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İÇİNDEKİLER — CONTENTS

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Kalınlığı lineer değişen dikdörtgen levhalarda yaklaşık düzlem gerilme analizi

An approximate plane stress analysis in rectangular plates with linearly varying thickness

By S. Erdoğan ŞUHUBİ
Division of Applied Mechanics, Faculty of Civil Engineering
Technical University of Istanbul

Bu çalışmada kalınlığı lineer değişen dikdörtgen levhalarda gerilme analizi, iki boyutlu elastisite teorisini kullanarak yaklaşık bir şekilde incelenmiştir. Bulunan diferansiyel denklem için, yüksek mertebeden terimleri ihmal etmek şartıyla kapalı bir çözüm verilmiş ve iki misâl yapılmıştır.

* * *

In this paper, the stress analysis in rectangular plates having linearly varying thickness is studied in an approximate way using two-dimensional theory of elasticity. A closed-form solution is given for the resulting differential equation provided that terms of higher order can be neglected and two examples are worked out.

* * *

1. Introduction. As was indicated by Conway in his previous papers [1,2], it is possible to extend methods of solution of the two-dimensional elasticity to problems in which the thickness of the plate varies slightly. In the present paper, using Airy stress function, stress distribution in a rectangular plate having linearly varying thickness has been studied. The solution of the case in which the variation of thickness is proportional to e^{ly} is due to Conway[2].

1) H.D. Conway, J. of Appl. Mech., September 1959, pp. 437-439.

2) H.D. Conway, Ing.-Arch. 29 (1960), pp. 219-222.

2. Analysis. Since we assume that the variation of thickness is so slight as to be able to apply equations of two-dimensional elesticity we can write the equilibrium equations in the following form :

$$\frac{\partial}{\partial x}(h\,\sigma_x)+\frac{\partial}{\partial y}(h\,\tau_{xy})=0.$$
$$\frac{\partial}{\partial x}(h\,\tau_{xy})+\frac{\partial}{\partial y}(h\,\sigma_y)=0. \tag{1}$$

These equations are identically satisfied by introducing a stress function $F(x, y)$ so that

$$h\,\sigma_x=\frac{\partial^2 F}{\partial y^2},\quad h\,\sigma_y=\frac{\partial^2 F}{\partial x^2},\quad h\,\tau_{xy}=-\frac{\partial^2 F}{\partial x \partial y}\,. \tag{2}$$

Using Hooke's relations and substituting (2) into compatibility equation

$$\frac{\partial^2 \varepsilon_x}{\partial y^2}+\frac{\partial^2 \varepsilon_y}{\partial x^2}=\frac{\partial^2 \gamma_{xy}}{\partial x \partial y}$$

we obtain, supposing that h depends only on y,

$$h^2\Delta\Delta F-2\,hh'\left(\frac{\partial^3 F}{\partial y^3}+\frac{\partial^3 F}{\partial x^2 \partial y}\right)+(hh''-2\,h'^2)\left[\nu\frac{\partial^2 F}{\partial x^2}-\frac{\partial^2 F}{\partial y^2}\right]=0 \tag{3}$$

where primes denote differentiations with respect to y and ν is Poisson's ratio.

Now let us consider the plate, shown in Fig. 1, which is loaded at the boundaries $y=0$ and $y=a$ by

$$q_1(x)=\sum_{n=1}^{\infty} a_{1n}\sin\lambda_n x$$

and

$$q_2(x)=\sum_{n=1}^{\infty} a_{2n}\sin\lambda_n x$$

where $\lambda_n=\dfrac{n\pi}{a}$, and having a thickness which varies according to the formula

$$h = h_0 (1 + \alpha y),$$

where α is a small constant number in accordance with our assumption. Introducing new variables such as

$$\xi = x \qquad \text{and} \qquad \eta = \frac{1 + \alpha y}{\alpha},$$

we find

$$\eta^2 \Delta\Delta F - 2\eta \left(\frac{\partial^3 F}{\partial \eta^3} + \frac{\partial^3 F}{\partial \xi^2 \partial \eta} \right) - 2 \left(\nu \frac{\partial^2 F}{\partial \xi^3} - \frac{\partial^2 F}{\partial \eta^2} \right) = 0 . \tag{4}$$

