_Y = 4$. The auto-covariance matrix $\mathfrak{R}_Y^{(1,1)}$ is computed as

$$\begin{aligned} \mathfrak{R}_Y^{(1,1)} &= E \left[\mathbf{Y}_{k,l}^{(1,1)} \mathbf{Y}_{k,l}^{(1,1)H} \right] \\ &= \begin{bmatrix} \mathfrak{R}_Y(0,0) & \mathfrak{R}_Y(0,1) & \mathfrak{R}_Y(1,1) & \mathfrak{R}_Y(1,0) \\ \mathfrak{R}_Y(0,1) & \mathfrak{R}_Y(0,0) & \mathfrak{R}_Y(1,0) & \mathfrak{R}_Y(1,1) \\ \mathfrak{R}_Y(1,1) & \mathfrak{R}_Y(1,0) & \mathfrak{R}_Y(0,0) & \mathfrak{R}_Y(0,1) \\ \mathfrak{R}_Y(1,0) & \mathfrak{R}_Y(1,1) & \mathfrak{R}_Y(0,1) & \mathfrak{R}_Y(0,0) \end{bmatrix}. \end{aligned} \quad (6.52)$$

After a vertical augmentation is applied, the backward prediction error field (6.30) obtained at the initial first order stage is updated by adding two new horizontal backward PEFs, $\tilde{r}_1^{(1,1)}$ and $\tilde{r}_2^{(1,1)}$. The new backward prediction error vector $\boldsymbol{\beta}^{(1,1)}$

will be

$$\boldsymbol{\beta}^{(1,1)} = \begin{bmatrix} \tilde{e}_1^{(0,0)} \\ \tilde{e}_1^{(0,1)} \\ \tilde{r}_1^{(1,1)} \\ \tilde{r}_2^{(1,1)} \end{bmatrix} \quad (6.53)$$

and $L_\beta = 4$. The auto-covariance matrix $\mathfrak{R}_\beta^{(1,1)}$ is

$$\begin{aligned} \mathfrak{R}_\beta^{(1,1)} &= E \left[\boldsymbol{\beta}^{(1,1)} \boldsymbol{\beta}^{(1,1)H} \right] \\ &= \begin{bmatrix} E \begin{bmatrix} \tilde{e}_1^{(0,0)} & \tilde{e}_1^{(0,0)*} \end{bmatrix} & E \begin{bmatrix} \tilde{e}_1^{(0,0)} & \tilde{e}_1^{(0,1)*} \end{bmatrix} & E \begin{bmatrix} \tilde{e}_1^{(0,0)} & \tilde{r}_1^{(1,1)*} \end{bmatrix} & E \begin{bmatrix} \tilde{e}_1^{(0,0)} & \tilde{r}_2^{(1,1)*} \end{bmatrix} \\ E \begin{bmatrix} \tilde{e}_1^{(0,1)} & \tilde{e}_1^{(0,0)*} \end{bmatrix} & E \begin{bmatrix} \tilde{e}_1^{(0,1)} & \tilde{e}_1^{(0,1)*} \end{bmatrix} & E \begin{bmatrix} \tilde{e}_1^{(0,1)} & \tilde{r}_1^{(1,1)*} \end{bmatrix} & E \begin{bmatrix} \tilde{e}_1^{(0,1)} & \tilde{r}_2^{(1,1)*} \end{bmatrix} \\ E \begin{bmatrix} \tilde{r}_1^{(1,1)} & \tilde{e}_1^{(0,0)*} \end{bmatrix} & E \begin{bmatrix} \tilde{r}_1^{(1,1)} & \tilde{e}_1^{(0,1)*} \end{bmatrix} & E \begin{bmatrix} \tilde{r}_1^{(1,1)} & \tilde{r}_1^{(1,1)*} \end{bmatrix} & E \begin{bmatrix} \tilde{r}_1^{(1,1)} & \tilde{r}_2^{(1,1)*} \end{bmatrix} \\ E \begin{bmatrix} \tilde{r}_2^{(1,1)} & \tilde{e}_1^{(0,0)*} \end{bmatrix} & E \begin{bmatrix} \tilde{r}_2^{(1,1)} & \tilde{e}_1^{(0,1)*} \end{bmatrix} & E \begin{bmatrix} \tilde{r}_2^{(1,1)} & \tilde{r}_1^{(1,1)*} \end{bmatrix} & E \begin{bmatrix} \tilde{r}_2^{(1,1)} & \tilde{r}_2^{(1,1)*} \end{bmatrix} \end{bmatrix}. \end{aligned} \quad (6.54)$$

The coefficient matrices of $\tilde{r}_1^{(1,1)}$ and $\tilde{r}_2^{(1,1)}$ are defined as $\tilde{R}_1^{(1,1)} = \begin{bmatrix} c1_{00}^{(1,1)} & c1_{01}^{(1,1)} \\ 0 & 1 \end{bmatrix}$,

and $\tilde{R}_1^{(1,1)} = \begin{bmatrix} c2_{00}^{(1,1)} & c2_{01}^{(1,1)} \\ 1 & 0 \end{bmatrix}$, respectively. The matrix $\mathbf{L}^{(1,1)}$ can be formed as

$$\mathbf{L}^{(1,1)} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ a_{00}^{(0,1)} & 1 & 0 & 0 \\ c1_{00}^{(1,1)} & c1_{01}^{(1,1)} & 1 & 0 \\ c2_{00}^{(1,1)} & c2_{01}^{(1,1)} & 0 & 1 \end{bmatrix}. \quad (6.55)$$

Entropies of the input data (6.51) and the backward prediction error vector (6.53) at this stage order can be calculated by using (6.26) and (6.27), respectively.

At the second step, the entropy by adding the vertical backward PEF into the backward prediction error vector (6.30) is calculated. In that case the input data will be written as

$$\mathbf{Y}_{k,l}^{(0,2)} = \begin{bmatrix} y(k,l) \\ y(k,l-1) \\ y(k,l-2) \end{bmatrix} \quad (6.56)$$

and $L_Y = 3$. The auto-covariance matrix $\mathfrak{R}_Y^{(0,2)}$ is computed as

$$\begin{aligned} \mathfrak{R}_Y^{(0,2)} &= E \left[\mathbf{Y}_{k,l}^{(0,2)} \mathbf{Y}_{k,l}^{(0,2)H} \right] \\ &= \begin{bmatrix} \mathfrak{R}_Y(0,0) & \mathfrak{R}_Y(0,1) & \mathfrak{R}_Y(0,2) \\ \mathfrak{R}_Y(0,1) & \mathfrak{R}_Y(0,0) & \mathfrak{R}_Y(0,1) \\ \mathfrak{R}_Y(0,2) & \mathfrak{R}_Y(0,1) & \mathfrak{R}_Y(0,0) \end{bmatrix}. \end{aligned} \quad (6.57)$$

After a vertical augmentation is applied to the PSR $S_{k,l}^{(0,0)}$, the backward prediction error field (6.30) obtained at the initial first order stage will be updated by adding the vertical backward PEF, $\tilde{e}_1^{(0,2)}$. The new backward prediction error vector $\boldsymbol{\beta}^{(0,2)}$ will be

$$\boldsymbol{\beta}^{(0,2)} = \begin{bmatrix} \tilde{e}_1^{(0,0)} \\ \tilde{e}_1^{(0,1)} \\ \tilde{e}_1^{(0,2)} \end{bmatrix} \quad (6.58)$$

and $L_\beta = 3$. The auto-covariance matrix $\mathfrak{R}_\beta^{(0,2)}$ is

$$\begin{aligned} \mathfrak{R}_\beta^{(0,2)} &= E \left[\boldsymbol{\beta}^{(0,2)} \boldsymbol{\beta}^{(0,2)H} \right] \\ &= \begin{bmatrix} E \begin{bmatrix} \tilde{e}_1^{(0,0)} \tilde{e}_1^{(0,0)*} \\ \tilde{e}_1^{(0,1)} \tilde{e}_1^{(0,0)*} \\ \tilde{e}_1^{(0,2)} \tilde{e}_1^{(0,0)*} \end{bmatrix} & E \begin{bmatrix} \tilde{e}_1^{(0,0)} \tilde{e}_1^{(0,1)*} \\ \tilde{e}_1^{(0,1)} \tilde{e}_1^{(0,1)*} \\ \tilde{e}_1^{(0,2)} \tilde{e}_1^{(0,1)*} \end{bmatrix} & E \begin{bmatrix} \tilde{e}_1^{(0,0)} \tilde{e}_1^{(0,2)*} \\ \tilde{e}_1^{(0,1)} \tilde{e}_1^{(0,2)*} \\ \tilde{e}_1^{(0,2)} \tilde{e}_1^{(0,2)*} \end{bmatrix} \\ \end{bmatrix}. \end{aligned} \quad (6.59)$$

The coefficient matrices of $\tilde{e}_1^{(0,2)}$ is defined as $\tilde{E}_1^{(0,2)} = \begin{bmatrix} d1_{00}^{(1,1)} & d1_{01}^{(1,1)} & 1 \end{bmatrix}$. The matrix $\mathbf{L}^{(0,2)}$ can be built up as

$$\mathbf{L}^{(0,2)} = \begin{bmatrix} 1 & 0 & 0 \\ a_{00}^{(0,1)} & 1 & 0 \\ d1_{00}^{(2,0)} & d1_{01}^{(2,0)} & 1 \end{bmatrix}. \quad (6.60)$$

By using (6.26) and (6.27), entropies of the input data (6.56) and the backward prediction error vector (6.58) can be calculated, respectively.

In this section, the calculation of the entropies of the input data and the backward prediction error vector from vertical and horizontal backward PEFs in case of vertical augmentation have been shown and two new prediction error vectors to be used in the new stage order after the augmentation method are determined

6.2.1.3 Entropy calculation for $S_{k,l}^{(1,1)}$

It is possible to reach the PSR $S_{k,l}^{(1,1)}$ by two scanning ways. The first one is realized by applying first horizontal augmentation and then vertical augmentation. In other words, a horizontal augmentation is applied to the PSR $S_{k,l}^{(0,0)}$, and the PSR $S_{k,l}^{(1,0)}$ is obtained. By applying a vertical augmentation on this PSR $S_{k,l}^{(1,0)}$, we can have the PSR $S_{k,l}^{(1,1)}$.

The second way is to apply a vertical augmentation first, and then a horizontal augmentation. That is to say, a vertical augmentation is applied to the PSR $S_{k,l}^{(0,0)}$,

then the PSR $S_{k,l}^{(0,1)}$ is obtained. By applying a horizontal augmentation to this PSR $S_{k,l}^{(0,1)}$, the new PSR will be $S_{k,l}^{(1,1)}$.

In fact, independent on the scanning direction, the entropy at the desired stage order will be the same. That means, regardless how to reach the PSR $S_{k,l}^{(1,1)}$, the entropy calculation of input data vector and the backward prediction error vector will be the same for each one. Now, we will demonstrate how to calculate the entropies for both scanning ways.

Entropy calculation for the first scanning way

Firstly, we think that we came to this stage order by applying first horizontal and then vertical augmentation. Hence, we will use (6.38) and (6.40) to determine new input data vector and the backward prediction error vector, respectively. We will calculate the entropies separately by using both vertical and horizontal backward PEFs.

First, we calculate the entropy by using the vertical backward PEFs. The input data vector will be defined as

$$\mathbf{Y}_{k,l}^{(1,2)} = \begin{bmatrix} y(k,l) \\ y(k-1,l) \\ y(k-1,l-1) \\ y(k,l-1) \\ y(k-1,l-2) \\ y(k,l-2) \end{bmatrix} \quad (6.61)$$

and $L_Y = 6$. The auto-covariance matrix $\mathfrak{R}_Y^{(1,2)}$ is computed as

$$\begin{aligned} \mathfrak{R}_Y^{(1,2)} &= E \left[\mathbf{Y}_{k,l}^{(1,2)} \mathbf{Y}_{k,l}^{(1,2)H} \right] \\ &= \begin{bmatrix} \mathfrak{R}_Y(0,0) & \mathfrak{R}_Y(1,0) & \mathfrak{R}_Y(1,1) & \mathfrak{R}_Y(0,1) & \mathfrak{R}_Y(1,2) & \mathfrak{R}_Y(0,2) \\ \mathfrak{R}_Y(1,0) & \mathfrak{R}_Y(0,0) & \mathfrak{R}_Y(0,1) & \mathfrak{R}_Y(1,1) & \mathfrak{R}_Y(0,2) & \mathfrak{R}_Y(1,2) \\ \mathfrak{R}_Y(1,1) & \mathfrak{R}_Y(0,1) & \mathfrak{R}_Y(0,0) & \mathfrak{R}_Y(1,0) & \mathfrak{R}_Y(0,1) & \mathfrak{R}_Y(1,1) \\ \mathfrak{R}_Y(0,1) & \mathfrak{R}_Y(1,1) & \mathfrak{R}_Y(1,0) & \mathfrak{R}_Y(0,0) & \mathfrak{R}_Y(1,1) & \mathfrak{R}_Y(0,1) \\ \mathfrak{R}_Y(1,2) & \mathfrak{R}_Y(0,2) & \mathfrak{R}_Y(0,1) & \mathfrak{R}_Y(1,1) & \mathfrak{R}_Y(0,0) & \mathfrak{R}_Y(1,0) \\ \mathfrak{R}_Y(0,2) & \mathfrak{R}_Y(1,2) & \mathfrak{R}_Y(1,1) & \mathfrak{R}_Y(0,1) & \mathfrak{R}_Y(1,0) & \mathfrak{R}_Y(0,0) \end{bmatrix} \end{aligned} \quad (6.62)$$

The new backward prediction error vector $\boldsymbol{\beta}^{(1,2)}$ will be

$$\boldsymbol{\beta}^{(1,2)} = \begin{bmatrix} \tilde{e}_1^{(0,0)} \\ \tilde{r}_1^{(1,0)} \\ \tilde{e}_1^{(1,1)} \\ \tilde{e}_2^{(1,1)} \\ \tilde{e}_1^{(1,2)} \\ \tilde{e}_2^{(1,2)} \end{bmatrix} \quad (6.63)$$

and $L_\beta = 6$. The auto-covariance matrix $\mathfrak{R}_\beta^{(1,2)}$ can be obtained as

$$\mathfrak{R}_\beta^{(1,2)} = E \left[\boldsymbol{\beta}^{(1,2)} \boldsymbol{\beta}^{(1,2)H} \right]. \quad (6.64)$$

The coefficient matrices of $\tilde{e}_1^{(1,2)}$ and $\tilde{e}_2^{(1,2)}$ are defined as

$$\tilde{E}_1^{(1,2)} = \begin{bmatrix} d1_{00}^{(1,2)} & d1_{01}^{(1,2)} & 0 \\ d1_{10}^{(1,2)} & d1_{11}^{(1,2)} & 1 \end{bmatrix}, \quad (6.65)$$

$$\tilde{E}_2^{(1,2)} = \begin{bmatrix} d2_{00}^{(1,2)} & d2_{01}^{(1,2)} & 1 \\ d2_{10}^{(1,2)} & d2_{11}^{(1,2)} & 0 \end{bmatrix}. \quad (6.66)$$

The matrix $\mathbf{L}^{(1,2)}$ can be written as

$$\mathbf{L}^{(1,2)} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ b_{00}^{(1,0)} & 1 & 0 & 0 & 0 & 0 \\ a1_{00}^{(1,1)} & a1_{10}^{(1,1)} & 1 & 0 & 0 & 0 \\ a2_{00}^{(1,1)} & a2_{10}^{(1,1)} & 0 & 1 & 0 & 0 \\ d1_{00}^{(1,2)} & d1_{10}^{(1,2)} & d1_{11}^{(1,2)} & d1_{10}^{(1,2)} & 1 & 0 \\ d2_{00}^{(1,2)} & d2_{10}^{(1,2)} & d2_{11}^{(1,2)} & d2_{10}^{(1,2)} & 0 & 1 \end{bmatrix}. \quad (6.67)$$

Entropies of the input data (6.61) and the backward prediction error vector (6.63) at this stage order can be calculated by using (6.26) and (6.27), respectively.

