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THE CHARACTERS OF S_6 AND S_7

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PREFACE

I owe a great dept to my supervisor Prof.Dr. Kadir AHRE, and I must thank him for his invaluable comments on the thesis. Thank are also due to Dr. Salih CELIK for typing the thesis.

CONTENTS

SUMMARY

This work is about matrix representations and characters of group, espectially the symmetric groups of degree 6 and 7,namely S_6 and S_7. Chapter I gives a brief history of the subject. Chapter II gives the very basic definitions such as group representations, characters, reducibility and complete reducibility. Most of the theory was invented by Frobenius, and Schur-Maschke's theorem on complete reducibility is also very important.

After having defined the character notion, in the previous chapter, chapter III gives the important properties characters which will clarify the importance of group characters in the study of group representations.

Then the work goes on with a brief recall of some basic information on symmetric group and the theory of group representations linked with symmetric groups by the generating functions of Frobenius and Schur which give the value of irreducible characters of symmetric groups as coefficients.

The concluding chapter is devoted to the study of some special Schur functions which will be used to construct the character table of S_6 and S_7.

OZET

S_6 VE S_7 NIN KARAKTERLERI

Bu tez, pratikte relativite ve kuantum teorisi gibi alanlarda yaygın bir şekilde kullanılan matris temsillerinin grup karakteri yoluyla incelenmektedir. Tezin amacı, S_6 ve S_7 simetrik gruplarının bütün karakterlerinin elde edilebileceği karakter tablolarını üretmektir.

Konunun hazırlanmasında kullanılan temel kaynaklar Ledermann [1] ve Murnaghan [2] tarafından verilmiştir. Keown [3] ve Littlewood [4] un yaklaşımları da yol gösterici olmuştur.

G boş olmayan bir grup ve $x \in G$ olmak üzere

$$A(x) = [a_{ij}(x)] \quad (i,j = 1,2,...,m)$$

olacak şekildeki katsayıların bir K cisminden seçildiği m-inci mertebeden bir singüler olmayan matris bulunabiliyorsa ve

$$A(x)A(y) = A(xy), \quad (x,y \in G)$$

şartı sağlanıyorsa, A(x) e G nin K üzerinde m-inci dereceden bir temsili denir. Simetrik grup üzerinde tanımlayabileceğimiz iki basit temsil aşağıdadır.

1) $\delta(x) = \begin{cases} 1, & \text{eğer } x \text{ çift ise} \\ 0, & \text{eğer } x \text{ tek ise} \end{cases}$

2) $N(x) = [\delta_{x_{ij}}], \quad \delta_{x_{ij}} = \begin{cases} 1 & \text{eğer } i = x_j \\ 0 & \text{eğer } i \neq x_j \end{cases}.$

Bu temsillerden [1] 2-incisine doğal temsil adı verilir.
A(x) ve B(x), G grubunun aynı dereceden K cismi üzerinde tanımlı iki matris temsili olsun. Katsayılarını K dan alan, singüler olmayan ve

$$B(x) = T^{-1}A(x)T$$

şartını sağlayan bir T sabit matrisi bulunabiliyorsa, A(x) ve B(x) temsillerini denk kabul edeceğiz ve bunu

$$A(x) \approx B(x)$$

şeklinde göstereceğiz.

A(x) bir matris temsili olmak üzere

$$\phi(x) = trA(x)$$

A(x) matrisinin izini verir ve A(x) matrisinin karakteri diye adlandırılır. Denk temsillerin karakterleri aynıdır ve grup manasında konjuge olan elemanların aynı temsil için karakterleri aynıdır[1]. Bu teorem 2.1 de ifade edilmiştir ve karakterlerin temsillerinin incelenmesinde önemli bir rol oynar.

Konu matris temsilleri olduğuna ve matrislerin üçgen formda yazılabilmesinin özellikle determinant hesabında kolaylık sağladığına göre, matris temsillerinin bu şekilde yazılıp yazılamadığını bilmenin faydası açıktır.

Herhangi bir A(x) temsili, eğer uygun şartlardaki bir T matrisi tarafından, her $x \in G$ için

$$B(x) = T^{-1}A(x)T = \begin{bmatrix} C(x) & 0 \\ D(x) & E(x) \end{bmatrix}$$

şekline getirilebilirse, A(x) matrisine, K üzerinde indirgenebilir, aksi takdirde indirgenemez denir.

Herhangi bir temsil, ya indirgenemezdir ya da köşegen elemanları indirgenemez temsiller olacak şekilde bir üçgen matris şeklinde yazılabilir[1]. Bu, temsillerin irdelenmesinde kolaylık sağlayacaktır. Fakat köşegen dışındaki terimler de yok edilebilseydi bu daha da faydalı olurdu.

Maschke'nin teoremi, bunun, her zaman olmasa da çok genel şartlarda bunun mümkün olduğunu göstermektedir [1]. Bu teoremin daha genel bir şekli, tam olarak, G boş olmayan sonlu bir grup, K karakteristiği 0 ya da $|G|$ ile aralarında asal olan bir cisim ise, G nin K cismi üzerindeki her matris temsilinin bu şekilde yazılabileceğini göstermektedir [1].

Grup karakterlerinin özelliklerinin incelenmesi için,

$\phi(x)$ ve $\psi(x)$ karakterleri için

$$\langle\phi,\psi\rangle = \frac{1}{g} \sum_{x\in G} \phi(x)\psi(x^{-1})$$

şeklinde tanımlı bir $\langle\phi,\psi\rangle$ iç çarpımı tanımlanmıştır.

Eğer A(x) ve B(x), bir G grubunun iki indirgenemez temsili ve $\chi(x)$ ve $\chi'(x)$, sırasıyla, bu temsillerin karakter fonksiyonları ise, birinci dereceden karakter bağıntıları

$$\langle\chi,\chi'\rangle = \begin{cases} 1 & \text{eğer } A(x) \approx B(x) \text{ ise} \\ 0 & \text{aksi takdirde} \end{cases}$$

şeklinde ifade edilir. Bu, temsillerin karakterlerinin incelenmesinde indirgenemez temsillerin karakterlerinin ortonormal bir taban oluşturduğunu gösterir[1]. Herhangi bir temsilin karakteri, Fourier analizinde kullanılana benzer bir teknikle indirgenemez temsillerin karakterlerinin lineer bir kombinasyonu olarak yazılabilir. Bir grubun k tane konjuge sınıfı varsa, k tane denk olmayan temsili vardır[1]. Bu teorem grubun konjuge sınıfları ile temsillerini birbirine bağlar.

n>1 olmak üzere S_n , n-inci dereceden simetrik grubu temsil etsin. Simetrik gruplar, üzerinde en çok çalışılmış grup türlerindendir. S_n nin konjuge sınıflarının sayısı, n sayısının parçalanışları sayısı kadardır[2]. O halde, S_n nin indirgenemez temsillerinin sayısı n nin parçalanışları sayısına eşittir. Bu gerçek, grup temsilleri gibi, nispeten yeni bir konuyu sayılar kadar eski olan sayılar teorisiyle bağlamaktadır.

Frobenius, grup karakterlerinin 1896 yılında bulunmasından 4 yıl sonra, bulduğu üreten fonksiyonlar ile S_n simetrik grubunun karakterlerini en azından prensip olarak bulma başarısını gösterdi. Prensip olarak diyoruz, çünki Frobenius'un fonksiyonları çok fazla hesaplama içerirler ve n nin büyük değerleri için hesaplama miktarı korkunç boyutlara ulaşır. Bu sebeple S_6 ve S_7 nin karakterlerini hesaplamak için Schur'un Frobenius'tan bir kaç yıl sonra geliştirdiği Schur'un üreten fonksiyonları (veya s-fonksiyonları) kullanılmıştır.

Bu metodun tercih sebeplerinden biri de herhangi bir

$n>1$ doğal sayısı için indirgenemeyen karakterlerin hesaplanmasında, değişik dereceden simetrik grupların karakterlerinin kullanılmasının gerekmemesidir[4].

Bölüm 5, S_6 ve S_7 nin karakterlerini verecek olan formüllerin s-fonksiyonları yardımı ile üretilmesine ayrılmıştır.

İlk olarak 2 eleman içeren parçalanışları verecek formüller λ parçalanışını $(\lambda) = (\lambda_1,\lambda_2)$; $\lambda_1 + \lambda_2 = n$ olacak şekilde düşünülecek olursa Schur'un fonksiyonları $\chi^{[\lambda_1,\lambda_2]}$ karakter fonksiyonunun değerleri

$$\begin{vmatrix} w_{\lambda_1} & w_{\lambda_1+1} \\ w_{\lambda_2-1} & w_{\lambda_2} \end{vmatrix}$$

determinantının açılımındaki katsayılar şeklinde elde edilir. Bu ifadenin

$$w_{\lambda_1} w_{\lambda_2} - w_{\lambda_1+1} w_{\lambda_2-1}.$$

şeklinde düşünülerek açılması ve gerekli sadeleştirmelerin yapılması sonucu

$$\chi_{(\alpha)}^{[\lambda_1,\lambda_2]} = \sum_{(\tau)} \begin{bmatrix} \alpha_1 \\ \tau_1 \end{bmatrix} \cdots \begin{bmatrix} \alpha_n \\ \tau_n \end{bmatrix} - \sum_{(\sigma)} \begin{bmatrix} \alpha_1 \\ \sigma_1 \end{bmatrix} \cdots \begin{bmatrix} \alpha_n \\ \sigma_n \end{bmatrix}$$

ana formulü elde edilir. Burada (σ) $\lambda_2 - 1$ inci mertebeden simetrik gurubun ve $(\tau_1,\tau_2,\ldots,\tau_n)$ ise (λ_2) nin bir parçalanış sınıfıdır.