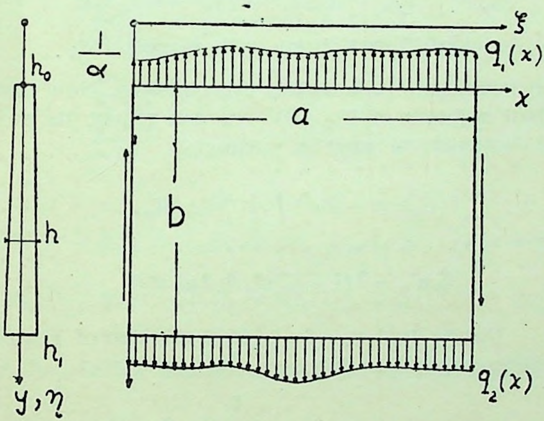

Fig. 1

Now we shall seek the solution of this differential equation in the following form:

$$F(\xi, \eta) = \sum_{n=1}^{\infty} f_n(\eta) \sin \lambda_n \xi .$$

Substituting this into (4) and using a new dimensionless variable

$$\zeta_n = \lambda_n \eta = \frac{\eta \pi}{a \alpha} (1 + \alpha y)$$

we obtain

$$\zeta^2_n \frac{d^4 f_n}{d\zeta^4_n} - 2\,\zeta_n \frac{d^3 f_n}{d\zeta^3_n} + 2\,(1-\zeta^2_n) \frac{d^2 f_n}{d\zeta^2_n} + 2\,\zeta_n \frac{d f_n}{d\zeta_n} + (\zeta^2_n + 2\nu)\, f_n = 0. \quad (5)$$

If we assume that α is so small as the following inequality will satisfy

$$\frac{h_1 - h_0}{h_0} << \frac{b}{a}\,\pi$$

we can neglect terms, such as 1 and 2ν, which are of second order compared to ζ_n^2. Hence we obtain a simplified differential equation as follows

$$\zeta_n f_n^{IV} - 2\, f_n''' - 2\,\zeta_n f_n'' + 2\, f_n' + \zeta_n f_n = 0 \quad (6)$$

where primes denote differentiations with respect to ζ_n.

It is easily seen that $e^{\pm\zeta_n}$ satisfy this equation. Now that we have two independent solution of Eq. (6) we can reduce its order to the second in the conventional way[1] by putting

$$f_n(\zeta_n) = -2\, e^{\zeta_n} \int e^{-2\zeta_n} v_n\, d\zeta_n . \quad (7)$$

We find

$$\zeta_n u''_n - 2\,(1+\zeta_n) u'_n + 2\, u_n = 0 \quad (8)$$

where $u_n = v'_n$. Noting that $u_n = 1 + \zeta_n$ is a solution of Eq. (8) we obtain the following expression for u_n:

$$u_n = C_{3n}\,(1+\zeta_n) + C_{4n}\,(1+\zeta_n) \int^{\zeta_n} \left(\frac{\zeta_n}{1+\zeta_n}\right)^2 e^{2\zeta_n} d\zeta_n .$$

Integrating this equation with respect to ζ_n and substituting the result into (7) we find

$$f_n(\zeta_n) = C_{1n}\, e^{\zeta_n} + C_{2n}\, e^{-\zeta_n} + C_{3n}\, e^{-\zeta_n} (\zeta^2_n + 3\,\zeta_n) +$$
$$+ C_{1n}\, e^{\zeta_n} \int^{\zeta_n} e^{-2\zeta_n} \left\{ \int^{\zeta_n} (1+\zeta_n) \left[\int^{\zeta_n} \left(\frac{\zeta_n}{1+\zeta_n}\right)^2 e^{2\zeta_n}\, d\zeta_n \right] d\zeta_n \right\} d\zeta_n$$

or integrating by parts in the last term we obtain

$$f_n(\zeta_n) = A_n\, e^{\zeta_n} + B_n\, e^{-\zeta_n} + C_n\, e^{\zeta_n} (\zeta^2_n - 3\,\zeta_n) + D_n\, e^{-\zeta_n} (\zeta^2_n + 3\,\zeta_n) . \quad (9)$$

[1] E. L. Ince, Ordinary Differential Equations, Dover Publication, p 121.

The four unknown constants are to be determined from the boundary conditions.

