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Elastik bir yarım uzay üzerinde hareketli pürüzlü blok

Moving rough punch on an elastic half-space

by

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Bu çalışma bir rijit blokun bir elastik yarım uzayın yüzeyi üzerinde sürtünmeli düzgün hareketi ile ilgilidir. Çözüm için bir kompleks fonksiyon tekniği kullanılmış ve problem düzlemde bir Hilbert problemine indirgenmiştir. Bunun çözümü de standard metodla elde edilmiştir. Genel çözüm düzlem ve dairesel tabanlı bloklara özelleştirilmiştir. Bazı ana sabitler için tablolar verilmiş ve yüzey üzerindeki basınç dağılımı gösterilmiştir.

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This paper is concerned with the investigation of the uniform motion of a rough rigid punch on the surface of an elastic half-space. A complex function technique is employed and the problem is reduced to a Hilbert problem in the plane, the solution of which is obtained by the standard method. The general solution is specialized to the cases of rectangular and circular punches. Tables for some fundamental constants are given and pressure distributions are shown.

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1. Introduction

It has been established that if the sources responsible for a plane motion of an elastic medium move with a constant velocity, the number of independent variables may be reduced from three to two under the assumption that a steady wave pattern exist relative to a frame rigidly attached to the moving sources. This, in turn, enables us to develop a complex function technique, similar to one used in anisotropic plane elastostatics, to attack such problems provided that the magnitude of the velocity is less than the shear wave velocity of the medium. This paper deals with the application of this method to the study of a uniform motion of a rough rigid punch over the surface of an elastic half-space. This problem has been previously investigated by Galin [1], briefly and in a somewhat limited framework employing a technique based on the solution of Riemann-Hibert problem. Also it appears that there are some computational errors in the evaluation of some fundamental constants. We believe that a thorough and exhaustive treatment of this technically important problem is presented here. Our analysis is based on the solution of a Hilbert problem for a plane cut along a finite straight line.

We derived the fundamental equations in Sec. 2 where we introduced the complex potentials and discussed briefly some of their properties. The problem of moving punch is formulated in Sec. 3 and the complete solution for a general profile is furnished. Three cases coresponding to the position of the corners of the punch relative to the surface of the half-space are distinguished and studied separately. The general solution is specialized to a rectangular punch in Sec. 4 and to a circular punch of a large radius (in other words, a parabolic punch) in Sec. 5. Tables for fundamental constants are provided for various values of the punch velocity, coefficient of friction between the punch and the half-space and Poisson's ratio of the elastic medium. Pressure distribution under the rectangular and cylindrical punches are also shown. These graphs also represent the distribution of tangential stresses under the punch within a scale factor.

2. Fundamental Equations

It is a well-known fact that the plane motion of a lineary elastic solid in the $x_1 x_2$ - plane is governed by the following two scalar wave equations:

$$\nabla^2\varphi=\frac{1}{c_1{}^2}\frac{\partial^2\varphi}{\partial t^2}, \qquad \nabla^2\psi=\frac{1}{c_2{}^2}\frac{\partial^2\psi}{\partial t^2} \tag{2.1}$$

$$c_1=\sqrt{\frac{\lambda+2\mu}{\rho}}, \quad c_2=\sqrt{\frac{\mu}{\rho}} \tag{2.2}$$

are the velocities of irrotational and equivoluminal waves, respectively, λ and μ are Lamé's constants and ρ is the density of the medium. In terms of the potentials φ and ψ the stress and displacement components are given as follows:

$$
\begin{aligned}
t_{11} &= \lambda\nabla^2\varphi + 2\mu\,\frac{\partial^2\varphi}{\partial x_1{}^2} + 2\mu\,\frac{\partial^2\psi}{\partial x_1\partial x_2} \\
t_{22} &= \lambda\nabla^2\varphi + 2\mu\,\frac{\partial^2\varphi}{\partial x_2{}^2} - 2\mu\,\frac{\partial^2\psi}{\partial x_1\partial x_2} \\
t_{12} &= 2\mu\,\frac{\partial^2\varphi}{\partial x_1\partial x_2} + \mu\left(\frac{\partial^2\psi}{\partial x_2{}^2} - \frac{\partial^2\psi}{\partial x_1{}^2}\right) \\
t_{33} &= \lambda\nabla^2\varphi \\
t_{13} &= t_{23} = 0
\end{aligned}
\tag{2.3}
$$

and

$$
\begin{aligned}
u_1 &= \frac{\partial\varphi}{\partial x_1} + \frac{\partial\psi}{\partial x_2} \\
u_2 &= \frac{\partial\varphi}{\partial x_2} - \frac{\partial\psi}{\partial x_1}
\end{aligned}
\tag{2.4}
$$

We define now the complex quantities

$$
\begin{aligned}
D &= u_1 + iu_2 \\
\Theta &= t_{11} + t_{22} \\
\Phi &= t_{11} - t_{22} + 2it_{12}
\end{aligned}
\tag{2.5}
$$

In view of (2.3) and (2.4) these are expressible in the form

$$
\begin{aligned}
D &= \left(\frac{\partial}{\partial x_1} + i\,\frac{\partial}{\partial x_2}\right)(\varphi - i\psi) \\
\Theta &= 2(\lambda+\mu)\nabla^2\varphi \\
\Phi &= 2\mu\left(\frac{\partial^2}{\partial x_1{}^2} - \frac{\partial^2}{\partial x_2{}^2} + 2i\,\frac{\partial^2}{\partial x_1\partial x_2}\right)(\varphi - i\psi) = 2\mu\left(\frac{\partial}{\partial x_1} + i\,\frac{\partial}{\partial x_2}\right)D
\end{aligned}
\tag{2.6}
$$

Let us now suppose that the elastic medium is set into motion by a source moving with a constant velocity U in the direction of one coordinate axis, say x_1 - axis. If the source has been moving for a sufficiently long time then it might be reasonably assumed that a steady wave pattern with respect to an observer situated in a frame rigidly attached to the moving source is formed in the medium. Hence the field quantities may be thought as the function of only two independent variables.

$$\xi_1 = x_1 - Ut, \quad \xi_2 = x_2 \tag{2.7}$$

If the source travels in the opposite direction to x_1 axis, ξ_1 should be taken as equal to $x_1 + Ut$. If we introduce (2.7) into (2.1) we have

$$\begin{aligned} (1-M_1^2)\frac{\partial^2\varphi}{\partial\xi_1^2} + \frac{\partial^2\varphi}{\partial\xi_2^2} = 0 \\ (1-M_2^2)\frac{\partial^2\psi}{\partial\xi_1^2} + \frac{\partial^2\psi}{\partial\xi_2^2} = 0 \end{aligned} \tag{2.8}$$

where

$$M_1 = \frac{U_1}{c_1}, \quad M_2 = \frac{U_2}{c_2} \tag{2.9}$$

represent *Mach numbers* of the moving source. Since $c_1 > c_2$ then $M_1 < M_2$ and we call the case $M_2 < 1$ as the *subsonic case*. It is clear that equations (2.8) are both of elliptic type in this case. Henceforth we shall always assume that we deal with elliptic equations. If we define now

$$z_1 = \xi_1 + i\beta_1\xi_2, \quad z_2 = \xi_1 + i\beta_2\xi_2 \tag{2.10}$$

where

$$\beta_1 = \sqrt{1-M_1^2}, \quad \beta_2 = \sqrt{1-M_2^2} \tag{2.11}$$

we see that (2.8) reduce to

$$\frac{\partial^2\varphi}{\partial z_1 \partial \bar{z}_1} = 0, \quad \frac{\partial^2\psi}{\partial z_2 \partial \bar{z}_2} = 0 \tag{2.12}$$

The solution of (2.12), recalling that φ and ψ are real functions, can be represented as

$$\begin{aligned} \varphi &= \Omega_1(z_1) + \overline{\Omega}_1(\bar{z}_1) \\ \psi &= i[\Omega_2(z_2) - \overline{\Omega}_2(\bar{z}_2)] \end{aligned} \tag{2.13}$$

where $\Omega_1(z_1)$ and $\Omega_2(z_2)$ are, respevtively, function of complex variables z_1 and z_2 only. The choice for ψ is for convenience.

If we define

$$z = \xi_1 + i\,\xi_2$$

we see that (2.6) becomes

$$\begin{aligned} D &= 2\frac{\partial}{\partial \bar{z}}(\varphi - i\psi) \\ \Theta &= 8(\lambda+\mu)\frac{\partial^2\varphi}{\partial z \partial \bar{z}} \\ \Phi &= 8\mu\frac{\partial^2}{\partial \bar{z}^2}(\varphi - i\psi) = 4\mu\frac{\partial D}{\partial \bar{z}} \end{aligned} \tag{2.14}$$

Noting that

$$\frac{\partial z_\alpha}{\partial z}=\frac{\partial \bar{z}_\alpha}{\partial \bar{z}}=\frac{1+\beta_\alpha}{2}\,,\quad \frac{\partial z_\alpha}{\partial \bar{z}}=\frac{\partial \bar{z}_\alpha}{\partial z}=\frac{1-\beta_\alpha}{2}\,,\quad (\alpha=1,\,2) \tag{2.15}$$

we evaluate (2.14), in view of (2.13), as

$$\begin{aligned} D&=(1-\beta_1)\omega_1(z_1)+(1+\beta_1)\bar{\omega}_1(\bar{z}_1)+(1-\beta_2)\omega_2(z_2)-(1+\beta_2)\bar{\omega}_2(\bar{z}_2)\\ \Theta&=2\mu(\beta_1^2-\beta_2^2)[\omega_1'(z_1)+\bar{\omega}_1'(\bar{z})]\\ \Phi&=2\mu[(1-\beta_1)^2\omega_1'(z_1)+(1+\beta_1)^2\bar{\omega}_1'(\bar{z}_1)+(1-\beta_2)^2\omega_2'(z_2)-(1+\beta_2)^2\bar{\omega}_2'(\bar{z}_2)] \end{aligned} \tag{2.16}$$

where we defined

$$\omega_1(z_1)=\Omega_1'(z_1)\,,\quad \omega_2(z_2)=\Omega_2'(z_2) \tag{2.17}$$

and employed the relation

$$\lambda+\mu=\mu\,\frac{\beta_1^2-\beta_2^2}{1-\beta_1^2} \tag{2.18}$$

Primes denote differentiation with respect to the argument. Hence the solution of the elastodynamic problem is reduced to the determination of two complex potentials $\omega_1(z_1)$ and $\omega_2(z_2)$ with distinct arguments z_1 and z_2 defined by (2.10). The complex function theory was first applied to the plane dynamic elasticity by Sneddon [2]. We also refer the reader to Radok [3], Teodorescu [4] and Gladwell [5] whose derivation we followed here to some extent.