Bu ana formulün irdelenmesi sonucu

$$\chi_{\alpha}^{[\lambda_1,\lambda_2]} = \sum_{(\sigma)} \left[\frac{\alpha_1 - 2\sigma_1 - 1}{\sigma_1 + 1}\right] \begin{pmatrix} \alpha_1 \\ \sigma_1 \end{pmatrix} \cdots \begin{pmatrix} \alpha_n \\ \sigma_n \end{pmatrix} + \sum_{(\beta)} \begin{pmatrix} \alpha_2 \\ \beta_2 \end{pmatrix} \cdots \begin{pmatrix} \alpha_n \\ \beta_n \end{pmatrix}.$$

formulü elde edilir. Burada (β) λ_2 inci dereceden simetrik gurubun sabit terimsiz bir parçalanışıdır.

Bu formulün S_6 ve S_7 de karşılaşabileceğimiz durumlara uygulanması sonucu iki elemanlı parçalanışlar için şu formulleri elde ederiz.

$$\chi^{[n-1,1]} = \alpha_1 - 1$$

$$\chi^{[n-2,2]} = \frac{1}{2}\alpha_1(\alpha_1 - 3) + \alpha_2 ,$$

$$\chi^{[n-3,3]}_{(\alpha)} = = \frac{1}{6}\alpha_1(\alpha_1 - 1)(\alpha_1 - 5) + \alpha_2(\alpha_1 - 1) + \alpha_3 ,$$

$$\chi^{[n-4,4]}_{(\alpha)} = = \frac{1}{24}\alpha_1(\alpha_1-1)(\alpha_1-2)(\alpha_1-7) + \frac{1}{2}\alpha_1(\alpha_1-3)\alpha_2 + $$

$$+ (\alpha_1 - 1)\alpha_3 + \frac{1}{2}\alpha_2(\alpha_2 - 1) + \alpha_4 .$$

Yukarıdaki eşitlikler, inceleyeceğimiz S_6 ve S_7 grupları için yeterlidir.

CHAPTER 1

1. Introduction

The three outstanding names in the theory of group representations are Frobenius, Schur and Weyl.It may be claimed that Maschke's theorem and Schur's lemma are two pillars on which the edifice of representation theory rests. The group characters discovered in 1896 and only 4 years later, Frobenius succeeded in obtaining all the characters of S_n. More precisely, he constructed a set of generating functions whose coefficients reveal the full character table. It must, however, be admitted that the expansion of the generating functions tends to be cumbersome. A few years later, Issai Schur developed an alternative version of the character theory for the symmetric group. He constructed a set of generating functions which, in a sense, are dual to those of Frobenius. These Schur Functions have a great deal of attention as they play an important part in the more advanced study of the symmetric group.

This abstract is prepared as follows:

Chapter 2 gives some basic information on group representations and characters.

Character 3 gives the elementary properties of group characters to show their importance and simplicity in the study of group representations.

Chapter 4 introduces the main results of Frobenius theory and Schur's functions together with some basic information on symmetric groups.

Chapter 5 utilizes chapter 4 in order to construct the full character table of S_6 and S_7, the symmetric group on 6 and 7 letters. Here we will get some formulae giving the value of any simple character of S_n in any class C_α of S_n where $\alpha : \alpha_1 + 2\alpha_2 + \ldots + n\alpha_n = n$ is a partition of n.

CHAPTER 2

2. Group Representations and Characters

2.1. Group Representations

Definition 2.1.1. Let G be a nonempty group and suppose that with each element x of the group G there is associated an m by m non-singular matrix

$$A(x) = \left[a_{ij}(x) \right] \quad (i,j = 1,2,\ldots,m) \tag{2.1}$$

with coefficients in the field K, in such a way that

$$A(x)A(y) = A(xy) \ , \ \forall x,y \in G. \tag{2.2}$$

Then $A(x)$ is called a matrix representation of G of degree(dimension) m over K.

Definition 2.1.2. Every group possesses the trivial representation given by the constant function

$$\lambda(x) = 1 \quad (x \in G). \tag{2.3}$$

Clearly this constitutes a representation.

Definition 2.1.3. A non-trivial example of a linear representation is furnished by the alternating character of the symmetric group S_n (for each $n>1$). This is defined by

$$\zeta(x) = \begin{cases} 1 & \text{if } x \text{ is even} \\ -1 & \text{if } x \text{ is odd} \end{cases} \tag{2.4}$$

The equation $\zeta(x)\zeta(y) = \zeta(xy)$ expresses a well-known fact about the parity of permutations.

Definition 2.1.4. (Permutation Representations)

Let G be a permutation group of degree m, that is a subgroup of S_m , possibly the whole of S_m , and let

$$x = \begin{bmatrix} 1 & 2 & \dots & m \\ x_1 & x_2 & \dots & x_m \end{bmatrix} \qquad (2.5)$$

be a typical element of G where $(x_1, x_2, \dots, x_m)$ is an ordering of $1, 2, \dots, m$.

We define a matrix representation

$$N(x) = \left[\delta_{x_{ij}} \right] \qquad (2.6)$$

where it means that in the ith row jth element is 1 if $i = x_j$, else 0.

As an example if

$$x = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 3 & 2 \end{bmatrix}$$

then

$$N(x) = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

<u>Lemma</u> 2.1. N(x) is a representation,which we call the natural representation of G [1].

It is easy to find the natural charecter $\nu(x)$ of G we will find the trace of N(x). But we get 1 on the trace iff $i = x_i$ for some $1 \le i \le m$, that means iff i is fixed by x. Hence here,

$$\nu(x) = \text{number of object fixed by } x. \qquad (2.7)$$

<u>Definition</u> 2.1.5. (Equivalent Representations)

Let A(x) be a representation of G and suppose that

$$B(x) = T^{-1}A(x)T \qquad (2.8)$$

where T is a fixed non-singular matrix with coefficients in K. Clearly

$$B(x)B(y) = T^{-1}A(x)TT^{-1}A(y)T$$
$$= T^{-1}A(x)A(y)T. \qquad (2.9)$$

But since A(x) is a group representation

$$A(x)A(y) = A(xy) \quad (\forall x,y) \qquad (2.10)$$

So we get

$$B(x)B(y) = T^{-1}A(xy)T$$
$$= B(xy) \qquad (2.11)$$

so that B(x), too, is a representation of G. We say that the representations A(x) and B(x) are equivalent over K and we write

$$A(x) \approx B(x). \qquad (2.12)$$

As a rule, we do not distinguish between equivalent representations, that is we are interested in equivalence classes of representations.

2.2. Characters

Let $A(x) = \left[a_{ij}(x) \right]$ be a matrix representation of G of degree m. We consider the charecteristic polynomial of $A(x)$, namely

$$\det(\lambda I - A(x)) = \begin{vmatrix} \lambda - a_{11}(x) & -a_{12}(x) & \cdots & -a_{1m}(x) \\ -a_{21}(x) & \lambda - a_{22}(x) & \cdots & -a_{2m}(x) \\ \cdots & \cdots & & \cdots \\ -a_{m1}(x) & -a_{m2}(x) & \cdots & \lambda - a_{mm}(x) \end{vmatrix}$$

This is a polynomial of degree m in λ, and when we expand this determinant, the coefficient of λ^{m-1} is equal to

$$\phi(x) = a_{11}(x) + a_{22}(x) + \ldots + a_{mm}(x). \tag{2.13}$$

The right-hand side is called the trace of the matrix $A(x)$, abbreviated to $trA(x)$, so that

$$\phi(x) = trA(x). \tag{2.14}$$

<u>Definition</u> 2.2.1. $\phi(x)$ is a function on G with values in K, and it is called the charecter of $A(x)$.

Now if $B(x) = T^{-1}A(x)T$ is a representation equivalent to $A(x)$, then

$$\det(\lambda I - B(x)) = \det(T^{-1}\lambda IT - T^{-1}A(x)T).$$

Then $T^{-1}\lambda IT = \lambda I$. Hence

$$\begin{aligned} \det(\lambda I - B(x)) &= \det\left[T^{-1}[\lambda I - A(x)]\, T\right] \qquad (2.15) \\ &= (\det T^{-1}).(\det(\lambda I - A(x)).\det T \end{aligned}$$

Since T is non-singular,

$$\det T^{-1} = \frac{1}{\det T} .$$

So we get

$$\det(\lambda I - B(x)) = \det(\lambda I - A(x)).$$

In particular on comparing the coefficients of λ^{m-1} in the above equation, we find that

$$b_{11}(x)+b_{22}(x)+ \dots +b_{mm}(x) = a_{11}(x)+a_{22}(x)+ \dots +a_{mm}(x).$$

That is, equivalent representations have the same character.

Theorem 2.2. Let $A(x)$ be a matrix representation of G. Then the character $\phi(x) = trA(x)$ has the following properties [1]:

(i) Equivalent representations have the same character

(ii) If x and y are conjugate in G, then $\phi(x) = \phi(y)$.

2.3. Reducibility

<u>Definition</u> 2.3.1 The matrix representation $A(x)$ is reducible over K if there exists a non-singular matrix T over K such that

$$B(x) = T^{-1}A(x)T = \begin{bmatrix} C(x) & 0 \\ E(x) & D(x) \end{bmatrix} \tag{2.16}$$

for all $x \in G$.

Otherwise, the representation is called irreducible.

<u>Theorem</u> 2.3. Let $A(x)$ be a matrix representation of G of degree m over K. Then either $A(x)$ is irreducible or else

$$A(x) \approx \begin{vmatrix} A_1(x) & 0 & 0 & \cdots & 0 \\ A_{21}(x) & A_2(x) & 0 & \cdots & 0 \\ A_{31}(x) & A_{32}(x) & A_3(x) & \cdots & 0 \\ \cdots & \cdots & \cdots & & \cdots \\ A_{l1}(x) & A_{l2}(x) & A_{l3}(x) & \cdots & A_l(x) \end{vmatrix} \tag{2.17}$$

where $A_1(x), \ldots , A_l(x)$ are irreducible over K [1].