Similarly, we can also calculate the entropy by using the horizontal backward PEFs.

For that case the input data vector will be written as

$$\mathbf{Y}_{k,l}^{(2,1)} = \begin{bmatrix} y(k, l) \\ y(k-1, l) \\ y(k, l-1) \\ y(k, l-1) \\ y(k-2, l-1) \\ y(k-2, l) \end{bmatrix} \quad (6.68)$$

and $L_Y = 6$. The auto-covariance matrix $\mathfrak{R}_Y^{(2,1)}$ is computed as

$$\begin{aligned} \mathfrak{R}_Y^{(2,1)} &= E \left[\mathbf{Y}_{k,l}^{(2,1)} \mathbf{Y}_{k,l}^{(2,1)H} \right] \\ &= \begin{bmatrix} \mathfrak{R}_Y(0,0) & \mathfrak{R}_Y(1,0) & \mathfrak{R}_Y(1,1) & \mathfrak{R}_Y(0,1) & \mathfrak{R}_Y(2,1) & \mathfrak{R}_Y(2,0) \\ \mathfrak{R}_Y(1,0) & \mathfrak{R}_Y(0,0) & \mathfrak{R}_Y(0,1) & \mathfrak{R}_Y(1,1) & \mathfrak{R}_Y(1,1) & \mathfrak{R}_Y(1,0) \\ \mathfrak{R}_Y(1,1) & \mathfrak{R}_Y(0,1) & \mathfrak{R}_Y(0,0) & \mathfrak{R}_Y(1,0) & \mathfrak{R}_Y(1,0) & \mathfrak{R}_Y(1,1) \\ \mathfrak{R}_Y(0,1) & \mathfrak{R}_Y(1,1) & \mathfrak{R}_Y(1,0) & \mathfrak{R}_Y(0,0) & \mathfrak{R}_Y(2,0) & \mathfrak{R}_Y(2,1) \\ \mathfrak{R}_Y(2,1) & \mathfrak{R}_Y(1,1) & \mathfrak{R}_Y(1,0) & \mathfrak{R}_Y(2,0) & \mathfrak{R}_Y(0,0) & \mathfrak{R}_Y(0,1) \\ \mathfrak{R}_Y(2,0) & \mathfrak{R}_Y(1,0) & \mathfrak{R}_Y(1,1) & \mathfrak{R}_Y(2,1) & \mathfrak{R}_Y(0,1) & \mathfrak{R}_Y(0,0) \end{bmatrix} \end{aligned} \quad (6.69)$$

The new backward prediction error vector $\boldsymbol{\beta}^{(2,1)}(k, l)$ will be defined as

$$\boldsymbol{\beta}^{(2,1)} = \begin{bmatrix} \tilde{e}_1^{(0,0)} \\ \tilde{r}_1^{(1,0)} \\ \tilde{e}_1^{(1,1)} \\ \tilde{e}_2^{(1,1)} \\ \tilde{r}_1^{(2,1)} \\ \tilde{r}_2^{(2,1)} \end{bmatrix} \quad (6.70)$$

and $L_\beta = 6$. The auto-covariance matrix $\mathfrak{R}_\beta^{(2,1)}$ is

$$\mathfrak{R}_\beta^{(2,1)} = E \left[\boldsymbol{\beta}^{(2,1)} \boldsymbol{\beta}^{(2,1)H} \right] \quad (6.71)$$

The coefficient matrices of $\tilde{r}_1^{(2,1)}$ and $\tilde{r}_2^{(2,1)}$ are defined as

$$\tilde{R}_1^{(2,1)} = \begin{bmatrix} f1_{00}^{(2,1)} & f1_{01}^{(2,1)} \\ f1_{10}^{(2,1)} & f1_{11}^{(2,1)} \\ 0 & 1 \end{bmatrix}, \quad (6.72)$$

$$\tilde{R}_2^{(2,1)} = \begin{bmatrix} f2_{00}^{(2,1)} & f2_{01}^{(2,1)} \\ f2_{10}^{(2,1)} & f2_{11}^{(2,1)} \\ 1 & 0 \end{bmatrix}. \quad (6.73)$$

The matrix $\mathbf{L}^{(1,2)}$ can be written as

$$\mathbf{L}^{(1,2)} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ b_{00}^{(1,0)} & 1 & 0 & 0 & 0 & 0 \\ a1_{00}^{(1,1)} & a1_{10}^{(1,1)} & 1 & 0 & 0 & 0 \\ a2_{00}^{(1,1)} & a2_{10}^{(1,1)} & 0 & 1 & 0 & 0 \\ f1_{00}^{(1,2)} & f1_{10}^{(1,2)} & f1_{11}^{(1,2)} & f1_{10}^{(1,2)} & 1 & 0 \\ f2_{00}^{(1,2)} & f2_{10}^{(1,2)} & f2_{11}^{(1,2)} & f2_{10}^{(1,2)} & 0 & 1 \end{bmatrix}. \quad (6.74)$$

Entropies of the input data (6.68) and the backward prediction error vector (6.70) at this stage order can be calculated by using (6.26) and (6.27), respectively.

Entropy calculation for the second scanning way

Now, let us assume that we came to this stage order by applying first vertical and then horizontal augmentation. Therefore, we will use (6.56) and (6.53) to determine new input data vector and the backward prediction error vector, respectively. We will calculate the entropies separately by using both vertical and horizontal backward PEFs.

First, we calculate the entropy by using the vertical backward PEFs. The input data vector will be defined as

$$\mathbf{Y}_{k,l}^{(1,2)} = \begin{bmatrix} y(k,l) \\ y(k,l-1) \\ y(k-1,l-1) \\ y(k-1,l) \\ y(k-1,l-2) \\ y(k,l-2) \end{bmatrix} \quad (6.75)$$

and $L_Y = 6$. The auto-covariance matrix $\mathfrak{R}_Y^{(1,2)}$ is computed as

$$\begin{aligned} \mathfrak{R}_Y^{(1,2)} &= E \left[\mathbf{Y}_{k,l}^{(1,2)} \mathbf{Y}_{k,l}^{(1,2)H} \right] \\ &= \begin{bmatrix} \mathfrak{R}_Y(0,0) & \mathfrak{R}_Y(0,1) & \mathfrak{R}_Y(1,1) & \mathfrak{R}_Y(1,0) & \mathfrak{R}_Y(1,2) & \mathfrak{R}_Y(0,2) \\ \mathfrak{R}_Y(0,1) & \mathfrak{R}_Y(0,0) & \mathfrak{R}_Y(1,0) & \mathfrak{R}_Y(1,1) & \mathfrak{R}_Y(1,1) & \mathfrak{R}_Y(0,1) \\ \mathfrak{R}_Y(1,1) & \mathfrak{R}_Y(1,0) & \mathfrak{R}_Y(0,0) & \mathfrak{R}_Y(0,1) & \mathfrak{R}_Y(0,1) & \mathfrak{R}_Y(1,1) \\ \mathfrak{R}_Y(1,0) & \mathfrak{R}_Y(1,1) & \mathfrak{R}_Y(0,1) & \mathfrak{R}_Y(0,0) & \mathfrak{R}_Y(0,2) & \mathfrak{R}_Y(1,2) \\ \mathfrak{R}_Y(1,2) & \mathfrak{R}_Y(1,1) & \mathfrak{R}_Y(0,1) & \mathfrak{R}_Y(0,2) & \mathfrak{R}_Y(0,0) & \mathfrak{R}_Y(1,0) \\ \mathfrak{R}_Y(0,2) & \mathfrak{R}_Y(0,1) & \mathfrak{R}_Y(1,1) & \mathfrak{R}_Y(1,2) & \mathfrak{R}_Y(1,0) & \mathfrak{R}_Y(0,0) \end{bmatrix} \end{aligned} \quad (6.76)$$

The new backward prediction error vector $\boldsymbol{\beta}^{(1,2)}(k,l)$ will be

$$\boldsymbol{\beta}^{(1,2)} = \begin{bmatrix} \tilde{e}_1^{(0,0)} \\ \tilde{e}_1^{(0,1)} \\ \tilde{r}_1^{(1,1)} \\ \tilde{r}_2^{(1,1)} \\ \tilde{e}_1^{(1,2)} \\ \tilde{e}_2^{(1,2)} \end{bmatrix} \quad (6.77)$$

and $L_\beta = 6$. The auto-covariance matrix $\mathfrak{R}_\beta$ can be obtained as

$$\mathfrak{R}_\beta^{(1,2)} = E \left[\boldsymbol{\beta}^{(1,2)} \boldsymbol{\beta}^{(1,2)H} \right]. \quad (6.78)$$

The coefficient matrices of $\tilde{e}_1^{(1,2)}$ and $\tilde{e}_2^{(1,2)}$ are defined as

$$\tilde{E}_1^{(1,2)} = \begin{bmatrix} g1_{00}^{(1,2)} & g1_{01}^{(1,2)} & 0 \\ g1_{10}^{(1,2)} & g1_{11}^{(1,2)} & 1 \end{bmatrix}, \quad (6.79)$$

and

$$\tilde{E}_2^{(1,2)} = \begin{bmatrix} g2_{00}^{(1,2)} & g2_{01}^{(1,2)} & 1 \\ g2_{10}^{(1,2)} & g2_{11}^{(1,2)} & 0 \end{bmatrix}, \quad (6.80)$$

respectively. The matrix $\mathbf{L}^{(1,2)}$ can be written as

$$\mathbf{L}^{(1,2)} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ a_{00}^{(0,1)} & 1 & 0 & 0 & 0 & 0 \\ c1_{00}^{(1,1)} & c1_{01}^{(1,1)} & 1 & 0 & 0 & 0 \\ c2_{00}^{(1,1)} & c2_{01}^{(1,1)} & 0 & 1 & 0 & 0 \\ g1_{00}^{(1,2)} & g1_{01}^{(1,2)} & g1_{11}^{(1,2)} & g1_{10}^{(1,2)} & 1 & 0 \\ g2_{00}^{(1,2)} & g2_{01}^{(1,2)} & g2_{11}^{(1,2)} & g2_{10}^{(1,2)} & 0 & 1 \end{bmatrix}. \quad (6.81)$$

Entropies of the input data (6.75) and the backward prediction error vector (6.77) at this stage order can be calculated by using (6.26) and (6.27), respectively.

Similarly, the entropy by using the horizontal backward PEFs can also be calculated. For that case the input data vector will be written as

$$\mathbf{Y}_{k,l}^{(2,1)} = \begin{bmatrix} y(k,l) \\ y(k,l-1) \\ y(k-1,l-1) \\ y(k-1,l) \\ y(k-2,l-1) \\ y(k-2,l) \end{bmatrix} \quad (6.82)$$

and $L_Y = 6$. The auto-covariance matrix $\mathfrak{R}_Y^{(2,1)}$ is computed as

$$\begin{aligned} \mathfrak{R}_Y^{(2,1)} &= E \left[\mathbf{Y}_{k,l}^{(2,1)} \mathbf{Y}_{k,l}^{(2,1)H} \right] \\ &= \begin{bmatrix} \mathfrak{R}_Y(0,0) & \mathfrak{R}_Y(0,1) & \mathfrak{R}_Y(1,1) & \mathfrak{R}_Y(1,0) & \mathfrak{R}_Y(2,1) & \mathfrak{R}_Y(2,0) \\ \mathfrak{R}_Y(0,1) & \mathfrak{R}_Y(0,0) & \mathfrak{R}_Y(1,0) & \mathfrak{R}_Y(1,1) & \mathfrak{R}_Y(2,0) & \mathfrak{R}_Y(2,1) \\ \mathfrak{R}_Y(1,1) & \mathfrak{R}_Y(1,0) & \mathfrak{R}_Y(0,0) & \mathfrak{R}_Y(0,1) & \mathfrak{R}_Y(1,0) & \mathfrak{R}_Y(1,1) \\ \mathfrak{R}_Y(1,0) & \mathfrak{R}_Y(1,1) & \mathfrak{R}_Y(0,1) & \mathfrak{R}_Y(0,0) & \mathfrak{R}_Y(1,1) & \mathfrak{R}_Y(1,0) \\ \mathfrak{R}_Y(2,1) & \mathfrak{R}_Y(2,0) & \mathfrak{R}_Y(1,0) & \mathfrak{R}_Y(1,1) & \mathfrak{R}_Y(0,0) & \mathfrak{R}_Y(0,1) \\ \mathfrak{R}_Y(2,0) & \mathfrak{R}_Y(2,1) & \mathfrak{R}_Y(1,1) & \mathfrak{R}_Y(1,0) & \mathfrak{R}_Y(0,1) & \mathfrak{R}_Y(0,0) \end{bmatrix} \end{aligned} \quad (6.83)$$

The new backward prediction error vector $\boldsymbol{\beta}^{(2,1)}$ will be defined as

$$\boldsymbol{\beta}^{(2,1)} = \begin{bmatrix} \tilde{e}_1^{(0,0)} \\ \tilde{e}_1^{(0,1)} \\ \tilde{r}_1^{(1,1)} \\ \tilde{r}_2^{(1,1)} \\ \tilde{r}_1^{(2,1)} \\ \tilde{r}_2^{(2,1)} \end{bmatrix} \quad (6.84)$$

and $L_\beta = 6$. The auto-covariance matrix $\mathfrak{R}_\beta^{(2,1)}$ is

$$\mathfrak{R}_\beta^{(2,1)} = E \left[\boldsymbol{\beta}^{(2,1)} \boldsymbol{\beta}^{(2,1)H} \right] \quad (6.85)$$

The coefficient matrices of $\tilde{r}_1^{(2,1)}$ and $\tilde{r}_2^{(2,1)}$ are defined as

$$\tilde{R}_1^{(2,1)} = \begin{bmatrix} h1_{00}^{(2,1)} & h1_{01}^{(2,1)} \\ h1_{10}^{(2,1)} & h1_{11}^{(2,1)} \\ 0 & 1 \end{bmatrix}, \quad (6.86)$$

and

$$\tilde{R}_2^{(2,1)} = \begin{bmatrix} h2_{00}^{(2,1)} & h2_{01}^{(2,1)} \\ h2_{10}^{(2,1)} & h2_{11}^{(2,1)} \\ 1 & 0 \end{bmatrix}, \quad (6.87)$$

respectively. The matrix $\mathbf{L}^{(2,1)}$ can be formed as

$$\mathbf{L}^{(2,1)} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ a_{00}^{(0,1)} & 1 & 0 & 0 & 0 & 0 \\ c1_{00}^{(1,1)} & c1_{01}^{(1,1)} & 1 & 0 & 0 & 0 \\ c2_{00}^{(1,1)} & c2_{01}^{(1,1)} & 0 & 1 & 0 & 0 \\ h1_{00}^{(2,1)} & h1_{01}^{(2,1)} & h1_{11}^{(2,1)} & h1_{10}^{(2,1)} & 1 & 0 \\ h2_{00}^{(2,1)} & h2_{01}^{(2,1)} & h2_{11}^{(2,1)} & h2_{10}^{(2,1)} & 0 & 1 \end{bmatrix}. \quad (6.88)$$

Entropies of the input data (6.82) and the backward prediction error vector (6.84) at this stage order can be calculated by using (6.26) and (6.27), respectively.