The expressions for stresses are from Eqs. (2)

$$h\sigma_x = \sum_{n=1}^{\infty} \lambda^2_n f''_n(\zeta_n) \sin \lambda_n \xi,$$

$$h\sigma_y = -\sum_{n=1}^{\infty} \lambda^2_n f_n(\zeta_n) \sin \lambda_n \xi,$$

$$h\tau_{xy} = -\sum_{n=1}^{\infty} \lambda^2_n f'_n(\zeta_n) \cos \lambda_n \xi,$$

where

$$f'_n(\zeta_n) = A_n e^{\zeta_n} - B_n e^{-\zeta_n} + C_n e^{\zeta_n} (\zeta^2_n - \zeta_n - 3) - D_n e^{-\zeta_n} (\zeta^2_n + \zeta_n - 3).$$

$$f''_n(\zeta_n) = A_n e^{\zeta_n} + B_n e^{-\zeta_n} + C_n e^{\zeta_n} (\zeta^2_n + \zeta_n - 4) + D_n e^{-\zeta_n} (\zeta^2_n - \zeta_n - 4).$$

At the boundaries the stresses must satisfy the following relations :

(a) $h_0 \sigma_y = q_1(\xi)$, $h_0 \tau_{xy} = 0$, for $y = 0$

(b) $h_1 \sigma_y = q_2(\xi)$, $h_1 \tau_{xy} = 0$, for $y = b$

(c) $h \sigma_x = 0$, for $x = 0$ and $x = a$,

or

(a) $\lambda^2_n f_n\left(\dfrac{\lambda_n}{\alpha}\right) = -c_{1n}$, $f'_n\left(\dfrac{\lambda_n}{\alpha}\right) = 0$,

(b) $\lambda^2_n f_n\left[\dfrac{\lambda_n}{\alpha}(1 + \alpha b)\right] = -a_{2n}$, $f'_n\left[\dfrac{\lambda_n}{\alpha}(1 + \alpha b)\right] = 0$. (10)

The conditions (c) are identically satisfied. Let us now calculate the resultants of the shear forces acting on edges $x = 0$ and $x = a$

$$T(0) = \int_0^b h\,\tau_{xy}\,d_y = \sum_{n=1}^{\infty} -\lambda_n \int_{\frac{\lambda_n}{\alpha}}^{\frac{\lambda_n}{\alpha}(1+\alpha b)} f'_n(\zeta)\,d\zeta = \sum_{n=1}^{\infty} \frac{1}{\lambda_n}\left[a_{2n} - a_{1n}\right],$$

$$T(a) = \sum_{n=1}^{\infty} \frac{(-1)^n}{\lambda_n}\left[a_{2n} - a_{1n}\right].$$

If loads acting on upper and lower boundaries are identical, it is seen that the resultant shear forces on $x=0$ and $x=a$ vanish as it should be.

3. **Examples.** a) Infinitely long plate (Fig. 2):

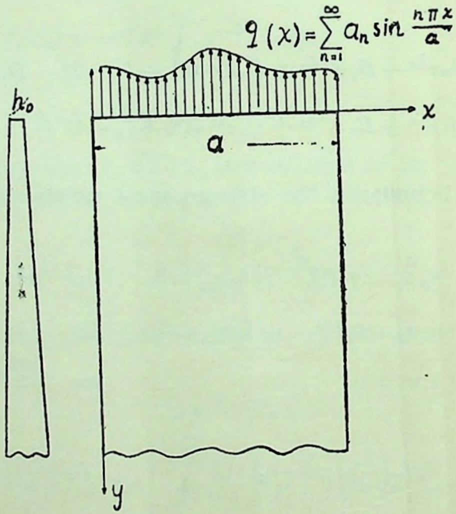

Fig. 2

In this case A_n and C_n should be set equal to zero since stress components must vanish at the vanish at the infinity. To satisfy boundary conditions we ought to take

$$B_n = -\frac{a_n}{\lambda_n^2}\,\frac{3-\frac{\lambda_n}{\alpha}\left(1+\frac{\lambda_n}{\alpha}\right)}{3+2\,\frac{\lambda_n}{\alpha}}\,e^{\frac{\lambda_n}{\alpha}},$$

$$D_n = -\frac{a_n}{\lambda_n^2}\,\frac{e^{\frac{\lambda_n}{\alpha}}}{3+2\,\frac{\lambda_n}{\alpha}},$$