2.4. Complete Reducibility

A reducible matrix representation $A(x)$ can be brought into the triangular shape

$$A(x) = \begin{bmatrix} C(x) & 0 \\ E(x) & D(x) \end{bmatrix}. \qquad (2.18)$$

But it would be more satisfactory if we could remove the off-diagonal block $E(x)$ by a further transformation. If this could be done, then we would have that

$$A(x) \approx \begin{bmatrix} C(x) & 0 \\ 0 & D(x) \end{bmatrix} \qquad (2.19a)$$

or written more concisely

$$A(x) \approx \mathrm{diag}(C(x),D(x)). \qquad (2.19b)$$

Though not always possible, the diagonal form is attainable under very general conditions.

<u>Theorem</u> 2.4 (Maschke's Theorem)

Let G be a finite group of order G and let K be a field whose charecteristic is either 0 or prime to G. Suppose that $A(x)$ is a matrix representation of G over K such that

$$A(x) \approx \begin{bmatrix} C(x) & 0 \\ E(x) & D(x) \end{bmatrix}.$$

Then [1]

$$A(x) \approx \mathrm{diag}(C(x),D(x)).$$

<u>Definition</u> 2.4.1. A matrix representation A(x) over K is said to be completely reducible if

$$A(x) \approx \mathrm{diag}(A_1(x), A_2(x), \ \ldots \ , A_l(x))$$

where $A_i(x)$ $(i = 1,2, \ \ldots \ ,l)$ are irreducible representations over K. So we have a more general version of Maschke's theorem.

<u>Theorem</u> 2.5. Let G be a finite group of order g, and let K be a field whose charecteristic is zero or else prime to g. Then every matrix representation of G over K is completely reducible [1].

<u>Theorem</u> 2.6 (Schur's Lemma)

Let A(x) and B(x) be two irreducible representations over K of a group G, and suppose that there exists a constant matrix T over K such that

$$TA(x) = B(x)T$$

for all $x \in G$. Then

(i) Either T = 0 or

(ii) T is non-singular so that $A(x) = T^{-1}B(x)T$,

which means A(x) and B(x) are equivalent [1].

<u>Corollary to Schur's Lemma</u>:

Let A(x) be an irreducible matrix representation of G over an algebraically closed fixed. Then the only matrices which commute with all the matrices A(x) $(x \in G)$ are the scalar multiples of the unit matrix.

<u>Proof</u>. Suppose that $A(x)$ is irreducible over the algebraically closed field K, and let T be a matrix (necessarily square) over K which satisfies

$$TA(x) = A(x)T$$

for all $x \in G$. Then if k is any scalar (element of K) we have that

$$(kI - T)A(x) = A(x)(T - kI) \quad (x \in G)$$

where I is the identity matrix.

Since K is algebraically closed there exists a scalar k_o such that

$$\det(k_o I - T) = 0$$

because $\det(k_o I - T)$ is a polynomial in K and any algebraically closed field has at least one root for any polynomial with coefficients chosen from the field.

So the matrix $k_o I - T$ is therefore singular. Applying Schur's Lemma to ( ) with $k = k_o$, we conclude that

$$T = k_o I.$$

Conversely, it is obvious that every matrix of the form kI $(k \in K)$ commutes with all the $A(x)$, which completes the proof.

CHAPTER 3

3. Elementary Properties of Group Characters

3.1. Orthogonality Relations

Definition 3.1.1.

Let $\phi(x)$ and $\psi(x)$ be functions where $x \in G$ with values in K and write

$$\langle \phi, \psi \rangle = \frac{1}{g} \sum_{x \in G} \phi(x) \psi(x^{-1}) \tag{3.1}$$

$\langle \phi, \psi \rangle$ is called the "inner product" for functions ϕ and ψ defined on G.

Since the sum is unaltered if we replace x by x^{-1}, it follows that

$$\langle \phi, \psi \rangle = \langle \psi, \phi \rangle. \tag{3.2}$$

We shall say that the functions ϕ and ψ are orthogonal if $\langle \phi, \psi \rangle = 0$.

Theorem 3.1. (Charecter Relations of the 1st kind)

Let $\chi(x)$ and $\chi'(x)$ be the charecters of the irreducible representations A(x) and B(x) respectively.

Then [1]

$$\langle \chi, \chi' \rangle = \begin{cases} 1 & \text{if } A(x) \approx B(x) \\ 0 & \text{if } A(x) \not\approx B(x) \end{cases} . \tag{3.3}$$

Definition 3.1.2.

The charecter of an irreducible representation is called a "simple charecter" while the charecter of a reducible representation is termed "compound" .

If G is a group of order g, then any group function ϕ on G may be regarded as g-tuple, thus

$$\phi = (\phi(x_1),\phi(x_2), \ldots ,\phi(x_g)), \tag{3.4}$$

where x_1 , x_2 , ... , x_g are the elements of G enumerated in some fixed manner.

When viewed as g-tuples, any collection

$$\chi^{(1)} , \chi^{(2)} , \ldots , \chi^{(s)} \tag{3.5}$$

of distinct simple charecters is linearly independent. For suppose that

$$\sum_{i=1}^{s} c_i\chi^{(i)} = 0 \tag{3.6}$$

where the right-hand side denotes the zero g-tuple.

Taking the inner product of this equation with $\chi^{(j)}$ where $1 \leq j \leq s$, we find that

$$\begin{aligned} \sum_{i=1}^{s} c_i<\chi^{(i)},\chi^{(j)}> &= \sum_{i=1}^{s} c_i\delta_{ij} \\ &= c_j \\ &= 0 \end{aligned} \tag{3.7}$$

which proves the linear independence.

It is known from linear algebra that there can be no set comprising more than g-linearly independent g-tuples. Therefore $s \le g$, and it follows that a group of order g can have at most g inequivalent irreducible representations. For the time being we shall denote the precise number of irreducible representations by r.

Let

$$\chi^{(1)} , \chi^{(2)} , \dots , \chi^{(r)}$$

be a complete set of simple charecters, and suppose that they correspond to the irreducible representations

$$F^{(1)} , F^{(2)} , \dots , F^{(r)} \tag{3.8}$$

which, of course, are determined up to equivalence. The degrees of the irreducible representations will be denoted by

$$f^{(1)} , f^{(2)} , \dots , f^{(r)} \tag{3.9}$$

respectively. By Maschke's theorem, if A is an arbitrary representation then

$$A \approx \mathrm{diag}(F, F', F'', \dots) \tag{3.10}$$

where F,F',F'', ... are irreducible. The constituents of A need not be distinct. If ϕ is the character of A, then we can write

$$\phi = \sum_{j=1}^{r} d_j \chi^{(j)} \tag{3.11}$$

where $d_j \ge 0$ is the multiplicity of $F^{(j)}$ in A. On taking the inner product with $\chi^{(i)}$ we obtain that

$$d_i = \langle \phi, \chi^{(i)} \rangle \qquad (i = 1,2, \ldots ,r) \tag{3.12}$$

This is analogous to the way in which the coefficients of a Fourier series are determined. For this reason (3.11) is called as the Fourier analysis of ϕ or of A.

Let B be another representation, and suppose that its character, ψ has the Fourier analysis

$$\psi = \sum_{j=1}^{r} e_j \chi^{(j)} \tag{3.13}$$

so that for $i = 1,2,\ldots,r$

$$e_i = \langle \psi, \chi^{(i)} \rangle \tag{3.14}$$

We are now in the position to establish the converse of the elementary result that equivalent representations have the same character. Indeed, if

$$\phi(x) = \psi(x) \quad (x \in G) \tag{3.15}$$

then (3.12) and (3.13) immediately show that

$$d_i = e_i \quad (i = 1,2,\ldots,r). \tag{3.16}$$

Hence A and B are equivalent to the same diagonal array of irreducible constituents and are therefore equivalent to each other.

So we proved the

Theorem 3.1.2.

Two representations of a finite group of a finite group over the complex field are equivalent if and only if they have the same charecter.

In other words, the trace of a representation furnishes us with complete information about the irreducible constituents into which this representation may be decomposed. In this sense, the character truly characterizes a representation up to equivalence.

Now

$$<\phi,\phi> = < \sum_{i=1}^{r} e_i \chi^{(i)} \ , \ \sum_{j=1}^{r} c_j \chi^{(j)} >. \qquad (3.17)$$

By using the orthogonality relations of the first kind

$$<\phi,\phi> = \sum_{k=1}^{r} (e_k)^2. \qquad (3.18)$$

Now if ϕ is irreducible, then only one of e_k $(1 \leq k \leq r)$ is 1 and the others are zero, so if ϕ is irreducible, then $<\phi,\phi> = 1$. So we proved

Proposition 3.1.3.

A representation with character ϕ is irreducible if and only if $<\phi,\phi> = 1$ [1].

Theorem 3.1.4.

Let G be a group of order g. If G has k conjugacy classes, then there are, up to equivalence, k distinct representations over $\mathbb{C}$, say

$$F^{(1)}, F^{(2)}, \ldots, F^{(k)}.$$

If $F^{(i)}$ is of degree $f^{(i)}$, then [1]

$$g = \sum_{i=1}^{k} (f^{(i)})^2 .$$

3.2. The Character Table

The complete information about the characters of G is conveniently displayed in a character table, which lists the values of the k simple characters for all the elements. We know, by theorem 2.1, that a character is constant on each of the conjugacy classes. If $x \in C_\alpha$, where C_α is one of the conjugacy classes of G, we put

$$\chi(x) = \chi_\alpha. \tag{3.19}$$

Thus it is sufficient to record the values χ_α $(1 \le \alpha \le k)$. Denoting the number of elements in C_α by h_α, we have the class equation

$$h_1 + h_2 + \ldots + h_k = g. \tag{3.20}$$

Unless the contrary is stated, we adhere to the convention that $C_1 = 1$ and that $F^{(1)}$ is the trivial representation

$$F^{(1)}(x) = 1 \ (x \in G). \tag{3.21}$$

As in theorem 3.1.2, the degree of $F^{(i)}$ will be denoted by $f^{(i)}$, so that

$$\chi_1^{(i)} = f^{(i)} \quad (i=1,2,\ldots,k). \tag{3.22}$$

Table 1 presents a typical character table. The body of the table is a kxk square matrix whose rows correspond to the different characters while each column contains the values of all simple characters for a particular conjugacy class.

