İSTANBUL TECHNICAL UNIVERSITY ★ INSTITUTE OF SCIENCE AND TECHNOLOGY

DEVELOPMENT OF A NEW FUZZY MULTIPLE ATTRIBUTE DECISION MAKING APPROACH AND ITS APPLICATION TO DECISION MAKING IN SHIP DESIGN AND SHIPBUILDING

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YENİ BİR BULANIK ÇOK ÖZ-NİTELİKLİ KARAR VERME TEKNİĞİNİN GELİŞTİRİLMESİ VE GEMİ İNŞAATI VE GEMİ DİZAYNI KARAR VERME PROBLEMLERİNE UYGULANMASI

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LIST OF ABBREVIATIONS

AHP	: Analytic hierarchy process
AM	: Agreement matrix
AMT	: Advanced manufacturing technology
DM	: Decision maker
DWT	: Deadweight ton
ELECTRE	: Elimination and (et) Choice Translating Reality
FDM	: Fuzzy decision making
FMADM	: Fuzzy multiple attribute decision making
FMCDM	: Fuzzy multiple criteria decision making
FMODM	: Fuzzy multiple objective decision making
FST	: Fuzzy set theory
GDM	: Group decision making
IFWA	: Improved fuzzy weighted average
MADM	: Multiple attribute decision making
MAUF	: Multiple attribute utility function
MC	: Manufacturing competence
MCDM	: Multiple criteria decision making
MODM	: Multiple objective decision making
OAR	: Overall alternative rating
OST	: Ordinary set theory
PROMETHEE	: Preference ranking organisation methods for enrichment evaluation
QFD	: Quality function deployment
SAW	: Simple additive weighting
SFOC	: Specific fuel oil consumption
SLOC	: Specific lubrication oil consumption
SMART	: Simple multi-attribute rating technique
TDİ	: Turkish maritime lines
TOPSIS	: Technique for order preference by similarity to ideal solution
WET	: Weighted evaluation technique

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NOMENCLATURE[1]

a_{ij}	: The element of a reciprocal matrix A
A	: Reciprocal matrix
A^*	: Positive-ideal solution
A^-	: Negative-ideal solution
A_i	: ith attribute
$AA(E_i)$	: Average degree of agreement of ith expert
C_i	: ith constraint
C_i^*	: Relative closeness (or similarity)
$CC(E_i)$	: Consensus degree coefficient of ith expert
D	: Decision
E_i	: ith expert
F_i	: Approx. fuzzy number of the fuzzy suitability index of ith alternative
G_i	: ith goal
J_1	: The set of benefit attributes
J_2	: The set of cost attributes
R_i	: Trapezoidal fuzzy number
R_{ij}, r_{ij}	: Performance rating (or performance score)
R_{AG}^{HM}	: Aggregated fuzzy number for homogeneous group of experts
R_{AG}^{HT}	: Aggregated fuzzy number for heterogeneous group of experts
$RA(E_i)$	: Relative degree of agreement of ith expert
S_i^*	: Separation from the positive-ideal solution
S_i^-	: Separation from the negative-ideal solution
$S(R_i, R_j)$	: The degree of agreement (or degree of similarity)
X^*	: The most preferred alternative
X_i	: ith alternative
U_i	: The performance of the ith alternative or fuzzy utility
W	: A set of weights of experts or attributes
w_i	: Weight of ith expert or attribute
$\mu(x)$	: Membership function
$\mu_{U_i}(u_i)$	: Membership function of the utility of the ith alternative
β	: The relaxation factor of the proposed method
λ_{max}	: Maximum eigenvalue of reciprocal matrix A
$\cancel{A}_{ij}$	: Relative suitability rating
Λ_{ij}	: Overall suitability rating

[1] Since there is no unified standards in nomenclature in Chapter 4 and related Appendices, the variables employed by their original contributors have been adopted as they appear in literature. Their definitions have been provided locally.

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SUMMARY

Decision making is the process of determining the best course of action from a finite set of available alternatives. The major concern is that almost all decision problems have multiple, usually conflicting criteria. Research on how to solve such multiple criteria decision making (MCDM) problems has been enormous. These problems are broadly classified into two categories:

- Multiple Attribute Decision Making (MADM) or multiple attribute analysis, and
- Multiple Objective Decision Making (MODM) or multiple criteria optimisation.

MADM is associated with problems whose number of alternatives has been predetermined and the MADM methods are management decision aids in evaluating and/or selecting a desired one from the finite number of alternatives, which are characterised by multiple attributes. The decision maker is to select/prioritise/rank a finite number of courses of action (or alternatives).

On the other hand, MODM is not associated with problems in which the alternatives have been predetermined. The decision maker's primary concern is to design a most promising alternative with respect to limited resources. An MADM problem can be expressed in matrix format as follows:

Attributes	Alternatives			
	X_1	X_2	......	X_N
A_1	R_{11}	R_{12}	......	R_{1N}
A_2	R_{21}	R_{22}	......	R_{2N}
......	......	......	......	......
A_K	R_{K1}	R_{K2}	......	R_{KN}

Where X_i, $i = 1, \ldots, N$ are possible alternatives; A_j, $j = 1, \ldots, K$ are attributes with which alternative performances are measured; R_{ij} is the performance score (or performance rating) of alternative X_i with respect to attribute A_j.

Current shipbuilding MADM situations are characterised by the following interrelated problems:

- Imprecise data,

Most of the real world decision making problems involve vagueness and fuzziness and the decision maker has the difficult task to choose among the many alternatives

and to specify the optimal alternative. In many cases the decision maker (or expert) has inexact information about the alternatives with respect to an attribute. The classical MADM methods cannot effectively handle problems with such imprecise information. It is obvious that the R_{ij} value (or rating) cannot be assessed precisely. The imprecision may come from different sources such as incomplete information, unquantifiable information, or nonobtainable information etc.

- The mixture of fuzzy and crisp data,

In real world decision making problems, decision data of MADM problems are usually fuzzy, crisp, or mixture of them.

- Involvement of multiple decision makers,

Most of the shipbuilding problems involve the work of a team of experts or specialists (technology experts, design engineers, ship owners, etc.) and are focused on an analysis and evaluation of attributes of decision making process.

- Attribute based expert weighting,

In general, the importance of each decision maker against an attribute is not equal. Sometimes there are important experts in decision group, such as the executive manager of a shipyard, or some experts who are more experienced than others, the final decision is influenced by the different importance of each expert.

Hence, a useful decision model is to provide the ability to handle above-mentioned problems.

It is obvious that much knowledge in the real world is fuzzy rather than precise. Decision making is one of the subjects to which Fuzzy Set Theory (FST), which was first introduced by Zadeh to deal with vague, imprecise, and uncertain problems, has been successfully applied to in the recent years. Various approaches to different aspects of decision problems with vague data have been published. It has been proved that FST provides a sophisticated framework for describing and processing uncertain or imprecise information in decision problems.

Fuzzy Multiple Attribute Decision Making (FMADM) methods have been developed to solve MADM problems, which contain fuzzy data. FMADM is a subcategory of Fuzzy Multiple Criteria Decision Making (FMCDM). FMCDM can be classified as Fuzzy Multiple Objective Decision Making (FMODM) and FMADM; the former emphasises on continuous decision making spaces and it mainly deals with multiple objective mathematical programming problems; the latter mainly deals with discrete decision making space problems.

The study of FMADM problems is still in its infancy and still has a lot of room for improvement. After a systematic and critical study of the existing FMADM methods, the drawbacks of them have been assessed from a practical point of view in this research. These drawbacks certainly limit their applicability to real world (shipbuilding) MADM problems.

The objective of this research is to overcome the difficulties found in FMADM methods and to contribute to the development of an MADM method with multiple decision makers, capable of working in a fuzzy environment.

The proposed FMADM method is designed to overcome the aforementioned difficulties so that MADM problems can be meaningfully and efficiently solved in a fuzzy environment. The basic assumption of the proposed method is that the MADM problem may contain fuzzy and crisp data and it may consist of multiple decision maker (or expert) with the difference degree of importance.

The thesis discusses the theoretical background of the proposed method and presents the application of it to two real shipbuilding case studies, demonstrating the versatility and potential of the proposed method for solving FMADM problems.

The proposed method is composed of three major states as described below:

- Rating state,

In the rating state of the proposed method, each expert (or decision maker) gives his/her opinions (or performance ratings) about alternatives with respect to each subjective attribute. The first state aims to convert fuzzy data into standardised positive trapezoidal fuzzy numbers. If the fuzzy data are linguistic terms, they are transformed into fuzzy numbers first by using appropriate conversion scale and then converted to standardised positive trapezoidal fuzzy numbers.

- Attribute based aggregation state,

In the second state, attribute based aggregation method for heterogeneous group of experts is employed. Aggregation is necessary only for subjective attributes. After the weights of attributes and the degree of importance of experts are assigned, under each subjective attribute all performance ratings are aggregated for each alternative.

- Selection state.

In the last state of the proposed approach, all fuzzy elements of the aggregated decision matrix are deffuzzified in the deffuzzification phase. The result of this phase is a decision matrix, which contains only crisp data. Then the alternatives of the problem are ranked by TOPSIS (Technique for Order Preference by Similarity to Ideal Solution), which is a classical MADM method.

In this dissertation, two real case studies are carried out. The first one is a system (propulsion/manoeuvring system) selection under fuzzy environment and the second one is a component (ship main engine) selection under semi-fuzzy environment.

From the work carried out in this thesis, the two main contributions have been reached. They are classified as contributions to "multiple attribute decision making theory" and contributions to "naval architecture" points of views.

Development of a new FMADM method is the first focus and contribution of this dissertation. From the decision theory point of view, proposed method has the following achievements:

- It is an entire MADM model which combines FMADM methodologies with GDM techniques,

- The proposed method is very suitable for solving the multiple attributive GDM problems under fuzzy environment,

- The proposed method enables the researchers to incorporate homo/heterogeneous group of experts with the different degrees of importance into the FMADM models,

- The majority of classical MADM methods are capable of handling large MADM problems. The proposed approach extends that ability to the fuzzy problems with multiple experts domain,

- It is a new FMADM method that is easy to use and to understand, and the algorithm of the proposed approach is also easy to be coded into a computer program due to the stepwise description,

The second concern and contribution of this dissertation is to show the applicability of the proposed method into the naval architecture MADM problems. From the naval architecture point of view, the following can be concluded:

- As illustrated in the real life examples, the proposed method is a generalised model which can be applied to great variety of practical problems encountered in the naval architecture from propulsion/manoeuvring system selection to warship requirements definition,

- As the application grows, the real value of fuzzy decision making tools will find more widespread use, as most of the practical problems from design to production involves the aggregation of rational and fuzzy elements in harmony,

- Such an approach will also assist the use of optimisation by placing them within the correct context in problem solving and hence will avoid sub-system or sub-attribute optimisation problems.

Finally, the proposed method can efficiently help the decision makers and engineers to make decisions in real world. And it can provide a useful way to solve the selection problems in a fuzzy environment. It is a versatile and flexible system, which covers a vast variety of FMADM problems.

This research also concludes by highlighting future directions for research in this area.

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ÖZET

Karar Verme (KV), günlük hayatımızın önemli bir parçasıdır ve sonlu sayıdaki mevcut alternatiflerden en iyisini belirleme prosesidir. Hemen hemen tüm KV problemleri genellikle birbiriyle çatışan kriterlere sahiptir. Bu tür problemler Çok Kriterli Karar Verme (MCDM) teknikleri olarak adlandırılır.

MCDM teknikleri temel olarak:

- Çok Amaçlı Karar Verme (MODM) ve
- Çok Öz-nitelikli Karar Verme (MADM) yaklaşımları

olmak üzere ikiye ayrılır. Aralarındaki fark, MODM teknikleri sistem dizayn ederken MADM teknikleri ise karar vericinin önündeki önceden belirlenmiş alternatifler arasından en iyi alternatifi seçmek için kullanılır.

Bir MADM problemi aşağıdaki matris formatında tanımlanabilir:

Öz-nitelikler	Alternatifler			
	X_1	X_2	......	X_N
A_1	R_{11}	R_{12}	......	R_{1N}
A_2	R_{21}	R_{22}	......	R_{2N}
......	......	......	......	......
A_K	R_{K1}	R_{K2}	......	R_{KN}

Burada X_i ($i = 1, \dots, N$) ile gösterilenler alternatifleri, A_j ($j = 1, \dots, K$) ile gösterilenler öz-nitelikleri ve R_{ij} ile gösterilenler ise performans değerlerini göstermektedir.

Gemi inşaatı MADM problemleri dört ana başlık altında aşağıda verildiği üzere gruplandırılabilir:

- Belirsiz veri,

Gerçek hayattaki KV problemlerinin çoğu belirsizlik ve bulanıklık içerir ve bu durumda karar vericinin en iyi alternatifi seçmesi zorlaşır. Çoğu zaman uzman (veya karar verici), alternatifleri herhangi bir öz-niteliğe göre değerlendirirken eksik veriyle hareket etmek durumundadır. Klasik MADM teknikleri bu tür problemlerin çözümünde yetersiz kalmaktadır. Bu tür problemlerdeki belirsizlik kaynakları,

tamamlanmamış bilgi, nümerik olarak ölçülemeyen bilgi veya elde edilemeyen bilgi vs. olabilir.

- Bulanık ve deterministik verinin karışımı,

Gerçek hayattaki MADM problemlerinin karar verisi genellikle bulanık, deterministik veya her ikisinin karışımıdır.

- Birden fazla karar vericinin probleme katılımı,

Gemi inşaatı KV problemlerinin çoğu genellikle uzmanların, dizayn mühendislerinin, armatörlerin vs. katılımıyla gerçekleşen bir takım çalışması gerektirir.

- Öz-nitelik bazlı uzman ağırlıklandırma,

Genellikle KV problemlerinde uzman ağırlıkları eşit değildir. Bazen bir tersane müdürü bir problemde işletme müdüründen daha deneyimliyken, bir başka problemde daha az deneyimli olabilir. Bu gibi durumlarda uzman ağırlıklarının öz-nitelik bazında model içerisine katılması gerekir.

Böylece, efektif ve yararlı bir KV modelinin yukarıda bahsedilen problemleri çözebilecek bir kabiliyete sahip olması gerekmektedir.

Zadeh 1965 yılında Bulanık Küme Teorisi (FST)'nin temellerini atmış ve ilk olarak Bellman ve Zadeh KV problemlerinin çözümünde FST'yi kullanmışlardır. KV bilimi, son yıllarda FST'nin başarıyla uygulandığı alanlardan biridir ve literatürde birçok bulanık MADM yöntemi ortaya atılmıştır. Kanıtlanmıştırki, FST KV problemlerinin belirsiz ve bulanık verisinin tanımlanması ve proses edilmesinde çok önemli bir teoridir.

Bulanık Çok Öz-nitelikli Karar Verme (FMADM) teknikleri, bulanık veri içeren MADM yöntemlerinin çözümü için geliştirilmiş tekniklerdir ve Bulanık Çok Kriterli Karar Verme (FMCDM) yaklaşımlarının bir alt sınıfıdırlar. FMCDM yöntemleri, Bulanık Çok Amaçlı Karar Verme (FMODM) ve FMADM teknikleri olmak üzere ikiye ayrılırlar.

FMADM çalışmaları daha çok yenidir ve gelişimleri için açılmamış birçok kapı mevcuttur. Bu çalışmada, literatürde mevcut olan FMADM yöntemleri sistematik olarak incelenerek dezavantajları belirlenmiştir. Bulunan bu dezavantajlar, onları gerçek hayattaki (gemi inşaatı) problemlere uygulanmalarını önemli ölçüde kısıtlamaktadır.

Çalışmanın amacı, yukarıda bahsedilen FMADM tekniklerindeki zorlukların üstesinden gelinmesi ve çok karar vericili yeni bir FMADM yöntemi geliştirilmesi olarak belirlenmiştir.

Bu çalışmada, önerilen FMADM yaklaşımının teorik altyapısı verilerek yöntemin iki adet gemi inşaatı KV problemine uygulaması gösterilmektedir.

Önerilen FMADM tekniği, aşağıda verilen üç adet ana safha içermektedir:

- Performans değerlendirme safhası,

Bu safhada, herbir uzman KV probleminin herbir alternatifini herbir öz-nitelik için performans açısından değerlendirmek suretiyle kendi görüşlerini belirtir. Safhanın amacı, bulanık halde olan görüşlerin standartlaştırılmış pozitif yamuk bulanık sayılarına dönüştürülmesidir.

- Öz-nitelik bazlı toplama safhası,

Farklı ağırlıktaki uzman grubu (heterojen uzman grubu) için öz-nitelik bazlı bir toplama yöntemi kullanılmaktadır. Toplama sadece subjektif öz-nitelikler için gereklidir. Öz-nitelik ve uzmanların ağırlıkları belirlendikten sonra bu safhada, herbir öz-nitelik bazında herbir alternatif için toplama yapılır.

- Seçim safhası.

Önerilen yöntemin son safhasında, toplanmış karar matrisinin tüm bulanık elemanları deterministik hale getirilir. Bunun sonucu, tüm elemanları deterministik olan bir karar matrisidir. Daha sonra, çok sıkça kullanılan klasik bir MADM yöntemi olan TOPSIS, alternatiflerin sıralamasını bulmak için kullanılır.

Vaka çalışması olarak iki adet gemi inşaatı vakası seçilmiştir. Bunlardan birincisi, bulanık bir ortamda sistem (sevk/manevra sistemi) seçimi, ikincisi ise yarı-bulanık bir ortamda eleman (gemi ana makine) seçimidir.

Geliştirilen yöntemin uygulama sonuçları itibariyle, MADM karar teorisi ve gemi inşaatı mühendisliği olmak üzere iki önemli alana katkı sağladığı görülmüştür.

Yeni bir FMADM yönteminin geliştirilmesi bu tezin ilk odak noktası ve katkısı olmuştur. Karar teorisi açısından, önerilen yöntem aşağıda verilen başarımları kazanmıştır:

- Geliştirilen yöntem, FMADM metodolojileriyle Grup Karar Verme (GDM) tekniklerini birleştiren bir yöntemdir,

- Geliştirilen yöntem, bulanık bir ortamda çok öz-nitelikli GDM problemlerinin çözümü için çok uygun bir yöntemdir,

- Yöntem, farklı ağırlıktaki uzman grubunun FMADM KV modeli içerisine katılmasına imkan veren bir yöntemdir,

- MADM tekniklerinin birçoğu, büyük boyuttaki KV problemlerinin çözümü için geliştirilmişlerdir. Geliştirilen yöntem bu özelliği, çok karar vericili bulanık KV yöntemi boyutuna çekmiştir.

- Geliştirilen yöntem, kullanımı ve anlaşılması çok kolay bir yöntem olup bilgisayarda kodlanması da, adımlar halinde tanımlanmış algoritması sayesinde oldukça basittir.

Çalışmanın ikinci katkısı, geliştirilen yöntemin gemi inşaatı MADM problemlerine uygulanabilirliğini göstermektir. Gemi inşaat mühendisliği açısından, elde edilen en önemli başarım aşağıda verilmektedir:

- Vaka çalışmalarından görülebileceği üzere, geliştirilen yöntem, savaş gemisi gereksinim tanımından sevk/manevra sistemi seçimine kadar olan geniş bir alanda gemi inşaatında pratikte karşılaşılan problemlerin çözümünde kullanılabilecek genelleştirilmiş bir modeldir,

Son olarak, önerilen yöntem, bulanık ortamda çok karar vericili MADM problemlerinin çözümünde kullanılabilecek çok yönlü ve esnek bir yöntemdir. Bu yönüyle, karar vericilere ve mühendislere KV problemlerinde önemli bir destek sağlayacağı gözükmektedir.

Araştırma ayrıca, bu alanda gelecekte yapılabilecek çalışmalara da ışık tutacak yönleri belirlemiştir.

1. INTRODUCTION

A decision maker working in the areas of engineering or social sciences is quite often faced with the problem of selecting an alternative from a given set of finite number of alternatives. The chosen alternative is an optimal or a compromise option that meets certain predefined objectives/goals.

Due to the categorical classification of variable types in engineering, there are basically two types of approaches for tackling any decision problem : a deterministic and a stochastic (or probabilistic) approach. In the deterministic approach, the parameters and the constraints are precisely known. In the stochastic approach, the parameters and the constraints are modelled as random variables and are not crisp. In this case, the decisions are taken under risk, and results are given with confidence levels.

Advancements in science and technology have made engineering and social problems quite complex. Many of our activities and the systems with which we deal each day cannot be modelled easily with the known classical tools and methods available in the aforementioned categories (deterministic or probabilistic). These conventional methods, both deterministic and random processes, tend to be less effective in conveying the imprecision and vagueness characteristics.

In fact, the human brain possesses some special characteristics that enable it to learn and reason in a fuzzy environment. It has the ability to arrive at decisions based on imprecise qualitative data, in contrast to formal and classical mathematical methods based on quantitative data. This has led to the development of Fuzzy Set Theory (FST) by Zadeh (1965), who proposed that the key elements in human thinking are not numbers but labels of fuzzy sets.

FST is a powerful tool to handle imprecise data. Fuzzy expressions are more natural for humans than rigid mathematical rules and equations. The fuzzy reasoning system makes no global assumptions about the data.

Zadeh (1965) formulated the initial statement of FST. Since then, this mathematical discipline has gone through substantial theoretical development. Correspondingly, there has been a proliferation of applications of this basic mathematical framework to a variety of fields.

Ordinary Set Theory (OST) principles underlie modern mathematics. Fundamental to basic set theory is the notion that an item is either a member or is not a member of a set. However, the fact is that in the real world, membership in a set is not always so well-defined. FST is based on recognition that certain sets imprecise boundaries. Fuzzy sets are those ill-specified and nondistinct collections of objects with unsharp boundaries in that the transition from membership to nonmembership in a subset of a reference set is gradual rather than abrupt.

Typically, we speak of tall men or expensive homes. Membership in such sets or classes of objects is not characterised by either/or, but are sets in that membership can be adequately considered in terms of degrees. A fuzzy set is characterised by a membership function, defined as a real number in the interval [0, 1]. For example, a membership measure $\mu_A(x_i) = 0.8$ suggests that member x_i of set X is a member of the fuzzy set A to a degree 0.8 on a scale where zero is no membership and one is complete membership. It should be clear that FST can be reduced to OST by constraining membership to the extremes of the range 0 to 1.

Before continuing, a fundamental clarification should be made that concerns how the imprecision of FST (or possibility theory) differs from the imprecision dealt with by probability theory. Basically, the difference is that probability theory deals with randomness of future events, whereas possibility theory deals with the imprecision of current or past events. Randomness deals with the uncertainty regarding the occurrence or non-occurrence of some event, while the imprecision of fuzzy sets deals with the membership or nonmembership of an object in a set with imprecise boundaries.

A typical probabilistic statement is "There is a 10 percent chance that the next person to enter the room will be over 1.70 m. tall". A typical possibilistic statement is "Aykut is tall". The probabilistic statement refers to a precise set of people over 1.70 m. The imprecision in this case has to do with the event relating to the next person in the room. The fuzzy statement is not imprecise about the event in question; it is "Aykut". The imprecision here has to do with the vagueness of the concept of "tall" itself.

In the study of the decision-making problem, the multiple objective, multiple attribute and multiple criteria decision models are useful. The multiple objective decision-making (MODM) consists of a set of goals that generally cannot be satisfied simultaneously. It also usually involves solving a problem on continuous space via a mathematical programming model, while multiple attribute decision-making

(MADM) deals with the problem of choosing an alternative from a set of candidate alternatives which are characterised in terms of some attributes.

In an MADM problem, a set of alternatives (or courses of action), or decisions, are considered. The performance rating (or performance score) of each alternative on each of a given set of attributes is the basis for final decision. These performance values on the attributes involved for each alternative are aggregated to form a preference rating and the alternative with the highest preference, indicating the best overall performance, is identified.

Multiple attribute, multiple alternative decision problems, which require the ranking of a set of alternatives, are of importance in a variety of fields including engineering, economics, etc. It is assumed that each alternative can be characterised by a set of attributes, that associated with each attribute is a weight which is a measure of its relative importance, and that each alternative can be rated with respect to each attribute. Frequently, such decision problems are encountered in the presence of uncertainty.

For a review of the various MADM methods the reader is referred to, for example, Hwang and Yoon (1981), Chen and Hwang (1992) and Yoon and Hwang (1995).

Real world decision making problems are defined in a domain which is shown in Figure 1.1.

Since the subjectivity, imprecision and vagueness in the estimates of a given quantity enter into MADM problems, FST, is helpful in dealing with the fuzziness of human judgement quantitatively.

FMADM methods have been developed due to the lack of precision in assessing the performance ratings of alternatives with respect to an attribute.

1.1 Statement of the Problem

The most of the shipbuilding MADM situations have the common problems as follows:

- Imprecise data,

In the real world, decisions are made every day and humans deal naturally with subjective or fuzzy information.

The most of the real world decision making problems involve vagueness and fuzziness and the decision maker has the difficult task to choose among the many alternatives and to specify the optimal alternative. The uncertainty of subjective judgement is present when this process is carried out.

Typically, military command and control, by its nature, deals extensively with imprecise knowledge and subjective goals. The state of a battlefield situation is usually not well known. There is never enough information or time to completely analyse a situation in order to make a decision. Yet humans tend to perform reasonably well under such circumstances, arriving at good decisions in spite of ambiguity and confusion.

The imprecision comes from a variety of sources such as i) Unquantifiable information, ii) Incomplete information, iii) Nonobtainable information, iv) Partial ignorance (Chen and Hwang (1992)). In many cases the decision maker (or expert) has inexact information about the alternatives with respect to an attribute. The classical MADM methods cannot effectively handle problems with such imprecise information.

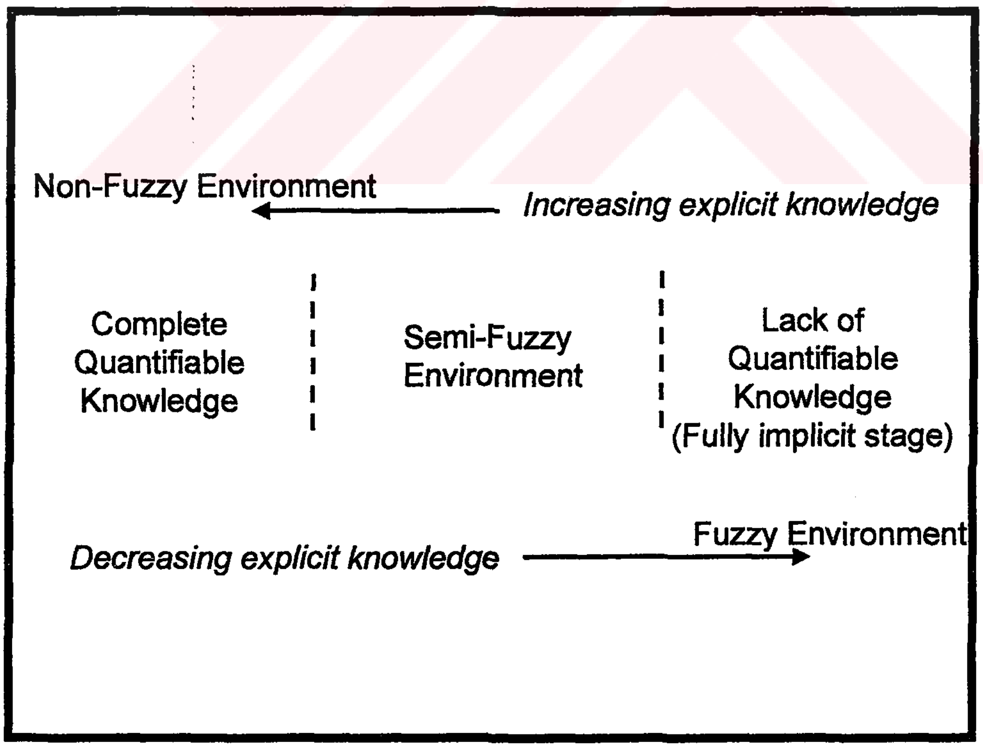

Figure 1.1 Real world decision making problems domain

- The mixture of fuzzy and crisp data,

One of the most crucial problems in many decision making methods is the precise evaluation of the pertinent data. Very often in real life decision making applications data are imprecise and fuzzy. For example, how can one quantify statements such as "What is the performance rating of the second propulsion system alternative in terms of manoeuvrability attribute?" A decision maker may encounter difficulty in quantifying and processing such linguistic statements. Therefore, it is desirable to develop decision making methods which use fuzzy data.

In real world decision making problems, decision data of MADM problems are usually fuzzy, crisp, stochastic[2] or mixture of them.

Hence, a useful decision model is to provide the ability to handle both fuzzy and crisp data.

- Involvement of multiple decision makers,

The most of the shipbuilding problems involve the work of a team of experts or specialists (technology experts, design engineers, shipowners, etc.) and are focused on an analysis and evaluation of attributes of decision making process. Consequently, they are, in fact, cases of multiple attribute based group decision making problems.

Human opinions often conflict because of group decision making in fuzzy environment. The important issue of fuzzy group decision making is to aggregate conflicting opinions.

- Attribute based expert weighting,

In general, the importance of each decision maker or expert against an attribute are not equal. Sometimes there are important experts in decision group, such as the executive manager of a shipyard, or some experts who are more experienced than others, the final decision is influenced by the different importance of each expert.

Therefore, a good method of aggregating multiple expert opinions must consider the degree of importance of each expert in the aggregation procedure.

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[2] In most cases stochastic variables are reduced to crisp values through the use of expected value, significant value and design equivalent or can be converted to fuzzy variables by the choice of appropriate scales.

1.2 Objectives and Scope of the Research

This research is devoted to solve the aforementioned problems. Research objectives have been identified at two stage. In the first stage, it was assumed that existing FMADM (Fuzzy Multiple Attribute Decision Making) methods in the literature were adequate to handle the types of problems mentioned before. Our review revealed that this assumption was incorrect.

In the second stage, therefore, research objectives have been changed and have been redefined as follows:

The aim of this study is to develop a method which combines the FMADM concepts with Group Decision Making (GDM) methodologies and to apply it in typical shipbuilding decision making problems.

To fulfil this aim the following objectives are identified:

i) To review critically the existing FMADM methods in order to identify their major advantages and shortcomings,

ii) To propose a new method which combines FMADM methodologies with GDM Techniques,

iii) To conduct real case studies to illustrate how the proposed method can be applied in shipbuilding decision problems.

iv) To make recommendations for the future research.

1.3 Thesis Organisation

The thesis consists of seven chapters and four appendices.

The first chapter is an introduction to explain the background of the MADM problems and to explain the objectives and the motivation of the research. The problem statements, the goal and the objectives of the research are given in this chapter.

A review of relevant literature is summarised in the second chapter. Chapter 2 discusses and reviews the literature and background knowledge of this research. Several concepts such as Decision Making, Fuzzy Ranking methods, and FMADM methods are described and some previous applications of MADM and fuzzy decision making (FDM) approaches are also given in this chapter.

The third chapter introduces the concepts of Multiple Objective Decision Making (MODM) and Multiple Attribute Decision Making (MADM) problems and presents the differences between MODM and MADM problems. This chapter also discusses the basic elements of MADM problems and a classification of the existing MADM methods is given. Fuzzy Set Theory (FST) needs for MADM problems is discussed in the last section of the chapter.

In Chapter 4, most of the existing FMADM methods in the literature are given and described in more detail. The classification of them is also given in this chapter.

The conceptual model of the proposed approach is given in the fifth chapter. FMADM methods, described in the fourth chapter, are reviewed and the drawbacks of them are presented in this chapter. Proposed method's states are explained and an illustrative selection problem is shown to demonstrate the computational process of the proposed method.

In Chapter 6, two real shipbuilding case studies are used to validate the proposed methodology and demonstrate its application.

Finally, Chapter 7 summarises this research and suggests future directions for further research.

Basic concepts of fuzzy sets, linguistic variables and linguistic hedges are explained with examples in Appendix A. Some special techniques and algorithms used in FMADM methods, discussed in Chapter 4, are given in Appendix B. Detailed aggregation state calculations and their figures of the first and second cases are given in Appendix C and Appendix D respectively.

2 LITERATURE REVIEW

The literature review is carried out for establishing a background for the proposed research. The review may be classified into six groups as follows.

The first attempt at applying FST (Fuzzy Set Theory) to multiple attribute decision analysis was conducted by Bellman and Zadeh (1970), who outlined one possible route toward fuzzy decision making. Another important approach was by Zadeh (1973) who outlined the possibility of using the max-min rule to combine relational matrices. Kickert (1978) summarised FST applications in relation to MADM problems. An in depth summary of FST and its application was completed by Dubois and Prade (1980). They classified the fuzzy MADM into a fuzzy rating phase, in which the fuzzy utility of each alternative was obtained, and a fuzzy ranking phase, in which the fuzzy utilities were compared. In addition, both fuzziness and randomness were accounted for as one of the possible fuzzy applications to decision analysis.

2.1 Fuzzy Set Theory

The FST was introduced by Zadeh (1965) to deal with vague, imprecise, and uncertain problems. The lack of data is the reason for uncertainty in many daily problems.

FST which was first proposed by Zadeh (1965) has been useful for dealing with vagueness or ambiguity, and has made remarkable development.

Some basic definitions of fuzzy sets (Zadeh (1972), (1973), (1975a), (1975b), Kaufmann (1975), Dubois and Prade (1980), Kandel (1986), and Zimmermann (1987), (1991)), linguistic variables and hedges (Zadeh (1972), Zimmermann (1987)), fuzzy numbers (Kandel (1986), Kaufmann and Gupta (1991)), and the traditional fuzzy arithmetic operations of fuzzy numbers (Schmucker (1983)) have been reviewed. As noted by Dubois and Prade (1980), the extension principle was introduced by Zadeh.

Basic concepts of fuzzy sets, linguistic variables and linguistic hedges are explained with examples in Appendix A.

The concept of using fuzzy sets in the formulation of decision problems under certainty appeared in the work of Bellman and Zadeh (1970) and also in Fung and Fu (1974).

A number of authors (Buckley (1985a), Kacprzyk (1986), Juan (1988), Roubens (1989), Zahariev (1987), Li (1999)) have provided interesting results on group decision making with the help of FST. The representations of the fuzzy individual preferences provided by those authors can be classified as the following three levels : fuzzy choice set, fuzzy preference relation, and fuzzy utility function. To obtain the fuzzy group preference from the fuzzy individual preferences, different aggregating methods have been used on the basis of consensus pooling, satisfying a number of conditions, such as reciprocity and max-min transitivity for group as well as individual preferences.

Contributions of FST to group decision making have been made in several directions. Blin (1974) proposed to represent a relative group preference as a fuzzy preference matrix from individual preferences. Fung and Fu (1975) discussed the aggregation of individual preferences into a group preference from an axiomatic point of view. Orlovsky (1978) introduced two types of linearity of a fuzzy relation and studied the equivalence of crisp nondominated alternatives. He showed that crisp nondominated solutions to the decision making problem exist if the original fuzzy relation satisfies some topological requirements. Kuz'min and Ovchinnikov (1980) introduced an appropriate distance in the space of fuzzy relation matrices and studied GDM on its basis. Nurmi (1981) developed ways of determining the best alternative(s) on the basis of a fuzzy preference relation and of deriving a (nonfuzzy) group preference relation from fuzzy individual preference relations.

2.2 Decision Making

Decision making has been studied in many domains. Classical decision making theory generally deals with a set of alternative actions comprising the decision space, a set of outcomes comprising the outcome space, a relation indicating the outcome to be expected from each alternative action, and an objective (utility) function which orders these outcomes according to their desirability. Several approaches exist for solving decision making problems. These include

1. Statistical decision theory,

2. Mathematical programming (linear or non-linear),

3. Multiple criteria decision making theory and,

4. Decision making in a fuzzy environment in which the outcome values as well as input parameters are fuzzy.

A decision is said to be made under certainty when the outcome for each action can be determined and ordered precisely. On the contrary, a decision is said to be made under uncertainty (risk) when the available information for the outcome is incomplete or imprecise. It is worth noting that real world decision making problems usually comprise fuzzy information, and the decision must thus be made under uncertainty. As Fang and Chen (1990) have stated, uncertainties are better handled by fuzzy sets and fuzzy arithmetic. To deal with real world problems, the fourth approach (decision making in a fuzzy environment) stated above embedded in FST has proven to have the ability to handle uncertain information and is more appropriate than the other approaches.

In the field of decision making, FST is very helpful in dealing with fuzziness of human judgement quantitatively, and a number of results have been published. The study of decision making in a fuzzy environment has gained much interest in the past few years. Briefly, fuzzy decision making problems may be subdivided into three aspects :

1. Converting linguistic terms to fuzzy sets,

2. Employing FMADM methods,

3. Ranking techniques.

Since converting between linguistic terms and fuzzy sets varies from circumstance to circumstance, existing work on this topic is few and appear to be quite arbitrary. As a result, the review in the following sections is restricted to the FMADM methods and ranking techniques only.

2.2.1 Classical Multiple Attribute Decision Making (MADM) Methods

Chen and Hwang (1992) classified fourteen classical MADM methods as follows :

1. Dominance (Hwang and Yoon (1981)),

2. Maximin (MacCrimmon (1968), Hwang and Yoon (1981)),

3. Maximax (Hwang and Yoon (1981)),

4. Conjunctive method (or satisfying method) (Hwang and Yoon (1981)),
5. Disjunctive method (Hwang and Yoon (1981)),
6. Lexicographic method (Hwang and Yoon (1981)),
7. Lexicographic semiorder (Hwang and Yoon (1981)),
8. Elimination by aspects (Hwang and Yoon (1981)),
9. Linear assignment method (Hwang and Yoon (1981)),
10. SAW (Simple Additive Weighting) method (Churchman and Ackoff (1954), Hwang and Yoon (1981)),
11. ELECTRE (Elimination and (et) Choice Translating Reality) (Roy (1971), Hwang and Yoon (1981)),
12. TOPSIS (Technique for Order Preference by Similarity to Ideal Solution) (Hwang and Yoon (1981)),
13. Weighted product,
14. Distance from target.

Some of them are simple and easy to use and understand. However, there are some disadvantages, such as noncompensatory trade-off among attributes. Therefore, the most widely used method TOPSIS is used to apply in this research (see Section 5.2.3.2.1).

2.3 Fuzzy Ranking Methods

Fuzzy ranking techniques have been studied extensively. Basically, the techniques are roughly divided into four categories : fuzzy preference relation, fuzzy mean and spread, centroid index, and linguistic expression. Techniques of applying fuzzy preference relation include those developed by Baas and Kwakernaak (1977), Baldwin and Guild (1979), Yager (1978), and Buckley and Chanas (1989). Fuzzy mean and spread, as proposed by Lee and Li (1988), uses a generalised mean and standard deviation based on the probability measures of fuzzy events to rank the fuzzy numbers. Centroid index measures the geometric centre of the fuzzy set. This technique has been heavily utilised to defuzzify/decode an inferred control action as studied in fuzzy control. All the ranking techniques mentioned above involve transforming the fuzzy sets or fuzzy numbers into a crisp scale. Some researchers have pointed out that these mathematical ranking procedures may generate counter-intuitive results during the defuzzification process. Therefore, linguistic rating methods should suffer less from this problem. One approach, proposed by Tong and Efstathiou (1982), uses the final fuzzy numbers to generate the dominance set, and based on this dominance set, to carry out rating.

In MADM applications, when the final ratings are fuzzy, it is very difficult to distinguish the best possible course of action from the mediocre ones, or even the worst one. To resolve this problem, many researchers have proposed fuzzy ranking methods that can be used to compare fuzzy numbers. In Table 2.1, twenty eight ranking methods are classified by Chen and Hwang (1992) into four major classes according to the means (or media) of each method used. Included in the chart are the "preference relation" methods, the "fuzzy mean and spread" method, the "fuzzy scoring (or direct comparison)" methods, and "linguistic" methods. Each main class is further divided according to the techniques used. For instance, methods using "degree of optimality" is a subclass of "preference relation", methods using "centroid index" is a subclass of "fuzzy scoring", and methods using "linguistic approximation" is a subclass of "linguistic expression". The following review is a summary of the ranking methods related with this research.

2.4 Fuzzy Multiple Attribute Decision Making (FMADM) Methods

The widely recognised classical work on FMADM was proposed by Baas and Kwakernaak (1977). It is known as the SAW method. Originally, this method was used to solve continuous functions only. Later, Baas and Kwakernaak utilised the α-cut to approximate the utility of the alternatives, $\mu_{U_i}(u_i)$, and thus enabled the computer to solve the problem in a discrete format. Before the SAW method was proposed, the first attempt to solve the FMADM problem using FST was made by Bellman and Zadeh (1970) who defined the fuzzy decision as the fuzzy set D resulting from the intersection of the goals and constraints. That is, given the fuzzy goals, G_i, $i = 1, \dots, m$, and the constraints, C_j, $j = 1, \dots, n$, solution D can be determined using

$$D = G_1(X) \cap G_2(X) \cap \dots\dots \cap G_m(X) \cap C_1(X) \cap C_2(X) \cap \dots\dots \cap C_n(X)$$

Table 2.1 Classification of fuzzy ranking methods by Chen and Hwang (1992)

	Comparison Medium	Technique involved	Approaches
Fuzzy Ranking	Preference Relation	Degree of optimality	Baas and Kwakernaak, Baldwin and Guild, Watson et al.,
		Hamming distance	Yager, Kerre, Nakamura, Kolodzijezyk,
		α-cut	Adamo, Buckley and Chanas, Mabuchi, Huang, Yuan,
		Comparison function	Dubois and Prade, Tsukamoto et al., Delgado et al.,
	Fuzzy mean and spread	Probability distribution	Lee and Li,
	Fuzzy scoring	Proportion to optimal	McCahone,
		Left/right score	Jain, Chen, Kim and Park, Chen et al.,
		Centroid index	Yager, Murakami et al,
		Area measurement	Yager, Choobineh and Li,
	Linguistic expression	Intuition	Efstathiou and Tong,
		Linguistic approximation	Tong and Bonissone,

The best decision is chosen from the maximum of the goals and the constraints. This method is the forerunner of the Maximin method (Hwang and Yoon (1981)). A modification to this method (Bellman and Zadeh) was proposed by Yager (1978). The modification involves utilising a decision matrix to facilitate the decision making process. Mathematically, a simple decision matrix with N alternatives, X_i, and K attribute, Aj, via a set of performance score (rating), R_{ij}, is defined as

Attributes	Alternatives			
	X_1	X_2	……	X_N
A_1	R_{11}	R_{12}	……	R_{1N}
A_2	R_{21}	R_{22}	……	R_{2N}
……	……	……	……	……
A_K	R_{K1}	R_{K2}	……	R_{KN}

One of the approaches that relies heavily on the decision matrix was proposed by Saaty (1978) and is classified as the Analytic Hierarchy Process (AHP) method. This method uses pairwise comparison between alternatives X_i for each attribute in a hierarchy, and also between attributes. The eigenvector method is used to solve the K+1 positive pairwise comparison matrices with a set of ratios, a_{ij}, representing the relative importance of X_i over X_j. Since this method uses crisp numbers only, Laarhoven and Pedrycz (1983) extended the method using both triangular fuzzy numbers and logarithmic least square method to obtain the fuzzy utilities, U_i. Buckley (1985b) also extended Saaty's method to incorporate fuzzy comparison ratios, and geometric means to derive fuzzy weights and performance scores.

There are two important ways to arrive at the decision outcome – alternative acceptance and alternative ranking. To dichotomise alternatives into acceptable/not acceptable categories, the fuzzy conjunctive/disjunctive method was proposed by Dubois et al. (1988). Since the data and attribute values used in decision making are fuzzy, the match between standard levels and attribute values become vague. To measure the fuzziness, the possibility measure and certainty (necessity) measure were used to compute the degree of matching, and acceptance is determined by the computed values. On the other hand, to compare the final rating of fuzzy numbers or fuzzy sets, fuzzy ranking must be used.

A thorough literature review on the existing methods of FMADM can be found in Chen and Hwang (1992). The application of fuzzy set theory to decision making can be found in (Bellman and Zadeh (1970), Zadeh (1994), Zimmermann (1987)).