Up to now, we showed how to calculate the entropies of the input data vector and the backward prediction error vector from vertical and horizontal backward PEFs by following the augmentation directions. The method explained here can be applied easily for the higher stage order to calculate the information loss between input data vector and the backward prediction error vector. To make it clear, the numerical examples will be given in the following sections.

6.2.1.4 Numerical example for entropy calculation

Here, we will calculate the information loss between the input vector and the backward prediction error vector for the AR process given in Example 3, in Section 5.3. The method is applied to the AR data generated by using the following coefficient matrix.

$$A_{org}^{(4,4)} = \begin{bmatrix} 1 & -0.4400 & 0.0074 & 0.1600 & -0.4000 \\ -0.4400 & 0.2200 & -0.0095 & -0.0700 & 0.1700 \\ -0.1200 & 0.0690 & 0.0140 & 0.0200 & -0.0480 \\ -0.0520 & 0.0200 & -0.0030 & 0.0015 & 0.0180 \\ -0.0060 & 0.0027 & 0.000012 & -0.0032 & 0.0070 \end{bmatrix} \quad (6.89)$$

The input and the backward prediction error vector entropies are calculated for each stage order by using vertical and horizontal backward PEFs. At each stage order, we computed two input and two backward prediction error vector, but one pair of them is taken into account when calculating the next stage order entropies as explained above sections. Table 6.1 gives the input and the backward prediction error vector entropies calculated at each stage order using the proposed method.

At each order state, the entropies calculated for both augmentation directions. The entropies of the input data and the backward prediction error vector for the vertical augmentation direction is illustrated in Figure 6.3 and the entropies of the input data

Table 6.1: Input and backward prediction error entropies for the proposed lattice structures.

Scan Steps	Vertical& Horizontal PEFs Order	Input Entropy (vertical)	BPEE ¹ (vertical)	Input Entropy (horizontal)	BPEE (horizontal)
0	(0,0)-(0,0)	1.5978	1.5978	1.5978	1.5978
1	(0,1)-(1,0)	3.06447	3.06312	3.14463	3.14414
2	(1,1)-(2,0)	6.03495	6.03371	4.69138	4.68929
3	(1,2)-(2,1)	8.92434	8.92096	9.00501	9.00263
4	(2,2)-(3,1)	13.3169	13.3112	11.9750	11.9706
5	(2,3)-(3,2)	17.6284	17.4825	17.7089	17.5649
6	(3,3)-(4,2)	23.4422	23.2941	22.1005	21.9599
7	(3,4)-(4,3)	29.1736	28.7909	29.2551	28.8804
8	(4,4)-(5,3)	36.4070	36.0326	35.0675	34.6944

¹ Backward Prediction Error Entropy

and the backward error vector for the horizontal augmentation direction is shown in Figure 6.4. In our proposed method the entropy loss is very minimal due two reasons: (i) the increase of the number of backward PEFs after each augmentation process, and (ii) the determinant of the matrix $\mathbf{L}^{(p,q)}$ (6.9) being equal to 1.

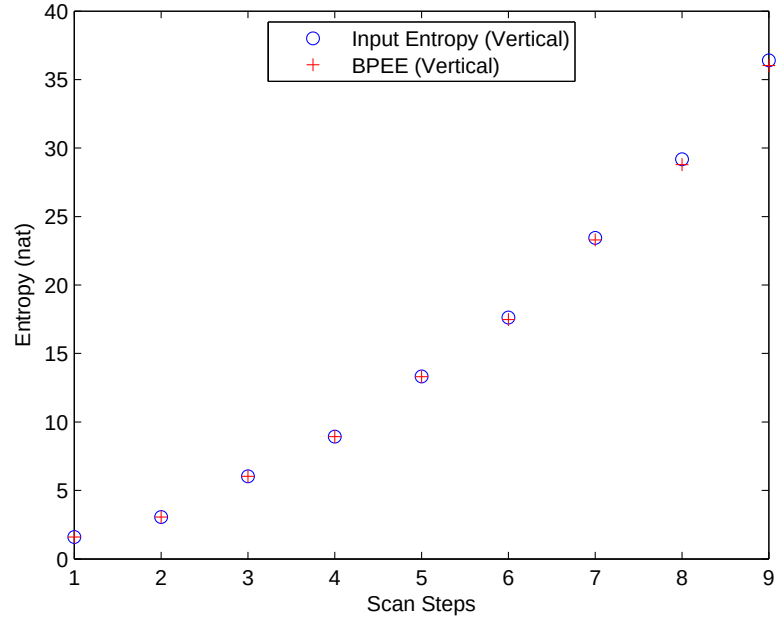


Figure 6.3: Entropy Comparisons between the input data and the backward prediction error vector entropies calculated for the vertical augmentation case.

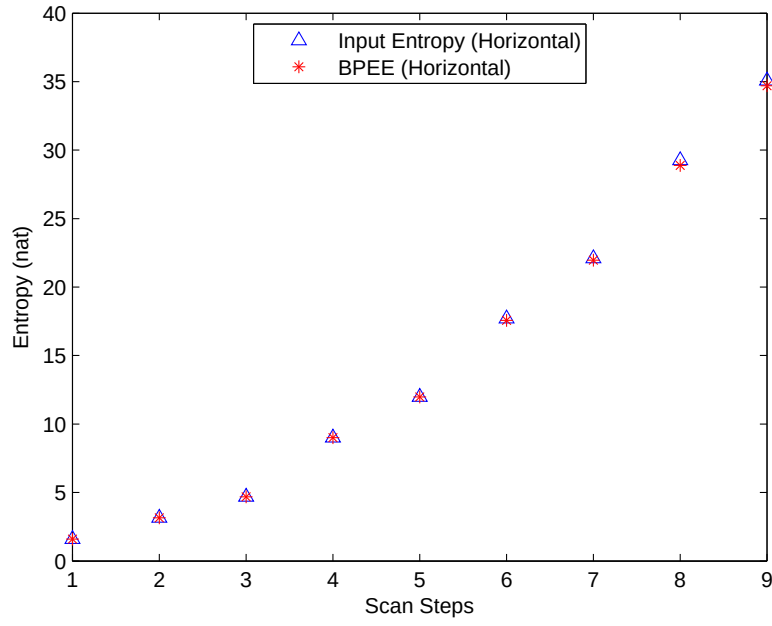


Figure 6.4: Entropy Comparisons between the input data and the backward prediction error vector entropies calculated for the horizontal augmentation case.

6.2.1.5 Discussion of example

This example shows how close the backward prediction error entropy is to the input entropy. It can be said that the proposed method conveys the information into the backward prediction errors conveniently. The increase in the number of error fields after each augmentation causes the backward prediction error entropy to be close to the input entropy. The proposed filter models approximate the input entropy more closely since the determinant of the matrix $\mathbf{L}^{(p,q)}$ (6.9) is always equal to 1.

7. CONCLUSIONS

In this thesis, we have presented a new efficient general 2-D AR lattice structure for modeling of 2-D AR random fields. We have introduced the auxiliary vertical and horizontal PEFs for an arbitrary rectangular PSR for the analysis and synthesis of 2-D systems, and explained the augmentation concept to scan the given data. In contrast with other existing 2-D lattice configurations [11, 14, 16–19, 29, 32, 34, 35, 39, 40, 42, 55, 57, 59, 64–67], the proposed structure generates all types of QP, ASHP and NC half-plane 2-D lattice models concurrently. It is also possible to obtain any arbitrary (M, N) th order stage 2-D AR filter models. On the other hand, it is interesting to note that for a given limited data size, the order of the 2-D lattice filters are bounded with the size of this observation.

For the proposed method, stability is not guaranteed but the QP model filters are found by experimentation to be almost always stable due to the fact that the 2-D lattice parameter factors are calculated by Burg's method [60] which definitely leads to the maximum entropy. We have compared the proposed method with the existing structures. The comparison results show that the proposed structure requires few numbers of lattice sections and lattice parameters.

We have shown that the information loss between the input data and backward prediction error vector is minimum due to the increase in the number of backward PEFs after each order increase. It can be said that the proposed method conveys the information into the backward prediction errors conveniently. We know that the increase in the number of error fields causes the information loss to decrease [34]. Since we have an increase in the number of error fields after each order incrementation, the backward prediction error entropy is getting close to the input entropy. It can be deduced that the input data and the backward prediction error vector contain nearly the same amount of information and one may be regained from the other.

The small difference between the input and backward prediction error entropies occurs due to round-off errors and the mandatory initiation of auxiliary PEFs for the ones newly created during the augmentation process.

In this thesis, the examples show that the proposed method gives good results for modeling 2-D AR process. The model parameters estimated are close to the original ones. It is seen that the dominant parameter values in the coefficient matrix are estimated correctly. In addition, spectrum estimation gives very good accuracy.

In some applications, noncausal half-plane models are preferable for the 2-D spectrum estimation with superior resolution [58]. The proposed method is also capable of producing the necessary 2-D noncausal half-plane models without any additional computational cost. In order to improve the spectrum geometry and estimation accuracy, we have combined all QP and NC models by computing their harmonic mean. Experimental results show the high resolution characteristics of the model in the frequency domain. Application of these results to the ISAR/SAR imaging will be presented subsequently.

In some examples, it has been shown that 2-D ARMA process can be modeled as a 2-D AR process by using the proposed method, but further investigations are necessary to prove this conjecture.

The comparison of the classical FFT transform and the results of the proposed method for the short data record has shown that the proposed method is superior to the classical FFT method. AR spectrum estimation calculated by the proposed method has better frequency resolution and the distortion coming out of the side lobes which are seen in the classical FFT method is not observed in AR spectrum estimation found by using the proposed method.

Since the proposed structure inherently generates a set of orthogonal backward prediction error vectors, linear adaptive algorithms can be applied to solve for 2-D system parameters. We also believe that this orthogonality property can be used to derive 2-D ARMA models and to solve the 2-D joint-process estimation problem.

We can list the following items as future work:

- A proof for the stability of each predicted 2-D lattice filter can be studied.

- The proposed method can be applied on different ISAR/SAR images.
- 2-D ARMA process can be modeled as a 2-D AR process by using the proposed method.
- A study can be done to derive 2-D ARMA models and to solve the 2-D joint-process estimation problem by using the proposed method.
- A study can be carried on designing 2-D IIR filters.

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APPENDICES

APPENDIX A.1 : The Calculation of the Lattice Reflection Coefficient Matrices

APPENDIX A.2 : Explanation of the Reduction of the Support Data

APPENDIX A.3 : Definitions of the Constant Matrix Ψ and Vector Π

APPENDIX A.4 : Itakura's LPC Distance Measure

APPENDIX A.5 : Spectrum Estimation

APPENDIX A.1: The Calculation of the Lattice Reflection Coefficient Matrices

In order to calculate lattice coefficients, we will take into account the sum of the mean squared values of the horizontal/vertical forward and backward PEFs depending on the augmentation type. For obtaining a general formula, we can write a set of PEFs propagation equation as follows

$$\mathbf{f} = \mathbf{f}_x - \mathbf{K}^H \mathbf{b}_y \quad (\text{A.1a})$$

$$\mathbf{b} = \mathbf{b}_x - \mathbf{K}^T \mathbf{f}_y \quad (\text{A.1b})$$

where $\mathbf{f}$ and $\mathbf{b}$ are the newly predicted forward and backward PEFs, respectively. Indices x and y represent the previous order PEF of the related forward and backward ones. K represents the coefficient matrix to be calculated. For simplification, stage order values are omitted during the construction of the formulas. Before applying the mean-square method, we should make some definitions and introduce some rules which are shown in the following tables.

Table A.1: Some definitions for expected values.

$\Delta_1 = E [\mathbf{b}_y \mathbf{f}_x^H]$	$\Sigma_{b_y} = E [\mathbf{b}_y \mathbf{b}_y^H]$
$\Delta_2 = E [\mathbf{f}_y \mathbf{b}_x^H]$	$\Sigma_{f_y} = E [\mathbf{f}_y \mathbf{f}_y^H]$

From the definition given in the Table (A.1), it is clearly seen the following equations can be defined:

$$\begin{aligned} \Sigma_{f_y}^H &= \Sigma_{f_y}, & \Sigma_{f_y}^T &= \Sigma_{f_y}^* \\ \Sigma_{b_y}^H &= \Sigma_{b_y}, & \Sigma_{b_y}^T &= \Sigma_{b_y}^* \end{aligned}$$

Table A.2: Some useful rules for derivations.