Table 3.1. The character table of G.

G:	C_1	C_2	$\cdots$	C_α	$\cdots$	C_k
	h_1	h_2	$\cdots$	h_α	$\cdots$	h_k
$\chi^{(1)}$	$f^{(1)}$	1	$\cdots$	1	$\cdots$	1
$\chi^{(2)}$	$f^{(2)}$	$\chi_2^{(2)}$	$\cdots$	$\chi_\alpha^{(2)}$	$\cdots$	$\chi_k^{(2)}$
$\vdots$	$\vdots$	$\vdots$		$\vdots$		$\vdots$
$\chi^{(i)}$	$f^{(i)}$	$\chi_2^{(i)}$	$\cdots$	$\chi_\alpha^{(i)}$	$\cdots$	$\chi_k^{(i)}$
$\vdots$	$\vdots$	$\vdots$		$\vdots$		$\vdots$
$\chi^{(k)}$	$f^{(k)}$	$\chi_2^{(k)}$		$\chi_\alpha^{(k)}$		$\chi_k^{(k)}$

CHAPTER 4

4. Permutation Groups

4.1. The Symmetric Group

We recall that each permutation of S_n can be resolved into a product of disjoint cycles in a unique manner save for the order of the cycle factors. A cycle involving a single symbol indicates that this symbol remains fixed. Two elements of S_n are conjugate if and only if they have the same cycle pattern. For example, in S_6 the permutations

$$x = (146)(35)(2) \quad \text{and} \quad y = (243)(16)(5)$$

have the same cycle pattern and are therefore conjugate; indeed,

$$t^{-1}xt = y$$

where

$$t = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 5 & 1 & 4 & 6 & 5 \end{pmatrix}.$$

Thus each conjugacy class C_α of S_n is determined by its cycle pattern comprising, say, α_1 cycles of degree 1, α_2 cycles of degree 2 , and so on. Accordingly, the specification of C_α will be described by the formula

$$\alpha : \alpha_1 + 2\alpha_2 + 3\alpha_3 + \dots + n\alpha_n = n \qquad (4.1)$$

where $\alpha_1, \alpha_2, \dots, \alpha_n$ are non-negative integers.

Alternatively, each permutation of C_α is the product of, say, u cycles whose degrees, in some order, are p_1, p_2, ... ,p_u respectively. Hence C_α is determined by the partition

$$p : p_1 + p_2 + \dots + p_n = n, \qquad (4.2a)$$

which is abbreviated to

$$|p| = n. \qquad (4.2b)$$

No distinction is made between partitions that differ merely by the arrangement of their terms. Hence in order to achieve uniqueness we may impose the conditions

$$p_1 \geq p_2 \geq \dots \geq p_n \geq 0.$$

Let k be the number of conjugacy classes of S_n. There is no simple expression for k as a function of n, but in view of the foregoing discusion we know that k is the number of solution of $\alpha : \alpha_1 + 2\alpha_2 + 3\alpha_3 + \dots + n\alpha_n = n$ or else that k is the number of solutions of $|p| = n$ and $p_1 \geq p_2 \geq \dots \geq p_n \geq 0$.

When C_α is specified by $\alpha : \alpha_1+2\alpha_2+3\alpha_3+\dots+n\alpha_n = n$, then

$$|C_\alpha| = h_\alpha \, , \qquad (4.3)$$

where h_α is given by Cauchy's formula, namely

$$h_\alpha = \frac{n!}{1^{\alpha_1}.\alpha_1! \dots n^{\alpha_n}.\alpha_n!} \, . \qquad (4.4)$$

4.2. Generating Functions

It is one of the most remarkable achievements of Frobenius that, only four years after the discovery of group characters in 1896, he succeeded in obtaining all the character of S_n, at least in principle. More precisely, he constructed a set of generating functions whose coefficients reveal the full character table.

The main results of Frobenius's theory may be summarised as follows:

Theorem 4.1.

Let $x_1, x_2, \ldots, x_n$ be indeterminates and put [1]

$$s_r = x_1^r + x_2^r + \ldots + x_n^r \quad (r=1,2,\ldots) \qquad (4.5)$$

$$\Delta = \prod_{i<j} (x_i - x_j). \qquad (4.6)$$

For each partition

$$p_1 \geq p_2 \geq \ldots \geq p_n \geq 0; \ |p| = n$$

let

$$V^{(p)} = \det\left[x_j^{p_i+n-i}\right]. \qquad (4.7)$$

If the conjugacy class C_α is specified by cycle pattern

$$\alpha_1 + 2\alpha_2 + 3\alpha_3 + \ldots + n\alpha_n = n,$$

define

$$s_\alpha = s_1^{\alpha_1} s_2^{\alpha_2} \ldots s_n^{\alpha_n}. \qquad (4.8)$$

Then the values of all simple characters of S_n appear coefficients in the generating functions

$$\diamond_\alpha \Delta' = \sum_p \chi_\alpha^{(p)} V^{(p)} . \qquad (4.9)$$

Moreover,

$$f^{(p)} = \chi_1^{(p)} = \frac{n! \prod_{i<j}(p_i - p_j + j - i)}{\prod_i (p_i + n - i)!} . \qquad (4.10)$$

Definition 4.2.1 (Schur's Function or S-Function).

The polynomial

$$F^{(p)} = \det\left[x_j^{p_i+n-i}\right]/\det\left[x_j^{n-i}\right] \qquad (4.11)$$

is called the Schur function, corresponding to the partition

$$p_1 \geq p_2 \geq \ldots \geq p_n \geq 0; \ |p| = n.$$

These functions were introduced by I.Schur in his treatment of the symmetric group. We can now write

$$\diamond_\alpha \Delta = \sum_p \chi_\alpha^{(p)} V^{(p)}$$

as

$$\diamond_\alpha = \sum_p \chi_\alpha^{(p)} F^{(p)} . \qquad (4.12)$$

Since the character s are real, the orthogonality relations of the first kind state that

$$\frac{1}{n!} \sum_\alpha h_\alpha \chi_\alpha^{(p)} \chi^{(q)} = \delta_{pq} .$$

Hence we deduce from $\phi_\alpha = \sum_p \chi_\alpha^{(p)} F^{(p)}$ that

$$F^{(q)} = \frac{1}{n!} \sum_p h_\alpha \chi_\alpha^{(q)} \phi_\alpha , \qquad (4.13)$$

which is Schur's formula. Thus while the generating functions of Frobenius furnish the values of all the characters for a particular class, a Schur function is associated with a particular character and displays its values for all classes. It is this sense that the two theories may be described as dual to each other.

The problem of determining the characters of S_n has now been reduced to the algebraic task of expanding the Schur functions in terms of $s_1, s_2, \ldots$.

Proposition 4.2.

The Schur function corresponding to the partition

$p_1 \geq p_2 \geq \ldots \geq p_u > 0$, $p_{u+1} = \ldots = p_n = 0$: $|p| = n$

is given by [1]

$$F^{(p)} = \det\left(w_{p_i - i + j}\right) \quad (i,j=1,2,\ldots,u) \qquad (4.14)$$

where

$$w_r = \sum_{(\mu)} x_{\mu_1} x_{\mu_2} \ldots x_{\mu_r} , \quad 1 \leq \mu_1 \leq \mu_2 \leq \ldots \leq \mu_r \leq n \qquad (4.15)$$

and

$$w_0 = 1 , \; w_m = 0 \text{ if } m < 0. \qquad (4.16)$$

Using these results we have the

<u>Theorem</u> 4.3.

i) $\chi_{\alpha}^{[n]} = 1$, that is, $\chi^{[n]}$ is the trivial character,

ii) $\chi_{\alpha}^{[1^n]} = (-1)^{\alpha_2+\alpha_4+\dots}$,that is, $\chi^{[1^n]}$ is the alternating character [1].

4.3. Conjugate Partitions

When zero terms are disregarded a partition

$$p: p_1 \geq p_2 \geq \ldots \geq p_u > 0 \ , \ |p| = n$$

may be conveniently be decribed by a graph (p). This consists of p_1 nodes in the first row, p_2 nodes in the second row, ..., p_u nodes in the uth row, the initial nodes in each row being vertically aligned. For example, the partition

$$p: 5 + 3 + 3 + 2 + 1 = 14$$

is shown by the graph

```
        o o o o o
        o o o
    (p) o o o
        o o
        o
```

With every partition p we associate its conjugate partition p′ obtained by interchanging rows and columns of the graph (p); in order words the terms of p′ are the number of nodes in the columns of (p). Evidently, the conjugate of p′ is p.

In our example

$$p' : 5 + 4 + 3 + 1 + 1 = 14.$$

A partition which is identical with its conjugate is said to be self conjugate. For example

$$3 + 2 + 1 = 6$$

is self-conjugate as is seen from its graph

o o o

o o

o

<u>Theorem</u> 4.4.

If p and q are conjugate partitions of n then [1]

$$\chi^{(p)} = \chi^{[1^n]}\chi^{(q)}.$$

CHAPTER 5

5. The Character Table of S_6 and S_7

5.1. A Formula for Charecters of Partitions with two Elements

We will find some formulae for two element partitions. Assume that $(\lambda) = (\lambda_1,\lambda_2)$ where $\lambda_1 + \lambda_2 = n$.