All of the FMADM methods in the literature will be discussed in a separate chapter (chapter 4) to criticise them for the proposed research.

2.5 Fuzzy Number Aggregation

To reach a group consensus for aggregating these estimates ratings to a common opinion is an important issue in handling multiple attribute based group decision making problems.

Some researchers (Bardossy et al. (1993), Chen and Lin (1995), Chen et al. (1989), Hsu and Chen (1996), Ishikawa et al. (1993), Kacprzyk and Fedrizzi (1988), Kacprzyk et al. (1992), Lee (1996), Nurmi (1981), Xu and Zhai (1992), Chen (1997)) have focused on the fuzzy opinion aggregation problem in the multiple attribute based group decision making environment to combine the individual opinions of experts, where each expert usually has his/her own opinion or estimated rating under each attribute for each alternative.

Kacprzyk et al. (1992) showed how fuzzy logic with linguistic quantifiers can be used in group decision making. Tanino (1984) discussed some use of fuzzy preference orderings in group decision making. Bardossy et al. (1993) represented expert opinions or imprecise estimates of a physical variable by using fuzzy numbers and proposed five aggregation techniques for combining these fuzzy numbers into a single fuzzy number estimate; namely crisp weighting, fuzzy weighting, minimal fuzzy extension, convex fuzzy extension and mixed linear extension method. The guidelines for the choice of combination technique are also provided. Ishikawa et al. (1993) proposed the max-min Delphi method and fuzzy Delphi method via fuzzy integration. Xu and Zhai (1992) presented extensions of the Analytic Hierarchy Process in a fuzzy environment, where each expert represents his/her subjective judgement by an interval value rating of each attribute for each alternative.

Lee (1996) presented a method for GDM using FST for evaluating the rate of aggregative risk in software development. Nurmi (1981) presented some approaches to collective decision making with fuzzy preference relations. Hsu and Chen (1996) proposed a similarity aggregation method for aggregating individual fuzzy opinions into a group fuzzy consensus opinion, where the estimates of experts are represented by positive trapezoidal fuzzy numbers.

2.6 Typical Published Applications of MADM and FDM Methods

A large amount of literature is available on applications of MADM and Fuzzy Decision Making (FDM) techniques.

Liang and Wang (1991) proposed a decision algorithm to solve the facility site selection problem under fuzzy environment. By utilising this algorithm, the decision makers' fuzzy assessments with various rating attitudes and the trade-off among various selection attribute can be taken into account in the aggregation process. Liang and Wang (1993) also applied this algorithm to the robot selection problem under fuzzy environment. Karsak (1998) proposed a two-phase decision method for robot selection problems. In the first phase of the method, Data Envelopment Analysis is used to determine technically efficient robot alternatives. In the second phase of the method, Liang and Wang's approach is utilised to rank the robots according to both objective and subjective attributes of the problem.

Machacha and Bhattacharya (2000) proposed a system based on fuzzy logic and applied it to the problem of selection for database software packages.

Chen (1994) developed a method for handling multiple attribute fuzzy decision making problems, in which the characteristics of the alternatives are represented by interval-valued fuzzy sets. Chen (1997) presented a new method to solve the tool steel materials selection problem under fuzzy environment, where the importance weights of different criteria and the ratings of various alternatives under different criteria are assessed in linguistic terms represented by fuzzy numbers.

Chang and Chen (1994) proposed a decision algorithm based on the FST and hierarchical structure analysis to solve the technology transfer strategy selection problem, where the linguistic variables and fuzzy numbers are used to aggregate the decision makers' subjective assessment about attribute weightings and appropriateness of alternative transfer strategies versus selection attribute to obtain the final scores called fuzzy appropriateness indices.

Ravi and Reddy (1999) used Yager's FMADM approach to rank both coking and non-coking coals of India for industrial use. In this study, three different kinds of membership functions in conjunction with four kinds of aggregators were used and the results were compared.

Wang (1997) modelled the imprecise preference structure of decision making in conceptual design based on the outranking approach and fuzzy preference relations and a valve selection problem was used to illustrate the concept. Wang (1999) considered the Quality Function Deployment (QFD) planning as a multiple criteria decision making problem and proposed a new fuzzy outranking approach to prioritise design requirements recognised in QFD and used an example of a car design to illustrate the proposed approach. Güngör and Arikan (2000) compared and ranked natural gas, imported coal, and nuclear power plant alternatives in terms of long term Turkish production economy. In this study, they used a fuzzy outranking approach (Wang's approach) for Turkish energy policy planning.

Maeda and Murakami (1988) proposed a new FMADM technique and applied it in a company choice problem under fuzzy environment. This new method's main features are the use of fuzzy connectives to represent the decision maker's preference structure and fuzzy probability to express imprecise and uncertain outcomes.

Perego and Rangone (1998) gave a reference framework for the application of a FMADM approach to Advanced Manufacturing Technologies (AMTs) selection. In particular, the implied conjunction methodology, the fuzzy linguistic model and the fuzzy hierarchical model based on pairwise comparisons were compared with respect to their application to AMTs selection.

Azzone and Rangone (1996) proposed a new framework of Manufacturing Competence (MC) and suggested a consistent measurement framework derived from fuzzy set theory. Their proposed fuzzy approach was applied to measure the MC of a company that operates in the plastic industry.

Ekel (1999) proposed a general approach to solving a wide class of optimisation problems with fuzzy coefficients in objective functions and constraints. This approach has been applied within the context of fuzzy discrete optimisation models.

Yoon and Hwang (1985) employed five developed MADM methods for different versions of manufacturing plant site selection problems. Kirkwood (1982) used MADM methods to evaluate and rank candidate sites for a nuclear power plant as well as water sources.

Jones et al. (1990) developed a multiple attribute value model, adapted from SMART (Simple Multi-Attribute Rating Technique) technique, to study UK energy policy options.

Vlacic et al. (1997) proposed an algorithm which can support the process of GDM relating to industrial automation, especially involving the selection of control and instrumentation equipment.

Tavana et al. (1996) proposed a group decision support system which combines the AHP with Delphi principles and applied it to rank the nurse manager candidates at a hospital in the United States.

Kuei et al. (1994) used AHP and adjusted priority method to evaluate and rank the advanced technologies.

Liberatore (1987) developed an AHP modelling framework for the Research and Development project selection decision and it was linked to a spreadsheet model to assist in the ranking of a large number of project alternatives.

Bard and Sousk (1990) identified the three next generation of rough terrain cargo handlers for the U.S. army and used AHP to rank and select them.

Korpela and Tuominen (1996) demonstrated how AHP could be used for supporting logistics benchmarking and applied AHP for three steps of the logistics generic benchmarking process.

Tadisina et al. (1991) examined the application of AHP for selecting a doctoral programme.

Rangone (1998) discussed the applicability to small and medium sized firms of major MADM methods (scoring method, and AHP) to advanced manufacturing technologies assessment and selection.

Stewart (1991) described the development of a multiple attribute decision support system for the selection of a portfolio of Research and Development projects, which was carried out for a large electricity utility corporation.

Bellehumeur et al. (1997) applied three multiple attribute decision making techniques (weighted sum, ELECTRE, and fuzzy set method) to an environmental (sewage sludge) management problem where various solutions are compared on the basis of several attributes.

Dyk and Smith (1990) developed a new MADM method QualScal and applied it to the problem of selecting an extramural Fisheries R&D portfolio in the Ministry of Agriculture, Fisheries and Food in England.

3. MULTIPLE ATTRIBUTE DECISION MAKING

This chapter is divided into three sections. The first section presents the differences between Multiple Objective Decision Making (MODM) and Multiple Attribute Decision Making (MADM) problems. The second section provides an overview of the underlying concepts and theories of MADM. FST needs for MADM problems are discussed in the third section.

3.1 Multiple Criteria Decision Making

Decision making is the process of determining a best course of action from a set of available alternatives. The major concern is that almost all decision problems have multiple, usually conflicting criteria. Research on how to solve such multiple criteria decision making (MCDM) problems has been enormous. These problems are basicly classified into two categories :

1. Multiple Attribute Decision Making (MADM) or multiple attribute analysis, and

2. Multiple Objective Decision Making (MODM) or multiple criteria optimisation.

MADM is associated with problems whose number of alternatives has been predetermined. The decision maker is to select/prioritise/rank a finite number of alternatives (or courses of action). On the other hand, MODM is not associated with problems in which the alternatives have been predetermined. The decision maker's primary concern is to design the most promising alternative with respect to limited resources. Table 3.1 shows the differences between these two classes.

3.2 Overview of Multiple Attribute Decision Making Problems

Multiple-attribute decision making (MADM) is the study of techniques that can be used by a decision maker to select the best alternative from a finite number of alternatives when faced with conflicting objectives.

In MADM the decision maker evaluates the alternatives based on several attributes which best characterise the alternatives. The decision maker chooses one or more alternatives from the set based on the assessments of the alternatives on the attribute and the relative importance of the attribute in his or her mind. The decision maker determines his or her preference structure and studies the characteristics of the available alternatives to select the suitable ones. An MADM problem with N alternatives and K attributes requires the decision maker to process data of an (NxK) – dimensional space. Thus a huge amount of information is involved in MADM.

Table 3.1 MODM vs. MADM

Criteria	MADM (Attributes)	MODM (Objectives)
Objective	Implicit (ill defined)	Explicit
Attribute	Explicit	Implicit
Constraint	Inactive (incorporated into attributes)	Active
Alternative	Finite number, discrete (prescribed)	Infinite number, continuous (emerging as process goes)
Interaction with decision maker	Not much	Much
Usage	Selection / Evaluation	Design

MADM deals with the problem of choosing an alternative from a set of alternatives which are characterised in terms of their attributes. Usually MADM consists of a single goal, but this may be of two different type.

The first type of goal is to select an alternative from a set of scored ones based on the values and importance of the attributes of each alternative.

The second type of goal is to classify alternatives, using a kind of role model or similar cases. The use of past cases to deduce answers or explanations is a recent field of research, termed case-based reasoning.

MADM is a qualitative approach due to the existence of attribute subjectivity. Both type of goals require information about the preferences among the instances of an attribute and the preferences across the existing attributes. The assessment of these preferences is either provided directly by the decision maker or based on past choices.

MADM refers to making selections among some courses of action in the presence of multiple, usually conflicting, attributes. Problems dealing with MADM are common occurrences in the real world. An MADM problem can be concisely expressed in matrix format as shown in Table 3.2.

Table 3.2 Decision Matrix

Attributes	Alternatives			
	X_1	X_2	......	X_N
A_1	R_{11}	R_{12}	......	R_{1N}
A_2	R_{21}	R_{22}	......	R_{2N}
......	......	......	......	......
A_K	R_{K1}	R_{K2}	......	R_{KN}

Where X_i, i = 1, ... , N are possible course of actions (referred to as alternatives); A_j, j = 1, ... , K are attributes with which alternative performances are measured; R_{ij} is the performance score (or performance rating) of alternative X_i with respect to attribute A_j.

It is obvious that the R_{ij} value (or rating) cannot be assessed precisely. The imprecision may come from different sources such as incomplete information etc.

MADM methods are used for selecting an alternative from a small, explicit list of alternatives, while MODM methods are used for an infinite set of options defined implicitly by the constraints. The various MADM methods have been reviewed extensively by Yoon and Hwang (1995).

3.2.1 Common Characteristics of MADM Problems

Methods and applications of MADM and MODM regarding a single decision maker have been thoroughly and systematically reviewed and classified by Hwang and Yoon (1981), (1995). The MADM problems considered here share the following common characteristics (Chen and Hwang (1992)):

1. Alternatives : A finite number of alternatives, from several to thousands, are to be screened, selected and ranked. Alternatives are mutually exclusive with each other.

2. Attributes : Attributes should provide a means of evaluating the levels of an objective. Each alternative is characterised by a number of attributes. The number of attributes can be very large. A decision maker must generate relevant attributes for each problem setting. Performance measures, components, characteristics are synonyms for attributes.

3. Conflict among attributes : Multiple attributes are usually in conflict with each other. For example, in selecting a ship, the higher voyage range might reduce the cargo carrying capacity.

4. Decision matrix : An MADM problem can be concisely expressed in a matrix format called a decision matrix. The decision matrix is constructed with information on the values of the attributes for alternatives. A decision matrix 'D' is an (KxN) matrix whose elements R_{ij} indicate the performance rating of alternatives i, X_i, with respect to an attribute j, A_j (see Table 3.2).

5. Incommensurable units : Each attribute has a different unit of measurement. In the ship selection case, fuel consumption is expressed in tons per mile, cargo capacity is expressed by cubic feet, cost is indicated by dollars, safety may be indicated in an non-numerical way, etc.

6. Decision weights : Almost all MADM problems require information regarding the relative importance of each attribute. The relative importance is usually given by a set of weights $W=\{w_j \mid j=1, 2, \ldots, K\}$, where K is the number of attributes and weights are generally normalised such that their total sum is equal to one. The assignment of weights plays a key role in the MADM process.

3.2.1.1 Attributes

MADM begins with the generation of attributes that should provide a means of evaluating goal accomplishments. All attributes are not likely to be considered equally important. The role of weight serves to express the importance of each attribute relative to the others. Hence, the assignment of weights plays a key role in the MADM process and may vary from decision maker to decision maker. Weights should reflect the purpose of the evaluation. Direct assignment, Weighted Evaluation Technique (WET), eigenvector method, entropy method, and minimal information method are some of the weighting techniques that are most widely used.

3.2.2 Classification of MADM methods

There has been no single, widely adopted classification of MADM methods. For generic classification of MADM approaches, several schemata have been proposed. For instance, Teghem et al. (1989) used six criteria to categorise MADM methods. Yoon and Hwang (1995) classified a group of 17 methods according to the type and salient features of information received from the decision maker. Sen and Yang

(1998) proposed a specific classification framework for application of MADM methods in engineering design. This framework is shown in Table 3.3.

Table 3.3 Classification of MADM Methods (Sen and Yang (1998))

Type of Information	Method	Preference Information
No Information	Dominance Maximin Maximax	
Standard Levels	Conjunctive Disjunctive	
Weight Assignment	Direct Assignment	
	Least Square Eigenvector	Pairwise Comparisons of All Attributes
	Entropy	
	MITA	Appropriate Comparisons of All Attributes
Weight Given Beforehand	Lexicographic	Ranking of All Attributes
	Simple Weighting	
	TOPSIS	Definition of Ideal and Negative Ideal Points
	Linear Assignment	
	Relative Positive Estimation	Pairwise Comparisons of All Attributes
	ELECTRE	
Weight Given Beforehand	AHP	Pairwise Comparisons of All Alternatives & Attributes
	LIMAP	Pairwise Comparisons & Ideal Points
Weight to be Generated	UTA	Ranking of a Subset of Alternatives
Local Utility Function	ILUTA	Pairwise Comparisons of Some Alternatives
Implicit Utility Function	EDMCM	Pairwise Comparisons & Trade-off Questions

3.3 The Need for Fuzzy Set Concepts in MADM Problems

As presented earlier, an MADM problem can be expressed in a matrix format. In most of the real world problems, some of the decision data R_{ij} can be precisely assessed while others cannot. Real (crisp) numbers are used to represent data which can be precisely measured. For those data which cannot be precisely assessed, Zadeh's fuzzy sets (numbers) are used to denote them. The use of FST allows us to incorporate unquantifiable information, incomplete information, nonobtainable information, and partially ignorant facts into the decision model.

Frequently, real world decision making problems are ill defined, i.e., their objectives and parameters are not precisely known. The obstacles of lack of precision have been dealt with using the probabilistic approach. But, due to the fact that the requirements on the data and on the environment are very high and many real world problems are fuzzy by nature and not random, the probability applications have not been very satisfactory in a lot of real world cases. On the other hand, the application of FST in real world decision making problems has given very good results. Its main feature is that it provides a more flexible framework, where it is possible to solve many of the obstacles of lack of precision satisfactorily.

FMADM methods have been developed due to the lack of precision in assessing the w_j and R_{ij} values. The imprecision may come from a variety of sources such as (Chen and Hwang (1992)):

1. Unquantifiable information : The price of a new ship can be easily determined while the safety or comfort of a ship is not easily quantifiable. Safety and comfort are usually expressed in linguistic terms such as good, fair, poor, etc. They are qualitative data (subjective judgement by an individual).

2. Incomplete information : The noise of a ship can be measured by some equipment as "about 65 dB" but not "exactly 67.11 dB". Such data may be represented as a fuzzy set because of incomplete information.

3. Nonobtainable information : Sometimes crisp data is obtainable but the cost is too high and the decision maker may wish to get an "approximation" of that crisp data. When the data is very sensitive (e.g. an individual's bank account, or a main engine price), some "approximated" data or linguistic descriptions are used. The information is fuzzy because of its unavailability.

4. Partial ignorance : Some fuzziness is attributed to partial ignorance of the phenomenon since one knows only part of the facts.

The classical MADM methods cannot effectively handle problems with such imprecise information. To solve this difficulty, FST, first introduced by Zadeh (1965), has been used and is one of the focuses of this research.

4. FUZZY MULTIPLE ATTRIBUTE DECISION MAKING (FMADM) METHODS

The general multiple attribute decision making (MADM) model is described as follows:

Let $X = \{X_j \mid j=1, \dots , N\}$ be a finite set of alternatives (courses of action, candidates) and $A = \{A_i \mid i=1, \dots , K\}$ be a finite set of attributes according to which the desirability of an alternative is to be judged. And let $R = \{R_{ij} \mid i=1, \dots , K; j=1, \dots , N\}$ be the KxN decision matrix, where R_{ij} is the performance rating of alternative X_j with respect to attribute A_i and w_i values $\{i=1, \dots , K\}$ are the weights of attributes.

The aim of MADM is to determine the optimal alternative with the highest degree of desirability with respect to all relevant attributes. The classical (or crisp) MADM techniques assume all R_{ij} and w_i values are crisp numbers. In real world MADM problems, R_{ij} values can be crisp and/or fuzzy (linguistic terms, fuzzy numbers) data.

Fuzzy Multiple Attribute Decision Making (FMADM) methods have been developed to solve MADM problems which contain fuzzy data. FMADM is a subcategory of Fuzzy Multiple Criteria Decision Making (FMCDM). FMCDM can be classified as Fuzzy Multiple Objective Decision Making (FMODM) and FMADM; the former emphasises on continuous decision making spaces and it mainly deals with multiple objective mathematical programming problems; the latter mainly deals with discrete decision making space problems.

A large number of articles in the literature on decision making analysis have addressed the FMADM methods. Zimmermann (1987), Dubois and Prade (1980), Chen and Hwang (1992), Ribeiro (1993), Ribeiro and Baldwin (1995) and Ribeiro (1996) have indicated that the FMADM methods basically consist of two phases :

Phase (I) The aggregation of the performance ratings (or the degree of satisfactions) with respect to all attributes for each alternative, and

Phase (II) The ranking of the alternatives according to the overall aggregated performance ratings.

The methods for solving phase (II) problems are referred to as "fuzzy ranking methods", and methods for solving phase (I) and/or both phases of MADM problems are referred to as "fuzzy multiple attribute decision making (FMADM) methods".

Riberio (1996) concentrated on phase (I) of FMADM methods and classified them into five categories. These are "non-fuzzy methods", "fuzzy hierarchical aggregation methods", "conjunction implication methods", "weighted average aggregation methods", and "weighted average aggregation with criteria assessment methods".

Perego and Rangone (1998) grouped FMADM techniques into four major categories, namely "fuzzy goal methodology", "fuzzy linguistic models", "fuzzy hierarchical models based on pairwise comparisons", and "heuristic models based on fuzzy logic".

The best one of the good surveys is done by Chen and Hwang (1992). They make distinctions between fuzzy ranking methods and FMADM methods. Chen and Hwang (1992) have reviewed and analysed most of the known FMADM methods.

In this research, FMADM methods are classified based on the classical MADM techniques used in these FMADM methods. This classification is given in Table 4.1. In the following, most of the existing FMADM methods in literature, related to this research, are given. They are described in more detail even though the proofs will be omitted because they are of more mathematical interest.

4.1 Non-Fuzzy Approaches

The methods in this category do not represent a survey of non-fuzzy approaches, they simply form the basis for later methods.

4.1.1 Kahne's Approach

Weighted average rating rule is used in Kahne's model (Kahne (1975)). This approach considers that the weights and criteria are stochastic variables and then uses random variables and Monte Carlo simulation to determine the optimal solution. This method does not deal specifically with fuzzy multiple attribute problems.

In Kahne's model, the w_i and the R_{ij} values are assumed to be stochastic variables and the optimal alternative is determined by using Monte Carlo simulation.

Table 4.1 Classification of the FMADM methods in the literature

FMADM Approaches	Performance Ratings		Attribute Weights		Result of the Phase I	GDM
	Crisp	Fuzzy	Crisp	Fuzzy		
Baas & Kwak.		√		√	Fuzzy	X
Kwakernaak		√		√	Fuzzy	X
Dubois & Pra.		√		√	Fuzzy	X
Cheng & McI.		√		√	Fuzzy	X
Bonissone	√	√	√	√	Fuzzy	X
Laarh. & Pedr.		√		√	Fuzzy	√
Buckley		√		√	Fuzzy	√
Ruon. & Xiao.		√		√	Fuzzy	√
Chang		√		√	Fuzzy	√
Roy	√		√		Crisp	X
Siskos et al.	√		√		Crisp	X
Brans et al.	√		√		Crisp	X
Takeda	√		√		Crisp	X
Wang		√		√	Crisp	X
Bell.& Zadeh		√			Crisp	X
Yager		√	√		Crisp	X
Liang & Wang		√		√	Fuzzy	√
Chang & Chen		√		√	Fuzzy	√
Wang &Chang		√		√	Fuzzy	√
Chen		√		√	Fuzzy	√
Rangone		√		√	Fuzzy	X
Efstathiou		√			Fuzzy	√
Dubois et al.		√	√		Crisp	X
Negi	√	√		√	Crisp	X
Chen&Hwang	√	√	√		Crisp	X

Where FMADM : Fuzzy Multiple Attribute Decision Making,

GDM : Group Decision Making

√ : Enable

X : Not enable

4.1.2 Saaty's Approach

Saaty (1977), (1978) states that there are two types of fuzziness :

- Fuzziness in perception, and
- Fuzziness in meaning

The first one is caused by complexity of objects or ideas which cannot be apprehended at once. The second one is attributed to relativism of meaning, i.e., the meaning of objects is tied to what function those objects perform in the fulfilment of different purposes. When the objects are decomposed, they appear fuzzy because they have different meanings according to the context of the decomposition.

Saaty proposed a method to give meaning to both kinds of fuzziness. This method measures relative fuzziness by structuring the criteria and objectives of a system, hierarchically, in a multiple attribute framework. In order to rate the alternatives, Saaty uses a hierarchical pairwise comparison between attributes and/or objectives and then solves them with eigenvectors of the reciprocal matrices.

Saaty developed a procedure for obtaining a ratio scale of importance for a group of p elements based upon paired comparisons.

Assume we have p objects, and we want to construct a scale, rating these objects with respect to each other. The objects could be the attribute and/or constraints characterising a decision problem. The decision maker compares the objects in paired comparisons. When comparing object i with object j, the decision maker is asked first to make a decision, which object is more important?. Having made that decision, s/he is then asked to assign a value taken from the scale 1 to 9 to the more important objects domination over the less important object. These scale values are given in Table 4.2 below with verbal hints on how to apply them. If object i dominates object j, the assigned value is denoted as a_{ij}. The paired comparison matrix has an interesting reciprocal property given by:

$a_{ij} = 1/a_{ji}$.

This property allows easy generation of a paired comparison matrix of dimension p by p, A, whose elements are

$a_{ii} = 1$, and $a_{ji} = 1/ a_{ij}$ for $i \neq j$.

Table 4.2 Judgement Scale used in Saaty's approach

ImportanceValue	Definition
1	Equal Importance
3	Weak importance of one over the other
5	Strong importance of one over the other
7	Demonstrated importance of one over the other
9	Absolute domination of one over the other
2, 4, 6, 8	Intermediate values between the two adjacent judgements

In Analytic Hierarchy Process (AHP), the eigenvector method (Saaty (1980)) is a very common method in deriving the priority vector from the matrix of pairwise comparisons. The eigenvector method that can be used to generate the relative importance of attributes and the performance scores can be found in Saaty (1980).

Saaty showed that the eigenvector corresponding to the maximum eigenvalue associated with A is a cardinal ratio scale for the elements compared.

Given that A is the matrix of pairwise comparison values, in order to find the priority vector, we can determine the vector W by solving the equation

$$A\,W = \lambda_{max}\,W$$

Where λ_{max} is the maximum eigenvalue of A, and the eigenvector W corresponding to λ_{max} can be used as the weight of elements $w_1, w_2, \ldots, w_p$. Saaty's eigenvector method is explained in Appendix B.

4.2 Simple Additive Weighting (SAW) Based FMADM Approaches

The classical Simple Additive Weighting (SAW) method was mathematically determined by Churchman and Ackoff (1954), MacCrimmon (1968), Keeney and Raiffa (1976) and Hwang and Yoon (1981). Several approaches based on the extension principle (Zadeh (1975)) have been proposed to solve the problems, when both the rating, R_{ij} of alternative X_i with respect to attribute A_j and importance of attribute j, w_j, are fuzzy sets (Baas&Kwakernaak (1977); Kwakernaak (1979), Cheng&McInnis (1980), Dubois&Prade (1983) and Bonissone (1982)).

The first four approaches utilise the α-cut technique to approximate $\mu_{U_i}(u_i)$, membership function of the utility of alternative X_i. On the other hand, Bonissone (1982) assumes that all piecewise continuously differentiable fuzzy numbers can be approximated by L-R type trapezoidal fuzzy numbers and applies special fuzzy arithmetic functions to calculate the fuzzy utility U_i for alternative X_i. Bonissone's approach is much easier to use than the other approaches in this category.

The classical SAW method is mathematically defined as follows. Suppose the decision maker assigns a set of weights, $w = (w_1, \ldots, w_n)$ to the attributes A_j, $j = 1, \ldots, n$. The performance of alternative, X_i, is calculated as:

$$U_i = \frac{\sum_{j=1}^{n} w_j r_{ij}}{\sum_{j=1}^{n} w_j}$$

Where r_{ij} is the rating of the ith alternative under the jth attribute with a numerically comparable scale. This is the simplest form in Multiple Attribute Utility Theory. The most preferred alternative, X^*, is then selected such that

$$X^* = \left\{ X_i \middle| \max_i U_i \right\}$$

When both w_j and r_{ij} are fuzzy sets defined as:

$w_j = \{(y_j, \mu_{wj}(y_j))\}, \forall j,$

and

$r_{ij} = \{(x_{ij}, \mu_{rij}(x_{ij}))\}, \forall i, j,$

Where y_j and x_{ij} take their numbers on the real line $\Re$ and $\mu_{wj}(y_j)$ and $\mu_{rij}(x_{ij})$ take values in [0, 1], the utility of alternative X_i, $U_i = \{(u_i, \mu_{ui}(u_i))\}$, can be calculated as follows. The variable u_i takes its value on the real line $\Re$ and can be obtained using

$$u_i = \frac{\sum_{j=1}^{n} y_j x_{ij}}{\sum_{j=1}^{n} y_j} \qquad (4.1)$$

The membership function $\mu_{ui}(u_i)$ can be calculated using

$$\mu_{Ui}(u_i) = \sup_{v} \left\{ \left[\bigwedge_{j=1}^{n} \mu_{wj}(y_j) \right] \wedge \left[\bigwedge_{j=1}^{n} \mu_{rij}(x_{ij}) \right] \right\}$$

where $v = (y_1, \ldots, y_n, x_{i1}, \ldots, x_{in})$. The membership function $\mu_{ui}(u_i)$ is not directly obtainable when $\mu_{wj}(y_j)$ and $\mu_{rij}(x_{ij})$ are piecewise continuously differentiable functions. To resolve that difficulty and preserve the simplicity of the SAW method, several approaches have been proposed by various authors as follows.

4.2.1 Baas and Kwakernaak's Approach

Baas and Kwakernaak's work was a reaction to a probabilistic suggestion by Kahne. This is the first approach for extending the classical weighted average rating formula to fuzzy numbers and has already become a "classic" in FMADM models.

Baas and Kwakernaak (1977) proposed a method to deal with multiple-aspect decision making in the presence of uncertainty. The method is based on a straightforward rating and ranking method, where the weights and ratings are fuzzy variables.

Further, they also recognise that their non-linear programming solution is equivalent to the max-min solution given by the extension principle of Zadeh. They also address the problem of comparing fuzzy sets for ranking of alternatives.

Baas and Kwakernaak identified the computational problem in calculating $\mu_{U_i}(u_i)$, membership function of the utility of alternative X_i. To overcome this difficulty, they proposed the use of the α-cut technique to obtain the fuzzy utility U_i. An α_0 value for $\mu_{U_i}(u_i)$ is assigned first. The corresponding u_i value(s) are then calculated using Equation (4.1). By setting different α_0 values and repeating Baas and Kwakernaak's algorithm, an approximated fuzzy utility U_i can be obtained. Steps of the Baas and Kwakernaak's algorithm are given in Appendix B.

Baas and Kwakernaak's approach was the first to use fuzzy variables with the weighted average rating method. It gives exact solutions but it is quite cumbersome and computationally inefficient.

4.2.2 Kwakernaak's Approach

Kwakernaak's approach is a modification of Baas and Kwakernaak's approach. Kwakernaak (1979) pointed out that the use of trial-and-error to identify the desired u_i values was not efficient. Therefore, an improved algorithm was proposed.

Given fuzzy weights $w_j = \{(y_{ij}, \mu_{w_j}(y_j))\}$ and fuzzy attribute $r_{ij} = \{(x_{ij}, \mu_{r_{ij}}(x_{ij}))\}$ for alternative X_i, steps, described in Appendix B, are used to derive fuzzy utility $U_i = \{(u_i, \mu_{U_i}(u_i))\}$.

4.2.3 Dubois and Prade's Approach

Dubois and Prade (1983) pointed out that Baas and Kwakernaak's approach can only effectively solve two-attribute problems. Since the trial-and-error technique, which was used in Bass and Kwakernaak's approach to derive fuzzy utilities, was not efficient, an alternative approach was proposed.

Dubois and Prade's approach also uses the α-cut technique, but provides a more efficient search procedure for obtaining u_i values. The algorithm assumes that all fuzzy weights w_j and fuzzy rating r_{ij} are normalised fuzzy numbers. Given an α level, an α-level set for each fuzzy weight and fuzzy rating can be obtained. The α-level sets are used to derive fuzzy utilities based on the classical SAW method. This approach's algorithm is also given in Appendix B.

4.2.4 Cheng and McInnis's Approach

Cheng and McInnis (1980) developed an algorithm, which is capable of handling large decision making problems, based on the concepts presented by Baas and Kwakernaak (1977). Instead of considering continuous functions, they discretise the membership functions, and the ranking function is obtained as the composite of the piecewise constant weight and rating membership functions. This approach leads to a simple computational algorithms.

They pointed out that continuous membership functions of r_{ij} and w_j are the cause of the complexity of obtaining fuzzy utilities. To avoid such difficulty, they suggested to first convert the continuous membership functions to discrete ones and then to compute the fuzzy utilities.

Cheng and McInnis's stepwise discrete membership functions can represent fuzzy information better than the simple piecewise linear membership functions as many pointed out that fuzzy data can be best described by either trapezoidal or triangular shaped fuzzy numbers. Also many approximated arithmetic operations can be easily applied to obtain approximated fuzzy utilities. The computational requirements are even less. This approach's algorithm is also explained in Appendix B.

4.2.5 Bonissone's Approach

Bonissone (1982) assumed that fuzzy/crisp information in decision problems can be approximated by a parameter-based representation. It is called the L-R type trapezoidal fuzzy number (a,b,α,β) (see Figure 4.1). It is this family of fuzzy numbers which allows approximated arithmetic operations on fuzzy numbers.

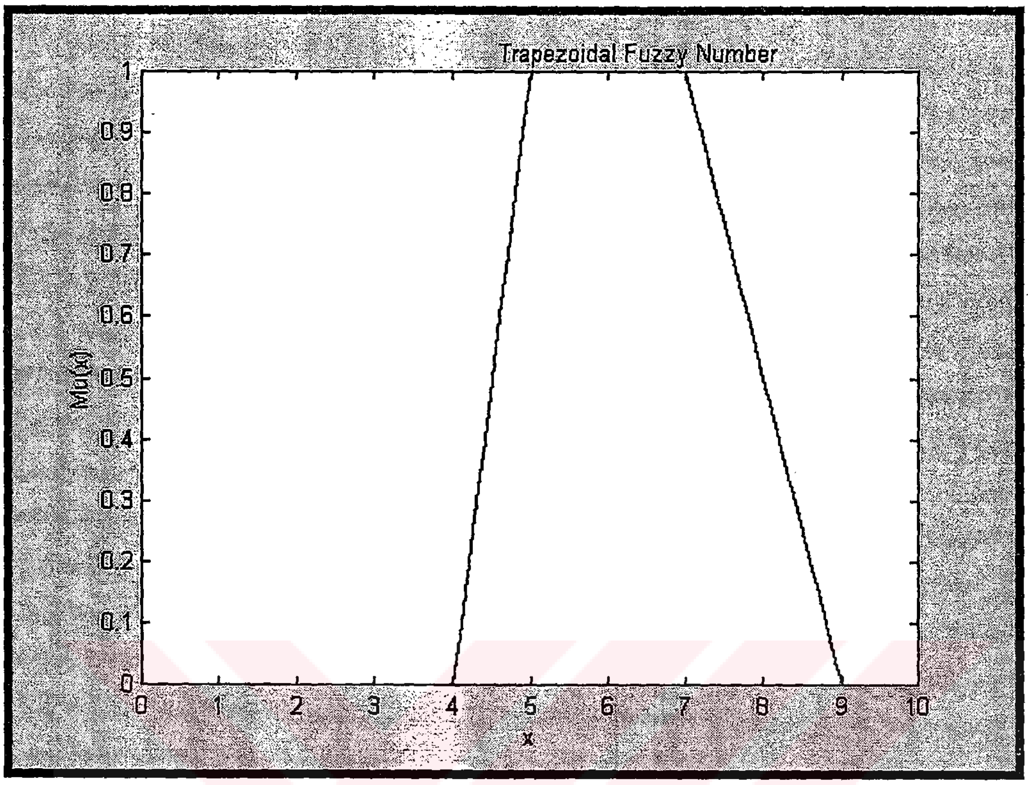

Figure 4.1 L-R type trapezoidal fuzzy number (for a=5,b=7,α=1,β=2)

With the help of the approximated algebraic operations, we can quickly compute the performance of alternative X_i with respect to attributes, A_j, j=1, …, n, using

$$U_i = \sum_{j=1}^{n} w_j r_{ij}$$

where w_j and r_{ij} may be crisp or fuzzy numbers represented in the L-R trapezoidal fuzzy number format.

Bonissone's approach is much simpler to use than other SAW based FMADM methods. It is applicable only when fuzzy concepts are represented by trapezoidal or triangular numbers. If that assumption does not hold, then other approaches discussed earlier may be appropriate. In addition, Bonissone's approach generates less precise fuzzy utility, i.e., the spreads generated by Bonissone's approach are larger than those which were generated by other SAW based FMADM methods.

Since L-R trapezoidal fuzzy number provides satisfactory explanations to fuzzy concepts, Bonissone's approach may be an appropriate method to use when larger spreads are tolerable. This approach is recommended for its simplicity.

4.3 Analytic Hierarchy Process (AHP) Based FMADM Approaches

The Analytic Hierarchy Process (AHP) developed by Saaty (1977) allows the decision makers to visually structure a complex problem in the form of a hierarchy having at least two levels : objectives (attributes for evaluation) and activities (alternatives, courses of action, etc.). Each factor or alternative on a given level can be identified and evaluated with respect to other related factors. This ability to structure a complex problem and then focus attention on specific components broadens one's decision making capabilities. Another advantage of AHP is its simplicity.

In the classical AHP method, the decision maker is asked to supply ratios a_{ij} for each pairwise comparison between alternatives $A_1, A_2, \dots, A_m$ for each attribute in a hierarchy, and also between attributes. This results in n+1 positive pairwise comparison matrices, where n is the number of attributes. Each matrix is represented as:

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix} = \begin{bmatrix} w_1/w_1 & w_1/w_2 & \dots & w_1/w_n \\ w_2/w_1 & w_2/w_2 & \dots & w_2/w_n \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ w_n/w_1 & w_n/w_2 & \dots & w_n/w_n \end{bmatrix}$$

which is a 'reciprocal matrix' with all a_{ij} being positive. The ratio a_{ij} represents, for the decision maker, the relative importance of A_i over A_j. For example, when the decision maker considers A_1 more important than A_5, a_{15} might equal 3/1, or 5/1, or 7/1, or 9/1. Since the numbers for the ratio are usually taken from the set {1, 2, ... , 9}, a_{15} could be s_1/s_2 where $s_1, s_2 \in \{1, 2, \dots, 9\}$. Note that if a_{15}= 3/1, then a_{51} must be 1/3. This is why matrix A is called a 'reciprocal matrix'.

The AHP method uses the pairwise comparison matrices for each attribute to compute the performance score of alternative A_i with respect to attribute X_j, r_{ij}. The pairwise comparison matrix for the attributes is used to compute the weights of the attributes. The performance scores and weight set are organised as:

$$
D = \begin{array}{c} \\ A_1 \\ A_2 \\ . \\ . \\ A_m \end{array}
\begin{array}{c}
\begin{array}{cccc} X_1 & X_2 & \cdots & X_n \end{array} \\
\begin{bmatrix} r_{11} & r_{12} & \cdots & r_{1n} \\ r_{21} & r_{22} & \cdots & r_{2n} \\ . & & & \\ . & & & \\ r_{m1} & r_{m2} & \cdots & r_{mn} \end{bmatrix}
\end{array}
\tag{4.2}
$$

$$
\underline{w} = (w_1, w_2, \ldots, w_n)
$$

where r_{ij} and w_j, $\forall i, j$, take their numbers on the real line $\Re$. The classical SAW method is used to find the utilities of A_i, U_i, $\forall i$:

$$
U_i = \sum_{j=1}^{n} w_j r_{ij} / \sum_{j=1}^{n} w_j \tag{4.3}
$$

The AHP method was first proposed by Saaty (1977), (1978). In Saaty's approach, the pairwise comparison ratios a_{ij}, $\forall i, j$, are real numbers. Each pairwise comparison matrix is solved using the eigenvector method. The resulting weights and performance scores are also crisp, real numbers. The classical SAW method is used to calculate the alternatives' utilities.

Saaty's AHP method was extended by Laarhoven and Pedrycz (1983). They argue that if a person considers A_1 more important than A_5, then the ratio a_{15} might be "approximated 3 to 1," or "about 5 to 1," or "between 5 to 1 and 7 to 1". These linguistic expressions are expressed by triangular fuzzy numbers. That is, a_{ij} is a triangular fuzzy number. In addition, Laarhoven and Pedrycz allow several decision makers to express their ratios on the same pair of alternatives (or attributes). In this case, the pairwise comparison ratio may be represented by a_{ijk} (k=0, 1, … , p_{ij}). The term p_{ij} denotes the number of experts who expressed their comparison ratios.

There are many methods one can use to derive performance scores and attributes' weight. According to Laarhoven and Pedrycz, because of the presence of fuzzy, multiple comparison ratios for the same pair of alternatives (or attributes), the most suitable method for their approach is Lootsma's logarithmic least square method. Once the fuzzy performance scores r_{ij} and the fuzzy weights w_j have been derived, fuzzy arithmetic operations that are suitable for triangular fuzzy numbers are used to obtain the fuzzy utilities, U_i, $\forall_i$, where r_{ij}, w_j, and U_i, are triangular fuzzy numbers.

Buckley (1984), (1985b) also extends Saaty's AHP method to the case in which decision makers can express their preference in fuzzy ratios instead of crisp ratios. The fuzzy ratios a_{ij} are given as a trapezoidal number (a, b, c, d) where $0 \leq a \leq b \leq c \leq d$ as shown in Figure 4.2. The geometric mean method is employed to calculate the fuzzy weights w_j and the fuzzy performance scores r_{ij}. Note that the derived w_j and r_{ij} may not be trapezoidal fuzzy numbers anymore. In this case, special fuzzy arithmetic formulas are needed in order to add and/or multiply them. Buckley (1984) has developed some special fuzzy arithmetic formulas for that purpose.

AHP has been developed to incorporate the difficulties to quantify criteria into the decision making process. One shortcoming of these methods is the failure to realistically represent the imprecision of the decision makers' judgements. In this aspect, the use of fuzzy logic and linguistic variables have attracted some attention.

The aim of these AHP based FMADM techniques is to include fuzzy concepts within a hierarchical decision making framework. In particular, fuzzy extensions of Saaty's priority theory are considered (Laarhoven and Pedrycz's Approach (1983), Buckley (1985b), Ruoning and Xiaoyan (1992)). These techniques are mainly based on the concept of a fuzzy judgmental matrix. Decision makers are asked to express pairwise comparisons of attribute and alternatives in fuzzy terms, resulting in a judgmental matrix. To derive priorities from the fuzzy judgmental matrix specific algorithms are introduced, i.e. logarithmic regression (Laarhoven and Pedrycz's Approach (1983)) and altered gradient eigenvector (Ruoning and Xiaoyan (1992)).

4.3.1 Laarhoven and Pedrycz's Approach

Laarhoven and Pedrycz (1983) concentrate on the Phase (I) of FMADM – that is, on the determination of fuzzy ratings for the decision alternatives, which can then be used for the ranking in Phase (II) of FMADM.

Laarhoven and Pedrycz (1983) proposed an algorithm which is a direct extension of Saaty's AHP method. In this extended version of AHP, the elements in the reciprocal matrix are represented by triangular fuzzy numbers. The computation steps are the same as those in AHP. The Lootsma's logarithmic least square method (see Appendix B for more details) is used to derive fuzzy weights and fuzzy performance scores. The arithmetic operations for fuzzy triangular numbers are applied to compute fuzzy utilities. The opinion of multiple decision makers can also be modelled in the reciprocal matrix.

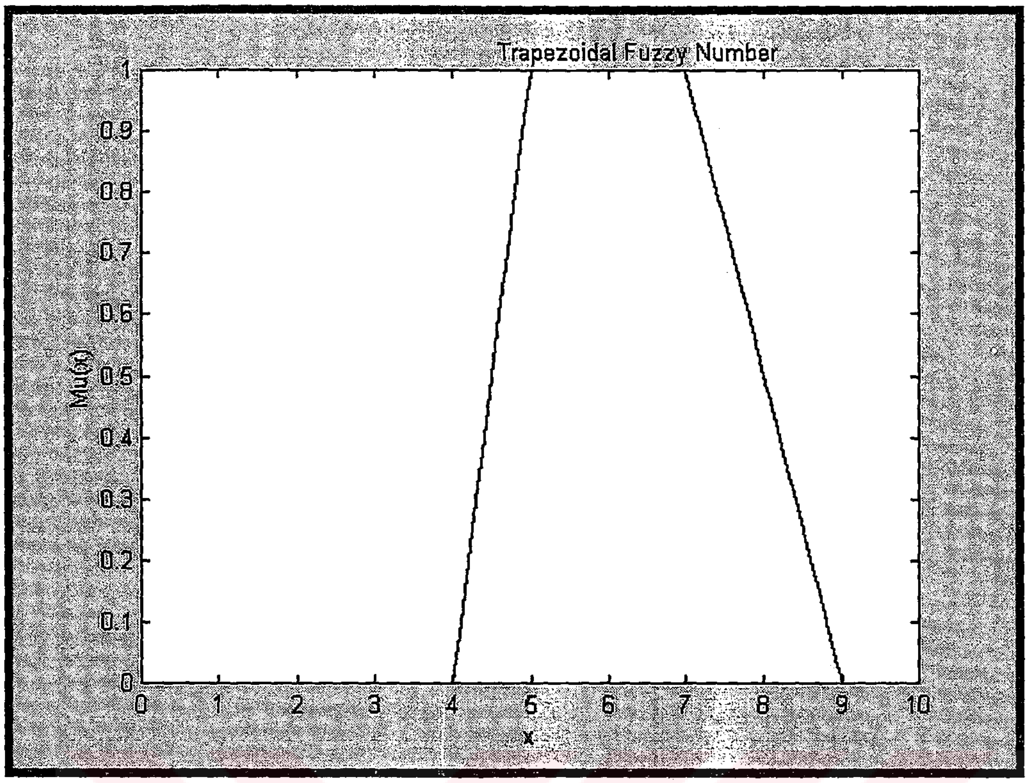

Figure 4.2 Trapezoidal fuzzy number (for a=4, b=5, c=7, d=9)

The main difference from the classical approach is that the weights are all considered to be fuzzy numbers and that the eigenvector of Saaty's reciprocal matrix also consists of fuzzy numbers.

