İSTANBUL TECHNICAL UNIVERSITY ⋆ INSTITUTE OF SCIENCE AND TECHNOLOGY

DELAY-DEPENDENT GUARANTEED COST CONTROL FOR T-S FUZZY SYSTEMS

M.Sc. Thesis by

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JUNE 2007

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T-S BULANIK SİSTEMLER İÇİN GECİKMEYE BAĞIMLI GARANTİLİ MALİYET DENETİMİ

YÜKSEK LİSANS TEZİ

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CONTENTS

ABBREVIATIONS

T-S	:	Takagi-Sugeno
PDC	:	Parallel Distributed Compensation
LMI	:	Linear Matrix Inequalities
GEVP	:	Generalized Eigenvalue Minimization Problem

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T-S BULANIK SİSTEMLER İÇİN GECİKMEYE BAĞIMLI GARANTİLİ MALİYET DENETİMİ

ÖZET

Bu tezde, Takagi-Sugeno bulanık modeli ile ifade edilen zamanla değişen gecikmeli nonlineer sistemler için durum geribeslemesi ile gecikmeye bağımlı garantili maliyet denetim probleminin çözümü yapılmıştır. Çıkış sinyali ile ölçülen bir garantili maliyet fonksiyonu ele alınmış, uygun bir Lyapunov-Krasovskii fonksiyoneli tanımlanarak problemin çözümü için yeter koşullar elde edilmiş, paralel dağıtılmış dengeleyici yöntemi kullanılarak durum geribeslemeli denetim kuralı tanımlanmıştır. Zamanla değişen gecikmenin üst sınırı ve garantili maliyetin suboptimal üst sınırı, sırasıyla, genelleştirilmiş özdeğer minimizasyon problemi (GEVP) ve bir suboptimal değer bulma yöntemi yardımıyla sunulmuştur. Tüm sonuçlar gecikmenin büyüklüğüne bağlı olarak lineer matris eşitsizlikleri biçiminde verilmiştir.

İkinci bölümde, bulanık kümeler ve bulanık sistemlerle ilgili kavramlar açıklanmıştır. Bulanık 'if–then' kurallarının değerlendirme yöntemi ve bulanık akıl yürütmede kullanılan yöntemler verilmiştir. Takagi-Sugeno bulanık modelinin özellikleri ve yapısı açıklanmıştır. Bulanık sistem modellemesiyle ilgili temel bilgiler anlatılmış ve iki yöntem bulanık model oluşturmak için önerilmiştir. Ayrıca, verilen bir nonlineer sistemden bulanık model oluşturulması örneklendirilmiştir.

Üçüncü bölümde, T-S bulanık modeli için paralel dağıtılmış dengeleme (PDC) yöntemi ile denetim tasarımı ve kararlılık koşullarının lineer matris eşitsizliği (LMI) cinsinden ifade edilmesi anlatılmıştır.

Dördüncü bölümde, durum geribesleme denetleyici yapısı temel alınarak zamanla değişen gecikmeli Takagi-Sugeno bulanık modeli için gecikmeye bağımlı garantili maliyet denetim yöntemi sunulmuş, bu amaçla, kontrol çıkış sinyali ile belirlenen bir garantili maliyet fonksiyonu ele alınmıştır. Tanımlanan problemin çözümü lineer matris eşitsizliği cinsinden verilmiştir.

Son olarak, sunulan yöntemlerin bir uygulaması verilmiş ve sonuçlar literatürdeki sonuçlar ile karşılaştırılmıştır.

DELAY-DEPENDENT GUARANTEED COST CONTROL FOR T-S FUZZY SYSTEMS

SUMMARY

In this thesis work, a solution of delay-dependent guaranteed cost control problem for nonlinear systems with time-varying delay represented by the Takagi-Sugeno fuzzy model is achieved by the state feedback controller. A guaranteed cost function that measured by controlled output is considered and the sufficient conditions for the solution are obtained by defining a suitable Lyapunov-Krasovskii functional. The state feedback control law is defined via parallel distributed compensation technique. The upper bound of time-varying delay and the suboptimal upper bound of the guaranteed cost is presented by generalized eigenvalue minimization problem (GEVP) method and an suboptimal value searching method, respectively. All results are presented in terms of linear matrix inequalities dependent on the size of time delay.

In the second chapter, some concepts about fuzzy sets and fuzzy systems are explained. The evaluation procedure for fuzzy if-then rules and the methods used in fuzzy reasoning are given. The properties and the structure of Takagi-Sugeno fuzzy model is also described. A basic introduction to fuzzy modeling is given, and two approaches are suggested for the design of fuzzy models. Furthermore, an example of fuzzy modeling from a given nonlinear system is presented.

In the third chapter, the presentation of the stability conditions in terms of LMIs and the controller design by PDC for T-S fuzzy models are explained.

In fourth chapter, a delay-dependent guaranteed cost control method for Takagi-Sugeno fuzzy model with time-varying delay is presented based on the state feedback controller structure and for this purpose, a guaranteed cost function that measured by controlled output is considered. The solution of the problem is given in terms of linear matrix inequalities.

Finally, an application of the presented methods is given and the results are compared with the results in the literature.

1. INTRODUCTION

1.1. General Background

The theory of fuzzy logic stems from Zadeh's work on fuzzy sets in [8]. Since the basis for fuzzy logic is the basis for human communication, fuzzy logic enables us to describe complexity and uncertainty in a mathematical form like the way human can describe complexity and uncertainty with natural language. Thus, by using fuzzy logic, it is possible to describe a model for systems that are difficult to be represented by analytical models.

The fuzzy logic technique was first applied to control applications by Mamdani in [36]. After that some fuzzy control systems design methods have appeared in fuzzy control field. Among various kinds of fuzzy control methods, Takagi and Sugeno proposed a fuzzy model in [3] with a design and analysis method for fuzzy systems. They introduced the concept of representing nonlinear systems using fuzzy models. After that many researches have focused on this model-based approach for controlling nonlinear systems.

T-S fuzzy model is described by fuzzy IF–THEN rules which represent local linear input-output relations of a nonlinear system. In T-S fuzzy model, the local dynamics of each rule is represented by a linear system model. The overall fuzzy model of the objective system is achieved by aggregation of the linear models.

For any given control system, the most important question about its various properties is the stability. The most frequently employed method for stability analysis of control systems is the well-known Lyapunov method. The idea of the method is to discuss the stability of a solution of the given system through the time-derivatives of a Lyapunov function along the trajectories of the given system [43]. Thus, it is possible to analyze the stability of a solution of the systems without solving the associated equations.

A fuzzy control system is a system with fuzzy controller. The control design is carried out based on the fuzzy model via the so-called parallel distributed compensation method [18, 23]. The idea is that for each local linear model, a linear feedback control is designed and the resulting overall controller, which is nonlinear in general, is aggregation of each individual linear controller. The appeal of PDC controller design is that the Lyapunov function based techniques can be directly employed for the stability analysis and control synthesis of T-S fuzzy models [11, 18].

In classical T-S fuzzy models, there is no delays in the state. However, nonlinear systems with time-delay are very common in real processes such as chemical processes, biological systems, network systems and so on. Time-delays are often a source of instability and degradation in control performance in many control systems.

In recent years, many authors investigated the stability and control of nonlinear systems with time-delays by using T-S fuzzy models [1, 2, 46, 47]. The stabilization problems for time-delay systems can be classified into two types: delay-independent stabilization [48, 49] and delay-dependent stabilization [50, 51]. Delay-independence, contrary to delay-dependence do not include any information on the sizes of delays. It is possible that the controller which is obtained independent of the size of the delay, cannot stabilize a time-delay system. In this case, a controller designed with the consideration of the size of the delay may work better. The delay-independent stabilization for linear time-delay system has been extensively studied, and it is considered more conservative in general than the delay-dependent case. T-S based fuzzy control for nonlinear time-delay systems is first considered in [46, 47] which is only for delay-independent stabilization. There are seldom literatures that consider the delay-dependent stabilization for T-S fuzzy systems with time-delays because of the difficulties in controlling the nonlinear dynamics and applying PDC [1].

The delay-dependent stability for T-S fuzzy systems with delay which is a function of time is studied under some constraints. Two of the constraints are the model transformations in the system and the upperbound 1 on the derivative of the time-delay function. First constraint makes the result conservative and the second constraint do not allow the solution for fast time-varying delays. Note that, in this thesis, we don't use any model transformation and also there is no upper bound on the derivative of the delay function.

In addition, it can be required to design a control system which is not only stable but also guarantees an adequate level of performance. An approach to this problem is the so-called guaranteed cost control approach which is introduced in [52]. The guaranteed cost control approach provides an upper bound on a given performance index while stabilizes the system. Thus the system performance degradation is guaranteed to be less than this bound.

Generally, a time-delay in a system is expressed by a constant, but a time delay can be a function of time. This type of systems are called time-varying delay systems and this can be considered as an extension of the constant delay case. In a control system, there can be time delays in both the state and control of the dynamic part. But we assume that there is no delay in control part.

1.2. Problem Statement

We consider a nonlinear time-delay system represented by the T-S fuzzy system with time-varying delays

$Rule \quad i:$

$$IF \quad M_1(t)\, is\, F_{i1} \quad and \quad M_2(t)\, is\, F_{i2} \quad and \ldots and \quad M_g(t)\, is\, F_{ig}$$

$$THEN \quad \begin{cases} \dot{x}(t) &= A_i x(t) + A_{di} x(t-\sigma(t)) + B_i u(t) \\ z(t) &= C_{zi} x(t) + C_{zdi} x(t-\sigma(t)) + D_{zi} u(t) \\ x(t) &= \varphi(t), \quad -\bar{\sigma} \le t \le 0 \end{cases} \tag{1.1}$$

where $i = 1, 2, \ldots, \bar{n}$; $\bar{n}$ is the number of the IF-THEN rules, $x \in \mathbb{R}^n$ denotes the state vector; $u \in \mathbb{R}^{n_u}$ and $z \in \mathbb{R}^{n_z}$ are the control input and controlled output, respectively; $\sigma(t)$ is the time-varying delay in the state and it is assumed to be $0 < \sigma(t) \leq \bar{\sigma}$ and $\dot{\sigma}(t) \leq \beta < \infty$ for $\beta, \bar{\sigma} \in \mathbb{R}$; F_{ij} is the fuzzy set, g is the number of the fuzzy sets F_{ij} and $M_1(t), M_2(t), \ldots, M_g(t)$ are the premise variables.

We consider the following cost function

$$J = \int_0^\infty \| z(t) \|_2^2 \, dt = \int_0^\infty z^T(t) z(t) dt \tag{1.2}$$

Then the guaranteed cost control is defined as follows.

Definition 1.1. Consider the system (1.1). If there exists a fuzzy control law $u(t)$ and a scalar $\delta(\bar{\sigma})$ such that the closed-loop system is asymptotically stable and the closed-value of the cost function (1.2) satisfies $J \leq \delta(\bar{\sigma})$, then $\delta(\bar{\sigma})$ is said to be a guaranteed cost and the control law $u(t)$ is said to be a guaranteed cost control law for (1.1).

Our objective is to provide some sufficient conditions for stability of the T-S fuzzy system with time-varying delay with a guaranteed cost performance.

1.3. Outline of the Thesis

In the first chapter, a brief introduction to the context of the work is given.

In chapter two, some concepts about fuzzy sets and fuzzy systems are explained. A basic introduction to fuzzy modeling is given. The properties and the structure of Takagi-Sugeno fuzzy model is described and two approaches are suggested for the design of fuzzy models.

In chapter three, the parallel distributed compensation and linear matrix inequality concepts are explained.

In the fourth chapter, the theorems for the stated problem are proved.

In the fifth chapter, an application of the methods is given. After that, conclusion of the study is summarized.

In this thesis, if not stated, matrices are assumed to have compatible dimensions. For the matrices S and T, $S > 0$ means that S is a positive definite matrix and $S > T$ means that $S - T > 0$.

2. FUZZY SETS AND SYSTEMS

2.1. Fuzzy Set Theory

This chapter contains only the basic information for fuzzy set theory that we will need in this thesis. For more and detailed information [6 − 8] and [31] can be referred.

Here, we use the notation $\underline{A}$ for a classical set A and the notation A for a fuzzy set A, and X represents the universe of discourse, that is the universe of all available information on a given problem.

2.1.1. Fuzzy sets

In classic set theory, a set is defined by certain properties and it has unambiguous boundaries. For this reason, a classical set can only represent certainty like "positive integers less then 10", that is $\{x \in Z \quad | \quad x < 10\}$. Thus an element x in the universe X is either a member of a set $\underline{A}$ or it is not as shown in Figure 2.1. For example, the point a in Figure 2.1 is clearly a member of the set $\underline{A}$ and its membership in the set can be represented by the value 1 and the point b in Figure 2.1 is clearly not a member of $\underline{A}$ and its membership in the set can be represented by the value 0.

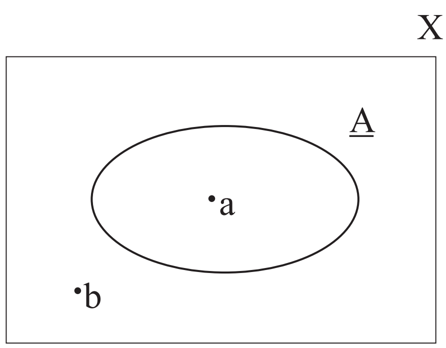

Figure 2.1: Diagram for a crisp set

This binary issue of membership can be represented mathematically with the characteristic function as follows:

$$\chi_{\underline{A}} = \begin{cases} 1, & x \in \underline{A} \\ 0, & x \notin \underline{A} \end{cases}$$

$$\chi_{\underline{A}} : X \to \{1, 0\}$$

In fuzzy set theory, a fuzzy set is defined by ambiguous properties; hence it has ambiguously specified boundaries as shown in Figure 2.2. Thus, a fuzzy set can represent uncertainties like the linguistic terms “tall”, “very soft” or “hot”. Elements of a fuzzy set have varying degrees of membership in the set. For example in Figure 2.2, the point a is clearly a full member of the fuzzy set and its membership in the set is represented by the value 1. The point b is clearly not a member of the fuzzy set and its membership in the set is represented by the value 0. However, the membership of the point c is ambiguous since it is on the boundary region and its membership is represented by an intermediate value on the interval $[0, 1]$.

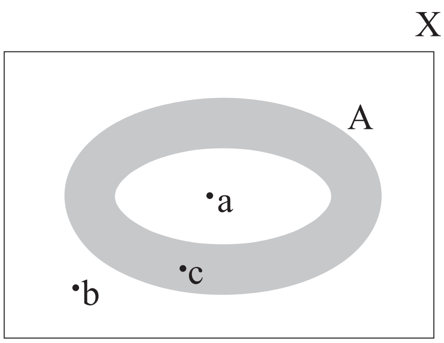

Figure 2.2: Diagram for a fuzzy set

There, elements of a fuzzy set are mapped to a universe of membership values using a function which maps elements of a fuzzy set A to a real number interval $[0, 1]$. This function is called the membership function and the membership function of a fuzzy set A is expressed by

$$\mu_A : X \to [0, 1]$$

In fuzzy set theory, standard sets are viewed as exceptional cases of fuzzy sets. The standard sets are called crisp sets and the word crisp indicates clearly defined boundaries. The characteristic function $\chi_{\underline{A}}$ of a crisp set $\underline{A}$ corresponds to the membership function of $\underline{A}$. A membership function graphic for the fuzzy set '$real\ \ numbers\ \ about\ \ \frac{a+b}{2}$' and the characteristic function graphic for the crisp set $\{x \in \mathbb{R} \mid a \leq x \leq b\}$ are illustrated in Figure 2.3.

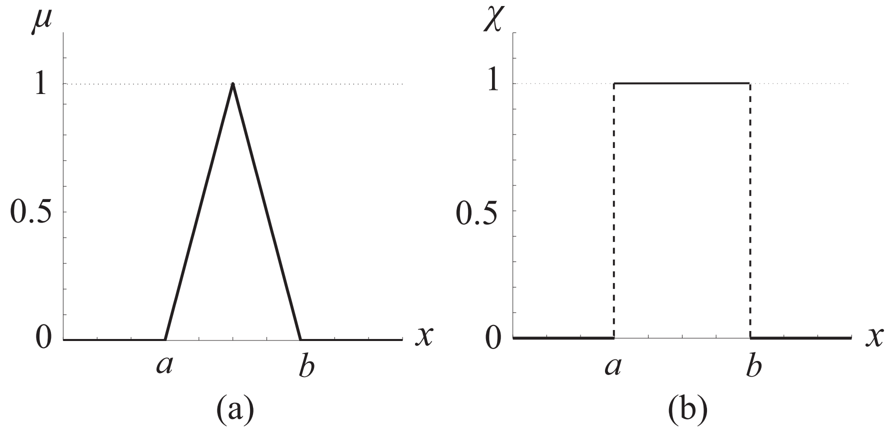

Figure 2.3: A membership function and a characteristic function

Therefore, the definition of a fuzzy set can be given as follows.