By Schur's study, we know that the values of $\chi^{[\lambda_1,\lambda_2]}$ can be obtained as coefficients in the expansion of

$$\begin{vmatrix} w_{\lambda_1} & w_{\lambda_1+1} \\ w_{\lambda_2-1} & w_{\lambda_2} \end{vmatrix} \tag{5.1}$$

In terms of the 1st row, we obtain

$$w_{\lambda_1}w_{\lambda_2} - w_{\lambda_1+1}w_{\lambda_2-1}.$$

A typical element of

$$w_{\lambda_1} = \frac{1}{\lambda_1!} \sum_{|\theta|=\lambda_1} \frac{\lambda_1!}{1^{\theta_1}.\theta_1! \ldots n^{\theta_n}.\theta_n!} s_1^{\theta_1} \ldots s_n^{\theta_n}$$

$$= \sum_{|\theta|=\lambda_1} \left(\frac{s_1}{1}\right)^{\theta_n} \ldots \left(\frac{s_n}{n}\right)^{\theta_n} \frac{1}{\theta_1!\ldots\theta_n!} . \tag{5.2}$$

Since $\lambda_1 < n$, some elements of $\theta_1, \ldots, \theta_n$ are zero.

Similarly a typical element of w_{λ_2} is

$$\frac{1}{\tau_1! \ldots \tau_n!} \left[\frac{s_1}{1} \right]^{\tau_1} \ldots \left[\frac{s_n}{n} \right]^{\tau_n} \quad (5.3)$$

where $(\tau_1, \ldots, \tau_n)$ is a class of (λ_2).

So, the product $w_{\lambda_1} . w_{\lambda_2}$ is obtained by adding the terms of the type

$$\frac{1}{\theta_1! \ldots \theta_n! \, \tau_1! \ldots \tau_n!} \left[\frac{s_1}{1} \right]^{\tau_1+\theta_1} \ldots \left[\frac{s_n}{n} \right]^{\tau_n+\theta_n}$$

over all classes of (θ) and (τ) of the symmetric group on λ_1 and λ_2 letters, respectively.

θ_1 writing $\alpha_1 = \theta_1 + \tau_1$, ... , $\alpha_n = \theta_n + \tau_n$ it is clear that

$(\alpha) = (\alpha_1, \ldots, \alpha_n)$ is a class on $\lambda_1 + \lambda_2 = n$ letters.

Hence the coefficient of

$$\frac{1}{\tau_1! \ldots \tau_n!} \left[\frac{s_1}{1} \right]^{\tau_1} \ldots \left[\frac{s_n}{n} \right]^{\tau_n}$$

(α) any class of the symmetric group on λ_2^n letters the product $w_{\lambda_1} . w_{\lambda_2}$ is the sum over all classes (τ) of the symmetric group on λ_2 letters of the product

$$\begin{bmatrix} \alpha_1 \\ \tau_1 \end{bmatrix} \cdots \begin{bmatrix} \alpha_n \\ \tau_n \end{bmatrix}$$

where $\begin{bmatrix} \alpha_j \\ \tau_j \end{bmatrix}$ denotes the binomial coefficient

$$\begin{bmatrix} \alpha_j \\ \tau_j \end{bmatrix} = \frac{\alpha_j}{\tau_j!.\theta_j!} . \qquad (5.4)$$

(we adopt that $\begin{bmatrix} \alpha_j \\ \tau_j \end{bmatrix} = 0$ if $\tau_j > \alpha_j$.)

This follows from the fact that

$$\frac{\alpha_1! \ldots \alpha_n!}{\theta_1!\tau_1! \ldots \theta_n!\tau_n!}\left[\frac{1}{\alpha_1! \ldots \alpha_n!}\left(\frac{s_1}{1}\right)^{\alpha_1} \cdots \left(\frac{s_n}{n}\right)^{\alpha_n}\right] =$$

$$= \frac{1}{\theta_1!\tau_1! \ldots \theta_n!\tau_n!}\left(\frac{s_1}{1}\right)^{\alpha_1} \cdots \left(\frac{s_n}{n}\right)^{\alpha_n} \qquad (5.5)$$

where the R.H.S is the term we have in our expansion.

Similarly for the form $w_{\lambda_1+1}.w_{\lambda_2-1}$ the coefficient of

$$\frac{1}{\alpha_1! \ldots \alpha_n!}\left(\frac{s_1}{1}\right)^{\alpha_1} \cdots \left(\frac{s_n}{n}\right)^{\alpha_n}$$

in this product is

$$\sum_{(\sigma)} \begin{bmatrix} \alpha_1 \\ \sigma_1 \end{bmatrix} \cdots \begin{bmatrix} \alpha_n \\ \sigma_n \end{bmatrix} \qquad (5.6)$$

where (σ) is any class of the symetric group on $\lambda_2 - 1$ letters. So we get

$$\chi_{(\alpha)}^{[\lambda_1,\lambda_2]} = \sum_{(\tau)} \begin{bmatrix} \alpha_1 \\ \tau_1 \end{bmatrix} \cdots \begin{bmatrix} \alpha_n \\ \tau_n \end{bmatrix} - \sum_{(\sigma)} \begin{bmatrix} \alpha_1 \\ \sigma_1 \end{bmatrix} \cdots \begin{bmatrix} \alpha_n \\ \sigma_n \end{bmatrix} \quad (5.7)$$

Since, when $\tau_1 \geq 1$, $\tau' = (\tau_1 - 1, \tau_2, \ \ldots \ ,\tau_n)$ is a class (σ) of the symmetric group on $\lambda_2 - 1$ letters, we may combine the terms

$$\begin{bmatrix} \alpha_1 \\ \sigma_1 + 1 \end{bmatrix} \begin{bmatrix} \alpha_2 \\ \sigma_2 \end{bmatrix} \cdots \begin{bmatrix} \alpha_n \\ \sigma_n \end{bmatrix} - \begin{bmatrix} \alpha_1 \\ \sigma_1 \end{bmatrix} \cdots \begin{bmatrix} \alpha_n \\ \sigma_n \end{bmatrix}. \quad (5.8)$$

Using

$$\begin{bmatrix} \alpha_1 \\ \sigma_1 + 1 \end{bmatrix} = \frac{\alpha_1 !}{(\alpha_1+1)!(\alpha_1-\sigma_1-1)!} = \frac{(\alpha_1 - \sigma_1)}{(\sigma_1 + 1)} \cdot \frac{\alpha_1 !}{\sigma_1 !(\alpha_1-\sigma_1)!}$$
$$= \frac{(\alpha_1-\sigma_1-1)}{(\sigma_1+1)} \cdot \begin{bmatrix} \alpha_1 \\ \sigma_1 \end{bmatrix}$$

in a more compact way:

$$\left[\frac{\alpha_1 - \sigma_1}{\sigma_1 + 1} - 1 \right] \left[\frac{\alpha_1}{\sigma_1} \right] \cdots \left[\frac{\alpha_n}{\sigma_n} \right] = \left[\frac{\alpha_1 - 2\sigma_1 - 1}{\sigma_1 + 1} \right] \left[\frac{\alpha_1}{\sigma_1} \right] \cdots \left[\frac{\alpha_1}{\sigma_1} \right]. \quad (5.9)$$

We can do this whenever (σ) has at least one fixed letter. So denoting by (β) an arbitrary class of the symmetric group on λ_2 letters which does not contain any unary cycles, we get the formula:

$$\chi_{\alpha}^{[\lambda_1,\lambda_2]} = \sum_{(\sigma)} \begin{bmatrix}\alpha_1-2\sigma_1-1\\ \hline \sigma_1+1\end{bmatrix}\begin{bmatrix}\alpha_1\\ \sigma_1\end{bmatrix}\cdots\begin{bmatrix}\alpha_n\\ \sigma_n\end{bmatrix} + \sum_{(\beta)} \begin{bmatrix}\alpha_2\\ \beta_2\end{bmatrix}\cdots\begin{bmatrix}\alpha_n\\ \beta_n\end{bmatrix}. \quad (5.10)$$

Here the first summation being over all classes (σ) of the symmetric group on $\lambda_2 - 1$ letters, and the second over those classes (β) of the symmetric group on λ_2 letters for which $\beta_1 = 0$.

Using this general formula, we will derive some formulae

I) $\chi_{(\alpha)}^{[n-1,1]} = ?$. This is a special case. Here $\lambda_2 = 1$, $\lambda_2 - 1 = 0$. But in this case we have to evaluate only

$$\begin{bmatrix}\alpha_1\\ 1\end{bmatrix}\begin{bmatrix}\alpha_2\\ 0\end{bmatrix}\cdots\begin{bmatrix}\alpha_n\\ 0\end{bmatrix} - \begin{bmatrix}\alpha_2\\ 0\end{bmatrix}\cdots\begin{bmatrix}\alpha_n\\ 0\end{bmatrix} = \alpha_1 - 1 \quad (5.11)$$

(by the 1st formula).

II) $\chi_{(\alpha)}^{[n-2,2]} = ?$ .Here $\lambda_2 = 2$,

$$\chi_{(\alpha)}^{[n-2,2]} = \sum_{|\sigma|=1} \begin{bmatrix}\alpha_1-2\sigma_1-1\\ \hline \sigma_1+1\end{bmatrix}\begin{bmatrix}\alpha_1\\ \sigma_1\end{bmatrix}\cdots\begin{bmatrix}\alpha_n\\ \sigma_n\end{bmatrix} + $$

$$+ \sum_{|\beta|=2} \begin{bmatrix}\alpha_2\\ \beta_2\end{bmatrix}\cdots\begin{bmatrix}\alpha_n\\ \beta_n\end{bmatrix}$$

$$= \frac{\alpha_1-2\sigma_1-1}{\sigma_1+1}\begin{bmatrix}\alpha_1\\ 1\end{bmatrix}\begin{bmatrix}\alpha_2\\ 0\end{bmatrix}\cdots\begin{bmatrix}\alpha_n\\ 0\end{bmatrix} + \begin{bmatrix}\alpha_2\\ 0\end{bmatrix}\cdots\begin{bmatrix}\alpha_n\\ 0\end{bmatrix} \quad (5.12)$$

$$\chi^{[n-2,2]} = \left[\frac{\alpha_1 -3}{2}\right]\alpha_1 + \alpha_2$$

$$= \frac{1}{2}\,\alpha_1(\alpha_1 - 3) + \alpha_2 \;. \quad (5.13)$$

III) $\chi^{[n-3,3]}_{(\alpha)} = ?$.