This approach is characterised by three features:

- They use fuzzy numbers with triangular membership functions to simplify the calculations.
- It is possible to handle decision situations in which there is either no information or multiple information available for certain pairs of factors.
- Using the principle of hierarchic composition, the authors apply priority theory on two levels: in assigning weights to the attribute and for weighting the alternatives under each of their attribute separately.

According to Laarhoven and Pedrycz's technique, the following steps are to be performed :

Step 1. The relative importance weightings W_j of attributes (j = 1, 2, … , m, with m = number of criteria) are obtained on the basis of fuzzy pairwise comparisons. Each pairwise comparison is a triangular fuzzy number. To pass from these fuzzy pairwise values to fuzzy estimates of the suitability ratings, the method of logarithmic regression is used, properly extended to fuzzy version by means of the extension principle and simplified operation rules on triangular fuzzy numbers,

Step 2. The relative suitability ratings A_{ij} of the alternatives with respect to each attribute (i = 1, 2, … , n, with n = number of alternatives) are attributed following the same procedure as that described in step 1,

Step 3. The overall suitability ratings A_i of the alternatives are finally calculated by weighting the suitability ratings relevant to each attribute with the corresponding importance weightings, on the basis of Zadeh's extension principle and the operation rules on triangular fuzzy numbers. Since the overall suitability ratings are fuzzy numbers, the 'maximising and minimising sets' algorithm is used to rank the alternatives (Liang and Wang (1993), (1994)). Laarhoven and Pedrycz's algorithm is also given and explained in Appendix B.

4.3.2 Buckley's Approach

Buckley (1985b) also extended Saaty's method to incorporate fuzzy comparison ratios a_{ij}. He pointed out that Laarhoven and Pedrycz's method is subject to two problems. First, the linear equations of Laarhoven and Pedrycz's method do not always have a unique solution. Secondly, they insist on obtaining triangular fuzzy numbers for their weights. Since algebraic operations on triangular fuzzy numbers do not necessarily produce a triangular fuzzy number, Laarhoven and Pedrycz are forced to employ approximate methods to preserve the shape of the fuzzy number.

To overcome these difficulties, Buckley uses the geometric mean method (see Appendix B) to derive fuzzy weights and performance scores. This method is used since it is easy to extend to the fuzzy case and guarantees a unique solution to the reciprocal comparison matrix. Instead of using a triangular fuzzy number, Buckley uses a trapezoidal fuzzy number (a,b,c,d) (see Figure 4.2) to represent the fuzzy ratio expressed by the decision makers.

For example, in Figure 4.3, (3, 4, 6, 7) represents the fuzzy ratio "between 4 to 1 and 6 to 1" and (2, 3, 3, 4) represents the fuzzy ratio "about 3 to 1". The fuzzy utilities, however, are not restricted to trapezoidal shape. Buckley believed that his approach avoids all the problems found in Laarhoven and Pedrycz's approach. Buckley's algorithm is also given in Appendix B.

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4.3.4 Chang's Approach

Chang (1996) introduced a new approach for handling fuzzy AHP problems, which is different from the above mentioned AHP based methods. But the ordering of a permutation with respect to elements is quite the same.

Triangular fuzzy numbers are also used for a pairwise comparison scale of fuzzy AHP.

4.4 Outranking Relation Based FMADM Approaches

Outranking relation

Given two alternatives A and B, the statement 'A outranks B' signifies that the decision maker has enough reasons to admit that A is at least as good as B. Through the successive assessments of the outranking relations of the other alternatives, the dominated alternatives defined by the outranking relation can be eliminated. To derive outranking relation between pairs of alternatives is the key issue in classical outranking method.

Fuzzy Outranking relation

A fuzzy outranking relation can be characterised by a degree of outranking (or membership function) which indicates the degree of outranking associated with each pair of alternatives A and B. The ranking of alternatives is conducted using the fuzzy outranking relations.

Roy (1977), Takeda (1982), Siskos et al. (1984), and Brans et al. (1984) have developed various procedures in deriving fuzzy outranking relations.

Roy (1977) used the degree of concordance and discordance to derive fuzzy outranking relations; Siskos et al. (1984) followed Roy's approach but used different formulas and threshold values in deriving the degree of concordance and discordance. Takeda (1982) proposed an interactive approach for building fuzzy outranking relations from which the decision maker's preference structure could be extracted as a fuzzy multilevel graph. Brans et al. (1984) proposed a method with six different formulas for computing the degree of outranking.

The Brans et al.'s approach is probably the simplest one (in terms of computational requirement) in this category.

4.4.1 Roy's Approach

The use of fuzzy outranking relations in MADM is first seen in Roy (1977). It is considered a fuzzy edition of the classical ELECTRE method. Roy (1977) proposed the use of the degree of concordance and the degree of discordance to construct fuzzy outranking relations.

ELECTRE (Roy (1971)) is a MADM method. This method consists of pairwise comparisons of alternatives based on the degree to which evaluations of the alternatives and preference weights confirm or contradict their pairwise dominance relationship.

4.4.2 Siskos et al.'s Approach

Siskos et al. (1984) present a fuzzy outranking method that is similar to Roy's approach. There are two major differences between these approaches. The formulas used in deriving concordance and discordance relations are different. Siskos et al. build a fuzzy dominance relation and subsequently a fuzzy nondominance relation. The alternative with the highest degree of nondominance is said to be the best.

4.4.3 Brans et al.'s Approach

Brans et al. (1984) presented a family of outranking methods called PROMETHEE (Preference Ranking Organisation Methods for Enrichment Evaluation).

PROMETHEE III is an outranking method and can result in the partial preordering of alternatives (PROMETHEE I) or the complete preordering of alternatives (PROMETHEE II).

The PROMETHEE methods consist of three steps: construction of generalised criteria or preference functions, calculation of the multiple criteria preference index, and determination and evaluation of an outranking relation to give an answer to the multiple-attribute problem of interest.

4.4.4 Takeda's Approach

Takeda (1982) proposed an interactive procedure for building fuzzy outranking relations from which the decision maker's preference structure could be extracted as a fuzzy multilevel graph, of which a vertex corresponds to an alternative.

Similar to Roy's approach, Takeda used concordance and discordance relations to obtain fuzzy outranking relation. The difference is that Roy assumes that in concordance analysis certain a priori weights about attributes are available. This assumption is not valid for many cases since the decision maker may not be certain about the weights. To resolve the difficulty, Takeda proposed an interactive procedure to obtain the weights of attribute.

4.4.5 Wang's Approach

Wang (1997) pointed out that most of the fuzzy outranking approaches (Roy et al., Siskos et al., and Brans et al.) developed for multiple attribute decision making problems are limited to evaluating the alternatives with a quantitative form for the performance of each attribute. To overcome this difficulty, Wang (1997) proposed an outranking approach to model the imprecise preference structure. Wang's approach is able to handle the linguistic representation of each attribute that is one of the major characteristics of MADM problems.

According to Wang (1997), the fuzzy preference relation is used to represent the imprecise preference relations between alternatives. Based on the outranking approach, three preference models are developed to discriminate the nondominance set from a set of alternatives for further development. These models are 'pseudo-order preference model', 'semi-order preference model', and 'complete-preorder preference model'.

The pseudo-order preference model discriminates the set of alternatives into nondominance and dominance sets without the information on relative importance among various attribute. If the information on relative importance is known, the semi-order preference model is used. The complete-preorder preference model is used to rank the set of alternatives in a complete order. The most promising 'best' alternative is selected.

The task of design evaluation in conceptual design is important, since a poor selection of design concept can rarely be compensated at later design stages. At that stage, it is difficult to determine the 'best' design alternatives, because information collected is too subjective or too incomplete to make a judgement. Wang's approach is more applicable in the imprecise and uncertain design environment.

4.5 Implied Conjunction Methods

The implied conjunction methods, also called maximin methods, don't handle fuzzy weights and the solution represents the support for each alternative. The best support corresponds to the alternative with higher support.

The classical maximin method is used to select an alternative A^* such that

$$A^* = \left\{A_i \middle| \ \max_i \min_j x_{ij} \right\} \ \mathrm{j=1,...,n;i=1,...,m.}$$

where x_{ij}'s are in a common scale.

The term "maximin" signals the selection of the maximum (across alternatives) of the minimum (across attributes) values. In this situation, where the overall performance of an alternative is determined by the weakest or poorest attribute, a decision maker would examine the attribute values for each alternative, note the lowest value for each alternative, and then select the alternative with the most acceptable value in its lowest attribute. In general, this method would be reasonable only if the decision maker is assumed to have a pessimistic nature in the decision making situation (Hwang and Yoon (1981)).

The decision matrix for the maximin method is given as:

$$\begin{array}{cc} & \begin{array}{ccccc} X_1 & \cdots & X_j & \cdots & X_n \end{array} \\ \begin{array}{c} A_1 \\ \cdot \\ \cdot \\ \cdot \\ A_i \\ \cdot \\ \cdot \\ \cdot \\ A_m \end{array} & \begin{bmatrix} \mu_1(x_1) & \cdots & \mu_1(x_j) & \cdots & \mu_1(x_n) \\ \cdot & & & & \\ \cdot & & & & \\ \cdot & & & & \\ \mu_i(x_1) & \cdots & \mu_i(x_j) & \cdots & \mu_1(x_n) \\ \cdot & & & & \\ \cdot & & & & \\ \cdot & & & & \\ \mu_1(x_1) & \cdots & \mu_m(x_j) & \cdots & \mu_m(x_n) \end{bmatrix} \end{array}$$

where $\mu_i(x_j) \in [0,1]$ is interpreted as how well A_i satisfies attribute X_j. It represents a subjective judgement of the decision maker, and hence, is fuzzy. The best alternative A^* is determined as:

$$A^* = \left\{A_i \middle| \ \max_i \min_j \mu_i(x)_j \right\} \ \mathrm{j=1,...,n;i=1,...,m.}$$

In a classical MADM problem, values of different attributes may be measured in different units. The values must be normalised such that interattribute values are comparable. However, in a fuzzy case, the values in the decision matrix are all given as degrees of "how one alternative satisfies a certain attribute." There is no need for normalisation when the decision data are fuzzy. The decision data $\mu_i(x_j)$ in the decision matrix is referred to as the fuzzy singleton (Zadeh (1973)).

The concept of maximin applied in a fuzzy environment was first seen in Bellman and Zadeh (1970). Although its original intention was for general fuzzy decision making, this concept is readily applicable to fuzzy multiple attribute decision making problems. Yager (1977), (1978) utilises this concept and develops an algorithm for FMADM problems with unequal weights.

4.5.1 Bellman and Zadeh's Approach

Bellman and Zadeh (1970) asserted that in the conventional approach to decision making, the principal ingredients of a decision process are:

1. a set of alternatives,
2. a set of constraints on the choice between different alternatives, and
3. a performance function which associates with each alternative the gain (or loss) resulting from the choice of that alternative.

In a fuzzy environment, the performance function may be replaced by the concept "fuzzy goal". A fuzzy goal, G, may be represented by a fuzzy set $\{(x,\mu_G(x)) \mid x \in U\}$ where U is the universe of the fuzzy set G. $\mu_G(x)$ is the membership function of the fuzzy goal and takes its values in [0,1].

The x value that makes the highest $\mu_G(x)$ value is the preferred one. Clearly, the membership function serves the same purpose as a conventional performance function. Furthermore, $\mu_G(x)$ may be considered a normalised performance function. Such normalisation provides a common denominator for the various fuzzy goals and fuzzy constraints and thereby makes it possible to treat them alike. This line of reasoning explains why it is perfectly appropriate to regard the concept of the "fuzzy goal" –rather than the performance function- as one of the major components for decision analysis in a fuzzy environment. Similar remarks can be made to fuzzy constraints.

The above definitions of goals and constraints in a fuzzy environment (i.e., when both sets are fuzzy) make it appropriate to treat the fuzzy goals and fuzzy constraints identically in the formulation of a decision. By contrast, in the conventional approach to decision-making, the use of Lagrangian multipliers and penalty functions makes it apparent that there is an intrinsic similarity between performance functions and constraints. This similarity is made explicit in the formulation of fuzzy decision analysis.

Thus, a fuzzy decision may be stated as the fuzzy set D resulting from the intersection of the goals and constraints. That is, given the fuzzy goals, G_i, i= 1,...,m, and the constraints, C_j, j= 1, ..., n, we can determine the solution D, using

$D = G_1 \cap \ldots \cap G_m \cap C_1 \cap \ldots C_n.$

Its membership function is defined as:

$\mu_D(x) = \mu_{G_1}(x) \cap \ldots \cap \mu_{G_m}(x) \cap \mu_{C_1}(x) \cap \ldots \cap \mu_{C_n}(x)$.

The selection of the most appropriate x value for fuzzy set D is then given as:

$\mu_{D^*}(x) = \max \mu_D(x), x \in K \subset U,$

$= 0$ elsewhere

where K is the set of points in U on which $\mu_D(x)$ attains its maximum, if it exits. Note that $\mu_{D^*}(x)$ is the optimal decision and any x in the support of D^* will be referred to as a maximising decision.

4.5.2 Yager's Approach

Bellman and Zadeh (1970) defined the concept of "decision" in a broad sense, as the confluence of goals and constraints. According to Yager (1977), (1978) the decision is the intersection of all fuzzy goals raised to powers w_i and is given as follows :

$D = G_1^{w_1} \cap G_2^{w_2} \cap \ldots G_m^{w_m}.$

Where w_i (i = 1, 2, ... , m) is the weight attached to the goal/criterion G_j (j = 1, 2, ... , m) depending on its importance and the optimal alternative is defined as that achieving the highest degree of membership in D. This approach is used when the goals or objective attributes as well as the constraints are not of equal importance to the decision-maker.

Yager (1977), (1978) proposes a method based on the idea of assigning to each attribute in a MADM problem a number indicating its importance to the decision maker. An exponential weighting method is used for weighting the importances of the decision components and the weights are obtained using a method of paired comparisons developed by Saaty (1977). This form of weighting make the fuzzy decision process more responsive to real-world needs.

When the fuzzy decision takes place, the performance data under all attributes for each alternative are raised to their appropriate power and the alternative that satisfies

$$\max_{i} \min_{i} \left(\mu_i(x_j)^{w_j}\right)$$

is preferred.

Algorithm of the Yager's approach is given as follows:

A fuzzy MADM problem is given as:

$$\begin{array}{cc} & \begin{array}{ccccc} X_1 & \dots & X_j & \dots & X_n \end{array} \\ \begin{array}{c} A_1 \\ . \\ . \\ . \\ A_i \\ . \\ . \\ . \\ A_m \end{array} & \begin{bmatrix} \mu_1(x_1) & \dots & \mu_1(x_j) & \dots & \mu_1(x_n) \\ . & & & & \\ . & & & & \\ . & & & & \\ \mu_i(x_1) & \dots & \mu_i(x_j) & \dots & \mu_i(x_n) \\ . & & & & \\ . & & & & \\ . & & & & \\ \mu_m(x_1) & \dots & \mu_m(x_j) & \dots & \mu_m(x_n) \end{bmatrix} \end{array}$$

where $\mu_i(x_j) \in [0,1]$ indicates how well alternative A_i satisfies criteria x_j. $\mu_i(x_j)$ is a measure of subjective judgement. The selection of the best alternative is done using the following steps:

Step 1. Compute the relative importance for each attribute. Saaty's method is used to calculate weight, w_j, $\forall j$.

Step 2. Obtain the weighted decision matrix. The weights are used to modify the decision matrix. A weighted decision matrix is:

$$
\begin{array}{c c}
 & \begin{array}{ccccc} X_1 & \cdots & X_j & \cdots & X_n \end{array} \\
\begin{array}{c} A_1 \\ \cdot \\ \cdot \\ \cdot \\ A_i \\ \cdot \\ \cdot \\ \cdot \\ A_m \end{array} &
\begin{bmatrix}
\mu_{11}^{(w_1)} & \cdots & \mu_{1j}^{(wj)} & \cdots & \mu_{1n}^{(w_n)} \\
\cdot & & & & \\
\cdot & & & & \\
\cdot & & & & \\
\mu_{i1}^{(w_1)} & \cdots & \mu_{ij}^{(wj)} & \cdots & \mu_{in}^{(w_n)} \\
\cdot & & & & \\
\cdot & & & & \\
\cdot & & & & \\
\mu_{m1}^{(w_1)} & \cdots & \mu_{mj}^{(w_j)} & \cdots & \mu_{mn}^{(w_n)}
\end{bmatrix}
\end{array}
$$

where $\mu_{ij} = \mu_i(x_j)$ and w_j is the weight obtained in Step 1.

Step 3. Select a compromise alternative. Ideally, our goal is to select the alternative which has the highest membership values with respect to all the criteria, X_j, $\forall j$. However, this rarely happens, because one alternative that has the highest membership value with respect to X_1 does not necessarily have the highest membership value under other criteria. Since an alternative must be chosen, some forms of compromise are to be made. In this case, Yager proposed the use of the max and the min operators to select the best alternative. The selected alternative is said to maximise the minimum membership values over all the criteria, i.e.,

$$\max_i \left[\min_i \ \mu_{ij}^{wj} \right]$$

4.6 Fuzzy Linguistic Approaches

Models in this category are the techniques that have already been proposed in the literature (Liang and Wang (1991), (1993), Chang and Chen (1994), Wang and Chang (1995), Chen (1997), Rangone (1998)). These models are based on the concept of linguistic variables.

These models require the use of two linguistic variables for assessing the importance weightings of the attributes and for assessing the suitability rating of the alternative with respect to each attribute.

According to the fuzzy linguistic approaches in the literature, the evaluation attribute can be distinguished into two categories: subjective and objective attributes.

4.6.1 Liang and Wang's Approach

Liang and Wang (1991), (1993) proposed an algorithm based on the concepts of FST and the hierarchical structure analysis to aggregate decision makers' linguistic assessments about attribute weightings and suitability of alternatives versus various selection attribute to obtain fuzzy suitability indices.

Two preference rating systems are used for assigning the importance weight of the attribute and the suitability of the alternatives versus the attribute. Decision makers employ a weighting set W, W = {Very Low (VL), Low (L), Medium (M), High (H), Very High (VH)}, to evaluate the importance of each attribute. The linguistic set S, S = {Very Poor (VP), Between Very Poor and Poor (B.VP&P), Poor (P), Between Poor and Fair (B.P&F), Fair (F), Between Fair and Good (B.F&G), Good (G), Between Good and Very Good (B.G&VG), Very Good (VG)}, are used to evaluate the suitability of alternatives versus various subjective attribute. The fuzzy numbers used in linguistic sets W and S are shown in Table 4.3.

Table 4.3 Linguistic variables and their corresponding fuzzy numbers used in Liang and Wang's approach

Linguistic set W	Fuzzy numbers	Linguistic set S	Fuzzy numbers
VL	Trap (0, 0, 0, 0.3)	VP	Trap (0, 0, 0, 0.2)
L	Trap (0, 0.3, 0.3, 0.5)	B.VP&P	Trap (0, 0, 0.2, 0.4)
M	Trap (0.2, 0.5, 0.5, 0.8)	P	Trap (0, 0.2, 0.2, 0.4)
H	Trap (0.5, 0.7, 0.7, 1)	B.P&F	Trap (0, 0.2, 0.5, 0.7)
VH	Trap (0.7, 1, 1, 1)	F	Trap (0.3, 0.5, 0.5, 0.7)
		B.F&G	Trap (0.3, 0.5, 0.8, 1)
		G	Trap (0.6, 0.8, 0.8, 1)
		B.G&VG	Trap (0.6, 0.8, 1, 1)
		VG	Trap (0.8, 1, 1, 1)
Where Trap indicates a trapezoidal fuzzy number			

Attribute weightings assigned by decision makers are then pooled to obtain their aggregated weightings. Similarly, decision makers' assessments for each alternative with respect to each attribute are aggregated to get the aggregated fuzzy ratings of an alternative under each attribute.

Finally, aggregated weightings and aggregated fuzzy ratings are aggregated to get the fuzzy suitability indices of all alternatives. The fuzzy suitability indices's ranking values are calculated and then ranked to select the best alternative.

They applied this approach to robot and facility site selection problems (Liang and Wang (1991), (1993)).

Karsak (1998) extended this approach for robot selection problems. He proposed a two phase methodology. In the first phase, data envelopment analysis is used. In the second phase of the proposed method, Liang and Wang's approach is employed to select the best alternative.

4.6.2 Chang and Chen's Approach

Chang and Chen (1994) proposed a FMADM algorithm based on the concepts of FST and the hierarchical structure analysis.

The authors used the concepts of hierarchical structure analysis with two distinct levels in their study. The first level is to evaluate fuzzy importance of the decision attribute and the second level is to assign ratings to various alternatives under each attribute.

The linguistic variables and fuzzy numbers are used to aggregate the decision makers' subjective assessment about attribute weightings and appropriateness of alternatives versus selection attribute to obtain the final scores called fuzzy appropriateness indices. Since the final scores of alternatives are fuzzy numbers, they also proposed a revised method for ranking fuzzy numbers with index of optimism which was proposed by Kim and Park (1990).

This revised fuzzy numbers ranking method is based on the stage of data input for computing the total index of optimism in GDM problem, instead of giving the index of optimism independently by a decision maker on the stage of information output.

4.6.3 Wang and Chang's Approach

Wang and Chang (1995) presented a fuzzy linguistic approach to solve the tool steel materials selection problem under fuzzy environment, where the importance weights of the attribute and the performance ratings of attribute assessed by the decision makers are described in linguistic terms represented by fuzzy numbers.

The importance weights of the attribute are assessed in linguistic terms represented by fuzzy numbers, such as Very Low (VL), Low (L), Medium (M), High (H), Very High (VH), and the membership functions of the five linguistic terms are shown in Figure 4.1.

They also assumed that the decision makers can assign the ratings of different alternatives under different selection attribute using linguistic terms represented by fuzzy numbers, such as worst (W), Poor (P), Fair (F), Good (G), and the Best (B), where the membership functions of the five linguistic terms are shown in Figure 4.2.

The n decision makers' opinions are aggregated by

$W_t = (1/n) \otimes (W_{t1} \oplus W_{t2} \oplus \dots \oplus W_{tn}), t = 1, 2, \dots , k$

Where W_t is the aggregated weighting for attribute t, W_{tn} is the importance weight given by decision maker n to attribute t, k is the number of attribute, $\oplus$ and $\otimes$ are the addition operator and the multiplication operator of fuzzy numbers, respectively.

If the ratings are assigned by different decision makers, then their opinions can be aggregated by

$R_{it} = (1/n) \otimes (R_{it1} \oplus R_{it2} \oplus \dots \oplus R_{itn}),$

Where $i = 1, 2, \dots , m$, $t = 1, 2, \dots , k$, R_{it} is the aggregated rating of alternative i under attribute t, R_{itn} is the assigned rating of alternative i under attribute t by decision maker n. After the weights and ratings have been assigned and aggregated, R_{it} can further be weighted by the aggregated weight W_t to obtain the final rating F_i,

$F_i = (1/k) \otimes [(R_{i1} \otimes W_1) \oplus (R_{i2} \otimes W_2) \oplus \dots \oplus (R_{ik} \otimes W_k)],$

Where F_i is the approximated fuzzy number of the fuzzy suitability index of alternative i. Finally, they used Chen's method (Chen (1985)) of maximising set and minimising set to rank the final ratings of different alternatives.

4.6.4 Chen's Approach

Chen (1997) also proposed a fuzzy linguistic approach to solve the tool steel materials selection problem under fuzzy environment. He pointed out that the method presented in Wang and Chang's approach is not efficient enough due to the fact that it needs to perform the complicated aggregation and ranking operations of fuzzy numbers to determine the most suitable alternative. Thus, he developed a more efficient fuzzy linguistic approach by using simple arithmetic operations rather than the complicated aggregation and ranking operations of fuzzy numbers mentioned in Wang and Chang (1995).

He considered the same linguistic terms shown in Figure 4.1 and Figure 4.2, where the linguistic terms and their corresponding quadruple representations of fuzzy numbers are shown in Table 4.4.

The methodology is the same as Wang and Chang's approach. However, the main difference is that he introduced a deffuzzification method of trapezoidal fuzzy numbers and used it to defuzzify all weightings and ratings before aggregation and ranking operations. Thus, Chen's approach yields simpler calculations than Wang and Chang's approach.

Table 4.4 Linguistic terms and their corresponding fuzzy numbers used in Chen's approach

Linguistic terms		Fuzzy numbers
Very Low (VL)	Worst (W)	Trap (0, 0, 0, 0.3)
Low (L)	Poor (P)	Trap (0, 0.3, 0.3, 0.5)
Medium (M)	Fair (F)	Trap (0.2, 0.5, 0.5, 0.8)
High (H)	Good (G)	Trap (0.5, 0.7, 0.7, 1)
Very High (VH)	Best (B)	Trap (0.7, 1, 1, 1)
Where Trap indicates a trapezoidal fuzzy number		

4.6.5 Rangone's Approach

Rangone (1998) used two linguistic sets similar to Liang and Wang's approach. The first one, W = {Very Low (VL), Low (L), Medium (M), High (H), Very High (VH)}, is used for assessing the importance weighting of the attribute. Second set A, A = {Very Poor (VP), Poor (P), Fair (F), Good (G), Very Good (VG)}, is used for assessing the suitability ratings of the alternatives with respect to every attribute. The fuzzy numbers used in linguistic sets W and A are all triangular fuzzy numbers.

According to Rangone's approach, the steps are performed as follows:

1. The relative weightings of the attribute are measured using the linguistic set W,
2. The assessments of the alternatives with respect to each attribute are made using linguistic set A,
3. The overall suitability rating Λ_i of each alternative is derived from a numeric operator that calculates the mean of the fuzzy linguistic assessments given in step 2, taking into account of the fuzzy linguistic weightings attributed in step 1.

Since Λ_i are fuzzy numbers, they must be converted into numeric values. This can be done by using any ranking algorithm proposed in the literature (e.g. Azzone and Rangone (1996)).

The methodology is the same as Wang and Chang's approach. However, the main difference is that he introduced a deffuzzification method of trapezoidal fuzzy numbers and used it to defuzzify all weightings and ratings before aggregation and ranking operations. Thus, Chen's approach yields simpler calculations than Wang and Chang's approach.

Table 4.4 Linguistic terms and their corresponding fuzzy numbers used in Chen's approach

Linguistic terms		Fuzzy numbers
Very Low (VL)	Worst (W)	Trap (0, 0, 0, 0.3)
Low (L)	Poor (P)	Trap (0, 0.3, 0.3, 0.5)
Medium (M)	Fair (F)	Trap (0.2, 0.5, 0.5, 0.8)
High (H)	Good (G)	Trap (0.5, 0.7, 0.7, 1)
Very High (VH)	Best (B)	Trap (0.7, 1, 1, 1)
Where Trap indicates a trapezoidal fuzzy number		

4.6.5 Rangone's Approach

Rangone (1998) used two linguistic sets similar to Liang and Wang's approach. The first one, W = {Very Low (VL), Low (L), Medium (M), High (H), Very High (VH)}, is used for assessing the importance weighting of the attribute. Second set A, A = {Very Poor (VP), Poor (P), Fair (F), Good (G), Very Good (VG)}, is used for assessing the suitability ratings of the alternatives with respect to every attribute. The fuzzy numbers used in linguistic sets W and A are all triangular fuzzy numbers.

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Since Λ_i are fuzzy numbers, they must be converted into numeric values. This can be done by using any ranking algorithm proposed in the literature (e.g. Azzone and Rangone (1996)).

4.7 Miscellaneous FMADM Methods

Techniques in this category cant be classified as above techniques because of their different structure.

4.7.1 Heuristic Approach Based On Fuzzy Decision Rules

Approaches in this category are based on the concepts of linguistic (fuzzy) decision rules (Dubois and Prade (1984), Lee (1990)), which are of the form :

IF (a set of conditions are satisfied) THEN (a consequent can be inferred).

Models based on fuzzy decision rules consist of the following major steps:

1. A set of fuzzy decision rules is developed which converts all possible combinations of fuzzy linguistic values of performance variables into the fuzzy linguistic values of the overall level of support to the company's goals. Various authors have suggested heuristic guidelines to obtain the set of fuzzy rules (e.g. Efstathiou and Rajkovic (1979)).

2. Each decision alternative is assessed with respect to performance variables and these assessments, if crisp, have to be converted into fuzzy measures (fuzzification).

3. The overall suitability rating of each decision alternative is calculated. It is necessary, for each alternative, to consider all those fuzzy rules whose antecedents correspond to the fuzzy assessments of step 2 and to translate the consequence of these rules into a crisp value suitable for ranking of alternatives (defuzzification).

Efstathiou (1979) and Efstathiou and Rajkovic (1979) argued that the Multiple Attribute Utility Function (MAUF) cannot be practically obtained by the combination of single attribute utility functions because of the dependency among attributes. Therefore, a heuristic approach is needed to define the MAUF. Since decision data may be numerically and/or linguistically expressed, fuzzy set theory must be incorporated in this heuristic approach. The utility function is represented in the "IF ... THEN..." decision rule format and the algorithm may be summarised by the following steps:

Step 1. Identify interest groups of people involved in the decision environment.

Step 2. Identify attributes and establish a universe of discourse for each attribute.

Step 3. Interact with the decision makers to identify their heuristics and ideal solutions for each interest group.

Step 4. Construct the utility relation for each group. The rules are constructed according to the heuristics obtained in Step 3.

Step 5. List all alternatives and assess them with respect to the attributes.

Step 6. Calculate the utilities of individual alternatives for each group.

Step 7. Rank alternatives on the basis of calculated utilities.

The decision makers are actively involved in the decision process in this method. For example, in Step 3 heuristic information is obtained through a question-and-answer system, where dialogue between the decision makers and analysts takes place; and in Step 4, where the decision rules are constructed through serious discussions between the decision makers and analysts.

The major drawbacks of Efstathiou and Rajkovic's approach are

- It is extremely time consuming to construct the decision rules, even with the help of the decision maker's heuristics. For a small problem, it may be useful to use this method. However, when the problem size increases to 10 attributes, each having four values in its universe of discourse, there will be a total of 4^{10} decision rules, and even with the help of the decision maker's heuristics, the number of rules is still high. That makes the algorithm impractical for application to large real world problems.

- The involvement of the decision maker is tremendous. The discussion between the decision makers and the analysts is time consuming. When the decision makers are not available for consultation, this approach is not applicable.

4.7.2 Fuzzy Conjunctive/Disjunctive Method

The classical conjunctive method is an intuitive approach used to dichotomise alternatives into acceptable/not acceptable categories (Hwang and Yoon (1981)). The decision maker sets up the minimum attribute values (standard levels) s/he will accept for each of the attributes. Any alternative which has an attribute value less than the standard level will be rejected (not acceptable).

The classical disjunctive method, on the other hand, is one in which an alternative is evaluated based on its greatest value of an attribute.

Dubois et al. (1988) pointed out that when data in a decision problem are fuzzy, the match between standard levels provided by the decision maker and attribute values becomes vague and, naturally, a matter of degree. The degree of matching is computed using the possibility measure and the necessity measure. The alternative that has the highest degree of matching is considered the best.

Dubois et al. (1988) proposed the fuzzy version of the conjunctive and disjunctive methods. They pointed out that when data in a decision matrix and the decision maker's standard levels are fuzzy, the matching between these two fuzzy data becomes vague and, naturally, a matter of degree. (see Appendix B for more details of this method)

4.7.3 Negi's Approach

Negi (1989) proposed a fuzzy multiple attribute decision making problem where the rating of alternatives and relative weights of criteria are modelled using fuzzy numbers.

In this research, methodologies to solve the linear programming, linear multiple objective decision making, multiple attribute decision making and queuing problems with fuzzy parameters, are presented. Modelling of uncertainty is achieved by using fuzzy numbers.

He provides methods for fuzzification of four already existing crisp methods: Simple Additive Weighting, Linear Assignment, Technique for Order Preference by Similarity to Ideal Solutions (TOPSIS), and Elimination et Choice Translating Reality (ELECTRE).

In addition, the proposed approach is very general and can be applied to any type of MADM problem involving fuzzy data.

4.7.4 Chen and Hwang's Approach

Chen and Hwang (1992) proposed an efficient approach to deal with fuzzy and crisp data together. They pointed out that the existing FMADM methods share one or more of the following pitfalls:

- Size of the MADM problems,

- Fuzzy versus crisp data, and
- Fuzzy singleton.

They overcame the above difficulties in their approach. The basic assumption of this approach is that MADM problem may contain fuzzy and crisp data. This method enables the MADM problem to involve fuzzy data as linguistic terms or fuzzy numbers. The proposed approach consists of two major phases. The first phase converts fuzzy data into crisp data. In the second phase of the method, classical MADM methods can be utilised to specify the ranking order of alternatives. TOPSIS method is used for ranking of alternatives in this method.

This method's main drawback is that the proposed method does not enable multiple experts participation into the MADM problems.

5. PROPOSED METHOD AND ITS METHODOLOGY

In this chapter, the arguments for a new approach to FMADM shall be summarised and shall be explained how it intends to use Fuzzy Set Theory (FST). Proposed method and its methodology shall be explained and discussed and shall be illustrated with an illustrative example. Real case studies will be given in the latter chapter.

5.1 Review of the Existing FMADM Methods

The study of FMADM problems is still in its infancy and still has a lot of room for improvement. After a systematic and critical study of the existing FMADM methods, as described in Chapter 4, the drawbacks of them have been assessed from a practical point of view as follows :

1. Unnecessary fuzzifying of crisp data,

Most approaches of the existing FMADM methods, such as all of the fuzzy linguistic approaches, Baas and Kwakernaak (1977), Dubois and Prade (1983), Bonissone (1982), Laarhoven and Pedrycz (1983), etc., require unnecessary fuzzifying of crisp data so that the each element of the decision matrix must be represented in a fuzzy format, even though they are crisp in real world.

Such an assumption violates the original intent of FST to cope with human subjective judgement. If the data is precisely known, there is no subjectivity involved in the decision problem. Such data should never be represented in any fuzzy format. Too much (or unnecessary) fuzzification does not imply better modelling of reality, on the contrary, many times it can be counterproductive. Over fuzzification can create unnecessary complexity and the fuzzifying of crisp data increases the computational requirements. This makes these methods cumbersome to use and incapable of solving many large decision making problems.

2. Excessive fuzzification,

Since the results of the Phase (I) of the majority of approaches of the FMADM methods are fuzzy, they concentrate on Phase (II) of the FMADM problems. In Phase (II), when the overall aggregated performance ratings are fuzzy numbers, a more sophisticated ranking procedure is required. When final results are crisp numbers, selecting the best alternative or ranking of the alternatives is simple. The best alternative will be the one with the highest overall alternative rating (OAR) value.

Therefore, the need for complex algorithm due to excessive fuzzification of the result of the Phase (I) for ranking of the alternatives could be considered an important drawback.

3. Cumbersome computations, and size of the MADM problem,

The majority of the existing approaches, such as AHP based methods and Takeda (1982), require cumbersome computations and complicated computer programming due to the above mentioned problems. The computational requirements for some of the existing FMADM approaches are tremendous. Therefore none of them is suitable for solving large decision making problems. This reduces the approaches' applicability to MADM problems in which there are approximately more than 10 attributes and more than 10 alternatives. For example, in AHP based approaches, exhaustive pairwise comparison is time consuming if there are many attributes in the MADM problem.

4. Little research on FMADM methods with GDM problems,

There has been little research on FMADM methods with multiple experts. All of the SAW based methods, outranking based techniques, and implied conjunction methods, and Rangone (1998), Dubois et al. (1988), Negi (1989), Chen and Hwang (1992)'s approaches don't handle fuzzy multiple attributive GDM problems. Although on the occasion of making an engineering decision, multiple experts' judgements are frequently required for a correct decision.

5. Outranking relation based FMADM approaches' drawbacks

All of the fuzzy outranking methods require involved and complex computations. They are suitable only during evaluation process for early product design stages, but not for all types of MADM problems.

6. Heuristic approach based on fuzzy decision rules's drawbacks

Most of the effort required by the application of fuzzy decision rules based models is due to the development of the set of fuzzy decision rules. Such a task can be reliably performed only if the attribute that have to be included in the decision are only a few, due to the limited capability of human beings to handle several items of information at once. Hence, these models are especially effective in decision contexts where:

- There aren't too many attributes to consider,
- Several decisions have to be taken on the basis of the same attribute. Hence, having once developed the set of fuzzy rules, each decision can be easily supported.

However, in decision contexts with many attributes to consider, developing the whole set of fuzzy decision rules would be hardly possible.

7. Fuzzy singleton,

Some approaches, such as Yager (1977), etc. , assume that fuzzy datum can be represented by a fuzzy singleton, which is a fuzzy set of only one element with its membership value $\in [0,1]$. Such an assumption is not practical. If a fuzzy singleton is only a real number in [0,1] then its fuzzy matrix is no fuzzier than a normalised decision matrix in the classical MADM study domain. Yager's method is not fuzzy at all.

These drawbacks certainly limit their applicability to real world MADM problems.

The proposed FMADM method is designed to overcome the aforementioned difficulties so that MADM problems can be meaningfully and efficiently solved in a fuzzy environment. The basic assumption of the proposed method is that the MADM problem may contain fuzzy and crisp data and it may consist of multiple expert with the difference degree of importance. The conceptual model of the proposed method is illustrated in Figure 5.1

5.2 States of Proposed Methodology

The proposed method is composed of three major states and the new algorithm will be developed in the following major states :

1. Rating state,
2. Attribute based aggregation state,
3. Selection state.

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5.2 States of Proposed Methodology

The proposed method is composed of three major states and the new algorithm will be developed in the following major states :

1. Rating state,
2. Attribute based aggregation state,
3. Selection state.

In the rating state of the proposed method, each expert (or decision maker) gives his/her opinions (or performance ratings) about alternatives with respect to each subjective attribute. For the subjective attributes these ratings can be fuzzy data. The fuzzy data can be linguistic terms or fuzzy numbers.

The first state aims to convert fuzzy data into standardised positive trapezoidal fuzzy numbers. If the fuzzy data are linguistic terms, they are transformed into fuzzy numbers first by using appropriate conversion scale and then converted to standardised positive trapezoidal fuzzy numbers.

In the second state, attribute based aggregation method for homo/heterogeneous group of experts is employed. Aggregation is necessary only for subjective attributes. After the weights of attributes and the degree of importance of experts are assigned, under each subjective attribute all performance ratings are aggregated for each alternative.

In the last state of the proposed approach, all fuzzy elements of the aggregated decision matrix are deffuzzified in the deffuzzification phase. The result of this phase is a decision matrix which contains only crisp data. Then the alternatives of the problem are ranked by using any classical MADM method such as TOPSIS and SAW.

There are basically two types of attributes, namely subjective and objective attributes. If an assessment for an alternative with respect to an attribute is crisp and identical for all experts of the problem, this attribute is called an "objective attribute".

The main assumption of the proposed method is that experts' opinions for each alternative with respect to each objective attribute are all the same and crisp numbers since subjectiveness is not involved into the MADM problem for these attributes. Therefore there is no need to aggregate (or to combine) experts' opinions for the objective attributes of the MADM problem.

5.2.1 Rating State

In this state, each expert expresses his/her opinions or estimated performance ratings (or performance scores) for each alternative with respect to each subjective attribute. This can be carried out by questionnaires applied to the experts. The questionnaires are used for soliciting expert opinions for each alternative with respect to each subjective attribute.

The estimates of experts of a subjective attribute for an alternative involve subjectiveness, imprecision, and vagueness. For example, these opinions can be linguistic terms such as good, medium, fair etc. or sentences such as "at least two", "the cycle time is about two weeks", or "approximately between 200 and 300". FST can provide us with a useful way to deal with the fuzziness of human judgements.

When experts are not able to give exact numerical values to express their opinions or when the subjectiveness and vagueness involves into the decision problem, then, a more realistic alternative option is using linguistic or fuzzy assessments instead of numerical values. In such a situation, for each variable of problem domain an appropriate linguistic label set is chosen and used by experts who participate in the decision making process to express their opinions. This setting is known as the linguistic setting.

The concept of linguistic variable is very useful in dealing with situations which are too complex or too ill-defined to be reasonably described in conventional quantitative expressions (Zadeh (1975)). A linguistic variable is a variable whose values are not numbers but words or sentences in a natural or artificial language (Zimmermann (1991)). Linguistic values can be words such as very high, low, medium etc. or they can be represented as triangular or trapezoidal fuzzy numbers by the approximate reasoning of FST.

5.2.1.1 Semantic Modelling of Linguistic Terms

The vague information in fuzzy environment can be frequently expressed in linguistic setting. Linguistic terms are not mathematically operable. To cope with that difficulty, each linguistic term is associated with a fuzzy set or a composition of fuzzy sets which represents the meaning of that linguistic term.

Since the meaning of each linguistic term varies from circumstance(s) to circumstance(s), to assign a fuzzy set to a linguistic term is a constant challenge. Existing studies on this topic are few and quite arbitrary. The determination of the membership functions of fuzzy numbers to match the linguistic variables is crucial for solving FMADM problems. It seems difficult to accept that all experts would agree on the same membership function associated to linguistic terms, and therefore there are not any universality in distribution concepts.

An environment is considered where experts can discriminate perfectly the same term set under a similar conception, taking into account that the concept of a linguistic variable serves the purpose of providing a means of approximated characterisation of imprecise preference information.

5.2.1.2 Converting Linguistic Terms to Fuzzy Numbers

If the decision matrix of the problem contains fuzzy data, which may be expressed in linguistic terms or as fuzzy numbers, linguistic terms must first be transformed into fuzzy numbers.

In the proposed method, a numerical approximation system proposed by Chen and Hwang (1992), which may be considered as the first step to a systematic and rational approach to associate fuzzy sets with linguistic terms, is used to systematically convert linguistic terms to their corresponding fuzzy numbers. This system, which was synthesised and modified from several authors' works contains eight conversion scales (see Figures 5.2, 5.3, 5.4, 5.5, 5.6, 5.7, 5.8, and 5.9). There are generic verbal terms (ranging from 2 to 11) in this system where Scale 1 contains only two verbal terms and Scale 8 contains 11 verbal terms. The meaning of each generic verbal term is represented by a fuzzy number.

The principle of this system is to pick a Scale that matches all the linguistic terms in a row (attribute) of the decision matrix and use the fuzzy sets on that scale to represent the meaning of these linguistic terms. The system is used on all rows which contain linguistic terms, one by one.

The linguistic terms used in those conversion scales and their corresponding representations of fuzzy numbers are given in (Table 5.1).

Note that even when the number of terms allowed is the same, the actual verbal terms may be slightly different. It is also worth noting that even when the same term such as "high" is used, the fuzzy numbers graphed are quite different from figure to figure. This reflects the fact that the same linguistic term may possess different meanings for different occasions.

As an example, assume the decision maker gives terms (medium, very high). Figure 5.4 (Scale 3) should be used because Figure 5.4 contains the terms medium and very high. Or assume the terms (medium, high) are used by a decision maker. Although all the scales contain these two terms, the simplest Scale (Scale 1 (Figure 5.2)) is chosen to be our conversion scale. If the terms (medium, high, excellent) are used, Figure 5.9 (Scale 8) is the only figure which matches all the terms given by the decision maker and should be used as the conversion scale.

