

ISTANBUL TECHNICAL UNIVERSITY ★ INSTITUTE OF SCIENCE AND TECHNOLOGY

**THE EFFECTS OF DIMENSION RATIO AND HORIZON IN THE
MICROPOLAR PERIDYNAMIC MODEL**

**M.Sc. Thesis by
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Programme : Interdisciplinary Programme

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MICROPOLAR PERIDYNAMIC TEORIDE BOYUT VE HORIZON ETKİSİ

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FOREWORD

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Yipaer Ferhat
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ABBREVIATIONS

MPD : Micropolar Peridynamic Theory
FEM : Finite Elements Methods

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THE EFFECTS OF DIMENSION RATIO AND HORIZON IN THE MICROPOLAR PERIDYNAMIC THEORY

SUMMARY

The basic difference of the classical continuum mechanics and the peridynamic theory is a new approach to the constitutive model. The forces between particles are defined using integration instead of spatial differentiation of displacement. Therefore this constitutive model does not fail in the body where discontinuities are come out. In addition to the constitutive equations, the peridynamic model is taking place in non-local continuum mechanics, in which particles affect each other even though they are separated by a finite distance. In the peridynamic theory the constitutive model contains only central forces and can be applied to only material with Poisson's ratio is $1/4$. This is the most important shortcoming of the peridynamic theory.

To overcome this strict shortcoming, the micropolar peridynamic theory is proposed. The model allows to peridynamic moment, besides peridynamic central force. This addition allows the model to apply to materials with different Poisson's ratio. By this improvement, the micropolar peridynamic model is stated as a generalization of the peridynamic theory. Furthermore, it is possible to use the micropolar theory in the finite element method with harmony. This gives easy application of boundary conditions to physical model in hand.

In this study, elastic behavior of plates, with various lengths and widths, and the effect of horizon selection have been analyzed. The mathematical model of plates has been provided by applying the theory to finite element methods, and the displacement fields have been computed for the different horizons and dimension ratios of plates. To compute the displacement field a program code has been developed using the software package MATHEMATICA. The results have been compared with the analytical solution of classical elasticity theory and with the solution of displacement based finite element methods. For FEM solution the software package ANSYS has been used. According to results, the validation of the theory has been discussed.

MICROPOLAR PERIDYNAMIC TEORİDE BOYUT VE HORIZON ETKİSİ

ÖZET

Klasik sürekli ortamlar mekaniği ile Peridynamic Teori arasındaki en temel fark Peridynamic teorisinin bünye denklemlerine olan yeni yaklaşımıdır. Parçacıklar arasındaki kuvvet yerdeğiřtirmenin uzaysal türevleri yerine integrasyonla ifade edilir. Dolayısıyla teori süreksizliklerin olduđu durumlarda çalışır. Bünye denklemlerindeki farkın yanı sıra Peridynamic Teori yerel olmayan sürekli ortamlar mekaniğine dahildir. Parçacıklar sonlu bir mesafeden de etkileşirler. Peridynamic Teoride bünye denklemi sadece merkezsiz kuvvetleri kapsar ve sadece Poisson Oranı $\frac{1}{4}$ olan malzemelere uygulanabilir. Teorisinin en önemli eksikliği budur.

Bu eksikliği gidermek için, Micropolar Peridynamic Teori önerilmiştir. Bu teori merkezsiz peridynamic kuvvetlerin yanı sıra, peridynamic momenti de kapsar. Bu ekleme teorisinin farklı Poisson oranına sahip malzemelere uygulanabilmesini sağlar. Bu geliştirme ile Micropolar Peridynamic Teori, Peridynamic Teorisinin genelleştirilmiş hali olarak kabul edilebilir. Ayrıca teori sonlu elemanlar metodu ile uygulanabilir. Bu durum sınır koşullarının daha kolay uygulanabilmesine olanak tanır.

Bu çalışmada teorisinin farklı boyutlarda plakların elastic davranışı ve farklı horizon seçimlerinin gerilme dağılımı üzerine nasıl bir etkisi olduđu incelenmiştir. Plakların matematiksel modeli teorisinin sonlu elemanlara uygulanmasıyla oluşturulmuş, farklı boyut ve horizonlar için yerdeğiřtirmeler hesaplanmıştır. Hesaplama için MATHEMATİCA programı ile kod yazılmıştır. Sonuçlar klasik elastisite ve sonlu elemanlar çözümleri ile kıyaslanarak yorumlanmıştır. Sonlu elemanlar yöntemi için ANSYS programı kullanılmıştır. Sonuçlara göre mikropolar peridynamic teorisinin ne ölçüde geçerli olduđu tartışılmıştır.

1. INTRODUCTION

1.1 Statement of The Problem

Continuum mechanics is a branch of science that analyzes the response of the materials to external impacts. These impacts can be classified such as force, temperature, chemical reactions and electric phenomena. The forces can be classified like gravitational forces, electromagnetic forces, and mechanical forces. Solids and fluids have different reactions to the impacts. Therefore, deformation of solids and velocity of fluids are studied in continuum mechanics. If a material is connected by internal forces that keep the particles together, the material is called continuous. The internal forces are different from chemical bonds between atoms. It will be focused on the macroscopic properties, instead of microscopic properties of the material. The material will be considered as homogeneous and continuous while it is modeled.

In classical theory of continuum mechanics, a representative volume element of the material is chosen and reaction of this material to homogeneous deformation is described as stress and strain relation. This reaction expresses the behavior of the whole body. The size of the representative volume is decided according to the body. It changes from Angstroms for crystal lattices to centimeter for concrete.