Definition 2.1. [8] A fuzzy set A in X is a set characterized by a membership function $\mu_A(x)$ which associates with each point x in X a real number in the interval $[0, 1]$, with the value of $\mu_A(x)$ at x representing the grade of membership of x in A.

2.1.2. Fuzzy set operations

Set operations similar to the crisp sets can be defined for fuzzy sets. But these operations are not uniquely defined as for crisp sets due to the fact that membership functions can have any value in the interval $[0, 1]$ for any element in a fuzzy set. Let us consider the fuzzy sets A, B and C on the universe X. For a given element x of the universe, the following operations of union, intersection and complement with most common forms are defined for A, B and C on X.

$$\begin{aligned} &Union &&: \mu_{A\cup B}(x) = \mu_A(x) \vee \mu_B(x) \\ &Intersection &&: \mu_{A\cap B}(x) = \mu_A(x) \wedge \mu_B(x) \\ &Complement &&: \mu_{\bar{A}} = 1 - \mu_A(x) \end{aligned}$$

where the symbol '$\vee$' denotes the maximum operator and the symbol '$\wedge$' denotes the minimum operator and $\bar{A}$ is the complement of a set A. Also, the intersection operation for fuzzy sets can be defined by $\mu_{A\cap B}(x) = \mu_A(x)\mu_B(x)$ which is the multiplication of the two membership grades.

The whole set X, a subset A and the null set $\emptyset$ has the following properties:

$$\begin{aligned} &Containment: \;\; A \subseteq X \Rightarrow \mu_A(x) \leq \mu_X(x) \\ &For\, all\, x \in X, \qquad \mu_\emptyset(x) = 0 \\ &For\, all\, x \in X, \qquad \mu_X(x) = 1 \end{aligned}$$

As seen above, the null set $\emptyset$ and the whole set X are crisp sets. Also, if the fuzzy sets are replaced by some crisp sets and the membership functions are replaced by characteristic functions of the above equations, the similarity between fuzzy and crisp set operations can be seen.

De Morgan's laws hold for fuzzy sets as denoted by

$$\begin{aligned} \overline{A \cap B} &= \bar{A} \cup \bar{B} \\ \overline{A \cup B} &= \bar{A} \cap \bar{B} \end{aligned}$$

All other operations on crisp sets also hold for fuzzy sets. Also fuzzy sets follow the same properties as crisp sets such as commutativity, associativity and distributivity, except for the excluded middle laws. Since fuzzy sets can overlap, a set and its complement can also overlap, thus these two laws do not hold for fuzzy sets. The excluded middle laws for fuzzy sets are expressed by

$$\begin{aligned} A \cup \bar{A} &\neq X \\ A \cap \bar{A} &\neq \emptyset \end{aligned}$$

If the collection of all fuzzy sets and fuzzy subsets on X is denoted as the fuzzy power set $P(X)$, then the cardinality of $P(X)$ is infinite, based on the fact that all fuzzy sets can overlap.

Some important definitions for fuzzy sets are given below.

Definition 2.2. The height of a fuzzy set A, $hgt(A)$, is the largest membership grade obtained by any element in A, that is

$$hgt(A) = \sup_{x \in X} \mu_A(x)$$

A fuzzy set A is called normal when $hgt(A) = 1$ and it is called subnormal when $hgt(A) < 1$.

Definition 2.3. The core of a fuzzy set A, $core(A)$, is the crisp set that contains all the elements of the universe such that $\mu_A(x) = 1$, that is

$$core(A) = \{x \in X \quad | \quad \mu_A(x) = 1\}$$

Definition 2.4. The support of a fuzzy set A, $supp(A)$, is the crisp set that contains all the elements of the universe that have nonzero membership grades in A, that is

$$supp(A) = \{x \in X \quad | \quad \mu_A(x) > 0\}$$

If $supp(A)$ is finite, it is called compact support.

Definition 2.5. If, for any elements x_1, x_2 and x_3 in a fuzzy set A, the relation $x_1 < x_2 < x_3$ implies that

$$\mu_A(x_2) \geq \min[\mu_A(x_1), \mu_A(x_3)]$$

then A is called a convex fuzzy set.

Definition 2.6. A fuzzy set A, a subset of $\mathbb{R}$, is a fuzzy number if the fuzzy set is convex and normal, membership function is piecewise continuous and the core consists of one point only. The fuzzy set A with the same restrictions but with a core that consists of more than one point is called fuzzy interval.

An example of fuzzy number which can be called "about 1" can be given by $\mu_A(x) = e^{-\beta(x-1)^2}$.

2.1.3. Fuzzy relations

We consider only the most common case for a relation which is given for two universes. However, the idea can be extended for more universes, easily. Although there are other methods [6, 30], we will use the Cartesian product method to assign values to characterize a fuzzy relation.

The Cartesian product of two universes X and Y is determined as

$$X \times Y \quad = \quad \{(x, y) \quad | \quad x \in X,\, y \in Y\}$$

which forms an ordered pair of every $x \in X$ with every $y \in Y$.

Fuzzy relations map elements of one universe to those of another universe through the Cartesian product of the two universes. For two universes X and Y, a fuzzy relation R can be defined as a mapping from the Cartesian space $X \times Y$ to the interval $[0, 1]$ where the strength of the mapping is expressed by the membership function $\mu_R(x, y)$ of the relation for ordered pairs $(x, y) \in X \times Y$.

Let A be a fuzzy set on universe X and B be a fuzzy set on universe Y, then the Cartesian product between fuzzy sets A and B will result in fuzzy relation R which is contained within the full Cartesian product space, that is

$$A \times B \quad = \quad R \subset X \times Y$$

where the fuzzy relation R has the membership function

$$\mu_R(x, y) = \mu_{A\times B}(x, y) = \mu_A \wedge \mu_B$$

The Cartesian product defined $A \times B = R$ is not the same operation as the arithmetic product. It is implemented in the same fashion as is the cross product of two vectors. Each of the fuzzy sets could be thought of as a vector of membership values. Each value is associated with a particular element in each set. Thus, for a fuzzy set A that has n elements and for a fuzzy set B that has m elements, the resulting fuzzy relation R can be represented by a matrix of size $n \times m$.

Let R and S be fuzzy relations on $X \times Y$. Since a fuzzy relation is also a fuzzy set, then some set operations for the fuzzy relations can be defined as follows:

$$\begin{aligned}
&Union &: \quad &\mu_{R\cup S}(x,y) &&= \mu_R(x,y) \vee \mu_S(x,y)\\
&Intersection &: \quad &\mu_{R\cap S}(x,y) &&= \mu_R(x,y) \wedge \mu_S(x,y)\\
&Complement &: \quad &\mu_{\bar{R}}(x,y) &&= 1 - \mu_R(x,y)
\end{aligned}$$

Similar to the fuzzy sets properties, all the properties of commutativity, associativity, distributivity, involution and idempotency, except the excluded middle laws, all hold for fuzzy relations. Since a fuzzy relation R is also a fuzzy set there is overlap between a relation and its complement, hence,

$$\begin{aligned}
R \cup \bar{R} &\neq E\\
R \cap \bar{R} &\neq 0
\end{aligned}$$

where 0 denotes the null relation and E denotes the complete relation. Also fuzzy relations has the property of containment as follows:

$$R \subset S \Rightarrow \mu_R(x,y) \leq \mu_S(x,y)$$

2.1.4. Fuzzy composition

For a fuzzy relation R in $X \times Y$ and a fuzzy relation S in $Y \times Z$, a relation can be defined which relates the same elements in universe X that R contains to the same elements in universe Z that S contains by using the composition of the relations R and S. There are many forms of the composition operation. Each of them can be used for different kind of situations or problems. Three forms of composition operations, the max-min, the min-max and the max-star compositions, are given below, respectively.

$$\begin{aligned}
R \circ S &: \quad \mu_{R\circ S}(x,z) &&= \bigvee\nolimits_{y\in Y} \{\mu_R(x,y) \wedge \mu_S(y,z)\}\\
R \square S &: \quad \mu_{R\square S}(x,z) &&= \bigwedge\nolimits_{y\in Y} \{\mu_R(x,y) \vee \mu_S(y,z)\}\\
R * S &: \quad \mu_{R*S}(x,z) &&= \bigvee\nolimits_{y\in Y} \{\mu_R(x,y) * \mu_S(y,z)\}
\end{aligned}$$

where $*$ on the right side of the max-star composition is defined as a binary operation. For example, if multiplication dot, "•", is used for the star, the max-product composition which is an important composition operation, is obtained. Also it is straightforward to see that

$$\overline{R \Box S} = \bar{R} \circ \bar{S}$$

We will only use the symbol "$\circ$" for any composition operation in this thesis.

2.1.5. Membership functions

Since the membership function is the underlying power of fuzzy sets, its description is the essence of a fuzzy property or operation. There are many methods described on literature to assign membership functions to fuzzy variables for certain types of problems and for their data or knowledge bases. The assignment methods are based on intuition ability of human or based on some algorithmic or logical operations. Some of the methods are intuition, inference, rank ordering, angular fuzzy sets, neural networks, genetic algorithms, inductive reasoning, soft partitioning, meta rules and fuzzy statistics $[6, 38 - 40]$.

Intuition method is derived from the capacity of humans to develop membership functions through their own intelligence, understanding and also experience. Intuition involves semantic knowledge about an issue and linguistic truth values about this knowledge. For example, if we consider the membership functions for the fuzzy variable temperature, we can define membership functions for 'cold', 'warm' and 'hot' by using our knowledge according to the range of human comfort as in Figure 2.4.

In the inference method, we use knowledge to perform deductive reasoning. Thus, we infer a conclusion for a given knowledge or data. One example of this method can be given as follows.

Consider an elliptic shape with parameters a and b as in Figure 2.5. Mathematically, we know that a circle results when $\frac{a}{b} = 1$ thus we can infer a membership function as in Figure 2.5 by using our knowledge.

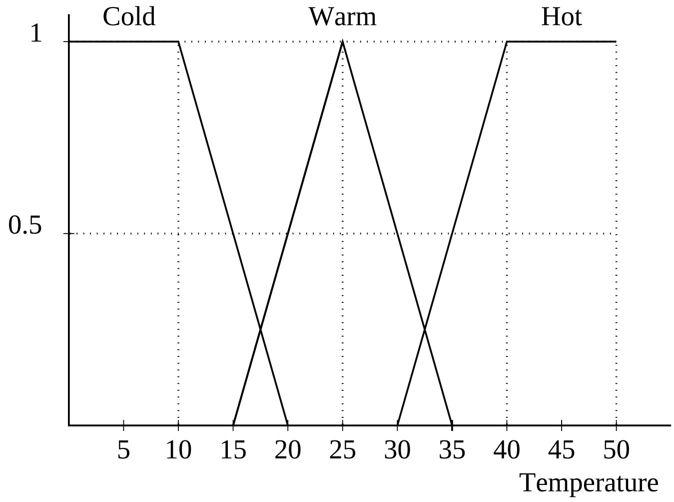

Figure 2.4: Example membership functions for temperature

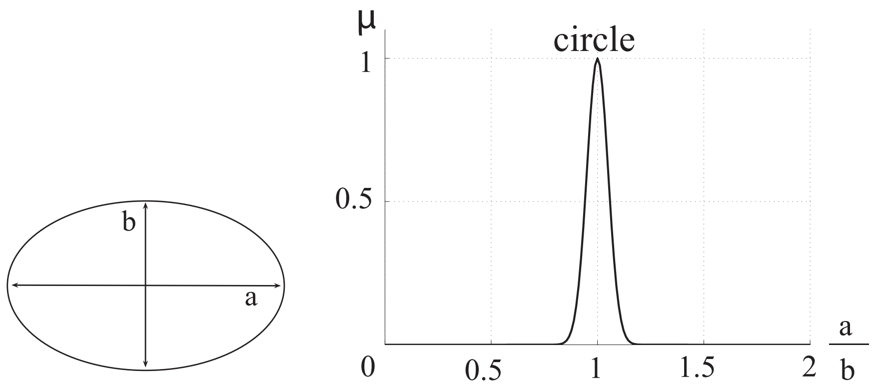

Figure 2.5: Elliptic shape and membership function for circle

For other methods and the details of these methods, [6], [31], [34] and the references cited therein can be seen. Here, four mostly used types of membership functions; singleton, trapezoidal, triangular and Gaussian, are described.

The simplest membership function type is the singleton function which is defined by

$$\mu(x;\acute{x}) = \begin{cases} 1, & if \quad x = \acute{x} \\ 0, & otherwise \end{cases}$$

Trapezoidal membership functions are used when piecewise linear membership functions are needed. Because of their simplicity and efficiency with respect to computability, they can be useful in many situations. This function defined by four parameters, a, b, c and d, can be described as follows:

$$\mu(x;a,b,c,d) = \begin{cases} 0, & x < a\,, \quad d < x \\ \frac{x-a}{b-a}, & a \leq x \leq b \\ 1, & b < x < c \\ \frac{d-x}{d-c}, & c \leq x \leq d \end{cases}$$

The trapezoidal function with parameters a, b, c and d can be illustrated as in Figure 2.6

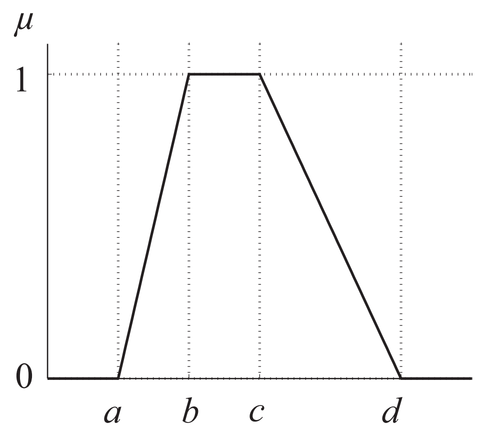

Figure 2.6: Trapezoidal membership function

For $b = c$ the trapezoidal function turns in to triangular membership function. An example of the triangular membership function is illustrated in Figure 2.3.a.

When smooth transitions are required for membership values, which the trapezoidal functions do not have, functions like Gaussian, bell and sigmoidal can be used with respect to the parameters of the application. The Gaussian membership function defined by two parameters σ and c can be characterized by

$$\mu(x;\sigma,c) = \exp(-\frac{(x-c)^2}{2\sigma^2})$$

An example for the Gaussian function for $\sigma = 0.05$ and $c = 1$ was illustrated for the fuzzy set circle in Figure 2.5.

2.2. Fuzzy Logic

A fuzzy logic proposition P is a statement involving some concept without clearly defined boundaries like linguistic statements such as "the temperature is very high" or "the weather is fine". These statements express subjective ideas but can be interpreted by anyone.

Generally, a fuzzy proposition is written as

$$x\ is\ A$$

where A is a fuzzy set and x is called the fuzzy variable. Fuzzy variables are also called linguistic variables and are expressed in terms of fuzzy sets.