Table 5.1 Linguistic terms and their corresponding fuzzy numbers used in the proposed approach

	Linguistic terms	Scale 1	Scale 2	Scale 3	Scale 4	Scale 5	Scale 6	Scale 7	Scale 8
1	None								(0, 0, 0.1)
2	Very Low			(0, 0, 0.1, 0.2)		(0, 0, 0.2)	(0, 0, 0.1, 0.2)	(0, 0, 0.2)	(0, 0.1, 0.2)
3	Low - Very Low							(0, 0, 0.1, 0.3)	
4	Low		(0, 0, 0.2, 0.4)	(0.1, 0.25, 0.4)	(0, 0, 0.3)	(0, 0.2, 0.4)	(0.1, 0.2, 0.3)	(0, 0.2, 0.4)	(0.1, 0.3, 0.5)
5	Fairly Low				(0, 0.25, 0.5)	(0.2, 0.4, 0.6)		(0.2, 0.35, 0.5)	(0.3, 0.4, 0.5)
6	Mol. Low						(0.2, 0.3, 0.4, 0.5)		(0.4, 0.45, 0.5)
7	Medium	(0.4, 0.6, 0.8)	(0.2, 0.5, 0.8)	(0.3, 0.5, 0.7)	(0.3, 0.5, 0.7)		(0.4, 0.5, 0.6)	(0.3, 0.5, 0.7)	(0.3, 0.5, 0.7)
8	Mol. High						(0.5, 0.6, 0.7, 0.8)		(0.5, 0.55, 0.6)
9	Fairly High				(0.5, 0.75, 1)	(0.4, 0.6, 0.8)		(0.5, 0.65, 0.8)	(0.5, 0.6, 0.7)
10	High	(0.6, 0.8, 1)	(0.6, 0.8, 1, 1)	(0.6, 0.75, 0.9)	(0.7, 1, 1)	(0.6, 0.8, 1)	(0.7, 0.8, 0.9)	(0.6, 0.8, 1)	(0.5, 0.7, 0.9)
11	High - Very High							(0.7, 0.9, 1, 1)	
12	Very High			(0.8, 0.9, 1, 1)		(0.8, 1, 1)	(0.8, 0.9, 1, 1)	(0.8, 1, 1)	(0.8, 0.9, 1)
13	Excellent								(0.9, 1, 1)

Where Mol. : more or less

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3	Low - Very Low							(0, 0, 0.1, 0.3)	
4	Low		(0, 0, 0.2, 0.4)	(0.1, 0.25, 0.4)	(0, 0, 0.3)	(0, 0.2, 0.4)	(0.1, 0.2, 0.3)	(0, 0.2, 0.4)	(0.1, 0.3, 0.5)
5	Fairly Low				(0, 0.25, 0.5)	(0.2, 0.4, 0.6)		(0.2, 0.35, 0.5)	(0.3, 0.4, 0.5)
6	Mol. Low						(0.2, 0.3, 0.4, 0.5)		(0.4, 0.45, 0.5)
7	Medium	(0.4, 0.6, 0.8)	(0.2, 0.5, 0.8)	(0.3, 0.5, 0.7)	(0.3, 0.5, 0.7)		(0.4, 0.5, 0.6)	(0.3, 0.5, 0.7)	(0.3, 0.5, 0.7)
8	Mol. High						(0.5, 0.6, 0.7, 0.8)		(0.5, 0.55, 0.6)
9	Fairly High				(0.5, 0.75, 1)	(0.4, 0.6, 0.8)		(0.5, 0.65, 0.8)	(0.5, 0.6, 0.7)
10	High	(0.6, 0.8, 1)	(0.6, 0.8, 1, 1)	(0.6, 0.75, 0.9)	(0.7, 1, 1)	(0.6, 0.8, 1)	(0.7, 0.8, 0.9)	(0.6, 0.8, 1)	(0.5, 0.7, 0.9)
11	High - Very High							(0.7, 0.9, 1, 1)	
12	Very High			(0.8, 0.9, 1, 1)		(0.8, 1, 1)	(0.8, 0.9, 1, 1)	(0.8, 1, 1)	(0.8, 0.9, 1)
13	Excellent								(0.9, 1, 1)

Where Mol. : more or less

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8	Mol. High						(0.5, 0.6, 0.7, 0.8)		(0.5, 0.55, 0.6)
9	Fairly High				(0.5, 0.75, 1)	(0.4, 0.6, 0.8)		(0.5, 0.65, 0.8)	(0.5, 0.6, 0.7)
10	High	(0.6, 0.8, 1)	(0.6, 0.8, 1, 1)	(0.6, 0.75, 0.9)	(0.7, 1, 1)	(0.6, 0.8, 1)	(0.7, 0.8, 0.9)	(0.6, 0.8, 1)	(0.5, 0.7, 0.9)
11	High - Very High							(0.7, 0.9, 1, 1)	
12	Very High			(0.8, 0.9, 1, 1)		(0.8, 1, 1)	(0.8, 0.9, 1, 1)	(0.8, 1, 1)	(0.8, 0.9, 1)
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2	Very Low			(0, 0, 0.1, 0.2)		(0, 0, 0.2)	(0, 0, 0.1, 0.2)	(0, 0, 0.2)	(0, 0.1, 0.2)
3	Low - Very Low							(0, 0, 0.1, 0.3)	
4	Low		(0, 0, 0.2, 0.4)	(0.1, 0.25, 0.4)	(0, 0, 0.3)	(0, 0.2, 0.4)	(0.1, 0.2, 0.3)	(0, 0.2, 0.4)	(0.1, 0.3, 0.5)
5	Fairly Low				(0, 0.25, 0.5)	(0.2, 0.4, 0.6)		(0.2, 0.35, 0.5)	(0.3, 0.4, 0.5)
6	Mol. Low						(0.2, 0.3, 0.4, 0.5)		(0.4, 0.45, 0.5)
7	Medium	(0.4, 0.6, 0.8)	(0.2, 0.5, 0.8)	(0.3, 0.5, 0.7)	(0.3, 0.5, 0.7)		(0.4, 0.5, 0.6)	(0.3, 0.5, 0.7)	(0.3, 0.5, 0.7)
8	Mol. High						(0.5, 0.6, 0.7, 0.8)		(0.5, 0.55, 0.6)
9	Fairly High				(0.5, 0.75, 1)	(0.4, 0.6, 0.8)		(0.5, 0.65, 0.8)	(0.5, 0.6, 0.7)
10	High	(0.6, 0.8, 1)	(0.6, 0.8, 1, 1)	(0.6, 0.75, 0.9)	(0.7, 1, 1)	(0.6, 0.8, 1)	(0.7, 0.8, 0.9)	(0.6, 0.8, 1)	(0.5, 0.7, 0.9)
11	High - Very High							(0.7, 0.9, 1, 1)	
12	Very High			(0.8, 0.9, 1, 1)		(0.8, 1, 1)	(0.8, 0.9, 1, 1)	(0.8, 1, 1)	(0.8, 0.9, 1)
13	Excellent								(0.9, 1, 1)

Where Mol. : more or less

Table 5.1 Linguistic terms and their corresponding fuzzy numbers used in the proposed approach

	Linguistic terms	Scale 1	Scale 2	Scale 3	Scale 4	Scale 5	Scale 6	Scale 7	Scale 8
1	None								(0, 0, 0.1)
2	Very Low			(0, 0, 0.1, 0.2)		(0, 0, 0.2)	(0, 0, 0.1, 0.2)	(0, 0, 0.2)	(0, 0.1, 0.2)
3	Low - Very Low							(0, 0, 0.1, 0.3)	
4	Low		(0, 0, 0.2, 0.4)	(0.1, 0.25, 0.4)	(0, 0, 0.3)	(0, 0.2, 0.4)	(0.1, 0.2, 0.3)	(0, 0.2, 0.4)	(0.1, 0.3, 0.5)
5	Fairly Low				(0, 0.25, 0.5)	(0.2, 0.4, 0.6)		(0.2, 0.35, 0.5)	(0.3, 0.4, 0.5)
6	Mol. Low						(0.2, 0.3, 0.4, 0.5)		(0.4, 0.45, 0.5)
7	Medium	(0.4, 0.6, 0.8)	(0.2, 0.5, 0.8)	(0.3, 0.5, 0.7)	(0.3, 0.5, 0.7)		(0.4, 0.5, 0.6)	(0.3, 0.5, 0.7)	(0.3, 0.5, 0.7)
8	Mol. High						(0.5, 0.6, 0.7, 0.8)		(0.5, 0.55, 0.6)
9	Fairly High				(0.5, 0.75, 1)	(0.4, 0.6, 0.8)		(0.5, 0.65, 0.8)	(0.5, 0.6, 0.7)
10	High	(0.6, 0.8, 1)	(0.6, 0.8, 1, 1)	(0.6, 0.75, 0.9)	(0.7, 1, 1)	(0.6, 0.8, 1)	(0.7, 0.8, 0.9)	(0.6, 0.8, 1)	(0.5, 0.7, 0.9)
11	High - Very High							(0.7, 0.9, 1, 1)	
12	Very High			(0.8, 0.9, 1, 1)		(0.8, 1, 1)	(0.8, 0.9, 1, 1)	(0.8, 1, 1)	(0.8, 0.9, 1)
13	Excellent								(0.9, 1, 1)

Where Mol. : more or less

The verbal terms used in these scales are in the universe U = {excellent, very high, high to very high, high, fairly high, mol.high, medium, mol.low, fairly low, low, low to very low, very low, none}. This universe of verbal terms may be appropriate to describe the reliability of a ship propulsion system but certainly is not suitable for describing the distance of two places or two objects. Fortunately, this system does not confine itself to that universe. Rather, the universe can be adjusted to fit the nature of attributes used in a decision problem. For example, if price is one of the attributes, the possible universe will be (extremely expensive, very expensive, ... , fair price, fairly cheap, ... , extremely cheap). Or if size is one of the attributes, the possible universe will be (extremely small, very small, ... , medium, medium large, ... , extremely large). For any type of the attributes, it can be always found that a pair of words that represents extreme meanings, such as high vs. low, good vs. poor, small vs. large, and so on.

Ultimately, the standard scales system is capable of converting linguistic terms into fuzzy numbers in a systematic manner. Such characteristics guarantee the consistency of translating linguistic terms to fuzzy numbers.

The determination of the number of conversion scales in this conversion system is rather intuitive. Too few conversion scales provide no more help than previous research results, while too many conversion scales may make the system too complex to be practical (Chen and Hwang (1992)).

This conversion system is employed in the proposed approach since this system is simple enough to be understood by the decision maker, and easy to use by engineers.

5.2.2 Attribute Based Aggregation State

Aggregation means to combine or to pool the experts' opinions.

In the MADM with GDM problems generally there arise situations of conflict and agreement among the experts as each expert has his/her own opinion or estimated rating under each attribute for each alternative. Hence, finding a group consensus function of aggregating these estimated ratings to represent a common opinion is an important issue. The purpose of this state is to establish a procedure to combine a group of experts' opinions to form a group consensus opinion.

Sometimes, one may admit that the various individuals that give the opinions are not equally important (e.g. reliable). In such a case, it is called heterogeneous (non-homogeneous) group decision making problem and, otherwise, homogeneous group decision making problem. One way of modelling this aspect is to consider the existence of a manager (or moderator) that assigns a weight to each expert.

In general, the relative importance of each decision maker or expert may not be equal. Sometimes there are important experts in decision group, such as the executive director of a shipyard, or some experts who are more experienced than others on the evaluation one or more attributes, the final decision is influenced by the different importance of each expert.

Therefore, a good method of aggregating multiple expert opinions must consider the degree of importance of each expert in the aggregation procedure.

In this state, an aggregation approach for homo/heterogeneous group of experts under each subjective attribute is employed. Since the aggregation is based on each subjective attribute, expert weighting is also determined separately for each subjective attribute. This yields more accurate and reliable models. When more than one expert involve into the selection problems, each expert must have a different weight. For ex: An expert, who is very experienced on finance, may not give good assessments for technical attributes as in finance. For that reason attribute based expert weighting is a necessity.

5.2.2.1 Trapezoidal Fuzzy Number Aggregation

In the aggregation state of the proposed approach, the method presented in Chen (1998) is used for dealing with fuzzy opinion aggregation with GDM problems. This algorithm essentially is a modification of the study proposed in Hsu and Chen (1996). Chen (1998) overcame the drawbacks of the study presented in Hsu and Chen (1996) due to the fact that

1. The experts' estimates do not necessarily have a common intersection at the α level, where $\alpha \in (0, 1]$. Thus, it is more flexible than the one presented in Hsu and Chen.

2. It does not need to use the Delphi method to adjust trapezoidal fuzzy numbers given by experts.

3. It can perform fuzzy opinion aggregation in a more efficient manner, since it can calculate the degree of similarity between the subjective estimates of experts in a more efficient manner.

Let U be the universe of discourse, U=[0, m]. Assume that each expert E_i (i=1, 2, … , M) constructs a positive trapezoidal fuzzy number $R_i=(a_i, b_i, c_i, d_i)$ to represent the subjective estimate of the rating to a given attribute and alternative, where $0 \le a_i \le b_i \le c_i \le d_i \le m$. Furthermore, assume that the degree of importance of expert E_i (i=1, 2, … , M) is w_i, where $w_i \in [0, 1]$ and $\sum_{i=1}^{M} w_i = 1$.

In some cases, the relative importance of experts is widely different. Some are more important than others, such as the production manager of a shipyard and productivity experts are more important than others, if the problem is determination of a shipbuilding startegy. Therefore, the relative importance weight of each expert is considered. First, the most important person is selected among experts and weight one is assigned him/her, i.e. $r_i=1$. Then the kth expert is compared with the most important person and a relative weight for the kth expert r_k, k=1, 2, … , M, is obtained. So we have $\max\{r_1, r_2, \dots, r_M\}=1$ and $\min\{r_1, r_2, \dots, r_M\}>0$. Finally, the degree of importance w_i is defined as follows:

$$w_i = \frac{r_i}{\sum_{i=1}^{M} r_i} \tag{5.1}$$

If the importance of each expert is equal then $w_1 = w_2 = \dots = w_M = 1/M$.

The aggregation algorithm for homo/heterogeneous group of experts is presented as follows :

Step 1. Translate each trapezoidal fuzzy number $R_i=(a_i, b_i, c_i, d_i)$ given by expert E_i into standardised trapezoidal fuzzy number R_i(i=1, 2, … , M), where

$$R_i=(a_i/m, b_i/m, c_i/m, d_i/m) = (a_i^*, b_i^*, c_i^*, d_i^*) \tag{5.2}$$

and $0 \le a_i^* \le b_i^* \le c_i^* \le d_i^* \le 1$.

Step 2. Calculate the degree of agreement (or degree of similarity) $S(R_i, R_j)$ of the opinions between each pair of experts E_i and E_j, where $S(R_i, R_j) \in [0, 1]$, $1 \le i \le M$, $1 \le j \le M$, and $i \ne j$.

A new method introduced by Chen and Lin (1995) is used for measuring the degree of similarity between trapezoidal fuzzy numbers. According to this new approach, let A and B be two standardised trapezoidal fuzzy numbers,

$A = (a_1, a_2, a_3, a_4)$ and

$B = (b_1, b_2, b_3, b_4)$

where $0 \le a_1 \le a_2 \le a_3 \le a_4 \le 1$ and $0 \le b_1 \le b_2 \le b_3 \le b_4 \le 1$. Then the degree of similarity between the standardised trapezoidal fuzzy numbers A and B can be measured by the similarity function S,

$$S(A,B) = 1 - \frac{|a_1 - b_1| + |a_2 - b_2| + |a_3 - b_3| + |a_4 - b_4|}{4} \tag{5.3}$$

where $S(A, B) \in [0, 1]$. Larger the value of S(A, B), greater the similarity between the standardised trapezoidal fuzzy numbers A and B. It should be noted that S(A, B) = S(B, A).

Step 3. Construct the agreement matrix (AM), after all the agreement (or similarity) degrees between experts are measured,

$$AM = \begin{bmatrix} 1 & S_{12} & \cdots & S_{1j} & \cdots & S_{1M} \\ \cdots & \cdots & & \cdots & & \cdots \\ S_{i1} & S_{i2} & \cdots & S_{ij} & \cdots & S_{iM} \\ \cdots & \cdots & & \cdots & & \cdots \\ S_{M1} & S_{M2} & \cdots & S_{Mj} & \cdots & 1 \end{bmatrix}$$

Where $S_{ij} = S(R_i, R_j)$, if $i \neq j$ and $S_{ij} = 1$, if $i = j$. By the definition of $S(R_i, R_j)$, the diagonal elements of AM are unity.

Step 4. Calculate the average degree of agreement $AA(E_i)$ of expert E_i (i=1, 2, … , M) by using the AM of the problem, where

$$AA(E_i) = \frac{1}{M-1} \sum_{\substack{j=1 \\ j \neq i}}^{M} S(R_i, R_j) \tag{5.4}$$

Step 5. Calculate the relative degree of agreement $RA(E_i)$ of expert E_i (i=1, 2, … , M), where

$$RA(E_i) = \frac{AA(E_i)}{\sum_{i=1}^{M} AA(E_i)} \tag{5.5}$$

Step 6. Calculate the consensus degree coefficient $CC(E_i)$ of expert E_i ($i=1, 2, \ldots, M$), where

$$CC(E_i) = \beta.w_i + (1-\beta).RA(E_i) \tag{5.6}$$

where β ($0 \leq \beta \leq 1$) is a relaxation factor of the proposed method. It shows the importance of the w_i over $RA(E_i)$. When $\beta = 0$, the problem is a homogeneous group of experts problem as shown below.

$$\beta = \begin{cases} 0, & \text{Homogeneous Group of Experts} \\ 0 < \beta \leq 1 & \text{Heterogeneous Group of Experts} \end{cases}$$

The consensus degree coefficient of each expert is a good measure for evaluating the relative worthiness of each expert's opinions.

Step 7. The aggregation result of the fuzzy opinions is R_{AG} as

$$R_{AG}=CC(E_1) \otimes R_1 \oplus CC(E_2) \otimes R_2 \oplus \ldots \oplus CC(E_M) \otimes R_M \tag{5.7}$$

Where operators $\otimes$ and $\oplus$ are the fuzzy multiplication operator and the fuzzy addition operator, respectively.

The proposed method is independent of the type of membership functions being used. Some other membership functions, for example triangular membership functions, are also applicable.

The reason of using trapezoidal or triangular fuzzy numbers is that it is intuitively easy to be used by the decision makers.

5.2.2.1.1.1 Properties of the Aggregation Approach

The aggregation method, which is used for the proposed approach, preserves some important properties. These properties are as follows:

(1) Agreement preservation (Bardossy et al. (1993)). If $R_i = R_j$ for all i, j, then $R_{AG} = R_i$ (or R_j). In other words, if all estimates are identical the combined result should be

the common estimate. Agreement preservation is a consistency requirement. Proof of this property can be found in Hsu and Chen (1996).

(2) If the fuzzy opinions of all experts can be represented by a positive trapezoidal fuzzy number, then the membership function of the combination is also a positive trapezoidal fuzzy number. This property reduces the complexity of mathematical analysis process in group decision making.

(3) The common intersection area of all experts' estimates is included in the aggregation result. It means that $\bigcap_{i=1}^{M} R_i \subseteq R_{AG}$. Proof of this property can also be found in Hsu and Chen (1996).

(4) If an expert's estimate is far from the others, then his estimate is less important.

(5) Order independence (Bardossy et al. (1993)). Obviously, the results of the aggregation method would not depend on order with which individual opinions or estimates are combined.

(6) Individual versus overall (combination) uncertainty (Bardossy et al. (1993)). Let the uncertainty measure $H(R_i)$ of each expert estimate's R_i be defined as the area under its membership function

$$H(R_i) = \int_{-\infty}^{\infty} \mu_{R_i}(x)dx$$

The uncertainty measure H defined in above equation fulfils the following equation :

$$H(R_{AG}) = \sum_{i=1}^{M} CC_i \times H(R_i)$$

This means that the uncertainty after combination is a mean of the uncertainties of each expert. Therefore, the uncertainty of the aggregation result can be computed between the uncertainties of all experts, i.e. $\min_i H(R_i) \leq H(R_{AG}) \leq \max_i H(R_i)$. This is a reasonable result for combining the opinions of all experts.

5.2.3 Selection State

Up to this state, we have aggregated all experts' performance ratings for each alternative under each subjective attribute. In order to rank the alternatives of the problem, we have to defuzzify all aggregated trapezoidal fuzzy numbers so that all components of the aggregated decision matrix are all crisp numbers and any classical MADM method can be used.

The selection state consists of two major phases. They are defuzzification, and ranking phases respectively.

5.2.3.1 Defuzzification Phase

This phase allows fuzzy numbers to be translated to crisp values early on, so that the arithmetic process of the proposed approach becomes easy.

Defuzzifying (or transforming) approaches (Chen and Hwang (1992), Kim and Park (1990), Tseng and Klein (1992)), which transform the fuzzy data into numeric data, have been developed so that MADM problems can be meaningfully and efficiently solved in a fuzzy environment.

Chen and Klein (1997) proposed an approach using six defuzzifying methods for the FMADM problems. The computational effectiveness and efficiency of six defuzzifying methods combined with the SAW method were evaluated based on a comparison to the IFWA (Improved Fuzzy Weighted Average) followed by four fuzzy ranking methods. Numerical examples were also discussed to demonstrate the implementation and effectiveness of the methods.

In this phase of the proposed approach, all the aggregated fuzzy numbers (or fuzzy sets) of linguistic terms (or ratings) are transformed into numeric ratings using assigned crisp scores approach (Chen and Hwang (1992)) as explained below. The result of this phase is a decision matrix which contains only numeric ratings or crisp data.

In general, mathematical computations are reduced to a minimum. The easy to use and easy to understand characteristics of this defuzzifying approach make it valuable to management and system analysts (Chen and Hwang (1992), Mabuchi (1988)).

A fuzzy scoring method proposed by Chen and Hwang (1992) converts fuzzy numbers to crisp scores. This method is a modification of Chen (1985)'s fuzzy ranking approach. The crisp score of a fuzzy number B is obtained as follows .

$$\mu_{max}(x) = \begin{cases} x, 0 \le x \le 1 \\ 0, \text{otherwise} \end{cases} \qquad (5.8)$$

$$\mu_{min}(x) = \begin{cases} 1 - x, 0 \le x \le 1 \\ 0, \text{otherwise} \end{cases} \quad (5.9)$$

The right score of B can be determined using .

$$\mu_R(B) = \sup_x [\mu_M(x) \Lambda \mu_{\max}(x)] \quad (5.10)$$

The left score of B can be determined using .

$$\mu_L(B) = \sup_x [\mu_M(x) \Lambda \mu_{\min}(x)] \quad (5.11)$$

Given the left and right scores of B, the total score of B can be computed using

$$\mu_T(B) = [\mu_R(B) + 1 - \mu_L(B)]/2 \quad (5.12)$$

5.2.3.2 Ranking Phase

In the ranking phase of the selection state, classical MADM methods can be utilised to determine the ranking order of the alternatives. "Technique for Order Preference by Similarity to Ideal Solution" (TOPSIS) method is used in this phase because of its general and broad acceptability in many problem domains. TOPSIS gives cardinal order of the alternatives.

TOPSIS is quite effective in identifying the best alternative quickly. The underlying logic premise of the TOPSIS method is that an alternative that is more like an ideal alternative (the best that could be imagined) and more unlike a negative-deal alternative (the worst that could be imagined) should be preferred. In the TOPSIS method, the ideal alternative is constructed out of exclusively the best attribute values attainable and therefore it is usually an `invented' alternative. The negative-ideal alternative is also usually an invented alternative that is constructed out of exclusively the worst attribute values attainable. The relative closeness (similarity) of each alternative to the ideal alternative is rated on the basis of its distances from both the ideal and the negative- ideal alternatives simultaneously. Finally, the preference order of the alternatives is obtained by their rank on a descending order of those ratings. The computational procedure of the TOPSIS method is quite straightforward.

5.2.3.2.1 TOPSIS

Hwang and Yoon (1981) developed the TOPSIS method based on the concept that the chosen alternative should have the shortest distance from the positive-ideal solution and the longest distance from the negative-ideal solution.

TOPSIS defines an index called similarity (or relative closeness) to the positive-ideal solution by combining the proximity to the positive-ideal solution and remoteness from the negative-ideal solution. Then the method chooses an alternative with the maximum similarity to the positive-ideal solution. TOPSIS assumes that each attribute takes either monotonically increasing or monotonically decreasing utility. That is, the larger the attribute outcome, the greater the preference for benefit attributes and the less the preference for cost attributes (Yoon and Hwang (1995)).

According to TOPSIS, following steps are to be performed:

Step 1. *Calculate Normalised Ratings*. This step tries to transform various attribute dimensions into the non-dimensional attribute, which allows comparison across the attributes. The vector normalisation technique is used for computing the element (r_{ij}) of the normalised decision matrix, which is given as

$$r_{ij} = \frac{x_{ij}}{\sqrt{\sum_{i=1}^{m} x_{ij}^2}}, i = 1, 2, \dots, m; j = 1, 2, \dots, n. \tag{5.13}$$

where x_{ij} is the value of alternative i with respect to attribute j.

Step 2. *Calculate Weighted Normalised Ratings*. A set of attribute weights assessed from the decision maker is accommodated to the normalised decision matrix in this step. The weighted normalised decision matrix can be calculated by multiplying each row of the normalised decision matrix with its associated attribute weight w_j. An element of the weighted normalised decision matrix is calculated as

$$v_{ij} = w_j r_{ij} \quad , i = 1, 2, \dots, m; j = 1, 2, \dots, n. \tag{5.14}$$

where w_j is the weight of the jth attribute.

There are many methods for assigning attribute weights such as weighted evaluation technique (WET), eigenvector method, entropy method, etc. In the proposed method, WET is used for finding the attribute weights. WET is a conventional and highly useful weighting technique.

According to WET, the moderator (or manager) begins by rank ordering attributes and attribute relative importances are assigned on a zero to 100 scale. The attribute perceived as most important is assigned a weight of 100; all other attribute relative importances are assigned relative to that.

The final step of the weighting procedure is to normalise the relative importances, $\{r_1, r_2, \dots, r_M\}$, to obtain the weights $\{w_1, w_2, \dots, w_M\}$. The standard normalisation is

$$w_i = \frac{r_i}{\sum_{i=1}^{M} r_i}, \text{i} = 1, 2, \dots, \text{M}, \tag{5.15}$$

where $0 \leq w_i \leq 1$ and $\sum_{i=1}^{M} w_i = 1$.

Step 3. *Identify Positive-Ideal and Negative-Ideal Solutions.* Let the positive-ideal solution, A^*, and the negative-ideal solution, A^-, be defined in terms of the weighted normalised values :

$A^* = \{v_1^*, v_2^*, \dots, v_j^*, \dots, v_n^*\}$, where

$$v_j^* = \left\{\max_i v_{ij}, j \in J_1; \quad \min_i v_{ij}, j \in J_2\right\} \tag{5.16}$$

$A- = \{v_1^-, v_2^-, \dots, v_j^-, \dots, v_n^-\}$, where

$$v_j^- = \left\{\min_i v_{ij}, j \in J_1; \quad \max_i v_{ij}, j \in J_2\right\} \tag{5.17}$$

where J_1 is the set of benefit attributes (the larger, the more preference) and J_2 is the set of cost attributes (the larger, the less preference).

Step 4. *Calculate Separation Measures.* Separation (distance) between alternatives can be measured by the n-dimensional Euclidean distance. Separation of each alternative from the positive-ideal solution is then given by

$$S_i^* = \sqrt{\sum_{j=1}^{n} (v_{ij} - v_j^*)^2} \qquad \text{i} = 1, 2, \dots, \text{m} \tag{5.18}$$

Similarly, separation from the negative-ideal solution is then given by

$$S_i^- = \sqrt{\sum_{j=1}^{n}(v_{ij} - v_j^-)^2} \qquad i = 1, 2, \ldots, m \tag{5.19}$$

Step 5. *Calculate Similarities to Positive-Ideal Solution.* Relative closeness (or similarity) of A_i with respect to A^* is defined as

$$C_i^* = \frac{S_i^-}{S_i^* + S_i^-}, 0 < C_i^* < 1; i = 1, 2, \ldots, m \tag{5.20}$$

When C_i^* is close to 1, the alternative is regarded as ideal; and when C_i^* is close to 0, the alternative is regarded as non-ideal.

Step 6. *Rank Preference Order.* Choose an alternative with the maximum C_i^* or rank alternatives according to C_i^* in descending order. It is clear that an alternative A_i is closer to A^* than to A^- as C_i^* approaches 1.

The proposed algorithm contains the following steps:

1. Form a committee of experts (or decision makers), then identify the selection attributes and list all possible alternatives to choose from them.

2. Collect each expert's opinion in linguistic, fuzzy or crisp settings and establish a decision matrix for each expert. This can be achieved by using questionnaires.

3. Transform the fuzzy data (linguistic expressions and fuzzy assessments) into standardised positive trapezoidal fuzzy numbers attribute by attribute by using Equation (5.2). The process continues until all linguistic terms under every attribute have been converted to standardised positive trapezoidal fuzzy numbers.

4. Define the attribute types (cost or benefit), and assign the relative importances of experts and attributes, and then calculate the weights of them by using Equations (5.1) and (5.15) respectively.

5. Under each subjective attribute, aggregate all experts' fuzzy opinions for each alternative by using the Equations (5.3), (5.4), (5.5), (5.6), and (5.7). This step gives us aggregated matrices for homo/heterogeneous group of experts.

6. Deffuzzify these matrices by applying Equations (5.8), (5.9), (5.10), (5.11), and (5.12) on every fuzzy number in question. Up to this point, we have transformed the aggregated decision matrices with fuzzy elements into ones with crisp numbers.

7. Construct the normalised ratings, and weighted normalised ratings of the deffuzzified matrices by using Equations (5.13) and (5.14).

8. Then calculate positive-ideal and negative-ideal solutions, seperation measures, and similarities of each alternative by using Equations (5.16), (5.17), (5.18), (5.19), and (5.20).

9. Order or rank the alternatives according to the Overall Alternative Rating (OAR) values (C_i^* values) and select the alternative with the maximum OAR value as the best alternative.

Based on the above stepwise algorithm, this methodology has been coded into an interactive PC-based computer program incorporating two modules. The rating and aggregation states of the proposed method is included in the first module of the computer program. For the first module of the program, Microsoft Excel has been used. Second module uses MATLAB 5.2 with its Fuzzy ToolBox to make the necessary calculations for the selection state of the proposed method. The output of the first module is the input of the Second module.

5.3 Illustrative Example

An illustrative selection problem is designed to demonstrate the computational process of the proposed method presented in this chapter.

Let E be the set of experts, let A be the set of attributes, and let X be the set of alternatives, where $E = \{E_1, E_2, E_3\}$, $A = \{A_1, A_2, A_3, A_4\}$, and $X = \{X_1, X_2, X_3\}$. E, A, and X sets are all non-empty and finite sets.

Assume that three experts (E_1, E_2, E_3) are based on the four selected attribute (A_1, A_2, A_3, A_4) in choosing the most appropriate alternative among the three alternatives (X_1, X_2, X_3). Let R_{ij}^k, $1 \leq k \leq 3$, $1 \leq i \leq 4$ and $1 \leq j \leq 3$, be the assigned rating of alternative X_j with respect to attribute A_i by the expert E_k.

5.3.1 Rating State Calculations

Attributes are classified as subjective and objective attributes. Let subjective attributes be A_1, A_2, and A_3 and the objective attribute be A_4. The experts utilise either the eight scales which contain linguistic rating sets described in Section 5.2.1.2 or fuzzy numbers by directly giving the triangular or trapezoidal type of fuzzy number to assess the rating of alternatives under each of the subjective attribute.

Experts' opinions are collected by the questionnaires and then decision matrices of each expert are formed. The ratings of alternatives versus each attribute given by each expert are presented in Tables 5.2, 5.3, and 5.4. Remember that since A_4 is an objective attribute, ratings for this attribute will be the same for each expert.

Table 5.2 First expert's evaluation of three alternatives under the four attributes

	X_1	X_2	X_3
A_1	Very good	Poor	Fair
A_2	Approximately equal to 50	Approximately equal to 45	Approximately equal to 48
A_3	Approximately between 150 and 170	Approximately between 155 and 165	Approximately between 150 and 175
A_4	1000	700	850

Table 5.3 Second expert's evaluation of three alternatives under the four attributes

	X_1	X_2	X_3
A_1	Good	Poor	Fair
A_2	Approximately equal to 49	Approximately equal to 49	Approximately equal to 47
A_3	Approximately between 145 and 175	Approximately between 145 and 160	Approximately between 150 and 175
A_4	1000	700	850

Table 5.4 Third expert's evaluation of three alternatives under the four attributes

	X_1	X_2	X_3
A_1	Poor	Good	Good
A_2	Approximately equal to 47	Approximately equal to 48	Approximately equal to 49
A_3	Approximately between 140 and 155	Approximately between 150 and 170	Approximately between 160 and 165
A_4	1000	700	850

'Approximately equal to 45' can be represented by the triangular fuzzy number of (44, 45, 46) or by the trapezoidal fuzzy number of (44, 45, 45, 46) and also 'approximately between 140 and 155' can be represented by the trapezoidal fuzzy number of (135, 140, 155, 160).

Since the ratings under A_1 are linguistic terms, they must be transformed into fuzzy numbers by using appropriate scale described in Section 5.2.1.2. For the first attribute, we match the linguistic terms with scale 3. The results are given in Tables 5.5, 5.6, and 5.7.

Table 5.5 First expert's decision matrix with fuzzy numbers

	X_1	X_2	X_3
A_1	(0.8, 0.9, 1, 1)	(0.1, 0.25, 0.4)	(0.3, 0.5, 0.7)
A_2	(49, 50, 51)	(44, 45, 46)	(46, 48, 49)
A_3	(149, 150, 170, 175)	(150, 155, 165, 170)	(145, 150, 175, 180)
A_4	1000	700	850

Table 5.6 Second expert's decision matrix with fuzzy numbers

	X_1	X_2	X_3
A_1	(0.6, 0.75, 0.9)	(0.1, 0.25, 0.4)	(0.3, 0.5, 0.7)
A_2	(48, 49, 51)	(47, 49, 51)	(46, 47, 48)
A_3	(140, 145, 175, 180)	(140, 145, 160, 165)	(145, 150, 175, 180)
A_4	1000	700	850

Table 5.7 Third expert's decision matrix with fuzzy numbers

	X_1	X_2	X_3
A_1	(0.1, 0.25, 0.4)	(0.6, 0.75, 0.9)	(0.6, 0.75, 0.9)
A_2	(45, 47, 48)	(47, 48, 49)	(47, 49, 52)
A_3	(135, 140, 155, 160)	(145, 150, 170, 175)	(155, 160, 165, 170)
A_4	1000	700	850

For the second and third attributes, all non-standardised fuzzy numbers must be converted to standardised trapezoidal fuzzy numbers by using Equation (5.2). These converted fuzzy numbers are shown in Tables 5.8, 5.9, and 5.10.

Table 5.8 First expert's decision matrix with standardised trapezoidal fuzzy numbers

	X_1	X_2	X_3
A_1	(0.8, 0.9, 1, 1)	(0.1, 0.25, 0.25, 0.4)	(0.3, 0.5, 0.5, 0.7)
A_2	(0.94, 0.96, 0.96, 0.98)	(0.85, 0.87, 0.87, 0.88)	(0.88, 0.92, 0.92, 0.94)
A_3	(0.81, 0.83, 0.94, 0.97)	(0.83, 0.86, 0.92, 0.94)	(0.81, 0.83, 0.97, 1)
A_4	1000	700	850

Table 5.9 Second expert's decision matrix with standardised trapezoidal fuzzy numbers

	X_1	X_2	X_3
A_1	(0.6, 0.75, 0.75, 0.9)	(0.1, 0.25, 0.25, 0.4)	(0.3, 0.5, 0.5, 0.7)
A_2	(0.92, 0.94, 0.94, 0.98)	(0.90, 0.94, 0.94, 0.98)	(0.88, 0.90, 0.90, 0.92)
A_3	(0.78, 0.81, 0.97, 1)	(0.78, 0.81, 0.89, 0.92)	(0.81, 0.83, 0.97, 1)
A_4	1000	700	850

Table 5.10 Third expert's decision matrix with standardised trapezoidal fuzzy numbers

	X_1	X_2	X_3
A_1	(0.1, 0.25, 0.25, 0.4)	(0.6, 0.75, 0.75, 0.9)	(0.6, 0.75, 0.75, 0.9)
A_2	(0.87, 0.90, 0.90, 0.92)	(0.90, 0.92, 0.92, 0.94)	(0.90, 0.94, 0.94, 1)
A_3	(0.75, 0.78, 0.86, 0.89)	(0.81, 0.83, 0.94, 0.97)	(0.86, 0.89, 0.92, 0.94)
A_4	1000	700	850

5.3.2 Aggregation State Calculations

The manager (or moderator) evaluates the importance weights and types of each selection attribute. The importance weight of each attribute is obtained by WET. The moderator also defines the degree of importance of each expert under each subjective attribute. Table 5.11 shows the weights of attributes and experts and type of attributes.

Table 5.11 Experts and attributes' weights and type of attributes

	Attributes			E_1		E_2		E_3	
	Type	R.I.	w.	R.I.	w.	R.I.	w.	R.I.	w.
A_1	Cost	20	0.1	0.2	0.11	1	0.53	0.7	0.37
A_2	Benefit	40	0.2	0.3	0.17	0.5	0.28	1	0.56
A_3	Cost	100	0.5	1	0.57	0.35	0.20	0.4	0.23
A_4	Benefit	40	0.2						
Where R.I. : Relative Importance									

Since we have three subjective attributes, we aggregate the estimate ratings of experts for each alternative under each subjective attribute. Attribute based aggregation calculations are given in Tables 5.12, 5.13, and 5.14. In these Tables, R_{AG}^{HM} values are aggregated fuzzy numbers for homogeneous group of experts and R_{AG}^{HT} values are aggregated fuzzy numbers for heterogeneous group of experts.

Remember that β is taken as 0.4 in this illustrative case. Experts' opinions for each alternative with respect to each subjective attribute and their aggregation calculations and results for homo/heterogeneous group of experts are given in Figures 5.10, 5.11, 5.12, 5.13, 5.14, 5.15, 5.16, 5.17, and 5.18.

