ISTANBUL TECHNICAL UNIVERSITY ★ GRADUATE SCHOOL OF SCIENCE
ENGINEERING AND TECHNOLOGY

CONFORMAL MAPPINGS PRESERVING THE EINSTEIN TENSOR OF WEYL SPACES

M.Sc. THESIS

Merve GÜRLEK

Department of Mathematics

Mathematics Engineering Programme

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WEYL UZAYLARINDA EINSTEIN TENSÖRÜNÜ KORUYAN KONFORM DÖNÜŞÜMLER

YÜKSEK LİSANS TEZİ

Merve GÜRLEK
(509111077)

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Merve GÜRLEK, a M.Sc. student of ITU Graduate School of Science Engineering and Technology 509111077 successfully defended the thesis entitled **"CONFORMAL MAPPINGS PRESERVING THE EINSTEIN TENSOR OF WEYL SPACES"**, which she prepared after fulfilling the requirements specified in the associated legislations, before the jury whose signatures are below.

Thesis Advisor :	**Doç. Dr. Gülçin ÇİVİ** Istanbul Technical University	....
Jury Members :	**Prof. Dr. Aynur UYSAL** Doğuş University	....
	Doç. Dr. Güler GÜRPINAR ARSAN Istanbul Technical University	....
	Doç. Dr. Gülçin ÇİVİ Istanbul Technical University	....

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To my grandparents,

FOREWORD

I would like to extend my sincere gratitude to my thesis supervisor Assoc. Prof. Gülçin Çivi, who has been a great supporter regarding any issue throughout this work, assisted and evaluated with utmost care and provided me the opportunity to present my studies on an international platform.

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June 2013 Merve GÜRLEK

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LIST OF SYMBOLS

g_{ij}	:	Metric tensor
g^{ij}	:	Anti-metric tensor
T_k	:	Components of the covariant vector field of the Weyl space
A	:	Satellite of the metric tensor
∇	:	Symmetric connection
$\dot{\nabla}$	:	Prolonged covariant derivative
Γ^h_{ijk}	:	Connection coefficients of the symmetric connection ∇
R^h_{ijk}	:	Mixed curvature tensor of the Weyl manifold
R_{hijk}	:	Covariant curvature tensor of the Weyl manifold
R_{ij}	:	Ricci tensor of the Weyl manifold
R	:	Scalar curvature tensor of the Weyl manifold
E_{ij}	:	Einstein tensor of the Weyl manifold
C^h_{ijk}	:	Conformal curvature tensor of the Weyl manifold
Z^h_{ijk}	:	Concircular curvature tensor of the Weyl manifold
W^h_{ijk}	:	Projective curvature tensor of the Weyl manifold
P	:	Covariant vector field field of the conformal mapping
∇f	:	Gradient of the scalar function f
Δf	:	Laplacian of the scalar function f
$\|\nabla f\|$	:	Length of the ∇f
K	:	Sectional curvature of the Weyl manifold

CONFORMAL MAPPINGS PRESERVING THE EINSTEIN TENSOR OF WEYL SPACES

SUMMARY

This work contains four chapters. In Chapter 1, the fundamental definitions and properties concerning the Weyl manifolds are given.

An n-dimensional differentiable manifold with a conformal metric g and a symmetric connection ∇ satisfying the compatibility condition

$$\nabla g - 2g \otimes T = 0 \tag{1}$$

or, in local coordinates

$$\nabla_k g_{ij} - 2T_k g_{ij} = 0 \ , \tag{2}$$

is called a Weyl space, where $T_k (k = 1,2,...,n)$ are the components of the covariant vector field of the Weyl space. Such a Weyl space will be denoted by $W_n(g,T)$.

Under the renormalization

$$\tilde{g} = \lambda^2 g \tag{3}$$

of the metric tensor g, T is transformed by the rule

$$\tilde{T}_k = T_k + \partial_k (\ln \lambda), \tag{4}$$

where $\partial_k = \dfrac{\partial}{\partial x^k}$ and λ is a scalar function.

If under the renormalization of the metric tensor g, the quantity A is changed according to the rule

$$\tilde{A} = \lambda^p A, \tag{5}$$

then A is called the satellite of g of weight $\{p\}$.

The prolonged covariant derivative of A with respect to ∇ is defined by

$$\dot{\nabla}_k A = \nabla_k A - p\,T_k A. \tag{6}$$

By expanding (1) it is found that

$$\partial_k g_{ij} - g_{hj}\Gamma^h_{ik} - g_{ih}\Gamma^h_{jk} - 2T_k g_{ij} = 0, \tag{7}$$

where Γ^i_{jk} are the connection coefficients of the form

$$\Gamma^i_{jk} = \left\{ \begin{array}{c} i \\ jk \end{array} \right\} - (\,\delta^i_j T_k + \delta^i_k T_j - g_{jk} g^{ih} T_h\,). \tag{8}$$

In this chapter, moreover, the definitions and the basic properties of the mixed curvature tensor and the definitions of the Ricci tensor, the scalar curvature tensor, the Einstein tensor, the conformal curvature tensor, the concircular curvature tensor and the projective curvature tensor (Weyl tensor) of the Weyl manifold $W_n(g,T)$ are also given.

In chapter 2, firstly, the conformal mappings of Weyl Spaces are defined.

Let τ be a conformal transformation from a Weyl manifold $W_n(g,T)$ onto another Weyl manifold $\tilde{W}_n(\tilde{g},\tilde{T})$. At the corresponding points $W_n(g,T)$ and $\tilde{W}_n(\tilde{g},\tilde{T})$, it can be taken as

$$g = \tilde{g}. \tag{9}$$

The covariant vector field P with weight zero which is defined by

$$P = T - \tilde{T} \tag{10}$$

is called the covector field of the conformal transformation.

Then, the relations between connection coefficients, the mixed curvature tensors, the Ricci tensors, the scalar curvature tensors, the Einstein tensors, the concircular curvature tensors and the projective curvature tensors of the Weyl manifolds $W_n(g,T)$ and $\tilde{W}_n(\tilde{g},\tilde{T})$ under the conformal mapping τ are studied. Next, by using this relations we obtain some results concerning with P_{ij}, L_{ij} and Z_{ij} under τ. After that it is proved that the covector field P is a gradient, under the conformal mapping preserving the concircular curvature tensor or the projective curvature tensor of the Weyl manifold $W_n(g,T)$. And it is also proved that, if the conformal flat Weyl manifold $W_n(g,T)$ is a concircular flat Weyl manifold, then $W_n(g,T)$ is an Einstein-Weyl manifold by considering the relation between the concircular curvature tensor and the conformal curvature tensor.

After giving the definition and the some properties of the conformal curvature tensor, we close this chapter by obtaining the new invariants under the conformal mapping τ.

In Chapter 3, the necessary and sufficient condition for a conformal mapping between two Weyl manifolds to preserve the Einstein tensor is obtained. Then, it is proved that, if a conformal transformation from a Weyl manifold $W_n(g,T)$ onto a flat Weyl manifold $\tilde{W}_n(\tilde{g},\tilde{T})$ preserves the Einstein tensor, then both Weyl manifolds are Einstein-Weyl manifolds.

In the second section of the Chapter 3, the conformal curvature tensor is expressed for a conformal mapping preserving the Einstein tensor. Next, by using the relation between the concircular curvature tensor and the conformal curvature tensor under such a mapping, it is proved that, if a conformally flat Weyl manifold $W_n(g,T)$, is a concircularly flat Weyl manifold $W_n(g,T)$, then it is a Riemannian manifold. After that, the concircular curvature tensor is expressed for a conformal mapping preserving the

Einstein tensor. Moreover, by considering the concircular curvature tensor under the conformal mapping and using the conditions for which the Einstein tensor is preserved, a new invariant tensor is obtained. Then, it is proved that, if the concircular curvature tensor or the projective curvature tensor of the Weyl manifold $W_n(g,T)$ are preserved for a conformal mapping preserving the Einstein tensor, then the covector field P is a gradient. Conversely, it is shown that if P is a gradient, the concircular curvature tensor or the projective curvature tensor are preserved under such a mapping.

In the last section of the Chapter 3, the isotropic Weyl manifols are examined.

For a Weyl manifold $W_n(g,T)$ with a point p and the tangent space $T_p(W_n)$ of W_n, the scalar which is defined by

$$K(\Pi) = \frac{R_{ijkl}X^iY^jX^kY^l}{(g_{ik}g_{jl} - g_{il}g_{jk})X^iY^jX^kY^l}, \tag{11}$$

is called the sectional curvature of $W_n(g,T)$ at p with respect to the plane spanned by two linearly independent vectors $X,Y \in T_p(W_n)$, where X^i and Y^i are respectively the components of X and Y.

If, at each point, the sectional curvature K of $W_n(g,T)$ is independent of the 2-plane chosen, then, $W_n(g,T)$ is named as an isotropic manifold.

In this chapter, it is stated that, an isotropic Weyl manifold is an Einstein-Weyl manifold with zero scalar curvature. Furthermore by virtue of this fact, it is proved that, if a Weyl manifold is locally conformal to an isotropic Weyl manifold under the conformal mapping preserving the Einstein tensor, then both manifolds are Einstein-Weyl manifolds.

In the fourth and the last chapter, the results we obtained in work are discussed.

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ÖZET

Dört bölümden oluşan bu çalışmanın birinci bölümünde bir Weyl manifoldu ile ilişkili temel tanım ve özellikler verilmiştir.

Simetrik bir ∇ konneksiyonuna ve konform g metrik tensörüne sahip

$$\nabla g - 2g \otimes T = 0 \qquad \textbf{(12)}$$

ya da lokal koordinatlarda

$$\nabla_k g_{ij} - 2T_k g_{ij} = 0\ , \qquad \textbf{(13)}$$

uygunluk koşulunu sağlayan n$_b oyutlumanifoldabirWeyluzay$ß$denir.Burada$T$_k(k = 1,2,...,n)$, $W_n(g,T)$ Weyl uzayının kovaryant vektör alanıdır ve T vektör alanına Weyl uzayının komplementer vektör alanı adı verilir. g metrik tensörünün

$$\tilde{g} = \lambda^2 g \qquad \textbf{(14)}$$

ile verilen renormalizasyonu altında T komplementer vektör alanı

$$\tilde{T}_k = T_k + \partial_k(\ln\lambda), \qquad \textbf{(15)}$$

şeklinde dönüşür. Burada $\partial_k = \dfrac{\partial}{\partial x^k}$ dir ve λ skaler bir fonksiyondur.

g metrik tensörü renormalizasyonu altında, A büyüklüğü

$$\tilde{A} = \lambda^p A, \qquad \textbf{(16)}$$

şeklinde değişirse, A büyüklüğüne g nin $\{p\}$ ağırlıklı bir uydusu adı verilir.

A uydusunun ∇ konneksiyonuna göre $\dot{\nabla}A$ ile gösterilen kovaryant türevi

$$\dot{\nabla}_k A = \nabla_k A - p\,T_k A \qquad \textbf{(17)}$$

şeklinde tanımlanmıştır.

(13) eşitliğinin

$$\partial_k g_{ij} - g_{hj}\Gamma^h_{ik} - g_{ih}\Gamma^h_{jk} - 2T_k g_{ij} = 0 \qquad \textbf{(18)}$$

şeklinde açık formda yazılıp i,j,k indislerinin devirsel değişimi ile elde edilen denklemlerden Γ^i_{kj} katsayıları

$$\Gamma^i_{jk} = \left\{ \begin{matrix} i \\ jk \end{matrix} \right\} - (\,\delta^i_j T_k + \delta^i_k T_j - g_{jk} g^{ih} T_h\,). \tag{19}$$

olarak elde edilir.

Bu bölümde ayrıca, karışık eğrilik tensörünün tanımı ve temel özellikleri ile $W_n(g,T)$ Weyl manifoldunun kovaryant eğrilik tensörü, Ricci tensörü, skaler eğrilik tensörü, Einstein tensörü, konform eğrilik tensörü, konsörkılır eğrilik tensörü ve projektif eğrilik tensörü tanımlanmıştır.

$W_n(g,T)$ Weyl manifoldu için

$$R^h_{ijk} = \partial_j \Gamma^h_{ik} - \partial_k \Gamma^h_{ij} + \Gamma^h_{mj} \Gamma^m_{ik} - \Gamma^h_{mk} \Gamma^m_{ij} \tag{20}$$

karışık eğrilik tensörü,

$$R_{ijkl} = g_{ih} R^h_{jkl} \tag{21}$$

kovaryant eğrilik tensörü,

$$R_{ij} = R^h_{ijh} \tag{22}$$

$\{0\}$ ağırlıklı Ricci tensörü ve

$$R = g^{ih} R_{ih}. \tag{23}$$

$\{-2\}$ ağırlıklı skaler eğrilik tensörü göstermektedir.

$R_{(ij)}$, Ricci tensörünün simetrik kısmını göstermek üzere

$$E_{ij} = R_{(ij)} - \frac{R}{n} g_{ij} \tag{24}$$

tensörü $W_n(g,T)$ Weyl manifoldunun Einstein tensörünü tanımlar.