In classical logic, propositions are assigned a value 1 or 0 according to the truth of the proposition. Similarly, a fuzzy proposition is assigned a truth value. But in fuzzy logic, the truth value assigned to a fuzzy logic proposition can be any value on the interval $[0, 1]$. Suppose fuzzy logic proposition P is assigned to fuzzy set A, then for a crisp point x the truth value of the proposition P, denoted by $T(P)$ is given by

$$T(P) = \mu_A(x)$$

where $\mu_A(x)$ is the grade of membership of x in A and $0 \leq \mu_A \leq 1$. This indicates that the degree of truth for the proposition "$P : x\ is\ A$" is equal to the membership grade of x in the fuzzy set A. If x is a fuzzy set with a membership function μ_x, then the truth value of P is defined as

$$T(P) = hgt(\mu_x \cap \mu_A)$$

Let P and Q be two fuzzy logic propositions on the same universe of discourse. P defined on fuzzy set A and Q defined on fuzzy set B can be combined using the following logical connectives to form logical expressions involving the two propositions:

$$\begin{aligned} Negation \quad : \quad & \bar{P} : x\ is\ NOT\ A \\ & T(\bar{P}) = 1 - T(P) \end{aligned}$$

$$\begin{aligned}
Disjunction &: P \vee Q : x \; is \; A \quad OR \quad x \; is \; B \\
& \quad T(P \vee Q) = \max(T(P), T(Q)) \\
Conjunction &: P \wedge Q : x \; is \; A \quad AND \quad x \; is \; B \\
& \quad T(P \wedge Q) = \min(T(P), T(Q)) \\
Implication &: P \rightarrow Q : x \; is \; A \quad THEN \quad x \; is \; B \\
& \quad T(P \rightarrow Q) = T(\bar{P} \vee Q) = \max(T(\bar{P}), T(Q))
\end{aligned}$$

Note that, these definitions are subject to change and can be customized according to the field that the fuzzy logic is applied. For example, in T-S fuzzy model generally product operation is used for conjunction operation. This will be shown in the next sections. The disjunction and the conjunction operations are also known as the "OR" and the "AND" operations, respectively.

The implication $P \rightarrow Q$ can be read as 'P implies Q' and can described as (either "x is not A" OR "x is B"). It can involve two different universes of discourse and can be also represented by a fuzzy relation R. The given implication operation above is the one presented by Zadeh in [32]. There are other techniques for obtaning the implication of two propositions in the literature [6].

The implication connective can be modeled in rule-based form, that is $P \rightarrow Q$ is

$$IF \quad x \; is \; A \quad THEN \quad y \; is \; B$$

and it is equivalent to the fuzzy relation R such that $R = (A \times B) \cup (\bar{A} \times Y)$ in set-theoretic form where Y is the universe that B belongs. The membership function of R can be expressed as follows:

$$\mu_R(x, y) = \max[(\mu_A(x) \wedge \mu_B(y)), (1 - \mu_A(x))]$$

When the logical implication is of the compound form

$$IF \quad x \; is \; A \quad THEN \quad y \; is \; B \quad ELSE \quad y \; is \; C$$

then the equivalent fuzzy relation, R, is expressed as $R = (A \times B) \cup (\bar{A} \times C)$ whose membership function is expressed by the following formula [6]:

$$\mu_R(x, y) = \max[(\mu_A(x) \wedge \mu_B(y)), (1 - \mu_A(x) \wedge \mu_C(y))]$$

The if-part of the rule "x is A" is called the antecedent or premise and the the then-part of the rule "y is B" is called the conclusion or consequence.

Consider the following rule-based format to represent fuzzy information:

$$Rule \quad 1: \quad IF \quad x \; is \; A \quad THEN \quad y \; is \; B$$

where A and B represent fuzzy sets in universes X and Y, respectively. Now consider a new rule with a new premise A' as below:

$$Rule \quad 2: \quad IF \quad x \; is \; A' \quad THEN \quad y \; is \; B'$$

Now we can derive the consequent in Rule 2, that is B', from the information derived from Rule 1 by using fuzzy compostion. The consequent B' can be found from

$$B' = A' \circ R$$

Note that, if we use the original premise A in the fuzzy composition, generally we don't get the original fuzzy consequent B [6].

If we use product implication and max-product composition the membership value of B' is given as follows.

$$\mu_{B'}(y) = \max_{x \in X}(\mu_{A'}(x)\mu_A(x)\mu_B(y)) \tag{2.1}$$

for an input A'.

2.3. Fuzzy If-Then Rules

Fuzzy logic is a convenient way and a powerful tool to map an input space to an output space when we need to develop a system to deal efficiently with imprecision and nonlinearity. The primary mechanism for doing this is the if-then rules which are also a list of conditional statements and fuzzy reasoning is the basic tool for a rule-based system.

Consider the following if-then rule:

$$IF \quad x \; is \; A \quad THEN \quad y \; is \; B$$

Here for a given input variable x the premise returns a single value between 0 and 1 then according to this value the consequent assigns the entire fuzzy set B to the output variable y.

The premise of an if-then rule can have multiple parts. For example, a rule for a simple air-conditioning system can be defined as

$$\begin{array}{ll} IF & temperature\ is\ high \quad AND \quad humidity\ is\ normal \\ THEN & cooling\ is\ high \end{array}$$

In this case, all parts of the premise are calculated simultaneously and resolved to a single value using the logical operators in the premise as explained in the previous sections.

Also, the consequent of a rule can have multiple parts such as

$$\begin{array}{ll} IF & temperature\ is\ high \quad AND \quad humidity\ is\ normal \\ THEN & cooling_1\ is\ high_1 \\ & cooling_2\ is\ normal_2 \end{array}$$

In this case, all consequents are affected equally by the result of the premises. The implication operation modifies the fuzzy set, that consequent assigns to the output, to the degree specified by the premise as mentioned in the previous section. Two common ways to modify the output fuzzy set are the minimum function and the product function.

Also, generally there is more than one rule in an if-then rule-based system and for this reason the output of each rule must be aggregated to obtain the total system's output. Generally, 'OR' operation is used to aggregate. In multiple rules case, note that the rules are evaluated in parallel and the order of the rules is not important. In a multiple rule system, every rule can have a weight which is a value between 0 and 1 and applied to the number given by the premise.

2.4. Fuzzification

Fuzzification is simply to map a crisp value into a fuzzy set. The fuzzy rule-based inference systems operate on fuzzy sets to produce fuzzy sets. Generally, the inputs to the fuzzy systems are crisp values. Thus, these must be converted to fuzzy sets. This operation is done by fuzzifiers. A fuzzifier maps a crisp point x into a fuzzy set A'.

There are two types of fuzzifiers. When an input variable need to be a single numerical value, the fuzzy set is given by a singleton

$$\mu_{A'}(x) = \begin{cases} 1, & if \quad x = x' \\ 0, & otherwise \end{cases}$$

where x' is the input. This is called a singleton fuzzifier.

If the input contains noise, uncertainty or inaccuracy, it can be modeled by using a fuzzy number. This type of fuzzifiers are called nonsingleton fuzzifiers. For example, a triangular fuzzification which maps a crisp value into a triangular membership function is a nonsingleton fuzzifier.

Nonsingleton fuzzification methods add computational complexity to the process. Thus, most often, singleton fuzzification is used because of simplicity. Also we will use this type of fuzzification in the later sections.

2.5. Defuzzification

In many applications, the output of a fuzzy system must be a crisp value. Since the outputs of the fuzzy if-then rules are fuzzy sets or values, these must be mapped into crisp numbers. This is done by defuzzification methods. For a defuzzification which is a mapping from the set B' in the universe Y to a point y' in Y, some of the methods can be listed as follows.

Consider the following if-then rules and assume that the singleton fuzzifier is used:

$Rule \quad i:$

$$IF \quad x_1 \, is \, A_{i1} \quad AND \quad x_2 \, is \, A_{i2} \quad \ldots AND \quad x_r \, is \, A_{ir}$$

$$THEN \quad y_i \, is \, B_i, \qquad i = 1, 2, \ldots, n$$

Center of gravity: $\quad y' = \dfrac{\int_Y \mu_{B'}(y) y \mathrm{d}y}{\int_Y \mu_{B'}(y)}$

This defuzzifier determines y' as the center of the area under the membership function $\mu_{B'}(y)$.

Center average: $\quad y' = \dfrac{\sum_{i=1}^{n} w_i y_i}{\sum_{i=1}^{n} w_i}$

where y_i is the center of ith output fuzzy set and w_i is its height.

For singleton fuzzification of the inputs x'_k and by using product inference as in (2.1), the height of the ith fuzzy set B'_i is obtained as

$$w_i = \prod_{k=1}^{r} \mu_{A_{ik}}(x'_k) \mu_{B_i}(y_i)$$

The advantage of product over minimum operator is the fact that all of the inputs will have an effect on the output in the case of multi dimensional input space. If the min operation is used, only one input has effect on the output.

These two methods can be referred by different names in literature. Center average defuzzifier can be considered as a special case of center of gravity defuzzifier in the case of symmetric output sets. There are more than two defuzzification methods in the literature and each of them has various advantages in different applications [6, 7, 35]. Note that, the given two methods do not just defuzzify the output sets, but also they aggregate the outputs of all the rules.

2.6. Fuzzy Systems

A fuzzy system can be viewed as a mapping from given inputs to outputs using fuzzy logic. It is a set of rules and involves all the tools that we explained in the previous sections. Fuzzy inference systems have been successfully applied in many fields and it is known by a number of different names such as fuzzy rule-based system, fuzzy expert system, fuzzy model, fuzzy associative memory or simply fuzzy system.

The inputs and outputs values of a fuzzy system can be fuzzy or crisp values. But we only consider the crisp values case for generality since in this case the inputs are first fuzzified and the outputs are defuzzified. Also, we explain the fuzzy inference process for one output for simplicity.

The fuzzy inference process can be described in six steps. Since, the methods are explained in the previous sections, we only give brief definitions of these steps as follows:

Step 1 Fuzzification: Since fuzzy inference system operates on fuzzy sets to produce fuzzy sets, the crisp input values are converted to fuzzy sets by using a suitable fuzzification method.

Step 2 Proposition matching: The truth values of each proposition in the premises are determined according to the inputs.

Step 3 Premise conjunction: Using the appropriate operations for the connectives in the premises, the firing strength of each rule is calculated.

Step 4 Implication: An implication operation is applied from the premise to the consequent for each rule.

Step 5 Aggregation of the consequents: The outputs of the rules are aggregated by using the OR operator. Each of the fuzzy output in the consequent of each rule is evaluated independently.

Step 6 Defuzzification: Since generally the output of a fuzzy system must be a crisp value, the fuzzy output of the system is defuzzified by using an appropriate defuzzifier, for example, by using center average defuzzifier.

Simply, a fuzzy system can be illustrated as in Figure 2.7.

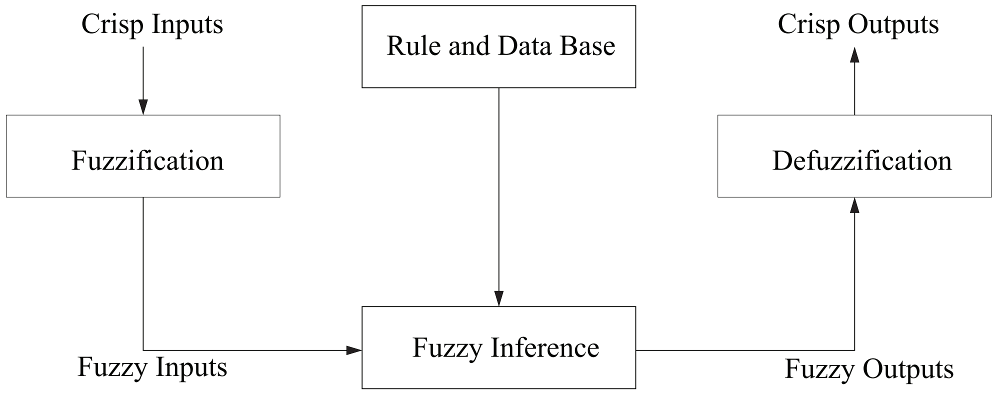

Figure 2.7: Diagram of a generic fuzzy system.

There exist two major types of fuzzy models, Mamdani fuzzy models [36] and Takagi-Sugeno fuzzy model [3], according to the different output formulations of the fuzzy rules. In Mamdani type fuzzy models, the consequence of each fuzzy rule is a fuzzy set. In T-S type fuzzy models, the consequence of each fuzzy rule is a function of the premise variables of each rule.

Mamdani type fuzzy model is also called linguistic or standard fuzzy model and it is first presented in [36]. Mamdani model is very useful for human-machine interfaces, because of its simple linguistic nature [33, 36]. The model rules have the structure of the form

$Rule \quad i:$

$$IF \quad x_1\, is\, A_{i1} \quad AND \quad x_2\, is\, A_{i2} \quad \ldots AND \quad x_r\, is\, A_{ir}$$

$$THEN \quad y_i\, is\, B_i, \qquad i = 1, 2, \ldots, n$$

where A_{ij} and B_i are fuzzy sets in the universes $X_j \subset \mathbb{R}$ and $Y \subset \mathbb{R}$, respectively. The fuzzy or linguistic variable $x = (x_1, \ldots, x_r)$ is an input to the fuzzy system and a vector of dimension r in $X_1 \times \ldots \times X_r$. y_i is the output of the ith fuzzy rule and a fuzzy variable in Y. For computational tools and examples of Mamdani type, Matlab fuzzy logic toolbox user's manual [42] can be referred.

T-S fuzzy model is first proposed in [3]. In the next section the T-S fuzzy model will be explained.

2.7. T-S Fuzzy Model Description

The ith rule for the continuous-time Takagi-Sugeno fuzzy system described by fuzzy IF-THEN rules is of the following form:

$Rule \quad i:$

$$\begin{aligned} &IF \quad z_1(t)\, is\, M_{i1} \quad and \ldots and \quad z_p(t)\, is\, M_{ip} \\ &THEN \quad \begin{cases} \dot{x}(t) &= A_i x(t) + B_i u(t) \\ y(t) &= C_i x(t), \qquad i = 1, 2, \ldots, r. \end{cases} \end{aligned} \tag{2.2}$$

Here, $x(t) \in \mathbb{R}^n$ is the state vector, $u(t) \in \mathbb{R}^m$ is the input vector, $y(t) \in \mathbb{R}^q$ is the output vector, M_{ij} is the fuzzy set and r is the number of model rules, $A_i \in \mathbb{R}^{n\times n}$,$B_i \in \mathbb{R}^{n\times m}$ and $C_i \in \mathbb{R}^{q\times n}$, $z_1(t), \ldots, z_p(t)$ are known premise variables that may be functions of the state variables, external disturbances and time. $z(t)$ is used to denote the vector containing all the individual elements $z_1(t), \ldots, z_p(t)$.

Given a pair of $(x(t), u(t))$ for the T-S fuzzy system, the final output of the system is inferred as follows:

$$\begin{aligned} \dot{x}(t) &= \frac{\sum_{i=1}^{r} w_i(z(t))\{A_i x(t) + B_i u(t)\}}{\sum_{i=1}^{r} w_i(z(t))} \\ &= \sum_{i=1}^{r} h_i(z(t))\{A_i x(t) + B_i u(t)\} \end{aligned} \tag{2.3}$$

$$\begin{aligned} y(t) &= \frac{\sum_{i=1}^{r} w_i(z(t)) C_i x(t)}{\sum_{i=1}^{r} w_i(z(t))} \\ &= \sum_{i=1}^{r} h_i(z(t)) C_i x(t) \end{aligned} \tag{2.4}$$

where

$$\begin{aligned} z(t) &= \begin{bmatrix} z_1(t) & z_2(t) & \ldots & z_p(t) \end{bmatrix} \\ w_i(z(t)) &= \prod_{j=1}^{p} M_{ij}(z_j(t)) \\ h_i(z(t)) &= \frac{w_i(z(t))}{\sum_{i=1}^{r} w_i(z(t))} \end{aligned}$$

for all t. The term $M_{ij}(z(t))$ is the grade of membership of $z_j(t)$ in the fuzzy set M_{ij}. Since

$$\begin{aligned}\sum_{i=1}^{r} w_i(z(t)) &> 0,\\ w_i(z(t)) &\geq 0, \qquad i = 1, 2, \ldots, r,\end{aligned}$$

we have

$$\begin{aligned}\sum_{i=1}^{r} h_i(z(t)) &= 1,\\ h_i(z(t)) &\geq 0, i = 1, 2, \ldots, r, \forall t.\end{aligned}$$

2.8. Design of T-S Fuzzy Systems

Two major applications of fuzzy systems are fuzzy control and fuzzy modeling. Modeling algorithms have been sufficiently developed for linear systems. But the most of the real processes are nonlinear and can be approximated by linear models only locally or, simplifying assumptions are made that all too often distort the realities of the processes [43, 45]. Also, there exist nonlinear systems with imprecise data, which cannot be adequately described mathematically or by analytical or physical models. These issues can be handled by using fuzzy models because of the nature of the fuzzy theory.

The design of a fuzzy system involves all the methods that explained in the previous sections and it can be described in the following six steps [37, 45]:

Step 1 Selection of the input and output fuzzy variables.