$\frac{\partial}{\partial \mathbf{X}} \mathbf{X}^T \mathbf{A} = \mathbf{A}$
$\frac{\partial}{\partial \mathbf{X}} \mathbf{A}^T \mathbf{X} = \mathbf{A}$
$\frac{\partial}{\partial \mathbf{X}} \mathbf{A} \mathbf{X} = \mathbf{A}^T$
$\frac{\partial}{\partial \mathbf{K}} \mathbf{K}^H \Sigma_{b_y} \mathbf{K} = (\mathbf{K}^H \Sigma_{b_y})^T = \Sigma_{b_y}^T \mathbf{K}^*$
$\frac{\partial}{\partial \mathbf{K}} \mathbf{K}^T \Sigma_{f_y} \mathbf{K}^* = \Sigma_{f_y}^T \mathbf{K}^*$

In order to find the coefficient matrix K , we can write the mean square propagation equation by using the sum of the mean squared of the newly predicted forward and backward PEFs given in (A.1),

$$\mathbf{P}_f + \mathbf{P}_b = E [\mathbf{f} \mathbf{f}^H] + E [\mathbf{b} \mathbf{b}^H] \quad (\text{A.2})$$

where $\mathbf{P}_f$ and $\mathbf{P}_b$ represent the mean squared for $\mathbf{f}$ and $\mathbf{b}$, respectively. If we substitute $\mathbf{f}$ and $\mathbf{b}$ into (A.2), we have,

$$\begin{aligned} \mathbf{P}_f &= E [(\mathbf{f}_x - \mathbf{K}^H \mathbf{b}_y)(\mathbf{f}_x - \mathbf{K}^H \mathbf{b}_y)^H] \\ &= E [(\mathbf{f}_x - \mathbf{K}^H \mathbf{b}_y)(\mathbf{f}_x^H - \mathbf{b}_y^H \mathbf{K})] \\ &= E [\mathbf{f}_x \mathbf{f}_x^H] - E [\mathbf{f}_x \mathbf{b}_y^H \mathbf{K}] - E [\mathbf{K}^H \mathbf{b}_y \mathbf{f}_x^H] + E [\mathbf{K}^H \mathbf{b}_y \mathbf{b}_y^H \mathbf{K}] \end{aligned} \quad (\text{A.3})$$

$$\begin{aligned}
\mathbf{P}_b &= E [(\mathbf{b}_x - \mathbf{K}^T \mathbf{f}_y)(\mathbf{b}_x - \mathbf{K}^T \mathbf{f}_y)^H] \\
&= E [(\mathbf{b}_x - \mathbf{K}^T \mathbf{f}_y)(\mathbf{b}_x^H - \mathbf{f}_y^H \mathbf{K}^*)] \\
&= E [\mathbf{b}_x \mathbf{b}_x^H] - E [\mathbf{b}_x \mathbf{f}_y^H \mathbf{K}] - E [\mathbf{K}^T \mathbf{f}_y \mathbf{b}_x^H] + E [\mathbf{K}^T \mathbf{f}_y \mathbf{f}_y^H \mathbf{K}^*] \quad (\text{A.4})
\end{aligned}$$

In that step, we can use the definitions given in the table A.1, and the equations (A.3) and (A.4) can be written as

$$\mathbf{P}_f = \mathbf{P}_{f_x} - \Delta_1^H \mathbf{K} - \mathbf{K}^H \Delta_1 + \mathbf{K}^H \Sigma_{b_y} \mathbf{K} \quad (\text{A.5})$$

$$\mathbf{P}_b = \mathbf{P}_{b_x} - \Delta_2^H \mathbf{K}^* - \mathbf{K}^T \Delta_2 + \mathbf{K}^T \Sigma_{f_y} \mathbf{K}^* \quad (\text{A.6})$$

where $\mathbf{P}_{f_x}$ and $\mathbf{P}_{b_x}$ are the mean square propagation values of the previous order of the forward and backward PEFs.

Now, we can minimize the equation (A.2) with respect to the $\mathbf{K}$,

$$\begin{aligned}
\frac{\partial}{\partial \mathbf{K}} \text{tr}(\mathbf{P}_f + \mathbf{P}_b) &= 0 \\
-\Delta_1^* + \Sigma_{b_y}^T \mathbf{K}^* - \Delta_2 + \Sigma_{f_y} \mathbf{K}^* &= 0 \\
\Sigma_{b_y}^T \mathbf{K}^* + \Sigma_{f_y} \mathbf{K}^* &= \Delta_1^* + \Delta_2 \\
\mathbf{K}^* &= [\Sigma_{b_y}^T + \Sigma_{f_y}]^{-1} [\Delta_1^* + \Delta_2] \quad (\text{A.7})
\end{aligned}$$

If we apply a conjugation on (A.7),

$$\mathbf{K} = [\Sigma_{b_y} + \Sigma_{f_y}^*]^{-1} [\Delta_1 + \Delta_2^*] \quad (\text{A.8})$$

After finding the equation (A.8), it is possible to derive a simple propagation equation by substituting (A.8) into the equation. (A.5) and (A.6).

$$\begin{aligned}
\min_{\mathbf{K}} \text{tr} [\mathbf{P}_f + \mathbf{P}_b] &= \text{tr} [\mathbf{P}_{f_x} + \mathbf{P}_{b_x}] \\
&\quad - \text{tr} [\Delta_1^H \mathbf{K}] - \text{tr} [\mathbf{K}^H \Delta_1] + [\mathbf{K}^H \Sigma_{b_y} \mathbf{K}] \\
&\quad - \text{tr} [\Delta_2^H \mathbf{K}^*] - \text{tr} [\mathbf{K}^T \Delta_2] + \text{tr} [\mathbf{K}^T \Sigma_{f_y} \mathbf{K}^*] \quad (\text{A.9})
\end{aligned}$$

By using the following definitions,

$$\text{tr} [\mathbf{A}] = \text{tr} [\mathbf{A}^T] \quad (\text{A.10})$$

$$\text{tr} [\mathbf{K}^H \Delta_1] = \text{tr} [\Delta_1^T \mathbf{K}^*] \quad (\text{A.11})$$

$$\text{tr} [\mathbf{K}^T \Delta_2] = \text{tr} [\Delta_2^T \mathbf{K}] \quad (\text{A.12})$$

the equation (A.9) can be rewritten as follows:

$$\begin{aligned}
\min_{\mathbf{K}} \text{tr} [\mathbf{P}_f + \mathbf{P}_b] &= \text{tr} [\mathbf{P}_{f_x} + \mathbf{P}_{b_x}] \\
&\quad - \text{tr} [\Delta_1^H \mathbf{K}] - \text{tr} [\Delta_1^T \mathbf{K}^*] + [\mathbf{K}^H \Sigma_{b_y} \mathbf{K}] \\
&\quad - \text{tr} [\Delta_2^H \mathbf{K}^*] - \text{tr} [\Delta_2^T \mathbf{K}] + \text{tr} [\mathbf{K}^T \Sigma_{f_y} \mathbf{K}^*] \\
&= \text{tr} [\mathbf{P}_{f_x} + \mathbf{P}_{b_x}] \\
&\quad - \text{tr} [(\Delta_1^H + \Delta_2^T) \mathbf{K}] - \text{tr} [(\Delta_1^T + \Delta_2^H) \mathbf{K}^*] \\
&\quad + \text{tr} [\mathbf{K}^H \Sigma_{b_y} \mathbf{K}] + \text{tr} [\mathbf{K}^T \Sigma_{f_y} \mathbf{K}^*] \quad (\text{A.13})
\end{aligned}$$

It is possible to write the following equations from the (A.8),

$$\mathbf{K}^H \left[\Sigma_{b_y} + \Sigma_{f_y}^T \right] = \left[\Delta_1^H + \Delta_2^T \right] \quad (\text{A.14})$$

$$\mathbf{K}^T \left[\Sigma_{b_y}^* + \Sigma_{f_y} \right] = \left[\Delta_1^T + \Delta_2^H \right] \quad (\text{A.15})$$

and we substitute them into the equation (A.13), then we obtain

$$\begin{aligned} \min_{\mathbf{K}} \text{tr} [\mathbf{P}_f + \mathbf{P}_b] &= \text{tr} [\mathbf{P}_{f_x} + \mathbf{P}_{b_x}] \\ &\quad - \text{tr} \left[\mathbf{K}^H \left(\Sigma_{b_y} + \Sigma_{f_y}^T \right) \mathbf{K} \right] - \text{tr} \left[\mathbf{K}^T \left(\Sigma_{b_y}^* + \Sigma_{f_y} \right) \mathbf{K}^* \right] \\ &\quad + \text{tr} [\mathbf{K}^H \Sigma_{b_y} \mathbf{K}] + \text{tr} [\mathbf{K}^T \Sigma_{f_y} \mathbf{K}^*] \\ &= \text{tr} [\mathbf{P}_{f_x} + \mathbf{P}_{b_x}] \\ &\quad - \text{tr} \left[\mathbf{K}^H \Sigma_{f_y}^T \mathbf{K} \right] - \text{tr} \left[\mathbf{K}^T \Sigma_{b_y}^* \mathbf{K}^* \right] \\ &= \text{tr} [\mathbf{P}_{f_x} + \mathbf{P}_{b_x}] - \text{tr} \left[\mathbf{K}^T \left(\Sigma_{f_y} + \Sigma_{b_y}^* \right) \mathbf{K}^* \right] \\ \text{or} \\ &= \text{tr} [\mathbf{P}_{f_x} + \mathbf{P}_{b_x}] - \text{tr} \left[\mathbf{K}^H \left(\Sigma_{f_y}^* + \Sigma_{b_y} \right) \mathbf{K} \right] \end{aligned} \quad (\text{A.16})$$

From the definition given in the Table (A.1), it is clearly seen the following equations can be defined:

$$\begin{aligned} \Sigma_{f_y}^H &= \Sigma_{f_y}, & \Sigma_{f_y}^T &= \Sigma_{f_y}^* \\ \Sigma_{b_y}^H &= \Sigma_{b_y}, & \Sigma_{b_y}^T &= \Sigma_{b_y}^* \end{aligned}$$

In some cases, $f_x = f_y$ and $b_x = b_y$ can occur as in the equation (A.17), then the calculation of the equation (A.5) can be computed as follows:

$$\mathbf{f} = \mathbf{f}_x - \mathbf{K}^H \mathbf{b}_y \quad (\text{A.17a})$$

$$\mathbf{b} = \mathbf{b}_y - \mathbf{K}^T \mathbf{f}_x \quad (\text{A.17b})$$

From the equation (A.17),

$$\Delta_1 = E [\mathbf{b}_y \mathbf{f}_x^H]$$

$$\Delta_2 = E [\mathbf{f}_x \mathbf{b}_y^H]$$

and it is easily seen that $\Delta_2 = \Delta_1^H$.

If we calculate the $\mathbf{K}$, we find that

$$\mathbf{K} = \left[\Sigma_{b_y} + \Sigma_{f_x}^* \right]^{-1} \left[\Delta_1^H + \Delta_1^* \right] \quad (\text{A.18})$$

The equation (A.8) can be used as a general form to calculate the lattice coefficients. It can be used in initial and augmentation cases easily.

Initial First Order Case

For the initial case, we should find two lattice coefficient values, $k_{e_1}^{(0,1)}$ and $k_{r_1}^{(1,0)}$ that will be calculated from vertical and horizontal forward/backward PEFs, respectively. If we define

$$\begin{aligned}\mathbf{f} &= e_1^{(0,1)}(k, l) \\ \mathbf{b} &= \tilde{e}_1^{(0,1)}(k, l) \\ \mathbf{f}_x &= e_1^{(0,0)}(k, l) \\ \mathbf{b}_x &= \tilde{e}_1^{(0,0)}(k, l-1) \\ \mathbf{f}_y &= e_1^{(0,0)}(k, l) \\ \mathbf{b}_y &= \tilde{e}_1^{(0,0)}(k, l-1) \\ \mathbf{K} &= k_{e_1}^{(0,1)}\end{aligned}$$

and write the lattice structure equations given in (A.1) again, we get the first order lattice structure equations in terms of vertical forward and backward PEFs.

$$e_1^{(0,1)}(k, l) = e_1^{(0,0)}(k, l) - k_{e_1}^{(0,1)H} \tilde{e}_1^{(0,0)}(k, l-1) \quad (\text{A.19a})$$

$$\tilde{e}_1^{(0,1)}(k, l) = \tilde{e}_1^{(0,0)}(k, l-1) - k_{e_1}^{(0,1)T} e_1^{(0,0)}(k, l) \quad (\text{A.19b})$$

Now we can easily write the following definitions:

$$\Sigma_{b_y}^{(0,0)} = E \left[\tilde{e}_1^{(0,0)}(k, l-1) \tilde{e}_1^{(0,0)H}(k, l-1) \right] \quad (\text{A.20a})$$

$$\Sigma_{f_y}^{(0,0)} = E \left[e_1^{(0,0)}(k, l) e_1^{(0,0)H}(k, l) \right] \quad (\text{A.20b})$$

$$\Delta_1^{(0,0)} = E \left[\tilde{e}_1^{(0,0)}(k, l-1) e_1^{(0,0)H}(k, l) \right] \quad (\text{A.20c})$$

$$\Delta_2^{(0,0)} = E \left[\tilde{e}_1^{(0,0)}(k, l) \tilde{e}_1^{(0,0)H}(k, l-1) \right]. \quad (\text{A.20d})$$

We know that at the initial state, the vertical forward and backward PEFs are initialized as

$$e_1^{(0,0)}(k, l) = \tilde{e}_1^{(0,0)}(k, l) = y(k, l)$$

and the definitions given by (A.20) can be reformed as

$$\Sigma_{b_y}^{(0,0)} = E \left[y(k, l-1) y^H(k, l-1) \right] \quad (\text{A.21a})$$

$$\Sigma_{f_y}^{(0,0)} = E \left[y(k, l) y^H(k, l) \right] \quad (\text{A.21b})$$

$$\Delta_1^{(0,0)} = E \left[y(k, l-1) y^H(k, l) \right] \quad (\text{A.21c})$$

$$\Delta_2^{(0,0)} = E \left[y(k, l) y^H(k, l-1) \right]. \quad (\text{A.21d})$$

It is well-known that the expected value operation of a random variable can be stated in terms of correlation values as follows;

$$\Re_y(0,0) = E [y(k,l-1)y^H(k,l-1)] \quad (\text{A.22a})$$

$$\Re_y(0,0) = E [y(k,l)y^H(k,l)] \quad (\text{A.22b})$$

$$\Re_y(0,-1) = E [y(k,l-1)y^H(k,l)] \quad (\text{A.22c})$$

$$\Re_y(0,1) = E [y(k,l)y^H(k,l-1)]. \quad (\text{A.22d})$$

Now, we can calculate $k_{e_1}^{(0,1)}$ by using the equation (A.8),

$$k_{e_1}^{(0,1)} = [\Re_y(0,0) + \Re_y(0,0)^*]^{-1} [\Re_y(0,-1) + \Re_y(0,1)^*] \quad (\text{A.23})$$

and we know that $\Re(0,-1)$ is equal to $\Re(0,1)^H$, then

$$k_{e_1}^{(0,1)} = [\Re_y(0,0) + \Re_y(0,0)^*]^{-1} [\Re_y(0,1)^H + \Re_y(0,1)^*]. \quad (\text{A.24})$$

Indeed, the result is accord to the equation (A.18).