$\{0\}$ ağırlıklı C^h_{ijk} konform eğrilik tensörü

$$C^h_{ijk} = R^h_{ijk} + \delta^h_k L_{ij} - \delta^h_j L_{ik} + g_{ij} L^h_k - g_{ik} L^h_j - 2\,\delta^h_i L_{[jk]}, \tag{25}$$

ile tanımlanır ve burada

$$L_{ij} = -\frac{R_{ij}}{n-2} + \frac{2}{n(n-2)} R_{[ij]} + \frac{R g_{ij}}{2(n-1)(n-2)} \tag{26}$$

olarak alınmıştır.

$$Z^h_{ijk} = R^h_{ijk} - \frac{R}{n(n-1)} (\delta^h_k g_{ij} - \delta^h_j g_{ik}), \tag{27}$$

ve

$$W^h_{ijk} = R^h_{ijk} + \frac{2}{(n+1)}\delta^h_i R_{[jk]} + \frac{1}{(n-1)}(\delta^h_j R_{ik} - \delta^h_k R_{ij}) + \frac{2}{(n^2-1)}(\delta^h_k R_{[ij]} - \delta^h_j R_{[ik]}) \quad \textbf{(28)}$$

tensörleri $W_n(g,T)$ manifoldunun, sırasıyla konsörkılır eğrilik ve projektif eğrilik tensörlerini tanımlar.

İkinci bölümde, öncelikle Weyl uzaylarında konform dönüşüm tanımı ve temel özellikleri verilmiştir.

Bir $W_n(g,T)$ Weyl manifoldundan bir diğer $\tilde{W}_n(\tilde{g},\tilde{T})$ Weyl manifoldu üzerine, $W_n(g,T)$ ve $\tilde{W}_n(\tilde{g},\tilde{T})$ nin belirli bir p noktasında tanımlanan konform bir dönüşüm için

$$g = \tilde{g}. \quad \textbf{(29)}$$

olarak alınabilir.

Sıfır ağırlıklı kovaryant vektör alanı P

$$P = T - \tilde{T} \quad \textbf{(30)}$$

şeklinde tanımlanır ve konform dönüşümün kovektör alanı adını alır.

Bu bölümde, daha sonra, Weyl manifoldları arasında konform bir dönüşüm altında konneksiyon katsayıları, karışık eğrilik tensörleri, Ricci tensörleri, skaler eğrilik tensörleri, Einstein tensörleri, konsörkılır eğrilik tensörleri ve projektif eğrilik tensörleri arasındaki ilişkiler verilip, bu ilişkiler kullanılarak Weyl manifoldlarının konform dönüşümü altında P_{ij}, L_{ij} ve Z_{ij} tensörleri ile ilgili bazı sonuçlar elde edildi. Sonrasında, Weyl manifoldları arasındaki konform bir dönüşüm altında konsökılır eğrilik tensörü ya da projektif eğrilik tensörü korunduğunda, P kovektör alanının gradiyent olduğu ispatlandı. Bunun ardından, konsörkılır eğrilik tensörü ile konform eğrilik tensörü arasındaki ilişki kullanılarak, konform düz bir $W_n(g,T)$ Weyl uzayı, konsörkılır düz olması için Einstein-Weyl manifoldu olması gerektiği gösterildi.

İkinci bölümün sonunda, konform eğrilik tensörü ve bazı özelliklerinin verilmesinin ardından, Weyl manifoldlarının konform dönüşümü altında bazı invaryant tensörler elde edildi.

Üçüncü bölüm üç alt başlık altında incelendi. Birinci kısımda, Weyl manifoldları arasındaki bir konform dönüşüm altinda Einstein tensörünün korunması için gerek ve yeter koşul elde edildi. Sonrasında, Einstein tensörünü koruyan konform bir dönüşüm altında, bir Weyl uzayının düz bir Weyl uzayına konform olması için her iki uzayın da Einstein-Weyl uzayı olması gerektiği ispatlandı.

Üçüncü bölümün ikinci kısmında, Einstein tensörünün korunumu koşulu yardımıyla konform eğrilik tensörü C^h_{ijk} nın Einstein tensörünü koruyan konform bir dönüşüm altındaki ifadesi ${C^*}^h_{ijk}$ bulundu. Sonrasında, Einstein tensörünün korunması durumunda konform eğrilik tensörü ${C^*}^h_{ijk}$ ile konsörkılır eğrilik tensörü Z^h_{ijk} arasındaki ilişki elde edildi. Ayrıca, Einstein tensörünü koruyan konform bir dönüşüm altında konform olarak düz bir Weyl manifoldunun konsörkılır düz olması halinde Riemann manifolduna indirgendiği ispatlandı. Daha sonra, genel durumda geçerli olan, konsörkılır eğrilik tensörü Z^h_{ijk} ile konform eğrilik tensörü C^h_{ijk} arasındaki ilişkide C^h_{ijk} yerine Einstein tensörünü koruyan konform dönüşüm altındaki ifadesi yazılıp buradan Z^h_{ijk} çekilerek, konsörkılır eğrilik tensörünün Einstein tensörünü koruyan konform dönüşüm altındaki ifadesi bulundu. Ayrıca, konform dönüşüm altında $\tilde{Z}^h_{ijk}$ ile Z^h_{ijk} arasındaki ilişkide, Einstein tensörünün korunması koşulu uygulanarak invaryant bir tensör elde edildi. Son olarak, bu kısımda "Einstein tensörünü koruyan konform bir dönüşüm altında konsörkılır eğrilik tensörü ve projektif eğrilik tensörünün korunması için gerek ve yeter koşul P kovektör alanının gradyent olmasıdır " şeklinde ifadesi verilen teorem ispatlandı.

Üçüncü bölümün son kısmında, izotropik Weyl manifoldları ele alındı. Bir p noktası ve $T_p(W_n) \in W_n$ tanjant uzayı ile tanımlı bir $W_n(g,T)$ Weyl manifoldunda, X^i ve Y^i sırasıyla X ve Y nin bileşenleri olmak üzere

$$K(\Pi) = \frac{R_{ijkl}X^iY^jX^kY^l}{(g_{ik}g_{jl} - g_{il}g_{jk})X^iY^jX^kY^l}, \tag{31}$$

ile tanımlanan skalere $W_n(g,T)$ nin p deki lineer bağımsız $X,Y \in T_p(W_n)$ vektörleri tarafından gerilen yüzeyin kesitsel eğriliği denir.

Eğer $W_n(g,T)$ nin K kesitsel eğriliği her noktada orayantasyondan bağımsız ise, $W_n(g,T)$ izotropik Weyl manifoldu adını alır.

Her izotropik Weyl manifoldunun sıfır skaler eğrilikli bir Einstein-Weyl manifoldu olduğunu ifade eden yardımcı teorem bu bölümde ispatlanmıştır. Ayrıca yardımcı teoremden faydalanarak bir Weyl manifoldu, Einstein tensörünü koruyan konform bir dönüşüm altında izotropik Weyl manifolduna yerel olarak konform ise her iki manifoldun Einstein-Weyl manifoldu olduğu da bu bölümde ispatlandı. Son olarak, konsörkılır ya da projektif eğrilik tensörlerinin Einstein tensörünü koruyan konform bir dönüşüm altında korunması durumunda, P kovektör alanının sağlaması gereken diferansiyel denklem elde edildi.

Dördüncü ve son bölümde ise elde edilen sonuçlar tartışıldı.

1. INTRODUCTION

An n dimensional manifold having a conformal metric tensor g and a symmetric connection ∇ satisfying the compatibility condition

$$\nabla g - 2g \otimes T = 0 \tag{1.1}$$

or, in local coordinates

$$\nabla_k g_{ij} - 2T_k g_{ij} = 0, \tag{1.2}$$

is called a Weyl space, where T is a 1-form. Such a Weyl space will be denoted by $W_n(g,T)$. [11], [13]

Under the renormalization

$$\tilde{g} = \lambda^2 g \tag{1.3}$$

of the metric tensor g, T is transformed by the rule

$$\tilde{T}_k = T_k + \partial_k(\ln \lambda), \tag{1.4}$$

where $\partial_k = \dfrac{\partial}{\partial x^k}$ and λ is a scalar function defined on $W_n(g,T)$. [11], [13]

Writing (1.2) in local coordinates and expanding it we find that

$$\dot{\nabla}_k g_{ij} = \partial_k g_{ij} - g_{hj}\Gamma^h_{ik} - g_{ih}\Gamma^h_{jk} - 2T_k g_{ij} = 0 \tag{1.5}$$

where Γ^h_{ik} are the connection coefficients of the connection ∇ given by

$$\Gamma^h_{ik} = \left\{ \begin{matrix} h \\ ik \end{matrix} \right\} - (\delta^h_i T_k + \delta^h_k T_i - g_{ik} g^{hm} T_m). \tag{1.6}$$

It is well known that the pair $(\tilde{g}, \tilde{T})$ generates the same Weyl manifold. The process of passing from (g,T) to $(\tilde{g}, \tilde{T})$ is called a gauge transformation.

Definition 1.1 An object A defined on $W_n(g,T)$ is called a satellite of weight $\{p\}$ of the tensor g_{ij}, if it admits a transformation of the form

$$\tilde{A} = \lambda^p A \tag{1.7}$$

under the renormalization (1.3) of the metric g_{ij}.

Definition 1.2 The prolonged derivative of a satellite A of the tensor g_{ij} with weight $\{p\}$ is defined by

$$\dot{\partial}_k A = \partial_k A - p T_k A. \tag{1.8}$$

By renormalization of the metric tensor we have

$$\dot{\partial}_k \tilde{A} = \lambda^p \dot{\partial}_k A. \tag{1.9}$$

Definition 1.3 The prolonged covariant derivative of a satellite A of the tensor g_{ij} with weight $\{p\}$ is defined by

$$\dot{\nabla}_k A = \nabla_k A - p T_k A. \tag{1.10}$$

We note that the prolonged derivative and the prolonged covariant derivative preserve the weight. [11]

$$\dot{\nabla}_k g_{ij} = \nabla_k g_{ij} - 2 T_k g_{ij}. \tag{1.11}$$

On the other hand, if A is a covariant vector with components A^h and of weight $\{p\}$ then, by (1.5), (1.11) can be expressed as

$$\dot{\nabla}_k A^h = \dot{\partial}_k A^h + \Gamma^h_{kl} A_l. \tag{1.12}$$

We note that, (1.12) can be generalized for any satellite A of the metric tensor g_{ij}. [22]

Let $R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$ denotes the curvature tensor associated with the connection ∇.

In local coordinates, the mixed curvature tensor R^h_{ijk} defined by

$$(\nabla_j \nabla_k - \nabla_k \nabla_j) v^h = v^i R^h_{ijk}, \tag{1.13}$$

which implies that

$$R^h_{ijk} = \partial_j \Gamma^h_{ik} - \partial_k \Gamma^h_{ij} + \Gamma^h_{mj} \Gamma^m_{ik} - \Gamma^h_{mk} \Gamma^m_{ij} \tag{1.14}$$

with weight $\{0\}$. It can be easily seen that, the connection Γ^h_{ik} is a gauge invariant. From (1.3), (1.4) and (1.6).

The mixed curvature tensor defined by (1.14) satisfies the 1^{st} and 2^{nd} Bianchi identities, respectively, [3], [7], [17]

$$R^h_{ijk} + R^h_{jki} + R^h_{kij} = 0, \tag{1.15}$$

$$\dot{\nabla}_l R^h_{ijk} + \dot{\nabla}_k R^h_{ilj} + \dot{\nabla}_j R^h_{ikl} = 0. \tag{1.16}$$

The tensor defined by

$$R_{ijkl} = g_{ih} R^h_{jkl} \tag{1.17}$$

is called the covariant curvature tensor of $W_n(g,T)$ and it satisfies the following relations, [6]

$$\begin{aligned} R_{mijk} - R_{jkmi} &= g_{im}\,(T_{k,j} - T_{j,k}) + g_{jm}\,(T_{k,i} - T_{i,k}) + g_{jk}\,(T_{m,i} - T_{i,m}) \\ &\quad + g_{km}\,(T_{i,j} - T_{j,i}) + g_{ij}\,(T_{m,k} - T_{k,m}) + g_{ik}\,(T_{j,m} - T_{m,j}), \end{aligned} \tag{1.18}$$

$$R_{ijkl} = -R_{ijlk} \tag{1.19}$$

and

$$R_{ijkl} + R_{ijlk} = 2\,g_{ij}\,(T_{k,l} - T_{l,k}). \tag{1.20}$$

The Ricci tensor with weight $\{0\}$ and the scalar curvature tensor $\{-2\}$ are, respectively, defined by

$$R_{ij} = R^h_{ijh} \tag{1.21}$$

and

$$R = g^{ih} R_{ih}. \tag{1.22}$$

Definition 1.4 The tensor

$$E_{ij} = R_{(ij)} - \frac{R}{n}\,g_{ij} \tag{1.23}$$

is defined as the Einstein tensor of $W_n(g,T)$, where $R_{(ij)}$ denotes the symmetric part of the Ricci tensor.

Definition 1.5 A Weyl manifold is said to be an Einstein-Weyl manifold, if the symmetric part of the Ricci tensor is proportional to the metric tensor. That is,

$$R_{(ij)} = \lambda\, g_{ij}, \tag{1.24}$$

where $\lambda = \dfrac{R}{n}$ is a scalar function. [10], [15], [17]

Let τ be a conformal mapping from a Weyl manifold onto another Weyl manifold.

The tensor defined by

$$C^h_{ijk} = R^h_{ijk} + \delta^h_k L_{ij} - \delta^h_j L_{ik} + g_{ij} L^h_k - g_{ik} L^h_j - 2\,\delta^h_i L_{[jk]}, \tag{1.25}$$

is an invariant under τ and it is said to be the conformal curvature tensor of the Weyl manifold $W_n(g,T)$, where L_{ij} is a tensor defined by

$$L_{ij} = -\frac{R_{ij}}{n-2} + \frac{2}{n(n-2)} R_{[ij]} + \frac{R\,g_{ij}}{2(n-1)(n-2)} \tag{1.26}$$

with weight $\{0\}$ and $L^h_k = g^{lh} L_{lk}$.