Step 2 Selection of the appropriate reasoning methods for the formalization of the fuzzy model.

Step 3 Determination of the universes of discourses.

Step 4 Determination of the fuzzy sets into which the fuzzy variables are partitioned.

Step 5 Formation of the if-then rules that represent the relationships between the input and output variables and determination of each rules weight.

Step 6 Evaluation of the adequacy of the system.

The first five steps can be regarded as the structure identification of the fuzzy system. Note that, fuzzy system design is completely application-dependent and an exact general design algorithm cannot be defined. If the adequacy of the system, which can be measured by a performance index such as root mean square error, is not as expected then an identification algorithm is also needed to obtain optimal parameters of membership functions, premise variables and consequent part of the system. Such an algorithm for parameter identification is given by Takagi and Sugeno in [3].

The T-S fuzzy model is described by fuzzy IF-THEN rules which represent local input-output relations of a nonlinear system [3]. The main feature of the T-S fuzzy model is to express the local dynamics of each fuzzy rule by a linear model. The overall fuzzy model of the objective system is achieved by fuzzy blending of the linear models [11].

In general, there are two main approaches for designing fuzzy models as illustrated in Figure 2.8.

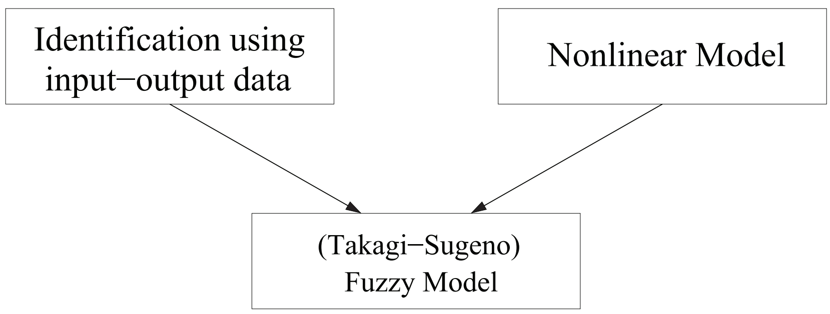

Figure 2.8: Fuzzy modeling

First approach is the identification of fuzzy models by using prior knowledge of some experts and recorded input-output data. Fuzzy identification means the acquisition or tuning of fuzzy systems by means of data. A number of fuzzy modeling techniques which identify a fuzzy model from input-output data of a nonlinear system have been proposed in [3],[19 − 22] and [29]. Introduction of T-S systems in [3] with a least-square method for the identification of parameters was a very important step in the direction of high-quality identification. This method is also related to the idea of multidimensional fuzzy reasoning [16] where a fuzzy implication is improved and reasoning is simplified. In [19] and [22], Sugeno and Kang extended the T-S procedure to consist both of the structure and parameter identification. This identification approach for fuzzy modeling is more suitable for plants that are unable or too difficult to be represented by analytical or physical models.

The second approach is the derivation of fuzzy models from given nonlinear system equations. The derivation method utilizes the idea of 'sector nonlinearity', 'local approximation' or a combination of them to construct fuzzy models. Fuzzy model construction by using sector nonlinearity method first appeared in [13]. Fuzzy modeling technique via sector nonlinearity concept [13, 14], produces a special type of fuzzy model which consists of local Takagi-Sugeno fuzzy models. Sector nonlinearity is based on the following idea [11]. Consider a simple nonlinear system $\dot{x}(t) = f(x(t))$ where $f(0) = 0$. Then find the global sector such that

$$\dot{x}(t) = f(x(t)) \in [a_1,\, a_2]x(t)$$

The sector nonlinearity approach is illustrated in Figure 2.9. This aproach guarantees an exact fuzzy model construction.

However, it is sometimes difficult to find global sectors for general nonlinear systems. In this case, local sector nonlinearity concept is used, since variables of physical systems are always bounded.

The local sector nonlinearity is illustrated in Figure 2.10 where two lines become the local sectors under $-d < x(t) < d$. Again, the fuzzy model exactly represents the nonlinear system in the local region, that is $-d < x(t) < d$.

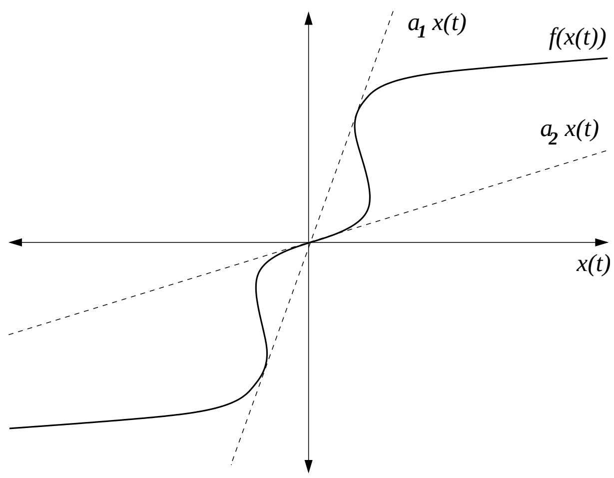

Figure 2.9: Global sector nonlinearity

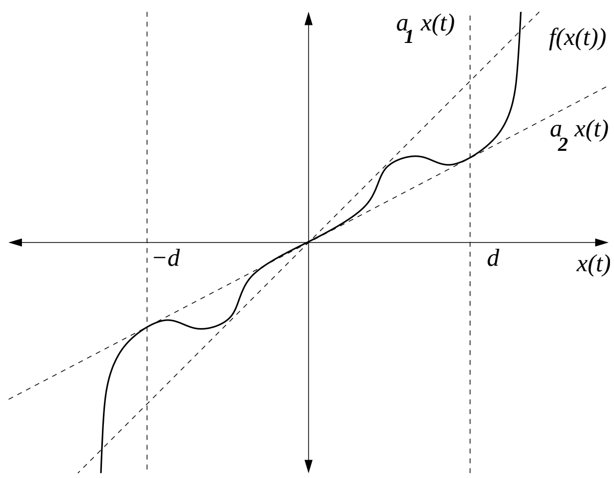

Figure 2.10: Local sector nonlinearity

Another approach to obtain Takagi-Sugeno fuzzy models is the local approximation in fuzzy partition spaces. Basically, the approach is to approximate nonlinear terms by chosen linear terms. This procedure reduces the number of model rules which is related to complexity of analysis. For other methods of fuzzy model design [11], [14] and the references cited therein can be seen.

An example of model derivation from nonlinear equations is given as follows [11]:

Example 2.1. Consider the following nonlinear system:

$$\begin{aligned}\dot{x_1}(t) &= -x_1(t) + x_1(t)x_2^3(t) \\ \dot{x_2}(t) &= -x_2(t) + (3 + x_2(t))x_1^3(t)\end{aligned} \tag{2.5}$$

For simplicity, it assumed that $x_1(t) \in [-1, 1]$ and $x_2(t) \in [-1, 1]$. Then, equation 2.5 can be written as

$$\dot{x}(t) = \begin{bmatrix} -1 & x_1(t)x_2^2(t) \\ (3 + x_2(t))x_1^2(t) & -1 \end{bmatrix} x(t)$$

where $x(t) = [x_1(t) \quad x_2(t)]^T$, and $x_1(t)x_2^2(t)$ and $(3 + x_2(t))x_1^2(t)$ are nonliear terms. Define $z_1(t)$ and $z_2(t)$ as

$$\begin{aligned}z_1(t) &= x_1(t)x_2^2(t) \\ z_2(t) &= (3 + x_2(t))x_1^2(t)\end{aligned}$$

Then, we have

$$\dot{x}(t) = \begin{bmatrix} -1 & z_1(t) \\ z_2(t) & -1 \end{bmatrix} x(t)$$

Next, the minimum and maximum values of $z_1(t)$ and $z_2(t)$ are calculated under $x_1(t) \in [-1, 1]$ and $x_2(t) \in [-1, 1]$:

$$\begin{aligned}\max_{x_1(t),x_2(t)} z_1(t) &= 1 \\ \min_{x_1(t),x_2(t)} z_1(t) &= -1 \\ \max_{x_1(t),x_2(t)} z_2(t) &= 4 \\ \min_{x_1(t),x_2(t)} z_2(t) &= 0\end{aligned}$$

Then, from the minimum and maximum values the $z_1(t)$ and $z_2(t)$ can be represented by

$$
\begin{aligned}
z_1(t) &= M_1(z_1(t)) \cdot 1 + M_2(z_1(t)) \cdot (-1) \\
z_2(t) &= N_1(z_2(t)) \cdot 4 + N_2(z_2(t)) \cdot 0
\end{aligned}
$$

where $M_1(z_1(t)) + M_2(z_1(t)) = 1$, $N_1(z_2(t)) + N_2(z_2(t)) = 1$. Therefore the membership functions of the IF–THEN rules can be calculated as

$$
M_1(z_1(t)) = \frac{z_1(t)+1}{2} \quad , \quad M_2(z_1(t)) = \frac{1-z_1(t)}{2}
$$
$$
N_1(z_2(t)) = \frac{z_2(t)}{4} \quad , \quad N_2(z_2(t)) = \frac{4-z_2(t)}{4}
$$

and we name the membership functions 'Positive' , 'Negative', 'Big' and 'Small', respectively. Figures 2.11 and 2.12 show the membership functions.

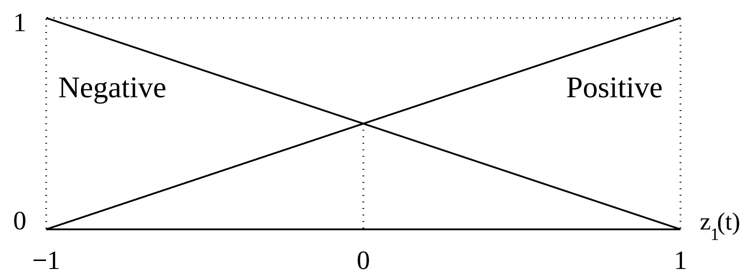

Figure 2.11: Membership functions $M_1(z_1(t))$ and $M_2(z_1(t))$

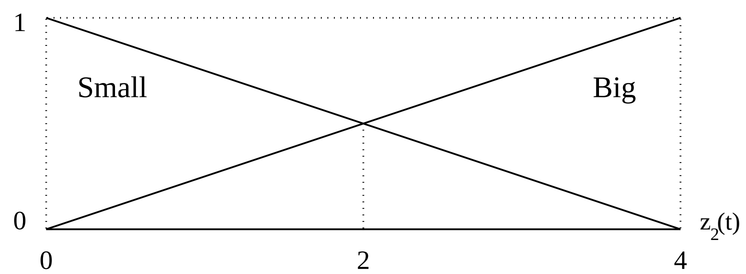

Figure 2.12: Membership functions $N_1(z_2(t))$ and $N_2(z_2(t))$

Then the nonlinear system (2.5) is represented by the following fuzzy model.

$Rule \quad 1:$

$$IF \quad z_1(t)\, is\, Positive \quad and \quad z_2(t)\, is\, Big$$
$$THEN \quad \dot{x}(t) = A_1 x(t)$$

$Rule \quad 2:$

$$IF \quad z_1(t)\, is\, Positive \quad and \quad z_2(t)\, is\, Small$$
$$THEN \quad \dot{x}(t) = A_2 x(t)$$

$Rule \quad 3:$

$$IF \quad z_1(t)\, is\, Negative \quad and \quad z_2(t)\, is\, Big$$
$$THEN \quad \dot{x}(t) = A_3 x(t)$$

$Rule \quad 4:$

$$IF \quad z_1(t)\, is\, Negative \quad and \quad z_2(t)\, is\, Small$$
$$THEN \quad \dot{x}(t) = A_4 x(t)$$

where

$$A_1 = \begin{bmatrix} -1 & 1 \\ 4 & -1 \end{bmatrix} \quad A_2 = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}$$

$$A_3 = \begin{bmatrix} -1 & -1 \\ 4 & -1 \end{bmatrix} \quad A_4 = \begin{bmatrix} -1 & -1 \\ 0 & -1 \end{bmatrix}$$

The overall output is calculated as

$$\dot{x}(t) = \sum_{i=1}^{4} h_i(z(t)) A_i x(t)$$

where

$$\begin{aligned} h_1(z(t)) &= M_1(z_1(t)) \cdot N_1(z_2(t)) \\ h_2(z(t)) &= M_1(z_1(t)) \cdot N_2(z_2(t)) \\ h_3(z(t)) &= M_2(z_1(t)) \cdot N_1(z_2(t)) \\ h_4(z(t)) &= M_2(z_1(t)) \cdot N_2(z_2(t)) \end{aligned}$$

This fuzzy model is the exact representation of the nonlinear system (2.5) in the region $[-1, 1] \times [-1, 1]$ on the $x_1 - x_2$ space.

Thus, we have suggested two useful approach to contruct a fuzzy model. In this thesis, the fuzzy modeling problem is out of our consideration. All the systems are assumed to have been identified and presented in the form of state space fuzzy models.

3. STABILITY OF T-S FUZZY SYSTEMS

In this section, stability will be explained only for continuous-time Takagi-Sugeno fuzzy models. For discrete time T-S fuzzy model [4], [18] and [23] can be referred.

3.1. Stability and Stability Analysis

For any given control system, the most important question about its various properties is the stability. An unstable control system is typically useless and dangerous. A system is described as stable if starting the system somewhere near its desired operating point implies that it will stay around the point ever after [43]. For example, the motions of a pendulum starting near the vertical up and down positions can be given as the unstable and stable behaviour of a system.

A nonlinear non-autonomous dynamic system can usually be represented by a set of nonlinear differential equations in the form

$$\dot{x} = f(x, t)$$

where f is a $n \times l$ nonlinear vector function, and x is the $n \times 1$ state vector. The number of states n is called the order of the system. The following definitions are cited from [43].

Definition 3.1. A state x_e is an equilibrium point of the system if $f(x_e, t) = 0$, $\forall t \geq t_0$ where t_0 is initial time.

For example, a linear time-varying system $\dot{x} = A(t)x$ has a single equilibrium point, the origin 0, unless the matrix $A(t)$ is always singular.

Definition 3.2. The equilibrium point 0 is said to be stable at t_0 if, for any $R > 0$, there exists a positive scalar $r(R, t_0)$, such that if $\|x(t_0)\| < r$, then $\|x(t)\| < R$ for all $t \geq t_0$. Otherwise, the equilibrium state is unstable.

Stability, which is also called stability in the sense of Lyapunov, means that the system trajectory can be kept arbitrarily close to the origin by starting sufficiently close to it.

Definition 3.3. The equilibrium point 0 is asymptotically stable at time t_0 if it is stable, and if in addition there exists some $r(t_0) > 0$ such that $\|x(t_0)\| < r(t_0)$ implies that $x(t) \to 0$ as $t \to \infty$.

The above definitions are formulated to characterize the local behavior of systems. Global stability concept is given in the following definition.

Definition 3.4. The equilibrium point 0 is globally asymptotically stable if $\forall x(t_0)$, $x(t) \to 0$ as $t \to \infty$. This is also called asymptotically stable in the large.

Here, the parallel distributed compensation is utilized to design fuzzy controllers to stabilize fuzzy system. The idea of parallel distributed compensation, abbreviated as PDC, first arised with a model-based design procedure proposed in [22]. Then the design procedure was improved and the stability of the control systems was analyzed in [4] and the procedure is defined and named parallel distributed compensation in [23].

In the PDC design, the idea is that for each local linear model, a linear feedback control rule is designed and the resulting overall controller, which is nonlinear in general, is fuzzy blending of each individual linear controller. The designed fuzzy controller uses the same fuzzy sets with the fuzzy model in the premise parts. [4]. Thus, for the fuzzy model (2.2) the following fuzzy controller is constructed by using PDC:

$Rule \quad i:$

$$\begin{aligned} IF \quad & z_1(t)\, is\, M_{i1} \quad and \ldots and \quad z_p(t)\, is\, M_{ip} \\ THEN \quad & u(t) \;=\; -K_i x(t), \qquad i = 1, 2, \ldots, r. \end{aligned} \tag{3.1}$$

Here, for the state feedback case, the fuzzy control rules have a linear controller in the consequent parts. Instead of the state feedback controllers, also other controllers such as output feedback and dynamic output feedback can be used [15]. But we only consider the state feedback controllers in this thesis.