The power of the auxiliary vertical PEFs can be indicated as

$$P_{e_1}^{(0,1)} = E [e_1^{(0,1)}(k,l)e_1^{(0,1)H}(k,l)] + E [\tilde{e}_1^{(0,1)}(k,l)\tilde{e}_1^{(0,1)H}(k,l)]. \quad (\text{A.25})$$

If we substitute (A.24) into (A.25), then we can rephrase it as

$$P_{e_1}^{(0,1)} = 2\Re_y(0,0) - k_{e_1}^{(0,1)*} (\Re_y(0,0) + \Re_y(0,0)^T) k_{e_1}^{(0,1)} \quad (\text{A.26})$$

$$= 2\Re_y(0,0) \left(1 - k_{e_1}^{(0,1)} k_{e_1}^{(0,1)*}\right) \quad (\text{A.27})$$

Similarly, for the horizontal forward and backward PEFs, lattice structure can be written as

$$r_1^{(1,0)}(k,l) = r_1^{(0,0)}(k,l) - k_{r_1}^{(1,0)H} \tilde{r}_1^{(0,0)}(k-1,l) \quad (\text{A.28a})$$

$$\tilde{r}_1^{(1,0)}(k,l) = \tilde{r}_1^{(0,0)}(k-1,l) - k_{r_1}^{(0,1)T} r_1^{(0,0)}(k,l) \quad (\text{A.28b})$$

where

$$\mathbf{f} = r_1^{(1,0)}(k,l)$$

$$\mathbf{b} = \tilde{r}_1^{(1,0)}(k,l)$$

$$\mathbf{f}_x = r_1^{(0,0)}(k,l)$$

$$\mathbf{b}_x = \tilde{r}_1^{(0,0)}(k-1,l)$$

$$\mathbf{f}_y = r_1^{(0,0)}(k,l)$$

$$\mathbf{b}_y = \tilde{r}_1^{(0,0)}(k-1,l)$$

$$\mathbf{K} = k_{r_1}^{(1,0)}.$$

The result for the $k_{r_1}^{(1,0)}$ by using the equation (A.8) can be found

$$k_{r_1}^{(0,1)} = [\Re_y(0,0) + \Re_y(0,0)^*]^{-1} [\Re_y(-1,0) + \Re_y(1,0)^*] \quad (\text{A.29})$$

and with the equality of $\Re(-1, 0) = \Re(1, 0)^H$, then we have

$$k_{r_1}^{(0,1)} = [\Re_y(0, 0) + \Re_y(0, 0)^*]^{-1} [\Re_y(1, 0)^H + \Re_y(1, 0)^*]. \quad (\text{A.30})$$

The power of the auxiliary horizontal PEFs can be rephrased as

$$\begin{aligned} P_{r_1}^{(1,0)} &= E \left[r_1^{(1,0)}(k, l) r_1^{(1,0)H}(k, l) \right] + E \left[\tilde{r}_1^{(1,0)}(k, l) \tilde{r}_1^{(1,0)H}(k, l) \right] \\ &= 2\Re_y(0, 0) - k_{e_1}^{(1,0)*} (\Re_y(0, 0) + \Re_y(0, 0)^T) k_{r_1}^{(1,0)} \\ &= 2\Re_y(0, 0) \left(1 - k_{r_1}^{(1,0)} k_{r_1}^{(1,0)*} \right). \end{aligned} \quad (\text{A.31})$$

$$= 2\Re_y(0, 0) \left(1 - k_{r_1}^{(1,0)} k_{r_1}^{(1,0)*} \right). \quad (\text{A.32})$$

Horizontal Augmentation Case

In case horizontal augmentation we have three propagation lattice structure equations: one for updated horizontal PEFs, one for updated vertical forward/backward PEFs, and one for the newly created vertical forward/backward PEFs. Now, the calculation of the lattice coefficients for each one will be explained in detail.

If we write the lattice form for the horizontal forward/backward PEFs,

$$\mathbf{r}^{(p+1,q-1)}(k, l) = \mathbf{r}^{(p,q-1)}(k, l) - \mathbf{K}_r^{(p+1,q-1)H} \tilde{\mathbf{r}}^{(p,q-1)}(k-1, l) \quad (\text{A.33a})$$

$$\tilde{\mathbf{r}}^{(p+1,q-1)}(k, l) = \tilde{\mathbf{r}}^{(p,q-1)}(k-1, l) - \mathbf{K}_r^{(p+1,q-1)T} \mathbf{r}^{(p,q-1)}(k, l) \quad (\text{A.33b})$$

then the following equations can be defined,

$$\Sigma_{b_y}^{(p,q-1)} = E \left[\tilde{\mathbf{r}}^{(p,q-1)}(k-1, l) \tilde{\mathbf{r}}^{(p,q-1)H}(k-1, l) \right] \quad (\text{A.34a})$$

$$\Sigma_{f_y}^{(p,q-1)} = E \left[\mathbf{r}^{(p,q-1)}(k, l) \mathbf{r}^{(p,q-1)H}(k, l) \right] \quad (\text{A.34b})$$

$$\Delta_1^{(p,q-1)} = E \left[\tilde{\mathbf{r}}^{(p,q-1)}(k-1, l) \mathbf{r}^{(p,q-1)H}(k, l) \right] \quad (\text{A.34c})$$

$$\Delta_2^{(p,q-1)} = E \left[\mathbf{r}^{(p,q-1)}(k, l) \tilde{\mathbf{r}}^{(p,q-1)H}(k-1, l) \right]. \quad (\text{A.34d})$$

Let us introduce a new notation for Σ and Δ according to the vectors.

$$\Sigma_{b_y}^{(p,q-1)} = \Sigma_{\tilde{\mathbf{r}}, \tilde{\mathbf{r}}}^{(p,q-1)} \quad (\text{A.35a})$$

$$\Sigma_{f_y}^{(p,q-1)} = \Sigma_{\mathbf{r}, \mathbf{r}}^{(p,q-1)} \quad (\text{A.35b})$$

$$\Delta_1^{(p,q-1)} = \Delta_{\tilde{\mathbf{r}}, \mathbf{r}}^{(p,q-1)} \quad (\text{A.35c})$$

$$\Delta_2^{(p,q-1)} = \Delta_{\mathbf{r}, \tilde{\mathbf{r}}}^{(p,q-1)} \quad (\text{A.35d})$$

It can be clear that $\Delta_2^{(p,q-1)}$ is equal to $\Delta_1^{(p,q-1)H}$, then we can write

$$\Delta_2^{(p,q-1)} = \Delta_{\mathbf{r}, \tilde{\mathbf{r}}}^{(p,q-1)H}. \quad (\text{A.36})$$

By using the definitions (A.35) and the equation (A.8), we can write

$$\mathbf{K}_r^{(p+1,q-1)} = \left[\Sigma_{\tilde{\mathbf{r}}, \tilde{\mathbf{r}}}^{(p,q-1)} + \Sigma_{\mathbf{r}, \mathbf{r}}^{(p,q-1)*} \right]^{-1} \left[\Delta_{\tilde{\mathbf{r}}, \mathbf{r}}^{(p,q-1)} + \Delta_{\mathbf{r}, \tilde{\mathbf{r}}}^{(p,q-1)T} \right]. \quad (\text{A.37})$$

The sum of the minimum mean squared error propagation equation can be defined for (A.33a),

$$P_r^{(p+1,q-1)} = tr \left[E \left[\mathbf{r}^{(p+1,q-1)}(k,l) \mathbf{r}^{(p+1,q-1)H}(k,l) \right] + E \left[\tilde{\mathbf{r}}^{(p+1,q-1)}(k,l) \tilde{\mathbf{r}}^{(p+1,q-1)H}(k,l) \right] \right] \quad (\text{A.38})$$

and we open the equation, it can be written as a simple form,

$$P_r^{(p+1,q-1)} = P_r^{(p,q-1)} - \mathbf{K}_r^{(p+1,q-1)H} \left(\Sigma_{\tilde{\mathbf{r}},\tilde{\mathbf{r}}}^{(p,q-1)} + \Sigma_{\mathbf{r},\mathbf{r}}^{(p,q-1)T} \right) \mathbf{K}_r^{(p+1,q-1)} \quad (\text{A.39})$$

$$= tr \left[\left(\Sigma_{\tilde{\mathbf{r}},\tilde{\mathbf{r}}}^{(p,q-1)} + \Sigma_{\mathbf{r},\mathbf{r}}^{(p,q-1)T} \right) \left(\mathbf{I} - \mathbf{K}_r^{(p+1,q-1)} \mathbf{K}_r^{(p+1,q-1)H} \right) \right] \quad (\text{A.40})$$

The lattice structure for the vertical forward/backward PEFs can be formed as follows,

$$\hat{\mathbf{e}}^{(p,q)}(k,l) = \mathbf{e}^{(p-1,q)}(k,l) - \hat{\mathbf{K}}_e^{(p,q)H} \tilde{\mathbf{r}}^{(p,q-1)}(k,l-1) \quad (\text{A.41a})$$

$$\hat{\tilde{\mathbf{e}}}^{(p,q)}(k,l) = \tilde{\mathbf{e}}^{(p-1,q)}(k-1,l) - \hat{\mathbf{K}}_e^{(p,q)T} \mathbf{r}^{(p,q-1)}(k,l). \quad (\text{A.41b})$$

The lattice coefficients are computed as

$$\hat{\mathbf{K}}_e^{(p,q)} = \left[\Sigma_{\tilde{\mathbf{r}},\tilde{\mathbf{r}}}^{(p,q-1)} + \Sigma_{\mathbf{r},\mathbf{r}}^{(p,q-1)*} \right]^{-1} \left[\Delta_{\tilde{\mathbf{r}},\mathbf{e}}^{(p,q-1)} + \Delta_{\mathbf{r},\tilde{\mathbf{e}}}^{(p,q-1)*} \right]. \quad (\text{A.42})$$

The power of the PEFs can be stated as

$$P_e^{(p,q)} = P_e^{(p-1,q)} - tr \left[\hat{\mathbf{K}}_e^{(p,q)H} \left(\Sigma_{\tilde{\mathbf{r}},\tilde{\mathbf{r}}}^{(p,q-1)} + \Sigma_{\mathbf{r},\mathbf{r}}^{(p,q-1)T} \right) \hat{\mathbf{K}}_e^{(p,q)} \right]. \quad (\text{A.43})$$

The natural result of the horizontal augmentation, a new set of vertical PEFs is created. For the newly created vertical PEFs, we can write the following order update lattice structure.

$$e_{p+1}^{(p,q)}(k,l) = e_p^{(p-1,q)}(k-1,l) - \mathbf{k}_{e_{p+1}}^{(p,q)H} \mathbf{r}^{(p,q-1)}(k,l-1) \quad (\text{A.44a})$$

$$\tilde{e}_{p+1}^{(p,q)}(k,l) = \tilde{e}_p^{(p-1,q)}(k,l) - \mathbf{k}_{e_{p+1}}^{(p,q)T} \tilde{\mathbf{r}}^{(p,q)}(k,l). \quad (\text{A.44b})$$

By using this lattice structure, we can write the following equations for the lattice coefficients $\mathbf{k}_{e_{p+1}}$.

$$\mathbf{k}_{e_{p+1}}^{(p,q)} = \left[\Sigma_{\mathbf{r},\mathbf{r}}^{(p,q-1)} + \Sigma_{\tilde{\mathbf{r}},\tilde{\mathbf{r}}}^{(p,q-1)*} \right]^{-1} \left[\Delta_{\mathbf{r},e_{p+1}}^{(p,q-1)} + \Delta_{\tilde{\mathbf{r}},\tilde{e}_{p+1}}^{(p,q-1)*} \right]. \quad (\text{A.45})$$

The power of the new PEFs can be indicated as

$$P_{e_{p+1}}^{(p,q)} = P_{e_p}^{(p-1,q)} - tr \left[\mathbf{k}_{e_{p+1}}^{(p,q)H} \left(\Sigma_{\mathbf{r},\mathbf{r}}^{(p,q-1)} + \Sigma_{\tilde{\mathbf{r}},\tilde{\mathbf{r}}}^{(p,q-1)T} \right) \mathbf{k}_{e_{p+1}}^{(p,q)} \right]. \quad (\text{A.46})$$

Vertical Augmentation Case

Similarly, the lattice structures, coefficients equations, and power of the PEFs can be written for the vertical augmentation. We know that as the natural result of the vertical augmentation, a set of new horizontal forward/backward PEFs is created.

The lattice structure for the vertical forward/backward PEFs is

$$\mathbf{e}^{(p-1,q+1)}(k,l) = \mathbf{e}^{(p-1,q)}(k,l) - \mathbf{K}_{\mathbf{e}}^{(p-1,q+1)H} \tilde{\mathbf{e}}^{(p-1,q)}(k,l-1) \quad (\text{A.47a})$$

$$\tilde{\mathbf{e}}^{(p-1,q+1)}(k,l) = \tilde{\mathbf{e}}^{(p-1,q)}(k,l-1) - \mathbf{K}_{\mathbf{e}}^{(p-1,q+1)T} \mathbf{e}^{(p-1,q)}(k,l). \quad (\text{A.47b})$$

The coefficients equation is

$$\mathbf{K}_{\mathbf{e}}^{(p-1,q+1)} = \left[\Sigma_{\tilde{\mathbf{e}},\tilde{\mathbf{e}}}^{(p-1,q)} + \Sigma_{\mathbf{e},\mathbf{e}}^{(p-1,q)*} \right]^{-1} \left[\Delta_{\tilde{\mathbf{e}},\mathbf{e}}^{(p-1,q)} + \Delta_{\mathbf{e},\tilde{\mathbf{e}}}^{(p-1,q)T} \right]. \quad (\text{A.48})$$

The power of PEFs is

$$P_e^{(p-1,q+1)} = P_e^{(p-1,q)} - \mathbf{K}_{\mathbf{e}}^{(p-1,q+1)H} \left(\Sigma_{\tilde{\mathbf{e}},\tilde{\mathbf{e}}}^{(p-1,q)} + \Sigma_{\mathbf{e},\mathbf{e}}^{(p-1,q)T} \right) \mathbf{K}_{\mathbf{e}}^{(p-1,q+1)} \quad (\text{A.49})$$

$$= \text{tr} \left[\left(\Sigma_{\tilde{\mathbf{e}},\tilde{\mathbf{e}}}^{(p-1,q)} + \Sigma_{\mathbf{e},\mathbf{e}}^{(p-1,q)T} \right) \left(\mathbf{I} - \mathbf{K}_{\mathbf{e}}^{(p-1,q+1)} \mathbf{K}_{\mathbf{e}}^{(p-1,q)H} \right) \right]. \quad (\text{A.50})$$