Here $\lambda_2 = 3$, so;

$$\chi^{[n-3,3]}_{(\alpha)} = \sum_{|\sigma|=2} \left[\frac{\alpha_1 -2\sigma_1 -1}{\sigma_1 + 1}\right]\begin{bmatrix}\alpha_1\\ \sigma_1\end{bmatrix}\cdots\begin{bmatrix}\alpha_n\\ \sigma_n\end{bmatrix} +$$

$$+ \sum_{|\beta|=3} \begin{bmatrix}\alpha_2\\ \beta_2\end{bmatrix}\cdots\begin{bmatrix}\alpha_n\\ \beta_n\end{bmatrix}$$

$$= \left[\frac{\alpha_1 - 2.2 - 1}{2 + 1}\right]\begin{bmatrix}\alpha_1\\ 2\end{bmatrix}\begin{bmatrix}\alpha_2\\ 0\end{bmatrix}\cdots\begin{bmatrix}\alpha_n\\ 0\end{bmatrix}$$

$$+ \left[\frac{\alpha_1 - 2.0 - 1}{0 + 1}\right]\begin{bmatrix}\alpha_1\\ 0\end{bmatrix}\begin{bmatrix}\alpha_2\\ 1\end{bmatrix}\cdots\begin{bmatrix}\alpha_n\\ 0\end{bmatrix}$$

$$+ \begin{bmatrix}\alpha_2\\ 0\end{bmatrix}\begin{bmatrix}\alpha_3\\ 1\end{bmatrix}\cdots\begin{bmatrix}\alpha_n\\ 0\end{bmatrix}$$

$$= \frac{\alpha_1 -5}{3} \cdot \frac{\alpha_1(\alpha_1 -1)}{2} + \frac{\alpha_1 -1}{1} \cdot \alpha_2 + \alpha_3$$

$$= \frac{1}{6}\,\alpha_1(\alpha_1 - 1)(\alpha_1 - 5) + \alpha_2(\alpha_1 - 1) + \alpha_3 \;. \quad (5.14)$$

$$\chi^{[n-4,4]}_{(\alpha)} = \sum_{|\sigma|=3} \left[\frac{\alpha_1 - 2\sigma_1 - 1}{\sigma_1 + 1}\right] \begin{bmatrix}\alpha_1\\ \sigma_1\end{bmatrix} \cdots \begin{bmatrix}\alpha_n\\ \sigma_n\end{bmatrix} + \sum_{|\beta|=4} \begin{bmatrix}\alpha_2\\ \beta_2\end{bmatrix} \cdots \begin{bmatrix}\alpha_n\\ \beta_n\end{bmatrix}$$

$$\chi^{[n-4,4]}_{(\alpha)} = \left[\frac{\alpha_1 - 2.3 - 1}{3 + 1}\right] \begin{bmatrix}\alpha_1\\ 3\end{bmatrix} \begin{bmatrix}\alpha_2\\ 0\end{bmatrix} \cdots \begin{bmatrix}\alpha_n\\ 0\end{bmatrix} +$$

$$+ \left[\frac{\alpha_1 - 2.1 - 1}{1 + 1}\right] \begin{bmatrix}\alpha_1\\ 1\end{bmatrix} \begin{bmatrix}\alpha_2\\ 1\end{bmatrix} \begin{bmatrix}\alpha_3\\ 0\end{bmatrix} \cdots \begin{bmatrix}\alpha_n\\ 0\end{bmatrix} +$$

$$+ \left[\frac{\alpha_1 - 2.0 - 1}{0 + 1}\right] \begin{bmatrix}\alpha_1\\ 0\end{bmatrix} \begin{bmatrix}\alpha_2\\ 0\end{bmatrix} \begin{bmatrix}\alpha_3\\ 1\end{bmatrix} \begin{bmatrix}\alpha_4\\ 0\end{bmatrix} \cdots \begin{bmatrix}\alpha_n\\ 0\end{bmatrix} +$$

$$+ \begin{bmatrix}\alpha_2\\ 2\end{bmatrix} \begin{bmatrix}\alpha_3\\ 0\end{bmatrix} \cdots \begin{bmatrix}\alpha_n\\ 0\end{bmatrix} + \begin{bmatrix}\alpha_2\\ 0\end{bmatrix} \begin{bmatrix}\alpha_3\\ 0\end{bmatrix} \begin{bmatrix}\alpha_4\\ 1\end{bmatrix} \begin{bmatrix}\alpha_5\\ 0\end{bmatrix} \cdots \begin{bmatrix}\alpha_n\\ 0\end{bmatrix}$$

$$= \frac{\alpha_1 - 7}{4}\, \alpha_1(\alpha_1 - 1)(\alpha_1 - 2) . \frac{1}{6} + \alpha_1 . \alpha_2 \frac{\alpha_1 - 3}{2} +$$

$$+ (\alpha_1 - 1)\alpha_3 + \frac{1}{2}\, \alpha_2(\alpha_2 - 1) + \alpha_4$$

$$= \frac{1}{24}\, \alpha_1(\alpha_1 - 1)(\alpha_1 - 2)(\alpha_1 - 7) + \frac{1}{2}\, \alpha_1(\alpha_1 - 3)\alpha_2 +$$

$$+ (\alpha_1 - 1)\alpha_3 + \frac{1}{2}\, \alpha_2(\alpha_2 - 1) + \alpha_4 . \qquad (5.15)$$

5.2. A Formula for Charecters of Partitions with three Elements

$$\lambda = (\lambda_1,\lambda_2,\lambda_3) \ , \ \lambda_1 + \lambda_2 + \lambda_3 = n,$$

$$\det\begin{bmatrix} w_{\lambda_1} & w_{\lambda_1+1} & w_{\lambda_1+2} \\ w_{\lambda_2-1} & w_{\lambda_2} & w_{\lambda_2+1} \\ w_{\lambda_3-2} & w_{\lambda_3-1} & w_{\lambda_3} \end{bmatrix} \qquad (5.16)$$

must be expanded.

The determinant is when expanded using the first row:

$$w_{\lambda_1} \det\begin{bmatrix} w_{\lambda_2} & w_{\lambda_2+1} \\ w_{\lambda_3-1} & w_{\lambda_3} \end{bmatrix} - w_{\lambda_1+1} \det\begin{bmatrix} w_{\lambda_2-1} & w_{\lambda_2+1} \\ w_{\lambda_3-2} & w_{\lambda_3} \end{bmatrix} +$$

$$+ w_{\lambda_1+2} \det\begin{bmatrix} w_{\lambda_2-1} & w_{\lambda_2} \\ w_{\lambda_3-2} & w_{\lambda_3-1} \end{bmatrix}.$$

The coefficient of w_{λ_1} and w_{λ_2} are easy to recognize.

$$\chi^{[\lambda_2,\lambda_3]} = \det\begin{bmatrix} w_{\lambda_2} & w_{\lambda_2+1} \\ w_{\lambda_3-1} & w_{\lambda_3} \end{bmatrix}$$

and

$$x^{[\lambda_2-1,\lambda_3-1]} = \det\begin{bmatrix} w_{\lambda_2-1} & w_{\lambda_2} \\ w_{\lambda_3-2} & w_{\lambda_3-1} \end{bmatrix} \qquad (5.17)$$

Now let us study the coefficient of the second term in the expansion of our determinant, namely

$$\det\begin{bmatrix} w_{\lambda_2-1} & w_{\lambda_2+1} \\ w_{\lambda_3-2} & w_{\lambda_3} \end{bmatrix} .$$

Here we need a lemma.

Lemma

For $n = 0,1,2, \ldots$ and $k = 1,2, \ldots , n$

$$k \frac{\partial w_n}{\partial s_k} = w_{n-k} . \qquad (5.18)$$

Proof . By definition we have

$$w_n = \sum_{(\alpha)} \frac{1}{\alpha_1! \ \ldots \ \alpha_n!} \left(\frac{s_1}{1} \right)^{\alpha_1} \ldots \left(\frac{s_n}{n} \right)^{\alpha_n} \qquad (5.19)$$

by diferentiating both sides with respect to s_k we get

$$\frac{\partial}{\partial s_k} w_n = \sum_{(\alpha')} \frac{1}{\alpha_1! \ldots \alpha'_k! \ldots \alpha_n!} \left(\frac{s_1}{1}\right)^{\alpha_1} \ldots \frac{1}{k} \left(\frac{s_k}{k}\right)^{\alpha'_k} \ldots \left(\frac{s_n}{n}\right)^{\alpha_n}$$

$$= \frac{1}{k} \sum_{(\alpha')} \frac{1}{\alpha_1! \ldots \alpha'_k! \ldots \alpha_n!} \left(\frac{s_1}{1}\right)^{\alpha_1} \ldots \left(\frac{s_k}{k}\right)^{\alpha'_k} \ldots \left(\frac{s_n}{n}\right)^{\alpha_n} \qquad (5.20)$$

Here $\alpha'_k = \alpha_k - 1$. So

$\alpha' = 1\alpha_1 + 2\alpha_2 + \dots + k(\alpha_k - 1) + \dots = n - k$.

Hence (α') is a class in $n - k$. So we proved that

$$\frac{\partial}{\partial s_k} w_n = \frac{1}{k} w_{n-k}$$

or (5.21)

$$k \frac{\partial}{\partial s_k} w_n = w_{n-k}.$$

So

$$\det \begin{bmatrix} w_{\lambda_2 - 1} & w_{\lambda_2 + 1} \\ w_{\lambda_3 - 2} & w_{\lambda_3} \end{bmatrix} = \begin{bmatrix} \frac{\partial}{\partial s_1} w_{\lambda_2 - 1} & w_{\lambda_2 + 1} \\ \frac{\partial}{\partial s_1} w_{\lambda_3 - 2} & w_{\lambda_3} \end{bmatrix}. \quad (5.22)$$

write it as $\frac{\partial}{\partial s_1}\chi^{[\lambda_2,\lambda_3]}$ meaning that the two rowed determinant furnishing $\chi^{[\lambda_2,\lambda_3]}$ being differentiated with respect to s_1 by differentiating the columns one at a time.