The stress and displacements components are readily obtained from (2.16) and (2.6):

$$\begin{aligned} u_1&=\omega_1(z_1)+\bar{\omega}_1(\bar{z}_1)-\beta_2[\omega_2(z_2)+\bar{\omega}_2(\bar{z}_2)]\\ u_2&=i\{\beta_1[\omega_1(z_1)-\bar{\omega}_1(\bar{z}_1)]-[\omega_2(z_2)-\bar{\omega}_2(\bar{z}_2)]\}\\ t_{11}&=\mu\{(2\beta_1^2-\beta_2^2+1)[\omega_1'(z_1)+\bar{\omega}_1'(\bar{z}_1)]-2\beta_2[\omega_2'(z_2)+\bar{\omega}_2'(\bar{z}_2)]\}\\ t_{22}&=-\mu\{(1+\beta_2^2)[\omega_1'(z_1)+\bar{\omega}_1'(\bar{z}_1)]-2\beta_2[\omega_2'(z_2)+\bar{\omega}_2'(\bar{z}_2)]\}\\ t_{12}&=i\mu\{2\beta_1[\omega_1'(z_1)-\bar{\omega}_1'(\bar{z}_1)]-(1+\beta_2^2)[\omega_2'(z_2)-\bar{\omega}_2'(\bar{z}_2)]\} \end{aligned} \tag{2.19}$$

Let us now consider a curve AB in the plane (x_1, x_2) which is stationary relative to the source and evaluate the resultant **F** of the contact forces exerted by the material in R_1 on the material in R_2 across this curve (Fig. 1). This force is given by

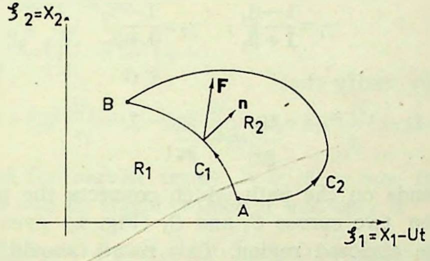


Fig. 1

$$F_\alpha = -\int_A^B t_{\alpha\beta}\, n_\beta\, ds$$

(Greek indices = 1, 2). We define a complex number F by

$$F = F_1 + iF_2$$

and recall that the unit normal **n** to the curve AB has components

$$n_1 = \frac{d\xi_2}{ds}, \quad n_2 = -\frac{d\xi_1}{ds}$$

Thus we have

$$F = -\int_A^B [(t_{11} + it_{12})d\xi_2 - i(t_{22} - it_{12})d\xi_1]$$

and using (2.5)

$$F = \frac{i}{2}\int_A^B (\Theta dz - \Phi d\bar{z}) \tag{2.20}$$

In (2.20), Θ and Φ are given by (2.16) and are function of $z_1, \bar{z}_1, z_2, \bar{z}_2$ which are related to z and $\bar{z}$ by

$$z_1=\frac{1+\beta_1}{2}z+\frac{1-\beta_1}{2}\bar{z}=\frac{1+\beta_1}{2}(z+\gamma_1\bar{z})$$

$$z_2=\frac{1+\beta_2}{2}(z+\gamma_2\bar{z})$$

with

$$\gamma_1=\frac{1-\beta_1}{1+\beta_1}, \quad \gamma_2=\frac{1-\beta_2}{1+\beta_2}$$

One the can easily verify that

$$\frac{\partial\Theta}{\partial\bar{z}}\neq-\frac{\partial\Phi}{\partial z}$$

Therefore F depends on the path which connects the points A and B and is different on two curves C_1 and C_2 (Fig. 1) even if there is no singularity in the enclosed region. This result should, of course, be expected since that region is not in the state of equilibrium. Thus, to study the algebraic structure of complex potentials ω_1 and ω_2 becomes rather difficult especially when the region is multiply-connected. Hence it deems that the practical applicability of this otherwise extremely elegant formulation is somewhat limited. However a number of problem have been successfully attacked by this method and the possibility of the full use of conformal mapping increases the merits of the technique. But it is in order to indicate that the occurence of two complex variables z_1 and z_2 in the complex potentials requires two conformal mapping functions and a boundary L in z - plane is mapped onto two boundaries L_1 and L_2 in z_1 - and z_2 - planes.

We have to emphasize once more that the method is strictly applicable only for subsonic cases. However, with proper modifications, the continuation to the the transonic and supersonic cases can be accomplished without too much difficulty.

Before concluding this section we investigate the singular parts of complex potentials corresponding to a concentrated force in the origin of z - plane in a simply - connected region and the behaviour of the solution at infinity.

We write

$$\begin{aligned}\omega_1(z_1)&=\frac{H_1}{2\pi}\log z_1+\tilde{\omega}_1(z_1)\\ \omega_2(z_2)&=\frac{H_2}{2\pi}\log z_2+\tilde{\omega}_2(z_2)\end{aligned}\tag{2.21}$$

where ω_1 and ω_2 are holomorphic in the z_1 and z_2 planes respectively and H_1, H_2 are complex constants. Hence we obtain from (2.16)

$$
\begin{aligned}
D&=\frac{1}{2\pi}[(1-\beta_1)H_1\log z_1+(1+\beta_1)\overline{H}_1\log \overline{z}_1+(1-\beta_2)H_2\log z_2 \\
&\quad -(1+\beta_2)\overline{H}_2\log \overline{z}_2]+\cdots \\
\Theta&=\mu\frac{\beta_1^2-\beta_2^2}{\pi}\left(\frac{H_1}{z_1}+\frac{\overline{H}_1}{\overline{z}_1}\right)+\cdots \\
\Phi&=\frac{\mu}{\pi}\left[(1-\beta_1)^2\frac{H_1}{z_1}+(1-\beta_2)^2\frac{H_2}{z_2}+(1+\beta_1)^2\frac{\overline{H}_1}{\overline{z}_1}-(1+\beta_2)^2\frac{\overline{H}_2}{\overline{z}_2}\right]+\cdots
\end{aligned}
\tag{2.22}
$$

where dots stand for regular terms. It is clear now from $(2.22)_1$ that the displacement will be single - valued if and only if

$$(1-\beta_1)H_1-(1+\beta_1)\overline{H}_1+(1-\beta_2)H_2+(1+\beta_2)\overline{H}_2=0 \tag{2.23}$$

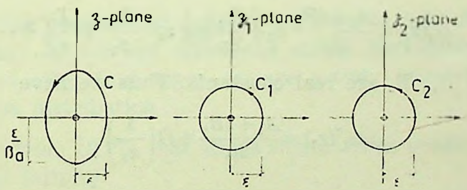


Fig. 2

Next we evaluate (2.20) along elliptical paths shrinking toward the origin in the z - plane. These paths correspond to circles with vanishing radii centred at the origin of the z_1 - and z_2 - planes as shown is Fig. 2. Considering the limiting case for $\varepsilon\to 0$ and noting that

$$\lim_{\varepsilon\to 0}\oint_{C_a}\frac{dz_a}{z_a}=2\pi i,\quad \lim_{\varepsilon\to 0}\oint_{C_a}\frac{d\overline{z}_a}{\overline{z}_a}=-2\pi i,\quad \lim_{\varepsilon\to 0}\oint_{C_a}\frac{dz_a}{\overline{z}_a}=0$$

we immediantely obtain

$$
\begin{aligned}
\frac{1+\beta_1}{\beta_1}\Big\{&[\beta_1^2-\beta_2^2+(1+\beta_1)^2\gamma_1^3]H_1+[\gamma_1(\beta_1^2-\beta_2^2)+(1+\beta_1)^2]\overline{H}_1 \\
&+\frac{(1+\beta_2)^3}{\beta_2}(\gamma_2^3H_2-\overline{H}_2)=-\frac{2F}{\mu}
\end{aligned}
\tag{2.24}
$$

It follows then from (2.23) and (2.24) that

$$H_1 = -(\alpha_1+\alpha_2)F - (\alpha_1-\alpha_2)\overline{F}$$

$$H_2 = -\frac{1+\beta_1\beta_2}{2\beta_2}[(\alpha_1+\alpha_2)F+(\alpha_1-\alpha_2)\overline{F}] \tag{2.25}$$

$$+\frac{1-\beta_1\beta_2}{2\beta_2}[(\alpha_1-\alpha_2)F+(\alpha_1+\alpha_2)\overline{F}]$$

where

$$\alpha_1 = \frac{\beta_1}{2\mu(1+\beta_1^2)(1-\beta_2^2)}, \quad \alpha_2 = \frac{\beta_2^2}{2\mu(1-\beta_2^4)} \tag{2.26}$$

When the region extends to infinity the complex potentials must admit the following representation for sufficiently large $|z|$ if the stresses are to remain bounded at infinity:

$$\omega_1(z_1) = \frac{A_1+iB_1}{2\mu}z_1 + \frac{H_1}{2\pi}\log z_1 + \frac{K_1}{z_1} + \frac{L_1}{z_1^2} + \cdots$$

$$\omega_2(z_2) = \frac{A_2+iB_2}{2\mu}z_2 + \frac{H_2}{2\pi}\log z_2 + \frac{K_2}{z_2} + \frac{L_2}{z_2^2} + \cdots \tag{2.27}$$

where A_1, B_1, A_2, B_2 are real constants. Thus we have

$$\omega_1'(z_1) = \frac{A_1+iB_1}{2\mu} + O\left(\frac{1}{z_1}\right)$$

$$\omega_2'(z_2) = \frac{A_2+iB_2}{2\mu} + O\left(\frac{1}{z_2}\right) \tag{2.28}$$

as $|z| \to \infty$. We conclude from (2.28) that the stress tensor should be constant at infinity. If we denote these components by t_{11}^{∞}, t_{22}^{∞} and t_{12}^{∞} we easily obtain

$$A_1 = \frac{t_{11}^{\infty}+t_{22}^{\infty}}{2(\beta_1^2-\beta_2^2)},$$

$$B_1 = -\frac{t_{12}^{\infty}}{2\beta_1} \tag{2.29}$$

$$A_2 = \frac{t_{22}^{\infty}}{2\beta_2} + \frac{1+\beta_2^2}{4\beta_2(\beta_1^2-\beta_2^2)}(t_{11}^{\infty}+t_{22}^{\infty})$$

Evidently the homogeneous state of stress at infinity cannot determine the constant B_2. In order to understand the physical meaning of B_2 let us consider the local rotation of the medium defined by

$$\omega=\frac{1}{2}\left(\frac{\partial u_2}{\partial x_1}-\frac{\partial u_1}{\partial x_2}\right)=\frac{1}{2}\left(\frac{\partial u_2}{\partial \xi_1}-\frac{\partial u_1}{\partial \xi_2}\right)$$

$$=\frac{1}{2i}\left(\frac{\partial D}{\partial z}-\frac{\partial \bar{D}}{\partial \bar{z}}\right)$$

which yields

$$\omega=\frac{1-\beta_2^2}{2i}\left[\omega_2'(z_2)-\bar{\omega}_2'(\bar{z}_2)\right] \tag{2.30}$$

Hence it follows from (2.28) that

$$\omega^{\infty}=\omega\Big|_{|z|\to\infty}=\frac{1-\beta_2^2}{2\mu}B_2$$

or

$$B_2=\frac{2\mu\omega^{\infty}}{1-\beta_2^2} \tag{2.31}$$

If the rotation vanishes at infinity we have to take $B_2=0$. At any rate, the constant B_2 will not effect the stress distribution so that it can be discarded altogether when we are mainly interested in determining the stress distribution

It is evident from (2.27) that stress singularities produce unbounded displacements at infinity.

It is also clear that the solution of the problem of a concentrated force moving with a subsonic velocity in the direction of the x_1-axis in an infinite medium free from stresses at infinity is given by

$$\omega_1(z_1)=\frac{H_1}{2\pi}\log z_1\,, \qquad \omega_2(z_2)=\frac{H_2}{2\pi}\log z_2 \tag{2.32}$$

where H_1 and H_2 are calculated in (2.25)

3. Moving Puch on the Half-Space

We consider a rigid punch moving with a uniform velocity U along the boundary of the half-space $x_2>0$ namely, in the direction of positive x_1 - axis. We assume that $M_2=U/c_2<1$. The profile of the punch is given, i. e., $g=g(\xi)$* (Fig. 3). We suppose that the profile of the punch is symmetrical with respect to its axis. The coefficient of friction between the punch and the half-space is k. It will further be assumed that the

* Henceforth we shall use $\xi=\xi_1$, $\eta=\xi_2$.

punch will always be kept in vertical position while it moves. Clearly the coordinate axes ξ and η are rigidly attached to the punch. In the sequel we shall reduce the solution of this problem to the solution of a standard Hilbert problem and be able to give explicit expressions for the complex potentials, or more truly for their derivatives.