Table 5.12 Aggregation under the first attribute (A_1)

	X_1	X_2	X_3
E_1	(0.8, 0.9, 1, 1)	(0.1, 0.25, 0.25, 0.4)	(0.3, 0.5, 0.5, 0.7)
E_2	(0.6, 0.75, 0.75, 0.9)	(0.1, 0.25, 0.25, 0.4)	(0.3, 0.5, 0.5, 0.7)
E_3	(0.1, 0.25, 0.25, 0.4)	(0.6, 0.75, 0.75, 0.9)	(0.6, 0.75, 0.75, 0.9)
Degree of Agreement (S)			
S_{12}	0.825	1.000	1.000
S_{13}	0.325	0.500	0.750
S_{23}	0.500	0.500	0.750
Average Degree of Agreement (AA)			
$AA(E_1)$	0.575	0.750	0.875
$AA(E_2)$	0.663	0.750	0.875
$AA(E_3)$	0.413	0.500	0.750
Relative Degree of Agreement (RA)			
$RA(E_1)$	0.348	0.375	0.350
$RA(E_2)$	0.402	0.375	0.350
$RA(E_3)$	0.250	0.250	0.300
Consensus Degree Coefficient (CC)			
$CC(E_1)$	0.251	0.267	0.252
$CC(E_2)$	0.451	0.436	0.421
$CC(E_3)$	0.297	0.297	0.327
R_{AG}^{HM}	(0.54, 0.68, 0.71, 0.81)	(0.23, 0.38, 0.38, 0.53)	(0.39, 0.58, 0.58, 0.76)
R_{AG}^{HT}	(0.50, 0.64, 0.66, 0.78)	(0.25, 0.40, 0.40, 0.55)	(0.40, 0.58, 0.58, 0.77)

Table 5.13 Aggregation under the second attribute (A_2)

	X_1	X_2	X_3
E_1	(0.94, 0.96, 0.96, 0.98)	(0.85, 0.87, 0.87, 0.88)	(0.88, 0.92, 0.92, 0.94)
E_2	(0.92, 0.94, 0.94, 0.98)	(0.90, 0.94, 0.94, 0.98)	(0.88, 0.90, 0.90, 0.92)
E_3	(0.87, 0.90, 0.90, 0.92)	(0.90, 0.92, 0.92, 0.94)	(0.90, 0.94, 0.94, 1)
Degree of Agreement (S)			
S_{12}	0.985	0.924	0.985
S_{13}	0.936	0.944	0.970
S_{23}	0.951	0.980	0.956
Average Degree of Agreement (AA)			
$AA(E_1)$	0.961	0.934	0.978
$AA(E_2)$	0.968	0.952	0.970
$AA(E_3)$	0.944	0.962	0.963
Relative Degree of Agreement (RA)			
$RA(E_1)$	0.334	0.328	0.336
$RA(E_2)$	0.337	0.334	0.333
$RA(E_3)$	0.328	0.338	0.331
Consensus Degree Coefficient (CC)			
$CC(E_1)$	0.267	0.263	0.268
$CC(E_2)$	0.313	0.312	0.311
$CC(E_3)$	0.419	0.425	0.421
R_{AG}^{HM}	(0.91, 0.94, 0.94, 0.96)	(0.88, 0.91, 0.91, 0.93)	(0.89, 0.92, 0.92, 0.95)
R_{AG}^{HT}	(0.90, 0.93, 0.93, 0.95)	(0.89, 0.91, 0.91, 0.94)	(0.89, 0.93, 0.93, 0.96)

Table 5.14 Aggregation under the third attribute (A_3)

	X_1	X_2	X_3
E_1	(0.81, 0.83, 0.94, 0.97)	(0.83, 0.86, 0.92, 0.94)	(0.81, 0.83, 0.97, 1)
E_2	(0.78, 0.81, 0.97, 1)	(0.78, 0.81, 0.89, 0.92)	(0.81, 0.83, 0.97, 1)
E_3	(0.75, 0.78, 0.86, 0.89)	(0.81, 0.83, 0.94, 0.97)	(0.86, 0.89, 0.92, 0.94)
Degree of Agreement (S)			
S_{12}	0.972	0.958	1.000
S_{13}	0.931	0.972	0.944
S_{23}	0.931	0.958	0.944
Average Degree of Agreement (AA)			
$AA(E_1)$	0.951	0.965	0.972
$AA(E_2)$	0.951	0.958	0.972
$AA(E_3)$	0.931	0.965	0.944
Relative Degree of Agreement (RA)			
$RA(E_1)$	0.336	0.334	0.337
$RA(E_2)$	0.336	0.332	0.337
$RA(E_3)$	0.328	0.334	0.327
Consensus Degree Coefficient (CC)			
$CC(E_1)$	0.430	0.429	0.430
$CC(E_2)$	0.281	0.279	0.282
$CC(E_3)$	0.288	0.292	0.288
R_{AG}^{HM}	(0.78, 0.81, 0.93, 0.95)	(0.81, 0.83, 0.92, 0.94)	(0.82, 0.85, 0.95, 0.98)
R_{AG}^{HT}	(0.78, 0.81, 0.93, 0.96)	(0.81, 0.84, 0.92, 0.94)	(0.82, 0.85, 0.96, 0.98)

Table 5.14 Aggregation under the third attribute (A_3)

	X_1	X_2	X_3
E_1	(0.81, 0.83, 0.94, 0.97)	(0.83, 0.86, 0.92, 0.94)	(0.81, 0.83, 0.97, 1)
E_2	(0.78, 0.81, 0.97, 1)	(0.78, 0.81, 0.89, 0.92)	(0.81, 0.83, 0.97, 1)
E_3	(0.75, 0.78, 0.86, 0.89)	(0.81, 0.83, 0.94, 0.97)	(0.86, 0.89, 0.92, 0.94)
Degree of Agreement (S)			
S_{12}	0.972	0.958	1.000
S_{13}	0.931	0.972	0.944
S_{23}	0.931	0.958	0.944
Average Degree of Agreement (AA)			
$AA(E_1)$	0.951	0.965	0.972
$AA(E_2)$	0.951	0.958	0.972
$AA(E_3)$	0.931	0.965	0.944
Relative Degree of Agreement (RA)			
$RA(E_1)$	0.336	0.334	0.337
$RA(E_2)$	0.336	0.332	0.337
$RA(E_3)$	0.328	0.334	0.327
Consensus Degree Coefficient (CC)			
$CC(E_1)$	0.430	0.429	0.430
$CC(E_2)$	0.281	0.279	0.282
$CC(E_3)$	0.288	0.292	0.288
R_{AG}^{HM}	(0.78, 0.81, 0.93, 0.95)	(0.81, 0.83, 0.92, 0.94)	(0.82, 0.85, 0.95, 0.98)
R_{AG}^{HT}	(0.78, 0.81, 0.93, 0.96)	(0.81, 0.84, 0.92, 0.94)	(0.82, 0.85, 0.96, 0.98)

Table 5.14 Aggregation under the third attribute (A_3)

	X_1	X_2	X_3
E_1	(0.81, 0.83, 0.94, 0.97)	(0.83, 0.86, 0.92, 0.94)	(0.81, 0.83, 0.97, 1)
E_2	(0.78, 0.81, 0.97, 1)	(0.78, 0.81, 0.89, 0.92)	(0.81, 0.83, 0.97, 1)
E_3	(0.75, 0.78, 0.86, 0.89)	(0.81, 0.83, 0.94, 0.97)	(0.86, 0.89, 0.92, 0.94)
Degree of Agreement (S)			
S_{12}	0.972	0.958	1.000
S_{13}	0.931	0.972	0.944
S_{23}	0.931	0.958	0.944
Average Degree of Agreement (AA)			
$AA(E_1)$	0.951	0.965	0.972
$AA(E_2)$	0.951	0.958	0.972
$AA(E_3)$	0.931	0.965	0.944
Relative Degree of Agreement (RA)			
$RA(E_1)$	0.336	0.334	0.337
$RA(E_2)$	0.336	0.332	0.337
$RA(E_3)$	0.328	0.334	0.327
Consensus Degree Coefficient (CC)			
$CC(E_1)$	0.430	0.429	0.430
$CC(E_2)$	0.281	0.279	0.282
$CC(E_3)$	0.288	0.292	0.288
R_{AG}^{HM}	(0.78, 0.81, 0.93, 0.95)	(0.81, 0.83, 0.92, 0.94)	(0.82, 0.85, 0.95, 0.98)
R_{AG}^{HT}	(0.78, 0.81, 0.93, 0.96)	(0.81, 0.84, 0.92, 0.94)	(0.82, 0.85, 0.96, 0.98)

Table 5.14 Aggregation under the third attribute (A_3)

	X_1	X_2	X_3
E_1	(0.81, 0.83, 0.94, 0.97)	(0.83, 0.86, 0.92, 0.94)	(0.81, 0.83, 0.97, 1)
E_2	(0.78, 0.81, 0.97, 1)	(0.78, 0.81, 0.89, 0.92)	(0.81, 0.83, 0.97, 1)
E_3	(0.75, 0.78, 0.86, 0.89)	(0.81, 0.83, 0.94, 0.97)	(0.86, 0.89, 0.92, 0.94)
Degree of Agreement (S)			
S_{12}	0.972	0.958	1.000
S_{13}	0.931	0.972	0.944
S_{23}	0.931	0.958	0.944
Average Degree of Agreement (AA)			
$AA(E_1)$	0.951	0.965	0.972
$AA(E_2)$	0.951	0.958	0.972
$AA(E_3)$	0.931	0.965	0.944
Relative Degree of Agreement (RA)			
$RA(E_1)$	0.336	0.334	0.337
$RA(E_2)$	0.336	0.332	0.337
$RA(E_3)$	0.328	0.334	0.327
Consensus Degree Coefficient (CC)			
$CC(E_1)$	0.430	0.429	0.430
$CC(E_2)$	0.281	0.279	0.282
$CC(E_3)$	0.288	0.292	0.288
R_{AG}^{HM}	(0.78, 0.81, 0.93, 0.95)	(0.81, 0.83, 0.92, 0.94)	(0.82, 0.85, 0.95, 0.98)
R_{AG}^{HT}	(0.78, 0.81, 0.93, 0.96)	(0.81, 0.84, 0.92, 0.94)	(0.82, 0.85, 0.96, 0.98)

Table 5.14 Aggregation under the third attribute (A_3)

	X_1	X_2	X_3
E_1	(0.81, 0.83, 0.94, 0.97)	(0.83, 0.86, 0.92, 0.94)	(0.81, 0.83, 0.97, 1)
E_2	(0.78, 0.81, 0.97, 1)	(0.78, 0.81, 0.89, 0.92)	(0.81, 0.83, 0.97, 1)
E_3	(0.75, 0.78, 0.86, 0.89)	(0.81, 0.83, 0.94, 0.97)	(0.86, 0.89, 0.92, 0.94)
Degree of Agreement (S)			
S_{12}	0.972	0.958	1.000
S_{13}	0.931	0.972	0.944
S_{23}	0.931	0.958	0.944
Average Degree of Agreement (AA)			
$AA(E_1)$	0.951	0.965	0.972
$AA(E_2)$	0.951	0.958	0.972
$AA(E_3)$	0.931	0.965	0.944
Relative Degree of Agreement (RA)			
$RA(E_1)$	0.336	0.334	0.337
$RA(E_2)$	0.336	0.332	0.337
$RA(E_3)$	0.328	0.334	0.327
Consensus Degree Coefficient (CC)			
$CC(E_1)$	0.430	0.429	0.430
$CC(E_2)$	0.281	0.279	0.282
$CC(E_3)$	0.288	0.292	0.288
R_{AG}^{HM}	(0.78, 0.81, 0.93, 0.95)	(0.81, 0.83, 0.92, 0.94)	(0.82, 0.85, 0.95, 0.98)
R_{AG}^{HT}	(0.78, 0.81, 0.93, 0.96)	(0.81, 0.84, 0.92, 0.94)	(0.82, 0.85, 0.96, 0.98)

Table 5.14 Aggregation under the third attribute (A_3)

	X_1	X_2	X_3
E_1	(0.81, 0.83, 0.94, 0.97)	(0.83, 0.86, 0.92, 0.94)	(0.81, 0.83, 0.97, 1)
E_2	(0.78, 0.81, 0.97, 1)	(0.78, 0.81, 0.89, 0.92)	(0.81, 0.83, 0.97, 1)
E_3	(0.75, 0.78, 0.86, 0.89)	(0.81, 0.83, 0.94, 0.97)	(0.86, 0.89, 0.92, 0.94)
Degree of Agreement (S)			
S_{12}	0.972	0.958	1.000
S_{13}	0.931	0.972	0.944
S_{23}	0.931	0.958	0.944
Average Degree of Agreement (AA)			
$AA(E_1)$	0.951	0.965	0.972
$AA(E_2)$	0.951	0.958	0.972
$AA(E_3)$	0.931	0.965	0.944
Relative Degree of Agreement (RA)			
$RA(E_1)$	0.336	0.334	0.337
$RA(E_2)$	0.336	0.332	0.337
$RA(E_3)$	0.328	0.334	0.327
Consensus Degree Coefficient (CC)			
$CC(E_1)$	0.430	0.429	0.430
$CC(E_2)$	0.281	0.279	0.282
$CC(E_3)$	0.288	0.292	0.288
R_{AG}^{HM}	(0.78, 0.81, 0.93, 0.95)	(0.81, 0.83, 0.92, 0.94)	(0.82, 0.85, 0.95, 0.98)
R_{AG}^{HT}	(0.78, 0.81, 0.93, 0.96)	(0.81, 0.84, 0.92, 0.94)	(0.82, 0.85, 0.96, 0.98)

5.3.3 Selection State Calculations

Aggregated decision matrices are deffuzzified according to sixth step of the proposed approach.The results are shown in Tables 5.17, and 5.18.

Table 5.17 Defuzzified aggregated matrix for homogeneous group of experts

	X_1	X_2	X_3
A_1	0.668	0.391	0.563
A_2	0.924	0.900	0.908
A_3	0.856	0.865	0.892
A_4	1000	700	850

Table 5.18 Defuzzified aggregated matrix for heterogeneous group of experts

	X_1	X_2	X_3
A_1	0.630	0.412	0.569
A_2	0.918	0.903	0.911
A_3	0.859	0.867	0.892
A_4	1000	700	850

The normalised and weighted normalised ratings for homo/heterogeneous group of experts are constructed according to the seventh step of the proposed approach. Tables 5.19, 5.20, 5.21 and 5.22 show the complete results of the normalised and weighted normalised decision matrices for homo/heterogeneous group of experts respectively.

Table 5.19 Normalised ratings for homogeneous group of experts

	X_1	X_2	X_3
A_1	0.6979	0.4085	0.5882
A_2	0.5858	0.5706	0.5756
A_3	0.5673	0.5733	0.5912
A_4	0.6723	0.4706	0.5714

Table 5.20 Normalised ratings for heterogeneous group of experts

	X_1	X_2	X_3
A_1	0.6676	0.4366	0.6030
A_2	0.5820	0.5725	0.5775
A_3	0.5682	0.5735	0.5901
A_4	0.6723	0.4706	0.5714

Table 5.21 Weighted normalised ratings for homogeneous group of experts

	X_1	X_2	X_3
A_1	0.0698	0.0409	0.0588
A_2	0.1172	0.1141	0.1151
A_3	0.2837	0.2866	0.2956
A_4	0.1345	0.0941	0.1143

Table 5.22 Weighted normalised ratings for heterogeneous group of experts

	X_1	X_2	X_3
A_1	0.0668	0.0437	0.0603
A_2	0.1164	0.1145	0.1155
A_3	0.2841	0.2868	0.2950
A_4	0.1345	0.0941	0.1143

Determination of the positive-ideal solution can easily be made by taking the largest element for each benefit attribute and the smallest element for each cost attribute. The negative-ideal solution is just the opposite formation of the positive-ideal solution.

Positive and negative ideal solutions for homo/heterogeneous group of experts are given in Tables 5.23 and 5.24 respectively.

Table 5.23 Positive and negative ideal solutions for homogeneous group of experts

	Positive-Ideal Solution	Negative-Ideal Solution
A_1	0.0409	0.0698
A_2	0.1172	0.1141
A_3	0.2837	0.2956
A_4	0.1345	0.0941

Table 5.24 Positive and negative ideal solutions for heterogeneous group of experts

	Positive-Ideal Solution	Negative-Ideal Solution
A_1	0.0437	0.0668
A_2	0.1164	0.1145
A_3	0.2841	0.2950
A_4	0.1345	0.0941

Table 5.25 and 5.26 show the separation measures and the relative closenesses to the positive-ideal solution.

Table 5.25 Relative closeness to the positive-ideal solution for homogeneous group of experts

	S_i^*	Rank	S_i^-	Rank	C_i^*	Rank
X_1	0.0289	1	0.0422	1	0.5930	1
X_2	0.0406	3	0.0303	2	0.4275	3
X_3	0.0296	2	0.0230	3	0.4371	2

Table 5.26 Relative closeness to the positive-ideal solution for heterogeneous group of experts

	S_i^*	Rank	S_i^-	Rank	C_i^*	Rank
X_1	0.0231	1	0.0418	1	0.6442	1
X_2	0.0405	3	0.0245	2	0.3775	3
X_3	0.0283	2	0.0212	3	0.4279	2

Three different rankings can be made based on S_i^*, S_i^-, and C_i^* respectively : higher ranking is given to an alternative which has smaller value of S_i^*, and larger values of S_i^-, and C_i^*. The preference ranking based on the positive-ideal alternative differs with the one based on the negative-ideal alternative. This conflict can be resolved by taking the C_i^* value. The TOPSIS ranking based on C_i^* is as follows :

Finally, according to last step of the proposed approach, the OAR values for each alternative can be obtained, and then the following overall ranking of alternatives for the decision is given as follows:

The ranking is $X_1 > X_3 > X_2$ for homogeneous group of experts and

The ranking is also $X_1 > X_3 > X_2$ for heterogeneous group of experts.

The ranking of alternatives for homo/heterogeneous group of experts are shown in Figures 5.19 and 5.20 respectively.

The alternative X_1 is the best alternative to choose. It should be realised that the ranking of alternatives can only be obtained and it can't be said that how much X_1 is better than X_3 and X_2.

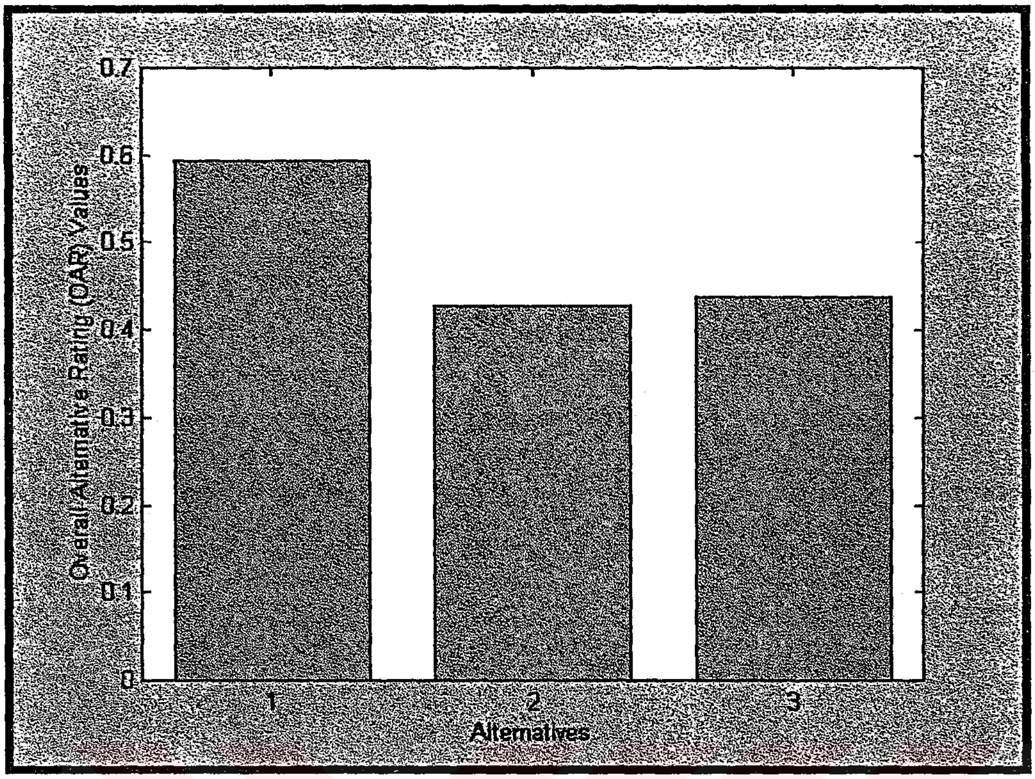


Figure 5.19 Ranking of alternatives for homogeneous group of experts

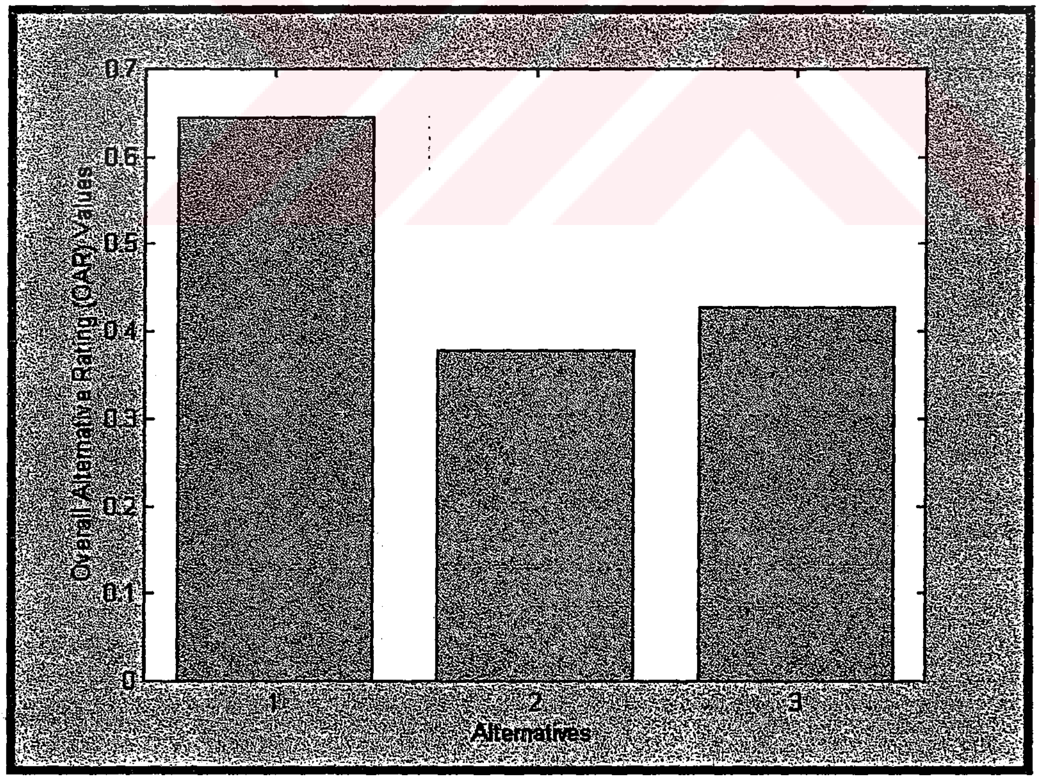


Figure 5.20 Ranking of alternatives for heterogeneous group of experts (for β=0.4)

6. CASE STUDIES

This chapter is intended to show that the proposed method can be applied to real world shipbuilding decision problems. In this chapter, two real case studies are carried out. The first one is a system (propulsion/manoeuvring system) selection under fuzzy environment and the second one is a component (ship main engine) selection under semi-fuzzy environment.

For the first case originated from a feasibility evaluation of Turkish Maritime Lines (TDİ) for Karaköy – Haydarpaşa – Kadıköy route, and the data was taken from a research project titled "Choice of Propulsion System for a Double Ended Ferry (Odabaşı et al. (1992))".

In the second case, Furtrans Shipping Incorporation was seeking an objective method for the selection of main engine for their new chemical tanker fleet, and the decision data was collected in a post-graduate thesis (Yaraş (1999)) and in an undergraduate study (HerişÇakar (1999)).

6.1 Case 1 - Multiple Attribute Evaluation of Propulsion/Manoeuvring System Alternatives

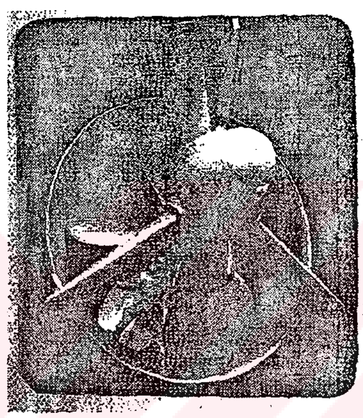
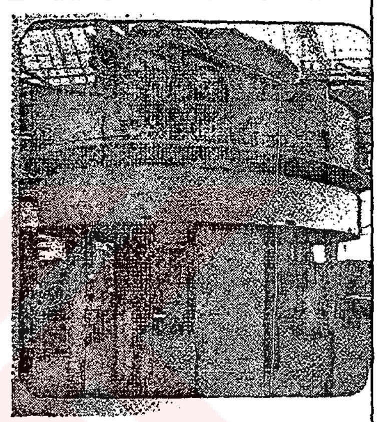
There has been growing interest in the assessment and selection of propulsion/manoeuvring system alternatives.

An inappropriate propulsion/manoeuvring system selection based on an inappropriate evaluation can lead to losses in propulsion/manoeuvring system capacity. There is a necessity to use logical or analytical decision support tools, especially when dealing with propulsion/manoeuvring systems, which are worth thousands of dollars in investment and more during the operations.

Propulsion/manoeuvring systems cannot be evaluated solely on the basis of one attribute such as cost and manoeuvrability. Multiple attribute evaluation procedures are required to structure, to focus and to improve the evaluation process. Proposed FMADM technique is a good evaluation procedure for that reason.

Imprecise and vague information in the multiple attribute evaluation must also be a consideration. It is a major flaw in human judgement and decision making. In particular, the FST seems to offer MADM techniques that fit well with the specific features of the assessment and selection of propulsion/manoeuvring system, as the vagueness and imprecision of many relevant effects can be explicitly considered.

The propulsion/manoeuvring system selection is based on the study that has been conducted for the selection of propulsion/manoeuvring system of a double ended passenger ferry to operate across the Bosphorus in Istanbul with the aim of reducing the journey time in highly congested seaway traffic (İnsel and Helvacıoğlu (1997)). In this case, the appropriate propulsion/manoeuvring system from among three alternatives, namely, conventional propeller and high lift rudder, Z drive and cycloidal propeller are being tried to choose. They are given in Figures 6.1, 6.2, and 6.3 respectively.

		
Figure 6.1 Conventional propeller and high lift rudder (2x1)	Figure 6.2 Z drive (2x2)	Figure 6.3 Cycloidal propeller (2x1)

Where 2x2 means two propulsion units at both ends, 2x1 means one propulsion unit at both ends.

6.1.1 Attribute Generation

Multiple attribute decision analysis starts with the generation of the attributes. These attributes should be complete and exhaustive, contain mutually exclusive items and be restricted to performance attributes of the highest degree of importance.

In more detail, there are seven subjective attributes to be considered in choosing among the above three alternative propulsion/manoeuvring systems. The subjective attributes, which are critical for the selection of propulsion/manoeuvring system of a double-ended passenger ferry, are the following:

1. Investment costs (IC),
2. Operating costs (OC), including handling, repair and maintenance costs,
3. Manoeuvrability (MV),
4. Vibration and noise (VN),
5. Reliability (RL), including mechanical safety, redundancy, service experience,
6. Propulsive Power Requirement (PPR), including ship geometry, ship resistance, power requirement, propulsion efficiency,
7. Propulsive Arrangement Requirement (PAR), including required propulsion engine.

It is useful to develop a hierarchical structure showing the overall objective, the attributes and alternatives. The hierarchy for the propulsion/manoeuvring system evaluation problem is shown in Figure 6.4.

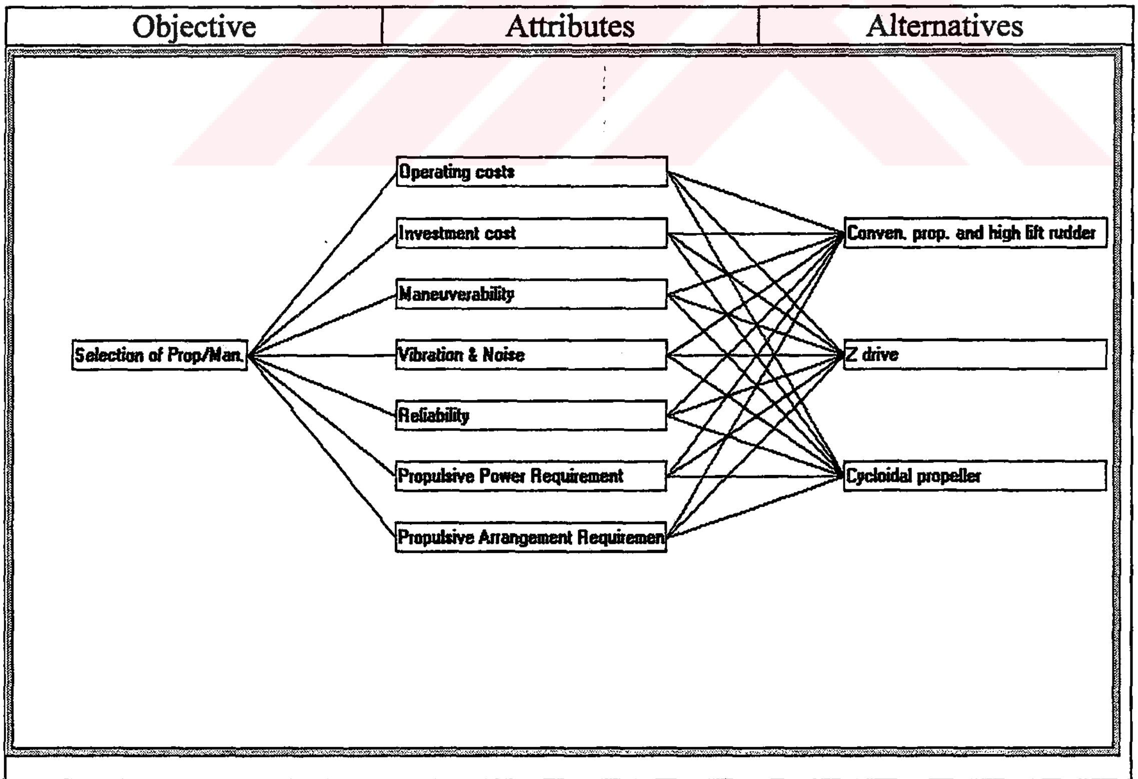


Figure 6.4 Decision hierarchy of Case - 1

(It should be noted that here the possibility of other intermediate levels of sub attribute between attributes and decision alternatives arent considered.)

Propulsion/manoeuvring system alternatives, attributes, and their abbreviations are shown as follows :

The set (of alternatives) X for the first case is given by:

X = { X_1 (Conventional propeller and high lift rudder (2x1)), X_2 (Z drive (2x2)), X_3 (Cycloidal propeller (2x1)) }.

The attributes of the decision are given by :

A_1 = IC, A_2 = OC, A_3 = MV, A_4 = VN, A_5 = RL, A_6 = PPR, A_7 = PAR,

For this case, attributes' properties such as type of attributes and type of assessments are summarised in Table 6.1.

Table 6.1 Attributes' properties of Case - 1

	Attribute properties		
Attributes	Type of assessment	Type of attribute	
Investment cost	Fuzzy (as interval)	Cost	Subjective
Operating costs	Fuzzy (as "approximately equal to")	Cost	Subjective
Manoeuvrability	Linguistic	Benefit	Subjective
Vibration & Noise	Linguistic	Cost	Subjective
Reliability	Linguistic	Benefit	Subjective
Propulsive Power Requirement	Linguistic	Cost	Subjective
Propulsive Arrangement Requirement	Linguistic	Cost	Subjective

6.1.2 Rating State Calculations

The alternatives are evaluated by a group of experts (managing director, designer, and operator) with respect to the seven subjective attributes as shown in Tables 6.2, 6.3, and 6.4. Note that all assessments for A_1 and A_2 attributes are in thousands of dollars.

Table 6.2 First expert's evaluation of three alternatives under the seven attributes

	E_1 (Managing Director)		
	X_1	X_2	X_3
A_1	Approximately between 70 and 72	Approximately between 75 and 79	Approximately between 120 and 125
A_2	Approximately equal to 6.6	Approximately equal to 7.2	Approximately equal to 11.4
A_3	Good	Very Good	Excellent
A_4	Low	Medium	Very Low
A_5	Good	Good	Good
A_6	Very Low	Very Low	Very Low
A_7	Very Low	Medium	Medium

Table 6.3 Second expert's evaluation of three alternatives under the seven attributes

	E_2 (Designer)		
	X_1	X_2	X_3
A_1	Approximately between 72 and 74	Approximately between 78 and 80	Approximately between 125 and 127
A_2	Approximately equal to 6.8	Approximately equal to 7.4	Approximately equal to 11.8
A_3	Good	Mol. Good	Good
A_4	Low	Mol. Low	Low
A_5	Good	Good – Very Good	Very Good
A_6	Low	Low	Very Low
A_7	Low	Medium	Low

Table 6.4 Third expert's evaluation of three alternatives under the seven attributes

	E$_3$ (Operator)		
	X$_1$	X$_2$	X$_3$
A$_1$	Approximately between 70 and 75	Approximately between 75 and 80	Approximately between 118 and 124
A$_2$	Approximately equal to 6.8	Approximately equal to 7.2	Approximately equal to 11.3
A$_3$	Fair	Mol. Good	Good
A$_4$	Medium	Low	Very Low
A$_5$	Fairly Good	Good	Fairly Good
A$_6$	Fairly Low	Low	Low
A$_7$	Very Low	Low	Medium

Experts' fuzzy assessments for the first and second attributes are first converted to fuzzy numbers. For example, 'approximately between 70 and 72' can be represented by the trapezoidal fuzzy number of (69, 70, 72, 73) and also 'approximately equal to 6.6' can be represented by the trapezoidal fuzzy number of (6.4, 6.6, 6.6, 6.8).

For the rest of the attributes, experts' linguistic assessments are transformed into fuzzy numbers by using appropriate Scale described in Section 5.2.1.2. Linguistic terms are matched with Scale 8 (for A_3), Scale 6 (for A_4), Scale 7 (for A_5), Scale 5 (for A_6), and Scale 3 (for A_7). These transformed fuzzy numbers are shown in Tables 6.5, 6.6, and 6.7.

Table 6.5 First expert's decision matrix with fuzzy numbers

	E$_1$ (Managing Director)		
	X$_1$	X$_2$	X$_3$
A$_1$	(69, 70, 72, 73)	(74, 75, 79, 80)	(119, 120, 125, 126)
A$_2$	(6.4, 6.9, 6.6, 6.8)	(7.0, 7.2, 7.2, 7.4)	(11.2, 11.4, 11.4, 11.6)
A$_3$	(0.5, 0.7, 0.7, 0.9)	(0.8, 0.9, 0.9, 1.0)	(0.9, 1.0, 1.0, 1.0)
A$_4$	(0.1, 0.2, 0.2, 0.3)	(0.4, 0.5, 0.5, 0.6)	(0.0, 0.0, 0.1, 0.2)
A$_5$	(0.6, 0.8, 0.8, 1.0)	(0.6, 0.8, 0.8, 1.0)	(0.6, 0.8, 0.8, 1.0)
A$_6$	(0.0, 0.0, 0.0, 0.2)	(0.0, 0.0, 0.0, 0.2)	(0.0, 0.0, 0.0, 0.2)
A$_7$	(0.0, 0.0, 0.1, 0.2)	(0.3, 0.5, 0.5, 0.7)	(0.3, 0.5, 0.5, 0.7)

Table 6.6 Second expert's decision matrix with fuzzy numbers

	E_2 (Designer)		
	X_1	X_2	X_3
A_1	(71, 72, 74, 75)	(77, 78, 80, 81)	(124, 125, 127, 128)
A_2	(6.6, 6.8, 6.8, 7.0)	(7.2, 7.4, 7.4, 7.6)	(11.6, 11.8, 11.8, 12)
A_3	(0.5, 0.7, 0.7, 0.9)	(0.5, 0.55, 0.55, 0.6)	(0.5, 0.7, 0.7, 0.9)
A_4	(0.1, 0.2, 0.2, 0.3)	(0.2, 0.3, 0.4, 0.5)	(0.1, 0.2, 0.2, 0.3)
A_5	(0.6, 0.8, 0.8, 1.0)	(0.7, 0.9, 1.0, 1.0)	(0.8, 1.0, 1.0, 1.0)
A_6	(0.0, 0.2, 0.2, 0.4)	(0.0, 0.2, 0.2, 0.4)	(0.0, 0.0, 0.0, 0.2)
A_7	(0.1, 0.25, 0.25, 0.4)	(0.3, 0.5, 0.5, 0.7)	(0.1, 0.25, 0.25, 0.4)

Table 6.7 Third expert's decision matrix with fuzzy numbers

	E_3 (Operator)		
	X_1	X_2	X_3
A_1	(69, 70, 75, 76)	(74, 75, 80, 81)	(117, 118, 124, 125)
A_2	(6.6, 6.8, 6.8, 7.0)	(7.0, 7.2, 7.2, 7.4)	(11.1, 11.3, 11.3, 11.5)
A_3	(0.3, 0.5, 0.5, 0.7)	(0.5, 0.55, 0.55, 0.6)	(0.5, 0.7, 0.7, 0.9)
A_4	(0.4, 0.5, 0.5, 0.6)	(0.1, 0.2, 0.2, 0.3)	(0.0, 0.0, 0.1, 0.2)
A_5	(0.5, 0.65, 0.65, 0.8)	(0.6, 0.8, 0.8, 1.0)	(0.5, 0.65, 0.65, 0.8)
A_6	(0.2, 0.4, 0.4, 0.6)	(0.0, 0.2, 0.2, 0.4)	(0.0, 0.2, 0.2, 0.4)
A_7	(0.0, 0.0, 0.1, 0.2)	(0.1, 0.25, 0.25, 0.4)	(0.3, 0.5, 0.5, 0.7)

Then for the first and second attributes, all non-standardised fuzzy numbers are converted to standardised fuzzy numbers. Results for each expert are given in Tables 6.8, 6.9, and 6.10.

Table 6.8 First expert's decision matrix with standardised trapezoidal fuzzy numbers

	E_1 (Managing Director)		
	X_1	X_2	X_3
A_1	(0.54, 0.55, 0.56, 0.57)	(0.58, 0.59, 0.62, 0.63)	(0.93, 0.94, 0.98, 0.98)
A_2	(0.53, 0.55, 0.55, 0.57)	(0.58, 0.60, 0.60, 0.62)	(0.93, 0.95, 0.95, 0.97)
A_3	(0.5, 0.7, 0.7, 0.9)	(0.8, 0.9, 0.9, 1.0)	(0.9, 1.0, 1.0, 1.0)
A_4	(0.1, 0.2, 0.2, 0.3)	(0.4, 0.5, 0.5, 0.6)	(0.0, 0.0, 0.1, 0.2)
A_5	(0.6, 0.8, 0.8, 1.0)	(0.6, 0.8, 0.8, 1.0)	(0.6, 0.8, 0.8, 1.0)
A_6	(0.0, 0.0, 0.0, 0.2)	(0.0, 0.0, 0.0, 0.2)	(0.0, 0.0, 0.0, 0.2)
A_7	(0.0, 0.0, 0.1, 0.2)	(0.3, 0.5, 0.5, 0.7)	(0.3, 0.5, 0.5, 0.7)

Table 6.9 Second expert's decision matrix with standardised trapezoidal fuzzy numbers

	E2 (Designer)		
	X_1	X_2	X_3
A_1	(0.55, 0.56, 0.58, 0.59)	(0.60, 0.61, 0.63, 0.63)	(0.97, 0.98, 0.99, 1.00)
A_2	(0.55, 0.57, 0.57, 0.58)	(0.60, 0.62, 0.62, 0.63)	(0.97, 0.98, 0.98, 1.00)
A_3	(0.5, 0.7, 0.7, 0.9)	(0.5, 0.55, 0.55, 0.6)	(0.5, 0.7, 0.7, 0.9)
A_4	(0.1, 0.2, 0.2, 0.3)	(0.2, 0.3, 0.4, 0.5)	(0.1, 0.2, 0.2, 0.3)
A_5	(0.6, 0.8, 0.8, 1.0)	(0.7, 0.9, 1.0, 1.0)	(0.8, 1.0, 1.0, 1.0)
A_6	(0.0, 0.2, 0.2, 0.4)	(0.0, 0.2, 0.2, 0.4)	(0.0, 0.0, 0.0, 0.2)
A_7	(0.1, 0.25, 0.25, 0.4)	(0.3, 0.5, 0.5, 0.7)	(0.1, 0.25, 0.25, 0.4)

Table 6.10 Third expert's decision matrix with standardised trapezoidal fuzzy numbers

	E3 (Operator)		
	X_1	X_2	X_3
A_1	(0.54, 0.55, 0.59, 0.59)	(0.58, 0.59, 0.63, 0.63)	(0.91, 0.92, 0.97, 0.98)
A_2	(0.55, 0.57, 0.57, 0.58)	(0.58, 0.60, 0.60, 0.62)	(0.93, 0.94, 0.94, 0.96)
A_3	(0.3, 0.5, 0.5, 0.7)	(0.5, 0.55, 0.55, 0.6)	(0.5, 0.7, 0.7, 0.9)
A_4	(0.4, 0.5, 0.5, 0.6)	(0.1, 0.2, 0.2, 0.3)	(0.0, 0.0, 0.1, 0.2)
A_5	(0.5, 0.65, 0.65, 0.8)	(0.6, 0.8, 0.8, 1.0)	(0.5, 0.65, 0.65, 0.8)
A_6	(0.2, 0.4, 0.4, 0.6)	(0.0, 0.2, 0.2, 0.4)	(0.0, 0.2, 0.2, 0.4)
A_7	(0.0, 0.0, 0.1, 0.2)	(0.1, 0.25, 0.25, 0.4)	(0.3, 0.5, 0.5, 0.7)

6.1.3 Aggregation State Calculations

In the aggregation state of the proposed method, all ratings are aggregated under each subjective attribute by taking into account the attribute based expert weights. Before aggregation, it is necessary to identify the weights of attributes and experts. Therefore, manager (or moderator) of the decision problem, TDİ (as Ship Owner), assigns relative importances of attributes and experts. Then weights of them are easily calculated. The relative importances and weights for attributes and experts are given in Table 6.11.

During the whole process of the aggregation state, β, showing the moderator's dominance on the problem, is taken as 0.4.

Table 6.11 Weights of attributes and experts

	Attributes		E_1		E_2		E_3	
	R.I.	w.	R.I.	w.	R.I.	w.	R.I.	w.
A_1	63	0.22	1	0.5	0.70	0.35	0.30	0.15
A_2	37	0.13	1	0.5	0.70	0.35	0.30	0.15
A_3	100	0.35	0.30	0.17	0.50	0.28	1	0.55
A_4	11	0.04	0.30	0.16	1	0.53	0.60	0.31
A_5	57	0.20	0.5	0.23	0.70	0.32	1	0.45
A_6	11	0.04	0.2	0.12	1	0.63	0.40	0.25
A_7	6	0.02	0.2	0.11	1	0.56	0.60	0.33
Where R.I. : Relative Importance.								

Detailed aggregation calculations related with this state and their figures are all given in Appendix C. These calculations involve attribute based necessary aggregation calculations, such as degree of agreement, relative degree of agreement of each expert, etc. After aggregation calculations, aggregation matrices for homo/heterogeneous group of experts can be constructed easily as shown in Tables 6.12 and 6.13 respectively.

Table 6.12 Aggregated matrix for homogeneous group of experts

	X_1	X_2	X_3
A_1	(0.54, 0.55, 0.58, 0.58)	(0.59, 0.59, 0.62, 0.63)	(0.94, 0.95, 0.98, 0.99)
A_2	(0.54, 0.56, 0.56, 0.58)	(0.59, 0.61, 0.61, 0.62)	(0.94, 0.96, 0.96, 0.97)
A_3	(0.44, 0.64, 0.64, 0.84)	(0.58, 0.65, 0.65, 0.71)	(0.61, 0.78, 0.78, 0.93)
A_4	(0.19, 0.29, 0.29, 0.39)	(0.23, 0.33, 0.37, 0.47)	(0.04, 0.07, 0.14, 0.24)
A_5	(0.57, 0.75, 0.75, 0.94)	(0.63, 0.83, 0.86, 1.00)	(0.63, 0.81, 0.81, 0.93)
A_6	(0.06, 0.20, 0.20, 0.40)	(0.00, 0.14, 0.14, 0.34)	(0.00, 0.07, 0.07, 0.27)
A_7	(0.03, 0.08, 0.15, 0.26)	(0.24, 0.43, 0.43, 0.61)	(0.23, 0.41, 0.41, 0.60)

Table 6.13 Aggregated matrix for heterogeneous group of experts

	X_1	X_2	X_3
A_1	(0.54, 0.55, 0.57, 0.58)	(0.59, 0.59, 0.62, 0.63)	(0.94, 0.95, 0.98, 0.99)
A_2	(0.54, 0.56, 0.56, 0.58)	(0.59, 0.61, 0.61, 0.62)	(0.94, 0.96, 0.96, 0.98)
A_3	(0.42, 0.62, 0.62, 0.82)	(0.57, 0.63, 0.63, 0.69)	(0.59, 0.77, 0.77, 0.92)
A_4	(0.19, 0.29, 0.29, 0.39)	(0.22, 0.32, 0.36, 0.46)	(0.04, 0.08, 0.14, 0.24)
A_5	(0.56, 0.74, 0.74, 0.93)	(0.63, 0.83, 0.87, 1.00)	(0.63, 0.81, 0.81, 0.92)
A_6	(0.06, 0.21, 0.21, 0.41)	(0.00, 0.15, 0.15, 0.35)	(0.00, 0.06, 0.06, 0.26)
A_7	(0.04, 0.10, 0.16, 0.28)	(0.24, 0.42, 0.42, 0.61)	(0.21, 0.39, 0.39, 0.57)

6.1.4 Selection State Calculations

Up to this state, experts' fuzzy and linguistic assessments have been transformed into standardised trapezoidal fuzzy numbers and then aggregated under each subjective attribute. In order to rank the alternatives, aggregated matrices' fuzzy elements should be defuzzified. Defuzzified aggregated matrices for homo/heterogeneous group of experts are shown in Tables 6.14 and 6.15 respectively.

The TOPSIS technique is applied to rank the alternatives for homo/heterogeneous group of experts.

Table 6.14 Defuzzified aggregated matrix for homogeneous group of experts

	X_1	X_2	X_3
A_1	0.563	0.607	0.959
A_2	0.560	0.604	0.951
A_3	0.615	0.640	0.741
A_4	0.307	0.364	0.141
A_5	0.713	0.787	0.760
A_6	0.253	0.201	0.144
A_7	0.155	0.437	0.426

Table 6.15 Defuzzified aggregated matrix for heterogeneous group of experts

	X_1	X_2	X_3
A_1	0.563	0.607	0.960
A_2	0.559	0.604	0.952
A_3	0.599	0.625	0.728
A_4	0.309	0.355	0.151
A_5	0.707	0.787	0.755
A_6	0.261	0.213	0.138
A_7	0.174	0.434	0.408

The defuzzified aggregated matrices are normalised to transform the measurement in various unit into nondimensional measurements which allow comparison across the attributes. Table 6.16 shows the normalised values of each attribute for homogeneous group of experts and Table 6.17 shows the normalised values of each attribute for heterogeneous group of experts.

Table 6.16 Normalised ratings for homogeneous group of experts

	X_1	X_2	X_3
A_1	0.4444	0.4791	0.7570
A_2	0.4451	0.4801	0.7559
A_3	0.5319	0.5535	0.6409
A_4	0.6182	0.7330	0.2839
A_5	0.5460	0.6027	0.5820
A_6	0.7152	0.5682	0.4071
A_7	0.2462	0.6940	0.6766

Table 6.17 Normalised ratings for heterogeneous group of experts

	X_1	X_2	X_3
A_1	0.4441	0.4788	0.7573
A_2	0.4442	0.4800	0.7565
A_3	0.5296	0.5526	0.6436
A_4	0.6252	0.7182	0.3055
A_5	0.5440	0.6055	0.5809
A_6	0.7169	0.5851	0.3791
A_7	0.2804	0.6994	0.6575

Weighted normalised ratings of each attribute can be calculated by multiplying each attribute with its associated weight: $(w_1, w_2, w_3, w_4, w_5, w_6, w_7) = (0.22, 0.13, 0.35, 0.04, 0.20, 0.04, 0.02)$. Tables 6.18 and 6.19 show the weighted normalised values for homo/heterogeneous group of experts respectively.

Table 6.18 Weighted normalised ratings for homogeneous group of experts

	X_1	X_2	X_3
A_1	0.0978	0.1054	0.1665
A_2	0.0579	0.0624	0.0983
A_3	0.1862	0.1937	0.2243
A_4	0.0247	0.0293	0.0114
A_5	0.1092	0.1205	0.1164
A_6	0.0286	0.0227	0.0163
A_7	0.0049	0.0139	0.0135

Table 6.19 Weighted normalised ratings for heterogeneous group of experts

	X_1	X_2	X_3
A_1	0.0977	0.1053	0.1666
A_2	0.0577	0.0624	0.0983
A_3	0.1853	0.1934	0.2253
A_4	0.0250	0.0287	0.0122
A_5	0.1088	0.1211	0.1162
A_6	0.0287	0.0234	0.0152
A_7	0.0056	0.0140	0.0131

Determination of the positive-ideal solution can easily be made by taking the largest element for each benefit attribute and the smallest element for each cost attribute. The negative-ideal solution is just the opposite formation of the positive-ideal solution. Positive and negative ideal solutions are given in Table 6.20 for homogeneous group of experts and in Table 6.21 for heterogeneous group of experts.