Lemma 1.1 For the tensor L_{ij} of $W_n(g,T)$ the following identities hold

i. $L_{[jk]} = \dfrac{1}{n} R_{[kj]} = \nabla_{[k} T_{j]}$

ii. $\dot{\nabla}_l L_{[jk]} + \dot{\nabla}_j L_{[kl]} + \dot{\nabla}_k L_{[lj]} = 0,$

where brackets denote the antisymmetric parts with respect to sub-indices. [6]

A conformal mapping is called a concircular mapping if it preserves the circles.

The tensor defined by

$$Z^h_{ijk} = R^h_{ijk} - \frac{R}{n(n-1)} (\delta^h_k g_{ij} - \delta^h_j g_{ik}), \tag{1.27}$$

is an invariant under a concircular mapping and it is said to be the concircular mapping of the Weyl manifold $W_n(g,T)$. [14]

If we take $h = k$, we find

$$Z_{ij} = R_{ij} - \frac{R}{n} g_{ij}. \tag{1.28}$$

Also, it is well known that the projective curvature tensor W^h_{ijk} defined as [9]

$$\begin{aligned} W^h_{ijk} &= R^h_{ijk} + \frac{2}{(n+1)} \delta^h_i R_{[jk]} + \frac{1}{(n-1)} (\delta^h_j R_{ik} - \delta^h_k R_{ij}) \\ &\quad + \frac{2}{(n^2-1)} (\delta^h_k R_{[ij]} - \delta^h_j R_{[ik]}), \end{aligned} \tag{1.29}$$

is an invariant with respect to the geodesic mapping of Weyl manifolds.

Contracting W^h_{ijk} with respect to h and k we get

$$W_{ij} = 0, \tag{1.30}$$

where $W^h_{ijk} = W_{ij}$.

2. CONFORMAL MAPPINGS OF WEYL SPACES

Conformal mappings of Riemannian manifolds were studied by many authors [5], [18], [19], [20]. Weyl and Schouten studied conformal mappings of a Riemannian space onto a flat space [18], [20]. In [8], Gribacheva obtained the necessary and sufficient conditions for a conformal flat Weyl space to admit a conformal mapping onto a flat Weyl space.

In this chapter, we consider a conformal mapping between two Weyl manifolds. Then, we study the relations between some basic tensors under such a conformal mapping. Next, by using this relations we obtain a necessary condition, for which the concircular or the projective curvature tensors are preserved. Finally, in this chapter we investigate the condition for a conformal flat Weyl manifold to be a concircular flat Weyl manifold.

2.1 Conformal Mappings

Let V_n and $\tilde{V}_n$ be two spaces referred to the same coordinates x^i, corresponding points being determined by the same values of the coordinates. If the components of the fundamental tensors g_{ij} and $\tilde{g}_{ij}$, of the two spaces are connected by the relations

$$\tilde{g}_{ij} = U^2 g_{ij}, \tag{2.1}$$

where U is any function of coordinates,the magnitudes of the elementary vectors with components dx^i,$(i = 1,...,n)$ at corresponding points of V_n and $\tilde{V}_n$ are

$$\sqrt{g_{ij}\, dx^i\, dx^j} \tag{2.2}$$

and

$$\sqrt{U^2 g_{ij}\, dx^i\, dx^j} \tag{2.3}$$

respectively, and these are in the ratio $\frac{1}{U}$. Also the inclination of the vectors dx^i and δx^i, at a point of V_n, is the same as the inclination of the corresponding vectors at the corresponding point of $\tilde{V}_n$. Because, if the vectors u and v have lengths $|u|$ and $|v|$ respectively, $\frac{u}{|u|}$ and $\frac{v}{|v|}$ are unit vectors having the directions of u and v. Their inclinations θ is therefore given by

$$|u|\,|v|\,cos\theta = u\,.\,v, \qquad \textbf{(2.4)}$$

so that

$$cos\theta = \frac{g_{ij}\,u^i\,v^j}{\sqrt{(g_{ij}\,u^i\,u^j)\,(g_{ij}\,u^i\,v^j)}}. \qquad \textbf{(2.5)}$$

If $cos\theta = 0$, then the vectors are said to be orthogonal. Hence the condition of orthogonality of two vectors u,v is

$$u.v = 0 \qquad \textbf{(2.6)}$$

or its equivalent

$$g_{ij}\,u^i\,v^j = 0. \qquad \textbf{(2.7)}$$

Thus, corresponding infinitesmall figures are similar. Such correspondence is conformal, and either space is said to be conformally represented on the other. [21]

Therefore, the transformation preserving the angles is called a conformal mapping.

Let τ be a conformal transformation of $W_n(g,T)$ into $\tilde{W}_n(\tilde{g},\tilde{T})$. At corresponding points of $W_n(g,T)$ and $\tilde{W}_n(\tilde{g},\tilde{T})$ we can make [13], [16]

$$g = \tilde{g}. \qquad \textbf{(2.8)}$$

The covariant vector field P defined by

$$P = T - \tilde{T} \tag{2.9}$$

is called the covector field of the conformal transformation. It can be shown that P has zero weight.

Let ∇ and $\tilde{\nabla}$ be the Weyl connections of $W_n(g,T)$ and $\tilde{W}_n(\tilde{g},\tilde{T})$, and the connection coefficients be denoted by Γ^h_{ik} and $\tilde{\Gamma}^h_{ik}$, respectively. From (1.6), (2.1) and (2.2) we can write the relation between Γ^h_{ik} and $\tilde{\Gamma}^h_{ik}$ as,

$$\begin{aligned}
\tilde{\Gamma}^h_{ik} &= \left\{ \begin{matrix} \tilde{h} \\ ik \end{matrix} \right\} - (\tilde{\delta}^h_i \tilde{T}_k + \tilde{\delta}^h_k \tilde{T}_i - \tilde{g}_{ik} \tilde{g}^{hm} \tilde{T}_m) \\
&= \left\{ \begin{matrix} h \\ ik \end{matrix} \right\} - (\delta^h_i (T_k - P_k) + \delta^h_k (T_i - P_i) - g_{ik} g^{hm} (T_m - P_m)) \\
&= \Gamma^h_{ik} + \delta^h_i P_k + \delta^h_k P_i - g_{ik} g^{hm} P_m.
\end{aligned} \tag{2.10}$$

After some calculations and by substituting (2.10) into (1.13) we find the relation between the the mixed curvature tensors of the Weyl manifold $W_n(g,T)$ and $\tilde{W}_n(\tilde{g},\tilde{T})$ as

$$\begin{aligned}
\tilde{R}^h_{ijk} &= \partial_j \tilde{\Gamma}^h_{ik} - \partial_k \tilde{\Gamma}^h_{ij} + \tilde{\Gamma}^h_{mj} \tilde{\Gamma}^m_{ik} - \tilde{\Gamma}^h_{mk} \tilde{\Gamma}^m_{ij} \\
&= R^h_{ijk} + 2\,\delta^h_i \nabla_{[j} P_{k]} \\
&\quad + \delta^h_k (\partial_j P_i - \Gamma^m_{ij} P_m - P_i P_j + \frac{1}{2} g^{mt} g_{ij} P_m P_t) \\
&\quad - \delta^h_j (\partial_k P_i - \Gamma^m_{ik} P_m - P_i P_k + \frac{1}{2} g^{mt} g_{ik} P_m P_t) \\
&\quad + g^{hl} g_{ij} (\partial_k P_l - \Gamma^m_{lk} P_m - P_l P_k + \frac{1}{2} g^{mt} g_{lk} P_m P_t) \\
&\quad - g^{hl} g_{ik} (\partial_j P_l - \Gamma^m_{lj} P_m - P_j P_l + \frac{1}{2} g^{mt} g_{lj} P_m P_t) \\
&\quad + g_i j P_l (\nabla_k g^{hl}) - g_i k P_l (\nabla_j g^{hl}) + g^{hl} P_l (\nabla_k g_{ij}) - g^{hl} P_l (\nabla_j g_{ik}) \\
&= R^h_{ijk} + 2\,\delta^h_i \nabla_{[j} P_{k]} + \delta^h_k P_{ij} - \delta^h_j P_{ik} + g^{hl} g_{ij} P_{lk} - g^{hl} g_{ik} P_{lj},
\end{aligned} \tag{2.11}$$

where P_{ij} is defined as

$$P_{ij} = \partial_j P_i - \Gamma^m_{ij} P_m - P_i P_j + \frac{1}{2} g^{ml} g_{ij} P_m P_l. \quad \textbf{(2.12)}$$

Contraction on (2.10) with respect to h and k, yields the relation between the Ricci tensor R_{ij} of $W_n(g,T)$ and $\tilde{R}_{ij}$ of $\tilde{W}_n(\tilde{g},\tilde{T})$ given by

$$\tilde{R}_{ij} = R_{ij} + 2\nabla_{[j} P_{i]} + (n-2) P_{ij} + g_{ij} P^h_h, \quad \textbf{(2.13)}$$

where $P^h_h = g^{ij} P_{ij}$.

Transvecting (2.13) by g^{ij} and using (2.8) we get the relation

$$\tilde{R} = R + 2(n-1) P^h_h, \quad \textbf{(2.14)}$$

which implies

$$P^h_h = \frac{\tilde{R} - R}{2(n-1)}. \quad \textbf{(2.15)}$$

On the other hand, from (2.13) it can be easily seen that antisymmetric and symmetric parts of the Ricci tensor of $W_n(g,T)$ and $\tilde{W}_n(\tilde{g},\tilde{T})$ are, respectively, related by

$$\tilde{R}_{[ij]} = R_{[ij]} + n\nabla_{[j} P_{i]} \quad \textbf{(2.16)}$$

and

$$\tilde{R}_{(ij)} = R_{(ij)} + (n-2) P_{(ij)} + g_{ij} g^{hl} P_{lh}. \quad \textbf{(2.17)}$$

By virtue of (2.15) and (2.16), (2.13) reduces to,

$$\tilde{R}_{ij} = R_{ij} + \frac{2}{n} (\tilde{R}_{[ij]} - R_{[ij]}) + (n-2) P_{ij} + \frac{1}{2(n-1)} (\tilde{R} - R) g_{ij}, \quad \textbf{(2.18)}$$

from which it follows that

$$P_{ij} = \frac{1}{n(n-2)} \left[(n-1)\,(\tilde{R}_{ij} - R_{ij}) + (\tilde{R}_{ji} - R_{ji}) - \frac{n}{2(n-1)}\, g_{ij}\,(\tilde{R} - R) \right]. \qquad \textbf{(2.19)}$$

By considering the symmetric part of the Ricci tensor of $\tilde{W}_n(\tilde{g}, \tilde{T})$ given by (2.17), we obtain the relation between the Einstein tensors E_{ij} and $\tilde{E}_{ij}$ of $\tilde{W}_n(\tilde{g}, \tilde{T})$ and $\tilde{W}_n(\tilde{g}, \tilde{T})$ as below

$$\begin{aligned} \tilde{E}_{ij} &= \tilde{R}_{(ij)} - g_{ij}\,\frac{\tilde{R}}{n} \\ &= R_{(ij)} + (n-2)\,P_{(ij)} + g_{ij}\,P^h_h - \frac{g_{ij}}{n}\,[2(n-2)\,P^h_h + R] \\ &= E_{ij} + (n-2)\,P_{(ij)} - \frac{(n-2)}{n}\,g_{ij}\,P^h_h. \qquad \textbf{(2.20)} \end{aligned}$$

From (2.11) and (2.14), we obtain that the concircular curvature tensor $\tilde{Z}^h_{ijk}$ of $\tilde{W}_n(\tilde{g}, \tilde{T})$ and the concircular curvature tensor Z^h_{ijk} of $W_n(g, T)$ are related by

$$\begin{aligned} \tilde{Z}^h_{ijk} &= \tilde{R}^h_{ijk} - \frac{1}{n(n-1)}\,\tilde{R}\,(\delta^h_k\,g_{ij} - \delta^h_j\,g_{ik}) \\ &= R^h_{ijk} + 2\,\delta^h_i\,\nabla_{[j}P_{k]} + \delta^h_k\,P_{ij} - \delta^h_j\,P_{ik} + g^{hl}\,g_{ij}\,P_{lk} - g^{hl}\,g_{ik}\,P_{lj} \\ &\quad - \frac{1}{n(n-1)} \left[R + 2(n-1)\,P^m_m)\,(\delta^h_k\,g_{ij} - \delta^h_j\,g_{ik} \right] \\ &= Z^h_{ijk} + 2\,\delta^h_i\,\nabla_{[j}P_{k]} + \delta^h_k\,P_{ij} - \delta^h_j\,P_{ik} + g^{hl}\,g_{ij}\,P_{lk} - g^{hl}\,g_{ik}\,P_{lj} \\ &\quad - \frac{2}{n}\,\delta^h_k\,g_{ij}\,P^h_h + \frac{2}{n}\,\delta^h_j\,g_{ik}\,P^h_h \\ &= Z^h_{ijk} + 2\,\delta^h_i\,\nabla_{[j}P_{k]} + \delta^h_k\,(P_{ij} - \frac{2}{n}\,g_{ij}\,P^m_m) - \delta^h_j\,(P_{ik} - \frac{2}{n}\,g_{ik}\,P^m_m) \\ &\quad + g^{hl}\,g_{ij}\,P_{lk} - g^{hl}\,g_{ik}\,P_{lj} \qquad \textbf{(2.21)} \end{aligned}$$

under the conformal mapping from $W_n(g, T)$ onto $\tilde{W}_n(\tilde{g}, \tilde{T})$.