But by Schur's formula

$$\chi^{[\lambda_2,\lambda_3]} = \sum_{(\beta)} \frac{1}{\alpha_1!\dots\alpha_n!} \chi^{[\lambda_2,\lambda_3]}_{(\beta)} \left(\frac{s_1}{1}\right)^{\beta_1} \dots \left(\frac{s_n}{n}\right)^{\beta_n} \quad (5.23)$$

where $\beta = (\beta_1, \dots, \beta_n)$ is an arbitrary class of the symmetric group on $\lambda_2 + \lambda_3$ letters, so the coefficient of

$$\frac{1}{\alpha_1!\ldots\alpha_n!}\left(\frac{s_1}{1}\right)^{\alpha_1}\ldots\left(\frac{s_n}{n}\right)^{\alpha_n}$$

in the product $w_{\lambda_1}\chi^{[\lambda_2,\lambda_3]}$ is

$$\sum_{(\beta)}\chi^{[\lambda_2,\lambda_3]}_{(\beta)}\begin{bmatrix}\alpha_1\\ \beta_1\end{bmatrix}\ldots\begin{bmatrix}\alpha_n\\ \beta_n\end{bmatrix}.$$

Also

$$\frac{\partial}{\partial s_1}\chi^{[\lambda_2,\lambda_3]}=\sum_{(\beta')}\frac{1}{\beta'_1!\ldots\beta'_n!}\chi^{[\lambda_2,\lambda_3]}_{(\beta)}\left(\frac{s_1}{1}\right)^{\beta'_1}\ldots\left(\frac{s_n}{n}\right)^{\beta'_n} \quad (5.24)$$

where $\beta'_1=\beta_1-1$ so that $(\beta')=(\beta'_1,\beta_2,\ \ldots\ ,\beta_n)$ is an arbitrary class of the symmetric group on $\lambda_2+\lambda_3-1$ letters. Hence the coefficient of

$$\frac{1}{\alpha_1!\ldots\alpha_n!}\left(\frac{s_1}{1}\right)^{\alpha_1}\ldots\left(\frac{s_n}{n}\right)^{\alpha_n}$$

in $w_{\lambda_1}\frac{\partial}{\partial s_1}\chi^{[\lambda_2,\lambda_3]}$ is $\sum_{(\beta')}\chi^{[\lambda_2,\lambda_3]}_{(\beta)}\begin{bmatrix}\alpha_1\\ \beta'_1\end{bmatrix}\ldots\begin{bmatrix}\alpha_n\\ \beta_n\end{bmatrix}$

the summation being over all classes (β') of the symmetric group on $\lambda_2+\lambda_3-1$ letters. Similarly, the coefficient of $\frac{1}{\alpha_1!\ldots\alpha_n!}\left(\frac{s_1}{1}\right)^{\alpha_1}\ldots\left(\frac{s_n}{n}\right)^{\alpha_n}$ in the product $w_{\lambda_1+1}\ \chi^{[\lambda_2-1,\lambda_3-1]}$ is

$$\sum_{(\gamma)} \chi_{(\gamma)}^{[\lambda_2-1,\lambda_3-1]} \begin{bmatrix} \alpha_1 \\ \gamma_1 \end{bmatrix} \cdots \begin{bmatrix} \alpha_n \\ \gamma_n \end{bmatrix}$$

the summation being over all classes (γ) of the symmetric group on $\lambda_2 + \lambda_3 - 2$ letters. Combining these results, we obtain

$$\chi_{(\alpha)}^{[\lambda_1,\lambda_2,\lambda_3]} = \sum_{(\beta)} \chi_{(\beta)}^{[\lambda_2,\lambda_3]} \begin{bmatrix} \alpha_1 \\ \beta_1 \end{bmatrix} \cdots \begin{bmatrix} \alpha_n \\ \beta_n \end{bmatrix}$$

$$- \sum_{(\beta')} \chi_{(\beta)}^{[\lambda_2,\lambda_3]} \begin{bmatrix} \alpha_1 \\ \beta'_1 \end{bmatrix} \cdots \begin{bmatrix} \alpha_n \\ \beta_n \end{bmatrix} \tag{5.25}$$

$$- \sum_{(\gamma)} \chi^{[\lambda_2-1,\lambda_3-1]} \begin{bmatrix} \alpha_1 \\ \gamma_1 \end{bmatrix} \cdots \begin{bmatrix} \alpha_n \\ \gamma_n \end{bmatrix}.$$

The summations on the right being respectively over all classes (β) of the symmetric group on $\lambda_2 + \lambda_3$ letters, all classes (β') of the symmetric group on $\lambda_2 + \lambda_3 - 1$ letters, and all classes (γ) of the symmetric group on $\lambda_2 + \lambda_3 - 2$ letters.

On seperating, in the first summation, those classes (β) which contain one more unary cycles $(\beta_1 \geq 1)$ and writing $\beta_1 = \beta'_1 + 1$, we obtain as we did when trying to find a formula for partition with two terms:

$$\chi_{(\alpha)}^{[\lambda_1,\lambda_2,\lambda_3]} = \sum_{(\beta')} \left(\frac{\alpha_1 - 2\beta'_1 - 1}{\beta'_1 + 1} \right) \chi_{(\beta)}^{[\lambda_2,\lambda_3]} \begin{bmatrix} \alpha_1 \\ \beta'_1 \end{bmatrix} \cdots \begin{bmatrix} \alpha_n \\ \beta_n \end{bmatrix}$$

$$+ \sum_{(\delta)} \chi_{(\delta)}^{[\lambda_2,\lambda_3]} \begin{bmatrix} \alpha_2 \\ \delta_2 \end{bmatrix} \cdots \begin{bmatrix} \alpha_n \\ \delta_n \end{bmatrix} \qquad (5.26)$$

$$+ \sum_{(\gamma)} \chi_{(\gamma)}^{[\lambda_2-1,\lambda_3-1]} \begin{bmatrix} \alpha_1 \\ \gamma_1 \end{bmatrix} \cdots \begin{bmatrix} \alpha_n \\ \gamma_n \end{bmatrix}$$

where (δ) is any class of the symmetric group on $\lambda_2+\lambda_3$ letters, for which $\delta_i = 0$.

As an example, suppose

$$\lambda_1 = n - 2 \ , \ \lambda_2 = 1 = \lambda_3 \ .$$

Then $\lambda_2 + \lambda_3 - 1 = 1$ and there is only one term in the first summation, that for which $\beta'_1 = 1$. There is also only one term in the second summation, that for which $\delta_2 = 1$. The third summation has only one term, that for which all $\gamma = 0$ and this yields unity.

Hence

$$\chi_{(\alpha)}^{[n-2,1,1]} = \frac{1}{2} \alpha_1(\alpha_1 - 3) - \alpha_2 + 1 \ . \qquad (5.27)$$

Because $\beta_1 = \beta'_1 + 1 = 2$ and $\chi_{(1^2)}^{[1^2]} = 1$ whilst $\delta_2 = 1$ and $\chi_{(2)}^{[1^2]} = -1$. Using this formula; we get

$$\chi_{(\alpha)}^{[n-2,1,1]} = \frac{1}{2} (\alpha_1 - 1)(\alpha_1 - 2) - \alpha_2 \qquad (5.28)$$

$$\chi^{[n-3,2,1]}_{(\alpha)} = \frac{1}{3}\,\alpha_1(\alpha_1 - 2)(\alpha_1 - 4) - \alpha_3 \qquad (5.29)$$

$$\chi^{[n-4,3,1]}_{(\alpha)} = \frac{1}{8}\alpha_1(\alpha_1 - 1)(\alpha_1 - 3)(\alpha_1 - 6) +$$

$$+ \frac{1}{2}\,\alpha_1(\alpha_1-3)\alpha_2 - \frac{1}{2}\,\alpha_2(\alpha_2-3) - \alpha_4\ . \qquad (5.30)$$

The formulae we found so far are enough to write down all the character table of S_6.

5.3. A Formula for $\chi^{[n-3,1^3]}$

The formulae we found for S_6 are sufficient for all partitions of S_7 except for the partition p: 4+1+1+1 which is self conjugate as its graph implies

```
     o o o o
     o
 (p) o
     o
```

In order to find the simple character $\chi^{[n-3,1^3]}$ we have to expand the fourth order determinant

$$\begin{bmatrix} w_{n-3} & w_{n-2} & w_{n-1} & w_n \\ w_0 & w_1 & w_2 & w_3 \\ 0 & w_0 & w_1 & w_2 \\ 0 & 0 & w_0 & w_1 \end{bmatrix}.$$

If we expand the determinant in terms of first row, recalling that $w_0 = 1$,

$$\mathrm{Det}\begin{bmatrix} w_{n-3} & w_{n-2} & w_{n-1} & w_n \\ w_0 & w_1 & w_2 & w_3 \\ 0 & w_0 & w_1 & w_2 \\ 0 & 0 & w_0 & w_1 \end{bmatrix} = w_{n-3}\mathrm{Det}\begin{bmatrix} w_1 & w_2 & w_3 \\ 1 & w_1 & w_2 \\ 0 & 1 & w_1 \end{bmatrix} - \tag{5.31}$$

$$- 1.\mathrm{Det}\begin{bmatrix} w_{n-2} & w_{n-1} & w_n \\ 1 & w_1 & w_2 \\ 0 & 1 & w_1 \end{bmatrix}$$