Table 6.20 Positive and negative ideal solutions for homogeneous group of experts

	Positive-Ideal Solution	Negative-Ideal Solution
A_1	0.0978	0.1665
A_2	0.0579	0.0983
A_3	0.2243	0.1862
A_4	0.0114	0.0293
A_5	0.1205	0.1092
A_6	0.0163	0.0286
A_7	0.0049	0.0139

Table 6.21 Positive and negative ideal solutions for heterogeneous group of experts

	Positive-Ideal Solution	Negative-Ideal Solution
A_1	0.0977	0.1666
A_2	0.0577	0.0983
A_3	0.2253	0.1853
A_4	0.0122	0.0287
A_5	0.1211	0.1088
A_6	0.0152	0.0287
A_7	0.0056	0.0140

The separation measures of each alternative from positive-ideal and negative-ideal solutions can be calculated by the n-dimensional Euclidean distance. Table 6.22 shows the values of separation measures and relative closeness to the positive-ideal solution for homogeneous group of experts and Table 6.23 shows the values of separation measures and relative closeness to the positive-ideal solution for heterogeneous group of experts.

Table 6.22 Values of separation measures and relative closeness to the positive-ideal solution for homogeneous group of experts

	X_1	X_2	X_3
S_i^*	0.0437	0.0382	0.0803
S_i^-	0.0804	0.0724	0.0445
C_i^*	0.6476	0.6547	0.3566

Table 6.23 Values of separation measures and relative closeness to the positive-ideal solution for heterogeneous group of experts

	X_1	X_2	X_3
S_i^*	0.0457	0.0388	0.0805
S_i^-	0.0805	0.0727	0.0459
C_i^*	0.6377	0.6520	0.3630

Finally, propulsion/manoeuvring system alternatives are ranked on the basis of C_i^* values.

For homogeneous group of experts, according to the descending order of C_i^*, the preference order is $X_2 > X_1 > X_3$, where the second alternative is the leader and the third alternative is the last contender.

Similarly, for heterogeneous group of experts, ranking of alternatives based on the C_i^* values is given as $X_2 > X_1 > X_3$. Figure 6.5 shows the ranking of alternatives for homogeneous group of experts and Figure 6.6 shows the ranking of alternatives for heterogeneous group of experts.

For homo/heterogeneous group of experts, the second alternative (Z drive) is ranked in the first position among the others and the first alternative's performance is ranked in the second position.

6.1.5 Sensitivity Analysis

Sensitivity analysis is performed to see the β effect on the OAR values. β values are taken as the range between 0.1 and 1 as shown in Table 6.24. In Table 6.24, OAR values with respect to each β value are also given.

According to the sensitivity analysis performed, this case is not sensitive for β coefficient. As shown in Figure 6.7, while the β values grow, ranking of alternatives doesn't change.

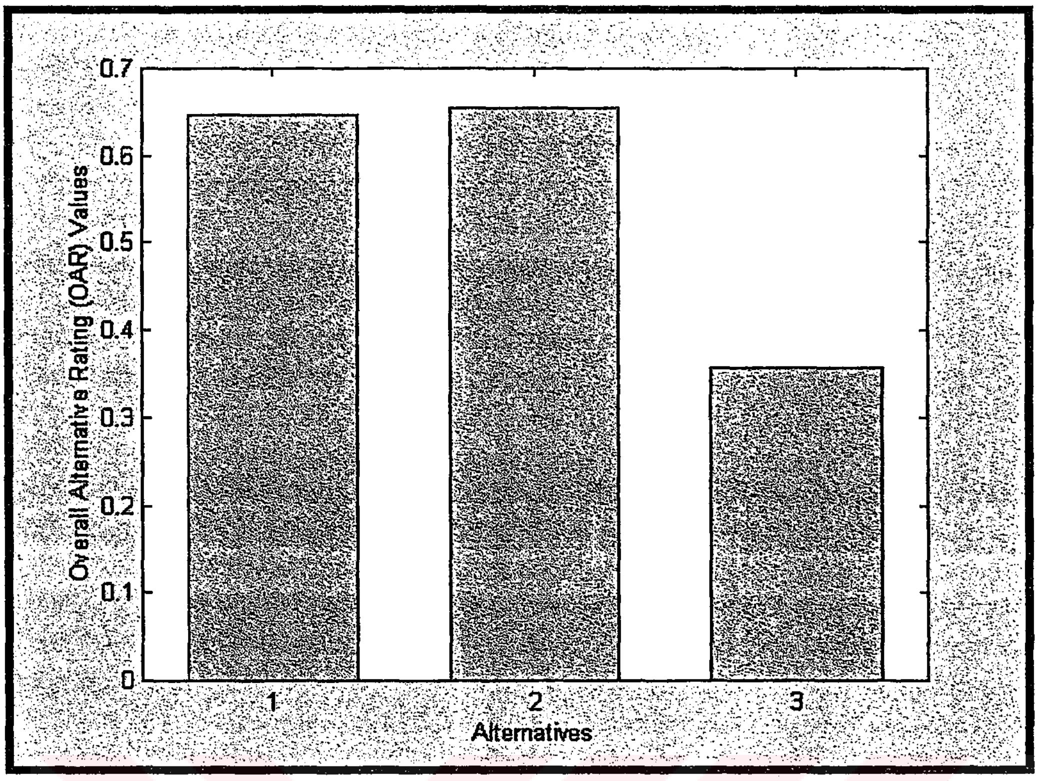


Figure 6.5 Ranking of alternatives for homogeneous group of experts

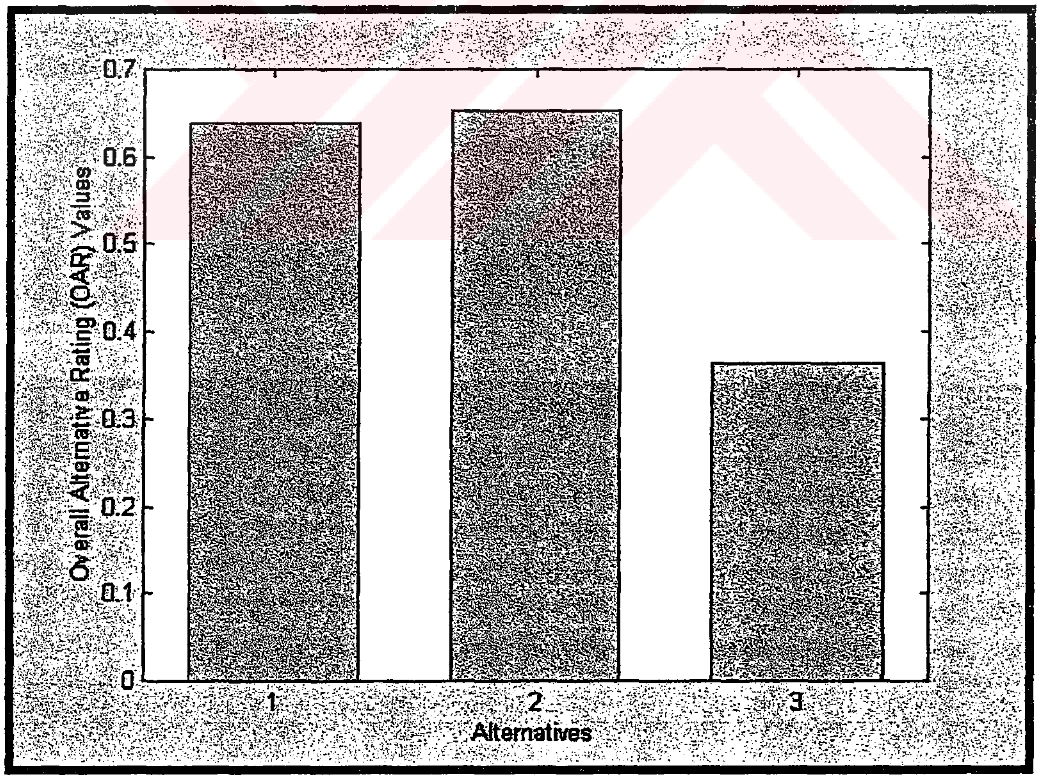


Figure 6.6 Ranking of alternatives for heterogeneous group of experts (for β=0.4)

Table 6.24 OAR values with respect to β values

β Coefficient	X_1	X_2	X_3
0 (Homogeneous)	0.6476	0.6547	0.3566
0.1	0.6447	0.6534	0.3585
0.2	0.6429	0.6532	0.3595
0.3	0.6401	0.6530	0.3613
0.4	0.6377	0.6520	0.3630
0.5	0.6349	0.6501	0.3649
0.6	0.6321	0.6480	0.3671
0.7	0.6278	0.6481	0.3705
0.8	0.6249	0.6466	0.3726
0.9	0.6221	0.6447	0.3746
1	0.6198	0.6425	0.3765

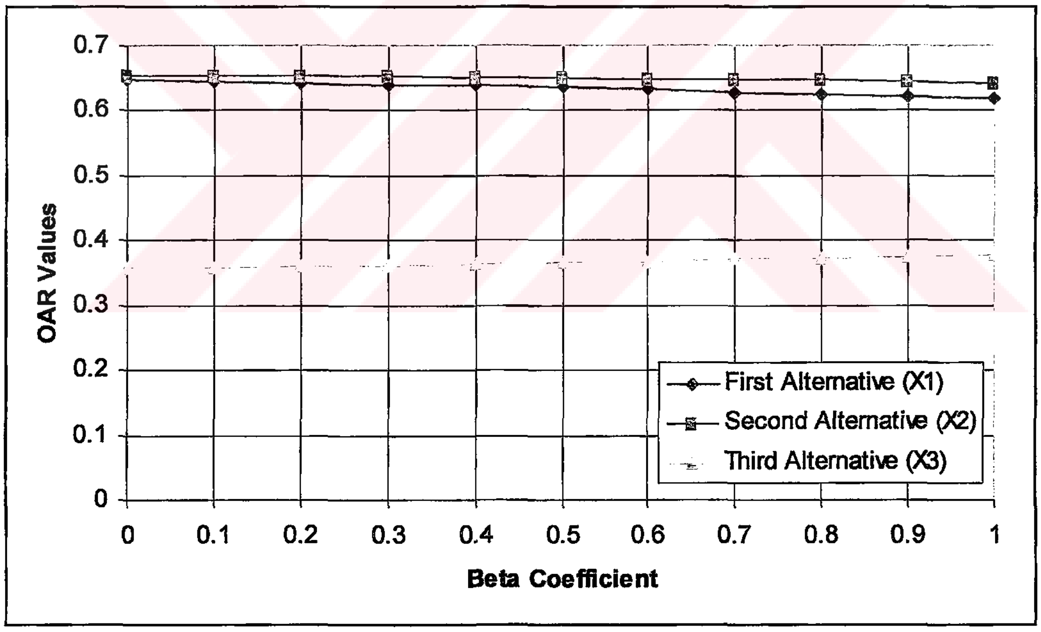


Figure 6.7 Sensitivity due to β coefficient

6.2 Case 2 - Ship Main Engine Selection

Choosing the main engines of the 6500 DWT chemical tankers fleet, projects of which were produced by Delta Marine between 1997-1999, is designated as main problem. The way of choosing these tankers' and others' main engines is formed by the help of some people who are experts on this subject. Consequently, building a model that is made up by asking to those experts who have different ideas and approaches with various positions on this work is aimed.

The data on attributes is obtained from direct interview with experts. An additional way to obtain data on the attributes, especially for objective attributes, is to look at published references and manufacturer's catalogue.

6.2.1 Attribute Generation

Two alternatives, namely MAN B&W 5S35MC and MAN B&W 8L32/40, have been designated for this case. After alternatives have been articulated, the next task is to develop a list of attributes by which each alternative could be evaluated. Although several factors were considered in this process, all of them were eventually grouped into nine attributes:

1. Price (A_1),
2. Running costs (A_2), including annual fuel bill (inc. lub. oil), spare part, maintenance and overhaul costs, the cost of manning, the ship's requirement for electrical power and heat,
3. Reliability and maintainability (A_3), including spare part availability, overhaul period, service, the number of cylinders,
4. Vibration, noise and other signatures (A_4),
5. Ease of operation (A_5),
6. Required power (A_6),
7. Weight (A_7),
8. SFOC (Specific Fuel Oil Consumption – A_8),
9. SLOC (Specific Lubrication Oil Consumption – A_9),

Attributes' properties such as type of attributes and type of assessments are given in Table 6.25.

Table 6.25 Attributes' properties of Case - 2

	Attribute properties		
Attributes	Type of assessment	Type of attribute	
Price (capital cost)	Fuzzy (as interval)	Cost	Subjective
Running (operating) costs	Linguistic	Cost	Subjective
Reliability and maintainability	Linguistic	Benefit	Subjective
Vibration, noise and other signatures	Linguistic	Cost	Subjective
Ease of operation	Linguistic	Benefit	Subjective
Required power	Crisp	Cost	Objective
Weight	Crisp	Cost	Objective
SFOC	Crisp	Cost	Objective
SLOC	Crisp	Cost	Objective

For the main engine selection case, the decision hierarchy of objective, attributes, and alternatives is shown in Figure 6.8.

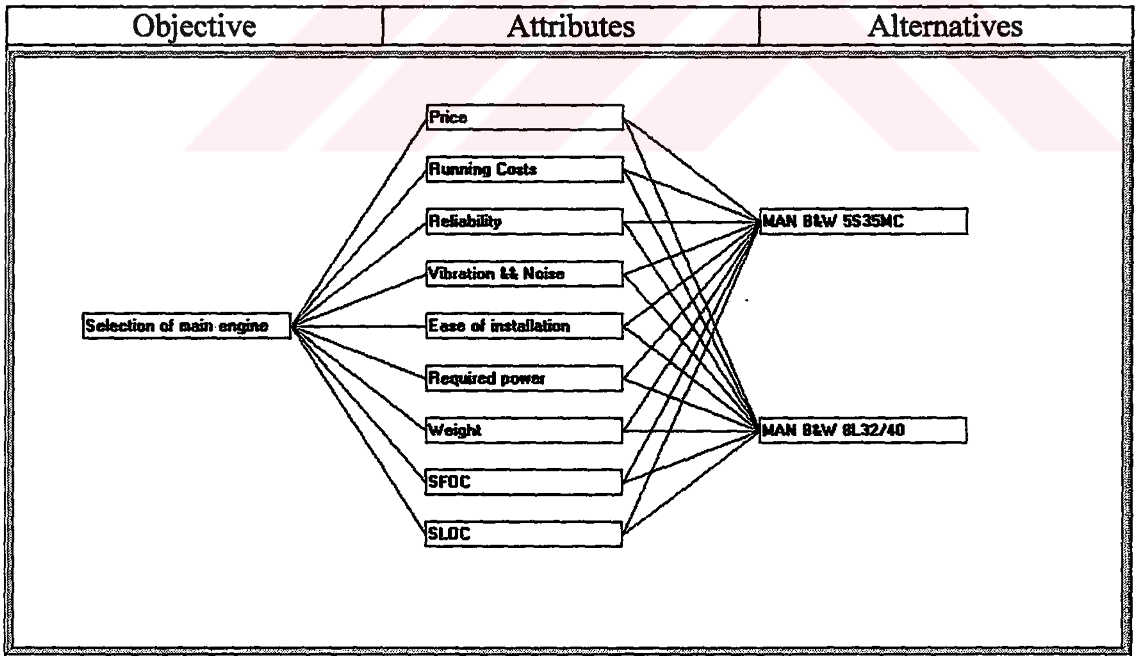


Figure 6.8 Decision hierarchy of Case - 2

6.2.2 Rating State Calculations

The alternatives of the second case study are evaluated by a group of experts (managing director, designer, and operator) with respect to the five subjective attributes as shown in Tables 6.26, 6.27, and 6.28. Note that all assessments for A_1 attribute are in thousands of dollars.

Table 6.26 First expert's evaluation of two alternatives under the five subjective attributes

	E_1 (Managing Director)	
	X_1	X_2
A_1	Approximately between 950 and 1100	Approximately between 900 and 1000
A_2	Medium	Medium
A_3	Good	Fair
A_4	Low	Low
A_5	Poor	Fair

Table 6.27 Second expert's evaluation of two alternatives under the five subjective attributes

	E_2 (Designer)	
	X_1	X_2
A_1	Approximately between 1100 and 1150	Approximately between 1000 and 1050
A_2	Medium	High
A_3	Very Good	Fair
A_4	Medium	Medium
A_5	Good	Good

Table 6.28 Third expert's evaluation of two alternatives under the five subjective attributes

	E_3 (Operator)	
	X_1	X_2
A_1	Approximately between 1100 and 1200	Approximately between 1000 and 1100
A_2	Very High	High
A_3	Mol. Poor	Mol. Good
A_4	Fairly Low	Mol. Low
A_5	Very Good	Good – Very Good

For the objective attributes of the problem, performance ratings for the alternatives have been obtained from their suppliers and all performance ratings are shown in Table 6.29. Remember that these values are all crisp numbers and they don't differ from one expert to other.

Table 6.29 Crisp ratings of two alternatives under the four objective attributes

	X_1	X_2
A_6	3500 Kw	3520 Kw
A_7	61.5 t	44 t
A_8	175 g/Kwh	182 g/Kwh
A_9	1 g/Kwh	1 g/Kwh

Experts' fuzzy assessments for the first subjective attribute (A_1) are all converted to fuzzy numbers. For example, 'approximately between 1100 and 1200' can be represented by the trapezoidal fuzzy number of (1050, 1100, 1200, 1250).

For the rest of the attributes (A_2, A_3, A_4, and A_5), experts' linguistic assessments are transformed into fuzzy numbers by using appropriate Scale described in Section 5.2.1.2. Linguistic terms are matched with Scale 3 (for A_2), Scale 6 (for A_3), Scale 8 (for A_4), and Scale 7 (for A_5). These transformed fuzzy numbers are shown in Tables 6.30, 6.31, and 6.32.

Table 6.30 First expert's decision matrix with fuzzy numbers

	E_1 (Managing Director)	
	X_1	X_2
A_1	(900, 950, 1100, 1150)	(850, 900, 1000, 1050)
A_2	(0.3, 0.5, 0.5, 0.7)	(0.3, 0.5, 0.5, 0.7)
A_3	(0.7, 0.8, 0.8, 0.9)	(0.4, 0.5, 0.5, 0.6)
A_4	(0.1, 0.3, 0.3, 0.5)	(0.1, 0.3, 0.3, 0.5)
A_5	(0.0, 0.2, 0.2, 0.4)	(0.3, 0.5, 0.5, 0.7)

Table 6.31 Second expert's decision matrix with fuzzy numbers

	E_2 (Designer)	
	X_1	X_2
A_1	(1050, 1100, 1150, 1200)	(950, 1000, 1050, 1100)
A_2	(0.3, 0.5, 0.5, 0.7)	(0.6, 0.75, 0.75, 0.9)
A_3	(0.8, 0.9, 1.0, 1.0)	(0.4, 0.5, 0.5, 0.6)
A_4	(0.3, 0.5, 0.5, 0.7)	(0.3, 0.5, 0.5, 0.7)
A_5	(0.6, 0.8, 0.8, 1.0)	(0.6, 0.8, 0.8, 1.0)

Table 6.32 Third expert's decision matrix with fuzzy numbers

	E_3 (Operator)	
	X_1	X_2
A_1	(1050, 1100, 1200, 1250)	(950, 1000, 1100, 1150)
A_2	(0.8, 0.9, 1.0, 1.0)	(0.6, 0.75, 0.75, 0.9)
A_3	(0.2, 0.3, 0.4, 0.5)	(0.5, 0.6, 0.7, 0.8)
A_4	(0.3, 0.4, 0.4, 0.5)	(0.4, 0.45, 0.45, 0.5)
A_5	(0.8, 1.0, 1.0, 1.0)	(0.7, 0.9, 1.0, 1.0)

Then for the first attribute, all non-standardised fuzzy numbers are converted to standardised fuzzy numbers. Results for each expert are given in Tables 6.33, 6.34, and 6.35.

Table 6.33 First expert's decision matrix with standardised trapezoidal fuzzy numbers

	E_1 (Managing Director)	
	X_1	X_2
A_1	(0.72, 0.76, 0.88, 0.92)	(0.68, 0.72, 0.80, 0.84)
A_2	(0.3, 0.5, 0.5, 0.7)	(0.3, 0.5, 0.5, 0.7)
A_3	(0.7, 0.8, 0.8, 0.9)	(0.4, 0.5, 0.5, 0.6)
A_4	(0.1, 0.3, 0.3, 0.5)	(0.1, 0.3, 0.3, 0.5)
A_5	(0.0, 0.2, 0.2, 0.4)	(0.3, 0.5, 0.5, 0.7)

Table 6.34 Second expert's decision matrix with standardised trapezoidal fuzzy numbers

	E_2 (Designer)	
	X_1	X_2
A_1	(0.84, 0.88, 0.92, 0.96)	(0.76, 0.80, 0.84, 0.88)
A_2	(0.3, 0.5, 0.5, 0.7)	(0.6, 0.75, 0.75, 0.9)
A_3	(0.8, 0.9, 1.0, 1.0)	(0.4, 0.5, 0.5, 0.6)
A_4	(0.3, 0.5, 0.5, 0.7)	(0.3, 0.5, 0.5, 0.7)
A_5	(0.6, 0.8, 0.8, 1.0)	(0.6, 0.8, 0.8, 1.0)

Table 6.35 Third expert's decision matrix with standardised trapezoidal fuzzy numbers

	E_3 (Operator)	
	X_1	X_2
A_1	(0.84, 0.88, 0.96, 1.00)	(0.76, 0.80, 0.88, 0.92)
A_2	(0.8, 0.9, 1.0, 1.0)	(0.6, 0.75, 0.75, 0.9)
A_3	(0.2, 0.3, 0.4, 0.5)	(0.5, 0.6, 0.7, 0.8)
A_4	(0.3, 0.4, 0.4, 0.5)	(0.4, 0.45, 0.45, 0.5)
A_5	(0.8, 1.0, 1.0, 1.0)	(0.7, 0.9, 1.0, 1.0)

6.2.3 Aggregation State Calculations

In this state, all ratings of decision matrices are aggregated under each subjective attribute by taking into account the attribute based expert weights.

The relative importances of attributes and experts are assigned according to importance observed through interview with manager (or moderator). In this case, Ship Owner has also been chosen as moderator. Then weights for attributes and experts are determined such that the sum of all weights is 1. The relative importances and weights of attributes and experts are given in Table 6.36.

Table 6.36 Weights of attributes and experts

	Attributes		E_1		E_2		E_3	
	R.I.	w.	R.I.	w.	R.I.	w.	R.I.	w.
A_1	100	0.30	1.00	0.32	0.70	0.19	0.30	0.09
A_2	67	0.20	1.00	0.32	0.70	0.19	0.30	0.09
A_3	50	0.15	0.30	0.10	0.50	0.14	1.00	0.31
A_4	33	0.10	0.30	0.10	1.00	0.28	0.60	0.19
A_5	17	0.05	0.5	0.16	0.7	0.19	1.0	0.31
A_6	17	0.05						
A_7	17	0.05						
A_8	17	0.05						
A_9	17	0.05						
Where R.I. : Relative Importance.								

During the whole process of the aggregation state, β, showing the moderator's dominance on the problem, is taken as 0.4.

Detailed aggregation calculations related with this state and their figures are all given in Appendix D. These calculations involve attribute based necessary aggregation calculations, such as degree of agreement, relative degree of agreement of each expert, etc. After aggregation calculations, aggregation matrices for homo/heterogeneous group of experts can be constructed easily as shown in Tables 6.37 and 6.38 respectively.

Table 6.37 Aggregated matrix for homogeneous group of experts

	X_1	X_2
A_1	(0.801, 0.841, 0.920, 0.960)	(0.734, 0.774, 0.840, 0.880)
A_2	(0.434, 0.607, 0.634, 0.780)	(0.510, 0.675, 0.675, 0.840)
A_3	(0.603, 0.703, 0.765, 0.830)	(0.431, 0.531, 0.563, 0.663)
A_4	(0.235, 0.400, 0.400, 0.565)	(0.270, 0.418, 0.418, 0.567)
A_5	(0.543, 0.743, 0.743, 0.870)	(0.545, 0.745, 0.780, 0.911)
A_6	3500	3520
A_7	61.5	44
A_8	175	182
A_9	1	1

Table 6.38 Aggregated matrix for heterogeneous group of experts

	X_1	X_2
A_1	(0.793, 0.833, 0.915, 0.955)	(0.728, 0.768, 0.835, 0.875)
A_2	(0.410, 0.588, 0.610, 0.766)	(0.486, 0.655, 0.655, 0.824)
A_3	(0.542, 0.642, 0.712, 0.780)	(0.441, 0.541, 0.582, 0.682)
A_4	(0.248, 0.415, 0.415, 0.581)	(0.282, 0.432, 0.432, 0.582)
A_5	(0.548, 0.748, 0.748, 0.867)	(0.558, 0.758, 0.797, 0.920)
A_6	3500	3520
A_7	61.5	44
A_8	175	182
A_9	1	1

6.2.4 Selection State Calculations

Up to this state, experts' fuzzy and linguistic assessments have been transformed into standardised trapezoidal fuzzy numbers and then aggregated under each subjective attribute. In order to rank the alternatives, aggregated matrices' fuzzy elements should be defuzzified. Defuzzified aggregated matrices for homo/heterogeneous group of experts are shown in Tables 6.39 and 6.40 respectively.

Table 6.39 Defuzzified aggregated matrix for homogeneous group of experts

	X_1	X_2
A_1	0.866	0.795
A_2	0.599	0.650
A_3	0.709	0.543
A_4	0.414	0.429
A_5	0.696	0.713
A_6	3500	3520
A_7	61.5	44
A_8	175	182
A_9	1	1

Table 6.40 Defuzzified aggregated matrix for heterogeneous group of experts

	X_1	X_2
A_1	0.859	0.790
A_2	0.581	0.633
A_3	0.657	0.556
A_4	0.427	0.441
A_5	0.699	0.725
A_6	3500	3520
A_7	61.5	44
A_8	175	182
A_9	1	1

TOPSIS procedure is applied to the two alternatives to obtain their OAR values and ranking orders. First, the defuzzified aggregated matrices are normalised. Table 6.41 shows the normalised values of each attribute for homogeneous group of experts and Table 6.42 shows the normalised values of each attribute for heterogeneous group of experts.

Table 6.41 Normalised ratings for homogeneous group of experts

	X_1	X_2
A_1	0.7367	0.6763
A_2	0.6777	0.7354
A_3	0.7939	0.6080
A_4	0.6944	0.7196
A_5	0.6985	0.7156
A_6	0.7051	0.7091
A_7	0.8133	0.5819
A_8	0.6931	0.7208
A_9	0.7071	0.7071

Table 6.42 Normalised ratings for heterogeneous group of experts

	X_1	X_2
A_1	0.7361	0.6769
A_2	0.6762	0.7367
A_3	0.7633	0.6460
A_4	0.6956	0.7184
A_5	0.6941	0.7199
A_6	0.7051	0.7091
A_7	0.8133	0.5819
A_8	0.6931	0.7208
A_9	0.7071	0.7071

Weighted normalised ratings of each attribute can be calculated by multiplying each attribute with its associated weight: $(w_1, w_2, w_3, w_4, w_5, w_6, w_7, w_8, w_9) = (0.30, 0.20, 0.15, 0.10, 0.05, 0.05, 0.05, 0.05, 0.05)$. Tables 6.43 and 6.44 show the weighted normalised values for homo/heterogeneous group of experts respectively.

Table 6.43 Weighted normalised ratings for homogeneous group of experts

	X_1	X_2
A_1	0.2210	0.2029
A_2	0.1355	0.1471
A_3	0.1191	0.0912
A_4	0.0694	0.0720
A_5	0.0349	0.0358
A_6	0.0353	0.0355
A_7	0.0407	0.0291
A_8	0.0347	0.0360
A_9	0.0354	0.0354

Table 6.44 Weighted normalised ratings for heterogeneous group of experts

	X_1	X_2
A_1	0.2208	0.2031
A_2	0.1352	0.1473
A_3	0.1145	0.0969
A_4	0.0696	0.0718
A_5	0.0347	0.0360
A_6	0.0353	0.0355
A_7	0.0407	0.0291
A_8	0.0347	0.0360
A_9	0.0354	0.0354

Positive and negative ideal solutions are given in Table 6.45 for homogeneous group of experts and in Table 6.46 for heterogeneous group of experts.

Table 6.45 Positive and negative ideal solutions for homogeneous group of experts

	Positive-Ideal Solution	Negative-Ideal Solution
A_1	0.2029	0.2210
A_2	0.1355	0.1471
A_3	0.1191	0.0912
A_4	0.0694	0.0720
A_5	0.0358	0.0349
A_6	0.0353	0.0355
A_7	0.0291	0.0407
A_8	0.0347	0.0360
A_9	0.0354	0.0354

Table 6.46 Positive and negative ideal solutions for heterogeneous group of experts

	Positive-Ideal Solution	Negative-Ideal Solution
A_1	0.2031	0.2208
A_2	0.1352	0.1473
A_3	0.1145	0.0969
A_4	0.0696	0.0718
A_5	0.0360	0.0347
A_6	0.0353	0.0355
A_7	0.0291	0.0407
A_8	0.0347	0.0360
A_9	0.0354	0.0354

Table 6.47 shows the values of separation measures and relative closeness to the positive-ideal solution for homo/heterogeneous group of experts.

Table 6.47 Values of separation measures and relative closeness to the positive-ideal solution for homo/heterogeneous group of experts

	S_i^*		S_i^-		C_i^*	
	Homog.	Heterog.	Homog.	Heterog.	Homog.	Heterog.
X_1	0.0215	0.0212	0.0303	0.0215	0.5849	0.5037
X_2	0.0303	0.0215	0.0215	0.0212	0.4151	0.4963

Finally, main engine alternatives are ranked on the basis of C_i^* values. For homogeneous group of experts, according to the descending order of C_i^*, the preference order is $X_1 > X_2$, where the first alternative is the leader and the second alternative is the last contender.

Similarly, for heterogeneous group of experts, ranking of alternatives based on the C_i^* values is given as $X_1 > X_2$.

Figure 6.9 shows the ranking of alternatives for homogeneous group of experts and Figure 6.10 shows the ranking of alternatives for heterogeneous group of experts.

For homogeneous group of experts, the first alternative is ranked in the first position among the others and the second alternative's performance is ranked in the second position.

6.2.5 Sensitivity Analysis

Sensitivity analysis is performed to see the β effect on the OAR values. β values are taken as the range between 0.1 and 1 as shown in Table 6.48. In Table 6.48, OAR values with respect to each β value are also given.

Table 6.48 OAR values with respect to β values

β Coefficient	X_1	X_2
0 (Homogeneous)	0.5849	0.4151
0.1	0.5694	0.4306
0.2	0.5480	0.4520
0.3	0.5238	0.4762
0.4	0.5037	0.4963
0.5	0.4738	0.5262
0.6	0.4500	0.5500
0.7	0.4238	0.5762
0.8	0.4031	0.5969
0.9	0.3859	0.6141
1	0.3763	0.6237

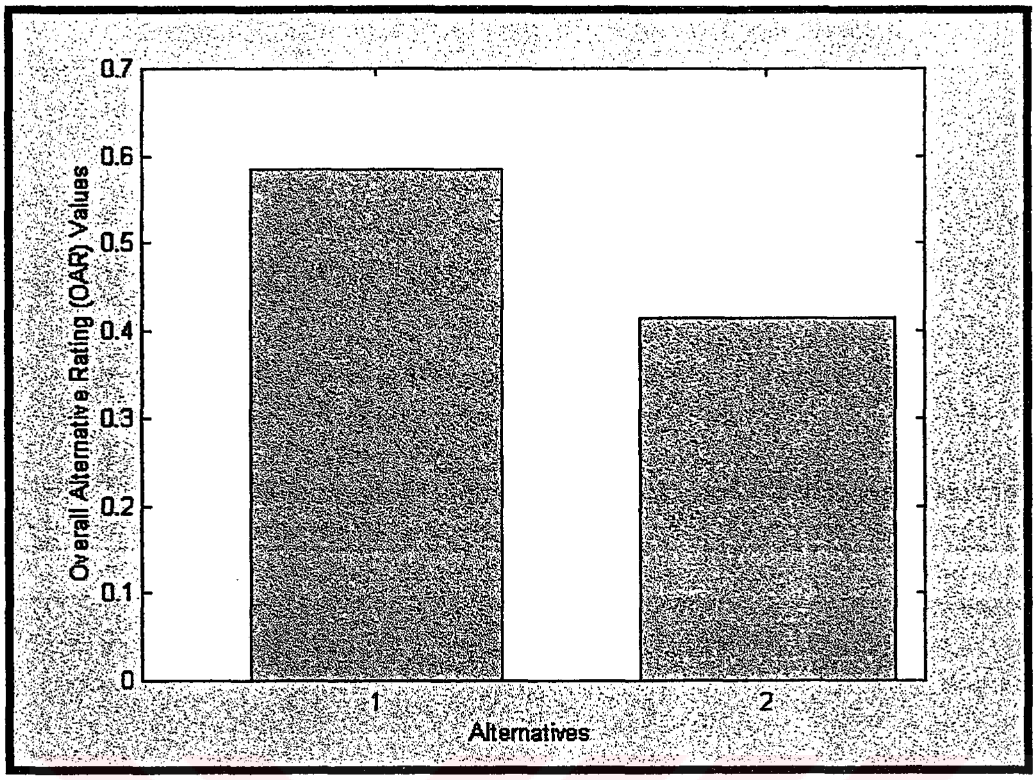


Figure 6.9 Ranking of alternatives for homogeneous group of experts

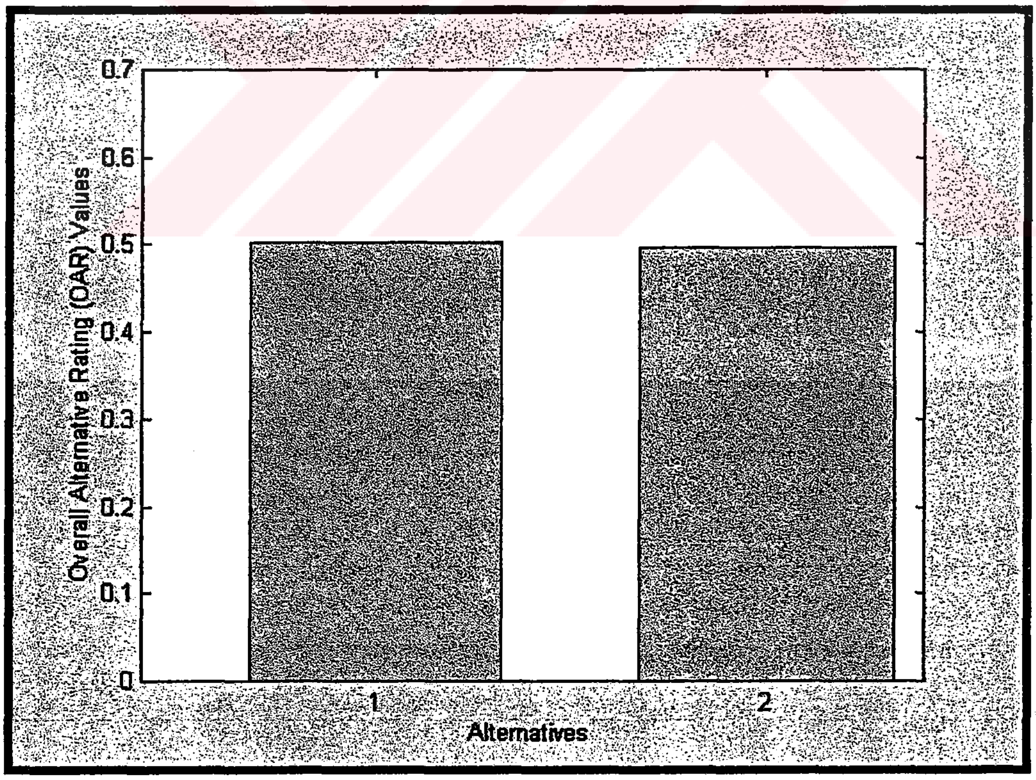


Figure 6.10 Ranking of alternatives for heterogeneous group of experts (for β=0.4)

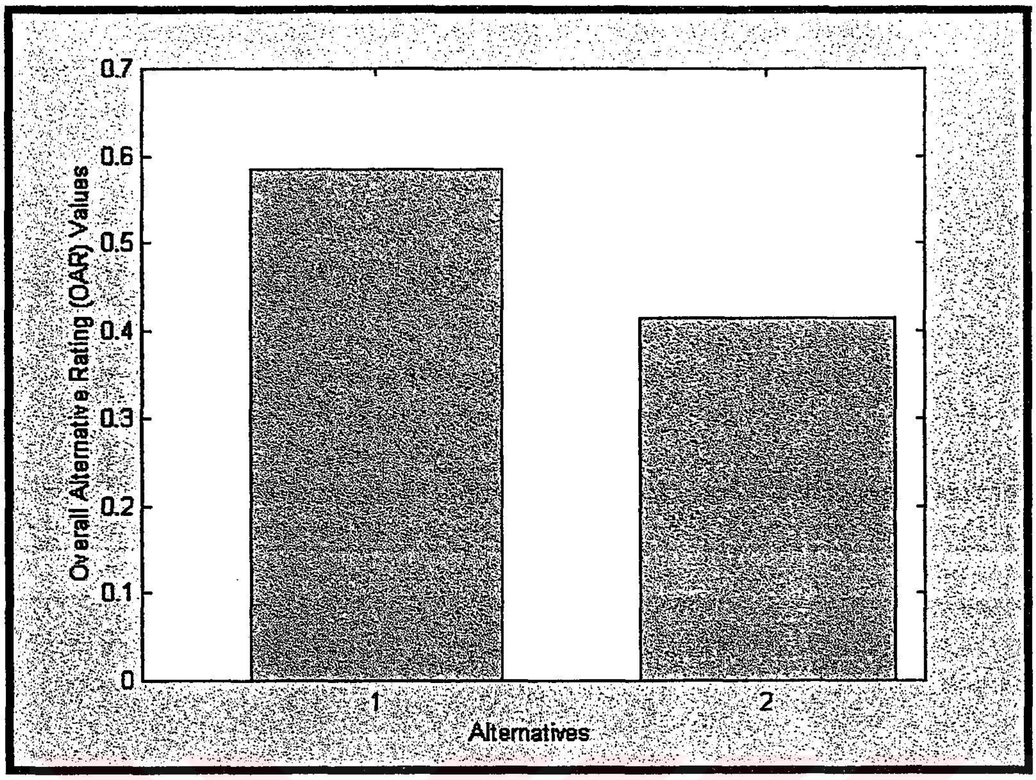


Figure 6.9 Ranking of alternatives for homogeneous group of experts

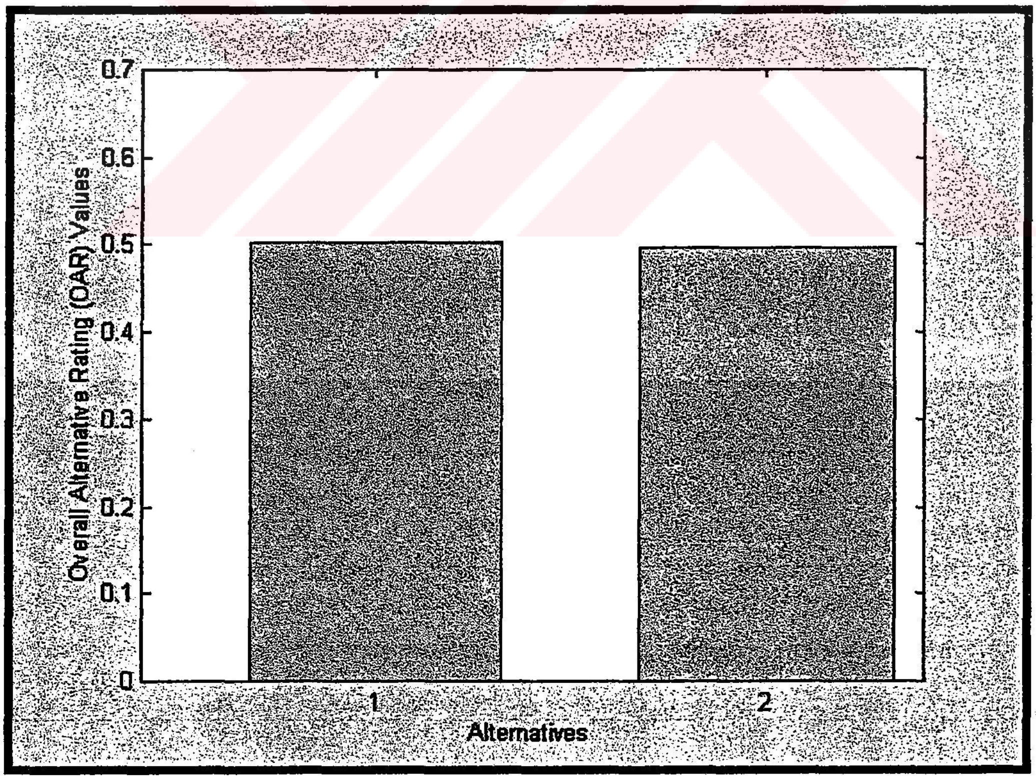


Figure 6.10 Ranking of alternatives for heterogeneous group of experts (for β=0.4)

7. CONCLUSIONS

This chapter summarises the achievements of the foregoing research. It brings together the elements discussed in previous chapters and shows how problems posed at the start of the thesis are addressed by the system constructed. It places the system in context, discussing the achievements and shortcomings of the proposed method. Finally, it points the way for further research in this area.

The problem statements, the goal and the objectives of the research were given in Chapter 1, the conceptual model of the proposed approach was given in Chapter 5. A review of related literature was summarised in Chapter 2.

Human beings are constantly making decisions in their daily activities. For this reason, the study of decision making processes has always been a field of great interest to many researchers.

Decision making is the process of determining the best course of action from a finite set of available alternatives. The major concern is the fact that almost all decision problems have multiple, usually conflicting criteria. Research on how to solve such multiple criteria decision making (MCDM) problems has been enormous. These problems are broadly classified into two categories :

1. Multiple Attribute Decision Making (MADM) or multiple attribute analysis, and

2. Multiple Objective Decision Making (MODM) or multiple criteria optimisation.

MADM is associated with problems whose number of alternatives has been predetermined. The decision maker is to select/prioritise/rank a finite number of courses of action (or alternatives). The MADM methods are management decision aids in evaluating and/or selecting a desired one from the finite number of alternatives, which are characterised by multiple attributes. On the other hand, MODM is not associated with problems in which the alternatives have been predetermined. The decision maker's primary concern is to design the most promising alternative with respect to limited resources.

Frequently, real world decision making problems are ill defined, i.e., their objectives, constraints, and parameters are not precisely known. These obstacles of lack of precision have been dealt with using the probabilistic approach. But, due to the fact that the requirements on the data and on the environment are very high and that many real world problems are fuzzy by nature and not random, the probability applications have not been very satisfactory in a lot of cases.

FST (Fuzzy Set Theory) is an alternative approach to solving MADM problems where available sources of information are inaccurate, unquantifiable, incomplete, nonobtainable, subjectively interpreted or uncertain.

This thesis discussed MADM problems under fuzzy environment for solving GDM (Group Decision Making) problems. The most important approaches and basic concepts were introduced. Since the focus is on FMADM (Fuzzy Multiple Attribute Decision Making) problems, a detailed discussion of the most important methods for solving these problems was presented and a new multiple attributive GDM method under fuzzy environment was developed.

7.1 Contributions of the Research

From the work carried out in this thesis, the two main contributions have been reached. They are classified as contributions to "multiple attribute decision making theory" and contributions to "naval engineering" points of views.

