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İÇİNDEKİLER — CONTENTS

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Mikroelastik Çubukların Eğilme Teorisi

Bending Theory of Microelastic Bars

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Bu çalışmada özel bir mikroelastik malzeme olan moment gerilmesi alan ortamların alan denklemleri kesit alanı üzerinde integre edilerek mikroelastik düz çubukların basit eğilme teorisinin ana denklemleri çıkarılmıştır. Elde edilen denklemler bir örneğe uygulanmıştır.

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In this paper the fundamental equations of the theory of simple bending of microelastic straight bars are derived by integrating the field equations of media having couple stresses, which are a special class of microelastic material, over the cross section area of the bar. The resulting equations are applied to a specific example.

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1. Introduction.

The theory of microelastic media has been developed in recent years and is becoming to draw the attention of researcher more and more [1], [2]. It can be shown that the theory of couple stress, which has been considered since the beginning of this century is but a special class of the general materials which are called microelastic bodies [2]. In the present paper we try to develop a one-dimensional theory of such a material by integrating the general equations over the cross section of a long, thin body. The investigation is confined only to the simple bending. To extend the theory is straightforward. The resulting equations are solved for the case of simply supported beam subjected to uniform load.

2. Fundamental Equations.

As it is well-known the motion of a medium having couple stresses is governed by the following equations in a cartesian system:

$$t_{kl,k} + \rho f_l = \rho a_l$$

$$m_{klm,k} + p_{lm} + \rho l_{lm} = \rho \dot{\sigma}_{lm} \quad t_{kl} = s_{kl} + p_{kl} \tag{2.1}$$

where t_{kl} is the stress tensor, s_{kl} is the symmetric part of the stress tensor, p_{kl} is the antisymmetric part of the stress tensor, **f** is the body force per unit mass, **a** is the acceleration of a material point, ρ is the density of the medium, m_{klm} is the couple stress tensor, l_{kl} is the body moment per unit mass and $\dot{\sigma}_{lm}$ is the internal spin tensor. The tensors **m**, **l** and $\dot{\sigma}$ are antisymmetric with respect to the indices l and m. Dots represent differentiation with respect to time. In the media having microisotropy the spin tensor is given by

$$\dot{\sigma}_{lm} = I_0 \ddot{\varphi}_{lm} \tag{2.2}$$

where I_0 is a material constant and the antisymmetric tensor φ_{lm} stands for the intrinsic rotation of the medium.

If we multiply Eq. $(2.1)_2$ by e_{ilm}, where e_{ilm} is the alternating tensor, we obtain

$$m_{kl,k} + p_l + \rho l_l = \rho I_0 \ddot{\varphi}_l \tag{2.3}$$

where **m**, **p**, **l** and φ_l are defined as

$$m_{kl} = e_{lmn} m_{kmn}\,, \quad p_l = e_{lmn} p_{mn}\,, \quad l_l = e_{lmn} l_{mn}\,, \quad \varphi_l = e_{lmn} \varphi_{mn} \tag{2.4}$$

and conversely one can verify that

$$m_{klm}=\frac{1}{2}\,e_{nlm}\,m_{kn}\,,\quad p_{lm}=\frac{1}{2}\,e_{nlm}\,p_n\,,\quad l_{lm}=\frac{1}{2}\,e_{nlm}\,p_n\,,\quad \varphi_{lm}=\frac{1}{2}e_{nlm}\varphi_n \tag{2.5}$$

By this transformation we establish a duality between the equations $(2.1)_1$ and (2.3). We shall call henceforth the second order tensor m_{kl} as the couple stress tensor. It is seen that the couple stress tensor is affected only by the antisymmetric part of the stress tensor.

We shall now proceed to obtain the macroscopic equations of motion over the cross section of a straight bar whose axis is taken as x_3-axis and x_α-axes $\alpha(=1, 2)$ are in the cross section, being perpendicular to the axis of the bar.

The integration of the equations

$$t_{3\alpha,3}+t_{\beta\alpha,\beta}+\rho f_\alpha=\rho\ddot{u}_\alpha$$

and

$$e_{\alpha\beta}\,x_\beta\,[t_{33,3}+t_{\gamma3,\gamma}+\rho f_3-\rho\ddot{u}_3]=0$$

over the cross section yields respectively

$$\begin{aligned}&\frac{\partial\bar{Q}_\alpha}{\partial x_3}+\bar{q}_\alpha=\rho\,\frac{\partial^2 U_\alpha}{\partial t^2}\\&\frac{\partial\bar{M}_\alpha}{\partial x_3}-e_{\alpha\beta}\,\bar{K}_\beta+\bar{m}_\alpha=\rho\,\frac{\partial^2 V_\alpha}{\partial t^2}\end{aligned} \tag{2.6}$$