Thus, the overall fuzzy controller is inferred as follows:

$$\begin{aligned} u(t) &= -\frac{\sum_{i=1}^{r} w_i(z(t))K_i x(t)}{\sum_{i=1}^{r} w_i(z(t))} \\ &= -\sum_{i=1}^{r} h_i(z(t))K_i x(t) \end{aligned} \tag{3.2}$$

The fuzzy controller design is to determine the local feedback gains K_i in the consequent parts. Although the fuzzy controller is constructed using the local design structure, the feedback gains K_i should be determined using global design conditions.

3.2. Stability Conditions and Stable Controller Design

First, the stability conditions for a fuzzy system will be given and then the stable fuzzy controller design for continuous time fuzzy systems will be presented.

A powerful and general approach for studying the stability of linear and nonlinear systems is the Lyapunov stability theory. This method is based on the determination of a function V which is called the Lyapunov function. From the properties of V, we can determine the stability or instability of the system. The main disadvantage of the Lyapunov functional approach is that it gives only the sufficient conditions for stability. Furthermore, in general, there is no method to construct a Lyapunov function. In this thesis, stability conditions of fuzzy models and fuzzy control systems are given in the sense of Lyapunov.

The following theorem is Lyapunov's stability theorem for continuous time systems:

Theorem 3.1. [44] Consider a continuous time system described by $\dot{x}(t) = f(x(t))$ where $x(t) \in \mathbb{R}^n$, $f(x(t))$ is an $n \times 1$ function vector with the property that $f(0) = 0$ for all t. Suppose that there exists a scalar function $V(x(t))$ continuous in $x(t)$ such that

a) $V(0) = 0$

b) $V(x(t)) > 0$ for $x(t) \neq 0$

c) $V(x(t)) \to \infty$ as $\|x(t)\| \to \infty$

d) $\dot{V}(x(t)) < 0$ for $x(t) \neq 0$

Then the equilibrium state $x(t) = 0$ for all t is asymptotically stable in the large and $V(x(t))$ is a Lyapunov function.

Now, consider the open-loop system of (2.3), that is,

$$\dot{x}(t) = \sum_{i=1}^{r} h_i(z(t))\{A_i x(t)\} \tag{3.3}$$

Note that, it is assumed that the premise variables are not functions of the input variables $u(t)$. However, the stability conditions, that will be given, can be applied even the case that the premise variables are functions of the input variables $u(t)$. A sufficient stability condition for the stability of the continuous time open-loop system (3.3) is given as follows.

Theorem 3.2. [11] The equilibrium of the continuous fuzzy system (2.3) with $u(t) = 0$ is globally asymptotically stable if there exists a common positive definite matrix P such that

$$A_i^T P + P A_i < 0, \qquad i = 1, 2, \ldots, r, \tag{3.4}$$

that is, a common matrix P has to exist for all subsystems.

The last theorem's proof is straightforward by using the respective Lyapunov's theorem and the Lyapunov function $V(x(t)) = x^T(t)Px(t)$. Also it can be seen that, this theorem reduces to the Lyapunov stability theorem for linear systems when $r = 1$.

A question naturally arises of whether the fuzzy system is stable if all the linear subsystems are stable. In general the answer is no and this is shown in [23].

By substituting (3.2) into (2.3), we obtain the equation (3.5), that is,

$$\begin{aligned}
\dot{x}(t) &= \sum_{i=1}^{r} h_i(z(t))\{A_i x(t) + B_i\{-\sum_{j=1}^{r} h_j(z(t))K_j x(t)\}\} \\
&= \sum_{i=1}^{r} h_i(z(t))\{A_i x(t) - \sum_{j=1}^{r} h_j(z(t))B_i K_j x(t)\} \\
&= \sum_{i=1}^{r}\sum_{j=1}^{r} h_i(z(t))h_j(z(t))\{A_i - B_i K_j\}x(t) \qquad (3.5)
\end{aligned}$$

Now, denote $G_{ij} = A_i - B_i K_j$. By using the simple equality

$$\sum_{i=1}^{r}\sum_{j=1}^{r} G_{ij} = \sum_{i=1}^{r} G_{ii} + \sum_{i=1}^{r-1}\sum_{j>i}^{r}\{G_{ij} + G_{ji}\} \qquad (3.6)$$

the equation (3.5) can be represented as the equation (3.7).

$$\begin{aligned}
\dot{x}(t) &= \sum_{i=1}^{r} h_i(z(t))h_i(z(t))\{G_{ii}\}x(t) \\
&\quad +2\sum_{i=1}^{r-1}\sum_{j>i}^{r}\{\frac{G_{ij} + G_{ji}}{2}\}x(t) \qquad (3.7)
\end{aligned}$$

By straightforward application of Theorem 3.2 to the equation (3.7), we have the following stability conditions for the continuous-time fuzzy system.

Theorem 3.3. [11] The equilibrium of the continuous fuzzy control system described (3.7) is globally asymptotically stable if there exists a common positive definite matrix P such that

$$G_{ii}^T P + P G_{ii} < 0 \tag{3.8}$$

$$(\frac{G_{ij} + G_{ji}}{2})^T P + P(\frac{G_{ij} + G_{ji}}{2}) \leq 0 \tag{3.9}$$

for $i < j \leq r$ s.t. $h_i(z(t)) \times h_j(z(t)) \neq 0$ for all $z(t)$.

If for a fuzzy control system, the number of IF-THEN rules, that is r, is very large, it might be difficult to find a common P satisfying the conditions of Theorem 3.3. Because of this, the relaxed stability conditions for fuzzy systems will be presented [15, 28].

Theorem 3.4. [11] Assume that the number of rules that fire for all t is less than or equal to s, where $1 < s \leq r$. The equilibrium of the continuous fuzzy control system described by (3.7) is globally asymptotically stable if there exist a common positive definite matrix P and a common positive semidefinite matrix Q such that

$$G_{ii}^T P + P G_{ii} + (s-1)Q < 0 \tag{3.10}$$

$$(\frac{G_{ij} + G_{ji}}{2})^T P + P(\frac{G_{ij} + G_{ji}}{2}) - Q \leq 0 \tag{3.11}$$

for $i < j \leq r$ s.t. $h_i(z(t)) \times h_j(z(t)) \neq 0$ for all $z(t)$ where $s > 1$.

It is assumed that the weight $h_i(z(t))$ of each rule in the fuzzy controller is equal to the weight of each rule in the fuzzy model for all t. If the assumption does not hold, the following stability conditions should be used in instead of the conditions of Theorem 3.3 and Theorem 3.4:

$$G_{ij}^T P + P G_{ji} \quad < \quad 0$$

where $G_{ij} = A_i - B_i K_j$.

Hence, the fuzzy control design problem is reduced to determine K_j's for $j = 1, 2, \ldots, r$ and a common positive definite matrix P which satisfy the conditions (3.8) with (3.9) for the fuzzy system.

In the stability analysis of fuzzy systems, most of the time a trial-and-error type of procedure has been used to find a common positive definite matrix P, that is to check the stability of fuzzy systems (3.7) [4, 25]. But then it is shown that the common P problem can be solved via convex optimization techniques for LMIs in [18],[23] and [26]. To check the stability of the fuzzy systems is to find a common P or to determine that no such P exists. This is called an LMI problem [17]. The LMI problems can be solved numerically and efficiently by using the tools in the mathematical programming literature. In this case, the LMI Control Toolbox in Matlab software is very useful tool for the solutions of the above LMI problems [27]. So the origin of the control design is the LMI-based design approach. Now our objective is to present stable fuzzy controller design via LMIs.

Next, two definition about LMI and the well-known Schur complement will be given.

Definition 3.5. [17] An LMI is a matrix inequality of the form

$$F(x) \quad = \quad F_0 + \sum_{i=1}^{r} x_i F_i > 0$$

where $x^T = (x_1, x_2, \ldots, x_r)$ is the variable and the symmetric matrices $F_i = F_i^T \in \mathbb{R}^{n \times n}, \quad i = 0, 1, \ldots, r$ are given.

Definition 3.6. [17] Given an LMI $F(x) > 0$, the corresponding LMI Problem is to find x_{feas} such that $F(x_{feas}) > 0$ or to determine that the LMI is infeasible. This is a convex feasibility problem. Hence, saying 'solving the LMI $F(x) > 0$' is to mean 'solving the corresponding LMI problem'.

Theorem 3.5. [17] (Schur Complement) Given matrices $Q(x)$, $R(x)$ and $S(x)$ where $Q(x) = Q(x)^T$, $R(x) = R(x)^T$ and $S(x)$ depend affinely on x then

$$R(x) > 0 \quad , \quad Q(x) - S(x)R(x)^{-1}S(x)^T > 0$$

if and only if the LMI

$$\begin{bmatrix} Q(x) & S(x) \\ S(x)^T & R(x) \end{bmatrix} > 0$$

holds.

We consider a fuzzy controller design problem for the continuous fuzzy system using the stability conditions of Theorem 3.3. The conditions (3.8) and (3.9) are not jointly convex in K_i and P. Multiplying the inequalities on the left and right by P^{-1} and defining a new variable $X = P^{-1}$, we rewrite the conditions as

$$-XA_i^T - A_iX + XK_i^TB_i^T + B_iK_iX > 0$$
$$-XA_i^T - A_iX - XA_j^T - A_jX + XK_j^TB_i^T + B_iK_jX + XK_i^TB_j^T + B_jK_iX \geq 0$$

Now define $M_i = K_iX$ so that for $X > 0$ we have $K_i = M_iX^{-1}$. Thus, by substituting into the above inequalities the LMI conditions are obtained and we define a stable fuzzy controller design problem for continuous fuzzy system (3.7) as follows:

Find $X > 0$ and M_i, $(i = 1, 2, \ldots, r)$, satisfying

$$-XA_i^T - A_iX + M_i^TB_i^T + B_iM_i > 0$$
$$-XA_i^T - A_iX - XA_j^T - A_jX + M_j^TB_i^T + B_iM_j + M_i^TB_j^T + B_jM_i \geq 0$$

for $i < j \leq r$ s.t. $h_i(z(t)) \times h_j(z(t)) \neq 0$ for all $z(t)$ where $X = P^{-1}$, $M_i = K_iX$.

We can find a positive definite matrix X and M_i satisfying the LMIs or determine that no such X and M_i exist.

The feedback gains K_i and the common matrix P can be obtained as $P = X^{-1}$ and $K_i = M_i X^{-1}$ from the solutions X and M_i.

By similar way, the fuzzy controller design problem for the continuous fuzzy system can be defined from the relaxed stability conditions of Theorem 3.4 as follows.

For the continuous fuzzy system:

Find $X > 0$, $Y \geq 0$ and M_i, $(i = 1, 2, \ldots, r)$, satisfying

$$
\begin{aligned}
-XA_i^T - A_iX + M_i^T B_i^T + B_iM_i - (s-1)Y &> 0 \\
2Y - XA_i^T - A_iX - XA_j^T - A_jX + M_j^T B_i^T + B_iM_j + M_i^T B_j^T + B_jM_i &\geq 0
\end{aligned}
$$

for $i < j \leq r$ s.t. $h_i(z(t)) \times h_j(z(t)) \neq 0$ for all $z(t)$ where $X = P^{-1}$, $M_i = K_iX$ and $Y = XQX$.

The feedback gains K_i, the common matrices P and Q can be obtained as $K_i = M_iX^{-1}$, $P = X^{-1}$ and $Q = PYP$ from the solutions X, Y and M_i.

4. DELAY-DEPENDENT GUARANTEED COST CONTROL FOR T-S FUZZY SYSTEMS

We consider a nonlinear time-delay system represented by the T-S fuzzy system (1.1). It is necessary to define the initial condition $\varphi(t)$ for $-\bar{\sigma} \leq t \leq 0$ as a constant scalar or differentiable function in order to obtain the upper bound of guaranteed cost performance in the following analysis.

By using center-average defuzzifier, product inference and singleton fuzzifier, the dynamic fuzzy model (1.1) can be expressed by the following global model:

$$\begin{aligned} \dot{x}(t) &= A(t)x(t) + A_d(t)x(t-\sigma(t)) + B(t)u(t) \\ z(t) &= C_z(t)x(t) + C_{zd}(t)x(t-\sigma(t)) \\ x(t) &= \varphi(t), -\bar{\sigma} \leq t \leq 0 \end{aligned} \tag{4.1}$$

where

$$\begin{aligned} A(t) &= \sum_{i=1}^{\bar{n}} h_i(t)A_i \quad , \quad A_d(t) = \sum_{i=1}^{\bar{n}} h_i(t)A_{di} \\ B(t) &= \sum_{i=1}^{\bar{n}} h_i(t)B_i \\ C_z(t) &= \sum_{i=1}^{\bar{n}} h_i(t)C_{zi} \quad , \quad C_{zd}(t) = \sum_{i=1}^{\bar{n}} h_i(t)C_{zdi} \end{aligned}$$

$h_i(M(t))$ denotes the normalized membership function which satisfies

$$\begin{aligned} h_i(M(t)) &= \frac{\mu_i(M(t))}{(\sum_{i=1}^{\bar{n}} \mu_i(M(t)))} \\ h_i(M(t)) &\geq 0 \quad , \quad \sum_{i=1}^{\bar{n}} h_i(M(t)) = 1 \end{aligned}$$

where

$$\begin{aligned}
\mu_i(M(t)) &= \prod_{j=1}^{g} F_{ij}(M_j(t)) \\
\mu_i(M(t)) &\geq 0
\end{aligned}$$

and $F_{ij}(M_j(t))$ is the grade of membership of $M_j(t)$ in the fuzzy set F_{ij}.

The following two lemmas reduce the computative complexity and play important roles in obtaining results in this study.

Lemma 4.1. [1] For any real matrices X_i, Y_i for $1 \leq i \leq \bar{n}$, and $S > 0$ with appropriate dimensions, we have

$$\begin{aligned}
2\sum_{i=1}^{\bar{n}}\sum_{j=1}^{\bar{n}} h_i h_j X_i^T S Y_j &\leq \sum_{i=1}^{\bar{n}} h_i (X_i^T S X_i + Y_i^T S Y_i) \\
2\sum_{i=1}^{\bar{n}}\sum_{j=1}^{\bar{n}}\sum_{k=1}^{\bar{n}}\sum_{l=1}^{\bar{n}} h_i h_j h_k h_l X_{ij}^T S Y_{kl} &\leq \sum_{i=1}^{\bar{n}}\sum_{j=1}^{\bar{n}} h_i h_j (X_{ij}^T S X_{ij} + Y_{ij}^T S Y_{ij})
\end{aligned}$$

where h_i, $(1 \leq i \leq \bar{n})$, are defined as

$$\begin{aligned}
h_i(M(t)) &\geq 0 \\
\sum_{i=1}^{\bar{n}} h_i(M(t)) &= 1
\end{aligned}$$

Lemma 4.2. [5] Let $x(t) \in \mathbb{R}^n$ be a vector-valued function with first-order continuous-derivative entries. Then, the following integral inequality holds for any matrices $M_1, M_2 \in \mathbb{R}^{n\times n}$ and $X = X^T > 0$, and a scalar function $h = h(t) \geq 0$:

$$\begin{aligned}
-\int_{t-h}^{t} \dot{x}^T(s) X \dot{x}(s) ds \leq\ & \xi^T(t) \begin{bmatrix} M_1^T + M_1 & -M_1^T + M_2 \\ * & -M_2^T - M_2 \end{bmatrix} \xi(t) \\
& + h\xi^T(t) \begin{bmatrix} M_1^T \\ M_2^T \end{bmatrix} X^{-1} \begin{bmatrix} M_1 & M_2 \end{bmatrix} \xi(t)
\end{aligned}$$

where

$$\xi(t) = \begin{bmatrix} x(t) \\ x(t-h) \end{bmatrix}$$

We consider the design of a guaranteed cost controller via state feedback. Also we consider the guaranteed cost function (1.2) and use the controller structure expressed in the form

Rule $\quad i:$

$$IF \quad M_1(t)\, is\, F_{i1} \quad and \quad M_2(t)\, is\, F_{i2} \, \ldots \quad and \quad M_g(t)\, is\, F_{ig}$$

$$THEN \quad u(t) = K_i x(t) \tag{4.2}$$

Hence, the overall fuzzy control law is represented by

$$u(t) = \sum_{i=1}^{\bar{n}} h_i(t) K_i x(t) \tag{4.3}$$

The design of the guaranteed cost controller is to determine the feedback gains K_i $(i = 1, 2, \ldots, \bar{n})$ and a positive scalar $\delta(\bar{\sigma})$ such that the resulting closed-loop system is asymptotically stable and the closed-loop value of the cost function (1.2) satisfies $J \leq \delta(\bar{\sigma})$.