For the horizontal forward/backward PEFs, the order update lattice form is

$$\hat{\mathbf{r}}^{(p,q)}(k,l) = \mathbf{r}^{(p,q-1)}(k,l) - \hat{\mathbf{K}}_{\mathbf{r}}^{(p,q)H} \tilde{\mathbf{e}}^{(p-1,q)}(k-1,l) \quad (\text{A.51a})$$

$$\tilde{\mathbf{r}}^{(p,q)}(k,l) = \tilde{\mathbf{r}}^{(p,q-1)}(k,l-1) - \hat{\mathbf{K}}_{\mathbf{r}}^{(p,q)T} \mathbf{e}^{(p,q)}(k,l). \quad (\text{A.51b})$$

The lattice coefficients $\hat{\mathbf{K}}_{\mathbf{r}}^{(p,q)}$ are

$$\hat{\mathbf{K}}_{\mathbf{r}}^{(p,q)} = \left[\Sigma_{\tilde{\mathbf{e}},\tilde{\mathbf{e}}}^{(p-1,q)} + \Sigma_{\mathbf{e},\mathbf{e}}^{(p-1,q)*} \right]^{-1} \left[\Delta_{\tilde{\mathbf{e}},\mathbf{r}}^{(p-1,q)} + \Delta_{\mathbf{e},\tilde{\mathbf{r}}}^{(p-1,q)*} \right]. \quad (\text{A.52})$$

The power of the PEFs can be stated as

$$P_r^{(p,q)} = P_r^{(p,q-1)} - \text{tr} \left[\hat{\mathbf{K}}_{\mathbf{r}}^{(p,q)H} \left(\Sigma_{\tilde{\mathbf{e}},\tilde{\mathbf{e}}}^{(p-1,q)} + \Sigma_{\mathbf{e},\mathbf{e}}^{(p-1,q)T} \right) \hat{\mathbf{K}}_{\mathbf{r}}^{(p,q)} \right]. \quad (\text{A.53})$$

For the newly created horizontal forward/backward PEFs, the order update lattice equations is

$$r_{q+1}^{(p,q)}(k,l) = r_q^{(p,q-1)}(k,l-1) - \mathbf{k}_{r_{q+1}}^{(p,q)H} \mathbf{e}^{(p-1,q)}(k-1,l) \quad (\text{A.54a})$$

$$\tilde{r}_{q+1}^{(p,q)}(k,l) = \tilde{r}_q^{(p,q-1)}(k,l) - \mathbf{k}_{r_{q+1}}^{(p,q)T} \tilde{\mathbf{e}}^{(p-1,q)}(k,l). \quad (\text{A.54b})$$

The following equations for the lattice coefficients $\mathbf{k}_{r_{q+1}}$ can be written.

$$\mathbf{k}_{r_{q+1}}^{(p,q)} = \left[\Sigma_{\mathbf{e},\mathbf{e}}^{(p-1,q)} + \Sigma_{\tilde{\mathbf{e}},\tilde{\mathbf{e}}}^{(p-1,q)*} \right]^{-1} \left[\Delta_{\mathbf{e},r_q}^{(p-1,q)} + \Delta_{\tilde{\mathbf{e}},\tilde{r}_q}^{(p-1,q)*} \right]. \quad (\text{A.55})$$

The power of the new PEFs can be indicated as

$$P_{r_{q+1}}^{(p,q)} = P_{r_q}^{(p,q-1)} - \text{tr} \left[\mathbf{k}_{r_{q+1}}^{(p,q)H} \left(\Sigma_{\mathbf{e},\mathbf{e}}^{(p-1,q)} + \Sigma_{\tilde{\mathbf{e}},\tilde{\mathbf{e}}}^{(p-1,q)T} \right) \mathbf{k}_{r_{q+1}}^{(p,q)} \right]. \quad (\text{A.56})$$

APPENDIX A.2: Explanation of the Reduction of the Support Data

In the proposed method, only the given data is taken into account during the calculation of the reflection coefficient matrices, and no calculation is done when computing any updated value for the data which is out of the support region. Therefore, after each stage order, updated PEFs support region will get smaller.

Actually, this reduction is corresponding with the stage order. In case of any (p, q) state order, the size of the forward/backward PEFs can be defined as $(M - p) \times (N - q)$ for the given $(M \times N)$ size initial support data. Depending on the order stage, uncalculated values are nullified for the updating auxiliary PEFs. This means, p rows from the beginning will be consisted of the zero vectors and similarly, q columns from the beginning will be zero vectors.

Let us assume that we have the data matrix of size 5×5 and demonstrate the data reduction step by step. The data matrix of size 5×5 is denoted as

$$y(k, l) = \begin{bmatrix} y(1,1) & y(1,2) & y(1,3) & y(1,4) & y(1,5) \\ y(2,1) & y(2,2) & y(2,3) & y(2,4) & y(2,5) \\ y(3,1) & y(3,2) & y(3,3) & y(3,4) & y(3,5) \\ y(4,1) & y(4,2) & y(4,3) & y(4,4) & y(4,5) \\ y(5,1) & y(5,2) & y(5,3) & y(5,4) & y(5,5) \end{bmatrix} \quad (\text{A.57})$$

At the *zero*-th order state, the PSR $S_{k,l}^{(0,0)}$ is empty set and the first order auxiliary vertical and horizontal forward/backward PEFs are initiated by the given 2-D observation data:

$$e_1^{(0,0)}(k, l) = r_1^{(0,0)}(k, l) = y(k, l) \quad (\text{A.58a})$$

$$\tilde{e}_1^{(0,0)}(k, l) = \tilde{r}_1^{(0,0)}(k, l) = y(k, l) \quad (\text{A.58b})$$

Lattice structure equations for the auxiliary vertical PEFs are given as:

$$e_1^{(0,1)}(k, l) = e_1^{(0,0)}(k, l) - k_{e_1}^{(0,1)*} \tilde{e}_1^{(0,0)}(k, l - 1) \quad (\text{A.59a})$$

$$\tilde{e}_1^{(0,1)}(k, l) = \tilde{e}_1^{(0,0)}(k, l - 1) - k_{e_1}^{(0,1)} e_1^{(0,0)}(k, l) \quad (\text{A.59b})$$

And, similarly, for the auxiliary horizontal PEFs, the lattice structure is

$$r_1^{(1,0)}(k, l) = r_1^{(0,0)}(k, l) - k_{r_1}^{(1,0)*} \tilde{r}_1^{(0,0)}(k - 1, l) \quad (\text{A.60a})$$

$$\tilde{r}_1^{(1,0)}(k, l) = \tilde{r}_1^{(0,0)}(k - 1, l) - k_{r_1}^{(1,0)} r_1^{(0,0)}(k, l). \quad (\text{A.60b})$$

By using these lattice structure equations, we are able to calculate, $e_1^{(0,1)}(k, l)$, $\tilde{e}_1^{(0,1)}(k, l)$, $r_1^{(1,0)}(k, l)$, and $\tilde{r}_1^{(1,0)}(k, l)$ after finding the lattice coefficient factors $k_{e_1}^{(0,1)}$ and $k_{r_1}^{(1,0)}$. The calculated vertical and horizontal forward/backward PEFs for the initial stage will be as follows:

$$e_1^{(0,1)}(k, l) = \begin{bmatrix} 0 & e_1^{(0,1)}(1,2) & e_1^{(0,1)}(1,3) & e_1^{(0,1)}(1,4) & e_1^{(0,1)}(1,5) \\ 0 & e_1^{(0,1)}(2,2) & e_1^{(0,1)}(2,3) & e_1^{(0,1)}(2,4) & e_1^{(0,1)}(2,5) \\ 0 & e_1^{(0,1)}(3,2) & e_1^{(0,1)}(3,3) & e_1^{(0,1)}(3,4) & e_1^{(0,1)}(3,5) \\ 0 & e_1^{(0,1)}(4,2) & e_1^{(0,1)}(4,3) & e_1^{(0,1)}(4,4) & e_1^{(0,1)}(4,5) \\ 0 & e_1^{(0,1)}(5,2) & e_1^{(0,1)}(5,3) & e_1^{(0,1)}(5,4) & e_1^{(0,1)}(5,5) \end{bmatrix} \quad (\text{A.61})$$

$$\tilde{e}_1^{(0,1)}(k,l) = \begin{bmatrix} 0 & \tilde{e}_1^{(0,1)}(1,2) & \tilde{e}_1^{(0,1)}(1,3) & \tilde{e}_1^{(0,1)}(1,4) & \tilde{e}_1^{(0,1)}(1,5) \\ 0 & \tilde{e}_1^{(0,1)}(2,2) & \tilde{e}_1^{(0,1)}(2,3) & \tilde{e}_1^{(0,1)}(2,4) & \tilde{e}_1^{(0,1)}(2,5) \\ 0 & \tilde{e}_1^{(0,1)}(3,2) & \tilde{e}_1^{(0,1)}(3,3) & \tilde{e}_1^{(0,1)}(3,4) & \tilde{e}_1^{(0,1)}(3,5) \\ 0 & \tilde{e}_1^{(0,1)}(4,2) & \tilde{e}_1^{(0,1)}(4,3) & \tilde{e}_1^{(0,1)}(4,4) & \tilde{e}_1^{(0,1)}(4,5) \\ 0 & \tilde{e}_1^{(0,1)}(5,2) & \tilde{e}_1^{(0,1)}(5,3) & \tilde{e}_1^{(0,1)}(5,4) & \tilde{e}_1^{(0,1)}(5,5) \end{bmatrix} \quad (\text{A.62})$$

$$r_1^{(1,0)}(k,l) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ r_1^{(1,0)}(2,1) & r_1^{(1,0)}(2,2) & r_1^{(1,0)}(2,3) & r_1^{(1,0)}(2,4) & r_1^{(1,0)}(2,5) \\ r_1^{(1,0)}(3,1) & r_1^{(1,0)}(3,2) & r_1^{(1,0)}(3,3) & r_1^{(1,0)}(3,4) & r_1^{(1,0)}(3,5) \\ r_1^{(1,0)}(4,1) & r_1^{(1,0)}(4,2) & r_1^{(1,0)}(4,3) & r_1^{(1,0)}(4,4) & r_1^{(1,0)}(4,5) \\ r_1^{(1,0)}(5,1) & r_1^{(1,0)}(5,2) & r_1^{(1,0)}(5,3) & r_1^{(1,0)}(5,4) & r_1^{(1,0)}(5,5) \end{bmatrix} \quad (\text{A.63})$$

$$\tilde{r}_1^{(1,0)}(k,l) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ \tilde{r}_1^{(1,0)}(2,1) & \tilde{r}_1^{(1,0)}(2,2) & \tilde{r}_1^{(1,0)}(2,3) & \tilde{r}_1^{(1,0)}(2,4) & \tilde{r}_1^{(1,0)}(2,5) \\ \tilde{r}_1^{(1,0)}(3,1) & \tilde{r}_1^{(1,0)}(3,2) & \tilde{r}_1^{(1,0)}(3,3) & \tilde{r}_1^{(1,0)}(3,4) & \tilde{r}_1^{(1,0)}(3,5) \\ \tilde{r}_1^{(1,0)}(4,1) & \tilde{r}_1^{(1,0)}(4,2) & \tilde{r}_1^{(1,0)}(4,3) & \tilde{r}_1^{(1,0)}(4,4) & \tilde{r}_1^{(1,0)}(4,5) \\ \tilde{r}_1^{(1,0)}(5,1) & \tilde{r}_1^{(1,0)}(5,2) & \tilde{r}_1^{(1,0)}(5,3) & \tilde{r}_1^{(1,0)}(5,4) & \tilde{r}_1^{(1,0)}(5,5) \end{bmatrix} \quad (\text{A.64})$$

After the finding all PEFs related to the initial stage, we can try to make a growth on the horizontal direction by realizing a vertical augmentation denoted in Fig. A.1. For this PSR $S_{k,l}^{(0,1)}$, we will have one vertical forward/backward PEF and two horizontal forward/backward PEFs. We know that the error propagation equations for the existing

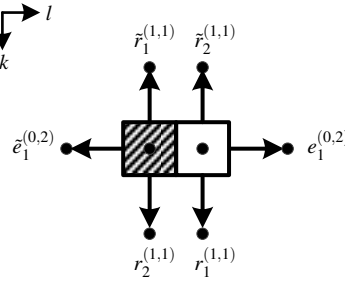


Figure A.1: Simple illustration of auxiliary vertical and horizontal forward/backward PEFs of the PSR $S_{k,l}^{(0,1)}$.

auxiliary vertical forward/backward PEFs are given as

$$e_1^{(0,2)}(k,l) = e_1^{(0,1)}(k,l) - k_{e_1}^{(0,2)*} \tilde{e}_1^{(0,2)}(k,l-1) \quad (\text{A.65a})$$

$$\tilde{e}_1^{(0,2)}(k,l) = \tilde{e}_1^{(0,1)}(k,l-1) - k_{e_1}^{(0,2)} e_1^{(0,1)}(k,l). \quad (\text{A.65b})$$

The calculated auxiliary PEFs will be as follows:

$$e_1^{(0,2)}(k,l) = \begin{bmatrix} 0 & 0 & e_1^{(0,2)}(1,3) & e_1^{(0,2)}(1,4) & e_1^{(0,2)}(1,5) \\ 0 & 0 & e_1^{(0,2)}(2,3) & e_1^{(0,2)}(2,4) & e_1^{(0,2)}(2,5) \\ 0 & 0 & e_1^{(0,2)}(3,3) & e_1^{(0,2)}(3,4) & e_1^{(0,2)}(3,5) \\ 0 & 0 & e_1^{(0,2)}(4,3) & e_1^{(0,2)}(4,4) & e_1^{(0,2)}(4,5) \\ 0 & 0 & e_1^{(0,2)}(5,3) & e_1^{(0,2)}(5,4) & e_1^{(0,2)}(5,5) \end{bmatrix} \quad (\text{A.66})$$

$$\tilde{e}_1^{(0,2)}(k,l) = \begin{bmatrix} 0 & 0 & \tilde{e}_1^{(0,2)}(1,3) & \tilde{e}_1^{(0,2)}(1,4) & \tilde{e}_1^{(0,2)}(1,5) \\ 0 & 0 & \tilde{e}_1^{(0,2)}(2,3) & \tilde{e}_1^{(0,2)}(2,4) & \tilde{e}_1^{(0,2)}(2,5) \\ 0 & 0 & \tilde{e}_1^{(0,2)}(3,3) & \tilde{e}_1^{(0,2)}(3,4) & \tilde{e}_1^{(0,2)}(3,5) \\ 0 & 0 & \tilde{e}_1^{(0,2)}(4,3) & \tilde{e}_1^{(0,2)}(4,4) & \tilde{e}_1^{(0,2)}(4,5) \\ 0 & 0 & \tilde{e}_1^{(0,2)}(5,3) & \tilde{e}_1^{(0,2)}(5,4) & \tilde{e}_1^{(0,2)}(5,5) \end{bmatrix} \quad (\text{A.67})$$