that is,

$$f_n(\zeta_n) = -a_n\,\frac{e^{\frac{\lambda_n}{\alpha}-\zeta_n}}{\lambda_n^2\left(3+2\,\frac{\lambda_n}{\alpha}\right)}\left[3-\frac{\lambda_n}{\alpha}\left(1+\frac{\lambda_n}{\alpha}\right)+3\,\zeta_n+\zeta_n^2\right]$$

or

$$F(x,y) = -\frac{a^3}{\pi^2}\sum_{n=1}^{\infty}\frac{a_n}{n^3}\,e^{\frac{-n\pi y}{a}}\left[1+n\pi\,\frac{y}{a}+\frac{n^2\pi^2}{3+2\,\frac{n\pi}{a\,\alpha}}\,\frac{y^2}{a^2}\right]\sin n\pi\,\frac{x}{a} \tag{11}$$

If α tends to zero Eq. (11) takes the well-known following form:

$$F(x,y) = -\frac{a^3}{\pi^2}\sum_{n=1}^{\infty}\frac{a_n}{n^2}\,e^{-n\pi\frac{y}{a}}\left[1+n\pi\,\frac{y}{a}\right]\sin\frac{n\pi x}{a}\,.$$

For a uniformly distributed load $a_n = \frac{4\,q}{\pi\,n}$ and we obtain

$$F(x,y) = \frac{4\,q\,a^2}{\pi^2}\sum_{n=1,3,5,\ldots}^{\infty}\frac{e}{n^3}^{-n\pi\frac{y}{a}}\left[1+n\pi\,\frac{y}{a}+\frac{n^2\pi^2}{3+2\,\frac{n\pi}{a\,\alpha}}\,\frac{y^2}{a^2}\right]\sin\frac{n\pi x}{a},$$

b) As another numerical example we give the solution of the problem shown in Fig. 3.

[1] K. Girkmann, Flächentragwerke, Springer Verlag (1954), p. 57.

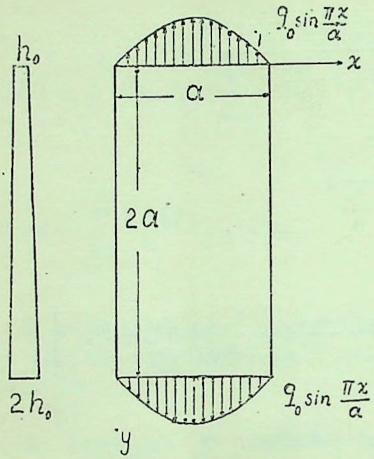

Fig. 3

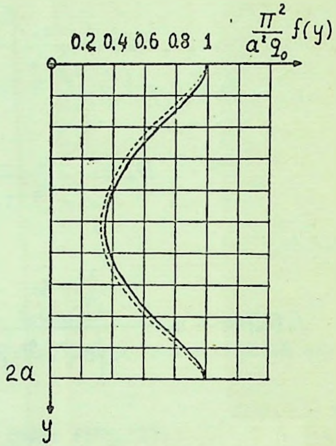

Fig 4

In this case

$$\alpha = \frac{1}{2a}$$

and using boundary conditions (10) we obtain

$$f(y) = \frac{q_0 a^2}{\pi^2}\Big[0.011644\, e^{\pi \frac{y}{a}} - 0.000082\, e^{\pi \frac{y}{a}} (\zeta^2 - 3\zeta) -$$

$$2.68529\, e^{-\pi \frac{y}{a}} + 0.063012\, e^{-\pi \frac{y}{a}} (\zeta^2 + 3\zeta)\Big]$$

where $\zeta = 2\pi\left(1 + \frac{y}{2a}\right)$. This function has been shown in Fig. 4. The dotted curve represents the solution of the exact differential equation (5) for $\nu = \frac{1}{4}$ which has been obtained by using finite differences method and dividing the interval 2a into ten equal parts.

4. Conclusion. In this paper it is tried to find an approximate solution of the stress distribution problem in rectangular plates with linearly varying thickness and loaded along two opposite sides. By neglecting some small terms it is obtained a closed-form solution for the differential equation (5) which governs the problem. It is obvious that the accuracy of this approximate solution depends on the ratio b/a and the coefficient α which characterizes the thickness variation. The degree of accuracy increases as this ratio increases or α decreases.