where the following notations are defined:

$$\begin{aligned}&\bar{Q}_\alpha=\int_S t_{3\alpha}\,dS,\quad \bar{K}_\alpha=\int_S t_{\alpha3}\,dS,\quad \bar{M}_\alpha=e_{\alpha\beta}\int_S x_\beta\,t_{33}\,dS\,,\\&\bar{q}_\alpha=\int_C t_{\beta\alpha}\,n_\beta\,ds+\int_S \rho f_\alpha\,dS\,,\quad \bar{m}_\alpha=\int_C e_{\alpha\beta}\,x_\beta\,t_{\gamma3}\,n_\gamma\,ds+\int_S e_{\alpha\beta}\,x_\beta\,f_3\,dS\\&U_\alpha=\int_S u_\alpha\,dS,\quad V_\alpha=e_{\alpha\beta}\int_S x_\beta\,u_3\,dS.\end{aligned} \tag{2.7}$$

In this expressions C represents the contour of the cross section, S cross section area and n unit normal to C and $e_{\alpha\beta}$ is the two dimensional permutation symbol. If we define as in the classical theory of bars

$$M_\alpha = e_{\alpha\beta} \int_S s_{33} x_\beta \, dS, \quad Q_\alpha = \int_S s_{3\alpha} \, dS \tag{2.8}$$

and furthermore

$$P_\alpha = 2 \int_S p_{3\alpha} \, dS = e_{i3\alpha} \int_S p_i \, dS = e_{\alpha\beta} \int_S p_\beta \, dS \tag{2.9}$$

we can easily verify the following relations:

$$\bar{M}_\alpha = M_\alpha \, , \quad \bar{Q}_\alpha = Q_\alpha + \frac{1}{2} P_\alpha \, , \quad \bar{K}_\alpha = Q_\alpha - \frac{1}{2} P_\alpha \, . \tag{2.10}$$

Then it is readily seen that Eqs. (2.6) take the forms

$$\begin{aligned} &\frac{\partial Q_\alpha}{\partial x_3} + \frac{1}{2} \frac{\partial P_\alpha}{\partial x_3} + \bar{q}_\alpha = \rho \frac{\partial^2 U_\alpha}{\partial t^2} \\ &\frac{\partial M_\alpha}{\partial x_3} - e_{\alpha\beta} Q_\beta + \frac{1}{2} e_{\alpha\beta} P_\beta + \bar{m}_\alpha = \rho \frac{\partial^2 V_\alpha}{\partial t^2} \, . \end{aligned} \tag{2.11}$$

Q_α can be eliminated from Eqs. (2.11) to give

$$\frac{\partial^2 M_\alpha}{\partial x_3^2} + e_{\alpha\beta} \frac{\partial P_\beta}{\partial x_3} = e_{\alpha\beta} \rho \frac{\partial^2 U_\beta}{\partial t^2} + \rho \frac{\partial^3 V_\alpha}{\partial t^2 \partial x_3} - e_{\alpha\beta} \bar{q}_\beta - \frac{\partial \bar{m}_\alpha}{\partial x_3} \tag{2.12}$$

We must add to the foregoing equations the integral of the equation

$$m_{3\alpha,3} + m_{\beta\alpha,\beta} + p_\alpha + \rho l_\alpha = \rho I_0 \ddot{\varphi}_\alpha$$

over the cross section which is

$$\frac{\partial M_{3\alpha}}{\partial x_3} - e_{\alpha\beta} P_\beta + L_\alpha = \rho I_0 \frac{\partial^2 \Phi_\alpha}{\partial t^2} \tag{2.13}$$

where

$$M_{3\alpha} = \int_S m_{3\alpha} \, dS \, , \quad L_\alpha = \int_C m_{\beta\alpha} n_\beta \, dS + \int_S \rho l_\alpha \, dS \, , \quad \Phi_\alpha = \int_S \varphi_\alpha \, dS \tag{2.14}$$

It is obvious that field equations (2.11) and (2.13) are not sufficient to determine all the involved quantities. Therefore the problem is hyperstatic as it is. We thus have resort to microelastic stress-strain reation in order to render problem determinate.