The closed-loop system of (4.1) with the control law (4.3) can be written as follows:

$$\begin{aligned} \dot{x}(t) &= \bar{A}(t)x(t) + A_d(t)x(t-\sigma(t)) \\ z(t) &= \bar{C}_z(t)x(t) + C_{zd}(t)x(t-\sigma(t)) \\ x(t) &= \varphi(t), \quad -\bar{\sigma} \leq t \leq 0 \end{aligned} \tag{4.4}$$

where

$$\begin{aligned} \bar{A}(t) &= \sum_{i=1}^{\bar{n}} \sum_{j=1}^{\bar{n}} h_i h_j \bar{A}_{ij} \quad , \quad \bar{C}_z(t) = \sum_{i=1}^{\bar{n}} \sum_{j=1}^{\bar{n}} h_i h_j \bar{C}_{zij} \\ \bar{A}_{ij} &= A_i + B_i K_j \quad , \quad \bar{C}_{zij} = C_{zi} + D_{zi} K_j \end{aligned}$$

The main result on the guaranteed cost control via state feedback for the T-S fuzzy model with time-varying delay is given in the following theorem.

Theorem 4.1. For given constants $\bar{\sigma}$ and β satisfying $0 < \sigma(t) \leq \bar{\sigma}$ and $\dot{\sigma}(t) \leq \beta$, respectively and given numbers λ_1 and μ_1 where $\mu_1 \neq 0$, if there exist matrices $X > 0$, $\bar{S} > 0$, $\bar{W} > 0$ and Y_i satisfying the following LMI

$$\begin{bmatrix} \Phi_{11} & \Phi_{12} & \bar{\sigma}\Phi_{13} & 0 & \Phi_{15} & 2X \\ * & \Phi_{22} & \bar{\sigma}\Phi_{23} & 2\bar{W} & \Phi_{25} & 0 \\ * & * & -2\bar{\sigma}\bar{W} & 0 & 0 & 0 \\ * & * & * & -2\bar{W} & 0 & 0 \\ * & * & * & * & -2I & 0 \\ * & * & * & * & * & -2\bar{S} \end{bmatrix} < 0 \tag{4.5}$$

where

$$\begin{aligned}
\Phi_{11} &= X(A_i + A_j)^T + (A_i + A_j)X + Y_j^T B_i^T + Y_i^T B_j^T + B_i Y_j + B_j Y_i \\
&\quad -2\lambda_1^2 \mu_1^{-2}(1-\beta)\bar{S} - \lambda_1 \mu_1^{-1} \bar{S}(A_{di}^T + A_{dj}^T) - \lambda_1 \mu_1^{-1}(A_{di} + A_{dj})\bar{S} \\
\Phi_{12} &= 2(X + \lambda_1 \mu_1^{-1}\bar{S} + \lambda_1 \mu_1^{-2}(1-\beta)\bar{S}) + \mu_1^{-1}(A_{di} + A_{dj})\bar{S} \\
\Phi_{13} &= XA_i^T + XA_j^T + Y_j^T B_i^T + Y_i^T B_j^T - \lambda_1 \mu_1^{-1}\bar{S}A_{di}^T - \lambda_1 \mu_1^{-1}\bar{S}A_{dj}^T \\
\Phi_{15} &= XC_{zi}^T + XC_{zj}^T + Y_i^T D_{zj}^T + Y_j^T D_{zi}^T - \lambda_1 \mu_1^{-1}\bar{S}C_{zdi}^T - \lambda_1 \mu_1^{-1}\bar{S}C_{zdj}^T \\
\Phi_{22} &= -4\mu_1^{-1}\bar{S} - 2\mu_1^{-2}(1-\beta)\bar{S} \\
\Phi_{23} &= \mu_1^{-1}\bar{S}A_{di}^T + \mu_1^{-1}\bar{S}A_{dj}^T \\
\Phi_{25} &= \mu_1^{-1}\bar{S}C_{zdi}^T + \mu_1^{-1}\bar{S}C_{zdj}^T
\end{aligned}$$

for $1 \leq i \leq j \leq \bar{n}$ then the closed loop system (4.4) is asymptotically stable with guaranteed cost performance $\delta(\bar{\sigma})$. Moreover, the controller parameters can be chosen as $K_i = Y_i X^{-1}$ and the guaranteed cost bound is

$$\delta(\bar{\sigma}) = \varphi^T(0)X^{-1}\varphi(0) + \int_{-\bar{\sigma}}^{0}\int_{\theta}^{0} \dot{\varphi}^T(s)\bar{W}^{-1}\dot{\varphi}(s)dsd\theta + \int_{-\bar{\sigma}}^{0} \varphi^T(s)\bar{S}^{-1}\varphi(s)ds$$

Proof. Consider the following Lyapunov function candidate for (4.4)

$$V(t) = V_1(t) + V_2(t) + V_3(t) \tag{4.6}$$

where

$$\begin{aligned}
V_1(t) &= x^T(t)Px(t) \\
V_2(t) &= \int_{-\bar{\sigma}}^{0}\int_{t+\theta}^{t} \dot{x}^T(s)W\dot{x}(s)dsd\theta \\
V_3(t) &= \int_{t-\sigma(t)}^{t} x^T(s)Sx(s)ds
\end{aligned}$$

By differentiating $V_1(t)$ we obtain,

$$\begin{aligned}
\dot{V}_1(t) &= x^T(t)P\dot{x}(t) + \dot{x}^T(t)Px(t) \\
&= 2x^T(t)P\dot{x}(t) \\
&= 2x^T(t)P(\bar{A}(t)x(t) + A_d(t)x(t-\sigma(t))) \qquad (4.7)
\end{aligned}$$

By differentiating $V_2(t)$,

$$\dot{V}_2(t) = \bar{\sigma}\dot{x}^T(t)W\dot{x}(t) - \int_{t-\bar{\sigma}}^{t} \dot{x}^T(s)W\dot{x}(s)ds \qquad (4.8)$$

It is clear that the following inequallity is true:

$$-\int_{t-\bar{\sigma}}^{t} \dot{x}^T(s)W\dot{x}(s)ds \leq -\int_{t-\sigma(t)}^{t} \dot{x}^T(s)W\dot{x}(s)ds$$

Applying the integral inequality given in Lemma 4.2 for any $M_1, M_2 \in \mathbb{R}^{n\times n}$ yields the following integral inequality:

$$\begin{aligned}
-\int_{t-\sigma(t)}^{t} \dot{x}^T(s)W\dot{x}(s)ds \leq\ & \eta^T(t)\begin{bmatrix} M_1^T + M_1 & -M_1^T + M_2 \\ * & -M_2^T - M_2 \end{bmatrix}\eta(t) \\
& +\eta^T(t)\begin{bmatrix} M_1^T \\ M_2^T \end{bmatrix} W^{-1}\begin{bmatrix} M_1 & M_2 \end{bmatrix}\eta(t) \qquad (4.9)
\end{aligned}$$

where $\eta^T(t) = \begin{bmatrix} x^T(t) & x^T(t-\sigma(t)) \end{bmatrix}$. We substitute (4.9) into (4.8) and then

$$\begin{aligned} \dot{V}_2(t) &\leq \bar{\sigma}\eta^T(t)\begin{bmatrix} \bar{A}^T(t) \\ A_d^T(t) \end{bmatrix} W \begin{bmatrix} \bar{A}(t) & A_d(t) \end{bmatrix} \eta(t) \\ &+\eta^T(t)(\begin{bmatrix} M_1^T + M_1 & -M_1^T + M_2 \\ * & -M_2^T - M_2 \end{bmatrix} \\ &+ \begin{bmatrix} M_1^T \\ M_2^T \end{bmatrix} W^{-1} \begin{bmatrix} M_1 & M_2 \end{bmatrix})\eta(t) \end{aligned} \tag{4.10}$$

$$\begin{aligned} \dot{V}_2(t) &\leq \bar{\sigma}\sum_{i=1}^{\bar{n}}\sum_{j=1}^{\bar{n}}\sum_{k=1}^{\bar{n}}\sum_{l=1}^{\bar{n}} h_i h_j h_k h_l \eta^T(t)\begin{bmatrix} \bar{A}_{ij}^T \\ A_{di}^T \end{bmatrix} W \begin{bmatrix} \bar{A}_{kl} & A_{dl} \end{bmatrix} \eta(t) \\ &+\eta^T(t)(\begin{bmatrix} M_1^T + M_1 & -M_1^T + M_2 \\ * & -M_2^T - M_2 \end{bmatrix} \\ &+ \begin{bmatrix} M_1^T \\ M_2^T \end{bmatrix} W^{-1} \begin{bmatrix} M_1 & M_2 \end{bmatrix})\eta(t) \end{aligned} \tag{4.11}$$

From Lemma 4.1 we obtain,

$$\sum_{i=1}^{\bar{n}}\sum_{j=1}^{\bar{n}}\sum_{k=1}^{\bar{n}}\sum_{l=1}^{\bar{n}} h_i h_j h_k h_l X_{ij}^T S X_{kl} \leq \sum_{i=1}^{\bar{n}}\sum_{j=1}^{\bar{n}} h_i h_j X_{ij}^T S X_{ij} \tag{4.12}$$

thus, we have

$$\begin{aligned} \dot{V}_2(t) &\leq \bar{\sigma}\sum_{i=1}^{\bar{n}}\sum_{j=1}^{\bar{n}} h_i h_j \eta^T(t)\begin{bmatrix} \bar{A}_{ij}^T \\ A_{di}^T \end{bmatrix} W \begin{bmatrix} \bar{A}_{ij}^T & A_{di} \end{bmatrix} \eta(t) \\ &+\eta^T(t)(\begin{bmatrix} M_1^T + M_1 & -M_1^T + M_2 \\ * & -M_2^T - M_2 \end{bmatrix} && (4.13) \\ &+ \begin{bmatrix} M_1^T \\ M_2^T \end{bmatrix} W^{-1} \begin{bmatrix} M_1 & M_2 \end{bmatrix})\eta(t) && (4.14) \end{aligned}$$

By differentiating $V_3(t)$ we obtain

$$\begin{aligned}
\dot{V}_3(t) &= x^T(t)Sx(t) - (1-\dot{\sigma}(t))x^T(t-\sigma(t))Sx(t-\sigma(t)) \\
&\le x^T(t)Sx(t) - (1-\beta)x^T(t-\sigma(t))Sx(t-\sigma(t)) \qquad (4.15)
\end{aligned}$$

It follows from (4.12) that

$$\begin{aligned}
z^T(t)z(t) &= \sum_{i=1}^{\bar{n}}\sum_{j=1}^{\bar{n}}\sum_{k=1}^{\bar{n}}\sum_{l=1}^{\bar{n}} h_i h_j h_k h_l \eta^T(t) \begin{bmatrix} \bar{C}_{zij}^T \\ C_{zdi}^T \end{bmatrix} \begin{bmatrix} \bar{C}_{zkl} & C_{zdk} \end{bmatrix} \eta(t) \\
&\le \sum_{i=1}^{\bar{n}}\sum_{j=1}^{\bar{n}} h_i h_j \eta^T(t) \begin{bmatrix} \bar{C}_{zij}^T \\ C_{zdi}^T \end{bmatrix} \begin{bmatrix} \bar{C}_{zij} & C_{zdi} \end{bmatrix} \eta(t) \qquad (4.16)
\end{aligned}$$

Consequently, the derivative of $V(t)$ can be presented as follows:

$$\begin{aligned}
\dot{V}(t) &\le \sum_{i=1}^{\bar{n}}\sum_{j=1}^{\bar{n}} h_i h_j \eta^T(t) (\begin{bmatrix} \bar{A}_{ij}^T P + P\bar{A}_{ij} & PA_{di} - M_1^T + M_2 \\ +S + M_1^T + M_1 & \\ * & -(1-\beta)S - M_2^T - M_2 \end{bmatrix} \\
&\quad + \bar{\sigma} \begin{bmatrix} \bar{A}_{ij}^T \\ A_{di}^T \end{bmatrix} W \begin{bmatrix} \bar{A}_{ij} & A_{di} \end{bmatrix} + \begin{bmatrix} M_1^T \\ M_2^T \end{bmatrix} W^{-1} \begin{bmatrix} M_1 & M_2 \end{bmatrix} \\
&\quad + \begin{bmatrix} \bar{C}_{zij}^T \\ C_{zdi}^T \end{bmatrix} \begin{bmatrix} \bar{C}_{zij} & C_{zdi} \end{bmatrix})\eta(t) - z^T(t)z(t) \qquad (4.17)
\end{aligned}$$

$$\begin{aligned}
\dot{V}(t) &\le \sum_{i=1}^{\bar{n}}\sum_{j=1}^{\bar{n}} h_i h_j \eta^T(t)\Sigma_{ij}\eta(t) - z^T(t)z(t) \\
&= \sum_{i=1}^{\bar{n}} h_i^2 \eta^T(t)\Sigma_{ii}\eta(t) + \sum_{i=1}^{\bar{n}-1}\sum_{j>i}^{\bar{n}} h_i h_j \eta^T(t)(\Sigma_{ij} + \Sigma_{ji})\eta(t) - z^T(t)z(t)
\end{aligned}$$

where

$$\Sigma_{ij} = H_{ij} + \bar{\sigma}\Gamma_{1ij}^T W \Gamma_{1ij} + \Gamma_2^T W^{-1}\Gamma_2 + \Gamma_{3ij}^T\Gamma_{3ij}$$

for

$$H_{ij} = \begin{bmatrix} \bar{A}_{ij}^T P + P\bar{A}_{ij} + S + M_1^T + M_1 & PA_{di} - M_1^T + M_2 \\ * & -(1-\beta)S - M_2^T - M_2 \end{bmatrix}$$

$$\begin{aligned} \Gamma_{1ij} &= \begin{bmatrix} \bar{A}_{ij} & A_{di} \end{bmatrix} \\ \Gamma_2 &= \begin{bmatrix} M_1 & M_2 \end{bmatrix} \\ \Gamma_{3ij} &= \begin{bmatrix} \bar{C}_{zij} & C_{zdi} \end{bmatrix} \end{aligned}$$

Thus we have

$$\begin{aligned} \Sigma_{ij} + \Sigma_{ji} &= H_{ij} + H_{ji} + \bar{\sigma}\Gamma_{1ij}^T W \Gamma_{1ij} + \bar{\sigma}\Gamma_{1ji}^T W \Gamma_{1ji} + 2\Gamma_2^T W^{-1} \Gamma_2 \\ &\quad + \Gamma_{3ij}^T \Gamma_{3ij} + \Gamma_{3ji}^T \Gamma_{3ji} \end{aligned}$$

By using the Schur complement in Theorem 3.5, $\Sigma_{ii} < 0$ is equivalent to the following inequality:

$$\begin{bmatrix} H_{ii} & \bar{\sigma}\Gamma_{1ii}^T & \Gamma_2^T & \Gamma_{3ii}^T \\ * & -\bar{\sigma}W^{-1} & 0 & 0 \\ * & * & -W & 0 \\ * & * & * & -I \end{bmatrix} < 0 \tag{4.18}$$

Similarly, $\Sigma_{ij} + \Sigma_{ji} < 0$ is equivalent to the following inequality:

$$\begin{bmatrix} H_{ij} + H_{ji} & \bar{\sigma}\Gamma_{1ij}^T & \bar{\sigma}\Gamma_{1ji}^T & 2\Gamma_2^T & \Gamma_{3ij}^T & \Gamma_{3ji}^T \\ * & -\bar{\sigma}W^{-1} & 0 & 0 & 0 & 0 \\ * & * & -\bar{\sigma}W^{-1} & 0 & 0 & 0 \\ * & * & * & -2W & 0 & 0 \\ * & * & * & * & -I & 0 \\ * & * & * & * & * & -I \end{bmatrix} < 0 \tag{4.19}$$

for $j > i$, $(i = 1, 2, \ldots, \bar{n} - 1)$, $(j = 1, 2, \ldots, \bar{n})$.