But the determinant $\mathrm{Det}\begin{bmatrix} w_1 & w_2 & w_3 \\ 1 & w_1 & w_2 \\ 0 & 1 & w_1 \end{bmatrix}$ is the one we use to get the simple character $\chi^{[1^3]}$ and the determinant $\begin{bmatrix} w_{n-2} & w_{n-1} & w_n \\ 1 & w_1 & w_2 \\ 0 & 1 & w_1 \end{bmatrix}$ is the one we use to get the simple character $\chi^{[n-2,1,1]}$. So

$$\chi^{[n-3,1^3]} = w_{n-3}\chi^{[1^3]} - \chi^{[n-2,1^2]}. \qquad (5.32)$$

By Schur's formula

$$\chi^{[1^3]} = \sum_{(\beta)} \frac{1}{\beta_1!\ldots\beta_n!}\, \chi^{[1^3]}_{(\beta)} \left(\frac{S_1}{1}\right)^{\alpha_1} \cdots \left(\frac{S_n}{n}\right)^{\alpha_n} \qquad (5.33)$$

where $\beta = (\beta_1,\ldots,\beta_n)$ is an arbitrary class of the symmetric group on 3 letters. So the coefficient of $w_{n-3}\ \chi^{[1^3]}$ is

$$\sum_{(\beta)} \chi^{[1^3]}_{(\beta)} \binom{\alpha_1}{\beta_1} \cdots \binom{\alpha_n}{\beta_n}.$$

$$w_{n-3}\ \chi^{[1^3]} = \sum_{|\beta|=3} \chi^{[1^3]}_{(\beta)} \binom{\alpha_1}{\beta_1} \cdots \binom{\alpha_n}{\beta_n} \qquad (5.34)$$

$$= \chi^{[1^3]}_{(1^3)} \binom{\alpha_1}{3}\binom{\alpha_2}{0}\binom{\alpha_3}{0} +$$

$$+ \chi^{[1^3]}_{(2,1)} \binom{\alpha_1}{1}\binom{\alpha_2}{1}\binom{\alpha_3}{0} + \chi^{[1^3]}_{(3)} \binom{\alpha_1}{0}\binom{\alpha_2}{0}\binom{\alpha_3}{1}.$$

Since $\chi^{[1^n]}_{(\alpha)} = (-1)^{\alpha_2+\alpha_4+\alpha_6+\ldots}$ by taking n=3, it becomes

$$\chi^{[1^3]}_{(\alpha)} = (-1)^{\alpha_2}. \qquad (5.35)$$

Hence

$$\chi^{[1^3]}_{(1^3)} = (-1)^0 = 1, \quad \chi^{[1^3]}_{(2,1)} = -1, \quad \chi^{[1^3]}_{(3)} = 1. \quad (5.36)$$

Inserting these values, we get

$$w_{n-3}\,\chi^{[1^3]} = 1.\binom{\alpha_1}{3} - 1.\binom{\alpha_1}{1}\binom{\alpha_2}{1} + 1.\binom{\alpha_3}{1}.$$

$$= \frac{1}{6}\alpha_1(\alpha_1 - 1)(\alpha_1 - 2) - \alpha_1\alpha_2 + \alpha_3 \ . \quad (5.37)$$

Since we found $\chi^{[n-2,1^2]} = \frac{1}{2}(\alpha_1 - 1)(\alpha_1 - 2) - \alpha_2$, combining all these we find that

$$\chi^{[n-3,1^3]} = \frac{1}{6}\alpha_1(\alpha_1 - 1)(\alpha_1 - 2) - \alpha_1\alpha_2 + \alpha_3 -$$

$$- \frac{1}{2}(\alpha_1 - 1)(\alpha_1 - 2) + \alpha_2$$

$$= \frac{1}{6}(\alpha_1-1)(\alpha_1-2)(\alpha_1-3)-(\alpha_1-1)\alpha_2 + \alpha_3 . \quad (5.38)$$

Now we are ready to construct the character table to S_6 and S_7 .

5.4. The character Table of S_6

6 has 11 partitions, namely (1^6), $(1^4 2)$, $(1^3 3)$, $(1^2 2^2)$ $(1^2 4)$, $(1\ 2\ 3)$ $(1\ 5)$ (2^3) $(2\ 4)$ (3^2) (6).

So every character will be on 11-tuble.

For easy of calculating, it is helpful to make a table for all these partitions

	1^6	$1^4 2$	$1^3 3$	$1^2 2^2$	$1^2 4$	123	15	2^3	24	3^2	6
α_1	6	4	3	2	2	1	1	0	0	0	0
α_2	0	1	0	2	0	1	0	3	1	0	0
α_3	0	0	1	0	0	1	0	0	0	2	0
α_4	0	0	0	0	1	0	0	0	1	0	0
α_5	0	0	0	0	0	0	1	0	0	0	0
α_6	0	0	0	0	0	0	0	0	0	0	1

$$\chi^{[6]} \equiv \chi^{[n]} \equiv (1,1,1,1,1,1,1,1,1,1,1)$$

$$\chi^{[5,1]} = \chi^{[n-1,1]} = \alpha_1 - 1$$

$$= (5,3,2,1,1,0,0,-1,-1,-1,-1)$$

$$\chi^{[4,2]} = \chi^{[n-2,2]} = \frac{1}{2}\ \alpha_1(\alpha_1 - 3) + \alpha_2$$

$$= (9,3,0,1,-1,0,-1,3,1,0,0) \qquad (5.39)$$

$$\chi^{[4,1^2]} = \chi^{[n-2,2]} = \frac{1}{2}\ (\alpha_1 - 1)(\alpha_1 - 2) - \alpha_2$$

$$= (10,2,1,-2,0,-1,0,-2,0,1,1)$$

$$\chi^{[3^2]} = \chi^{[n-3,3]} = \frac{1}{6}\ \alpha_1(\alpha_1-1)(\alpha_1-5)-(\alpha_1-1)\alpha_2+\alpha_3$$
$$= (5,1,-1,1,-1,1,0,-3,-1,2,0)$$

$$\chi^{[321]} = \chi^{[n-3,2,1]} = \frac{1}{3}\ \alpha_1(\alpha_1 - 2)(\alpha_1 - 4) - \alpha_3$$
$$= (16,0,-2,0,0,0,1,0,0,-2,0)$$

$$\chi^{[1^6]} = (-1)^{\alpha_2+\alpha_4+\alpha_6} = (1,-1,1,1,-1,-1,1,-1,1,1,-1). \qquad (5.40)$$

Now, we calculate the other characters in terms of their conjugates:

$$\chi^{[31^3]} = \chi^{[1^6]} \quad \chi^{[4,1^2]}$$
$$= (10,-2,1,-2,0,1,0,2,0,1,-1)$$

$$\chi^{[2^3]} = \chi^{[1^6]} \quad \chi^{[3^2]}$$
$$= (5,-1,-1,1,1,-1,0,3,-1,2,0) \qquad (5.41)$$

$$\chi^{[2^2 1^2]} = \chi^{[1^6]} \quad \chi^{[4,2]}$$
$$= (9,-3,0,1,1,0,-1,-3,1,0,0)$$

$$\chi^{[21^4]} = \chi^{[1^6]} \quad \chi^{[5,1]}$$
$$= (5,-3,2,1,-1,0,0,1,-1,-1,1).$$

To find the class sizes, we use

$$h_\alpha = \frac{n!}{1^{\alpha_1}\ \alpha_1!\ \ldots\ n^{\alpha_n}\alpha_n!} \qquad (5.42)$$

Then we get:

$$h_\alpha = (1,15,40,45,90,120,144,15,90,40,120). \quad (5.43)$$

Hence we have already constructed the character table for S_6

5.5. The Character Table of S_7

7 has 15 partitions, namely:
$(1^7),(1^5 2),(1^4 3)$, $(1^3 4),(1^3 2^2),(1^2 23),(1^2 5),(16),(124)$, $(12^3),(25),(2^2 3),(34),(7)$.

By using the class formula, we find the corresponding class sizes 1,21,70,210,105,420,504,840,630,105,280,504, 210,420 and 720 respectively. We use

$$\chi^{[7]} = \chi^{[n]}$$
$$\chi^{[6,1]} = \chi^{[n-1,1]}$$
$$\chi^{[5,2]} = \chi^{[n-2,2]}$$
$$\chi^{[5,1^2]} = \chi^{[n-2,1,1]}$$
$$\chi^{[4,3]} = \chi^{[n-3,3]}$$
$$\chi^{[4,2,1]} = \chi^{[n-3,2,1]}$$
$$\chi^{[3^2,1]} = \chi^{[n-4,3,1]}$$

to find these characters directly by using our set of formulae and use the theorem on conjugate partitions to calculate:

$$\chi^{[32^2]} = \chi^{[1^7]} . \chi^{[3^2,1]}$$
$$\chi^{[321^2]} = \chi^{[1^7]} . \chi^{[421]}$$
$$\chi^{[2^3 1]} = \chi^{[1^7]} . \chi^{[43]}$$
$$\chi^{[31^4]} = \chi^{[1^7]} . \chi^{[5,1^2]}$$
$$\chi^{[2^2 1^3]} = \chi^{[1^7]} . \chi^{[52]}$$
$$\chi^{[21^5]} = \chi^{[1^7]} . \chi^{[61]}.$$

$\chi^{[41^3]}$ is self conjugate and is of the type $\chi^{[n-3,1^3]}$. By using these relations we are ready to construct the character table of S_7 .

RESULTS AND CONCLUSION

In this thesis, some special s-functions have been used to construct the character tables of S_6 and S_7. The use of s-functions leads to some formulae which evaluates the character in terms of the spesification of any class of S_n that means the number of repetitions of any number in the partition which characterizes the cycle structure of that class.

From this point of view it is clear that s-functions are very easy to use when the character tables are needed and this method is self-contained in the sense that the use of smaller order of symmetric groups is not necessary, which is not the case for some other methods.

The other methods and different approaches can be found in Refs. [3] and [4].

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