If we contract (2.21) with respect to h and k we get

$$\tilde{Z}_{ij} = Z_{ij} + 2\,\nabla_{[j}P_{i]} + (n-2)\,P_{ij} - \frac{(n-2)}{n}\,g_{ij}\,P^h_h. \qquad \textbf{(2.22)}$$

On the other hand, by using (2.13), (2.14) and (2.16), we obtain that L_{ij} and $\tilde{L}_{ij}$ are related by the equation

$$
\begin{aligned}
\tilde{L}_{ij} &= \frac{1}{n-2}(-\tilde{R}_{ij} + \frac{2}{n}\tilde{R}_{[ij]} + \frac{1}{2(n-1)} g_{ij}\tilde{R}) \\
&= \frac{1}{(n-2)}\left[-R_{ij} - 2\nabla_{[j}P_{i]} - (n-2)P_{ij} - g_{ij}P_h^h\right] \\
&\quad + \frac{1}{(n-2)}\left[\frac{2}{n}R_{[ij]} + 2\nabla_{[j}P_{i]} + \frac{1}{2(n-1)} g_{ij}R + g_{ij}P_h^h\right] \\
&= L_{ij} - P_{ij} \qquad \textbf{(2.23)}
\end{aligned}
$$

and by, (2.11), (2.13) and (2.16) we find that the projective curvature tensors W^h_{ijk} of $W_n(g,T)$ and $\tilde{W}^h_{ijk}$ of $\tilde{W}_n(\tilde{g},\tilde{T})$ are related by

$$
\begin{aligned}
\tilde{W}^h_{ijk} &= \tilde{R}^h_{ijk} + \frac{2}{(n+1)}\delta^h_i\tilde{R}_{jk} + \frac{1}{(n-1)}\left[\delta^h_j\tilde{R}_{ik} - \delta^h_k\tilde{R}_{ij}\right] \\
&\quad + \frac{2}{(n^2-1)}\left[\delta^h_k\tilde{R}_{[ij]} - \delta^h_j\tilde{R}_{[ik]}\right] \\
&= R^h_{ijk} + 2\delta^h_i\nabla_{[j}P_{k]} + \delta^h_k P_{ij} - \delta^h_j P_{ik} + g^{hl}g_{ij}P_{lk} - g^{hl}g_{ik}P_{lj} \\
&\quad + \frac{2}{(n+1)}\delta^h_i(R_{[jk]} + n\nabla_{[k}P_{j]}) \\
&\quad + \frac{1}{(n-1)}\delta^h_j(R_{ik} + 2\nabla_{[k}P_{i]} + (n-2)P_{ik} + g_{ik}P^m_m) \\
&\quad - \frac{1}{(n-1)}\delta^h_k(R_{ij} + 2\nabla_{[j}P_{i]} + (n-2)P_{ij} + g_{ij}P^m_m) \\
&\quad + \frac{2}{(n^2-1)}\delta^h_k(R_{[ij]} + n\nabla_{[j}P_{i]}) \\
&\quad - \frac{2}{(n^2-1)}\delta^h_j(R_{[ik]} + n\nabla_{[k}P_{i]}) \\
&= W^h_{ijk} + \frac{1}{(n-1)}\delta^h_k\left[P_{ij} - \frac{2}{(n+1)}\nabla_{[j}P_{i]} - g_{ij}P^m_m\right] \\
&\quad - \frac{1}{(n-1)}\delta^h_j\left[P_{ik} - \frac{2}{(n+1)}\nabla_{[k}P_{i]} - g_{ik}P^m_m\right] \\
&\quad + \frac{2}{(n+1)}\delta^h_i\nabla_{[j}P_{k]} + g^{hl}g_{ij}P_{lk} - g^{hl}g_{ik}P_{lj}. \qquad \textbf{(2.24)}
\end{aligned}
$$

Theorem 2.1 Let $\tau : W_n(g,T) \to \tilde{W}_n(\tilde{g},\tilde{T})$ be a conformal mapping preserving the concircular curvature tensor or the projective curvature tensor of the Weyl manifold $W_n(g,T)$ then the covector field P is a gradient.

Proof: From (2.21) we have

$$
\tilde{Z}^h_{ijk} = Z^h_{ijk} + X^h_{ijk}, \qquad \textbf{(2.25)}
$$

where

$$\begin{aligned} X^h_{ijk} &= 2\,\delta^h_i\,P_{[kj]} + \delta^h_k\,P_{ij} - \delta^h_j\,P_{ik} \\ &\quad - g_{ik}\,g^{hm}\,P_{mj} + g_{ij}\,g^{hm}\,P_{mk} \\ &\quad - \frac{2}{n}\,P^h_h\,(\delta^h_k\,g_{ij} - \delta^h_j\,g_{ik}) = 0. \end{aligned} \tag{2.26}$$

Let $\tilde{Z}^h_{ijk} = Z^h_{ijk}$. Then we have $X^h_{ijk} = 0$. By contracting h and i in (2.26) we get

$$P_{[kj]} = 0, \tag{2.27}$$

which implies that P_k is a gradient.

By similar calculations, and by using (2.24) we obtain

$$\tilde{W}^h_{ijk} = W^h_{ijk} + P^h_{ijk}, \tag{2.28}$$

where

$$\begin{aligned} P^h_{ijk} = \frac{2}{(n+1)}\,\delta^h_i\,P_{[kj]} &+ \frac{1}{(n-1)}\,\delta^h_k\left[P_{ij} - \frac{2}{(n+1)}\,P_{[ij]} - g_{ij}\,P^h_h\right] \\ &- \frac{1}{(n-1)}\,\delta^h_j\left[P_{ik} - \frac{2}{(n+1)}\,P_{[ik]} - g_{ik}\,P^h_h\right] \\ &- g_{ik}\,g^{hl}\,P_{lj} + g_{ij}\,g^{hl}\,P_{lk}. \end{aligned} \tag{2.29}$$

Suppose that $\tilde{W}^h_{ijk} = W^h_{ijk}$. Then we have $P^h_{ijk} = 0$. Contraction on h and i in (2.29) gives that

$$P_{[kj]} = 0, \tag{2.30}$$

or P_k is a gradient.

Thus, the proof is completed.

Lemma 2.2 The concircular curvature tensor Z^h_{ijk} and the conformal curvature tensor C^h_{ijk} of the Weyl manifold $W_n(g,T)$ is related by [1]

$$\begin{aligned} C^h_{ijk} &= Z^h_{ijk} + \frac{2}{n(n-2)}\big(\delta^h_k Z_{[ij]} - \delta^h_j Z_{[ik]} + g^{hl} g_{ij} Z_{[lk]} \\ &\quad - g^{hl} g_{ik} Z_{[lj]} - (n-2)\,\delta^h_i Z_{[kj]}\big) \\ &\quad - \frac{1}{(n-2)}\big(\delta^h_k Z_{ij} - \delta^h_j Z_{ik} + g^{hl} g_{ij} Z_{lk} - g^{hl} g_{ik} Z_{lj}\big), \end{aligned} \quad \textbf{(2.31)}$$

where $Z_{[ij]}$ denotes the antisymmetric part of the Z_{ij} .

Proof: By using (1.28) we obtain that

$$L_{ij} = \frac{1}{(n-2)}\big(-Z_{ij} + \frac{2}{n} Z_{[ij]}\big) - \frac{1}{2n(n-1)} g_{ij} R. \quad \textbf{(2.32)}$$

Solving (1.27) for R^h_{ijk} and substituting R^h_{ijk} and L_{ij} into (1.25) we get

$$\begin{aligned} C^h_{ijk} &= Z^h_{ijk} + \frac{R}{n(n-1)}[\delta^h_k g_{ij} - \delta^h_j g_{ik}] - \frac{2}{n}\delta^h_i Z_{[kj]} \\ &\quad + \frac{1}{(n-2)}\delta^h_k [-Z_{ij} + \frac{2}{n} Z_{[ij]} - \frac{R}{2n(n-1)}\delta^h_k g_{ij} \\ &\quad - \frac{1}{(n-2)}\delta^h_j [-Z_{ik} + \frac{2}{n} Z_{[ik]} - \frac{R}{2n(n-1)}\delta^h_j g_{ik} \\ &\quad + \frac{1}{(n-2)} g^{hl} g_{ij} [-Z_{lk} + \frac{2}{n} Z_{[lk]} - \frac{R}{2n(n-1)} g^{hl} g_{ij} g_{lk} \\ &\quad - \frac{1}{(n-2)} g^{hl} g_{ik} [-Z_{lj} + \frac{2}{n} Z_{[lj]} - \frac{R}{2n(n-1)} g^{hl} g_{ik} g_{lj}, \end{aligned} \quad \textbf{(2.33)}$$

or

$$\begin{aligned} C^h_{ijk} &= Z^h_{ijk} + \frac{2}{n(n-2)}\big(\delta^h_k Z_{[ij]} - \delta^h_j Z_{[ik]} + g^{hl} g_{ij} Z_{[lk]} \\ &\quad - g^{hl} g_{ik} Z_{[lj]} - (n-2)\,\delta^h_i Z_{[kj]}\big) \\ &\quad - \frac{1}{(n-2)}\big(\delta^h_k Z_{ij} - \delta^h_j Z_{ik} + g^{hl} g_{ij} Z_{lk} - g^{hl} g_{ik} Z_{lj}\big). \end{aligned} \quad \textbf{(2.34)}$$

Thus, the proof is completed.

Theorem 2.3 Let $W_n(g,T)$ is a conformal flat Weyl manifold. If $W_n(g,T)$ is concircular flat Weyl manifold, then $W_n(g,T)$ is an Einstein-Weyl manifold.

Proof: For a conformal flat Weyl manifold $W_n(g,T)$ the conformal curvature tensor C^h_{ijk} of $W_n(g,T)$ is zero. Assume that the conformal flat Weyl manifold $W_n(g,T)$ is a concircular Weyl manifold, then the concirclar curvature tensor Z^h_{ijk} of $W_n(g,T)$ is also zero. Hence by considering the relation, which is given in Lemma 2.2, we have

$$\begin{aligned}&\frac{2}{n(n-2)}\left(\delta^h_k Z_{[ij]} - \delta^h_j Z_{[ik]} + g^{hl} g_{ij} Z_{[lk]} - g^{hl} g_{ik} Z_{[lj]} - (n-2)\,\delta^h_i Z_{[kj]}\right)\\ &-\frac{1}{(n-2)}\left(\delta^h_k Z_{ij} - \delta^h_j Z_{ik} + g^{hl} g_{ij} Z_{lk} - g^{hl} g_{ik} Z_{lj}\right) = 0. \qquad \textbf{(2.35)}\end{aligned}$$

By contracting (2.35) with respect to h and k and by inserting $Z_{ij} = R_{ij} - \dfrac{R}{n} g_{ij}$ we obtain that

$$Z_{ij} = 0, \qquad \textbf{(2.36)}$$

which implies that $Z_{[ij]} = Z_{(ij)} = 0$.

Therefore, according to Definition 1.5, $W_n(g,T)$ is an Einstein-Weyl manifold, which completes the proof.

2.2 Conformal Invariance

In this section, we will define the tensors which are invariants under the conformal mapping of the Weyl manifolds $W_n(g,T)$ onto $\tilde{W}_n(g,T)$ with weight $\{0\}$.