The inequalities (4.18) and (4.19) could be equivalent to

$$\Psi_{ij} = \begin{bmatrix} H_{ij}+H_{ji} & \bar{\sigma}(\Gamma_{1ij}^T+\Gamma_{1ji}^T) & 2\Gamma_2^T & \Gamma_{3ij}^T+\Gamma_{3ji}^T \\ * & -2\bar{\sigma}W^{-1} & 0 & 0 \\ * & * & -2W & 0 \\ * & * & * & -2I \end{bmatrix} < 0 \tag{4.20}$$

for $1 \leq i \leq j \leq \bar{n}$ where

$$H_{ij} = \begin{bmatrix} P(A_i+B_iK_j)+(A_i+B_iK_j)^TP & PA_{di}-M_1^T+M_2 \\ +S+M_1^T+M_1 & \\ * & -(1-\beta)S-M_2^T-M_2 \end{bmatrix}$$

Now we define the two matrices below

$$\widetilde{M} = \begin{bmatrix} P & 0 \\ M_1 & M_2 \end{bmatrix} \tag{4.21}$$

$$\widetilde{A}_{ij} = \begin{bmatrix} A_i+B_iK_j & A_{di} \\ I & -I \end{bmatrix} \tag{4.22}$$

Thus we have the following equalities

$$H_{ij} = \widetilde{M}^T\widetilde{A}_{ij}+\widetilde{A}_{ij}^T\widetilde{M}+diag\{S,-(1-\beta)S\} \tag{4.23}$$

$$\begin{aligned} \widetilde{M}^T\widetilde{A}_{ij} &= \begin{bmatrix} P & M_1^T \\ 0 & M_2^T \end{bmatrix}\begin{bmatrix} \bar{A}_{ij} & A_{di} \\ I & -I \end{bmatrix} \\ &= \begin{bmatrix} P\bar{A}_{ij}+M_1^T & PA_{di}-M_1^T \\ M_2^T & -M_2^T \end{bmatrix} \end{aligned} \tag{4.24}$$

$$
\begin{aligned}
\Gamma_2^T &= \begin{bmatrix} M_1^T \\ M_2^T \end{bmatrix} \\
&= \begin{bmatrix} P & M_1^T \\ 0 & M_2^T \end{bmatrix} \begin{bmatrix} 0 \\ I \end{bmatrix} \\
&= \widetilde{M}^T \begin{bmatrix} 0 \\ I \end{bmatrix}
\end{aligned} \tag{4.25}
$$

If we let $M_1 = \lambda_1 P$ and $M_2 = \mu_1 S$ where $\mu_1 \neq 0$, then $\widetilde{M}$ is invertible.

$$
\widetilde{M} = \begin{bmatrix} P & 0 \\ \lambda_1 P & \mu_1 S \end{bmatrix} \tag{4.26}
$$

$$
\begin{aligned}
\widetilde{M}^{-1} &= \begin{bmatrix} P^{-1} & 0 \\ -\lambda_1 \mu_1^{-1} S^{-1} & \mu_1^{-1} S^{-1} \end{bmatrix} \\
&= \begin{bmatrix} X & 0 \\ -\lambda_1 \mu_1^{-1} \bar{S} & \mu_1^{-1} \bar{S} \end{bmatrix}
\end{aligned} \tag{4.27}
$$

Now let the matrix T as follows

$$
T = \begin{bmatrix} \widetilde{M}^{-1} & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & W^{-1} & 0 \\ 0 & 0 & 0 & I \end{bmatrix} \tag{4.28}
$$

Since

$$
\Gamma_{1ij}^T = \begin{bmatrix} \bar{A}_{ij}^T \\ A_{di} \end{bmatrix} \tag{4.29}
$$

then we have

$$\begin{aligned}
\widetilde{M}^{-T}\Gamma_{1ij}^{T} &= \begin{bmatrix} P^{-1} & -\lambda_1\mu_1^{-1}S^{-1} \\ 0 & \mu_1^{-1}S^{-1} \end{bmatrix} \begin{bmatrix} \bar{A}_{ij}^{T} \\ A_{di}^{T} \end{bmatrix} \\
&= \begin{bmatrix} P^{-1}\bar{A}_{ij}^{T} - \lambda_1\mu_1^{-1}S^{-1}A_{di}^{T} \\ \mu_1^{-1}S^{-1}A_{di}^{T} \end{bmatrix} \\
&= \begin{bmatrix} P^{-1}(A_i^{T} + K_j^{T}B_i^{T}) - \lambda_1\mu_1^{-1}S^{-1}A_{di}^{T} \\ \mu_1 S^{-1}A_{di}^{T} \end{bmatrix} \\
&= \begin{bmatrix} XA_i^{T} + Y_j^{T}B_i^{T} - \lambda_1\mu_1^{-1}\bar{S}A_{di}^{T} \\ \mu_1^{-1}\bar{S}A_{di}^{T} \end{bmatrix} \\
&= \Pi_{1ij}^{T}
\end{aligned} \tag{4.30}$$

where $X = P^{-1}$, $Y_j^T = P^{-1}K_j^T$ and $\bar{S} = S^{-1}$.

Since

$$\Gamma_{3ij}^{T} = \begin{bmatrix} \bar{C}_{zij}^{T} \\ C_{zdi}^{T} \end{bmatrix} \tag{4.31}$$

we have

$$\begin{aligned}
\widetilde{M}^{-T}\Gamma_{3ij}^{T} &= \begin{bmatrix} P^{-1} & -\lambda_1\mu_1^{-1}S^{-1} \\ 0 & \mu_1^{-1}S^{-1} \end{bmatrix} \begin{bmatrix} C_{zdi}^{T} + K_j^{T}D_{zi}^{T} \\ C_{zdi}^{T} \end{bmatrix} \\
&= \begin{bmatrix} P^{-1}C_{zi}^{T} + P^{-1}K_j^{T}D_{zi}^{T} - \lambda_1\mu_1^{-1}S^{-1}C_{zdi}^{T} \\ \mu_1^{-1}S^{-1}C_{zdi}^{T} \end{bmatrix} \\
&= \begin{bmatrix} XC_{zi}^{T} + Y_j^{T}D_{zi}^{T} - \lambda_1\mu_1^{-1}\bar{S}C_{zdi}^{T} \\ \mu_1^{-1}\bar{S}C_{zdi}^{T} \end{bmatrix} \\
&= \Pi_{3ij}^{T}
\end{aligned} \tag{4.32}$$

Now, let us consider the equation

$$\begin{aligned}
U_{ij} &= \widetilde{M}^{-T}H_{ij}\widetilde{M}^{-1} \\
&= \widetilde{A}_{ij}\widetilde{M}^{-1} + \widetilde{M}^{-T}\widetilde{A}_{ij}^{T} + \widetilde{M}^{-T}diag\left\{S, -(1-\beta)S\right\}\widetilde{M}^{-1}
\end{aligned} \tag{4.33}$$

$$\widetilde{M}^{-T}\widetilde{A}_{ij}^{T} = \begin{bmatrix} X & -\lambda_1\mu_1^{-1}\bar{S} \\ 0 & \mu_1^{-1}\bar{S} \end{bmatrix} \begin{bmatrix} A_{ij}^T & I \\ A_{di}^T & -I \end{bmatrix}$$

$$= \begin{bmatrix} XA_{ij}^T - \lambda_1\mu_1^{-1}\bar{S}A_{di}^T & X + \lambda_1\mu_1^{-1}\bar{S} \\ \mu_1^{-1}\bar{S}A_{di}^T & -\mu_1^{-1}\bar{S} \end{bmatrix} \quad (4.34)$$

$$\widetilde{M}^{-T} diag\,\{S, -(1-\beta)S\}\,\widetilde{M}^{-1} =$$

$$= \begin{bmatrix} X & -\lambda_1\mu_1^{-1}\bar{S} \\ 0 & \mu_1^{-1}\bar{S} \end{bmatrix} \begin{bmatrix} S & 0 \\ 0 & -(1-\beta)S \end{bmatrix} \begin{bmatrix} X & 0 \\ -\lambda_1\mu_1^{-1}\bar{S} & \mu_1^{-1}\bar{S} \end{bmatrix}$$

$$= \begin{bmatrix} XSX - \lambda_1^2\mu_1^{-2}(1-\beta)\bar{S} & \lambda_1\mu_1^{-2}(1-\beta)\bar{S} \\ \lambda_1\mu_1^{-2}(1-\beta)\bar{S} & -\mu_1^{-2}(1-\beta)\bar{S} \end{bmatrix} \quad (4.35)$$

Thus we have

$$U_{ij} = \begin{bmatrix} \tilde{u}_{11} & X + \lambda_1\mu_1^{-1}\bar{S} + \mu_1^{-1}A_{di}\bar{S} + \lambda_1\mu_1^{-2}(1-\beta)\bar{S} \\ * & -2\mu_1^{-1}\bar{S} - \mu_1^{-2}(1-\beta)\bar{S} \end{bmatrix} \quad (4.36)$$

where

$$\tilde{u}_{11} = XA_{ij}^T + A_{ij}X + XSX - \lambda_1^2\mu_1^{-2}(1-\beta)\bar{S}$$
$$-\lambda_1\mu_1^{-1}\bar{S}A_{di}^T - \lambda_1\mu_1^{-1}A_{di}\bar{S}$$

Pre-and postmultiplying (4.20) by T yields

$$T\Psi_{ij}T = \begin{bmatrix} U_{ij} + U_{ji} & \bar{\sigma}(\Pi_{1ij}^T + \Pi_{1ji}^T) & 2\begin{bmatrix} 0 \\ I \end{bmatrix} W^{-1} & \Pi_{3ij}^T + \Pi_{3ji}^T \\ * & -2\bar{\sigma}W^{-1} & 0 & 0 \\ * & * & * - 2W^{-1} & 0 \\ * & * & * & -2I \end{bmatrix} < 0 \quad (4.37)$$

$$
U_{ij} + U_{ji} = \begin{bmatrix} \tilde{v}_{11} & 2(X + \lambda_1 \mu_1^{-1}\bar{S} + \lambda_1 \mu_1^{-2}(1-\beta)\bar{S}) + \mu_1^{-1}(A_{di} + A_{dj})\bar{S} \\ * & -4\mu_1^{-1}\bar{S} - 2\mu_1^{-2}(1-\beta)\bar{S} \end{bmatrix}
$$

where

$$
\begin{aligned}
\tilde{v}_{11} &= X(A_{ij} + A_{ji})^T + (A_{ij} + A_{ji})X + 2XSX - 2\lambda_1^2 \mu_1^{-2}(1-\beta)\bar{S} \\
&\quad -\lambda_1 \mu_1^{-1}\bar{S}(A_{di}^T + A_{dj}^T) - \lambda_1 \mu_1^{-1}(A_{di} + A_{dj})\bar{S}
\end{aligned}
$$

Also we have the following simple equality:

$$
\begin{aligned}
XA_{ij}^T + A_{ij}X + XA_{ji}^T + A_{ji}X &= X(A_{ij} + A_{ji})^T + (A_{ij} + A_{ji})X \\
&= X(A_i + A_j)^T + Y_j^T B_i^T + Y_i^T B_j^T \\
&\quad +(A_i + A_j)X + B_i Y_j + B_j Y_i
\end{aligned}
$$

Thus we obtain by using the inequality (4.37) and by the Schur Complement,

$$
\begin{bmatrix} \Phi_{11} & \Phi_{12} & \bar{\sigma}\Phi_{13} & 0 & \Phi_{15} & 2X \\ * & \Phi_{22} & \bar{\sigma}\Phi_{23} & 2\bar{W} & \Phi_{25} & 0 \\ * & * & -2\bar{\sigma}\bar{W} & 0 & 0 & 0 \\ * & * & * & -2\bar{W} & 0 & 0 \\ * & * & * & * & -2I & 0 \\ * & * & * & * & * & -2\bar{S} \end{bmatrix} < 0 \tag{4.38}
$$

where $\bar{W} = W^{-1}$ and

$$
\begin{aligned}
\Phi_{11} &= X(A_i + A_j)^T + (A_i + A_j)X + Y_j^T B_i^T + Y_i^T B_j^T + B_i Y_j + B_j Y_i \\
&\quad -2\lambda_1^2 \mu_1^{-2}(1-\beta)\bar{S} - \lambda_1 \mu_1^{-1}\bar{S}(A_{di}^T + A_{dj}^T) - \lambda_1 \mu_1^{-1}(A_{di} + A_{dj})\bar{S} \\
\Phi_{12} &= 2(X + \lambda_1 \mu_1^{-1}\bar{S} + \lambda_1 \mu_1^{-2}(1-\beta)\bar{S}) + \mu_1^{-1}(A_{di} + A_{dj})\bar{S} \\
\Phi_{22} &= -4\mu_1^{-1}\bar{S} - 2\mu_1^{-2}(1-\beta)\bar{S} \\
\Phi_{13} &= XA_i^T + XA_j^T + Y_i^T B_j^T + Y_j^T B_i^T - \lambda_1 \mu_1^{-1}\bar{S}A_{di}^T - \lambda_1 \mu_1^{-1}\bar{S}A_{dj}^T \\
\Phi_{23} &= \mu_1^{-1}\bar{S}A_{di}^T + \mu_1^{-1}\bar{S}A_{dj}^T \\
\Phi_{15} &= XC_{zi}^T + XC_{zj}^T + Y_i^T D_{zj}^T + Y_j^T D_{zi}^T - \lambda_1 \mu_1^{-1}\bar{S}C_{zdi}^T - \lambda_1 \mu_1^{-1}\bar{S}C_{zdj}^T \\
\Phi_{25} &= \mu_1^{-1}\bar{S}C_{zdi}^T + \mu_1^{-1}\bar{S}C_{zdj}^T
\end{aligned}
$$

Besides that, we obtain

$$
\begin{aligned}
J &= \int_0^\infty z^T(t)z(t)dtz \le V(0) \\
J &\le \varphi^T(0)P\varphi(0) + \int_{-\bar{\sigma}}^0 \int_\theta^0 \dot{\varphi}^T(s)W\dot{\varphi}(s)dsd\theta + \int_{-\bar{\sigma}}^0 \varphi^T(s)S\varphi(s)ds \\
&= \delta(\bar{\sigma}) \qquad (4.39)
\end{aligned}
$$

□

Theorem 4.1 can be solved for given scalars $\bar{\sigma}$, β, λ_1 and μ_1. Since there is too many parameters and the stability of the system is dependent on the value of $\bar{\sigma}$, a method to find the value of $\bar{\sigma}$ is needed. In general, the guaranteed cost depends on the given upper bound $\bar{\sigma}$ to some extend. If the estimation of $\bar{\sigma}$ is too large, it must result in very conservative cost bound $\delta(\bar{\sigma})$. It requires larger cost to guarantee the desired system performance. So, we desire the delay upper bound of $\bar{\sigma}$, denoted by $\bar{\sigma}_{max}$ can be estimated, but not given. In this sense, $\bar{\sigma}_{max}$ can show the capability of the state-feedback controller (4.3) in stabilizing the system (4.1) when the conditions in Theorem 4.1 is satisfied. We also desire to get the suboptimal value of the upper bound of guaranteed cost function with respect to $\bar{\sigma}_{max}$. The following two theorems are given for this purpose.