3. Microelastic Stress-Strain Relations Equations of Motion.

It has been shown that the lineer constitutive equations of a microelastic medium experiencing only couple stress as far as microstructure is concerned can be written as follow [2]:

$$
\begin{aligned}
s_{kl} &= \lambda u_{m,m}\,\delta_{kl} + \mu\,(u_{k,l} + u_{l,k}) \\
p_{kl} &= 2\varkappa\left[\varphi_{kl} - \frac{1}{2}\,(u_{k,l} - u_{l,k})\right] = \varkappa\,[e_{ikl}\,\varphi_i - (u_{k,l} - u_{l,k})] \\
m_{kl} &= -(\alpha + \beta - \delta)\,\varphi_{l,k} + \alpha\,\varphi_{k,l} - \delta\,\varphi_{m,m}\,\delta_{kl}
\end{aligned}
\tag{3.1}
$$

where λ and μ correspond to the classical Lamé constants while $\varkappa$, α, β and δ are new microelastic constants. These constitutive equations clearly show that the antisymmetric part of stress tensor is proportional to the difference between the intrinsic rotation produced by the global deformation of the medium.

Let us now assume that there exists in the domain of the bar a deformation field such as

$$
\begin{aligned}
u_\alpha &= w_\alpha(x_3), \quad u_3 = \theta_\alpha(x_3)\,x_\alpha \\
\varphi_\alpha &= \phi_\alpha(x_3), \quad \varphi_3 = 0
\end{aligned}
\tag{3.2}
$$

whence one obtains the stress field

$$
\begin{aligned}
s_{33} &= t_{33} \cong E u_{3,3} = E\theta'_\alpha\, x_\alpha \\
s_{3\alpha} &= \mu(u_{3,\alpha} + u_{\alpha,3}) = \mu(\theta_\alpha + w'_\alpha), \quad s_{\alpha\beta} = 0 \\
p_{3\alpha} &= -p_{\alpha 3} = \varkappa(w'_\alpha - \theta_\alpha) + \varkappa e_{\alpha\beta}\,\phi_\beta, \; p_{\alpha\beta} = 0 \\
m_{3\alpha} &= -(\alpha + \beta - \delta)\phi'_\alpha \\
m_{\alpha 3} &= \alpha\,\phi'_\alpha, \; m_{\alpha\beta} = m_{33} = 0
\end{aligned}
\tag{3.3}
$$

where primes denote differentiation with respect to the variable x_3. The macroscopic field quantities corresponding to this stress field are given by

$$
\begin{aligned}
Q_\alpha &= \mu S(\theta_\alpha + w'_\alpha) \\
P_\alpha &= 2\varkappa S[w'_\alpha - \theta_\alpha + e_{\alpha\beta}\phi_\beta] \\
M_\alpha &= E\vartheta'_\delta\, e_{\alpha\beta}\int_S x_\beta\, x_\delta\, dS = e_{\beta\alpha} E I_{\alpha\beta}\,\theta'_\delta \\
M_{3\alpha} &= -(\alpha+\beta-\delta)S\,\phi'_\alpha
\end{aligned}
\tag{3.4}
$$

and

$$
U_\alpha = S\,w_\alpha,\quad V_\alpha = e_{\beta\delta} I_{\alpha\beta}\,\theta_\delta,\quad w_\alpha = S\phi_\alpha \tag{3.5}
$$

where $I_{\alpha\beta}$ represents the tensor of moments of inertia and S is the area of cross section. If we substitute these expressions into Eqs. (2.11) and (2.13) we obtain the following system of equations for the unknowns θ_β, w_β and ϕ_β after some manipulations:

$$
\begin{aligned}
&(\mu-\varkappa)\theta'_\alpha + (\mu+\varkappa)w'_\alpha + \varkappa\, e_{\alpha\beta}\phi'_\beta + \frac{q_\alpha}{S} = \rho\,\ddot{w}_\alpha \\
&\theta''_\alpha - \frac{SI_{\alpha\beta}}{EI}[(\mu-\varkappa)\theta'_\beta + (\mu-\varkappa)w''_\beta] - \frac{\varkappa S}{EI} I_{\alpha\beta}\, e_{\gamma\beta}\phi'_\gamma + \\
&\qquad\qquad \frac{e_{\alpha\beta} I_{\alpha\beta}}{EI} m'_\gamma = \frac{\rho}{E}\ddot{\theta}_\alpha \\
&\phi''_\alpha + \frac{2\varkappa}{\alpha+\beta-\delta} e_{\alpha\beta}(w'_\beta - \theta_\beta) - \frac{2\varkappa}{\alpha+\beta-\delta}\phi_\alpha - \frac{L_\alpha}{(\alpha+\beta-\delta)S} = \\
&\qquad\qquad -\frac{\rho I_0}{\alpha+\beta-\delta}\ddot{\phi}_\alpha
\end{aligned}
\tag{3.6}
$$

where we made use of the identity $(I^{-1})_{\alpha\beta} = \frac{1}{I} e_{\alpha\lambda} e_{\beta\mu} I_{\lambda\mu}$, I being equal to the determinant $|I_{\alpha\beta}|$