By substituting (2.16) and (2.19) into (2.11) we have

$$
\begin{aligned}
\tilde{R}^h_{ijk} \;=\; & R^h_{ijk} + \frac{2}{n}\,[\tilde{R}_{[kj]} - R_{[kj]}] \\
& + \frac{1}{n(n-2)}\,\delta^h_k \left[(n-1)\,(\tilde{R}_{ij} - R_{ij}) + (\tilde{R_{ji}} - R_{ji}) - \frac{n}{2(n-1)}\,g_{ij}\,(\tilde{R} - R)\right] \\
& - \frac{1}{n(n-2)}\,\delta^h_j \left[(n-1)\,(\tilde{R}_{ik} - R_{ik}) + (\tilde{R_{ki}} - R_{ki}) - \frac{n}{2(n-1)}\,g_{ik}\,(\tilde{R} - R)\right] \\
& + \frac{1}{n(n-2)}\,g^{hl}\,g_{ij} \left[(n-1)\,(\tilde{R}_{lk} - R_{lk}) + (\tilde{R_{kl}} - R_{kl}) - \frac{n}{2(n-1)}\,g_{lk}\,(\tilde{R} - R)\right] \\
& - \frac{1}{n(n-2)}\,g^{hl}\,g_{ik} \left[(n-1)\,(\tilde{R}_{ik} - R_{ik}) + (\tilde{R_{ki}} - R_{ki}) - \frac{n}{2(n-1)}\,g_{ik}\,(\tilde{R} - R)\right].
\end{aligned}
\tag{2.37}
$$

Then, collecting the terms with "˜" on the left side and the terms without "˜" on the right side, we can write that

$$
\begin{aligned}
\tilde{R}^h_{ijk} - \frac{2}{n}\,\delta^h_i\,\tilde{R}_{[kj]} \quad + \quad & \frac{1}{(n-2)}\,\delta^h_k \left[-\tilde{R}_{ij} + \frac{2}{n}\,\tilde{R}_{[ij]} + \frac{1}{2(n-1)}\,g_{ij}\,\tilde{R}\right] \\
& - \frac{1}{(n-2)}\,\delta^h_j \left[-\tilde{R}_{ik} + \frac{2}{n}\,\tilde{R}_{[ik]} + \frac{1}{2(n-1)}\,g_{ik}\,\tilde{R}\right] \\
& + \frac{1}{(n-2)}\,g^{hl}\,g_{ij} \left. -\tilde{R}_{lk} + \frac{2}{n}\,\tilde{R}_{[lk]} + \frac{1}{2(n-1)}\,g_{lk}\,\tilde{R}\right] \\
& - \frac{1}{(n-2)}\,g^{hl}\,g_{ik} \left[-\tilde{R}_{lj} + \frac{2}{n}\,\tilde{R}_{[lj]} + \frac{1}{2(n-1)}\,g_{lj}\,\tilde{R}\right] \\
= R^h_{ijk} - \frac{2}{n}\,\delta^h_i\,R_{[kj]} \quad + \quad & \frac{1}{(n-2)}\,\delta^h_k \left[-R_{ij} + \frac{2}{n}\,R_{[ij]} + \frac{1}{2(n-1)}\,g_{ij}\,R\right] \\
& - \frac{1}{(n-2)}\,\delta^h_j \left[-R_{ik} + \frac{2}{n}\,R_{[ik]} + \frac{1}{2(n-1)}\,g_{ik}\,R\right] \\
& + \frac{1}{(n-2)}\,g^{hl}\,g_{ij} \left[-R_{lk} + \frac{2}{n}\,R_{[lk]} + \frac{1}{2(n-1)}\,g_{lk}\,R\right] \\
& - \frac{1}{(n-2)}\,g^{hl}\,g_{ik} \left[-R_{lj} + \frac{2}{n}\,R_{[lj]} + \frac{1}{2(n-1)}\,g_{lj}\,R\right].
\end{aligned}
\tag{2.38}
$$

Then, by considering L_{ij}, which is given in (1.26) we obtain

$$
\begin{aligned}
& \tilde{R}^h_{ijk} - \frac{2}{n}\,\delta^h_i\,\tilde{R}_{[kj]} + \delta^h_k\,\tilde{L}_{ij} - \delta^h_j\,\tilde{L}_{ik} + g^{hl}\,g_{ij}\,\tilde{L}_{lk} - g^{hl}\,g_{ik}\,\tilde{L}_{ij} \\
& = R^h_{ijk} - \frac{2}{n}\,\delta^h_i\,R_{[kj]} + \delta^h_k\,L_{ij} - \delta^h_j\,L_{ik} + g^{hl}\,g_{ij}\,L_{lk} - g^{hl}\,g_{ik}\,L_{ij},
\end{aligned}
\tag{2.39}
$$

and putting

$$C^h_{ijk} = R^h_{ijk} - \frac{2}{n}\delta^h_i R_{[kj]} + \delta^h_k L_{ij} - \delta^h_j L_{ik} + g^{hl} g_{ij} L_{lk} - g^{hl} g_{ik} L_{ij} \qquad \textbf{(2.40)}$$

we get

$$\tilde{C}^h_{ijk} = C^h_{ijk}, \qquad \textbf{(2.41)}$$

which denotes that the conformal curvature tensor is invariant under a conformal mapping of Weyl manifold. [14], [16]

Lemma 2.4 For a Weyl manifold $W_n(g,T)$ the following identities hold [6].

i. $C_{ijkl} - C_{klij} = 0$,

ii. $C_{ijkl} + C_{ijlk} = 0$,

iii. $C_{ijkl} + C_{jikl} = 0$,

iv. $C^h_{ijk} + C^h_{jki} + C^h_{kij} = 0$,

v. $C^h_{ijh} = C^h_{ihj} = C^h_{hij} = 0$.

Proof: Transvecting (1.25) with g_{hm} we get

$$C_{mijk} = R_{mijk} + g_{km} L_{ij} - g_{jm} L_{ik} + g_{ij} L_{mk} - g_{ik} L_{mj} - 2g_{im} L_{[jk]}. \qquad \textbf{(2.42)}$$

i. If we consider (1.18) and (1.26), it can be easily seen that

$$\begin{aligned} C_{ijkl} - C_{klij} &= R_{ijkl} - R_{klij} \\ &\quad + 2g_{kj} L_{[il]} + 2g_{li} L_{[jk]} - 2g_{ik} L_{[jl]} - 2g_{jl} L_{[ik]} - 2g_{ij} L_{[kl]} - 2g_{kl} L_{[ij]} \\ &= 0. \end{aligned} \qquad \textbf{(2.43)}$$

ii. By using (1.19) we get

$$C_{ijkl} + C_{ijlk} = R_{ijkl} + R_{ijlk} = 0. \qquad \textbf{(2.44)}$$

iii. Combining (2.43) and (2.44) we obtain

$$C_{ijkl} = C_{klij} = -C_{klji} = -C_{jikl}, \qquad \textbf{(2.45)}$$

which follows that

$$C_{ijkl} + C_{jikl} = 0. \qquad \textbf{(2.46)}$$

iv. The first Bianchi identity implies that

$$C^h_{ijk} + C^h_{jki} + C^h_{kij} = 0. \qquad \textbf{(2.47)}$$

v. If take $h = k$, we get

$$C^h_{ijh} = R_{ij} - \frac{2}{n}R_{[ij]} + (n-2)\,L_{ij} + g^{hl}\,g_{ij}. \qquad \textbf{(2.48)}$$

Substituting (1.26) into (2.48) we obtain

$$C^h_{ijh} = 0. \qquad \textbf{(2.49)}$$

From (1.15) and (1.19), it is clear that, $C^h_{ihk} = 0$ and $C^h_{hjk} = 0$.

Theorem 2.5 Let $\tau : W_n(g,T) \to \tilde{W}_n(\tilde{g},\tilde{T})$ be a conformal mapping. The following tensors with weight $\{0\}$ are invariants under τ.

i.
$$\begin{aligned} Q^h_{ijk} = W^h_{ijk} \;&-\; \frac{2}{n(n+1)}\,\delta^h_i\,R_{[kj]} - \frac{1}{(n-1)(n-2)}\,(\delta^h_k\,R_{ij} - \delta^h_j\,R_{ik}) \\ &+ \frac{4}{n(n^2-1)}\,(\delta^h_k\,R_{[ij]} - \delta^h_j\,R_{[ik]}) + \frac{(n-1)}{2(n-2)}\,(\delta^h_k\,g_{ij} - \delta^h_j\,g_{ik})R \\ &- g^{hl}\,g_{ij}\,(nR_{lk} - 2R_{[lk]}) - g^{hl}\,g_{ik}\,(nR_{lj} - 2R_{[lj]}). \end{aligned} \qquad \textbf{(2.50)}$$

ii.
$$\begin{aligned} U^h_{ijk} = Z^h_{ijk} - \frac{2}{n}\,\delta^h_i\,R_{[kj]} \;&+\; \frac{1}{(n-2)}\left[\delta^h_k\,L_{ij} - \delta^h_j\,L_{ik} + g^{hl}\,g_{ij}\,L_{lk} - g^{hl}\,g_{ik}\,L_{lj}\right] \\ &+ \frac{R}{n(n-1)}\left[\delta^h_k\,g_{ij} - \delta^h_j\,g_{ik}\right]. \end{aligned} \qquad \textbf{(2.51)}$$

Proof:

i. By substituting (2.14), (2.16) and (2.19) into (2.24) we get

$$
\begin{aligned}
\tilde{W}^h_{ijk} \;=\; & W^h_{ijk} + \frac{2}{n(n+1)}\,\delta^h_i\,(\tilde{R}_{[kj]} - R_{[kj]}) \\
& + \frac{1}{(n-1)(n-2)}\,\delta^h_k\,(\tilde{R}_{ij} - R_{ij}) - \frac{4}{n(n^2-1)}\,\delta^h_k\,(\tilde{R}_{[ij]} - R_{[ij]}) \\
& - \frac{(n-1)}{2(n-2)}\,\delta^h_k\,g_{ij}(\tilde{R} - R) - \frac{1}{(n-1)(n-2)}\,\delta^h_j\,(\tilde{R}_{ik} - R_{ik}) \\
& + \frac{4}{n(n^2-1)}\,\delta^h_j\,(\tilde{R}_{[ik]} - R_{[ik]}) + \frac{(n-1)}{2(n-2)}\,\delta^h_j\,g_{ik}(\tilde{R} - R) \\
& + g^{hl}\,g_{ij}\,\left[n(\tilde{R}_{lk} - R_{lk}) - 2(\tilde{R}_{lk]} - R_{[lk]})\right] - g^{hl}\,g_{ik}\,\left[n(\tilde{R}_{lj} - R_{lj}) - 2(\tilde{R}_{[lj]} - R_{[lj]})\right]
\end{aligned}
\tag{2.52}
$$

then, collecting the terms with "˜" on the left side and the terms without "˜" on the right side, we have

$$
\begin{aligned}
\tilde{W}^h_{ijk} \;-\; & \frac{2}{n(n+1)}\,\delta^h_i\,\tilde{R}_{[kj]} - \frac{1}{(n-1)(n-2)}\,(\delta^h_k\,\tilde{R}_{ij} - \delta^h_j\,\tilde{R}_{ik}) \\
& + \frac{4}{n(n^2-1)}\,(\delta^h_k\,\tilde{R}_{[ij]} - \delta^h_j\,\tilde{R}_{[ik]}) + \frac{(n-1)}{2(n-2)}\,(\delta^h_k\,g_{ij} - \delta^h_j\,g_{ik})(\tilde{R} - R) \\
& - g^{hl}\,g_{ij}\,(n\tilde{R}_{lk} - 2\tilde{R}_{lk]}) - g^{hl}\,g_{ik}\,(n\tilde{R}_{lj} - 2\tilde{R}_{[lj]}) \\
= W^h_{ijk} \;-\; & \frac{2}{n(n+1)}\,\delta^h_i\,R_{[kj]} - \frac{1}{(n-1)(n-2)}\,(\delta^h_k\,R_{ij} - \delta^h_j\,R_{ik}) \\
& + \frac{4}{n(n^2-1)}\,(\delta^h_k\,R_{[ij]} - \delta^h_j\,R_{[ik]}) + \frac{(n-1)}{2(n-2)}\,(\delta^h_k\,g_{ij} - \delta^h_j\,g_{ik})(R - R) \\
& - g^{hl}\,g_{ij}\,(nR_{lk} - 2R_{lk]}) - g^{hl}\,g_{ik}\,(nR_{lj} - 2R_{[lj]}).
\end{aligned}
\tag{2.53}
$$

If we put

$$
\begin{aligned}
Q^h_{ijk} = W^h_{ijk} \;-\; & \frac{2}{n(n+1)}\,\delta^h_i\,R_{[kj]} - \frac{1}{(n-1)(n-2)}\,(\delta^h_k\,R_{ij} - \delta^h_j\,R_{ik}) \\
& + \frac{4}{n(n^2-1)}\,(\delta^h_k\,R_{[ij]} - \delta^h_j\,R_{[ik]}) + \frac{(n-1)}{2(n-2)}\,(\delta^h_k\,g_{ij} - \delta^h_j\,g_{ik})\,R \\
& - g^{hl}\,g_{ij}\,(nR_{lk} - 2R_{lk]}) - g^{hl}\,g_{ik}\,(nR_{lj} - 2R_{[lj]}).
\end{aligned}
\tag{2.54}
$$

Then, we obtain

$$Q^h_{ijk} = \tilde{Q}^h_{ijk}. \tag{2.55}$$

ii. Let us consider the relation between the concircular curvature tensors of the Weyl manifolds $W_n(g,T)$ and $\tilde{W}_n(\tilde{g},\tilde{T})$ under the conformal mapping. By substituting (2.15), (2.16) and (2.19) into (2.21) we obtain

$$\begin{aligned}
\tilde{Z}^h_{ijk} \;=\;& Z^h_{ijk} + \frac{2}{n}\,\delta^h_i\,(\tilde{R}_{[kj]} - R_{[kj]}) \\
&+ \frac{1}{(n-2)}\,\delta^h_k \left[(\tilde{R}_{ij} - R_{ij}) - \frac{2}{n}\,(\tilde{R}_{[ij]} - R_{[ij]}) - \frac{2}{n}\,g_{ij}\,(\tilde{R} - R)\right] \\
&- \frac{1}{(n-2)}\,\delta^h_j \left[(\tilde{R}_{ik} - R_{ik}) - \frac{2}{n}\,(\tilde{R}_{[ik]} - R_{[ik]}) - \frac{2}{n}\,g_{ik}\,(\tilde{R} - R)\right] \\
&+ \frac{1}{(n-2)}\,g^{hl}\,g_{ij} \left[(\tilde{R}_{lk} - R_{lk}) - \frac{2}{n}\,(\tilde{R}_{[lk]} - R_{[lk]})\right] \\
&- \frac{1}{(n-2)}\,g^{hl}\,g_{ik} \left[(\tilde{R}_{lj} - R_{lj}) - \frac{2}{n}\,(\tilde{R}_{[lj]} - R_{[lj]})\right]
\end{aligned} \tag{2.56}$$

then, collecting the terms with "˜" on the left side and the terms without "˜" on the right side, we get

$$\begin{aligned}
\tilde{Z}^h_{ijk} - \frac{2}{n}\,\delta^h_i\,\tilde{R}_{[kj]} \;&+\; \frac{1}{(n-2)}\,\delta^h_k \left[-\tilde{R}_{ij} + \frac{2}{n}\,\tilde{R}_{[ij]} + \frac{2}{n}\,g_{ij}\,\tilde{R}\right] \\
&-\; \frac{1}{(n-2)}\,\delta^h_j \left[-\tilde{R}_{ik} + \frac{2}{n}\,\tilde{R}_{[ik]} + \frac{2}{n}\,g_{ij}\,\tilde{R}\right] \\
&+\; \frac{1}{(n-2)}\,g^{hl}\,g_{ij} \left[-\tilde{R}_{lk} + \frac{2}{n}\,\tilde{R}_{[lk]}\right] \\
&-\; \frac{1}{(n-2)}\,g^{hl}\,g_{ik} \left[-\tilde{R}_{lj} + \frac{2}{n}\,\tilde{R}_{[lj]}\right] \\
= Z^h_{ijk} - \frac{2}{n}\,\delta^h_i\,R_{[kj]} \;&+\; \frac{1}{(n-2)}\,\delta^h_k \left[-R_{ij} + \frac{2}{n}\,R_{[ij]} + \frac{2}{n}\,g_{ij}\,R\right] \\
&-\; \frac{1}{(n-2)}\,\delta^h_j \left[-R_{ik} + \frac{2}{n}\,R_{[ik]} + \frac{2}{n}\,g_{ij}\,R\right] \\
&+\; \frac{1}{(n-2)}\,g^{hl}\,g_{ij} \left[-R_{lk} + \frac{2}{n}\,R_{[lk]}\right] \\
&-\; \frac{1}{(n-2)}\,g^{hl}\,g_{ik} \left[-R_{lj} + \frac{2}{n}\,R_{[lj]}\right].
\end{aligned} \tag{2.57}$$