For the existing auxiliary horizontal forward/backward PEFs, the error propagation structures are

$$r_1^{(1,1)}(k,l) = r_1^{(1,0)}(k,l) - k_{r_1}^{(1,1)*} \tilde{r}_1^{(1,0)}(k-1,l) \quad (\text{A.68a})$$

$$\tilde{r}_1^{(1,1)}(k,l) = \tilde{r}_1^{(1,0)}(k-1,l) - k_{r_1}^{(1,1)} r_1^{(1,0)}(k,l). \quad (\text{A.68b})$$

The updated horizontal PEFs will be as follows:

$$r_1^{(1,1)}(k,l) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & r_1^{(1,1)}(2,2) & r_1^{(1,1)}(2,3) & r_1^{(1,1)}(2,4) & r_1^{(1,1)}(2,5) \\ 0 & r_1^{(1,1)}(3,2) & r_1^{(1,1)}(3,3) & r_1^{(1,1)}(3,4) & r_1^{(1,1)}(3,5) \\ 0 & r_1^{(1,1)}(4,2) & r_1^{(1,1)}(4,3) & r_1^{(1,1)}(4,4) & r_1^{(1,1)}(4,5) \\ 0 & r_1^{(1,1)}(5,2) & r_1^{(1,1)}(5,3) & r_1^{(1,1)}(5,4) & r_1^{(1,1)}(5,5) \end{bmatrix} \quad (\text{A.69})$$

$$\tilde{r}_1^{(1,0)}(k,l) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & \tilde{r}_1^{(1,1)}(2,2) & \tilde{r}_1^{(1,1)}(2,3) & \tilde{r}_1^{(1,1)}(2,4) & \tilde{r}_1^{(1,1)}(2,5) \\ 0 & \tilde{r}_1^{(1,1)}(3,2) & \tilde{r}_1^{(1,1)}(3,3) & \tilde{r}_1^{(1,1)}(3,4) & \tilde{r}_1^{(1,1)}(3,5) \\ 0 & \tilde{r}_1^{(1,1)}(4,2) & \tilde{r}_1^{(1,1)}(4,3) & \tilde{r}_1^{(1,1)}(4,4) & \tilde{r}_1^{(1,1)}(4,5) \\ 0 & \tilde{r}_1^{(1,1)}(5,2) & \tilde{r}_1^{(1,1)}(5,3) & \tilde{r}_1^{(1,1)}(5,4) & \tilde{r}_1^{(1,1)}(5,5) \end{bmatrix} \quad (\text{A.70})$$

The error propagation equations for the newly initiated horizontal forward/backward PEFs are defined as

$$r_2^{(1,1)}(k,l) = r_1^{(1,0)}(k,l) - k_{r_2}^{(1,1)*} \tilde{e}_1^{(0,1)}(k-1,l) \quad (\text{A.71a})$$

$$\tilde{r}_1^{(1,1)}(k,l) = \tilde{r}_1^{(1,0)}(k,l-1) - k_{r_2}^{(1,1)} e_1^{(0,1)}(k,l). \quad (\text{A.71b})$$

The newly initiated horizontal PEFs are computed as

$$r_2^{(1,1)}(k,l) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & r_2^{(1,1)}(2,2) & r_2^{(1,1)}(2,3) & r_2^{(1,1)}(2,4) & r_2^{(1,1)}(2,5) \\ 0 & r_2^{(1,1)}(3,2) & r_2^{(1,1)}(3,3) & r_2^{(1,1)}(3,4) & r_2^{(1,1)}(3,5) \\ 0 & r_2^{(1,1)}(4,2) & r_2^{(1,1)}(4,3) & r_2^{(1,1)}(4,4) & r_2^{(1,1)}(4,5) \\ 0 & r_2^{(1,1)}(5,2) & r_2^{(1,1)}(5,3) & r_2^{(1,1)}(5,4) & r_2^{(1,1)}(5,5) \end{bmatrix} \quad (\text{A.72})$$

$$\tilde{r}_2^{(1,1)}(k,l) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & \tilde{r}_2^{(1,1)}(2,2) & \tilde{r}_2^{(1,1)}(2,3) & \tilde{r}_2^{(1,1)}(2,4) & \tilde{r}_2^{(1,1)}(2,5) \\ 0 & \tilde{r}_2^{(1,1)}(3,2) & \tilde{r}_2^{(1,1)}(3,3) & \tilde{r}_2^{(1,1)}(3,4) & \tilde{r}_2^{(1,1)}(3,5) \\ 0 & \tilde{r}_2^{(1,1)}(4,2) & \tilde{r}_2^{(1,1)}(4,3) & \tilde{r}_2^{(1,1)}(4,4) & \tilde{r}_2^{(1,1)}(4,5) \\ 0 & \tilde{r}_2^{(1,1)}(5,2) & \tilde{r}_2^{(1,1)}(5,3) & \tilde{r}_2^{(1,1)}(5,4) & \tilde{r}_2^{(1,1)}(5,5) \end{bmatrix} \quad (\text{A.73})$$

After that point, if we realize a horizontal augmentation on the PSR $S_{k,l}^{(0,1)}$, the new PSR will be $S_{k,l}^{(1,1)}$, shown in Fig. A.2. Natural result of the horizontal augmentation, a new set of vertical forward/backward PEFs will be introduced.

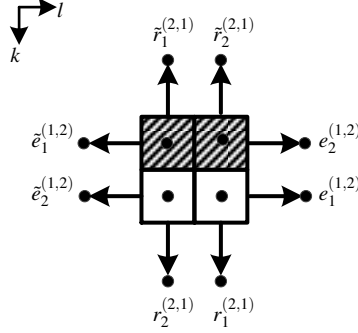


Figure A.2: Simple illustration of auxiliary vertical and horizontal forward/backward PEFs of the PSR $S_{k,l}^{(1,1)}$.

$$\mathbf{r}^{(2,1)}(k, l) = \mathbf{r}^{(1,1)}(k, l) - \mathbf{K}_{\mathbf{r}}^{(2,1)*} \tilde{\mathbf{r}}^{(1,1)}(k-1, l) \quad (\text{A.74a})$$

$$\tilde{\mathbf{r}}^{(2,1)}(k, l) = \tilde{\mathbf{r}}^{(1,1)}(k-1, l) - \mathbf{K}_{\mathbf{r}}^{(2,1)} \mathbf{r}^{(1,1)}(k, l). \quad (\text{A.74b})$$

The auxiliary horizontal forward/backward PEFs are calculated by using (A.74) and they are:

$$r_i^{(2,1)}(k, l) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & r_i^{(2,1)}(3, 2) & r_i^{(2,1)}(3, 3) & r_i^{(2,1)}(3, 4) & r_i^{(2,1)}(3, 5) \\ 0 & r_i^{(2,1)}(4, 2) & r_i^{(2,1)}(4, 3) & r_i^{(2,1)}(4, 4) & r_i^{(2,1)}(4, 5) \\ 0 & r_i^{(2,1)}(5, 2) & r_i^{(2,1)}(5, 3) & r_i^{(2,1)}(5, 4) & r_i^{(2,1)}(5, 5) \end{bmatrix} \quad (\text{A.75})$$

$$\tilde{r}_i^{(2,1)}(k, l) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & \tilde{r}_i^{(2,1)}(3, 2) & \tilde{r}_i^{(2,1)}(3, 3) & \tilde{r}_i^{(2,1)}(3, 4) & \tilde{r}_i^{(2,1)}(3, 5) \\ 0 & \tilde{r}_i^{(2,1)}(4, 2) & \tilde{r}_i^{(2,1)}(4, 3) & \tilde{r}_i^{(2,1)}(4, 4) & \tilde{r}_i^{(2,1)}(4, 5) \\ 0 & \tilde{r}_i^{(2,1)}(5, 2) & \tilde{r}_i^{(2,1)}(5, 3) & \tilde{r}_i^{(2,1)}(5, 4) & \tilde{r}_i^{(2,1)}(5, 5) \end{bmatrix} \quad (\text{A.76})$$

where $i = 1, 2$.

Similarly, for the auxiliary vertical forward/backward PEFs are computed by using (A.77) and the newly initiated auxiliary vertical forward/backward PEFs are calculated by using (A.78).

$$e_1^{(1,2)}(k, l) = e_1^{(0,2)}(k, l) - k_{e_1}^{(1,2)*} \tilde{\mathbf{r}}^{(1,1)}(k, l-1) \quad (\text{A.77a})$$

$$\tilde{e}_1^{(1,2)}(k, l) = \tilde{e}_1^{(0,2)}(k-1, l) - k_{e_1}^{(1,2)} \mathbf{r}^{(1,1)}(k, l). \quad (\text{A.77b})$$

$$e_2^{(1,2)}(k,l) = e_2^{(0,2)}(k,l) - k e_2^{(1,2)*} \mathbf{r}^{(1,1)}(k,l-1) \quad (\text{A.78a})$$

$$\tilde{e}_2^{(1,2)}(k,l) = \tilde{e}_2^{(0,2)}(k,l) - k e_2^{(1,2)} \tilde{\mathbf{r}}^{(1,1)}(k,l) \quad (\text{A.78b})$$

The auxiliary vertical forward/backward PEFs for the PSR $S_{k,l}^{(2,2)}$:

$$e_i^{(1,2)}(k,l) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & e_i^{(1,2)}(2,3) & e_i^{(1,2)}(2,4) & e_i^{(1,2)}(2,5) \\ 0 & 0 & e_i^{(1,2)}(3,3) & e_i^{(1,2)}(3,4) & e_i^{(1,2)}(3,5) \\ 0 & 0 & e_i^{(1,2)}(4,3) & e_i^{(1,2)}(4,4) & e_i^{(1,2)}(4,5) \\ 0 & 0 & e_i^{(1,2)}(5,3) & e_i^{(1,2)}(5,4) & e_i^{(1,2)}(5,5) \end{bmatrix} \quad (\text{A.79})$$

$$\tilde{e}_i^{(1,2)}(k,l) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \tilde{e}_i^{(1,2)}(2,3) & \tilde{e}_i^{(1,2)}(2,4) & \tilde{e}_i^{(1,2)}(2,5) \\ 0 & 0 & \tilde{e}_i^{(1,2)}(3,3) & \tilde{e}_i^{(1,2)}(3,4) & \tilde{e}_i^{(1,2)}(3,5) \\ 0 & 0 & \tilde{e}_i^{(1,2)}(4,3) & \tilde{e}_i^{(1,2)}(4,4) & \tilde{e}_i^{(1,2)}(4,5) \\ 0 & 0 & \tilde{e}_i^{(1,2)}(5,3) & \tilde{e}_i^{(1,2)}(5,4) & \tilde{e}_i^{(1,2)}(5,5) \end{bmatrix} \quad (\text{A.80})$$

where $i = 1, 2$.

Now we can apply horizontal and vertical augmentation sequentially on the PSR $S_{k,l}^{(2,2)}$, and then we will have $S_{k,l}^{(2,1)}$, $S_{k,l}^{(2,2)}$, $S_{k,l}^{(3,2)}$, and $S_{k,l}^{(3,3)}$, shown in Fig. A.3, Fig. A.4, Fig. A.5, and Fig. A.6, respectively. The PSR $S_{k,l}^{(3,3)}$ is the last augmentation step for the data matrix size 5×5 .

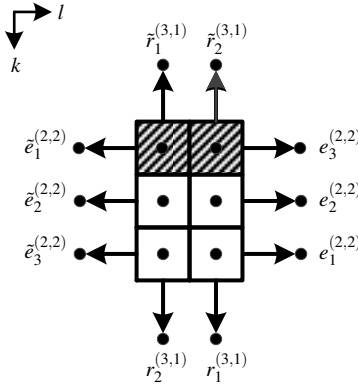


Figure A.3: Simple illustration of auxiliary vertical and horizontal forward/backward PEFs of the PSR $S_{k,l}^{(2,1)}$.

The updated vertical and horizontal forward/backward PEFs for the $S_{k,l}^{(2,1)}$:

$$e_i^{(2,2)}(k,l) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & e_i^{(2,2)}(3,3) & e_i^{(2,2)}(3,4) & e_i^{(2,2)}(3,5) \\ 0 & 0 & e_i^{(2,2)}(4,3) & e_i^{(2,2)}(4,4) & e_i^{(2,2)}(4,5) \\ 0 & 0 & e_i^{(2,2)}(5,3) & e_i^{(2,2)}(5,4) & e_i^{(2,2)}(5,5) \end{bmatrix} \quad (\text{A.81})$$

$$\tilde{e}_i^{(2,2)}(k,l) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \tilde{e}_i^{(2,2)}(3,3) & \tilde{e}_i^{(2,2)}(3,4) & \tilde{e}_i^{(2,2)}(3,5) \\ 0 & 0 & \tilde{e}_i^{(2,2)}(4,3) & \tilde{e}_i^{(2,2)}(4,4) & \tilde{e}_i^{(2,2)}(4,5) \\ 0 & 0 & \tilde{e}_i^{(2,2)}(5,3) & \tilde{e}_i^{(2,2)}(5,4) & \tilde{e}_i^{(2,2)}(5,5) \end{bmatrix} \quad (\text{A.82})$$

where $i = 1, 2, 3$.

$$r_i^{(3,1)}(k,l) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & r_i^{(3,1)}(4,2) & r_i^{(3,1)}(4,3) & r_i^{(3,1)}(4,4) & r_i^{(3,1)}(4,5) \\ 0 & r_i^{(3,1)}(5,2) & r_i^{(3,1)}(5,3) & r_i^{(3,1)}(5,4) & r_i^{(3,1)}(5,5) \end{bmatrix} \quad (\text{A.83})$$

$$\tilde{r}_i^{(3,1)}(k,l) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & \tilde{r}_i^{(3,1)}(4,2) & \tilde{r}_i^{(3,1)}(4,3) & \tilde{r}_i^{(3,1)}(4,4) & \tilde{r}_i^{(3,1)}(4,5) \\ 0 & \tilde{r}_i^{(3,1)}(5,2) & \tilde{r}_i^{(3,1)}(5,3) & \tilde{r}_i^{(3,1)}(5,4) & \tilde{r}_i^{(3,1)}(5,5) \end{bmatrix} \quad (\text{A.84})$$

where $i = 1, 2$.