Equations (3.6) can be solved for θ'_α and the result can be introduced to Eqs. (3.6) and the differentiated $(3.6)_3$ to yield

$$\overset{\mathrm{IV}}{w}_\alpha + \frac{\rho S I_{\alpha\beta}}{EI}\ddot{w}_\beta - \frac{4\mu\varkappa S I_{\alpha\beta}}{(\mu+\varkappa)EI}\, w''_\beta + \frac{2\mu\varkappa\rho I_{\alpha\beta}}{(\mu+\varkappa)EI}\, e_{\gamma\beta}\,\phi'_\gamma + \frac{\varkappa}{\mu+\varkappa}\, e_{\alpha\beta}\,\phi'''_\beta$$

$$-\rho\left[\frac{1}{E}+\frac{1}{\mu+\varkappa}\right]\ddot{w}''_\alpha + \frac{\rho^2}{E(\mu+\varkappa)}\,\overset{::}{w}_\alpha - \frac{\varkappa\rho}{E(\mu+\varkappa)}\, e_{\alpha\beta}\ddot{\phi}'_\beta =$$

$$\frac{I_{\alpha\beta}}{EI}\,(\bar{q}_\beta + e_{\gamma\beta} m'_\gamma) + \frac{\rho\ddot{\bar{q}}_\alpha}{E(\mu+\varkappa)} \qquad (3.7)$$

$$\phi''_\alpha + \frac{4\varkappa\mu}{(\mu-\varkappa)(\alpha+\beta-\delta)}\, e_{\alpha\beta} w'_\beta - \frac{2\varkappa\mu}{(\mu-\varkappa)(\alpha+\beta-\delta)}\,\phi'_\alpha + \frac{\rho I_0}{\alpha+\beta-\delta}\,\ddot{\phi}'_\alpha$$

$$-\frac{2\varkappa\rho}{\mu-\varkappa}\, e_{\alpha\beta}\,\ddot{w}_\beta = \frac{2\varkappa}{(\mu-\varkappa)S}\, e_{\beta\alpha}\,\bar{q}_\beta + \frac{L'_\alpha}{(\alpha+\beta-\delta)S}$$

To neglect of the effect of shearing strains on the deformation results in the relation

$$\theta_\alpha = -\frac{\mu+\varkappa}{\mu-\varkappa}\, w'_\alpha - \frac{\varkappa}{\mu-\varkappa}\, e_{\alpha\beta}\,\phi_\beta \qquad (3.8)$$

Hence, using this in the above expression, we may put Eqs. (3.7) into simpler form:

$$\overset{\mathrm{IV}}{w}_\alpha + \frac{\rho S I_{\alpha\beta}}{EI}\ddot{w}_\beta - \frac{2\mu\varkappa S I_{\alpha\beta}}{(\mu+\varkappa)EI}(2\, w''_\beta + e_{\beta\gamma}\,\phi'_\gamma) + \frac{\varkappa}{\mu+\varkappa}\, e_{\alpha\beta}\,\phi'''_\beta$$

$$-\frac{\rho}{E}\left[\ddot{w}''_\alpha + \frac{\varkappa}{\mu+\varkappa}\, e_{\alpha\beta}\,\ddot{\phi}'_\beta\right] = \frac{I_{\alpha\beta}}{EI}(\bar{q}_\beta + e_{\gamma\beta}\,\bar{m}'_\gamma) \qquad (3.9)$$

$$\phi''_\alpha + \frac{2\varkappa\mu}{(\mu-\varkappa)(\alpha+\beta-\delta)}\,[2\, e_{\alpha\beta}\, w'_\beta - \phi_\alpha] + \frac{\rho I_0}{\alpha+\beta-\delta}\,\ddot{\phi}_\alpha = \frac{L_\alpha}{(\alpha+\beta-\delta)S}$$

The macroscopic field quantities, in this case, can be written as

$$M_\alpha = - e_{\beta\delta}\, EI_{\alpha\beta}\left[\frac{\mu+\varkappa}{\mu-\varkappa}\, w''_\delta + \frac{\varkappa}{\mu-\varkappa}\, e_{\delta\gamma}\, \phi'_\gamma\right]$$

$$= - EI_{\alpha\beta}\left[\frac{\mu-\varkappa}{\mu-\varkappa}\, e_{\beta\gamma}\, w''_\delta - \frac{\varkappa}{\mu-\varkappa}\, \phi'_\beta\right] \tag{3-10}$$

$$M_{3\alpha} = -(\alpha+\beta-\delta) S \phi'_\alpha$$

$$P_\alpha = 2\varkappa S\,[2w'_\alpha + e_{\alpha\beta}\phi_\beta]$$

The above equations may be further simplified by choosing x_α — axes as the principal axes of the cross section.