Considering (1.26) in (2.57) and setting

$$U^h_{ijk} = Z^h_{ijk} - \frac{2}{n}\,\delta^h_i\,R_{[kj]} \quad + \quad \frac{1}{(n-2)}\left[\delta^h_k\,L_{ij} - \delta^h_j\,L_{ik} + g^{hl}\,g_{ij}\,L_{lk} - g^{hl}\,g_{ik}\,L_{lj}\right] + \frac{R}{n(n-1)}\left[\delta^h_k\,g_{ij} - \delta^h_j\,g_{ik}\right] \tag{2.58}$$

we obtain that

$$U^h_{ijk} = \tilde{U}^h_{ijk}. \tag{2.59}$$

3. CONFORMAL MAPPINGS PRESERVING THE EINSTEIN TENSOR OF WEYL SPACES

In [5], the authors obtained the necessary and sufficient condition for a Riemannian space V_n to admit a conformal mapping preserving the Einstein tensor onto some Riemannian space $\tilde{V}_n$ and in [2], the authors studied geodesic mappings preserving the Einstein tensor of Weyl spaces.

In this chapter, we consider a conformal mapping under which the Einstein tensor of the Weyl manifold is preserved. Then, we investigate the necessary and sufficient condition for a conformal mapping between two Weyl manifolds to preserve the Einstein tensor. Next, we study the conformal mapping between a Weyl manifold and a flat Weyl manifold and we obtain the relations between some basic tensors of the Weyl manifolds which are in a conformal correspondence preserving the Einstein tensor. Finally, we consider the isotropic Weyl manifolds and we obtain that if a Weyl manifold and an isotropic Weyl manifold which are in a conformal correspondence then both manifolds are Einstein-Weyl manifolds.

3.1 Conformal Invariance of the Einstein Tensor of Weyl Manifolds

Let $\tau : W_n(g,T) \to \tilde{W}_n(\tilde{g},\tilde{T})$ be a conformal mapping preserving the Einstein tensor. Thus, we have

$$\tilde{E}_{ij} = E_{ij}, \tag{3.1}$$

Since every 2-dimensional Weyl manifold is an Einstein-Weyl manifold, the Einstein tensor vanishes for such a Weyl manifold [16]. So, we assume that $n > 2$ in this work.

By considering (2.20) and (3.1) we get

$$(n-2)\left[P_{(ij)} - \frac{g_{ij}}{n} P^h_h\right] = 0, \tag{3.2}$$

which implies that

$$P_{(ij)} = \frac{g_{ij}}{n} P^h_h. \tag{3.3}$$

Conversely, suppose that the condition (3.3) is hold. By (2.20) it is clear that

$$\tilde{E}_{ij} = E_{ij}. \tag{3.4}$$

Thus, we proved that

Theorem 3.1 The conformal mapping of $W_n(g,T)$ onto $\tilde{W}_n(\tilde{g},\tilde{T})$ $(n > 2)$ preserves the Einstein tensor, if and only if the condition

$$P_{(ij)} = \frac{1}{2n(n-1)} g_{ij} (\tilde{R} - R) \tag{3.5}$$

holds.

From (2.19), the symmetric and the antisymmetric parts of P_{ij} are follow

$$P_{(ij)} = \frac{1}{(n-2)} (\tilde{R}_{(ij)} - R_{(ij)}) - \frac{1}{2(n-2)(n-1)} g_{ij} (\tilde{R} - R) \tag{3.6}$$

and

$$P_{[ij]} = \nabla_{[j} P_{i]} = \frac{1}{n} (\tilde{R}_{[ij]} - R_{[ij]}), \tag{3.7}$$

respectively.

Substituting (3.5) into (3.6) we can rewrite (3.6) as

$$P_{(ij)} = \frac{1}{2(n-1)} (\tilde{R}_{(ij)} - R_{(ij)}), \tag{3.8}$$

for the mapping τ.

Since $P_{ij} = P_{(ij)} + P_{[ij]}$, by using (3.8) and (3.7), P_{ij} can be written as

$$P_{ij} = \frac{1}{2(n-1)}\,(\tilde{R}_{(ij)} - R_{(ij)}) + \frac{1}{n}\,(\tilde{R}_{[ij]} - R_{[ij]}). \tag{3.9}$$

Suppose that the Weyl space $W_n(g,T)$ and the flat Weyl space $\tilde{W}_n(\tilde{g},\tilde{T})$ are related by a conformal mapping preserving the Einstein tensor. Then, we have

$$\tilde{R}^h_{ijk} = 0, \tag{3.10}$$

which implies that

$$\tilde{R}_{ij} = 0, \quad \tilde{R} = 0, \quad \tilde{E}_{ij} = 0. \tag{3.11}$$

Since $E_{ij} = \tilde{E}_{ij}$,

$$E_{ij} = 0. \tag{3.12}$$

Similarly, if the flat Weyl space $W_n(g,T)$ transforms onto the Weyl space $\tilde{W}_n(\tilde{g},\tilde{T})$ by a conformal mapping preserving the Einstein tensor, $\tilde{E}_{ij}$ becomes zero.

Thus, we get that a Weyl manifold and a flat Weyl manifold, which are conformal correspondence preserving the Einstein tensor are Einstein-Weyl manifold. So, we proved that

Theorem 3.2 If the conformal mapping of the Weyl manifold $W_n(g,T)$ onto a flat Weyl manifold $\tilde{W}_n(\tilde{g},\tilde{T})$ preserves the Einstein tensor, then both Weyl manifolds are Einstein-Weyl manifolds.

3.2 Invariants Under the Conformal Mappings Preserving the Einstein Tensor

In this section we investigate some basic invariants under the condition given by the Theorem 3.1 in section 3.1 and we study in the necessary and sufficient condition for which the conformal mapping preserving the Einstein tensor on the Weyl manifolds, preserves the projective or the concircular curvature tensors.

Lemma 3.3 The conformal curvature tensor C^h_{ijk} of $W_n(g,T)$ is in the form

$$C^{*h}_{\ ijk} = R^h_{ijk} - \frac{2}{n}\,\delta^h_i\,R_{[kj]} - \frac{1}{2n(n-1)}\,[\delta^h_k A_{ij} - \delta^h_j A_{ik} + g^{hl}\,g_{ij} A_{lk} - g^{hl}\,g_{ik} A_{lj}], \quad \textbf{(3.13)}$$

for the conformal mapping preserving the Einstein tensor, where $A_{ij} = nR_{ij} + (n-2)\,R_{[ij]}$.

Proof: By substituting (3.7) and (3.9) into (2.11) we get

$$\begin{aligned}
\tilde{R}^h_{ijk} \;\; = \;\; & R^h_{ijk} + \frac{2}{n}\,\delta^h_i\,(\tilde{R}_{[kj]} - R_{[kj]}) \\
& + \frac{1}{2(n-1)}\,\delta^h_k\,(\tilde{R}_{(ij)} - R_{(ij)}) + \frac{1}{n}\,\delta^h_k\,(\tilde{R}_{[ij]} - R_{[ij]}) \\
& - \frac{1}{2(n-1)}\,\delta^h_j\,(\tilde{R}_{(ik)} - R_{(ik)}) - \frac{1}{n}\,\delta^h_j\,(\tilde{R}_{[ik]} - R_{[ik]}) \\
& + \frac{1}{2(n-1)}\,g^{hl}\,g_{ij}\,(\tilde{R}_{(lk)} - R_{(lk)}) + \frac{1}{n}\,g^{hl}\,g_{ij}\,(\tilde{R}_{[lk]} - R_{[lk]}) \\
& - \frac{1}{2(n-1)}\,g^{hl}\,g_{ik}\,(\tilde{R}_{(lj)} - R_{(lj)}) - \frac{1}{n}\,g^{hl}\,g_{ik}\,(\tilde{R}_{[lj]} - R_{[lj]}), \qquad \textbf{(3.14)}
\end{aligned}$$

then collecting the terms with "˜" on the left side and the terms without "˜" on the right side, we have the following equation

$$\begin{aligned}
\tilde{R}^h_{ijk} - \frac{2}{n}\,\delta^h_i\,\tilde{R}_{[kj]} \;\; - \;\; & \frac{1}{2n(n-1)}\,[\delta^h_k\,(n\tilde{R}_{ij} + (n-2)\,\tilde{R}_{[ij]}) - \delta^h_j\,(n\tilde{R}_{ik} + (n-2)\,\tilde{R}_{[ik]}) \\
& + g^{hl}\,g_{ij}\,(n\tilde{R}_{lk} + (n-2)\,\tilde{R}_{[lk]}) - g^{hl}\,g_{ik}\,(n\tilde{R}_{lj} + (n-2)\,\tilde{R}_{[lj]})] \\
= R^h_{ijk} - \frac{2}{n}\,\delta^h_i\,R_{[kj]} \;\; - \;\; & \frac{1}{2n(n-1)}\,[\delta^h_k\,(nR_{ij} + (n-2)\,R_{[ij]}) - \delta^h_j\,(nR_{ik} + (n-2)\,R_{[ik]}) \\
& + g^{hl}\,g_{ij}\,(nR_{lk} + (n-2)\,R_{[lk]}) - g^{hl}\,g_{ik}\,(nR_{lj} + (n-2)\,R_{[lj]})]
\end{aligned}$$

(3.15)

and then putting $A_{ij} = nR_{ij} + (n-2)R_{[ij]}$ we can write

$$\tilde{R}^h_{ijk} - \frac{2}{n}\delta^h_i \tilde{R}_{[kj]} - \frac{1}{2n(n-1)}[\delta^h_k \tilde{A}_{ij} - \delta^h_j \tilde{A}_{ik} + g^{hl} g_{ij}\tilde{A}_{lk} - g^{hl} g_{ik}\tilde{A}_{lj}]$$
$$= R^h_{ijk} - \frac{2}{n}\delta^h_i R_{[kj]} - \frac{1}{2n(n-1)}[\delta^h_k A_{ij} - \delta^h_j A_{ik} + g^{hl} g_{ij} A_{lk} - g^{hl} g_{ik} A_{lj}]. \quad \mathbf{(3.16)}$$

Thus, setting

$$C^{*h}_{ijk} = R^h_{ijk} - \frac{2}{n}\delta^h_i R_{[kj]} - \frac{1}{2n(n-1)}[\delta^h_k A_{ij} - \delta^h_j A_{ik} + g^{hl} g_{ij} A_{lk} - g^{hl} g_{ik} A_{lj}], \quad \mathbf{(3.17)}$$

we obtain the invariant tensor

$$C^{*h}_{ijk} = \tilde{C}^{*h}_{ijk}, \quad \mathbf{(3.18)}$$

which express the conformal curvature tensor of $W_n(g,T)$ for the mapping τ,

where A_{ij} is defined as

$$A_{ij} = nR_{ij} + (n-2)R_{[ij]}. \quad \mathbf{(3.19)}$$

Lemma 3.4 The concircular curvature tensor Z^h_{ijk} of $W_n(g,T)$ is in the form

$$Z^{*h}_{ijk} = R^h_{ijk} - \delta^h_k B_{ij} + \delta^h_j B_{ik} - g^{hl} g_{ij} B_{lk} + g^{hl} g_{ik} B_{lj}, \quad \mathbf{(3.20)}$$

for the conformal mapping preserving the Einstein tensor, where

$$B_{ij} = \frac{-1}{(n-2)}\left[\frac{n}{2(n-1)} R_{(ij)} + \frac{R}{n} g_{ij}\right].$$