Development of a new FMADM method is the first focus and contribution of this dissertation. From the decision theory point of view, the proposed method has the following achievements that will be given in the states of the proposed method :

<u>In General</u>

- It is an entire MADM model which combines FMADM methodologies with GDM techniques,
- The proposed method is very suitable for solving the multiple attributive GDM problems under fuzzy environment,
- The proposed method enables the researchers to incorporate homo/heterogeneous group of experts with the different degrees of importance into the FMADM models,
- The majority of classical MADM methods are capable of handling large MADM problems. The proposed approach extends that ability to the fuzzy problems with multiple experts domain,

- It is a new FMADM method that is easy to use and understand. The algorithm of the proposed approach is also easy to be coded into a computer program due to its stepwise description,

Rating State

- It is capable of solving large, real world MADM problems which possibly contain a mixture of fuzzy and crisp data,

- The method is flexible enough to handle both fuzzy and non fuzzy information. The proposed approach allows MADM problems to take data in the forms of linguistic terms, fuzzy numbers, and/or crisp numbers. This yields more realistic, accurate and reliable decision models than the existing ones,

- The concepts of fuzzy numbers and linguistic variables are used to evaluate the subjective attributes in such a manner that the viewpoints of an entire MADM body can be expressed without any constraints,

- The proposed method is independent of the type of membership functions being used. Some other membership functions, for example triangular membership functions, are also applicable,

Aggregation State

- Through the use of FST and positive trapezoidal fuzzy numbers, the aggregation method provides a systematic way to aggregate the expert opinions in GDM, which may be objective or quasi-subjective depending on the preferences of the moderator,

- In addition, by using this aggregation procedure, one can obtain the consensus information and construct the judgement matrix for MADM with GDM problems,

- It is the only method that enables the attribute based aggregation for heterogeneous group of experts. In the proposed method, expert weights are assigned by a moderator (or manager) for each attribute to reflect the reality of the selection problems,

- Using β coefficient, moderator is included into consensus process of the method, Selection State

- The proposed method gives the decision maker the flexibility of selecting of any other classical MADM methods such as SAW, SMART etc.,

- The proposed method eliminates the difficulties resulting from ranking of fuzzy numbers by deffuzzification of fuzzy OAR values.

The second concern and contribution of this dissertation is to show the applicability of the proposed method into the naval engineering MADM problems. From the naval engineering point of view, the following can be concluded:

- As illustrated in the real life examples, the proposed method is a generalised model which can be applied to great variety of practical problems encountered in the naval engineering from propulsion/manoeuvring system selection to warship requirements definition,

- As the application grows, the real value of fuzzy decision making tools will find more widespread use, as most of the practical problems from design to production involves the aggregation of rational and fuzzy elements in harmony,

- Such an approach will also assist the use of optimisation by placing them within the correct context in problem solving and hence will avoid sub-system or sub-attribute optimisation problems.

However no decision support tool will have universal applicability nor will any system be able to represent all categories of problems fully. Hence, the proposed method has a number of shortcomings. These can be summarised as follows :

- Objectivity of the decision cannot be assured, since the moderator's view does have significant effect on the outcome. Similarly, the choice of experts, their degree of importances as well as the weight selection of the attributes are quasi-subjective,

- Due to the limitations of time, other possible techniques which could be incorporated into the general frame of the approach have not been tested. Hence, their influence on the recommended outcome are not known,

- While possible, probabilistic attribute has not been included.

7.2 Recommendations for Future Research

This research provided a theoretical and practical foundation for carrying out future research and development on more powerful and mature systems. It can be seen from the above that the proposed approach is open for improvement. Some of the improvements that may enhance the proposed methodology include the following:

- Semantic modelling of linguistic terms,

Since the meaning of each linguistic term varies from circumstance(s) to circumstance(s), assessing a fuzzy number or numbers to a linguistic term is a constant challenge. Existing works on this topic are few and seem quite arbitrary. Assigning an appropriate fuzzy number to a set of linguistic terms is a curious problem. Therefore, research may be further extended as follows :

Explore methods for constructing appropriate linguistic models for the semantic conversion systems. Matching fuzzy numbers with linguistic terms will be very difficult.

- Aggregation of expert' opinions under different assessment settings for the same subjective attribute,

Although each expert expresses his/her opinions under the same assessment settings in the proposed method, they may use different assessment settings (linguistic, fuzzy or crisp settings) in some selection problems to give his/her opinions (or performance ratings) about the alternatives with respect to each subjective attribute. Therefore, the extension could be to as follows :

Extend the research for solving the situations where experts express their opinions under different assessment settings for the same subjective attribute.

- Systematic investigation for different types of weighting, defuzzification and ranking techniques,

Further studies may be undertaken to investigate the effect of different weighting, defuzzification techniques on the final rankings as well as to examine the computational efficiency in terms of computing time of different aggregation operations of fuzzy sets. These will require extensive computer programming efforts.

Finally, from the case studies, it can be seen that the proposed method can efficiently help the decision makers and engineers to make decisions in real world. It can provide a useful way to solve the selection problems in a fuzzy environment. It is a versatile and flexible system which covers a vast variety of FMADM problems.

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APPENDIX A. BASICS OF FUZZY SET THEORY AND LINGUISTIC VARIABLES AND LINGUISTIC HEDGES

In the following sections, we will review some basic definitions of fuzzy sets from a mathematical perspective. Also, linguistic variables and hedges, the fuzzy numbers such as $\tilde{S}$ fuzzy numbers, $\tilde{\Pi}$ fuzzy numbers, L-R fuzzy numbers, triangular fuzzy numbers, and trapezoidal fuzzy numbers, and the extension operations of fuzzy numbers will be introduced. The Extension Principle introduced by (Zadeh (1972); (1975a), (1975b)) is used to generalise non-fuzzy (crisp) mathematical concepts into fuzzy quantities. An important field of applications for the Extension, Principle is found in arithmetic operations such as addition, subtraction, multiplication and division. We will review the definition of the Extension Principle first and extend from it to fuzzy arithmetic operations.

A.1 Basic Concepts of Fuzzy Sets

The basic concepts of fuzzy set theory can be found in the following (Zadeh (1972), (1973), (1975a), (1975b), Kaufmann (1975), Dubois and Prade (1980), Kandel (1986) and Zimmermann (1987), (1991)):

Definition A.1

Let *U* be a universe of discourse (or domain), a fuzzy set (or fuzzy subset) $\tilde{X}$ of *U* is characterised by a membership function $\mu_{\tilde{x}} : U \rightarrow [0,1]$, which associates with each element *x* of *U* and a number $\mu_{\tilde{x}}(x)$ in the interval [0,1]. $\tilde{X}$ is denoted as $\left\{(x, \mu_{\tilde{x}}(x)) \middle| x \in U\right\}$.

Obviously, the characteristic function of ordinary set theory is a special case of the membership function. Let *U* be an ordinary set and X be the subset of *U*. The characteristic function of X can be defined as $\mu_x : U \rightarrow [0,1]$. Whether or not an element in U belongs to X is determined by the value of the characteristic function (1 or 0). In general, fuzzy set theory is a generalisation of ordinary set theory; the definition, theorems and proofs for a fuzzy set always hold for the classic set.

Definition A.2

The support $S(\tilde{X})$, of a fuzzy set $\tilde{X}$ is the set of points in $\tilde{X}$ at which $\mu_{\tilde{x}}(x)$ is positive, i.e.,

$$S(\tilde{X}) = \left\{ x \middle| x \in \tilde{X}, \mu_{\tilde{x}}(x) > 0 \right\} \tag{A.1}$$

Definition A.3

The height $h_{\tilde{x}}(x)$ of a fuzzy set $\tilde{X}$ is the supremum of $\mu_{\tilde{x}}(x)$ over $\tilde{X}$, i.e.,

$$h_{\tilde{x}}(x) = \operatorname*{Sup}_{x \in \tilde{X}} \mu_{\tilde{x}}(x). \tag{A.2}$$

Definition A.4

A fuzzy set $\tilde{X}$ is said to be normal if its height is unitary, that is if $\operatorname*{Sup}_{x \in \tilde{X}} \mu_{\tilde{x}}(x) = 1$.

Otherwise, $\tilde{X}$ is non-normal.

Definition A.5

If $\tilde{X}$ is a fuzzy set of universe *U*, then an α-level set or α-cut of $\tilde{X}$ is a non-fuzzy set denoted by $\tilde{X}_\alpha$, which comprises, all elements of *U* whose grade of membership in $\tilde{X}$ is greater than or equal to α, i.e.

$$\tilde{X}_\alpha = \left\{ x \middle| \ \mu_{\tilde{x}}(x) \geq \alpha \right\} \tag{A.3}$$

Definition A.6

A fuzzy set is $\tilde{X}$ convex if only if $\forall_{x_1} \in \tilde{X}$, $\forall_{x_2} \in \tilde{X}$, and $\forall \lambda \in [0,1]$,

$$\mu_{\tilde{x}}(\lambda x_1 + (1-\lambda)x_2) \geq \min\,(\mu_{\tilde{x}}(x_1), \mu_{\tilde{x}}(x_2)) \tag{A.4}$$

Definition A.7

A fuzzy singleton is a fuzzy set whose support is a single point *x* in *U*, Suppose $\tilde{X}$ is a fuzzy singleton whose support is the point *x*. We then express $\tilde{X}$ as

$\tilde{X} = \mu/x$,

where μ is the grade of membership of x in $\tilde{X}$.

However, if $\tilde{X}$ has finite support, say $x_1, x_2, \ldots, x_n$, and for each $i = 1,2,\ldots,n$; μ_i is the grade of membership of x_i in $\tilde{X}$, then $\tilde{X}$ may be represented symbolically as a combination of fuzzy singletons by

$$\tilde{X} = \mu_1/x_1 + \mu_2/x_2 + \ldots + \mu_n/x_n,$$

or more completely, by

$$\tilde{X} = \sum_{i=1}^{n} \mu_i / x_i \, .$$

A similar notation may be used in the case where the support is countable infinite. If the support is a continuum, then $\tilde{X}$ may be denoted by

$$\tilde{X} = \int_U \mu_i / x_i$$

Let $\tilde{X}$ and $\tilde{Y}$ be fuzzy subsets of a universe *U*, with membership functions μ_x and μ_y. Three of the major set operations are defined as follows:

Definition A.8

The complement of $\tilde{X}$, denoted by $-\tilde{X}$ or $\tilde{X}$, is defined by

$$-\tilde{X} = \tilde{X}' = \int_U (1 - \mu_x(x)) / x \text{, for all } x \in U. \tag{A.5}$$

Definition A.9

The union of $\tilde{X}$ and $\tilde{Y}$, denoted by $\tilde{X} \cup \tilde{Y}$, is defined by

$$\tilde{X} \cup \tilde{Y} = \int_U \left\{ \max \left[\mu_{\tilde{x}}(x), \mu_{\tilde{Y}}(x) \right], \right\} / x, \text{ for all } x \in U. \tag{A.6}$$

Definition A.10

The intersection of $\tilde{X}$ and $\tilde{Y}$, denoted by $\tilde{X} \cap \tilde{Y}$, is defined by

$$\tilde{X} \cap \tilde{Y} = \int_U \left\{ \min \left[\mu_{\tilde{x}}(x), \mu_{\tilde{Y}}(x) \right], \right\} / x, \text{ for all } x \in U. \tag{A.7}$$

The union corresponds to the connective "OR" while the intersection corresponds to the connective "AND" the operation of complementation corresponds to negation.

So, $\tilde{X}$ AND $\tilde{Y} = \tilde{X} \cap \tilde{Y}$

$\tilde{X}$ OR $\tilde{Y} = \tilde{X} \cup \tilde{Y}$.

Definition A.11

$\tilde{X}$ contains $\tilde{Y}$, denoted as $\tilde{X} \supset \tilde{Y}$ if and only if

$\mu_{\tilde{X}}(x) \geq \mu_{\tilde{Y}}(x)$, for all $x \in U$.

And, $\tilde{X}$ and $\tilde{Y}$ are equal, denoted by $\tilde{X} = \tilde{Y}$, if and only if

$\mu_{\tilde{X}}(x) = \mu_{\tilde{Y}}(x)$, for all $x \in U$.

Definition A.12

The product of $\tilde{X}$ and $\tilde{Y}$, denoted by $\tilde{X}$ (x) $\tilde{Y}$, is defined by

$$\tilde{X}(x)\tilde{Y} = \int_U \left[\mu_{\tilde{X}}(x) \times \mu_{\tilde{Y}}(x) \right] / x, \text{ for all } x \in U. \tag{A.8}$$

Thus, $\tilde{X}^n$, where n is any positive number, is defined by

$$\tilde{X}^n = \int_U \left[\mu_{\tilde{X}}(x) \right]^n / x, \text{ for all } x \in U.$$

Definition A.13

The operation of concentration is defined as:

$$\text{CON}(\tilde{X}) = \tilde{X}^2 \tag{A.9}$$

Applying this operation to $\tilde{X}$ results in a fuzzy subset of $\tilde{X}$ such that the reduction in magnitude of the grade of membership of x in $\tilde{X}$ is relatively small for those x which have a high grade of membership in $\tilde{X}$ and relatively large for these x with low grade of membership.

Definition A.14

The operation of dilation is defined by

$$\text{DIL}(\tilde{X}) = \tilde{X}^{0.5} \tag{A.10}$$

The effect of this operation is the opposite of that concentration.

The operations of concentration and dilation are useful in the representation of linguistic hedges (see Section A.2.2).

Definition A.15

Let S be an interval in R. The Hamming distance between two fuzzy subsets $\tilde{X}$ and $\tilde{Y}$ in S is defined by

$$D(\tilde{X},\tilde{Y}\setminus S)=\int_{x\in S}\left|\mu_{\tilde{X}}(x)-\mu_{\tilde{Y}}(x)dx.\right. \tag{A.11}$$

If S = R, the $D(\tilde{X}, \tilde{Y} \setminus R) = D(\tilde{X}, \tilde{Y})$.

A.2 Linguistic Variables and Hedges (Zadeh (1972), Zimmermann (1987))

A variable is normally thought of as a notion that can be specified by assigning certain numerical values to it. If we define the variable x to mean "SPEED" and specify $0 \le x \le 100$, we know that the variable x can have all numbers between 0 and 100 assigned to it. The notion of a linguistic variable, $\tilde{X}$, is to regard it either as a variable whose numerical values are "fuzzy numbers", or as a variable whose range is not defined by numerical values, but by linguistic terms. In the following section, we will define linguistic variables and their relative terms.

A.2.1 Linguistic Variables

Definition A.16

A linguistic variable is characterised by the quintuplet ($\tilde{X}$, T($\tilde{X}$), U, G, M($\tilde{X}$)) in which $\tilde{X}$ is the name of the variable. T($\tilde{X}$) denotes the term-set of $\tilde{X}$, that is, the set of names of "linguistic values" of $\tilde{X}$, with each value being a fuzzy variable denoted generically by x and ranging over a universe of discourse U. The variable G is a syntactic rule (which usually has the form of a grammatical element) for generating the name, $\tilde{X}$, for values of $\tilde{X}$. M is a semantic rule for associating a meaning with each $\tilde{X}$; M($\tilde{X}$), which is a fuzzy subset of U, is characterised by a membership function (or compatibility function), $\mu_{\tilde{x}} : U \to [0,1]$ (i.e., a fuzzy restriction of U). A particular $\tilde{X}$, that is, a name generated by G; is called a term. It should be noted that the base variable x can also be vector valued (Zadeh (1978a), (1978b)).

Example A.1

Let $\tilde{X}$ be a linguistic variable with the label "SPEED" with U= [0,100]. Terms, T($\tilde{X}$), of this linguistic variable, which are again fuzzy sets, could be called "high", "medium", "low", and so on. The base variable x is the speed in miles per hour of a car driving on a highway. $M(\tilde{X})$ is the rule that assigns a meaning, that is, a fuzzy set, to these terms.

M(high)= $\{(\mu_{high}(x)|x \in [0,100]\}$

$$\text{Where } \mu_{high}(x) = \begin{cases} 0, & \text{for } 0 \le x \le 65, \\ \{1 + [(x-65)/5]^{-2}\}^{-1} & \text{for } 65 \le x \le 100. \end{cases}$$

$T(\tilde{X})$ will define the term set of the variable $\tilde{X}$. In this case, $\tilde{X}$= "SPEED", then T(SPEED)= {very low, low, medium, high, very high}

Where G is a rule which generates the (labels of) terms in the term set.

A.2.2 Linguistic Hedges

A linguistic hedge, or modifier, is an operation, which modifies the meaning of a term, or more generally, of a fuzzy set. Linguistic hedges are usually adjectives, such as "very", more or "less", "rather", and "approximately". By means of the above mentioned operations on fuzzy sets, the linguistic hedges can be defined. A typical example is the interpretation of the "very" hedge in terms of the concentration operation on [0,1]. This is generally defined as:

$$\text{very } \tilde{X} = \tilde{X}^2$$

where $\tilde{X}$ is a fuzzy subset of a universe U. Example A.2 illustrate the concept of a linguistic hedge.

Example A.2

If $\tilde{X}$ = "close to 0.5", with

$$\mu_{\text{close to } 0.5}(x) = \begin{cases} 0; & x \le 0.3, \\ [1 - (0.5 - x)/0.2]; & 0.3 \le x \le 0.5, \\ [1 - (x - 0.5)/0.2]; & 0.5 \le x \le 0.7, \\ 0; & x \ge 0.7. \end{cases}$$

then

$$\mu_{\text{veryclosето}0.5}(x) = \begin{cases} 0; & x \le 0.3, \\ [1 - (0.5 - x)/0.2]^2; & 0.3 \le x \le 0.5, \\ [1 - (x - 0.5)/0.2]^2; & 0.5 \le x \le 0.7, \\ 0; & x \ge 0.7. \end{cases}$$

On the other hand, the linguistic hedges, “rather”, “more or less”, or “approximately” may be defined by the dilation operation.

APPENDIX B. TECHNIQUES AND ALGORITHMS USED IN FMADM METHODS

In this Appendix, some special techniques and algorithms used in FMADM methods, discussed in Chapter 4, are given as follows:

B.1 Saaty's Eigenvector Method

Let the positive reciprocal matrix A be

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdot & & & \\ \cdot & & & \\ a_{n1} & a_{n2} & \cdots & \end{bmatrix} \tag{B.1}$$

where

$$a_{ij} \geq 0 \text{ and } a_{ij} = 1 / a_{ji}, \forall i, j, \tag{B.2}$$

$$a_{ij} = a_{ik} / a_{jk}, \tag{B.3}$$

$$a_{ij} = w_i / w_j. \tag{B.4}$$

Matrix A is called a 'reciprocal matrix'.

Multiplying A by $\underline{w} = (w_1, w_2, ..., w_n)^T$ yields

$$A\underline{w} = \begin{bmatrix} w_1/w_1 & w_1/w_2 & \cdots & w_1/w_n \\ w_2/w_1 & w_2/w_2 & \cdots & w_2/w_n \\ \cdot & & & \\ \cdot & & & \\ w_n/w_1 & w_n/w_2 & \cdots & w_n/w_n \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ \cdot \\ \cdot \\ w_n \end{bmatrix} = n \begin{bmatrix} w_1 \\ w_2 \\ \cdot \\ \cdot \\ w_n \end{bmatrix} = n\underline{w} \tag{B.5}$$

or

$$(A - n\ I) = \underline{w} = 0 \tag{B.6}$$

Due to the consistency property of Equation (B.3), the system of homogeneous linear equation, Equation (B.6), has only trivial solutions.

In general, the precise values of w_i/w_j are unknown and must be estimated. In other words, human judgements cannot be so accurate that Equation (B.3) be satisfied completely. We know that in any matrix, small permutations in the coefficients imply small permutations in the eigenvalues. If we define A' as the DM's estimate of A and $\underline{w}'$ corresponds to A', then

$$A'\underline{w}' = \lambda_{max}\,\underline{w}' \qquad \text{(B.7)}$$

where λ_{max} is the largest eigenvalue of A'. $\underline{w}'$ can be obtained by solving the system of linear equations, Equation (B.7). The consistency of the estimates in the matrix, A, is guaranteed when $\lambda_{max} \cong n$. When λ_{max} is not close to n, we must modify the estimates in A so that consistency is preserved.

The comparison scale uses ranges 1 to 9, each representing fuzzy entries as follows:

Table B.1 Judgement Scale

ImportanceValue	Definition
1	Equal Importance
3	Weak importance of one over the other
5	Strong importance of one over the other
7	Demonstrated importance of one over the other
9	Absolute domination of one over the other
2, 4, 6, 8	Intermediate values between the two adjacent judgements

Given any a_{ij}, the reciprocal value $a_{ji} = 1/a_{ij}$. Obviously, $a_{ii} = 1$ is always true.

Hierarchical Decision Structure

A hierarchy structure can be best described by the following example. Three job offers are considered by a new Ph.D. The attributes considered are research, growth, benefits, colleagues, location, and reputation. Since the entries of this MADM problem are only vaguely known, we cannot form a MADM decision matrix. To resolve this problem, a three-level hierarchy is constructed. The data of this MADM problem can be derived from this hierarchy decision structure.

The first level consists of a single objective, to have a successful career. The importance of it is assumed unity. The second level consists of six attributes. Their relative importance is determined using the eigenvector method with respect to the objective in the first level. The third level consists of the three jobs being considered. Their relative performances are derived using the eigenvector method with respect to each attribute. The weights and performance scores are then combined using the SAW method. The results are the final ratings of the three jobs. The job with the highest final rating can best fulfil the ultimate goal (successful career).

Formal Hierarchy : It is essentially a formalisation in terms of partially ordered sets of our intuitive understanding of the idea. It has levels: the top level consists of a single element and each element of a given level dominates or covers (serves as a property of a purpose for) some or all of the elements in the level immediately below. The pairwise comparison matrix approach may then be applied to compare elements in a single level with respect to a purpose from the adjacent higher level. The process in repeated up the hierarchy and the problem is to compose the resulting priorities (obtained by the eigenvector method) in such a way as to obtain one overall priority vector of the impact of the lowest elements on the top element of the hierarchy by successive weighting and composition.

Let the symbol L_k represent the k^{th} level of a hierarchy of h levels. Assume that $Y=(y_1, y_2, \ldots, y_k) \in L_k$ and that $X=(X_1, X_2, \ldots, X_{k+1}) \in L_{k+1}$. Also assume that there is an element $z \in L_{k-1}$ such that Y is covered by z. We can then consider the priority functions

$$w_z = Y \to [0, 1] \text{ and } w_y: X \to [0, 1] \quad \text{(B.8)}$$

We construct the "priority function of the elements in X with respect to z" denoted w, w: $X \to [0, 1]$ by

$$w(x_i) = \sum_{j=1}^{k} w_{y_j}(x_i)\, w_z(y_j), i = 1,2,\ldots,k+1. \quad \text{(B.9)}$$

It is obvious that this is no more than the process of weighting the influence of the element y_j on the priority of x_i by multiplying it with the importance of y_j with respect to z.

The algorithms involved will be simplified if one combines the $w_{y_j}(x_i)$ into a matrix B by setting $b_{ij} = w_{y_j}(x_i)$. If we further set $w_i = w(x_i)$ and $w_j' = w_z(y_j)$, then the above formula becomes

$$w_i = \sum_{j=1}^{k} b_{ij} w_j', i = 1,2,\ldots,k+1. \quad \text{(B.10)}$$

Thus, we may speak of the priority vector w and, indeed, of the priority matrix B; this gives the final formulation W= BW'.

A hierarchy is complete if all $x \in L_k$ are dominated by every element in L_{k-1}, $k = 2, \ldots, h$. Let H be a complete hierarchy with lowest element b and h levels. Let B_k be the priority matrix of k^{th} level, k= 1, 2 , ... , h. If W' is the priority vector of the p^{th} level with respect to some element z in the $(p-1)^{st}$ level, then the priority vector w of the q^{th} level ($p < q$) with respect to z is given as:

$$W = B_q B_{q-1} \ldots B_{p+1} W'. \quad \text{(B.11)}$$

Thus, the priority vector of the lowest level with respect to the element b is given as:

$$W = B_h B_{h-1} \ldots B_2 W'. \quad \text{(B.12)}$$

If L_1 has a single element, as usual, W' is just a scalar; if it has more elements, it is a vector.

B.2 SAW Based FMADM Methods

In the following, Baas and Kwakernaak's Algorithm, Kwakernaak's Algorithm, Dubois and Prade's Algorithm, and Cheng and McInnis's Algorithm are given.

B.2.1 Baas and Kwakernaak's Algorithm

It is assumed that $\mu_{w_j}(y_j)$ and $\mu_{r_{ij}}(x_{ij})$ are normalised membership functions. We can determine the approximated fuzzy utility u_i for alternative A_i, $\forall i$, using the following steps :

Step 1. Set an α_0 level for $\mu_{U_i}(u_i)$.

Step 2. After setting $\mu_{U_i}(u_i) = \alpha_{0'}$ identify the y_j and x_{ij} values that satisfy

$$\mu_{w_j}(y_j) = \mu_{r_{ij}}(x_{ij}) = \alpha_{0'}, \forall i, j \tag{B.13}$$

There may be more than one y_j value for $\mu_{w_j}(y_j) = \alpha_0$ and more than one x_{ij} value for $\mu_{r_{ij}}(x_{ij}) = \alpha_{0'}$.

Step 3. There are many u_i values such that $\mu_{u_i}(u_i) = \alpha_0$. We want to know the extreme ones, $u_{i_{min}}$ and $u_{i_{max}}$. For example, given two fuzzy attributes and two fuzzy weights, there will be a total of 2^4= 16 u_i values, i.e., 16 possible combinations of y_j and x_{ij}, i,j= 1,2. We simply pick the highest u_i, $u_{i_{max}}$,and the lowest u_i, $u_{i_{min}}$, and drop all other u_i's.

If the size of the problem increases, such as six attributes and six weights, there will be 2^{12}= 4096 u_i values. When problem size increases to 10 attributes and 10 weights, there will be over a million u_i values. Again, it is impossible to identify $u_{i_{max}}$ and $u_{i_{min}}$ values without the help of a computer.

To avoid calculating all the u_i values such that $\mu_{U_i}(u_i) = \alpha_{0'}$ Baas and Kwakernaak suggest the following. Given a set of real numbers $(\hat{y}_1,..., \hat{y}_n, \hat{x}_{i1},..., \hat{x}_{in})$ such that $\mu_{r_i}, (\hat{x}_{ij})$ and $\left[\mu'_{w_j}(\hat{y}_j)/(\hat{x}_{ij} - u_i)\right], \forall i, j,$ where

$$\mu'_{r_i}(x_{ij}) = d\mu_{r_i}(x_{ij})/dx_{ij} \tag{B.14}$$

and

$$\mu'_{wj}(y_j) = d\mu_{wj}(y_j)/dy_j \tag{B.15}$$

have the same sign, the resulting u_i will either be $u_{i_{max}}$ and $u_{i_{min}}$. Step 3 is complete when both $u_{i_{max}}$ and $u_{i_{min}}$ have been found.

To check if $\mu_{r_i}'(\hat{x}_{ij})$ and $\left[\mu_{w_j}'(\hat{y}_j)/(\hat{x}_{ij}-u_i)\right], \forall i, j,$ have the same sign, one would have to use the set of numbers $(\hat{x}_{i1},\ldots,\hat{x}_{in},\hat{y}_1,\ldots,\hat{y}_n)$ to compute its corresponding u_i value using Equation (4.1). Consequently, the values $(\hat{x}_{i1},\ldots,\hat{x}_{in},\hat{y}_1,\ldots,\hat{y}_n)$ are used in Equations (B.14) and (B.15). If $\mu_{r_i}'(\hat{x}_{ij})$ and $\left[\mu_{w_j}'(\hat{y}_j)/(\hat{x}_{ij}-u_i)\right]$, have the same sign, i.e., both are either positive or negative, the u_i value determined by $(\hat{x}_{i1},\ldots,\hat{x}_{in},\hat{y}_1,\ldots,\hat{y}_n)$ will be $u_{i_{max}}$ and $u_{i_{min}}$. If the signs are not the same, the corresponding u_i would be dropped.

The algorithm loops back to Step 1 for another α value. One must give general α values in order to get an approximated $\mu_{U_i}(u_i)$ function. The number of α values needed to construct $\mu_{U_i}(u_i)$ is a subjective matter. If more α values are given, the approximated function will be closer to the real one but will require much more computational effort.

B.2.2 Kwakernaak's Algorithm

Step 1. Choose an α_0 level.

Step 2. For alternative A_i, determine the following real numbers:

$$x_{ij}^- = \min\{x_{ij} \in R \mid \mu_{r_{ij}}(x_{ij}) \geq \alpha_0\} \; \forall j, \tag{B.16}$$

$$x_{ij}^* = \max\{x_{ij} \in R \mid \mu_{r_{ij}}(x_{ij}) \geq \alpha_0\} \; \forall j, \tag{B.17}$$

$$y_j^- = \min\{y_j \in R \mid \mu_{w_j}(y_j) \geq \alpha_0\} \; \forall j, \tag{B.18}$$

$$y_j^* = \max\{y_j \in R \mid \mu_{w_j}(y_j) \geq \alpha_0\} \; \forall j, \tag{B.19}$$

Step 3. At the α_0 level, the r_{ij} may be represented by $[x_{ij}^-, x_{ij}^*]$. Put $x_{ij}^-, \forall j$, in an order such that

$$m_1^- \leq m_2^- \leq \ldots \leq m_n^-, \tag{B.20}$$

where $m_1^- = \min_j x_{ij}^-$ and $m_n^- = \max_j x_{ij}^-$. The corresponding y_j^- values will also be put in an order such that

$$z_1^- \le z_2^- \le \ldots \le z_n^-. \tag{B.21}$$

Similarly, x_{ij}^* are rearranged in an order such that

$$m_1^* \le m_2^* \le \ldots \le m_n^*, \tag{B.22}$$

where $m_1^* = \min_j x_{ij}^*$ and $m_n^* = \max_j x_{ij}^*$. The corresponding y_j^* will be put in an order such that

$$z_1^* \le z_2^* \le \ldots \le z_n^*. \tag{B.23}$$

Step 4. Let fuzzy utility U_i at the α_0 level be $c_{\alpha_0} = \lfloor a_{\alpha_0}, b_{\alpha_0} \rfloor$. a_{α_0} and b_{α_0} are defined as:

$$a_{\alpha_0} = \min_{0 \le j \le n} \left[\frac{\sum_{k=1}^{j} z_k^* m_k^- + \sum_{k=j+1}^{n} z_k^- m_k^-}{\sum_{k=1}^{j} z_k^* + \sum_{k=j+1}^{n} z_k^-} \right], \tag{B.24}$$

$$b_{\alpha_0} = \max_{0 \le j \le n} \left[\frac{\sum_{k=1}^{j} z_k^- m_k^* + \sum_{k=j+1}^{n} z_k^* m_k^*}{\sum_{k=1}^{j} z_k^- + \sum_{k=j+1}^{n} z_k^*} \right]. \tag{B.25}$$

The lower bound a_{α_0} is computed according to the following concept. Since x_{ij} appears only in the numerator of the function,

$$u_i = \sum_{j=1}^{n} y_j x_{ij} / \sum_{j=1}^{n} y_j, \tag{B.26}$$

the minimum of u_i is quaranteed when x_{ij}, $\forall j$, take x_{ij}^- as their values. Thus, we need only be concerned with the combinations of y_j^* and y_j^- such that u_i is minimum. Similar remarks can be applied to Equation (B.25).

The algorithm may be applied several times to different α levels. Eventually, an approximated fuzzy utility U_i can be obtained.

Kwakernaak finds that by taking the maximum values of x_{ij}, we are guaranteed to have a maximum u_i. Similarly, by taking the minimum of x_{ij}, we get a minimum u_i. However, since w_j appears both in the numerator and denominator, a maximum w_j does not guarantee a maximum u_i and a minimum w_j does not guarantee a minimum u_i. Hence, the number of possible combinations one needs to test is C_n^{2n} instead of 2^{2n} as in Baas and Kwakernaak's approach where n is the number of weights.

For example, for a problem with five attributes and five weights, Baas and Kwakernaak's approach requires 2^{10}=1024 tries, while Kwakernaak's approach needs only C_5^{10}=252 tries. For a problem of 10 attributes and 10 weights, Baas and Kwakernaak's approach requires a maximum of 1,048,576 tries while Kwakernaak's approach requires 184,756 tries. The improvement is obvious.

B.2.3 Dubois and Prade's Algorithm

Step 1. Set an α-level and determine α-level sets for w_j and r_{ij} to be:

$$w_{j\alpha} = [y_j^-, y_j^*], \forall j, \tag{B.27}$$

$$r_{ij\alpha} = [x_{ij}^-, x_{ij}^*], \forall j,\ j. \tag{B.28}$$

Step 2. Compute normalised fuzzy weights, P_j, $\forall_j$. Given the α-level sets of w_j, $[y_j^-, y_j^*]$, j= 1,…,n, we can obtain n α-level sets $[p_j^-, p_j^*]$ of the normalised fuzzy weights P_j, $\forall_j$, as:

$$P_j^* = y_j^* / (y_j^* + \sum_{k \neq j} y_k^-) \tag{B.29}$$

and

$$P_j^- = y_j^- / (y_j^- + \sum_{k \neq j} y_k^*) \tag{B.30}$$

Let $q_j \in [p_j^-, p_j^*]$, ∀j, the condition,

$$\sum_{j=1}^{n} q_j = 1 \tag{B.31}$$

must hold.

Step 3. For alternative, A_i, the rating r_{ij} may be represented by an α-level set as in Equation (B.28). That is,

$$r_{ij\alpha} = [x_{ij}^-, x_{ij}^*], j = 1, \ldots, n.$$

We are going to order x_{ij}^- and x_{ij}^*, ∀j, respectively, as:

$$m_1^- \le m_2^- \le \ldots \le m_n^- \tag{B.32}$$

in which $m_1^- = \min_j x_{ij}^-$ and $m_n^- = \max_j x_{ij}^-$, and

$$m_1^* \le m_2^* \le \ldots \le m_n^* \tag{B.33}$$

in which $m_1^* = \min_j x_{ij}^*$ and $m_n^* = \max_j x_{ij}^*$. Equations (B.32) and (B.33) facilitate the construction of Equations (B.32) and (B.33) in the later steps.

Step 4. The smallest upper and the largest lower bound of U_i, $\lfloor U_{i_{min}}, U_{i_{max}} \rfloor$, are computed as:

$$U_{i_{min}} = \left(\sum_{j=1}^{d-1} p_j^* m_j^-\right) + \left[1 - \sum_{j=1}^{d-1} p_j^* - \sum_{j=d+1}^{n} p_j^-\right] m_d^- + \sum_{j=d+1}^{n} p_j^- m_j^- \tag{B.34}$$

$$U_{i_{max}} = \left(\sum_{j=1}^{e-1} p_j^- m_j^*\right) + \left[1 - \sum_{j=1}^{e-1} p_j^- - \sum_{j=e+1}^{n} p_j^*\right] m_e^* + \sum_{j=e+1}^{n} p_j^* m_j^* \tag{B.35}$$

The only unknowns in Equations (B.34) and (B.35) are parameters d and e. The parameter d can be determined when condition,

$$1 - \sum_{j=1}^{d-1} p_j^* - \sum_{j=d+1}^{n} p_j^- = z_d \in \lfloor p_d^-, p_d^* \rfloor, \tag{B.36}$$

is satisfied. The search process is carried out in the following manner. By substituting d with 1, the value of z_1 can be obtained. We can easily determine if $z_1 \in \lfloor p_1^-, p_1^* \rfloor$ is true. If the answer is yes, we can set d=1 and compute $U_{i_{min}}$ using Equation (B.34); otherwise, we need to substitute d with 2 and compute a z_2 value. Again, we need to determine if $z_2 \in \lfloor p_1^-, p_1^* \rfloor$. The search process goes on with the value of d increasing by one each time until the condition Equation (B.36) is met. The resulting z_d will be the value assumed by weight w_d, and

$$\sum_{j=1}^{d-1} p_j^* + \sum_{j=d+1}^{n} p_j^- = z_d = 1$$

which satisfies Equation (B.31). Dubois and Prade have shown that there is only one value of d such that condition (B.36) is satisfied. Similarly, we can determine the value of e when the condition,

$$1 - \sum_{j=1}^{e-1} p_j^- - \sum_{j=e+1}^{n} p_j^* = z_e \in \lfloor p_e^-, p_e^* \rfloor, \tag{B.37}$$

is satisfied. The resulting e value is used in Equation (B.35) to compute $U_{i_{max}}$. The value assumed by w_j, $\forall j$, must satisfy Equation (B.31), i.e.,

$$\sum_{j=1}^{e-1} p_j^- + \sum_{j=e+1}^{n} p_j^* + z_e = 1$$

Step 5. At any α level, the fuzzy utility U_i can be represented by the interval $\lfloor U_{i_{min}}, U_{i_{max}} \rfloor$. The DM can set several α levels and repeat the algorithm several times to derive an approximated fuzzy utility U_i.

The total number of testing for this algorithm is 2n at most. Comparing this number with $(2)^{2n}$ (in Baas and Kwakernaak's approach) and C_n^{2n} (in Kwakernaak's approach), we conclude that this algorithm is the least time-consuming one. For example, we need to test 1024 combinations for a five attributes, five weights problem using the Baas and Kwakernaak's algorithm. It takes 252 tries using the Kwakernaak's algorithm, while only 10 tries are needed using the Dubois and Prade algorithm.

B.2.4 Cheng and McInnis's Algorithm

The following steps are taken for deriving fuzzy utilities.

Step 1. The continuous membership function is converted to a discrete one. This is done by having the DM specify the number of α levels s/he wants to use. The width of intervals is determined according to the DM's preference.

Step 2. For each α-level, we need to perform steps 3 and 4. The first α level to be considered is the largest one among all the w_j and r_{ij} graphs.

Step 3. Given α_o, we can obtain the α-level set for each r_{ij} and each w_j as:

$$r_{ij}\alpha_o = \left[x_{ij}^{-}, x_{ij}^{*}\right] \forall i, j \tag{B.38}$$

and

$$w_{j\alpha_o} = \left[y_j^{-}, y_j^{*}\right] \forall j. \tag{B.39}$$

That is, at α_o, r_{ij} can take any value in the interval $\left[x_{ij}^{-}, x_{ij}^{*}\right]$ and w_j can take any value in $\left[y_j^{-}, y_j^{*}\right]$.

Step 4. Given the upper and lower bounds of r_{ij} and w_j at the α_o level as shown in Equations (B.38) and (B.39), we can compute the upper and lower bounds of the fuzzy utility at α_o, $U_{i\alpha_o} = \left[U_{i_{min}}, U_{i_{max}}\right]$ using the following process.

Step 4.1. Compute $U_{i_{max}}$. To obtain the upper bound of U_i at the α_o level, $U_{i_{max}}$, the upper bound of r_{ij}, $\forall j$, i.e., x_{ij}^{*}, must be used. Taking the equation

$$U_i = \sum_j y_j x_{ij} / \sum_j y_j , \tag{B.40}$$

since the x_{ij} value appears only in the denominator, a higher x_{ij} value will guarantee a larger U_i value; on the other hand, since y_j, $\forall j$, appear in both numerator and denominator, increasing y_j may not give a larger U_i value. Thus to maximise u_i, we must decide whether y_j^{-} or y_j^{*} should be used.

Cheng and McInnis proposed a search process to test whether y_j^{*} or y_j^{-} should be used by comparing the maximum values of all r_{ij}. First of all, x_{ij}^{*}, $\forall j$, are rearranged as:

$$m_1^{*} \le m_2^{*} \le \ldots \le m_n^{*} \tag{B.41}$$

in which $m_1^* = \min_j x_{ij}^*$ and $m_n^* = \max_j x_{ij}^*$. Assume $m_1^* = x_{ik}^*$, the corresponding w_k should take y_k^- as its value. Assume $m_n^* = x_{il}$, the corresponding w_l should take y_l^* as its value. For some m_p^* such that $m_1^* < m_p^* < m_n^*$, if the condition

$$\frac{\sum_{j=1}^{n} w_j r_{ij}}{\sum_{j=1}^{n} w_j} = \frac{\sum_{\substack{j=1 \\ j \neq p}}^{n} w_j r_{ij} + (w_p + \lambda) r_{ip}}{\sum_{\substack{j=1 \\ j \neq p}}^{n} w_j + (w_p + \lambda)}, \tag{B.42}$$

where λ is any positive real number, holds, then the upper bound of w_p, i.e., y_p^*, should be selected. Otherwise, y_p^- is selected.

Given the right combinations of y_j^* and y_j^-, $\forall j$, we can easily compute $u_{i_{max}}$ using Equation (B.40).

Step 4.2. After finding $u_{i_{max}}, u_{i_{min}}$ can be easily identified. First of all, we will use x_{ij}^- for all r_{ij}. Secondly, for those w_j whose upper bounds were used for deriving $u_{i_{max}}$, we will use their lower bounds in computing $u_{i_{min}}$ and vice versa.

Steps 3 and 4 are used for the next largest α-level until all α levels are exhausted. The resulting fuzzy utilities are also discrete and have several "steps" in it.

B.2.5 Bonissone's Fuzzy Arithmetic Operations

Here, the formulas shall only be listed regarding the addition, subtraction, multiplication, and division operations. Let the fuzzy numbers M= (a,b,α,β), and N=(c,d,γ,δ), M>0 and N>0. Their arithmetic operations can be displayed as:

M(+)N= $(a+c,\ b+d,\ \alpha+\gamma,\ \beta+\delta)$

M(-)N= $(a-d,\ b-c,\ \alpha+\delta,\ \beta+\gamma)$

M(.)N = $(ac,\ bd,\ a\gamma+c\alpha-\alpha\gamma,\ b\delta+d\beta+\beta\delta)$

$$M(\div)= \left(\frac{a}{d}, \frac{b}{c}, \frac{a\delta + d\alpha}{d(d+\delta)}, \frac{b\gamma + c\beta}{c(c-\gamma)}\right)$$

B.3 AHP Based FMADM Methods

In the following, Lootsma's Logarithmic Least Square Method, Laarhoven and Pedrycz's Algorithm, Geometric Mean Method, and Buckley's Algorithms are given.

B.3.1 Lootsma's Logarithmic Least Square Method

This weight-assessing method was chosen because it is suitable for handling multiple decision maker's opinions and is easily extended to the fuzzy case.

Let the positive reciprocal matrix A be represented as

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ . & . & & . \\ . & . & & . \\ . & . & & . \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}$$

where a_{ij} are real numbers. The estimated vector $\underline{w} = (w_1, w_2, \dots, w_n)$ is derived by minimising

$$\sum_{i<j} (\ln a_{ij} - \ln(w_i / w_j))^2. \tag{B.43}$$

When there are multiple decision makers, the weight vector $\underline{w}$ is derived by minimising

$$\sum_{i<j} \sum_{k=1}^{p_{ij}} (\ln a_{ijk} - \ln(w_i / w_j))^2. \tag{B.44}$$

where a_{ijk}, $k = 1,2,\dots,p_{ij}$, are p_{ij} estimates for w_i/w_j. Note that p_{ij} can be 0 (if no comparison ratios are expressed), equal to one, or greater than one, in which case there are several decision makers who have expressed their comparison ratios.

If we put $y_{ijk} = \ln a_{ijk}$, $z_i = \ln w_i$, and $z_j = \ln w_j$, we can minimise

$$\sum_{i<j} \sum_{k=1}^{p_{ij}} (y_{ijk} - z_i + z_j))^2. \tag{B.45}$$

by solving the associated normal equations

$$z_i \sum_{\substack{j=1 \\ j\neq 1}}^{n} p_{ij} - \sum_{\substack{j=1 \\ j\neq 1}}^{n} p_{ij} z_j = \sum_{\substack{j=1 \\ j\neq 1}}^{n} \sum_{k=1}^{p_{ij}} y_{ijk}, \forall i, \tag{B.46}$$

for z_i. Taking the exponentials of the z_i and normalising them, we can obtain estimates for the weight vector $\underline{w} = (w_1, w_2, \dots, w_n)$.