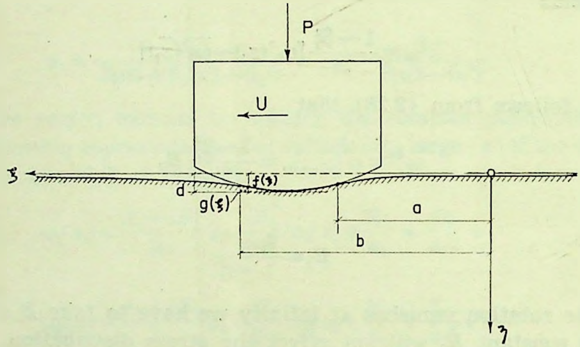


Fig. 3

The contact area under the punch is the interval $I_1 = [a, b]$. The complement of I_1 on the real axis is denoted by I_2. Obviously $I_2 = (-\infty, a) \cup (b, \infty)$. Hence the boundary conditions on the surface $\eta = 0$ become

$$\begin{aligned} &t_{12} = t_{22} = 0\,, \quad \xi \in I_2 \\ &t_{12} = kt_{22}\,, \quad u_2 = f(\xi)\,, \quad \xi \in I_1 \end{aligned} \tag{3.1}$$

where $f(\xi) = d - g(\xi)$. It is straightforward to see that the above conditions can be replaced by the following ones:

$$\begin{aligned} &t_{12} - kt_{22} = 0\,, \quad t_{22} = 0\,, \quad \xi \in I_2 \\ &t_{12} - kt_{22} = 0\,, \quad u_2 = f(\xi), \quad \xi \in I_1 \end{aligned} \tag{3.2}$$

If the punch moves in the opposite direction all one to do is to replace k by $-k$. In view of (2.19) we can express the boundary conditions as follows

$$\xi \in I_2 \left\{ \begin{aligned} &i\{2\beta_1[\omega_1'(\xi) - \bar{\omega}_1'(\xi)] - (1+\beta_2^2)[\omega_2'(\xi) - \bar{\omega}_2'(\xi)] \\ &+ k\,\{(1+\beta_2^2)[\omega_1'(\xi) + \bar{\omega}_1'(\xi)] - 2\beta_2[\omega_2'(\xi) + \bar{\omega}_2'(\xi)]\} = 0 \\ &(1+\beta_2^2)[\omega_1'(\xi) + \bar{\omega}_1'(\xi)] - 2\beta_2[\omega_2'(\xi) + \bar{\omega}_2'(\xi)] = 0 \end{aligned} \right. \tag{3.3}$$

$$\xi \in I_1 \begin{cases} i\{2\beta_1[\omega_1'(\xi)-\bar{\omega}_1'(\xi)]-(1+\beta_2^2)[\omega_2'(\xi)-\bar{\omega}_2'(\xi)\} \\ +k\{(1+\beta_2^2)[\omega_1'(\xi)+\bar{\omega}_1'(\xi)]-2\beta_2[\omega_2'(\xi)+\bar{\omega}_2'(\xi)]\}=0 \\ \beta_1[\omega_1'(\xi)-\bar{\omega}_1'(\xi)]-[\omega_2'(\xi)-\bar{\omega}_2'(\xi)]=if'(\xi) \end{cases} \tag{3.4}$$

where primes denote differentiations with respect to ξ. The last expression in (3.4) is the differentiated form of the displacement boundary condition in (3.2). Our problem now is to determine the complex functions $\omega_1'(z_1)$ and $\omega_2'(z_2)$ satisfying the mixed boundary conditions (3.3) and (3.4) and giving rise to vanishing stresses as $|z|\to\infty$, $\arg z\in[0,\pi]$. It follows from $(3.3)_1$ and $(3.4)_1$ that the function

$$i\{2\beta_1[\omega_1'(\xi)-\bar{\omega}_1'(\xi)]-(1+\beta_2^2)[\omega_2'(\xi)-\bar{\omega}_2'(\xi)]\}$$
$$+k\{(1+\beta_2^2)[\omega_1'(\xi)+\bar{\omega}_1'(\xi)]-2\beta_2[\omega_2'(\xi)+\bar{\omega}_2'(\xi)]\}$$

vanishes for $-\infty<\xi<\infty$. This implies that there is a functional relation such as

$$\omega_2'(\xi)=K\omega_1'(\xi) \tag{3.5}$$

where K is a complex constant. This substitution leads to

$$\{2\beta_1 i+(1+\beta_2^2)k-K[(1+\beta_2^2)i+2\beta_2 k]\}\omega_1'(\xi)+\{-2\beta_1 i+(1+\beta_2^2)k$$
$$-\bar{K}[-(1+\beta_2^2)i+2\beta_2 k]\}\bar{\omega}_1'(\xi)=0$$

whence we obtain the complex factor K:

$$K=\frac{(1+\beta_2^2)k+2\beta_1 i}{2\beta_2 k+(1+\beta_2^2)i}=\frac{2(1+\beta_2^2)(k^2\beta_2+\beta_1)+ik[4\beta_1\beta_2-(1+\beta_2^2)^2]}{(1+\beta_2^2)^2+4k^2\beta_2^2} \tag{3.6}$$

To satisfy the remaining conditions we introduce (3.5) into (3.3) and (3.4)

$$\begin{aligned} &(1+\beta_2^2-2\beta_2 K)\omega_1'(\xi)+(1+\beta_2^2-2\beta_2\bar{K})\bar{\omega}_1'(\xi)=0, \quad \xi\in I_2 \\ &(\beta_1-K)\omega_1'(\xi)-(\beta_1-\bar{K})\bar{\omega}_1'(\xi)=-if'(\xi). \quad \xi\in I_1 \end{aligned} \tag{3.7}$$

We now define a function $\chi(\xi)$ through

$$\chi(\xi)=(1+\beta_2^2-2\beta_2 K)\omega_1'(\xi) \tag{3.8}$$

in terms of which (3.7) becomes

$$\begin{aligned} &\chi(\xi)+\bar{\chi}(\xi)=0 \qquad \xi\in I_2 \\ &\chi(\xi)-\frac{H}{\bar{H}}\bar{\chi}(\xi)=-iHf'(\xi), \qquad \xi\in I_1 \end{aligned} \tag{3.9}$$

where

$$H=\frac{1+\beta_2^2-2\beta_2K}{\beta_1-K} \tag{3.10}$$

or

$$H=\frac{4\beta_1\beta_2-(1+\beta_2^2)^2}{\beta_1^2(1-\beta_2^2)^2+k^2(1+\beta_2^2-2\beta_1\beta_2)^2}[\beta_1(1-\beta_2^2)+ik(1+\beta_2^2-2\beta_1\beta_2)] \tag{3.11}$$

$$=he^{i\omega}$$

where

$$h=\frac{4\beta_1\beta_2-(1+\beta_2^2)^2}{\sqrt{\beta_1^2(1-\beta_2^2)^2+k^2(1+\beta_2^2-2\beta_1\beta_2)^2}}=\frac{4\beta_1\beta_2-(1+\beta_2^2)^2}{\beta_1(1-\beta_2^2)}\cos\omega \tag{3.12}$$

$$\omega=\operatorname{arctg} k\,\frac{1+\beta_2^2-2\beta_1\beta_2}{\beta_1(1-\beta_2^2)}\,,\qquad 0\leq\omega<\frac{\pi}{2}\quad(0\leq k<\infty) \tag{3.13}$$

We now consider the analytic continuation of the function $\chi(z)$ defined in the plane $Im\, z>0$ to the plane $Im\, z<0$ across the real axis;

$$\chi(\bar{z})=-\bar{\chi}(\bar{z})\,,\qquad Im\; z>0 \tag{3.14}$$

Let us also define

$$\chi(z)\,|_{z=\xi+i0}=\chi^+(\xi)\,,$$
$$\chi(\bar{z})\,|_{\bar{z}=\xi-i0}=-\bar{\chi}(\xi)=\chi^-(\xi)$$

which are values taken by the function $\chi(z)$ when is approached the real axis from the upper and lower planes, respectively, Hence (3.9) is now transformed to

$$\chi^+(\xi)-\chi^-(\xi)=0,\qquad \xi\in I_2$$
$$\chi^+(\xi)+e^{2i\omega}\chi^-(\xi)=-iHf'(\xi),\qquad \xi\in I_1$$

or

$$\chi^+(\xi)-\chi^-(\xi)=0\qquad \xi\in I_2$$
$$\chi^+(\xi)-e^{2i\pi\theta}\chi^-(\xi)=-iHf'(\xi),\qquad \xi\in I_1 \tag{3.15}$$

where

$$\theta=\frac{1}{2}+\frac{\omega}{\pi}$$
$$=\frac{1}{2}+\frac{1}{\pi}\operatorname{arctg}\frac{k}{M_2^2}\left[\frac{2-M_2^2}{\sqrt{1-\frac{1-2\nu}{2(1-\nu)}M_2^2}}-2\sqrt{1-M_2^2}\right] \tag{3.16}$$

Here obviously

$$\frac{1}{2} \leq \theta < 1, \qquad (0 \leq k < \infty)$$

If $M_2 \rightarrow 0$ (3.16) reduces to [6]

$$\theta = \frac{1}{2} + \frac{1}{\pi} \operatorname{arctg} k \frac{1-2\nu}{2(1-\nu)}$$

On the other hand if the friction is absent

$$\theta = \frac{1}{2}$$

for all $M_2 < 1$.

Subbstituting θ in (3.11) and (3.12) we find that

$$\begin{gathered} H = he^{i\left(\pi\theta - \frac{\pi}{2}\right)} = -ihe^{i\pi\theta} \\ h = \frac{4\beta_1\beta_2 - (1+\beta_2^2)^2}{\beta_1(1-\beta_2^2)} \sin \pi\theta = h_0 \sin \pi\theta \end{gathered} \tag{3.17}$$

The values of θ and h for various k, M_2 and Poisson's ratio ν are given in Tables 1-10.

Equations (3.15) express the fact that $\chi(z)$ so defined is a sectionally analytic function in the whole plane continuous across the line L_2 (Fig. 4). The right and left values of $\chi(z)$ along the line L_1 are related by $(3.15)_2$.

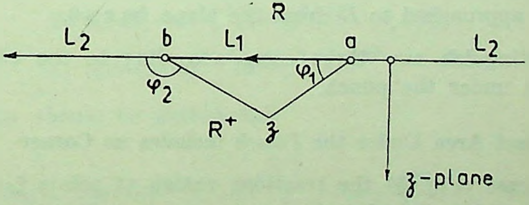


Fig. 4

Henceforth we shall assume that $f'(\xi)$ satisfies a Hölder condition. The determination of $\chi(z)$ requires the solution of a special case of the well-known Hilbert problem. For the investigation of this problem we refer to Muskhelishvili [6] or Gakhov [7].

A particular solution of the homogeneous Hilbert problem in (3.15) is given by*

$$X_0(z) = (z-a)^{-\theta}(z-b)^{\theta-1} \tag{3.18}$$

whereas the general homogeneous solution is expressible as

$$X(z) = X_0(z)P_0(z) \tag{3.19}$$

where $P_0(z)$ is an arbitrary entire function.

Since concentrated forces are not present under the punch the order of singularity should be less than unity. Although $X_0(z)$ is singular at the points a and b it meets the requirement since $\theta<1$. If we want $X(z)$ to remain bounded at the end points all we have to do is to choose $P_0(z)$ as $(z-a)(z-b)$ which results in the fundamental function

$$X_1(z) = (z-a)^{1-\theta}(z-b)^{\theta} \tag{3.20}$$

which bounded at both ends (actually zero!).

Having obtained the homogeneous solution of desired form we can write the solution of non-homogeneous problem as follows:

$$\chi(z) = -\frac{H}{2\pi}X(z)\int_a^b \frac{f'(\xi)}{X^+(\xi)(\xi-z)}d\xi + X(z)P(z) \tag{3.21}$$

where $P(z)$ is again an entire function and $X^+(\xi)$ denotes the value taken by $X(z)$ as approached to L_1 from the plane $Im\, z>0$.