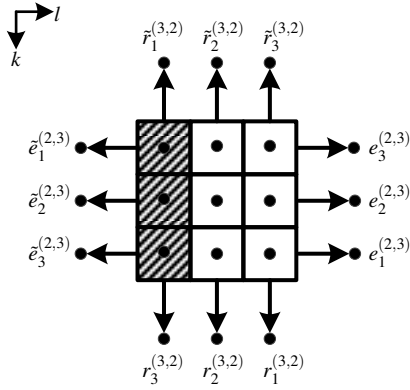


Figure A.4: Simple illustration of auxiliary vertical and horizontal forward/backward PEFs of the PSR $S_{k,l}^{(2,2)}$.

The updated vertical and horizontal forward/backward PEFs for the $S_{k,l}^{(2,2)}$:

$$e_i^{(2,3)}(k,l) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & e_i^{(2,3)}(3,4) & e_i^{(2,3)}(3,5) \\ 0 & 0 & 0 & e_i^{(2,3)}(4,4) & e_i^{(2,3)}(4,5) \\ 0 & 0 & 0 & e_i^{(2,3)}(5,4) & e_i^{(2,3)}(5,5) \end{bmatrix} \quad (\text{A.85})$$

$$\tilde{e}_i^{(2,3)}(k,l) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \tilde{e}_i^{(2,3)}(3,4) & \tilde{e}_i^{(2,3)}(3,5) \\ 0 & 0 & 0 & \tilde{e}_i^{(2,3)}(4,4) & \tilde{e}_i^{(2,3)}(4,5) \\ 0 & 0 & 0 & \tilde{e}_i^{(2,3)}(5,4) & \tilde{e}_i^{(2,3)}(5,5) \end{bmatrix} \quad (\text{A.86})$$

where $i = 1, 2, 3$.

$$r_i^{(3,2)}(k,l) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & r_i^{(3,2)}(4,3) & r_i^{(3,2)}(4,4) & r_i^{(3,2)}(4,5) \\ 0 & 0 & r_i^{(3,2)}(5,3) & r_i^{(3,2)}(5,4) & r_i^{(3,2)}(5,5) \end{bmatrix} \quad (\text{A.87})$$

$$\tilde{r}_i^{(3,2)}(k,l) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \tilde{r}_i^{(3,2)}(4,3) & \tilde{r}_i^{(3,2)}(4,4) & \tilde{r}_i^{(3,2)}(4,5) \\ 0 & 0 & \tilde{r}_i^{(3,2)}(5,3) & \tilde{r}_i^{(3,2)}(5,4) & \tilde{r}_i^{(3,2)}(5,5) \end{bmatrix} \quad (\text{A.88})$$

where $i = 1, 2, 3$.

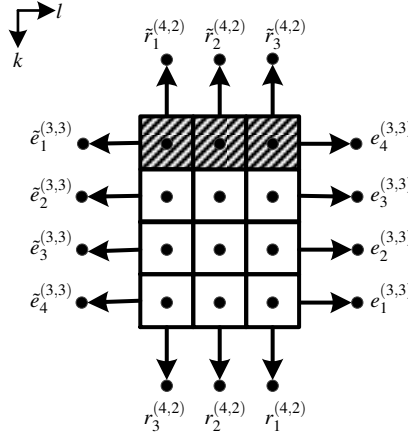


Figure A.5: Simple illustration of auxiliary vertical and horizontal forward/backward PEFs of the PSR $S_{k,l}^{(3,2)}$.

The updated vertical and horizontal forward/backward PEFs for the $S_{k,l}^{(3,2)}$:

$$e_i^{(3,3)}(k,l) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & e_i^{(3,3)}(4,4) & e_i^{(3,3)}(4,5) \\ 0 & 0 & 0 & e_i^{(3,3)}(5,4) & e_i^{(3,3)}(5,5) \end{bmatrix} \quad (\text{A.89})$$

$$\tilde{e}_i^{(3,3)}(k,l) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \tilde{e}_i^{(3,3)}(4,4) & \tilde{e}_i^{(3,3)}(4,5) \\ 0 & 0 & 0 & \tilde{e}_i^{(3,3)}(5,4) & \tilde{e}_i^{(3,3)}(5,5) \end{bmatrix} \quad (\text{A.90})$$

where $i = 1, 2, 3$.

$$r_i^{(4,2)}(k,l) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & r_i^{(4,2)}(5,3) & r_i^{(4,2)}(5,4) & r_i^{(4,2)}(5,5) \end{bmatrix} \quad (\text{A.91})$$

$$\tilde{r}_i^{(4,2)}(k,l) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \tilde{r}_i^{(4,2)}(5,3) & \tilde{r}_i^{(4,2)}(5,4) & \tilde{r}_i^{(4,2)}(5,5) \end{bmatrix} \quad (\text{A.92})$$

where $i = 1, 2, 3, 4$.

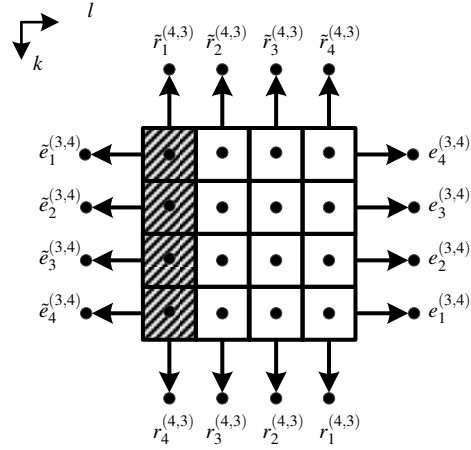


Figure A.6: Simple illustration of auxiliary vertical and horizontal forward/backward PEFs of the PSR $S_{k,l}^{(3,3)}$.

The updated vertical and horizontal forward/backward PEFs for the $S_{k,l}^{(3,3)}$:

$$e_i^{(3,4)}(k,l) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & e_i^{(3,4)}(4,5) \\ 0 & 0 & 0 & 0 & e_i^{(3,4)}(5,5) \end{bmatrix} \quad (\text{A.93})$$

$$\tilde{e}_i^{(3,4)}(k,l) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \tilde{e}_i^{(3,4)}(4,5) \\ 0 & 0 & 0 & 0 & \tilde{e}_i^{(3,4)}(5,5) \end{bmatrix} \quad (\text{A.94})$$

where $i = 1, 2, 3, 4$.

$$r_i^{(4,3)}(k,l) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & r_i^{(4,3)}(5,4) & r_i^{(4,3)}(5,5) \end{bmatrix} \quad (\text{A.95})$$

$$\tilde{r}_i^{(4,3)}(k,l) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \tilde{r}_i^{(4,3)}(5,4) & \tilde{r}_i^{(4,3)}(5,5) \end{bmatrix} \quad (\text{A.96})$$

where $i = 1, 2, 3, 4$.

As seen that the prediction support data is getting smaller with respect to the stage order. We look at the PEFs matrices for the $S_{k,l}^{(3,3)}$, it is seen that we have (3,4) stage order for the auxiliary vertical forward/backward PEFs. If we apply the rule for the left data $(M-p) \times (N-q)$, and then we can find $(5-3) \times (5-4) = 2 \times 1$ data. Indeed, if we look at the auxiliary vertical PEFs, we can see just only two values exist vertically in the related PEF data.

APPENDIX A.3: Definitions of the Constant Matrix Ψ and Vector Π

When modeling the lattice filter we need definitions of constant matrix Ψ and constant vector Π to simplify the equations. The constant matrix $\Psi_{1,2,\dots,j}$ is produced by removing the lines given in the subscript from the identity matrix having the appropriate order. The constant vector Π_j is also produced by getting the line given in the subscript.

For the PSR, $S_{k,l}^{(M-1,N-1)}$, we can define the following equations for using to model type-1 and type-2 ASHP lattice filters from the auxiliary horizontal PEFs,

$$\begin{aligned}\Psi_{1,2,\dots,j} &= \Psi_{ASHP_{1,j}} \\ &= \begin{bmatrix} \mathbf{O}_{(N-j) \times j} & \vdots & \mathbf{I}_{N-j} \end{bmatrix}_{(N-j) \times (N)},\end{aligned}\quad (\text{A.97})$$

and

$$\begin{aligned}\Psi_{j,j+1,\dots,N} &= \Psi_{ASHP_{2,j}} \\ &= \begin{bmatrix} \mathbf{I}_{j-1} & \vdots & \mathbf{O}_{(j-1) \times (N+1-j)} \end{bmatrix}_{(j-1) \times (N)},\end{aligned}\quad (\text{A.98})$$

respectively. For the type-1 and type-2 ASHP lattice model, only N changes to M .

It is possible to represent both constant matrix definitions of Ψ and the constant vector Π in the one matrix form as follows

$$\begin{aligned}\mathbf{I}_N &= \begin{bmatrix} \Psi_{ASHP_{2,j}} \\ \vdots \\ \Pi_j \\ \vdots \\ \Psi_{ASHP_{1,j}} \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{I}_{j-1} & \vdots & \mathbf{O}_{(j-1) \times (N+1-j)} \\ \vdots & \vdots & \vdots \\ \mathbf{O}_{(N-j) \times j} & \vdots & \mathbf{I}_{N-j} \end{bmatrix}_{N \times N}.\end{aligned}\quad (\text{A.99})$$

For example, let us assume that $N = 5$ and now let us define constant matrices $\Psi_{1,\dots,j} = \Psi_{ASHP_{1,j}}$ and $\Psi_{j,\dots,N-1} = \Psi_{ASHP_{2,j}}$, and constant vector Π_j for $j = 3$.

First, let us write the identity matrix $\mathbf{I}_5$:

$$\mathbf{I}_5 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}\quad (\text{A.100})$$

From the above definitions, we can write

$$\begin{aligned}\Psi_{ASHP_{1,3}} &= \begin{bmatrix} \mathbf{O}_{2 \times 3} & \vdots & \mathbf{I}_2 \end{bmatrix}_{2 \times 5} \\ &= \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}_{2 \times 5}\end{aligned}\quad (\text{A.101})$$

and

$$\begin{aligned}
 \Psi_{ASHP_{2,3}} &= \begin{bmatrix} \mathbf{I}_2 & \vdots & \mathbf{O}_{2 \times 3} \end{bmatrix}_{2 \times 5} \\
 &= \left[\begin{array}{cc|ccc} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{array} \right]_{2 \times 5}.
 \end{aligned} \tag{A.102}$$

It is clearly seen that

$$\Pi_3 = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \end{bmatrix}_{1 \times 5}. \tag{A.103}$$

Now, if we want to show all these in the identity matrix, it will be as

$$\begin{aligned}
 \mathbf{I}_5 &= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \\
 &= \begin{bmatrix} \Psi_{ASHP_{2,3}} \\ \Pi_3 \\ \Psi_{ASHP_{1,3}} \end{bmatrix} \\
 &= \begin{bmatrix} \mathbf{I}_2 & \vdots & \mathbf{O}_{2 \times 3} \\ \Pi_3 \\ \mathbf{O}_{2 \times 3} & \vdots & \mathbf{I}_2 \end{bmatrix}_{5 \times 5} \\
 &= \left[\begin{array}{cc|ccc} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array} \right]_{5 \times 5}.
 \end{aligned} \tag{A.104}$$

APPENDIX A.4: Itakura's LPC Distance Measure

Itakura's LPC distance measure [68] is used to find the similarity of given two sets of AR model parameters. In this measure calculation, the estimated coefficient matrix, denoted by $\mathbf{A}_{est}$ and the original coefficient matrix, denoted by $\mathbf{A}_{org}$ are used as an inputs. Itakura's LPC distance measure provides an effective comparison between the related spectra of both set of parameters. In other words, It is a measure of the perceptual difference between the estimated and the original spectrum. It can be defined for the 2-D case as follows,

$$d(\mathbf{A}_{est}, \mathbf{A}_{org}) = \log \left[\int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \left| \frac{A_{org}(\omega_1, \omega_2)}{A_{est}(\omega_1, \omega_2)} \right|^2 \frac{d\omega_1}{2\pi} \frac{d\omega_2}{2\pi} \right] \quad (\text{A.105})$$

The distance will be zero, if the model parameters are the same. Increase the difference between the spectra increases the distance. This method is not symmetric. That means, if we change the place of the $\mathbf{A}_{est}$ and $\mathbf{A}_{org}$ in the equation (A.105), we may not get the same result.

APPENDIX A.5: Spectrum Estimation

In the study, we need to estimate the power spectral density of a random data field. In the literature, the approaches to spectral estimation can be grouped as parametric and non-parametric methods. Non-parametric methods are known as classical methods using Fourier transform. The classical methods are based on periodogram and called as periodogram spectrum estimation that can be calculated by using the following equation

$$\hat{P}_y(\omega_1, \omega_2) = \frac{1}{N} |Y(\omega_1, \omega_2)|^2 \quad (\text{A.106})$$

where N is the number of elements in the data field and $Y(\omega_1, \omega_2)$ is Fourier transform of the data given as $y(k_1, k_2)$.

Parametric methods use the coefficients of the model predicted from the data by using the convenient process. For example, in this study we used our proposed method for modeling the data as AR process. The predicted power spectral density $\hat{P}_x(\omega_1, \omega_2)$ from the process $x(n_1, n_2)$ can be expressed as

$$\hat{P}_x(\omega_1, \omega_2) = \frac{\sigma_w^2}{\left| 1 + \sum_{k_1}^p \sum_{k_2}^q a(k_1, k_2) \exp(-j\omega_1 n_1) \exp(-j\omega_2 n_2) \right|^2} \quad (\text{A.107})$$

where p, q are the order of the AR model and $a(k_1, k_2)$ s are the coefficient values of the AR model. σ^2 is the variance of the white noise with zero mean.

In some application, the equation (A.107) is not enough to locate the sinusoidal peaks in the data when using a QP or ASHP or NC half-plane model. Therefore, harmonic mean estimates of the filters obtained by using the method give the satisfactory results. The harmonic mean estimate method which is used in the study is explained in [58].

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List of Publications and Patents:

PUBLICATIONS/PRESENTATIONS ON THE THESIS

- Kayran A.H., **Camcioglu E.**, Eksioglu E.M., Tanc A.K. and Seker S., 2011: Generalized 2D Lattice Structure for Causal and Noncausal Modeling of Random Fields. *IEEE International Symposium on Circuit and Systems (ISCAS) 2011*, pages: 1756-1759, May 15-18, 2011, Rio de Janeiro, Brazil.
DOI: 10.1109/ISCAS.2011.5937923
- **Camcioglu E.**, Kayran A.H., 2013: Spectrum Estimation using New Efficient 2-D AR Lattice Modeling of Random Fields. *IEEE AFRICON 2013 Proceedings*, September 9-12, 2013, Pointe-Aux-Piments, Mauritius.
DOI: 10.1109/AFRCON.2013.6757705
- Kayran A.H., **Camcioglu E.**, 2014: New Efficient 2-D Lattice Structures for General Autoregressive Modeling of Random Fields. *IEEE Transactions on Signal Processing*, issue date: March 15, 2014, Volume: 62, Issue:6, pages: 1590-1602.
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