B.3.2 Laarhoven and Pedrycz's Algorithm

The algorithm is shown in the following steps.

Step 1. Consult with the decision makers and obtain n+1 fuzzy reciprocal matrices that take the following form.

Where $a_{ijp_{ij}}$ are fuzzy ratios estimated by multiple decision makers. Note that p_{ij} may be 0 when no decision makers express their comparison ratios or greater than 1 when more than one DM expresses his/her comparison ratios.

Step 2. Let $z_i = (l_i, m_i, u_i)$. Solve the following linear equations:

$$l_i\left(\sum_{\substack{j=1\\ j\neq 1}}^{n} p_{ij}\right) - \sum_{\substack{j=1\\ j\neq 1}}^{n} p_{ij} u_j = \sum_{\substack{j=1\\ j\neq 1}}^{n} \sum_{k=1}^{p_{ij}} \left[\ln l_{ijk}\right], \forall i, \tag{B.47}$$

$$m_i\left(\sum_{\substack{j=1\\ j\neq 1}}^{n} p_{ij}\right) - \sum_{\substack{j=1\\ j\neq 1}}^{n} p_{ij} m_j = \sum_{\substack{j=1\\ j\neq 1}}^{n} \sum_{k=1}^{p_{ij}} \left[\ln m_{ijk}\right], \forall i, \tag{B.48}$$

$$u_i\left(\sum_{\substack{j=1\\ j\neq 1}}^{n} p_{ij}\right) - \sum_{\substack{j=1\\ j\neq 1}}^{n} p_{ij} l_j = \sum_{\substack{j=1\\ j\neq 1}}^{n} \sum_{k=1}^{p_{ij}} \left[\ln u_{ijk}\right], \forall i, \tag{B.49}$$

As $\ln(l_{ijk})$ and $\ln(u_{ijk})$ are lower an upper values of $\ln(a_{ijk}) = -\ln(a_{jik})$, the following must hold true.

$\ln(l_{ijk}) + \ln(l_{jik}) = \ln(u_{ijk}) + \ln(u_{jik}) = 0, \forall i, j, k.$

Thus Equations (B.47) and (B.49) are linear dependent. The same holds for Equation (B.48). Generally, a solution for Equations (B.47), (B.48), and (B.49) is given as

$$z_i = (l_i + t_1, m_i + t_2, u_i + t_1), \forall i \tag{B.50}$$

where t_1 and t_2 can be chosen arbitrarily.

Step 3. Recall that in the above linear system, all right hand sides have taken logarithmic operations. We have to take exponentials on l_i, m_i, and u_i and compute the fuzzy weight, w_i, as follows:

$$w_i = (\lambda_1 \exp(l_i), \lambda_2 \exp(m_i), \lambda_3 \exp(u_i)), \tag{B.51}$$

where

$$\lambda_1 = \left(\sum_{i=1}^{n} \exp(u_i)\right)^{-1}, \quad \lambda_2 = \left(\sum_{i=1}^{n} \exp(m_i)\right)^{-1},$$

$$\lambda_3 = \left(\sum_{i=1}^{n} \exp(l_i)\right)^{-1}.$$

Equation (B.51) can also be used to determine the performance score r_{ij}.

Step 4. Steps 1 through 3 are repeated several time until all reciprocal matrices have been solved. With the fuzzy weights and performance scores, we can compute the fuzzy utility for alternative A_i as

$$U_i = \sum_{j=1}^{n} w_j r_{ij} \qquad \text{(B.52)}$$

Note that the multiplication and summation of two triangular fuzzy numbers are based on Equations (B.53) and (B.54), respectively.

B.3.2.1 Arithmetic Operations for Triangular Fuzzy Numbers

Triangular fuzzy numbers are assumed throughout Laarhoven and Pedrycz's approach. Some arithmetic operations performed on triangular fuzzy numbers, such as addition and multiplication are presented here to facilitate the computation of fuzzy utilities.

Assume that we have two triangular fuzzy numbers $M_1= (l_1, m_1, u_1)$ and $M_2= (l_2, m_2, u_2)$. Their multiplication and addition are defined as:

$$(l_1, m_1, u_1)\ (.)\ (l_2, m_2, u_2)= (l_1 l_2, m_1 m_2, u_1 u_2) \qquad \text{(B.53)}$$

$$(l_1, m_1, u_1)\ (+)\ (l_2, m_2, u_2)= (l_1+l_2, m_1+m_2, u_1+u_2) \qquad \text{(B.54)}$$

B.3.3 Geometric Mean Method

This weight assessing method is chosen for its simplicity and ease in its application to the fuzzy case.

Given the positive comparison matrix as:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdot & & & \\ \cdot & & & \\ \cdot & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

The geometric mean of each row is calculated as:

$$z_i = \left[\prod_{j=1}^{n} a_{ij} \right]^{1/n}$$

The weight w_i is calculated as:

$w_i = z_i/(z_i + \ldots + z_n)$, $\forall i$.

To facilitate the calculation of fuzzy weights, fuzzy performance scores, and fuzzy utilities, the following arithmetic operations are presented.

B.3.4 Buckley's Algorithm

The algorithm may be applied to single or multiple decision makers. A Single DM is assumed for the following steps. The case of multiple decision makers shall be explained in the Note section.

Step 1. Consult the DM and obtain the comparison matrix A whose elements $\bar{a}_{ij} = (a_{ij}, b_{ij}, c_{ij}, d_{ij}), \forall i, j$, are trapezoidal fuzzy numbers.

Step 2. The fuzzy weights w_i can be calculated as follows. The geometric mean for each row is determined as:

$$z_i = \left(\bar{a}_{i1} \Theta \ldots \Theta \bar{a}_{in}\right)^{1/n}, \forall i, \tag{B.55}$$

where the sign Θ represent fuzzy multiplication. The fuzzy weight w_i is given as:

$$w_i = z_i \,\Theta (z_I \oplus \ldots \oplus z_n)^{-1} \tag{B.56}$$

where the sign $\oplus$ is for fuzzy addition.

The following will detail the derivation of fuzzy weight w_i. Let the left leg and right leg of $\bar{a}_{ij}$ be defined as:

$$f_i(\alpha) = \left[\mathop{\pi}_{j=1}^{n}((b_{ij} - a_{ij})\alpha + a_{ij})\right]^{1/n}, \alpha \in [0,1], \tag{B.57}$$

$$g_i(\alpha) = \left[\mathop{\pi}_{j=1}^{n}((c_{ij} - d_{ij})\alpha + b_{ij})\right]^{1/n}, \alpha \in [0,1], \tag{B.58}$$

respectively, Furthermore, let

$$a_i = \left[\mathop{\pi}_{j=1}^{n} a_{ij}\right]^{1/n}, \tag{B.59}$$

and

$$a = \sum_{i=1}^{m} a_i \,. \tag{B.60}$$

Similarly, we can define b_i and b, c_i and c, and d_i and d. The fuzzy weight w_i is determined as:

$$w_i = \left(\frac{a_i}{d}, \frac{b_i}{c}, \frac{c_i}{b}, \frac{d_i}{a}\right), \forall i, \qquad \text{(B.61)}$$

where membership function $\mu_{w_i}(x)$ is defined as follows. Let x be a real number on the horizontal axis. The $\mu_{w_i}(x)$ can be summarised as:

x	$\mu_{w_i}(x)$
$\leq (a_i/d_2)$	0
$\geq (d_i/a)$	0
$[b_i/c, c_i/b]$	1
$[a_i/d, b_i/c]$	$\alpha \in [0,1]$
$[c_i/b, d_i/a]$	$\alpha \in [0,1]$

When $x \in [a_i/d, b_i/c]$, the x is calculated as:

$$x = f_i(\alpha)/g(\alpha); \qquad \text{(B.62)}$$

and when $\in [c_i/b, d_i/a]$, the x is determined as:

$$x = g_i(\alpha)/f(\alpha); \qquad \text{(B.63)}$$

where

$$f(\alpha) = \sum_{i=1}^{m} f_i(\alpha) \qquad \text{(B.64)}$$

$$g(\alpha) = \sum_{i=1}^{m} g_i(\alpha) \qquad \text{(B.65)}$$

Step 2 is repeated until the fuzzy performance scores r_{ij}, $\forall i, j$, are obtained in a similar manner.

Step 3. The fuzzy weights and fuzzy performance scores are aggregated as a fuzzy MADM problem. The fuzzy utilities U_i, $\forall i$, are obtained based on

$$U_i = \sum_{j=1}^{n} w_j r_{ij}, \forall i. \qquad \text{(B.66)}$$

The multiplication and addition of fuzzy numbers is done according to the equations presented earlier.

B.3.4.1 Fuzzy Arithmetic on Trapezoidal Fuzzy Numbers

Assume that the comparison ratios $\bar{a}_{ij}, \forall i, j$ take trapezoidal fuzzy numbers (a_{ij}, b_{ij}, c_{ij}, d_{ij}). Their addition and multiplication are defined in this section.

Let $M_1 = (a_1, b_1, c_1, d_1)$ and $M_2 = (a_2, b_2, c_2, d_2)$ be two trapezoidal fuzzy numbers.

1. Addition

$Q = M_1 + M_2 = (a_1+a_2, b_1+b_2, c_1+c_2, d_1+d_2)$. (B.67)

$\mu_Q(x)$ is still a trapezoidal fuzzy number.

2. Multiplication

$Q = M_1 \times M_2 = [a(L_1,L_2), b, c, d(R_1,R_2)$ (B.68)

Where $a = a_1a_2$, $b = b_1b_2$, $c = c_1c_2$, $d = d_1d_2$,

$L_1 = (b_1-a_1)(b_2-a_2)$, $L_2 = a_2(b_1-a_1) + a_1(b_2-a_2)$,

$R_1 = (d_1-c_1)(d_2-c_2)$, $R_2 = -[d_2(d_1-c_1) + d_1(d_2-c_2)]$.

$\mu_Q(x)$ is no longer a trapezoidal fuzzy number and is defined as follows. For any unique x on the horizontal axis, $\mu_Q(x)$ can be:

x	$\mu_Q(x)$
$\leq a$	0
$\geq d$	0
$b \leq x \leq c$	1
$a \leq x \leq b$	$\alpha \in [0,1]$
$c \leq x \leq d$	$\alpha \in [0,1]$

when $a \leq x \leq b$, x defined as follows, Given $x_1 = [a_1,b_1]$ and $x_2 = [a_2,b_2]$ where

$x_i = (b_i - a_i)\alpha + a_i$, $i = 1,2$, (B.69)

the product $x = x_1x_2$ will take the form of

$x = L_1\alpha^2 + L_2\alpha + a$, $\alpha \in [0,1]$ (B.70)

Similarly, when $c \leq x \leq d$, we can define

$x = R_1\alpha^2 + R_2\alpha + d$, $\alpha \in [0,1]$ (B.71)

Fuzzy Addition of $(a_i[L_{i1}, L_{i2}], b_i, c_i, d_i[R_{i1}, R_{i2}])$ Fuzzy Numbers

Fuzzy addition involving fuzzy numbers taking the form of

$(a_i[L_{i1}, L_{i2}], b_i, c_i, d_i\ [R_{i1}, R_{i2}])$, can be determined as follows.

Let two fuzzy numbers Q_1 and Q_2 be:

$Q_1 = (a_1[L_{11}, L_{12}], b_1, c_1, d_1[R_{11}, R_{12}])$

$Q_2 = (a_2[L_{21}, L_{22}], b_2, c_2, d_2[R_{21}, R_{22}])$.

The addition of Q_1 and Q_2 is defined as:

$$Q' = \{(a_1 + a_2)\ [L_{11}+L_{21}, L_{12}+L_{22}], (b_1+b_2), (c_1+c_2), (d_1+d_2)[R_{11}+R_{21}, R_{12}+R_{22}])\} \quad (B.72)$$

The membership function $\mu_{Q'}(x)$ is defined as follows. Let x be some real numbers on the horizontal axis. $\mu_{Q'}(x)$ is defined as:

x	$\mu_Q(x)$
$\leq (a_1+a_2)$	0
$\geq (d_1+d_2)$	0
$(b_1+b_2) \leq x \leq (c_1+c_2)$	1
$(a_1+a_2) \leq x \leq (b_1+b_2)$	$\alpha\in[0,1]$
$(c_1+c_2) \leq x \leq (c_1+c_2)$	$\alpha\in[0,1]$

When $(a_1+a_2) \leq x \leq (b_1+b_2)$, x is determined as follows, Since

$x_i = L_{i1}\alpha^2 + L_{i2}\alpha + a_i$, i= 1,2,

the addition $x = x_1 + x_2$ will take the form

$$x = (L_{11} + L_{21})\ \alpha^2 + (L_{12} + L_{22})\ \alpha + (a_1 + a_2). \quad (B.73)$$

Similarly, when $(c_1+c_2) \leq x \leq (d_1+d_2)$, x is calculated as:

$$x = (R_{11} + R_{21})\ \alpha^2 + (R_{12} + R_{22})\ \alpha + (d_1 + d_2). \quad (B.74)$$

B.4 Dubois, Prade, and Testemale's Approach

The degree of matching is measured by the following membership function:

$$\mu_{P|Q}(\alpha) = \sup\ \{\pi_Q(x) \mid \mu_P(x) = \alpha\ \}, \forall\alpha. \quad (B.75)$$

where $\pi_Q(x)$ represents the degree of possibility that x is the (unique) value which describes an object modelled by Q: $\mu_p(x)$ is the degree of compatibility between the value x and the meaning of P. Thus, $\mu_{p|Q}(\alpha)$ denotes the degree of compatibility of Q with respect to P. Equation (B.75) was first introduced by Zadeh who interpreted $\mu_{p|Q}(\alpha)$ as the fuzzy truth value of predicate P, given a referential predicate Q describing a true state of facts.

Although the interpretation is clear from a theoretical point of view, Dubois et al. Believed that Equation (B.75) is not easily understood by users, and difficult to manipulate at an operational level. As a consequence, two scalar indices are used to approximate the $\mu_{p|Q}(\alpha)$

Measure so that compatibility between fuzzy sets can be estimated. The two indices are (1) the possibility of matching $\pi(P;Q)$, and (2) the necessity of matching $N(P;Q)$.

<u>Possibility and Necessity of Matching</u>

The possibility of matching is defined as:

$$\pi(P;Q) = \sup_{x \in U} \min\,(\mu_p(x), \mu_Q(x)) \tag{B.76}$$

which estimates to what extent it is possible that P and Q refer to the same x value. In other words, the possibility of matching is the degree of overlapping of he fuzzy set of values compatible with P, with the fuzzy set of possible values of Q.

The necessity of matching is defined as:

$$N(P;Q) = \inf_{x \in U} \max(\mu_p(x), 1 - \mu_Q(x)) \tag{B.77}$$

which estimates to what extent it is certain that the value to which Q refers is among the ones compatible with P. In other words, the necessity of matching is the degree of inclusion of the set of possible values of Q into the set of values compatible with P.

The necessity of an event corresponds to the impossibility of the opposite event, i.e.,

$$N(P;Q) = 1 - \pi(\bar{P};Q) \tag{B.78}$$

where $\mu_{\bar{p}}(x) = 1 - \mu_p(x)$ is the membership function of the complement of the fuzzy set of values compatible with P. Clearly, we always have

$$\pi(P;Q) \geq N(P;Q)\,. \tag{B.79}$$

Generally, if Q is a crisp umber, then $\pi(P;Q)=N(P;Q)= \mu_p(Q)$ which is also a crisp number in [0,1]. When both P and Q are fuzzy, then the following relation holds (given τ is a modal where $\mu_{P/Q}(\tau)=1$):

$$N(P;Q) \leq \tau \leq \pi(P;Q) \tag{B.80}$$

This relation is constructed based on the following equations:

$$\pi(P;Q) = \sup \min\,(x, \mu_{p/Q}(x)) \geq \min(\tau, \mu_{p/Q}(\tau)) = \tau\,, \tag{B.81}$$

and

$$N(P;Q) = \inf_{x} \max(x, \mu_{p/Q}(x)) \le \max(\tau, 1 - \mu_{p/Q}(\tau)) = \tau, \tag{B.82}$$

Hence [N(P;Q), π(P;Q)] is bracketing of τ, which provides information about the imprecision of $\mu_p(Q)$. We may conclude this section by stating that π(P;Q) and N(P;Q) together are reasonable approximations of $\mu_p(Q)$.

Equations (B.75) and (B.76) can only be used in single-attribute conditions. When there are multiple attributes involved, as in most real world problems, Equations (B.75) and (B.76) can be modified using the min operator:

$$\pi(A^0; A_i) = \min_{j=1,\dots,n} \quad \pi(x_j^0, x_{ij}) \tag{B.83}$$

$$N(A^0; A_i) = \min_{j=1,\dots,n} \quad N(x_j^0, x_{ij}) \tag{B.84}$$

where $A^0 = (x_1^0, \dots, x_n^0)$, $A_i = (x_{i1}, \dots, x_{in})$, and x_j^0 and x_{ij}^0 are defined on the same domain U. The vector A^0 is the cut-off vector specified by the DM, while A_i, i= 1,…,m, is the vector that contains the performance scores of the ith alternative under all attributes.

Equations (B.83) and (B.84) suggest that the matching is done attribute by attribute. These matching results are to be aggregated using the min operator to preserve the respective semantics of possibility and necessity of the indices. Equations (B.83) and (B.84) implicitly suggest that all attributes are of equal importance. If unequal weights are used, one of the following formulas can be used.

$$S = \min_{j} \max(1 - w_j, S_j), \text{ (for conjunctive case)} \tag{B.85}$$

$$S = \max_{j} \min(w_i, S_j), \text{ (for disjunctive case)} \tag{B.86}$$

where w_j denotes the relative importance of the attributes and

$$\overset{\max}{j = 1, \dots, n} \quad w_j = 1, \tag{B.87}$$

i.e., the most important attributes are rated 1; and S expresses to what extent we are certain that the fuzzy set of importance is included in the fuzzy set of requirements x_j^0 possibly (or necessarily) satisfied by the performance score x_{ij} defined by the equation

$$S_j = \pi(x_j^0, x_{ij}), \quad j = 1, \dots, n, \tag{B.88}$$

or

$$S_j = N(x_j^0, x_{ij}), \quad j = 1, \dots, n, \tag{B.89}$$

Thus, for the conjunctive case, Equation (B.85) may be rewritten (given different S_j's) as the aggregated π and N indices:

$$\min_j \max(1 - w_j, \pi(x_j^0; x_{ij})) = \pi(A^0; A_i), \tag{B.90}$$

$$\min_j \max(1 - w_j, N(x_j^0; x_{ij})) = N(A^0; A_i), \tag{B.91}$$

For the disjunctive case, Equation (B.86) may be rewritten as the aggregated π and N indices:

$$\max_j \min(w_j, \pi(x_j^0; x_{ij})) = \pi(A^0; A_i), \tag{B.92}$$

$$\max_j \min(w_j, N(w_j^0; x_{ij})) = \pi(A^0; A_i), \tag{B.93}$$

The reason for constructing Equations (B.85) and (B.86) is as follows. In the case of aggregation via the arithmetic mean, importance can easily be accommodated in the aggregation (given $P_1, \ldots, P_n$ fuzzy sets) through

$$\mu_p(x) = \sum_{j=1}^{n} w_j \mu_{p_j}(x), \forall\ x \in U \tag{B.94}$$

Note that Equation (B.94) does not have the intersection or union operations. In order to get a weighted counterpart of the minimum and maximum operation, we view Equation (B.94) as the probability of a fuzzy event Φ_x defined on the crisp set $\{1,\ldots,n\}$. Φ_x is the fuzzy set of P_j's containing x, and the w_j's define a probability allocation on the crisp set $\{1,\ldots,n\}$. Hence

$$\mu_p(x) = \text{Prob}(\Phi_x). \tag{B.95}$$

Changing probability into a possibility or a necessity measure leads us to consider the following analogs of Equation (B.94), namely,

$$\text{Poss}(\Phi_x) = \max_j \min(\mu_{p_j}(x), w_j), \tag{B.96}$$

$$\text{Ness}(\Phi_x) = \min_j \max(\mu_{p_j}(x), 1 - w_j), \tag{B.97}$$

It is clear that Equation (B.96) yields a weighted disjunction of the P_i's while Equation (B.97) yields a weighted conjunction. Thus Equations (B.95) and (B.96) are the direct applications of Equations (B.90), (B.91), (B.92), and (B.93) respectively.

Generally, the ideal ordering is used to rank the alternatives according to the following:

If	Then	
$[\pi(A^0;A_i) > \pi(A^0;A_j)$ and $N(A^0;A_i) \geq N(A^0;A_j)]$	$A_i > A_j$	(B.98)
$[\pi(A^0;A_i) > \pi(A^0;A_j)$ and $N(A^0;A_i) \geq N(A^0;A_j)]$	$A_i > A_j$	(B.99)
$[\pi(A^0;A_i) \geq \pi(A^0;A_j)$ and $N(A^0;A_i) > N(A^0;A_j)]$	$A_i > A_j$	(B.100)
$[\pi(A^0;A_i) - N(A^0;A_j) < \pi(A^0;A_i) - N(A^0;A_j)]$	$A_i > A_j$	

However, when the stated rule is not followed, we can't say that $A_i > A_j$ or $A_j > A_i$. Also note that the necessity index, N, is more important than the possibility index, π, because when the N index is positive we can be certain that he alternative (more or less) matches the requirements set by the DM.

APPENDIX C. AGGREGATION STATE CALCULATIONS OF CASE – 1

In this Appendix, detailed aggregation state calculations for the first case study discussed in the sixth Chapter are given in Tables and Figures as follows.

Aggregation state calculations of case – 1 under each subjective attribute are shown in Tables C.1, C.2, C.3, C.4, C.5, C.6 and C.7 respectively. In each Table, trapezoidal fuzzy number type of experts' opinions for each alternative and their related aggregation calculations for homo/heterogeneous group of experts are also given. Last two rows of each Table are aggregated trapezoidal fuzzy numbers for homo/heterogeneous group of experts respectively. For each attribute, they are shown in Figures C.1, C.2, and C.3 (for A_1), in Figures C.4, C.5, and C.6 (for A_2), in Figures C.7, C.8, and C.9 (for A_3), in Figures C.10, C.11, and C.12 (for A_4), in Figures C.13, C.14, and C.15 (for A_5), in Figures C.16, C.17, and C.18 (for A_6), and in Figures C.19, C.20, and C.21 (for A_7). In each Figure, trapezoidal fuzzy number type of experts' opinions (upper drawing of that figure) and their aggregated fuzzy numbers for homo/heterogeneous group of experts (lower drawing of that figure) are shown in different colours.

Table C.1 Aggregation under the first attribute (A_1)

	X_1	X_2	X_3
E_1	(0.54, 0.55, 0.56, 0.57)	(0.58, 0.59, 0.62, 0.63)	(0.93, 0.94, 0.98, 0.98)
E_2	(0.55, 0.56, 0.58, 0.59)	(0.60, 0.61, 0.63, 0.63)	(0.97, 0.98, 0.99, 1.00)
E_3	(0.54, 0.55, 0.59, 0.59)	(0.58, 0.59, 0.63, 0.63)	(0.91, 0.92, 0.97, 0.98)
Degree of Agreement (S)			
S_{12}	0.984	0.984	0.973
S_{13}	0.988	0.996	0.988
S_{23}	0.988	0.988	0.961
Average Degree of Agreement (AA)			
$AA(E_1)$	0.986	0.990	0.980
$AA(E_2)$	0.986	0.986	0.967
$AA(E_3)$	0.988	0.992	0.975
Relative Degree of Agreement (RA)			
$RA(E_1)$	0.333	0.334	0.336
$RA(E_2)$	0.333	0.332	0.331
$RA(E_3)$	0.334	0.334	0.334
Consensus Degree Coefficient (CC)			
$CC(E_1)$	0.400	0.400	0.201
$CC(E_2)$	0.340	0.339	0.199
$CC(E_3)$	0.260	0.261	0.200
R_{AG}^{HM}	(0.54, 0.55, 0.58, 0.58)	(0.59, 0.59, 0.62, 0.63)	(0.94, 0.95, 0.98, 0.99)
R_{AG}^{HT}	(0.54, 0.55, 0.57, 0.58)	(0.59, 0.59, 0.62 0.63)	(0.94, 0.95, 0.98, 0.99)

Table C.2 Aggregation under the second attribute (A_2)

	X_1	X_2	X_3
E_1	(0.53, 0.55, 0.55, 0.57)	(0.58, 0.60, 0.60, 0.62)	(0.93, 0.95, 0.95, 0.97)
E_2	(0.55, 0.57, 0.57, 0.58)	(0.60, 0.62, 0.62, 0.63)	(0.97, 0.98, 0.98, 1.00)
E_3	(0.55, 0.57, 0.57, 0.58)	(0.58, 0.60, 0.60, 0.62)	(0.93, 0.94, 0.94, 0.96)
Degree of Agreement (S)			
S_{12}	0.983	0.983	0.967
S_{13}	0.983	1.000	0.992
S_{23}	1.000	0.983	0.958
Average Degree of Agreement (AA)			
$AA(E_1)$	0.983	0.992	0.979
$AA(E_2)$	0.992	0.983	0.963
$AA(E_3)$	0.992	0.992	0.975
Relative Degree of Agreement (RA)			
$RA(E_1)$	0.331	0.334	0.336
$RA(E_2)$	0.334	0.331	0.330
$RA(E_3)$	0.334	0.334	0.334
Consensus Degree Coefficient (CC)			
$CC(E_1)$	0.399	0.401	0.201
$CC(E_2)$	0.341	0.339	0.198
$CC(E_3)$	0.261	0.261	0.201
R_{AG}^{HM}	(0.54, 0.56, 0.56, 0.58)	(0.59, 0.61, 0.61, 0.62)	(0.94, 0.96, 0.96, 0.97)
R_{AG}^{HT}	(0.54, 0.56, 0.56, 0.58)	(0.59, 0.61, 0.61, 0.62)	(0.94, 0.96, 0.96, 0.98)

Table C.3 Aggregation under the third attribute (A_3)

	X_1	X_2	X_3
E_1	(0.50, 0.70, 0.70, 0.90)	(0.80, 0.90, 0.90, 1.00)	(0.90, 1.00, 1.00, 1.00)
E_2	(0.50, 0.70, 0.70, 0.90)	(0.50, 0.55, 0.55, 0.60)	(0.50, 0.70, 0.70, 0.90)
E_3	(0.30, 0.50, 0.50, 0.70)	(0.50, 0.55, 0.55, 0.60)	(0.50, 0.70, 0.70, 0.90)
Degree of Agreement (S)			
S_{12}	1.000	0.650	0.725
S_{13}	0.800	0.650	0.725
S_{23}	0.800	1.000	1.000
Average Degree of Agreement (AA)			
$AA(E_1)$	0.900	0.650	0.725
$AA(E_2)$	0.900	0.825	0.863
$AA(E_3)$	0.800	0.825	0.863
Relative Degree of Agreement (RA)			
$RA(E_1)$	0.346	0.283	0.296
$RA(E_2)$	0.346	0.359	0.352
$RA(E_3)$	0.308	0.359	0.352
Consensus Degree Coefficient (CC)			
$CC(E_1)$	0.274	0.236	0.178
$CC(E_2)$	0.319	0.326	0.211
$CC(E_3)$	0.407	0.437	0.211
R_{AG}^{HM}	(0.44, 0.64, 0.64, 0.84)	(0.58, 0.65, 0.65, 0.71)	(0.61, 0.78, 0.78, 0.93)
R_{AG}^{HT}	(0.42, 0.62, 0.62, 0.82)	(0.57, 0.63, 0.63, 0.69)	(0.59, 0.77, 0.77, 0.92)

Table C.4 Aggregation under the fourth attribute (A_4)

	X_1	X_2	X_3
E_1	(0.10, 0.20, 0.20, 0.30)	(0.40, 0.50, 0.50, 0.60)	(0.00, 0.00, 0.10, 0.20)
E_2	(0.10, 0.20, 0.20, 0.30)	(0.20, 0.30, 0.40, 0.50)	(0.10, 0.20, 0.20, 0.30)
E_3	(0.40, 0.50, 0.50, 0.60)	(0.10, 0.20, 0.20, 0.30)	(0.00, 0.00, 0.10, 0.20)
Degree of Agreement (S)			
S_{12}	1.000	0.850	0.875
S_{13}	0.700	0.700	1.000
S_{23}	0.700	0.850	0.875
Average Degree of Agreement (AA)			
$AA(E_1)$	0.850	0.775	0.938
$AA(E_2)$	0.850	0.850	0.875
$AA(E_3)$	0.700	0.775	0.938
Relative Degree of Agreement (RA)			
$RA(E_1)$	0.354	0.323	0.341
$RA(E_2)$	0.354	0.354	0.318
$RA(E_3)$	0.292	0.323	0.341
Consensus Degree Coefficient (CC)			
$CC(E_1)$	0.276	0.257	0.205
$CC(E_2)$	0.423	0.423	0.191
$CC(E_3)$	0.301	0.320	0.205
R_{AG}^{HM}	(0.19, 0.29, 0.29, 0.39)	(0.23, 0.33, 0.37, 0.47)	(0.04, 0.07, 0.14, 0.24)
R_{AG}^{HT}	(0.19, 0.29, 0.29, 0.39)	(0.22, 0.32, 0.36, 0.46)	(0.04, 0.08, 0.14, 0.24)

Table C.5 Aggregation under the fifth attribute (A_5)

	X_1	X_2	X_3
E_1	(0.60, 0.80, 0.80, 1.00)	(0.60, 0.80, 0.80, 1.00)	(0.60, 0.80, 0.80, 1.00)
E_2	(0.60, 0.80, 0.80, 1.00)	(0.70, 0.90, 1.00, 1.00)	(0.80, 1.00, 1.00, 1.00)
E_3	(0.50, 0.65, 0.65, 0.80)	(0.60, 0.80, 0.80, 1.00)	(0.50, 0.65, 0.65, 0.80)
Degree of Agreement (S)			
S_{12}	1.000	0.900	0.850
S_{13}	0.850	1.000	0.850
S_{23}	0.850	0.900	0.700
Average Degree of Agreement (AA)			
$AA(E_1)$	0.925	0.950	0.850
$AA(E_2)$	0.925	0.900	0.775
$AA(E_3)$	0.850	0.950	0.775
Relative Degree of Agreement (RA)			
$RA(E_1)$	0.343	0.339	0.354
$RA(E_2)$	0.343	0.321	0.323
$RA(E_3)$	0.315	0.339	0.323
Consensus Degree Coefficient (CC)			
$CC(E_1)$	0.296	0.294	0.213
$CC(E_2)$	0.333	0.320	0.194
$CC(E_3)$	0.371	0.385	0.194
R_{AG}^{HM}	(0.57, 0.75, 0.75, 0.94)	(0.63, 0.83, 0.86, 1.00)	(0.63, 0.81, 0.81, 0.93)
R_{AG}^{HT}	(0.56, 0.74, 0.74, 0.93)	(0.63, 0.83, 0.86, 1.00)	(0.63, 0.81, 0.81, 0.92)

Table C.6 Aggregation under the sixth attribute (A_6)

	X_1	X_2	X_3
E_1	(0.00, 0.00, 0.00, 0.20)	(0.00, 0.00, 0.00, 0.20)	(0.00, 0.00, 0.00, 0.20)
E_2	(0.00, 0.20, 0.20, 0.40)	(0.00, 0.20, 0.20, 0.40)	(0.00, 0.00, 0.00, 0.20)
E_3	(0.20, 0.40, 0.40, 0.60)	(0.00, 0.20, 0.20, 0.40)	(0.00, 0.20, 0.20, 0.40)
Degree of Agreement (S)			
S_{12}	0.850	0.850	1.000
S_{13}	0.650	0.850	0.850
S_{23}	0.800	1.000	0.850
Average Degree of Agreement (AA)			
$AA(E_1)$	0.750	0.850	0.925
$AA(E_2)$	0.825	0.925	0.925
$AA(E_3)$	0.725	0.925	0.850
Relative Degree of Agreement (RA)			
$RA(E_1)$	0.326	0.315	0.343
$RA(E_2)$	0.359	0.343	0.343
$RA(E_3)$	0.315	0.343	0.315
Consensus Degree Coefficient (CC)			
$CC(E_1)$	0.246	0.239	0.206
$CC(E_2)$	0.465	0.456	0.206
$CC(E_3)$	0.289	0.306	0.189
R_{AG}^{HM}	(0.06, 0.20, 0,20, 0.40)	(0.00, 0.14, 0,14, 0.34)	(0.00, 0.07, 0,07, 0.27)
R_{AG}^{HT}	(0.06, 0.21, 0,21, 0.41)	(0.00, 0.15, 0,15, 0.35)	(0.00, 0.06, 0,06, 0.26)

Table C.7 Aggregation under the seventh attribute (A_7)

	X_1	X_2	X_3
E_1	(0.00, 0.00, 0.10, 0.20)	(0.30, 0.50, 0.50, 0.70)	(0.30, 0.50, 0.50, 0.70)
E_2	(0.10, 0.25, 0.25, 0.40)	(0.30, 0.50, 0.50, 0.70)	(0.10, 0.25, 0.25, 0.40)
E_3	(0.00, 0.00, 0.10, 0.20)	(0.10, 0.25, 0.25, 0.40)	(0.30, 0.50, 0.50, 0.70)
Degree of Agreement (S)			
S_{12}	0.825	1.000	0.750
S_{13}	1.000	0.750	1.000
S_{23}	0.825	0.750	0.750
Average Degree of Agreement (AA)			
$AA(E_1)$	0.913	0.875	0.875
$AA(E_2)$	0.825	0.875	0.750
$AA(E_3)$	0.913	0.750	0.875
Relative Degree of Agreement (RA)			
$RA(E_1)$	0.344	0.350	0.350
$RA(E_2)$	0.311	0.350	0.300
$RA(E_3)$	0.344	0.300	0.350
Consensus Degree Coefficient (CC)			
$CC(E_1)$	0.251	0.254	0.210
$CC(E_2)$	0.409	0.432	0.180
$CC(E_3)$	0.340	0.313	0.210
R_{AG}^{HM}	(0.03, 0.08, 0,15, 0.26)	(0.24, 0.43, 0,43, 0.61)	(0.23, 0.41, 0,41, 0.60)
R_{AG}^{HT}	(0.04, 0.10, 0,16, 0.28)	(0.24, 0.42, 0,42, 0.61)	(0.21, 0.39, 0,39, 0.57)

Table C.7 Aggregation under the seventh attribute (A_7)

	X_1	X_2	X_3
E_1	(0.00, 0.00, 0.10, 0.20)	(0.30, 0.50, 0.50, 0.70)	(0.30, 0.50, 0.50, 0.70)
E_2	(0.10, 0.25, 0.25, 0.40)	(0.30, 0.50, 0.50, 0.70)	(0.10, 0.25, 0.25, 0.40)
E_3	(0.00, 0.00, 0.10, 0.20)	(0.10, 0.25, 0.25, 0.40)	(0.30, 0.50, 0.50, 0.70)
Degree of Agreement (S)			
S_{12}	0.825	1.000	0.750
S_{13}	1.000	0.750	1.000
S_{23}	0.825	0.750	0.750
Average Degree of Agreement (AA)			
$AA(E_1)$	0.913	0.875	0.875
$AA(E_2)$	0.825	0.875	0.750
$AA(E_3)$	0.913	0.750	0.875
Relative Degree of Agreement (RA)			
$RA(E_1)$	0.344	0.350	0.350
$RA(E_2)$	0.311	0.350	0.300
$RA(E_3)$	0.344	0.300	0.350
Consensus Degree Coefficient (CC)			
$CC(E_1)$	0.251	0.254	0.210
$CC(E_2)$	0.409	0.432	0.180
$CC(E_3)$	0.340	0.313	0.210
R_{AG}^{HM}	(0.03, 0.08, 0,15, 0.26)	(0.24, 0.43, 0,43, 0.61)	(0.23, 0.41, 0,41, 0.60)
R_{AG}^{HT}	(0.04, 0.10, 0,16, 0.28)	(0.24, 0.42, 0,42, 0.61)	(0.21, 0.39, 0,39, 0.57)

Table C.7 Aggregation under the seventh attribute (A_7)

	X_1	X_2	X_3
E_1	(0.00, 0.00, 0.10, 0.20)	(0.30, 0.50, 0.50, 0.70)	(0.30, 0.50, 0.50, 0.70)
E_2	(0.10, 0.25, 0.25, 0.40)	(0.30, 0.50, 0.50, 0.70)	(0.10, 0.25, 0.25, 0.40)
E_3	(0.00, 0.00, 0.10, 0.20)	(0.10, 0.25, 0.25, 0.40)	(0.30, 0.50, 0.50, 0.70)
Degree of Agreement (S)			
S_{12}	0.825	1.000	0.750
S_{13}	1.000	0.750	1.000
S_{23}	0.825	0.750	0.750
Average Degree of Agreement (AA)			
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Consensus Degree Coefficient (CC)			
$CC(E_1)$	0.251	0.254	0.210
$CC(E_2)$	0.409	0.432	0.180
$CC(E_3)$	0.340	0.313	0.210
R_{AG}^{HM}	(0.03, 0.08, 0,15, 0.26)	(0.24, 0.43, 0,43, 0.61)	(0.23, 0.41, 0,41, 0.60)
R_{AG}^{HT}	(0.04, 0.10, 0,16, 0.28)	(0.24, 0.42, 0,42, 0.61)	(0.21, 0.39, 0,39, 0.57)

Table C.7 Aggregation under the seventh attribute (A_7)

	X_1	X_2	X_3
E_1	(0.00, 0.00, 0.10, 0.20)	(0.30, 0.50, 0.50, 0.70)	(0.30, 0.50, 0.50, 0.70)
E_2	(0.10, 0.25, 0.25, 0.40)	(0.30, 0.50, 0.50, 0.70)	(0.10, 0.25, 0.25, 0.40)
E_3	(0.00, 0.00, 0.10, 0.20)	(0.10, 0.25, 0.25, 0.40)	(0.30, 0.50, 0.50, 0.70)
Degree of Agreement (S)			
S_{12}	0.825	1.000	0.750
S_{13}	1.000	0.750	1.000
S_{23}	0.825	0.750	0.750
Average Degree of Agreement (AA)			
$AA(E_1)$	0.913	0.875	0.875
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$AA(E_3)$	0.913	0.750	0.875
Relative Degree of Agreement (RA)			
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$RA(E_3)$	0.344	0.300	0.350
Consensus Degree Coefficient (CC)			
$CC(E_1)$	0.251	0.254	0.210
$CC(E_2)$	0.409	0.432	0.180
$CC(E_3)$	0.340	0.313	0.210
R_{AG}^{HM}	(0.03, 0.08, 0,15, 0.26)	(0.24, 0.43, 0,43, 0.61)	(0.23, 0.41, 0,41, 0.60)
R_{AG}^{HT}	(0.04, 0.10, 0,16, 0.28)	(0.24, 0.42, 0,42, 0.61)	(0.21, 0.39, 0,39, 0.57)

Table C.7 Aggregation under the seventh attribute (A_7)

	X_1	X_2	X_3
E_1	(0.00, 0.00, 0.10, 0.20)	(0.30, 0.50, 0.50, 0.70)	(0.30, 0.50, 0.50, 0.70)
E_2	(0.10, 0.25, 0.25, 0.40)	(0.30, 0.50, 0.50, 0.70)	(0.10, 0.25, 0.25, 0.40)
E_3	(0.00, 0.00, 0.10, 0.20)	(0.10, 0.25, 0.25, 0.40)	(0.30, 0.50, 0.50, 0.70)
Degree of Agreement (S)			
S_{12}	0.825	1.000	0.750
S_{13}	1.000	0.750	1.000
S_{23}	0.825	0.750	0.750
Average Degree of Agreement (AA)			
$AA(E_1)$	0.913	0.875	0.875
$AA(E_2)$	0.825	0.875	0.750
$AA(E_3)$	0.913	0.750	0.875
Relative Degree of Agreement (RA)			
$RA(E_1)$	0.344	0.350	0.350
$RA(E_2)$	0.311	0.350	0.300
$RA(E_3)$	0.344	0.300	0.350
Consensus Degree Coefficient (CC)			
$CC(E_1)$	0.251	0.254	0.210
$CC(E_2)$	0.409	0.432	0.180
$CC(E_3)$	0.340	0.313	0.210
R_{AG}^{HM}	(0.03, 0.08, 0,15, 0.26)	(0.24, 0.43, 0,43, 0.61)	(0.23, 0.41, 0,41, 0.60)
R_{AG}^{HT}	(0.04, 0.10, 0,16, 0.28)	(0.24, 0.42, 0,42, 0.61)	(0.21, 0.39, 0,39, 0.57)

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	X_1	X_2	X_3
E_1	(0.00, 0.00, 0.10, 0.20)	(0.30, 0.50, 0.50, 0.70)	(0.30, 0.50, 0.50, 0.70)
E_2	(0.10, 0.25, 0.25, 0.40)	(0.30, 0.50, 0.50, 0.70)	(0.10, 0.25, 0.25, 0.40)
E_3	(0.00, 0.00, 0.10, 0.20)	(0.10, 0.25, 0.25, 0.40)	(0.30, 0.50, 0.50, 0.70)
Degree of Agreement (S)			
S_{12}	0.825	1.000	0.750
S_{13}	1.000	0.750	1.000
S_{23}	0.825	0.750	0.750
Average Degree of Agreement (AA)			
$AA(E_1)$	0.913	0.875	0.875
$AA(E_2)$	0.825	0.875	0.750
$AA(E_3)$	0.913	0.750	0.875
Relative Degree of Agreement (RA)			
$RA(E_1)$	0.344	0.350	0.350
$RA(E_2)$	0.311	0.350	0.300
$RA(E_3)$	0.344	0.300	0.350
Consensus Degree Coefficient (CC)			
$CC(E_1)$	0.251	0.254	0.210
$CC(E_2)$	0.409	0.432	0.180
$CC(E_3)$	0.340	0.313	0.210
R_{AG}^{HM}	(0.03, 0.08, 0,15, 0.26)	(0.24, 0.43, 0,43, 0.61)	(0.23, 0.41, 0,41, 0.60)
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E_1	(0.00, 0.00, 0.10, 0.20)	(0.30, 0.50, 0.50, 0.70)	(0.30, 0.50, 0.50, 0.70)
E_2	(0.10, 0.25, 0.25, 0.40)	(0.30, 0.50, 0.50, 0.70)	(0.10, 0.25, 0.25, 0.40)
E_3	(0.00, 0.00, 0.10, 0.20)	(0.10, 0.25, 0.25, 0.40)	(0.30, 0.50, 0.50, 0.70)
Degree of Agreement (S)			
S_{12}	0.825	1.000	0.750
S_{13}	1.000	0.750	1.000
S_{23}	0.825	0.750	0.750
Average Degree of Agreement (AA)			
$AA(E_1)$	0.913	0.875	0.875
$AA(E_2)$	0.825	0.875	0.750
$AA(E_3)$	0.913	0.750	0.875
Relative Degree of Agreement (RA)			
$RA(E_1)$	0.344	0.350	0.350
$RA(E_2)$	0.311	0.350	0.300
$RA(E_3)$	0.344	0.300	0.350
Consensus Degree Coefficient (CC)			
$CC(E_1)$	0.251	0.254	0.210
$CC(E_2)$	0.409	0.432	0.180
$CC(E_3)$	0.340	0.313	0.210
R_{AG}^{HM}	(0.03, 0.08, 0,15, 0.26)	(0.24, 0.43, 0,43, 0.61)	(0.23, 0.41, 0,41, 0.60)
R_{AG}^{HT}	(0.04, 0.10, 0,16, 0.28)	(0.24, 0.42, 0,42, 0.61)	(0.21, 0.39, 0,39, 0.57)

Table C.7 Aggregation under the seventh attribute (A_7)

	X_1	X_2	X_3
E_1	(0.00, 0.00, 0.10, 0.20)	(0.30, 0.50, 0.50, 0.70)	(0.30, 0.50, 0.50, 0.70)
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