We distinguish now three cases according to the allowed stress singularities under the punch.

i. Contact Area Under the Punch includes no Corner

In this case (Fig. 5) the tractions vanish at points $\xi=a$ and $\xi=b$. Hence the fundamental solution must be chosen as $X_1(z)$ given in (3.20). Since the stresses must die out as $|z|\to\infty$ and since $X_1(z)\to z$ as $|z|\to\infty$ we have to put $P(z)\equiv 0$ in (3.21). Therefore the corresponding solution becomes

* The positive sens along L_1 is denoted in Fig. 4.

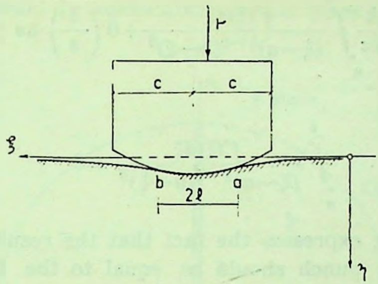


Fig. 5

$$\chi(z)=-\frac{H}{2\pi}(z-a)^{1-\theta}(z-b)^{\theta}\int_a^b\frac{f'(\xi)d\xi}{[(\xi-a)^{1-\theta}(\xi-b)^{\theta}]^{+}(\xi-z)} \tag{3.22}$$

where

$$X_1^{+}(\xi)=[(\xi-a)^{1-\theta}(\xi-b)^{\theta}]^{+}=[(z-a)^{1-\theta}(z-b)^{\theta}]_{z=\xi+i0} \tag{3.23}$$

It is easily seen from Fig. 4 that

$$(z-a)^{1-\theta}(z-b)^{\theta}=|z-a|^{1-\theta}|z-b|^{\theta}\,e^{i(1-\theta)\varphi_1+i\theta\varphi_2}$$

Since $\varphi_1=0$, $\varphi_2=\pi$ as $z\to\xi+i0$ and $\varphi_1=2\pi$, $\varphi_2=\pi$ as $z\to\xi-i0$ when $\xi\in L_1$ we have

$$\begin{aligned} X_1^{+}(\xi)&=(\xi-a)^{1-\theta}(b-\xi)^{\theta}\,e^{i\pi\theta} \qquad \xi\in I_1\\ X_1^{-}(\xi)&=(\xi-a)^{1-\theta}(b-\xi)^{\theta}\,e^{-i\pi\theta} \end{aligned} \tag{3.24}$$

Hence (3.22) should be written as

$$\chi(z)=\frac{ih}{2\pi}(z-a)^{1-\theta}(z-b)^{\theta}\int_a^b\frac{f'(\xi)d\xi}{(\xi-a)^{1-\theta}(b-\xi)^{\theta}(\xi-z)}\,. \tag{3.25}$$

It should be noted that a and b are unknown beforehand and they could be obtained by imposing additional conditions on (3.25). One obvious condition is that $\chi(z)\to 0$ as $|z|\to\infty$. If we consider that ξ in (3.25) is finite and expand (3.25) in increasing powers of $1/z$ we obtain

$$\chi(z) = -\frac{ih}{2\pi}\int_a^b \frac{f'(\xi)d\xi}{(\xi-a)^{1-\theta}(b-\xi)^{\theta}} + 0\left(\frac{1}{z}\right) \text{ as } |z| \to \infty$$

Hence we have

$$\int_a^b \frac{f'(\xi)d\xi}{(\xi-a)^{1-\theta}(b-\xi)^{\theta}} = 0 \tag{3.26}$$

The other condition expresses the fact that the resultant of the normal stresses under the punch should be equal to the force P pressing it downward, i. e.,

$$P = -\int_a^b t_{22}(\xi)d\xi \tag{3.27}$$

Equations (3.26) and (3.27) are sufficient to determine a and b, and consequently the contact area. We now try to give (3.27) a more explicit form. It follows from $(2.19)_4$ that

$$t_{22}(\xi) = -\mu[\chi(\xi) + \overline{\chi}(\xi)] = -\mu[\chi^+(\xi) - \chi^-(\xi)] \tag{3.28}$$

or from (3.25)

$$t_{22}(\xi) = 0, \qquad \xi \in L_2$$

$$t_{22}(\xi) = -\frac{i\mu h}{2\pi}(e^{i\pi\theta} - e^{-i\pi\theta})(\xi-a)^{1-\theta}(b-\xi)^{\theta}\int_a^b \frac{f'(u)du}{(u-a)^{1-\theta}(b-u)^{\theta}(u-\xi)}$$

$$= \frac{\mu h \sin \pi\theta}{\pi}(\xi-a)^{1-\theta}(b-\xi)^{\theta}\int_a^b \frac{f'(u)du}{(u-a)^{1-\theta}(b-u)^{\theta}(u-\xi)} \tag{3.29}$$

$$\xi \in L_1$$

Hence (3.27) yields

$$\int_a^b\int_a^b \frac{f'(u)(\xi-a)^{1-\theta}(b-\xi)^{\theta}\,du\,d\xi}{(u-a)^{1-\theta}(b-u)^{\theta}(u-\xi)} = -\frac{\pi}{\sin \pi\theta}\,\frac{P}{\mu h}\,. \tag{3.30}$$

We put now

$$I(\xi) = \int_a^b \frac{(\xi-a)^{1-\theta}(b-\xi)^{\theta}}{\xi-u}\,d\xi$$

This integral should be understood in the sense of Cauchy's principal value.

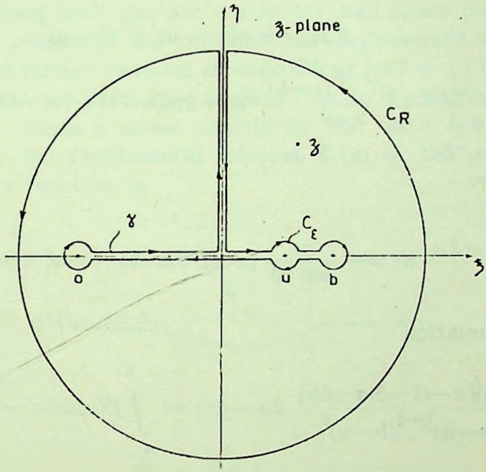


Fig. 6

In order to evaluate this integral we consider the path shown in Fig. 6 where C_R is a circle with a large radius R. The function

$$\frac{(z-a)^{1-\theta}(z-b)^{\theta}}{z-u}$$

in regular in region enclosed by that contour. Therefore one can write

$$\lim_{\varepsilon\to 0}\oint_{\gamma-C_\varepsilon} = \lim_{\varepsilon\to 0}\oint_{C_\varepsilon} - \lim_{R\to\infty}\oint_{C_R}$$

on observing that the integrals over vanishingly small circles around branch points a and b are zero.

On the other hand

$$\lim_{\varepsilon\to 0}\oint_{\gamma-C_\varepsilon}\frac{(z-a)^{1-\theta}(z-b)^{\theta}}{z-u}\,dz = 2i\sin\pi\theta\int_a^b\frac{(\xi-a)^{1-\theta}(b-\xi)^{\theta}}{\xi-u}\,d\xi$$

$$\lim_{\varepsilon\to 0}\oint_{C_\varepsilon}\frac{(z-a)^{1-\theta}(z-b)^{\theta}}{z-u}\,dz=\pi i\{[(u-a)^{1-\theta}(u-b)^{\theta}]^{+}+[(u-a)^{1-\theta}(u-b)^{\theta}]^{-}\}$$

$$=2\pi i\cos\pi\theta\,(u-a)^{1-\theta}(b-u)^{\theta}$$

$$\lim_{R\to\infty}\oint_{C_R}\frac{(z-a)^{1-\theta}(z-b)^{\theta}}{z-u}\,dz=2\pi i[u-(1-\theta)a-\theta b]$$

Hence we have

$$\int_a^b\frac{(\xi-a)^{1-\theta}(b-\xi)^{\theta}}{\xi-u}\,d\xi=-\frac{\pi}{\sin\pi\theta}\,[u-(1-\theta)a-\theta b-\cos\pi\theta(u-a)^{1-\theta}(b-u)^{\theta}]$$

and we find relation

$$\int_a^b\frac{f'(u)[u-(1-\theta)a-\theta b]}{(u-a)^{1-\theta}(b-u)^{\theta}}\,du-\cos\pi\theta\int_a^b f'(u)du=-\frac{P}{\mu h}$$

If we also consider (3.26) we reduce the above expression to

$$-\int_a^b\frac{uf'(u)}{(u-a)^{1-\theta}(b-u)^{\theta}}\,du+\cos\pi\theta[f(b)-f(a)]=\frac{P}{\mu h}\tag{3.31}$$

Employing a similar technique we can show that

$$\int_a^b\frac{f'(\xi)}{(\xi-a)^{1-\theta}(b-\xi)^{\theta}}\,d\xi=\frac{1}{2i\sin\pi\theta}\lim_{R\to\infty}\oint\frac{f'(z)}{(z-a)^{1-\theta}(z-b)^{\theta}}\,dz$$

Thus the equation (3.26) is transformed to

$$\lim_{R\to\infty}\oint_{C_R}\frac{f'(z)}{(z-a)^{1-\theta}(z-b)^{\theta}}\,dz=O\tag{3.32}$$

After having determined $\chi(z)$ completely the complex potentials are obtained as

$$\omega_1'(z_1)=\frac{\chi(z_1)}{1+\beta_2^2-2\beta_2 K}\,,\qquad \omega_2'(z_2)=\frac{K\,\chi(z_2)}{1+\beta_2^2-2\beta_2 K}\,.\tag{3.33}$$

ii. Contact Area Includes one Corner.

If we increase the vertical force one of the corners of the punch comes into contact with the surface of the half-space before the other does the same because of the asymmetrical displacement of the surface due to frictional forces. Physical considerations lead us to the conclusion that the corner at the point $\xi=a$ touches first to the surface of the half-space (Fig. 7). Since a stress singularity will occur now at $\xi=a$ we have to modify the fundamental solution $X_1(z)$ of the preceding case. The appropriate function is

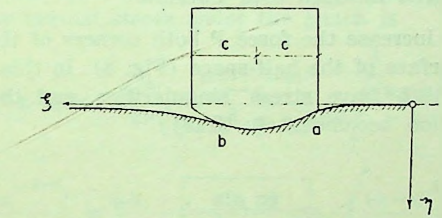


Fig. 7

$$X_2(z)=X_0(z)(z-b)=(z-a)^{-\theta}(z-b)^{\theta} \tag{3.34}$$

It is easy to see that

$$\left.\begin{aligned} X_2^+(\xi)&=(\xi-a)^{-\theta}(b-\xi)^{\theta}\, e^{i\pi\theta} \\ X_2^-(\xi)&=(\xi-a)^{-\theta}(b-\xi)^{\theta}\, e^{-i\pi\theta} \end{aligned}\right. \quad \xi\in L_1$$

Thus the solution of Hilbert problem becomes [cf. (3.21)]

$$\chi(z)=\frac{ih}{2\pi i}\left(\frac{z-b}{z-a}\right)^{\theta}\int_a^b\left(\frac{\xi-a}{b-\xi}\right)^{\theta}\frac{f'(\xi)}{\xi-z}\,d\xi \tag{3.35}$$

Since $X_2(z)\to 1$ as $|z|\to\infty$ we again had to put $P(z)=0$ in (3.21). To determine the contact area we have to find b. To this end we employ (3.27). The normal stresses under the punch are

$$t_{22}(\xi)=-\mu[\chi^+(\xi)-\chi^-(\xi)]=\mu h\,\frac{\sin\pi\theta}{\pi}\left(\frac{b-\xi}{\xi-a}\right)^{\theta}\int_a^b\left(\frac{u-a}{b-u}\right)^{\theta}\frac{f'(u)}{u-\xi}\,du$$

By a contour integration similar to the one which is used in obtaining (3.31) we get

$$-\int_a^b \left(\frac{u-a}{b-u}\right)^{\theta} f'(u)du+\cos \pi\theta[f(b)-f(a)]=\frac{P}{\mu h} \tag{3.36}$$

It is clear that $\chi(z) \to 0$ as $|z| \to \infty$.

The complex potentials are then written as in (3.33).

iii. Contact Area Includes both Corners.