The expressions (3.10) can be used in formulating the boundary conditions on the ends of the bar.

It is seen that if $\varphi_\alpha = 2e_{\alpha\beta}\, w'_\beta$, that is, if the rotation takes the value given in the classical theory of elasticity, then $P_\alpha = 0$ and all the results coincide with those given the classical theory of bars.

4. Simply Supported Beam Subjected to Uniform Load.

As a simple example to the theory which is developed above we consider a simply supported beam with a span l subjected to a uniform load q_0 acting in the principal direction 2. Simply supported beam will be defined by the boundary conditions.

$$w_2 = M_1 = M_{31} = 0 \qquad at\ x = \mp l/2 \tag{4.1}$$

or

$$w_2 = w_2'' = \phi_1' = 0 \qquad at\ x = \mp l/2 \tag{4.2}$$

The differential equations of the problem in this case are

$$\overset{\mathrm{IV}}{w_2} - 2\varepsilon\,\omega^2 w_2'' - \varepsilon\,(\phi_1''' - \omega^2 \phi_1') = \frac{q_0}{EI_{11}}$$

$$\phi_1'' - k^2\phi_1 + 2k^2 w_2' = 0 \tag{4.3}$$

where we introduce the notations

$$\varepsilon = \frac{\varkappa}{\mu+\varkappa}\,,\quad \omega^2 = \frac{2\mu S}{EI_{11}}\,,\quad k^2 = \frac{2\varkappa\mu}{(\mu-\varkappa)(\alpha+\beta-\delta)}\,. \tag{4.4}$$

It may be easily seen that the classical theory corresponds to the case $\varepsilon \to 0$, $k \to \infty$. If we try to look for the solution of the system of equations (4.3) with constant coefficients in the form $e^{\nu x}$ where we replaced x_3 by x the characteristic equation becomes

$$\nu^4(\nu^2 - a^2) = 0 \tag{4.5}$$

where

$$a^2 = (1-2\varepsilon)k^2 + 2\varepsilon\omega^2 \tag{4.6}$$

Therefore the solution of Eqs. (4.3) satisfying the boundary conditions (4.2) can be written as follows:

$$w_1 = \frac{q_0 l^4}{384\, EI_{11}} \left\{ \frac{k^2}{a^2}\left(16\frac{x^4}{l^4} - 24\frac{x^2}{l^2} + 5\right) - \frac{384}{a^4 l^4}\left(1-\frac{k^2}{a^2}\right)\left[1 - \frac{\cosh ax}{\cosh\frac{al}{2}}\right.\right.$$

$$\left.\left. - \frac{a^2 l^2}{8}\left(1-\frac{4x^2}{l^2}\right)\right]\right\} \tag{4.7}$$

$$\phi_1 = \frac{k^2}{a^2}\,\frac{q_0 l^3}{12\, EI_{11}}\left[4\frac{x^3}{l^3} - 3\left(1 - \frac{8}{a^2 l^2}\right)\frac{x}{l} - \frac{24}{a^3 l^3}\,\frac{\sinh ax}{\cosh\frac{al}{2}}\right]$$

and it follows from Eqs. (3.10) that

$$M_1 = \frac{q_0 l^2}{8}\left[\frac{k^2}{a^2}\left(1-\frac{4x^2}{l^2}\right) + \frac{8}{a^2 l^2}\left(\frac{k^2}{a^2} - \frac{\mu-\varkappa}{\mu+\varkappa}\right)\left(1 - \frac{\cosh ax}{\cosh\frac{al}{2}}\right)\right]$$

$$M_{31} = \frac{(\alpha+\beta-\delta)S}{EI_{11}}\,\frac{k^2}{a^2}\,\frac{q_0 l^2}{4}\left[1 - 4\frac{x^2}{l^2} - \frac{8}{a^2 l^2}\left(1 - \frac{\cosh ax}{\cosh\frac{al}{2}}\right)\right] \tag{4.8}$$

In the limiting case we have $k^2/a^2 \to 1$, $a^2 \to \infty$ and $\alpha+\beta-\delta \to 0$ and we find that the above expressions reduces to the classical results.

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