Proof: By substituting C^{*h}_{ijk} obtained in Lemma 3.3 into (2.31), we obtain

$$
\begin{aligned}
& R^h_{ijk} - \frac{2}{n}\,\delta^h_i\,R_{[kj]} - \frac{1}{2n(n-1)}\,\left[\delta^h_k A_{ij} - \delta^h_j A_{ik} + g^{hl}\,g_{ij}A_{lk} - g^{hl}\,g_{ik}A_{lj}\right] \\
= \;& Z^h_{ijk} + \frac{2}{n(n-2)}\,\left[\delta^h_k Z_{[ij]} - \delta^h_j Z_{[ik]}\right] \\
& + \frac{2}{n(n-2)}\,\left[+g^{hl}\,g_{ij}Z_{[lk]} - g^{hl}\,g_{ik}Z_{[lj]} - (n-2)\,\delta^h_i Z_{[kj]}\right] \\
& - \frac{1}{(n-2)}\,\left[\delta^h_k Z_{ij} - \delta^h_j Z_{ik} + g^{hl}\,g_{ij}Z_{lk} - g^{hl}\,g_{ik}Z_{lj}\right]. \qquad \textbf{(3.21)}
\end{aligned}
$$

By extracting Z^h_{ijk} from above equation, we get

$$
\begin{aligned}
Z^{*h}_{\;ijk} = R^h_{ijk} \quad - \quad & \delta^h_k\left[\frac{1}{2n(n-1)}A_{ij} + \frac{2}{n(n-2)}Z_{[ij]} - \frac{1}{(n-2)}Z_{ij}\right] \\
& + \delta^h_j\left[\frac{1}{2n(n-1)}A_{ik} + \frac{2}{n(n-2)}Z_{[ik]} - \frac{1}{(n-2)}Z_{ik}\right] \\
& - g^{hl}\,g_{ij}\left[\frac{1}{2n(n-1)}A_{lk} + \frac{2}{n(n-2)}Z_{[lk]} - \frac{1}{(n-2)}Z_{lk}\right] \\
& + g^{hl}\,g_{ik}\left[\frac{1}{2n(n-1)}A_{lj} + \frac{2}{n(n-2)}Z_{[lj]} - \frac{1}{(n-2)}Z_{lj}\right] \quad \textbf{(3.22)}
\end{aligned}
$$

and setting

$$
\begin{aligned}
B_{ij} \;&=\; \frac{1}{2n(n-1)}A_{ij} + \frac{2}{n(n-2)}Z_{[ij]} - \frac{1}{(n-2)}Z_{ij} \\
&=\; \frac{1}{2n(n-1)}\,(nR_{ij} + (n-2)\,R_{[ij]}) + \frac{2}{n(n-2)}\,R_{[ij]} - \frac{1}{(n-2)}\,(R_{ij} - \frac{R}{n}\,g_{ij}) \\
&=\; \frac{-1}{(n-2)}\,\left[\frac{n}{2(n-1)}R_{(ij)} + \frac{R}{n}\,g_{ij}\right], \qquad \textbf{(3.23)}
\end{aligned}
$$

we have

$$
Z^{*h}_{\;ijk} = R^h_{ijk} - \delta^h_k\,B_{ij} + \delta^h_j\,B_{ik} - g^{hl}\,g_{ij}\,B_{lk} + g^{hl}\,g_{ik}\,B_{lj}, \qquad \textbf{(3.24)}
$$

The tensor obtained by (3.24) states the concircular curvature tensor of $W_n(g,T)$ in the case of the conformal mapping between the Weyl manifolds $W_n(g,T)$ and $\tilde{W}_n(\tilde{g},\tilde{T})$ preserves the Einstein tensor.

Now, we proceed to obtain the new invariants of the Weyl manifold $W_n(g,T)$, which is related to a Weyl manifold $W_n(g,T)$ and $\tilde{W}_n(\tilde{g},\tilde{T})$ by a conformal mapping preserving the Einstein tensor.

Theorem 3.5 Let $\tau : W_n(g,T) \to \tilde{W}_n(\tilde{g},\tilde{T})$ be a conformal mapping preserving the Einstein tensor. The following tensors with weight $\{0\}$ are invariants under τ.

$$\begin{aligned}
\textbf{i.}\quad Y^h_{ijk} &= Z^h_{ijk} - 2\,\delta^h_i \nabla_{[j}T_{k]} - \delta^h_k \nabla_{[i}T_{j]} + \delta^h_j \nabla_{[i}T_{k]},\\
\textbf{ii.}\quad V^h_{ijk} &= W^h_{ijk} - \frac{2}{n(n+1)}\,\delta^h_i R_{kj} + \frac{2}{n(n^2-1)}\,(\delta^h_k R_{[ij]} - \delta^h_j R_{[ik]}) - \frac{1}{n}\,g^{hl}\,(g_{ij}R_{[lk]} - g_{ik}R_{[lj]}).
\end{aligned} \tag{3.25}$$

Proof:

i. The relation obtained as (2.21) in Chapter 2 is between the concircular curvature tensors Z^h_{ijk} and $\tilde{Z}^h_{ijk}$. By substituting (3.5), (3.7) and (3.9) into (2.21) we obtain the relation

$$\begin{aligned}
\tilde{Z}^h_{ijk} &= Z^h_{ijk} + \frac{2}{n}\,\delta^h_i\,(\tilde{R}_{[jk]} - R_{[jk]})\\
&\quad + \delta^h_k \left[\frac{-1}{2(n-1)}\,(\tilde{R}_{(ij)} - R_{(ij)}) + \frac{1}{n}\,(\tilde{R}_{[ij]} - R_{[ij]})\right]\\
&\quad - \delta^h_j \left[\frac{-1}{2(n-1)}\,(\tilde{R}_{(ik)} - R_{(ik)}) + \frac{1}{n}\,(\tilde{R}_{[ik]} - R_{[ik]})\right]\\
&\quad + g^{hl}\,g_{ij} \left[\frac{1}{2(n-1)}\,(\tilde{R}_{(lk)} - R_{(lk)}) + \frac{1}{n}\,(\tilde{R}_{[lk]} - R_{[lk]})\right]\\
&\quad - g^{hl}\,g_{ik} \left[\frac{1}{2(n-1)}\,(\tilde{R}_{(lj)} - R_{(lj)}) + \frac{1}{n}\,(\tilde{R}_{[lj]} - R_{[lj]})\right],
\end{aligned} \tag{3.26}$$

between the concircular curvature tensors ${Z^*}^h_{ijk}$ and Z^h_{ijk} under the mapping τ. Then collecting the terms with "~" on the left side and the terms without "~" on the right side we find

$$\begin{aligned}
&\tilde{Z}^h_{ijk} - \frac{2}{n}\,\delta^h_i\,\tilde{Z}_{[jk]} - \frac{1}{2n(n-1)}\left[\delta^h_k\,\tilde{D}_{ji} - \delta^h_j\,\tilde{D}_{ki} - g^{hl}\,g_{ij}\,\tilde{D}_{lk} + g^{hl}\,g_{ik}\,\tilde{D}_{lj}\right]\\
&= Z^h_{ijk} - \frac{2}{n}\,\delta^h_i\,Z_{[jk]} - \frac{1}{2n(n-1)}\left[\delta^h_k\,D_{ji} - \delta^h_j\,D_{ki} - g^{hl}\,g_{ij}\,D_{lk} + g^{hl}\,g_{ik}\,D_{lj}\right],
\end{aligned} \tag{3.27}$$

and setting

$$Y^h_{ijk} = Z^h_{ijk} - \frac{2}{n}\,\delta^h_i\,Z_{[jk]} - \frac{1}{2n(n-1)}\left[\delta^h_k\,D_{ji} - \delta^h_j\,D_{ki} - g^{hl}\,g_{ij}\,D_{lk} + g^{hl}\,g_{ik}\,D_{lj}\right], \tag{3.28}$$

where

$$D_{ij} = n R_{(ij)} + 2(n-1) R_{[ij]}. \quad \textbf{(3.29)}$$

we obtain

$$Y^h_{ijk} = \tilde{Y}^h_{ijk}. \quad \textbf{(3.30)}$$

Thus, we obtain an invariant tensor

$$Y^h_{ijk} = Z^h_{ijk} - \frac{2}{n}\,\delta^h_i\, Z_{[jk]} - \frac{1}{2n(n-1)}\left[\delta^h_k D_{ji} - \delta^h_j D_{ki} - g^{hl}\, g_{ij} D_{lk} + g^{hl}\, g_{ik} D_{lj}\right] \quad \textbf{(3.31)}$$

with weight $\{0\}$ for a conformal mapping preserving the Einstein tensor.

If we apply the equality $R_{[ij]} = n\nabla_{[i}T_{j]}$ given by Lemma 1.1 , i., we can rewrite $P_{[ij]}$ as

$$P_{[ij]} = \frac{1}{n}\,(\tilde{R}_{[ij]} - R_{[ij]}) = (\nabla_{[i}\tilde{T}_{j]} - \nabla_{[i}T_{j]}). \quad \textbf{(3.32)}$$

By using (3.5) and (3.8) in (3.26), Y^h_{ijk} can be written as,

$$Y^h_{ijk} = Z^h_{ijk} - 2\,\delta^h_i \nabla_{[j}T_{k]} - \delta^h_k \nabla_{[i}T_{j]} + \delta^h_j \nabla_{[i}T_{k]} \quad \textbf{(3.33)}$$

in terms of Z^h_{ijk} and the complementary vector field T_k.

ii. By considering the projective curvature tensor of $W_n(g,T)$ and by substituting (3.5), (3.7) and (3.9) into (2.34), we get

$$\begin{aligned} \tilde{W}^h_{ijk} &= W^h_{ijk} + \frac{2}{n(n+1)}\,\delta^h_i\,(\tilde{R}_{[kj]} - R_{[kj]}) \\ &\quad - \frac{2}{n(n^2-1)}\,\delta^h_k\,(\tilde{R}_{[ij]}) - R_{[ij]}) + \frac{2}{n(n^2-1)}\,\delta^h_j\,(\tilde{R}_{[ik]}) - R_{[ik]}) \\ &\quad + \frac{1}{n}\,g^{hl}\,g_{ij}\,(\tilde{R}_{[lk]}) - R_{[lk]}) - \frac{1}{n}\,g^{hl}\,g_{ik}\,(\tilde{R}_{[lj]}) - R_{[lj]}) \end{aligned} \quad \textbf{(3.34)}$$

then collecting the terms with "~" on the left side and the terms without "~" on the right side, we obtain

$$\tilde{W}^h_{ijk}-\frac{2}{n(n+1)}\,\delta^h_i\,\tilde{R}_{kj}+\frac{2}{n(n^2-1)}\,(\delta^h_k\,\tilde{R}_{[ij]}-\delta^h_j\,\tilde{R}_{[ik]})-\frac{1}{n}\,g^{hl}\,(g_{ij}\,\tilde{R}_{[lk]}-g_{ik}\,\tilde{R}_{[lj]})$$
$$=W^h_{ijk}-\frac{2}{n(n+1)}\,\delta^h_i\,R_{kj}+\frac{2}{n(n^2-1)}\,(\delta^h_k\,R_{[ij]}-\delta^h_j\,R_{[ik]})-\frac{1}{n}\,g^{hl}\,(g_{ij}\,R_{[lk]}-g_{ik}\,R_{[lj]}) \tag{3.35}$$

and setting

$$V^h_{ijk}=W^h_{ijk}-\frac{2}{n(n+1)}\,\delta^h_i\,R_{kj}+\frac{2}{n(n^2-1)}\,(\delta^h_k\,R_{[ij]}-\delta^h_j\,R_{[ik]})-\frac{1}{n}\,g^{hl}\,(g_{ij}\,R_{[lk]}-g_{ik}\,R_{[lj]}) \tag{3.36}$$

we find

$$V^h_{ijk}=\tilde{V}^h_{ijk}. \tag{3.37}$$

with weight $\{0\}$ for a conformal mapping preserving the Einstein tensor.

Lemma 3.6 The conformal curvature tensor ${C^*}^h_{ijk}$ and the concircular curvature tensor Z^h_{ijk} of $W_n(g,T)$ are related by

$${C^*}^h_{ijk}=Z^h_{ijk}-\frac{2}{n}\,\delta^h_i\,Z_{[kj]}-\frac{1}{2n(n-1)}\left[\delta^h_k\,F_{ij}-\delta^h_j\,F_{ik}+g^{hl}\,g_{ij}\,F_{lk}-g^{hl}\,g_{ik}\,F_{lj}\right], \tag{3.38}$$

where $F_{ij}=n\,Z_{ij}-(n-2)\,Z_{[ij]}$.

Proof: By considering the steps applied in Lemma 2.2, it can be easily seen that (3.38) is satisfied.

Theorem 3.7 Let $W_n(g,T)$ be a conformally flat Weyl manifold, which is related to a Weyl manifold $\tilde{W}_n(\tilde{g},\tilde{T})$ by the conformal mapping preserving the Einstein tensor. If the Weyl manifold $W_n(g,T)$ is a concircularly flat Weyl manifold, then it reduces to a Riemannian manifold.

Proof: For a conformally flat Weyl manifold $W_n(g,T)$, the conformal curvature tensor C^{*h}_{ijk} of $W_n(g,T)$ is zero.