If we further increase the force P both corners of the punch begin to touch to the surface of the half-space (Fig. 8). In this case both the points $\xi=a$ and $\xi=b$ have stress singularities and the appropriate fundamental solution becomes [cf. (3.18)]

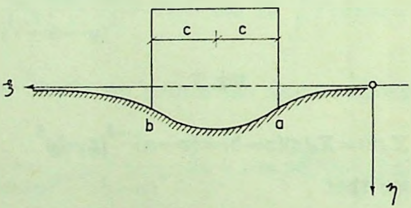


Fig. 8

$$X_0(z)=(z-a)^{-\theta}(z-b)^{\theta-1} \tag{3.37}$$

It is straightforward to see that

$$\left.\begin{aligned} X_0^{+}(\xi)&=-(\xi-a)^{-\theta}(b-\xi)^{\theta-1}e^{i\pi\theta} \\ X_0^{-}(\xi)&=-(\xi-a)^{-\theta}(b-\xi)^{\theta-1}e^{-i\pi\theta} \end{aligned}\right\} \xi\in L_1 \tag{3.38}$$

Therefore the solution for $\chi(z)$ can be written at once from (3.21):

$$\chi(z)=-\frac{ih}{2\pi}\,\frac{1}{(z-a)^{\theta}(z-b)^{1-\theta}}\int_a^b \frac{(\xi-a)^{\theta}(b-\xi)^{1-\theta}f'(\xi)}{\xi-z}\,d\xi \tag{3.39}$$

$$+\frac{A}{(z-a)^{\theta}(z-b)^{1-\theta}}$$

Since $X_0(z) \to 1/z$ as $|z| \to \infty$ we take $P(z) \equiv A$ in (3.21) where A is a complex constant. This constant will be so chosen that $(3.9)_1$ will be satisfied to guarantee the analytic continuation across L_2. Thus

$$\chi(\xi)+\bar{\chi}(\xi)=\frac{A+\bar{A}}{(\xi-a)(\xi-b)^{1-\theta}}=0, \qquad \xi \in L_2$$

Hence

$$A=-\bar{A} \tag{3.40}$$

Therefore A should be a pure imaginary number. Since a and b are now known beforehand we have to determine only A. For this purpose we use (3.27). The normal stress under the punch is

$$t_{22}(\xi)=-\mu[\chi^+(\xi)-\chi^-(\xi)]=-\frac{i\mu h}{2\pi}\frac{e^{i\pi\theta}-e^{-i\pi\theta}}{(\xi-a)^{\theta}(\xi-b)^{1-\theta}}\int_a^b\frac{(u-a)^{\theta}(b-u)^{1-\theta}f'(u)}{(u-\xi)}du$$

$$+\frac{\mu A(e^{i\pi\theta}-e^{-i\pi\theta})}{(\xi-a)^{\theta}(b-\xi)^{1-\theta}}=\frac{\mu h}{\pi}\frac{\sin\pi\theta}{(\xi-a)^{\theta}(b-\xi)^{1-\theta}}\int_a^b\frac{(u-a)^{\theta}(b-u)^{1-\theta}}{u-\xi}f'(u)du$$

$$+\frac{2i\mu A\sin\pi\theta}{(\xi-a)^{\theta}(b-\xi)^{1-\theta}}, \qquad \xi\in L_1 \tag{3.41}$$

Hence

$$-\frac{\pi}{\sin\pi\theta}P=\mu h\int_a^b\int_a^b\frac{(u-a)^{\theta}(b-u)^{1-\theta}f'(u)}{(\xi-a)^{\theta}(b-\xi)^{1-\theta}(u-\xi)}du\,d\xi+2i\mu\pi A\int_a^b\frac{d\xi}{(\xi-a)^{\theta}(b-\xi)^{1-\theta}}$$

on the other hand we can easily show that

$$\int_a^b\frac{d\xi}{(\xi-a)^{\theta}(b-\xi)^{1-\theta}(u-\xi)}=\frac{\pi\cos\pi\theta}{\sin\pi\theta(u-a)^{\theta}(b-u)^{1-\theta}}$$

$$\int_a^b\frac{d\xi}{(\xi-a)^{\theta}(b-\xi)^{1-\theta}}=\frac{\pi}{\sin\pi\theta}$$

Thus

$$-\frac{\pi}{\sin\pi\theta}P=\frac{\pi\cos\pi\theta}{\sin\pi\theta}\mu h\int_a^b f'(u)du+\frac{2i\mu\pi^2A}{\sin\pi\theta}$$

Since the profile of the punch is supposed to be symmetric with respect to its axis, then

$$\int_a^b f'(u)du=0$$

and

$$A=\frac{iP}{2\pi\mu} \tag{3.42}$$

Hence (3.39) is written as

$$\chi(z)=\frac{iP}{2\pi\mu(z-a)^{\theta}(z-b)^{1-\theta}}-\frac{ih}{2\pi(z-a)^{\theta}(z-b)^{1-\theta}}\int_a^b\frac{(\xi-a)^{\theta}(b-\xi)^{1-\theta}f'(\xi)}{\xi-z}d\xi \tag{3.43}$$

The complex potentials $\omega_1'(z_1)$ and $\omega_2'(z_2)$ are written then as in (3.33). The normal stress under the punch is obtained from (3.41) by substituting (3.42):

$$t_{22}(\xi)=-\frac{\sin\pi\theta}{\pi}\frac{P}{(\xi-a)^{\theta}(b-\xi)^{1-\theta}}$$

$$\left[1-\mu\frac{h}{P}\int_a^b\frac{(u-a)^{\theta}(b-u)^{1-\theta}f'(u)}{u-\xi}du\right] \tag{3.44}$$

In the next two sections we shall consider solutions for special punch profiles.

4. Rectangular Punch.

In case of a rectangular punch we write (Fig. 9)

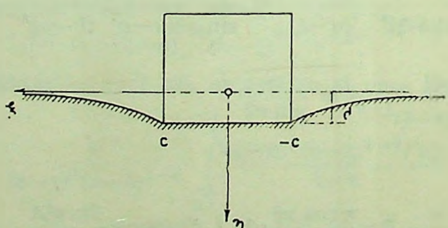


Fig. 9

$$f(\xi)=d\,, \qquad f'(\xi)=0 \tag{4.1}$$

This obviously falls into the case (iii) discussed in Sec. 3. Introducing (4.1) into (3.43), placing the η - axis into the middle of the punch, denoting the width of the punch by $2c$ with $a=-c$, $b=c$ we obtain

$$\chi(z)=\frac{iP}{2\pi\mu(z+c)^{\theta}(z-c)^{1-\theta}} \tag{4.2}$$

and consequently we get from (3.33)

$$\omega_1'(z_1)=-\frac{[2\beta_2 k+(1+\beta_2^2)i]\,P}{2\pi\mu[4\beta_1\beta_2-(1+\beta_2^2)^2](z_1+c)^{\theta}(z_2-c)^{1-\theta}} \tag{4.3}$$

$$\omega_2'(z_2)=-\frac{[(1+\beta_2^2)k+2\beta_1 i]P}{2\pi\mu[4\beta_1\beta_2-(1+\beta_2^2)^2](z_2+c)^{\theta}(z_2-c)^{1-\theta}}$$

The pressure under the punch is obtained from (3.44):

$$\begin{aligned} t_{22}(\xi)&=-\frac{\sin\pi\theta}{\pi}\frac{P}{(\xi+c)^{\theta}(c-\xi)^{1-\theta}} \\ &=-\frac{\sin\pi\theta}{\pi}\frac{P}{c}\frac{1}{(1+\zeta)^{\theta}(1-\zeta)^{1-\theta}} \qquad |\zeta|<1 \end{aligned} \tag{4.4}$$

where

$$\zeta=\frac{\xi}{c} \tag{4.5}$$

The minimum value of the pressure occurs at

$$\zeta=2\theta-1$$

The pressure distributions for $M_2=0$ (static case) and $M_2=0.5$ are shown in Fig. 10 for $k=0.6$, $\nu=0.3$ and tabulated in Table 11. The stress formula (4.4) for the former case coincides with the one given in |6|.

5. Circular Punch

We consider a circular punch with a large radius R_0. Therefore its profile is approximately parabolic and the displacement under it can be represented by (Fig. 11)

$$f(\xi)=d-\frac{\xi^2}{2R_0} \tag{5.1}$$

Hence

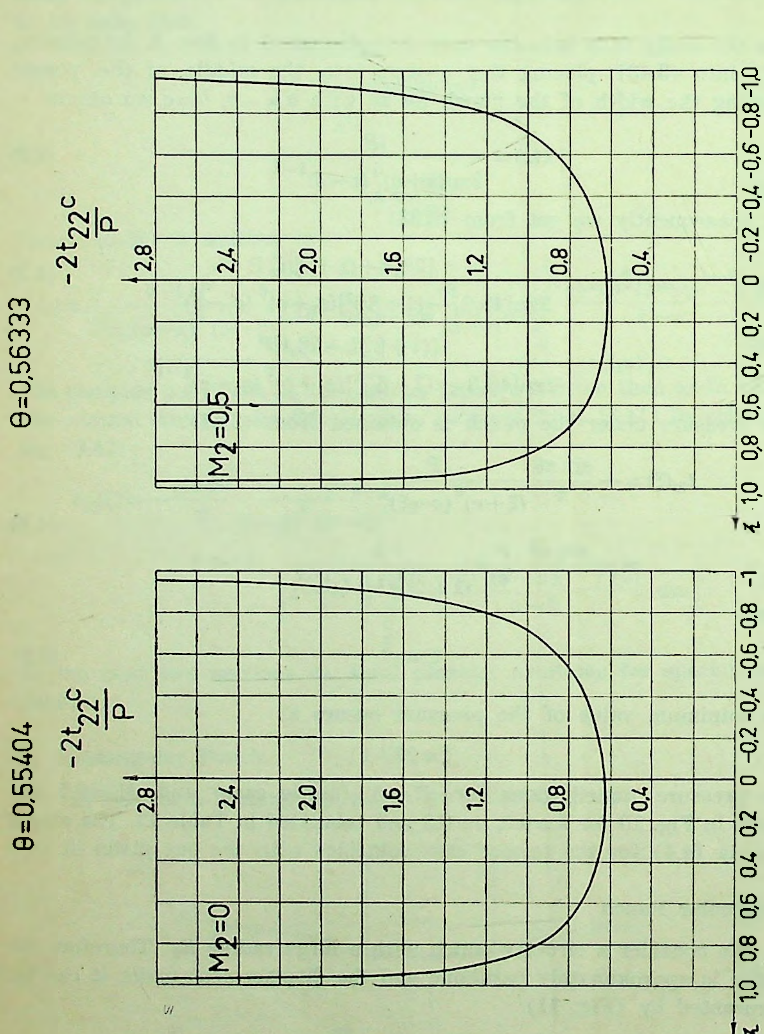

θ=0,56333
-2t22c/P
M2=0,5
2.8
2,4
2,0
1,6
1,2
0,8
0,4
1,0 0,8 0,6 0,4 0,2 0 -0,2 -0,4 -0,6 -0,8 -1,0
θ=0,55404
-2t22c/P
M2=0
2.8
2,4
2,0
1,6
1,2
0,8
0,4
1,0 0,8 0,6 0,4 0,2 0 -0,2 -0,4 -0,6 -0,8 -1

Fig. 10

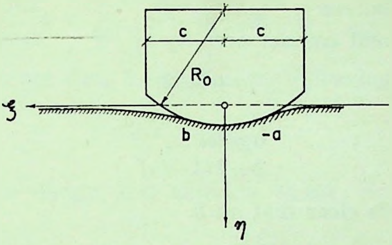


Fig. 11

$$f'(\xi) = -\frac{\xi}{R_0} \tag{5.2}$$

We distinguish now three cases studied in Sec. 3:

Case i. (Fig. 12a)

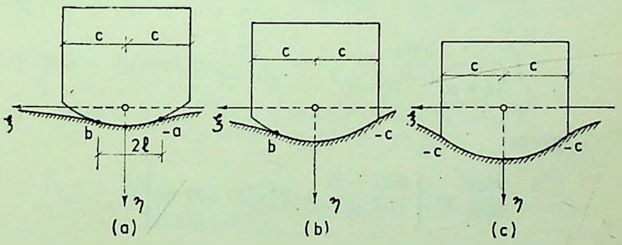


Fig. 12

We have to determine first a and b. Noting the position of the coordinate axes we write (3.31) and (3.32) in the following form by substituting (5.2):

$$\frac{1}{R_0}\int_{-a}^{b} \frac{u^2}{(u+a)^{1-\theta}(b-u)^{\theta}}\,du + \frac{\cos\pi\theta}{2R_0}\,(a^2-b^2) = \frac{P}{\mu h}$$

$$\lim_{R\to\infty} \oint_{C_R} \frac{z}{(z+a)^{1-\theta}(z-b)^{\theta}}\,dz = 0 \tag{5.3}$$

The second of (5.3) yields immediately

$$\theta b-(1-\theta)a=0$$

If we denote the total contact area by

$$a+b=2l$$

then we find

$$\begin{aligned} a&=2\theta l \\ b&=2(1-\theta)l \end{aligned} \tag{5.4}$$

Since $1/2\leqslant\theta<1$, it is clear that $a\geqslant b$.