Appendix

The foregoing calculation can be considered as a first step of a perturbation scheme. We can indeed introduce a perturbation parameter ε near to, but less than the unity into Eq. (5) and write it in the following form:

$$\zeta_n f_n^{IV} - 2 f_n''' - 2\zeta_n f_n'' + 2 f_n' + \zeta_n f_n = -\frac{2\varepsilon}{\zeta_n}[f'_n + \nu f_n] \qquad (a)$$

Now let us expand $f_n(\zeta_n)$ into a power series of ε :

$$f_n(\zeta_n) = \sum_{k=0}^{\infty} \varepsilon^k f_{n,k}(\zeta_n) \qquad (b)$$

Substituting (b) into (a) and equating coefficients of like powers of ε in the resulting equation, we obtain

$$\zeta_n f^{IV}_{n,0} - 2 f'''_{n,0} - 2\zeta_n f''_{n,0} + 2 f'_{n,0} + \zeta_n f_{n,0} = 0$$

$$\zeta_n f^{IV}_{n,1} - 2 f'''_{n,1} - 2\zeta_n f''_{n,1} + 2 f'_{n,1} + \zeta_n f_{n,1} = -\frac{2}{\zeta_n}[f'_{n,0} + \nu f_{n,0}]$$

....

$$\zeta_n f^{IV}_{n,p} - 2 f'''_{n,p} - 2\zeta_n f''_{n,p} + 2 f'_{n,p} + \zeta_n f_{n,p} = -\frac{2}{\zeta_n}[f'_{n,p-1} + f_{n,p-1}]$$

....

It is seen that our solution correspond to the zeroth order perturbation function $f_{n,0}$. Having this function we can obtain first order function $f_{n,1}$. And continuing in this way we can get p-th order function $f_{n,p}$ from (p—1)-*th* order function $f_{n,p-1}$ by solving non-homogeneous equations. We know that

$$f_{n,0} = A_{n,0}\, e^{\zeta_n} + B_{n,0}\, e^{-\zeta_n} + C_{n,0}\, e^{\zeta_n}(\zeta_n^2 - 3\zeta_n) + D_{n,0}\, e^{-\zeta_n}(\zeta_n^3 + 3\zeta_n)$$

and we find

$$f_{n,1} = A_{n,1}\, e^{\zeta_n} + B_{n,1}\, e^{-\zeta_n} + C_{n,1}\, e^{\zeta_n} (\zeta_n^2 - 3\,\zeta_n) + D_{n,1}\, e^{-\zeta_n} (\zeta_n^2 + 3\,\zeta_n)$$
$$- 2 \int^{\xi} K(z,\zeta_n) \frac{1}{z} \, [f''_{n,0}(z) + \nu f_{n,0}(z)]\, dz$$

where $K(z,\zeta_n)$ is the kernel function of the homogeneous differential equation and is the same in every step of the perturbation. Continuing in this way we may give an explicit expression for $f_{n,p}$.

If we substitute (b) into boundary conditions (10) and compare the coefficients of ε we see that

$$\begin{aligned} &\lambda_n^2 f_{n,0}\left(\frac{\lambda_n}{\alpha}\right) = -\alpha_{1n}, && f'_{n,0}\left(\frac{\lambda_n}{\alpha}\right) = 0, \\ &\lambda_n^2 f_{n,0}\left[\frac{\lambda_n}{\alpha}(1+\alpha b)\right] = -\alpha_{2n}, && f'_{n,0}\left[\frac{\lambda_n}{\alpha}(1+\alpha b)\right] = 0, \end{aligned} \tag{c}$$

and

$$\begin{aligned} &f_{n,p}\left(\frac{\lambda_n}{\alpha}\right) = 0, && f'_{n,p}\left(\frac{\lambda_n}{\alpha}\right) = 0, \\ &f_{n,p}\left[\frac{\lambda_n}{\alpha}(1+\alpha b)\right] = 0, && f'_{n,p}\left[\frac{\lambda_n}{\alpha}(1+\alpha b)\right] = 0. \end{aligned} \tag{d}$$

The coefficients $A_{n,0}$,...., $D_{n,0}$ can be found from the conditions (c). Having these, we can evaluate step by step $A_{n,p}$,...., $D_{n,p}$ using the conditions (d).

Finally if ε tends to the unity, we obtain as a solution of our original differential equation the sum of the perturbation functions :

$$f_n = f_{n,0} + f_{n,1} + f_{n,2} + \cdots$$

Although this method is simple, it is very obvious that calculations will be rather tedious mainly because of the complicated character of the kernel function which is necessary to evaluate perturbation functions in every step.

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