Suppose that $W_n(g,T)$ is a concircularly flat Weyl manifold. Then the concircularly curvature tensor Z^h_{ijk} of $W_n(g,T)$ is zero. Hence, from (3.38) we have

$$\begin{aligned}\frac{2}{n}Z_{[kj]} \quad - \quad & \frac{1}{2n(n-1)}\left[\delta^h_k\left(nZ_{ij}-(n-2)Z_{[ij]}\right)-\delta^h_j\left(nZ_{ik}-(n-2)Z_{[ik]}\right)\right.\\ & +g^{hl}g_{ij}\left(nZ_{lk}-(n-2)Z_{[lk]}\right)\\ & \left.-g^{hl}g_{ik}\left(nZ_{lj}-(n-2)Z_{[lj]}\right)\right]=0. \qquad \textbf{(3.39)}\end{aligned}$$

By contracting (3.39) with respect to h and k we obtain

$$Z_{[kj]}=R_{[kj]}=n\nabla_{[k}T_{j]}=0, \qquad \textbf{(3.40)}$$

from which it follows that

$$\nabla_k T_j-\nabla_j T_k=\partial_k T_j-\Gamma^m_{jk}T_m-\partial_j T_k+\Gamma^m_{kj}T_m=0, \qquad \textbf{(3.41)}$$

or,

$$\partial_k T_j=\partial_j T_k, \qquad \textbf{(3.42)}$$

which means T is a gradient, in this case the manifold $W_n(g,T)$ reduces to a Riemannian manifold. Thus, theorem is proved.

Theorem 3.8 Let $\tau: W_n(g,T)\to\tilde{W}_n(\tilde{g},\tilde{T})$ be a conformal mapping preserving the Einstein tensor. The concircular curvature tensor or the projective curvature tensor of the Weyl manifold $W_n(g,T)$ is preserved if and only if the covector field P is a gradient.

Proof: Assume that τ be a conformal mapping preserving the Einstein tensor. In the Theorem 2.1 we proved that, when the concircular curvature tensor or the projective curvature tensor of the Weyl manifold $W_n(g,T)$ preserved, then the covector field P is a gradient. But if the covector field P is a gradient, the concircular and the projective curvature tensors can not be invariant for a conformal mapping in the general case. So, in this section we consider a conformal mapping preserving the Einstein tensor. Then we have

$$P_{(ij)} = \frac{g_{ij}\,(\tilde{R}-R)}{2n(n-1)}. \tag{3.43}$$

Under this condition X^h_{ijk} and P^h_{ijk} reduces to

$$\begin{aligned} X^h_{ijk} \;=\;& 2\,\delta^h_i\,P_{[kj]} + \delta^h_k\,P_{[ij]} - \delta^h_j\,P_{[ik]} \\ & - g_{ik}\,g^{hm}\,P_{[mj]} + g_{ij}\,g^{hm}\,P_{[mk]} \\ & - \frac{2}{n}\,g^{mh}\,P_{[mh]}\,(\delta^h_k\,g_{ij} - \delta^h_j\,g_{ik}) \end{aligned} \tag{3.44}$$

and

$$\begin{aligned} P^h_{ijk} \;=\;& \frac{2}{(n+1)}\,\delta^h_i\,P_{[kj]} + \frac{1}{(n-1)}\left[\delta^h_k\,P_{[ij]} - \delta^h_j\,P_{[ik]}\right] \\ & - g_{ik}\,g^{hl}\,P_{[lj]} + g_{ij}\,g^{hl}\,P_{lk}, \end{aligned} \tag{3.45}$$

respectively.

It can be easily seen that, the condition (3.43) implies that

$$X^h_{ijk} = 0 \quad and \quad P^h_{ijk} = 0 \tag{3.46}$$

or, we have

$$\tilde{Z}^h_{ijk} = Z^h_{ijk} \quad and \quad \tilde{W}^h_{ijk} = W^h_{ijk}. \tag{3.47}$$

Thus, concerning the invariance of the concircular or the projective curvature tensors of $W_n(g,T)$ and combining the Theorem 2.1 and the Theorem 3.5 we can state the following corollary.

Corollary 3.9 Let $\tau : W_n(g,T) \to \tilde{W}_n(\tilde{g},\tilde{T})$ be a conformal transformation preserving the Einstein tensor. Then the following cases are equivalent:

i. The concircular curvature tensor is an invariant.
ii. The covector field of the mapping is a locally gradient.
iii. The projective curvature tensor is an invariant.

Theorem 3.10 Let $\tau : W_n(g,T) \to \tilde{W}_n(\tilde{g},\tilde{T})$, be a conformal mapping preserving the Einstein tensor of the Weyl manifold $W_n(g,T)$. If the concircular or the projective curvature tensors are preserved by τ then the scalar curvatures R and $\tilde{R}$ of the Weyl manifolds $W_n(g,T)$ and $\tilde{W}_n(\tilde{g},\tilde{T})$ are related by

$$\tilde{R} = R + 2(n-1)\left[\Delta f + \frac{(n-2)}{2}(|\nabla f|^2 - 2g(T,\nabla f))\right], \quad (n>2) \qquad \textbf{(3.48)}$$

where $f \in C^2(W_n)$ and $|\nabla f|$ denotes the length of the ∇f and Δf is the laplacian of f.

Proof: Suppose that $\tilde{Z}^h_{ijk} = Z^h_{ijk}$ or $\tilde{W}^h_{ijk} = W^h_{ijk}$. From Corollary 3.9 we have

$$P = \nabla f \qquad \textbf{(3.49)}$$

for any scalar $f \in C^2(W_n)$.

Transvection (2.12) with g^{ij} gives

$$P_{ij}g^{ij} = (\dot{\nabla}_j P_i)g^{ij} - P_i P_j g^{ij} + \frac{n}{2}g^{mh}P_m P_h. \qquad \textbf{(3.50)}$$

By using (3.43) and setting

$$
\begin{aligned}
g^{ij}\dot{\nabla}_j P_i &= \dot{\nabla}_j P^j \\
&= \nabla_j P^j + 2T_j P^j \\
&= P^j_j + (2-n)\,T_k P^k \\
&= \nabla f - (n-2)\,g(T,\nabla f)
\end{aligned} \tag{3.51}
$$

we obtain

$$
\frac{\tilde{R}-R}{2(n-1)} = \Delta f + \frac{(n-2)}{2}\,[|\nabla f|^2 - 2g(T,\nabla f)] \tag{3.52}
$$

and solving $\tilde{R}$ we get

$$
\tilde{R} = R + 2(n-1)\left[\Delta f + \frac{(n-2)}{2}(|\nabla f|^2 - 2g(T,\nabla f))\right], \quad (n>2), \tag{3.53}
$$

where $g^{ij}P_i = P^j$, $|\nabla f| = g^{ij}P_iP_j$ and $\dot{\nabla}_j P^j$ denotes the length of P and the generalized divergence of P^j, respectively.

3.3 Isotropic Weyl Manifolds

Let p be any point of $W_n(g,T)$ and $T_p(W_n)$ be the tangent space of W_n. The scalar defined by [12]

$$
K(\Pi) = \frac{R_{ijkl}X^iY^jX^kY^l}{(g_{ik}g_{jl} - g_{il}g_{jk})X^iY^jX^kY^l}, \tag{3.54}
$$

is called the sectional curvature of $W_n(g,T)$ at p with respect to the plane spanned by two linearly independent vectors $X, Y \in T_p(W_n)$, where X^i and Y^i are respectively the components of X and Y. [4]

If, at each point, the sectional curvature K of $W_n(g,T)$ is independent of the 2-plane chosen, then, $W_n(g,T)$ is named as an isotropic manifold.

Lemma 3.11 Suppose that S is any 4-covariant tensor and that X and Y are two arbitrary linearly independent vectors. If, for all X and Y

$$S_{ijlk}X^iY^jX^kY^l = 0, \tag{3.55}$$

then

$$S_{ijkl} + S_{klij} + S_{ilkj} + S_{kjil} = 0, \tag{3.56}$$

where X^i and Y^j are respectively the components of X and Y. [4], [15]

Lemma 3.12 An isotropic Weyl manifold is an Einstein-Weyl manifold with zero scalar curvature.

Proof: Suppose that $W_n(g,T)$ is an isotropic Weyl manifold, then S_{ijkl} can be defined as,

$$S_{ijkl} = R_{ijkl} - R(g_{ik}g_{jk} - g_{il}g_{jk}). \tag{3.57}$$

By considering S_{ijkl}, S_{klij}, S_{ilkj} and S_{kjil} and using Lemma 3.11, we get

$$R_{ijkl} + R_{klij} + R_{ilkj} + R_{kjil} - R(4g_{ik}g_{lj} - 2g_{il}g_{jk} - 2g_{ij}g_{lk}) = 0. \tag{3.58}$$

Transvecting (3.58) by g^{ik} and then using the property $R_{ijkl} = -R_{ijlk}$ of R_{ijkl} we obtain

$$2(R_{jl} + R_{lj}) - 4(n-1)g_{jl}R = 0, \tag{3.59}$$

which implies that

$$R_{(jl)} = \lambda\, g_{jl}, \tag{3.60}$$

where $\lambda = (1-n)R$.

Hence the symmetric part of the Ricci tensor is proportional to the conformal metric tensor g, $W_n(g,T)$ is an Einstein-Weyl manifold and by virtue of (3.60), we have

$$R = 0 \tag{3.61}$$

which completes the proof.

From Lemma 3.12 it is clear that, for an isotropic Weyl manifold the Einstein tensor

$$E_{ij} = 0. \tag{3.62}$$

Thus, we prove

Theorem 3.13 If the Weyl manifold $W_n(g,T)$ is locally conformal to the isotropic Weyl manifold $\tilde{W}_n(\tilde{g},\tilde{T})$ under the mapping preserving the Einstein tensor, then both manifolds are Einstein-Weyl manifolds.

Combining Theorem 3.10 and Theorem 3.13 we can state the following corollary.

Corollary 3.14 Let $\tau : W_n(g,T) \to \tilde{W}_n(\tilde{g},\tilde{T})$, be a conformal mapping preserving the Einstein tensor of the Weyl manifold $W_n(g,T)$. If the concircular or the projective curvature tensors are preserved by τ then the covector field P of the mapping satisfies the following differential equation for any scalar function $f \in C^2(W_n)$

$$\Delta f + \frac{(n-2)}{2}\left[|\nabla f|^2 - 2g(T,\nabla f)\right] = 0. \tag{3.63}$$

Proof: For the Einstein-Weyl manifolds

$$E_{ij} = \tilde{E}_{ij} = 0 \tag{3.64}$$

and

$$R = \tilde{R} = 0. \tag{3.65}$$

Thus, from Theorem 3.10 the function f satisfies the differential equation.

$$\Delta f+\frac{(n-2)}{2}\left[|\nabla f|^2-2g(T,\nabla f)\right]=0. \tag{3.66}$$

4. CONCLUSIONS AND RECOMMENDATIONS

The purpose of the present thesis is to consider the conformal mappings preserving the Einstein tensor of Weyl manifolds. In this manner, in the first chapter, we introduced the Weyl manifolds and some fundamental tensors of the Weyl manifolds.

In the second chapter, we considered the conformal mappings of Weyl manifolds and we found the relations between some basic tensors, under the conformal mapping from a Weyl manifold $W_n(g,T)$ onto another Weyl manifold $\tilde{W}_n(\tilde{g},\tilde{T})$. By using this relations we proved that if a conformal flat Weyl manifold $W_n(g,T)$ is a concircular flat Weyl manifold, then $W_n(g,T)$ is an Einstein-Weyl manifold. After that, we investigated the invariance of the conformal curvature tensor, the concircular curvature tensor and the projective curvature tensor under the conformal mapping of Weyl manifolds and we proved that if the cocircular or the projective curvature tensors are preserved under the conformal mapping of Weyl manifolds, the covector field P is a gradient.

In the third chapter, we obtained the necessary and sufficient condition for a conformal mapping between two Weyl manifolds to preserve the Einstein tensor and we proved that, if a conformal transformation from a Weyl manifold $W_n(g,T)$ onto a flat Weyl manifold $\tilde{W}_n(\tilde{g},\tilde{T})$ preserves the Einstein tensor, then both Weyl manifolds are Einstein-Weyl manifolds. Furthermore, we obtained the new invariants for a conformal mapping preserving the Einstein tensor is preserved. However, we proved that, the a conformal flat Weyl manifold $W_n(g,T)$ which is related to a Weyl manifold $\tilde{W}_n(\tilde{g},\tilde{T})$ by the conformal mapping preserving the Einstein tensor, reduces to Riemannian manifold, when the Weyl manifold $W_n(g,T)$ is a concircular flat Weyl manifold. Also, we obtained the necessary and sufficient condition for the concircular or the projective curvature tensors of $W_n(g,T)$ to be preserved under the conformal mapping preserving the Einstein tensor. Next, by considering isotropic Weyl manifolds, we proved that, if the Weyl manifold $W_n(g,T)$ is locally conformal to the isotropic Weyl manifold $\tilde{W}_n(\tilde{g},\tilde{T})$ under the mapping preserving the Einstein tensor, then both manifolds are Einstein-Weyl manifolds. Finally, we obtained a differential equation of the covector field P is satisfied, when the concircular or the projective curvature tensors are preserved under the conformal mapping preserving the Einstein tensor of Weyl manifolds.

It will be interesting to study in the conformal mappings preserving the Einstein tensor between conformally flat, conformally symmetric, concircularly flat, concircularly symmetric, projectively flat or projectively symmetric Weyl manifolds.

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CURRICULUM VITAE

Name Surname: Merve Gürlek

Place and Date of Birth: Osmangazi/BURSA 22.06.1989

E-Mail:mgurlek@itu.edu.tr , evremsel@gmail.com

B.Sc.: Istanbul University, Faculty of Science, Department of Mathematics

M.Sc.: Istanbul Technical University, Instıtute of Science and Technology, Department of Mathematics, Mathematics Engineering Programme

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