To evaluate the integral in $(5.3)_1$ we consider

$$\oint_{\gamma}\frac{z^2}{(z+a)^{1-\theta}(z-b)^{\theta}}\,dz=-2i\sin\pi\theta\int_{-a}^{b}\frac{\xi^2 d\xi}{(\xi+a)^{1-\theta}(b-\xi)^{\theta}}=$$

$$-\lim_{R\to\infty}\oint\frac{z^2dz}{(z+a)^{1-\theta}(z-b)^{\theta}}=-2\pi i\,\frac{(1-\theta)(2-\theta)a^2-2\theta(1-\theta)ab+\theta(1+\theta)b^2}{2}$$

whence on using (5.4)

$$\int_{-a}^{b}\frac{\xi^2 d\xi}{(\xi+a)^{1-\theta}(b-\xi)^{\theta}}=\frac{2\pi}{\sin\pi\theta}\,\theta(1-\theta)l^2$$

Thus $(5.3)_1$ gives

$$\frac{R_0P}{2\mu h}=\left[\frac{\pi\theta(1-\theta)}{\sin\pi\theta}+(2\theta-1)\cos\pi\theta\right]l^2$$

and

$$l=\left(\frac{R_0P}{2\mu h}\right)^{1/2}\left[\frac{\pi\theta(1-\theta)}{\sin\pi\theta}+(2\theta-1)\cos\pi\theta\right]^{-1/2} \tag{5.5}$$

a and b are then determined by (5.4). Of course (5.5) can only be valid if $a<c$, that is, if the contact area has not reached the corner of the punch. This condition implies, in view of (5.4)

$$l<\frac{c}{2\theta}$$

Employing this in (5.6) we find that

$$\frac{PR_0}{\mu c^2 h}<\frac{1}{2\theta^2}\left[\frac{\theta(1-\theta)\pi}{\sin\pi\theta}+(2\theta-1)\cos\pi\theta\right]$$

or using (3.17)

$$\frac{PR_0}{\mu h_0 c^2} < \frac{\pi}{2\theta^2}\left[\theta(1-\theta)+(2\theta-1)\frac{\sin 2\pi\theta}{2\pi}\right] \tag{5.6}$$

It is also trivial to see that b satisfies the following inequality:

$$\beta=\frac{b}{c}<\frac{1-\theta}{\theta} \tag{5.7}$$

Finally we can now obtain $\chi(z)$ through (3.25):

$$\chi(z)=-\frac{ih}{2\pi R_0}(z+a)^{1-\theta}(z-b)^{\theta}\int_{-a}^{b}\frac{\xi d\xi}{(\xi+a)^{1-\theta}(b-\xi)^{\theta}(\xi-z)}$$

This integral can be evaluated by the technique described in Sec. 3. However we must note now that the integral has a pole at the point z so that the residue at that point should also be added. Thus one finds

$$\int_{-a}^{b}\frac{\xi d\xi}{(\xi+a)^{1-\theta}(b-\xi)^{\theta}(\xi-z)}=\frac{\pi}{\sin\pi\theta}\left[1-\frac{z}{(z+a)^{1-\theta}(z-b)^{\theta}}\right]$$

and consequently

$$\chi(z)=-\frac{ih}{2R_0\sin\pi\theta}\left[(z+a)^{1-\theta}(z-b)^{\theta}-z\right] \tag{5.8}$$

Then

$$\begin{aligned}\omega_1'(z_1)&=\frac{2\beta_2k-(1+\beta_2^2)i}{2R_0\beta_1(1-\beta_2^2)}\left[(z_1+a)^{1-\theta}(z_1-b)^{\theta}-z_1\right]\\ \omega_2'(z_2)&=\frac{(1+\beta_2^2)k+2\beta_1i}{2R_0\beta_1(1-\beta_2^2)}\left[(z_2+a)^{1-\theta}(z_2-b)^{\theta}-z_2\right]\end{aligned} \tag{5.9}$$

where we used (3.17).

The normal stress under the punch is found from (3.28)

$$t_{22}(\xi)=-\frac{\mu h}{R_0}(\xi+a)^{1-\theta}(b-\xi)^{\theta}$$

or employing (5.5) and (5.7)

$$t_{22}(\xi)=-\frac{P}{2\left[\frac{\theta(1-\theta)\pi}{\sin\pi\theta}+(2\theta-1)\cos\pi\theta\right]l}(2\theta+\zeta)^{1-\theta}[2(1-\theta)-\zeta]^{\theta} \tag{5.10}$$

$$-2\theta\leq\zeta\leq2(1-\theta)$$

where $\zeta = \xi / l$. The maximum pressure occurs at $\zeta = -2(2\theta - 1)$.

The pressure distribution is shown in Fig. 13 for $k = 0.6$, $v = 0.3$, $M_2 = 0.5$ and tabulated in Table 12.

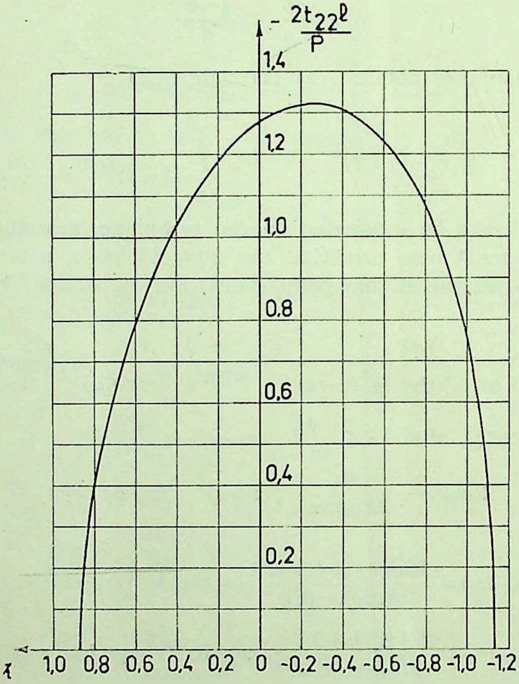


Fig. 13

Case ii. (Fig. 12b)

We have to determine first b. We put $a = -c$ Equation (3.36) becomes

$$\frac{1}{R_0} \int_{-c}^{b} \left(\frac{u+c}{b-u} \right)^{\theta} u du + \frac{\cos \pi\theta}{2R_0} (c^2 - b^2) = \frac{P}{\mu h} \tag{5.11}$$

It is now straightforward to see that

$$\int_{-c}^{b}\left(\frac{u+c}{b-u}\right)^{\theta} u\,du=\frac{\pi}{2\sin\pi\theta}\left[\theta(1+\theta)b^{2}+2\theta^{2}bc-\theta(1-\theta)c^{2}\right]$$

and the relation

$$\frac{\pi}{\sin\pi\theta}\left[\theta(1+\theta)\beta^{2}+2\theta^{2}\beta-\theta(1-\theta)\right]+\cos\pi\theta(1-\beta^{2})=\frac{2PR_{0}}{\mu hc^{2}} \tag{5.12}$$

where $\beta=b/c$, gives b as a function of P. Of course (5.12) is valid for $\beta<1$. In view of (5.7) β satisfies $\frac{1-\theta}{\theta}\leqslant\beta<1$. The solution of (5.12) yields

$$\beta=\frac{\left\{\left(\theta-\frac{\sin 2\pi\theta}{2\pi}\right)^{2}+\frac{PR_{0}}{\mu h_{0}c^{2}}\,\frac{2}{\pi}\left[\theta(1+\theta)-\frac{\sin 2\pi\theta}{2\pi}\right]\right\}^{1/2}-\theta^{2}}{\theta(1+\theta)-\frac{\sin 2\pi\theta}{2\pi}} \tag{5.13}$$

where h_0 was defined in (3.17). For (5.13) to be valid it is clear from (5.12) that the following inequality should be satisfied

$$\frac{PR_{0}}{\mu h_{0}c^{2}}<2\pi\theta^{2} \tag{5.14}$$

The function $\chi(z)$ is then determined from (3.35) as

$$\chi(z)=-\frac{ih}{2\pi R_{0}}\left(\frac{z-b}{z+c}\right)^{\theta}\int_{-c}^{b}\left(\frac{\xi+c}{b-\xi}\right)^{\theta}\frac{\xi}{\xi-z}\,d\xi \tag{5.15}$$

The integral in (5.15) is calculated as

$$\int_{-c}^{b}\left(\frac{\xi+c}{b-\xi}\right)^{\theta}\frac{\xi}{\xi-z}\,d\xi=\frac{\pi}{\sin\pi\theta}\left\{[z+(b+c)\theta]-z\left(\frac{z+c}{z-b}\right)^{\theta}\right\}$$

and (5.15) becomes

$$\chi(z)=-\frac{ih}{2R_{0}\sin\pi\theta}\left\{[z+(b+c)\theta]\left(\frac{z-b}{z+c}\right)^{\theta}-z\right\} \tag{5.16}$$

The normal stress distribution under the punch is

$$t_{22}(\xi)=-\mu[\chi^{+}(\xi)-\chi^{-}(\xi)]$$
$$=-\frac{\mu h}{R_0}\,[\xi+(b+c)\theta]\left(\frac{b-\xi}{\xi+c}\right)^{\theta}$$

or

$$t_{22}(\xi)=-\frac{\mu hc}{R_0}\,[\zeta+(1+\beta)\theta]\left(\frac{\beta-\zeta}{1+\zeta}\right)^{\theta}, \quad -1\leq\zeta\leq\beta \tag{5.17}$$

where $\zeta=\xi/c$ and β is given by (5.13). The stress distributions for $k=0.6$, $\nu=0.3$, $M_2=0.5$ and $PR_0/\mu hc^2=1.6$ and 1.95 are plotted in Fig. 14 and tabulated in Table 13.