Theorem 4.2. For any given numbers λ_1, μ_1 and β where $\dot{\sigma}(t) \le \beta$ and $\mu_1 \neq 0$, there exist an upper bound $\bar{\sigma}_{max} = \frac{1}{\xi}$ such that for any $0 \le \sigma(t) \le \bar{\sigma}_{max}$, the controller (4.2) can stabilize system (4.4) with guaranteed cost performance $\delta(\bar{\sigma})$, if the following GEVP is feasible for $Y > 0$, $X > 0$, $\bar{S} > 0$, $\bar{W} > 0$ and real matrices Y_i for $1 \le i \le \bar{n}$

$$
\begin{aligned}
&Minimize \quad \xi = \frac{1}{\bar{\sigma}} > 0 \\
&subject \quad to \begin{cases} U < \begin{bmatrix} Y & 0 \\ 0 & 0 \end{bmatrix} \\ Y < \xi V \end{cases} \qquad (4.40)
\end{aligned}
$$

for $1 \le i \le j \le \bar{n}$ where

$$U = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \Phi_{13} \\ * & 0 & 0 & 0 & 0 & \Phi_{23} \\ * & * & 0 & 0 & 0 & 0 \\ * & * & * & 0 & 0 & 0 \\ * & * & * & * & 0 & 0 \\ * & * & * & * & * & -2\bar{W} \end{bmatrix} \tag{4.41}$$

and

$$V = \begin{bmatrix} -\Phi_{11} & -\Phi_{12} & -2X & 0 & -\Phi_{15} \\ * & -\Phi_{22} & 0 & -2\bar{W} & -\Phi_{25} \\ * & * & 2\bar{S} & 0 & 0 \\ * & * & * & 2\bar{W} & 0 \\ * & * & * & * & 2I \end{bmatrix} \tag{4.42}$$

Proof. Consider the inequality (4.5). We exchange column 3 with column 6 and row 3 with row 6. Then we have

$$\begin{bmatrix} \Phi_{11} & \Phi_{12} & 2X & 0 & \Phi_{15} & \bar{\sigma}\Phi_{13} \\ * & \Phi_{22} & 0 & 2\bar{W} & \Phi_{25} & \bar{\sigma}\Phi_{23} \\ * & * & -2\bar{S} & 0 & 0 & 0 \\ * & * & * & -2\bar{W} & 0 & 0 \\ * & * & * & * & -2I & 0 \\ * & * & * & * & * & -2\bar{\sigma}\bar{W} \end{bmatrix} < 0$$

Moreover, we have

$$U < \xi\bar{V}$$

where

$$\bar{V} = \begin{bmatrix} V & 0 \\ 0 & 0 \end{bmatrix}$$

and $\bar{\sigma} = \frac{1}{\xi}$.

Then, in order to search the upper bound $\bar{\sigma}_{max}$, we formulate the following GEVP problem to obtain ξ_{min}:

$$\begin{aligned} Minimize \quad & \xi > 0 \\ subject \quad to \quad & U < \xi \bar{V} \end{aligned}$$

The GEVP function in Matlab LMI-Toolbox will search proper parameters to satisfy $\bar{V} > 0$. Technically, the positivity of $\bar{V}$ is not strictly feasible, but there is simple remedy in Matlab software which consists of replacing the constraints $U < \xi \bar{V}$ by

$$\begin{aligned} U &< \begin{bmatrix} Y & 0 \\ 0 & 0 \end{bmatrix} \\ Y &> 0 \end{aligned}$$

and

$$Y < \xi V$$

Based on Theorem 4.1, we know the T-S fuzzy system (4.4) is asymptotically stable for any $0 < \sigma(t) \leq \bar{\sigma}_{max}$ with guaranteed cost performance $\delta(\bar{\sigma})$ in the form of (4.39) where the upper bound $\bar{\sigma}_{max} = \frac{1}{\xi_{min}}$ is given by the feasible solution of the modified GEVP conditions. Thus, the proof is completed. $\square$

Now, we are interested in finding the least upper bound of $\delta(\bar{\sigma})$ in the form of (4.39). In order to obtain a state-feedback controller (4.2) which can achieve the least upper bound value of guaranteed cost function with respect to $\bar{\sigma}_{max}$, we should solve the following minimization problem:

$$\begin{aligned} Minimize \quad & \delta(\bar{\sigma}_{max}) \\ subject \quad to \quad & (4.5) \end{aligned}$$

However, it is hard to find in general the global minimum of the minimization problem. So, we are going to present an easy method to find a suboptimal value $\delta_{min}(\bar{\sigma}_{max})$ for the bound

Consider

$$\begin{aligned}\delta(\bar{\sigma}_{max}) &= \varphi^T(0)X^{-1}\varphi(0) + \int_{-\bar{\sigma}_{max}}^{0}\int_{\theta}^{0}\dot{\varphi}^T(s)\bar{W}^{-1}\dot{\varphi}(s)dsd\theta \\ &+ \int_{-\bar{\sigma}_{max}}^{0}\varphi^T(s)\bar{S}^{-1}\varphi(s)ds\end{aligned}$$

and let

$$\begin{aligned}\delta_1(\bar{\sigma}_{max}) &= \varphi^T(0)X^{-1}\varphi(0) \\ \delta_2(\bar{\sigma}_{max}) &= \int_{-\bar{\sigma}_{max}}^{0}\int_{\theta}^{0}\dot{\varphi}^T(s)\bar{W}^{-1}\dot{\varphi}(s)dsd\theta \\ \delta_3(\bar{\sigma}_{max}) &= \int_{-\bar{\sigma}_{max}}^{0}\varphi^T(s)\bar{S}^{-1}\varphi(s)ds\end{aligned}$$

Now, we define three positive-definite matrices $\digamma_1$, $\digamma_2$ and $\digamma_3$:

$$\begin{aligned}\digamma_1 &= \varphi(0)\varphi^T(0) \\ \digamma_2 &= \int_{-\bar{\sigma}_{max}}^{0}\int_{\theta}^{0}\dot{\varphi}(s)\dot{\varphi}^T(s)dsd\theta \\ \digamma_3 &= \int_{-\bar{\sigma}_{max}}^{0}\varphi(s)\varphi^T(s)ds\end{aligned}$$

By using $tr(AB) = tr(BA)$, we have

$$\begin{aligned}\delta_1(\bar{\sigma}_{max}) &= \varphi^T(0)X^{-1}\varphi(0) \\ &= tr(\digamma_1 X^{-1}) \\ &= tr(\digamma_1^{\frac{1}{2}}X^{-1}\digamma_1^{\frac{1}{2}})\end{aligned}$$

$$\begin{aligned}
\delta_2(\bar{\sigma}_{max}) &= \int_{-\bar{\sigma}_{max}}^{0}\int_{\theta}^{0}\dot{\varphi}^T(s)\bar{W}^{-1}\dot{\varphi}(s)dsd\theta \\
&= tr(\mathcal{F}_2\bar{W}^{-1}) \\
&= tr(\mathcal{F}_2^{\frac{1}{2}}\bar{W}^{-1}\mathcal{F}_2^{\frac{1}{2}})
\end{aligned}$$

$$\begin{aligned}
\delta_3(\bar{\sigma}_{max}) &= \int_{-\bar{\sigma}_{max}}^{0}\varphi^T(s)\bar{S}^{-1}\varphi(s)ds \\
&= tr(\mathcal{F}_3\bar{S}^{-1}) \\
&= tr(\mathcal{F}_3^{\frac{1}{2}}\bar{S}^{-1}\mathcal{F}_3^{\frac{1}{2}})
\end{aligned}$$

Now, let Σ_1 be a positive-definite matrix variable satisfying

$$\mathcal{F}_1^{\frac{1}{2}}X^{-1}\mathcal{F}_1^{\frac{1}{2}} < \Sigma_1 \tag{4.43}$$

By the Schur complement (4.43) is equivalent to

$$\begin{bmatrix} -\Sigma_1 & \mathcal{F}_1^{\frac{1}{2}} \\ \mathcal{F}_1^{\frac{1}{2}} & -X \end{bmatrix} < 0 \tag{4.44}$$

If there exists one Σ_1 satisfying (4.44), then $\delta_1(\bar{\sigma}_{max}) < tr(\Sigma_1)$.

By the same way, to get the upper bound of $\delta_2(\bar{\sigma}_{max})$, we introduce another matrix variable Σ_2 to satisfy

$$\begin{bmatrix} -\Sigma_2 & \mathcal{F}_2^{\frac{1}{2}} \\ \mathcal{F}_2^{\frac{1}{2}} & -\bar{W} \end{bmatrix} < 0 \tag{4.45}$$

so that $\delta_2(\bar{\sigma}_{max}) < tr(\Sigma_2)$ holds.

Similarly, we introduce another matrix variable Σ_3 to satisfy

$$\begin{bmatrix} -\Sigma_3 & \digamma_3^{\frac{1}{2}} \\ \digamma_3^{\frac{1}{2}} & -\bar{S} \end{bmatrix} < 0 \tag{4.46}$$

so that $\delta_3(\bar{\sigma}_{max}) < tr(\Sigma_3)$ holds.

We change the minimization problem of $\delta(\bar{\sigma}_{max})$ into the minimization problem

$$\begin{aligned} &\qquad Minimize \quad \{tr(\Sigma_1) + tr(\Sigma_2) + tr(\Sigma_3)\} \\ &subject \quad to \quad (4.5), (4.44), (4.45) \quad and \quad (4.46) \end{aligned}$$

Thus we have proven the following theorem:

Theorem 4.3. For a given constant upper bound $\bar{\sigma}_{max}$ and some given scalars $\mu_1 \neq 0$, λ_1 and β, suppose that the optimization problem

$$\begin{aligned} &\qquad Minimize \quad \{tr(\Sigma_1) + tr(\Sigma_2) + tr(\Sigma_3)\} \\ &subject \quad to \quad (4.5), (4.44), (4.45) \quad and \quad (4.46) \end{aligned}$$

has solutions $X > 0$, $\bar{W} > 0$, $\bar{S} > 0$ and Y_i for $1 \leq i \leq j \leq \bar{n}$ where

$$\begin{aligned} \digamma_1 &= \varphi(0)\varphi^T(0) \\ \digamma_2 &= \int_{-\bar{\sigma}_{max}}^{0} \int_{\theta}^{0} \dot{\varphi}(s)\dot{\varphi}^T(s) ds d\theta \\ \digamma_3 &= \int_{-\bar{\sigma}_{max}}^{0} \varphi(s)\varphi^T(s) ds \end{aligned}$$

then the guaranteed cost controller (4.2) is suboptimal and the upper bound on the closed-loop cost function (1.2) is minimal with respect to μ_1 and λ_1.

5. APPLICATION

In this chapter, an application of the developed method to a T-S fuzzy model with time-delay is given. In this model, the time-delay is constant. We set $\sigma(t) = \sigma$, that is, $\sigma(t)$ is a constant function. The example is first presented in [1]. It will be shown in the example that the methods in the previous chapter gives better results than the methods presented in [1].

Example 5.1. Consider the unstable nonlinear system with the following differential equation:

$$\ddot{s}(t) + f(s(t), \dot{s}(t)) - 0.1s(t) = F(t)$$

where

$$f(s(t), \dot{s}(t)) = 0.5s(t) + 0.75\sin(\frac{\dot{s}(t)}{0.5})$$

Choose the state variable and the input variable as $x(t) = [s(t) \quad \dot{s}(t)]^T$ and $u(t) = F(t)$, respectively. We assume that the delay state matrix is

$$A_d = \begin{bmatrix} 0.1 & 0 \\ 0.1 & -0.2 \end{bmatrix}$$

Also we assign the time-varying delay function as a constant function and assume that $\sigma(t) = \sigma = 0.5$ is the constant delay assumed to be unknown. Here, we use the same T-S fuzzy model representation of the system as in [1]:

Rule 1 :

$$IF \quad \tfrac{x_2(t)}{0.5}\, is \quad about \quad 0$$
$$THEN \quad \dot{x}(t) = A_1x(t) + A_{d1}x(t-\sigma) + B_1u(t)$$

Rule 2 :

$$IF \quad \tfrac{x_2(t)}{0.5}\, is \quad about \quad \pi \quad or \quad -\pi$$
$$THEN \quad \dot{x}(t) = A_2x(t) + A_{d2}x(t-\sigma) + B_2u(t)$$

where

$$A_1 = \begin{bmatrix} 0 & 1 \\ 0.1 & -2 \end{bmatrix} \quad A_2 = \begin{bmatrix} 0 & 1 \\ 0.1 & -0.5 - 1.5 \cdot \alpha \end{bmatrix}$$

$$B_1 = B_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad A_{d1} = A_{d2} := A_d$$

and $\alpha = \frac{0.01}{\pi}$. Here, α is used to avoid system matrices being singular.

We choose the other parameters in model (1.1) as follows:

$$C_{z1} = C_{z2} = \begin{bmatrix} 0.02 & -0.03 \end{bmatrix}$$

$$C_{zd1} = C_{zd2} = \begin{bmatrix} 0.03 & 0.01 \end{bmatrix} \quad D_{z1} = D_{z2} = 0.1$$

The membership functions are set as

$$\begin{aligned} h_1(t) &= (1 - \frac{1}{1 + \exp\{-3(\frac{x_2}{0.5} - \frac{\pi}{2})\}}) \times (\frac{1}{1 + \exp\{-3(\frac{x_2}{0.5} + \frac{\pi}{2})\}}) \\ h_2(t) &= 1 - h_1(t) \end{aligned}$$

Table 5.1: Guaranteed cost values

$\bar{\sigma}$	$\delta(\bar{\sigma})$		$\delta_{min}(\bar{\sigma})$	
	Theorem 4.1	[1]	Theorem 4.3	[1]
0.5	0.0905	0.3068	0.0037	0.2773
1.5	0.2971	0.3367	0.0085	0.3145
2.5	0.3032	19.9974	0.0840	1.4074
3.0	0.3370	58.2634	0.0975	3.8431
3.7836	1.6279	72.3411	0.2741	10.9987

Table 5.1 shows the comparison results of $\delta(\bar{\sigma})$ based on Theorem 4.1, Theorem 4.3 and the corresponding methods in [1]. The $\delta(\bar{\sigma})$ and the $\delta_{min}(\bar{\sigma})$ in Table 5.1 can be obtained by Theorem 4.1 and Theorem 4.3 for given initial condition $\varphi(t) = [1.8 \quad 0.5]^T$, respectively.

The parameters that are used to obtain the values in Table 5.1 are given in Table 5.2. Note that, since the delay is a constant function, the values in Table 5.1 are obtained for $\beta = 0$.

Table 5.2: The parameters that are used to find $\delta(\bar{\sigma})$ and $\delta_{min}(\bar{\sigma})$.

$\bar{\sigma}$	μ_1	λ_1
0.5	80	-4
1.5	80	-1
2.5	80	-1.3
3.0	80	-1.1
3.7836	80	-1.03

The feedback gains are determined based on Theorem 4.1 for $\bar{\sigma} = 0.5$ and $\varphi(t) = [1.8 \quad 0.5]^T$ as follows:

$$\begin{aligned} K_1 &= \begin{bmatrix} -0.6453 & 0.8598 \end{bmatrix} \\ K_2 &= \begin{bmatrix} -0.6484 & -0.5870 \end{bmatrix} \end{aligned}$$

The feedback gains are determined based on Theorem 4.3 for $\bar{\sigma} = 0.5$ and $\varphi(t) = [1.8 \quad 0.5]^T$ as follows:

$$\begin{aligned} K_1 &= \begin{bmatrix} -0.5087 & 0.2551 \end{bmatrix} \\ K_2 &= \begin{bmatrix} -0.5087 & 0.1199 \end{bmatrix} \end{aligned}$$

The guaranteed cost values get larger as $\bar{\sigma}$ is chosen to be larger. Thus the system time-response performance can be guaranteed by degradation of some other performance, such as the cost function J.

Also it can be easily seen in Table 5.1 that the results based on Theorem 4.1, Theorem 4.3 are better than the results based on the methods in [1].

Here, the same problem is considered for time-varying delay case. The following results in Table 5.3 are obtained for $\beta = 2$ with the same parameters in Table 5.2:

Table 5.3: The guaranteed cost values for $\beta = 2$.

$\bar{\sigma}$	$\delta(\bar{\sigma})$	$\delta_{min}(\bar{\sigma})$
0.5	0.1837	0.0062
1.5	0.3151	0.0091
2.5	0.3666	0.0952
3.0	4.7095	0.1103
3.7836	2.1204	0.3635

6. CONCLUSIONS

In the present study, we consider the delay-dependent guaranteed cost control problem for nonlinear systems with time-varying delay which can be represented by T-S fuzzy fuzzy model with time-varying delay. A guaranteed cost function that measured by controlled output is considered and the sufficient conditions for the solution are obtained by defining a suitable Lyapunov-Krasovskii functional. These conditions are presented in terms of LMIs, depending on the size of time delay. The state feedback control law is defined via PDC technique.

In addition, the upper bound of time-varying delay and the suboptimal upper bound of the guaranteed cost are given by GEVP method and a suboptimal value searching method, respectively.

In this study, no model transformation is used and also the restriction $\beta < 1$ given on the derivative of the time-varying delay is removed. The removal of this restriction allows fast time varying delays.

The presented method is also compared with other methods in the literature given for the constant delay case. It is shown that the results based on the theorems in this thesis are much better than the previous ones.

Further studies on the stability analysis of delay-dependent cost control for T-S fuzzy model with time-varying delay may be considered based on the output feedback controller. Also the studies may be extended to the case of the delay in the input vectors.

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