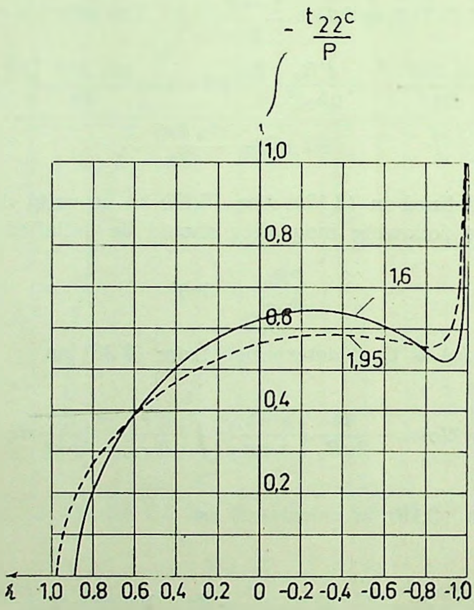


Fig. 14

With the knowledge of $\chi(z)$ we can express the complex potentials as follows:

$$\omega_1'(z_1)=\frac{2\beta_2 k+(1+\beta_2^2)\,i}{2R_0\beta_1(1-\beta_2^2)}\left\{[z_1+(b+c)\theta]\left(\frac{z_1-b}{z_1+c}\right)^{\theta}-z_1\right\}$$

$$\omega_2'(z_2)=\frac{(1+\beta_2^2)k+2\beta_1 i}{2R_0\beta_1(1-\beta_2^2)}\left\{[z_2+(b+c)\theta]\left(\frac{z_2-b}{z_2+c}\right)^{\theta}-z_2\right\} \tag{5.18}$$

Case iii. (Fig. 12c)

If $P>2\pi\mu h_0\theta^2/R_0$ [cf. (5.14)], then both corners of the punch come into contact with the surface of the half-space and we have to take

$$a=-c, \quad b=c$$

and (3.43) becomes

$$\chi(z)=\frac{iP}{2\pi\mu(z+c)^{\theta}(z-c)^{1-\theta}}+\frac{ih}{2\pi R_0(z+c)^{\theta}(z-c)^{1-\theta}}\int_{-c}^{c}\frac{(\xi+c)^{\theta}(c-\xi)^{1-\theta}\xi}{\xi-z}\,d\xi \tag{5.19}$$

The integral in (5.19) can be evaluated as

$$\int_{-c}^{c}\frac{(\xi+c)^{\theta}(c-\xi)^{1-\theta}\xi}{\xi-z}\,d\xi=\frac{\pi}{\sin\pi\theta}\left[z(z+c)^{\theta}(z-c)^{1-\theta}+2\theta(1-\theta)c^2-(2\theta-1)cz-z^2\right]$$

and (5.20) takes the following form

$$\chi(z)=\frac{iP}{2\pi\mu(z+c)^{\theta}(z-c)^{1-\theta}}-\frac{ih}{2R_0\sin\pi\theta}\left[\frac{z^2+(2\theta-1)cz-2\theta(1-\theta)c^2}{(z+c)^{\theta}(z-c)^{1-\theta}}-z\right] \tag{5.20}$$

The stress distribution under the punch is now

$$t_{22}(\xi)=-\mu[\chi^+(\xi)-\chi^-(\xi)]$$

$$=-\frac{\sin\pi\theta}{\pi}\frac{P}{(\xi+c)^{\theta}(c-\xi)^{1-\theta}}+\frac{\mu h}{R_0}\frac{\xi^2+(2\theta-1)c\xi-2\theta(1-\theta)c^2}{(\xi+c)^{\theta}(c-\xi)^{1-\theta}}$$

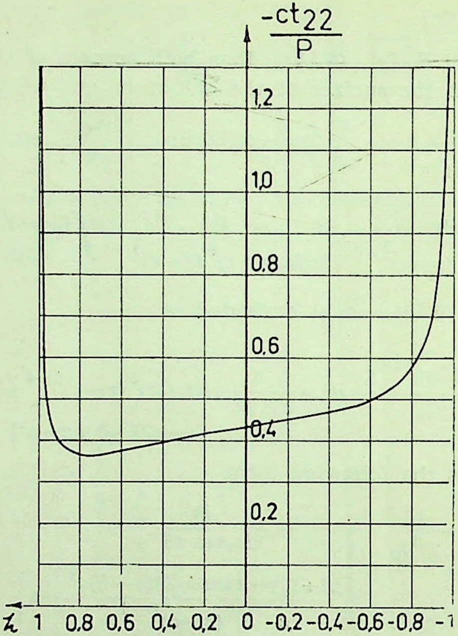
or

$$t_{22}(\zeta)=-\frac{P\sin\pi\theta}{\pi c}\frac{1}{(1+\zeta)^{\theta}(1-\zeta)^{1-\theta}} \qquad -1<\zeta<1)$$

$$\left\{1-\frac{\mu h_0 c^2\pi}{PR_0}\left[\zeta^2+(2\theta-1)\zeta-2\theta(1-\theta)\right]\right\} \tag{5.21}$$

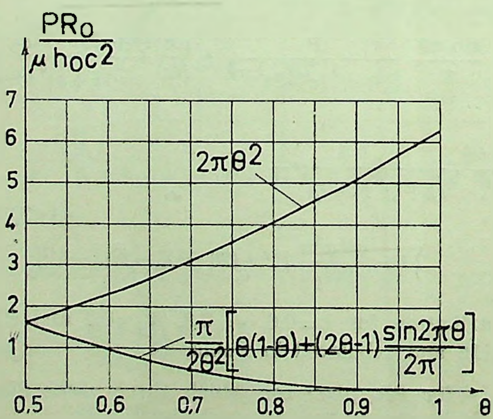
The stress distribution for $k=0.6$, $\nu=0.3$, $M_2=0.5$ and $PR_0/\mu hc^2=4$ is shown in Fig. 15 and tabulated in Table 14.

The bounds for $PR_0/\mu h_0c^2$ for which three cases discussed above are applicable are shown in Fig. 16.



$\frac{-ct_{22}}{P}$
1,2
1,0
0,8
0,6
0,4
0,2
z 1 0,8 0,6 0,4 0,2 0 -0,2 -0,4 -0,6 -0,8 -1

Fig. 15



$\frac{PR_0}{\mu h_0 c^2}$
7
6
5
4
3
2
1
$2\pi\theta^2$
$\frac{\pi}{2\theta^2}\left[\theta(1-\theta)+(2\theta-1)\frac{\sin 2\pi\theta}{2\pi}\right]$
0,5 0,6 0,7 0,8 0,9 1 θ

Fig. 16

After having determined $\chi(z)$ in (5.20) we can express the complex potentials as follows:

$$\omega_1'(z_1) = -\frac{[2\beta_2 k + (1+\beta_2^2)i]P}{2\pi\mu[4\beta_1\beta_2 - (1+\beta_2^2)^2](z_1+c)^{\theta}(z_1-c)^{\theta}} + \frac{2\beta_2 k + (1+\beta_2^2)i}{2R_0\beta_1(1-\beta_2^2)}\left[\frac{z_1^2 + (2\theta-1)cz_1 - 2\theta(1-\theta)c^2}{(z_1+c)^{\theta}(z_1-c)^{1-\theta}} - z_1\right]$$

$$\omega_2'(z_2) = -\frac{[(1+\beta_2^2)k + 2\beta_1 i]P}{2\pi\mu[4\beta_1\beta_2 - (1+\beta_2^2)^2](z_2+c)^{\theta}(z_2-c)^{1-\theta}} + \frac{(1+\beta_2^2)k + 2\beta_1 i}{2R_0\beta_1(1-\beta_2^2)}\left[\frac{z_2^2 + (2\theta-1)cz_2 - 2\theta(1-\theta^2)c^2}{(z_2+c)^{\theta}(z_2-c)^{1-\theta}} - z_2\right] \tag{5.22}$$

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TABLE 1. θ for $\nu=0.15$

$k \backslash M_2$	0.0	0.1	0.2	0.3	0.4	0.5	0.7	0.9
0.0	0.50000	0.50000	0.50000	0.50000	0.50000	0.50000	0.50000	0.50000
0.1	0.51310	0.51315	0.51332	0.51362	0.51407	0.51471	0.51699	0.52298
0.2	0.52615	0.52626	0.52660	0.52719	0.52808	0.52936	0.53389	0.54573
0.3	0.53912	0.53928	0.53978	0.54066	0.54198	0.54389	0.55059	0.56802
0.4	0.55196	0.55217	0.55283	0.55398	0.55572	0.55823	0.56702	0.58965
0.5	0.56463	0.56489	0.56570	0.56712	0.56926	0.57234	0.58310	0.61046
0.6	0.57710	0.57741	0.57836	0.58002	0.58255	0.58617	0.59875	0.63033
0.7	0.58933	0.58968	0.59077	0.59267	0.59555	0.59970	0.61393	0.64919
0.8	0.60129	0.60169	0.60290	0.60503	0.60823	0.61281	0.62858	0.66698
0.9	0.61297	0.61340	0.61473	0.61706	0.62056	0.62556	0.64268	0.68368
1.0	0.62433	0.62480	0.62624	0.62875	0.63253	0.63790	0.65620	0.69933

TABLE 2. h for $\nu=0.15$

$k \backslash M_2$	0.0	0.1	0.2	0.3	0.4	0.5	0.7	0.9
0.0	1.17647	1.16714	1.13877	1.09023	1.01939	0.92262	0.62148	0.01104
0.1	1.17547	1.16614	1.13777	1.08923	1.01840	0.92164	0.62060	0.01101
0.2	1.17250	1.16317	1.13480	1.08626	1.01543	0.91870	0.61796	0.01093
0.3	1.16760	1.15826	1.12989	1.08135	1.01054	0.91387	0.61365	0.01079
0.4	1.16083	1.15149	1.12313	1.07459	1.00381	0.90723	0.60776	0.01061
0.5	1.15230	1.14296	1.11460	1.06608	0.99536	0.89890	0.60042	0.01038
0.6	1.14213	1.13279	1.10444	1.05596	0.98531	0.88902	0.59181	0.01013
0.7	1.13045	1.12112	1.09278	1.04435	0.97381	0.87776	0.58210	0.00985
0.8	1.11741	1.10808	1.07978	1.03142	0.96103	0.86529	0.57146	0.00956
0.9	1.10316	1.09385	1.06559	1.01734	0.94714	0.85177	0.56088	0.00925
1.0	1.08786	1.07857	1.05038	1.00226	0.93231	0.83739	0.54814	0.00895

TABLE 3. θ for $\nu=0.25$

$k \backslash M_2$	0.0	0.1	0.2	0.3	0.4	0.5	0.7	0.9
0.0	0.50000	0.50000	0.50000	0.50000	0.50000	0.50000	0.50000	0.50000
0.1	0.51061	0.51066	0.51082	0.51112	0.51155	0.51219	0.51445	0.52045
0.2	0.52119	0.52129	0.52162	0.52220	0.52307	0.52434	0.52883	0.54073
0.3	0.53173	0.53188	0.53237	0.53323	0.53453	0.53642	0.54310	0.56068
0.4	0.54219	0.54240	0.54305	0.54418	0.54591	0.54840	0.55720	0.58016
0.5	0.55257	0.55283	0.55363	0.55503	0.56716	0.56024	0.57108	0.59905
0.6	0.56283	0.56314	0.56409	0.56575	0.56827	0.57191	0.58468	0.61724
0.7	0.57297	0.57332	0.57441	0.57632	0.57922	0.58339	0.59798	0.63467
0.8	0.58295	0.58335	0.58458	0.58672	0.58998	0.59465	0.61093	0.65127
0.9	0.59277	0.59321	0.59457	0.59694	0.60053	0.60568	0.62351	0.66704
1.0	0.60242	0.60290	0.60437	0.60695	0.61086	0.61644	0.63570	0.68194

TABLE 4. h for $\nu = 0.25$

$k \backslash M_2$	0.0	0.1	0.2	0.3	0.4	0.5	0.7	0.9
0.0	1.33333	1.32331	1.29285	1.24081	1.16499	1.06170	0.74251	0.10635
0.1	1.33259	1.32257	1.29210	1.24005	1.16422	1.06092	0.74174	0.10613
0.2	1.33038	1.32035	1.28987	1.23779	1.16193	1.05860	0.73946	0.10548
0.3	1.32672	1.31667	1.28617	1.23405	1.15814	1.05476	0.73571	0.10442
0.4	1.32164	1.31158	1.23104	1.22887	1.15289	1.04945	0.73055	0.10299
0.5	1.31519	1.30512	1.27454	1.22231	1.14626	1.04274	0.72407	0.10124
0.6	1.30744	1.29736	1.26673	1.21443	1.13829	1.03472	0.71638	0.09921
0.7	1.29845	1.28836	1.25768	1.20531	1.12910	1.02547	0.70761	0.09697
0.8	1.28831	1.27820	1.24748	1.19504	1.11875	1.01510	0.69787	0.09456
0.9	1.27710	1.26697	1.23621	1.18371	1.10737	1.00372	0.68731	0.09204
1.0	1.26491	1.25477	1.22397	1.17142	1.09505	0.99145	0.67605	0.08944

TABLE 5. θ for $\nu = 0.30$

$k \backslash M_2$	0.0	0.1	0.2	0.3	0.4	0.5	0.7	0.9
0.0	0.50000	0.50000	0.50000	0.50000	0.50000	0.50000	0.50000	0.50000
0.1	0.50909	0.50915	0.50931	0.50960	0.51005	0.51069	0.51298	0.51906
0.2	0.51817	0.51828	0.51861	0.51919	0.52008	0.52136	0.52593	0.53799
0.3	0.52722	0.52738	0.52787	0.52874	0.53007	0.53198	0.53878	0.55664
0.4	0.53622	0.53643	0.53709	0.53824	0.54000	0.54253	0.55151	0.57492
0.5	0.54517	0.54543	0.54624	0.54767	0.54985	0.55299	0.56408	0.59270
0.6	0.55404	0.55436	0.55532	0.55702	0.55960	0.56333	0.57644	0.60991
0.7	0.56283	0.56320	0.56431	0.56627	0.56925	0.57354	0.58858	0.62648
0.8	0.57153	0.57194	0.57320	0.57541	0.57877	0.58360	0.60046	0.64235
0.9	0.58012	0.58057	0.58197	0.58442	0.58814	0.59348	0.61205	0.65750
1.0	0.58859	0.58908	0.59062	0.59330	0.59136	0.60319	0.62335	0.67191

TABLE 6. h for $\nu = 0.30$

$k \backslash M_2$	0.0	0.1	0.2	0.3	0.4	0.5	0.7	0.9
0.0	1.42857	1.41804	1.38604	1.33143	1.25198	1.14398	0.81199	0.15835
0.1	1.42799	1.41745	1.38545	1.33082	1.25136	1.14334	0.81132	0.15807
0.2	1.42624	1.41570	1.38368	1.32901	1.24949	1.14141	0.80930	0.15723
0.3	1.42335	1.41280	1.38073	1.32600	1.24640	1.13821	0.80597	0.15585
0.4	1.41933	1.40876	1.37664	1.32183	1.24211	1.13378	0.80138	0.15399
0.5	1.41421	1.40362	1.37144	1.31652	1.23666	1.12817	0.79560	0.15168
0.6	1.40803	1.39741	1.36516	1.31012	1.23010	1.12141	0.78869	0.14900
0.7	1.40083	1.39018	1.35785	1.30267	1.22247	1.11359	0.78075	0.14601
0.8	1.39266	1.38198	1.34956	1.29424	1.21385	1.10476	0.77189	0.14278
0.9	1.38356	1.37285	1.34034	1.28487	1.20429	1.09500	0.76220	0.13936
1.0	1.37361	1.36286	1.33026	1.27464	1.19387	1.08439	0.75179	0.13581

TABLE 7. θ for $\nu = 0.40$

$k \backslash M_2$	0.0	0.1	0.2	0.3	0.4	0.5	0.7	0.9
0.0	0.50000	0.50000	0.50000	0.50000	0.50000	0.50000	0.50000	0.50000
0.1	0.50530	0.50536	0.50555	0.50587	0.50637	0.50708	0.50957	0.51601
0.2	0.51601	0.51073	0.51110	0.51174	0.51273	0.51415	0.51913	0.53194
0.3	0.51590	0.51608	0.51663	0.51761	0.51908	0.52120	0.52865	0.54771
0.4	0.52119	0.52143	0.52216	0.52346	0.52541	0.52824	0.53812	0.56324
0.5	0.52646	0.52676	0.52768	0.52929	0.53173	0.53524	0.54753	0.57878
0.6	0.53173	0.53208	0.53318	0.53510	0.53802	0.54222	0.55685	0.59336
0.7	0.53697	0.53738	0.53866	0.54089	0.54428	0.54915	0.56607	0.60783
0.8	0.54219	0.54266	0.54412	0.54666	0.55051	0.55604	0.57519	0.61185
0.9	0.54739	0.54792	0.54955	0.55239	0.55670	0.56287	0.58417	0.63540
1.0	0.55257	0.55315	0.55495	0.55809	0.56284	0.56964	0.59303	0.64844

TABLE 8. h for $\nu = 0.40$

$k \backslash M_2$	0.0	0.1	0.2	0.3	0.4	0.5	0.7	0.9
0.0	1.66666	1.65456	1.61785	1.55538	1.46497	1.34294	0.97397	0.27279
0.1	1.66644	1.65432	1.61760	1.55512	1.46468	1.34260	0.97353	0.27244
0.2	1.66574	1.65362	1.61686	1.55432	1.46380	1.34161	0.97221	0.27142
0.3	1.66459	1.65245	1.61564	1.55300	1.46234	1.33996	0.97003	0.26973
0.4	1.66298	1.65081	1.61393	1.55116	1.46031	1.33766	0.96670	0.26742
0.5	1.66091	1.64871	1.61173	1.54880	1.45770	1.33471	0.96313	0.26454
0.6	1.65840	1.64616	1.60907	1.54593	1.45454	1.33114	0.95848	0.26114
0.7	1.65544	1.64316	1.60593	1.54256	1.45082	1.32696	0.95307	0.25729
0.8	1.65205	1.63972	1.60233	1.53870	1.44657	1.32218	0.94693	0.25304
0.9	1.64823	1.63584	1.59829	1.53436	1.44180	1.31683	0.94012	0.24848
1.0	1.64399	1.63155	1.59380	1.52955	1.43652	1.31092	0.93267	0.24366

TABLE 9. θ for $\nu = 0.5$

$k \backslash M_2$	0.0	0.1	0.2	0.3	0.4	0.5	0.7	0.9
0.0	0.50000	0.50000	0.50000	0.50000	0.50000	0.50000	0.50000	0.50000
0.1	0.50000	0.50008	0.50032	0.50075	0.50139	0.50229	0.50531	0.51250
0.2	0.50000	0.50016	0.50065	0.50150	0.50277	0.50457	0.51061	0.52496
0.3	0.50000	0.50024	0.50097	0.50225	0.50416	0.50686	0.51591	0.53734
0.4	0.50000	0.50032	0.50130	0.50300	0.50555	0.50914	0.52120	0.54962
0.5	0.50000	0.50040	0.50162	0.50375	0.50693	0.51142	0.52648	0.56174
0.6	0.50000	0.50048	0.50195	0.50450	0.50832	0.51370	0.53174	0.57369
0.7	0.50000	0.50056	0.50227	0.50525	0.50970	0.51598	0.53699	0.58543
0.8	0.50000	0.50064	0.50260	0.50600	0.51109	0.51826	0.54222	0.59693
0.9	0.50000	0.50072	0.50292	0.50675	0.51247	0.52054	0.54742	0.60818
1.0	0.50000	0.50080	0.50325	0.50750	0.51386	0.52281	0.55260	0.61916

TABLE 10. h for $\nu = 0.5$

$k \backslash M_2$	0.0	0.1	0.2	0.3	0.4	0.5	0.7	0.9
0.0	2.00000	1.98497	1.93959	1.86285	1.75288	1.60641	1.17647	0.40427
0.1	2.00000	1.98497	1.93959	1.86285	1.75286	1.60636	1.17631	0.40396
0.2	2.00000	1.98497	1.93958	1.86283	1.75281	1.60624	1.17582	0.40303
0.3	2.00000	1.98497	1.93958	1.86281	1.75273	1.60603	1.17500	0.40149
0.4	2.00000	1.98497	1.93957	1.86277	1.75261	1.60574	1.17386	0.39937
0.5	2.00000	1.98497	1.93956	1.86272	1.75246	1.60537	1.17240	0.39669
0.6	2.00000	1.98497	1.93955	1.86267	1.75228	1.60492	1.17063	0.39349
0.7	2.00000	1.98497	1.93954	1.86260	1.75206	1.60438	1.16854	0.38980
0.8	2.00000	1.98497	1.93952	1.86252	1.75181	1.60376	1.16614	0.38567
0.9	2.00000	1.98496	1.93951	1.86243	1.75153	1.60306	1.16344	0.38115
1.0	2.00000	1.98496	1.93949	1.86234	1.75123	1.60228	1.16045	0.37627

TABLE 11. STRESS DISTRIBUTION UNDER A RECTANGULAR PUNCH

ζ	$-2t_{22}c/P$	
	$M_2 = 0$	$M_2 = 0.5$
-0.95	2.44946	2.52050
-0.90	1.68779	1.72518
-0.80	1.17763	1.19539
-0.70	0.96497	0.97533
-0.60	0.84535	0.85166
-0.50	0.76885	0.77253
-0.40	0.71670	0.71844
-0.30	0.68014	0.68035
-0.20	0.65459	0.65350
-0.10	0.63750	0.63523
0.00	0.62747	0.62406
0.10	0.62383	0.61928
0.1081	0.62381	-
0.1237	-	0.61906
0.20	0.62653	0.62078
0.30	0.63612	0.62904
0.40	0.65398	0.64533
0.50	0.68277	0.67217
0.60	0.72772	0.71451
0.70	0.80001	0.78295
0.80	0.92869	0.90499
0.90	1.22775	1.18814
0.95	1.64857	1.58476

TABLE 12. STRESS DISTRIBUTION UNDER A CIRCULAR PUNCH CASE i

ζ	$-2\ t_{22}/P$
-0.1127	0
-0.11	0.39465
-0.10	0.75686
-0.90	0.94615
-0.80	1.07416
-0.70	1.16584
-0.60	1.23170
-0.50	1.27726
-0.40	1.30575
-0.30	1.31918
-0.2533	1.32065
-0.20	1.31876
-0.10	1.30519
0.00	1.27873
0.10	1.23924
0.20	1.18615
0.30	1.11838
0.40	1.03406
0.50	0.93007
0.60	0.80086
0.70	0.63504
0.80	0.40038
0.8733	0

TABLE 13. STRESS DISTRIBUTION UNDER A CIRCULAR PUNCH. CASE ii.

ζ	$-ct_{22}/P$	
	$PR_o/\mu hc^2=1.6$	$PR_o/\mu hc^2=1.95$
-0.98	0.67148	0.91469
-0.95	0.53992	0.66159
-0.90	0.51835	0.57498
-0.80	0.54741	0.55295
-0.70	0.58068	0.56123
-0.60	0.60703	0.57218
-0.50	0.62582	0.58080
-0.40	0.63753	0.58579
-0.30	0.64267	0.58678
-0.20	0.64160	0.58367
-0.10	0.63452	0.57639
0.00	0.62151	0.56439
0.10	0.60245	0.54903
0.20	0.57707	0.52858
0.30	0.54487	0.50321
0.40	0.50497	0.47239
0.50	0.45599	0.43531
0.60	0.39548	0.39066
0.70	0.31854	0.33623
0.80	0.21234	0.26751
0.8577	0	-
0.90	-	0.17221
0.9770	-	0

TABLE 14. STRESS DISTRIBUTION UNDER A CIRCULAR PUNCH.CASE iii.

$-ct_{22}/P$	
ζ	$-ct_{22}/P$
-0.95	0.96724
-0.90	0.72158
-0.80	0.57533
-0.70	0.52308
-0.60	0.49678
-0.50	0.48075
-0.40	0.46935
-0.30	0.46009
-0.20	0.45170
-0.10	0.44349
0.00	0.43503
0.10	0.42607
0.20	0.41649
0.30	0.40624
0.40	0.39539
0.50	0.38419
0.60	0.37328
0.70	0.36429
0.80	0.36210
0.90	0.38844
0.95	0.45537