Elementary formulation of classical theory depends on deformation gradients and their derivatives. From mathematical aspect when there is a discontinuity, such as crack or phase boundary, the deformation gradients and their derivatives can not be expressed. So the characteristic length of the deformation has to be large enough respect to the representative element of the body in order to be valid in classical theory [1].

To solve the problem the discontinuous area is reformulated in classical approach. But recovering approaches depend on the severity of the problem and these are unique solutions for the problem. For example, “crack lips are considered body surfaces with particular boundary conditions. Crack tips satisfy a particular energy balance that is different from the rest of the body. Phase boundaries are surfaces

inside the body that satisfy particular jump conditions and kinetic relations. ” [2]. When ever defects happen, a new treatment is needed to solve the problem. While the major region of the body is expressed with regular continuum mechanics, the discontinuity is expressed using physical laws according to basic response of the body. This may be unmanageable and expensive for every time. Especially analytical solution can be too complex to solve while numerical solutions are expensive due to remeshing. As this techniques may not be enough, also can not be used for instant defects.

1.2 Applied Solution Techniques

There are some different approaches to solve the discontinuity problem, such as non-local theories, handshake methods and macroscopic treatment of defects [2].

Non-local theories: The material models that do not depend on the characteristic length are called simple or local theories. Changing the size of the representative volume does not affect the model. Opposite of these material, are called non-local and they have an internal length. In small scales this model works better. The non-local model approach can be divided in two parts, weakly non-local and strictly non-local or integral type non-local model. In weakly non-local model the higher order gradients are used to describe the fundamental formulation. In strictly non-local or integral type non-local models the stress depends on the deformation of a finite environment of a continuum point. In both model the non-local character of inter atomic forces are defined with the non-locality of the model.

Handshake methods: This approach divides the body into different region and describes them assuming suitable scale for every region. This method is used basically conjugating the Finite Element model and Molecular Dynamics. The difficulties of this method all interfaces must be modeled separately and existence of some unphysical forces at the interfaces of different scales.

Macroscopic Treatment of Defects: Cracks in arbitrary direction can be created by cohesive surface models. Macroscopically behavior of the body can be described with this model effectively. So this method can express the friction on crack faces but all crack tip must be modeled separately.

A macroscopic approach gained using atomistic laws is called The Virtual Internal Bond Model. For small scaled parts atomistic structure can be applied accurately by Finite Element aspect. However, the area must be remeshed for adequate numerical solution.

1.3 Peridynamic as a New Approach

Although there have been some remedies to achieve the discontinuity problem in continuum mechanics, these are not adequate for many situation and only some special conditions have considerable solution. As indicated all above, these techniques can not avoid using spatial differential equations in mathematical structure. And whenever happens a spontaneous formation of discontinuities, the mathematical modeling must be convert to suitable form after creation of discontinuity. For constitutive relation, the position of the discontinuity and boundary condition must be known. Therefore spontaneous formation of discontinuities can not be modeled before it exists such as cracks. These are the main reasons of necessity for a theory that works off and on defects and avoids using differential equation which is undefined at discontinuity.

To provide the necessity, Silling has purposed a new theory called peridynamic [3]. The term peridynamic comes from the Greek roots as near and force. This continuum model does not use well-defined deformation gradients or any spatial derivatives at discontinuity. Integral equation is used as constitutive model instead of differential equation and this is the basic feature of the theory so the constitutive relation is always valid at every point. The main structure of the theory derived from molecular dynamics and it sums the forces between particles. This method is classified as non-local model due to the interaction of particles from a finite distance. Although some non-local models used to solve the problem this theory is absolutely different from the others because other methods use stress and strain relation, that need to be differentiated in the equation of motion, while peridynamic does not use these properties.

In the peridynamic theory, the constitutive model contains only central forces and therefore it can be applied to only material having Poisson's ratio is $1/4$. This is the most important shortcoming of the peridynamic theory.

1.4 Micropolar Peridynamic Theory

To overcome strict Poisson's ratio condition of the peridynamic theory, Gerstle, Sau and Silling generalized the peridynamic theory and the new approach is called "The Micropolar Peridynamic Model" [4].

In the micropolar peridynamic model, besides peridynamic central forces, the peridynamic moments are also considered, and they interact with particles inside the material horizon. The introduction of the moments to the theory allows us to deal with the materials having Poisson's ratio different from $1/4$. This modification can be seen as the generalization of the peridynamic theory. Furthermore, the micropolar peridynamic theory can be easily implemented using the finite element methods. This provides easy application of the boundary conditions to the physical model in hand.

2. FUNDAMENTAL THEOREMS

2.1 Conventional Theory of Continuum Mechanics

2.1.1 Motion and deformation

The stress is described as intensity of force in unit of force over area. When a force applied to two area with different sizes, the results will be different locally. This is explained by stress consideration. The stress is modeled by a second order tensor.

The strain is expressed as the alteration in the geometry of the particle in the material due to stress. When strain is described the term of deformation is usually used. The stress and strain terms are used to consider motion of the body.

Configuration of a body is described as the complete specification of the positions of all particles of a body. Because the position of the particles of the body can reside in several position in a three dimensional space. Reference configuration of body that the configuration of the body at the reference time (or initial or undeformed state) is symbolized by R_0 and the current configuration at the given instant time (or deformed state) is symbolized by R_t . To explain the concept of deformation, it is considered that a material point P within a neighborhood N_0 at the reference configuration. As shown in Figure2.1, point P moves to point p within a neighborhood N_t at the current configuration due to load applied to the body. It can be understood clearly from the figure that the element is at the reference configuration and also at the current configuration. When compared the two states of the element it is distinguished that the size and the shape of two states are not the same. Deformation of the material is the change of size and shape.

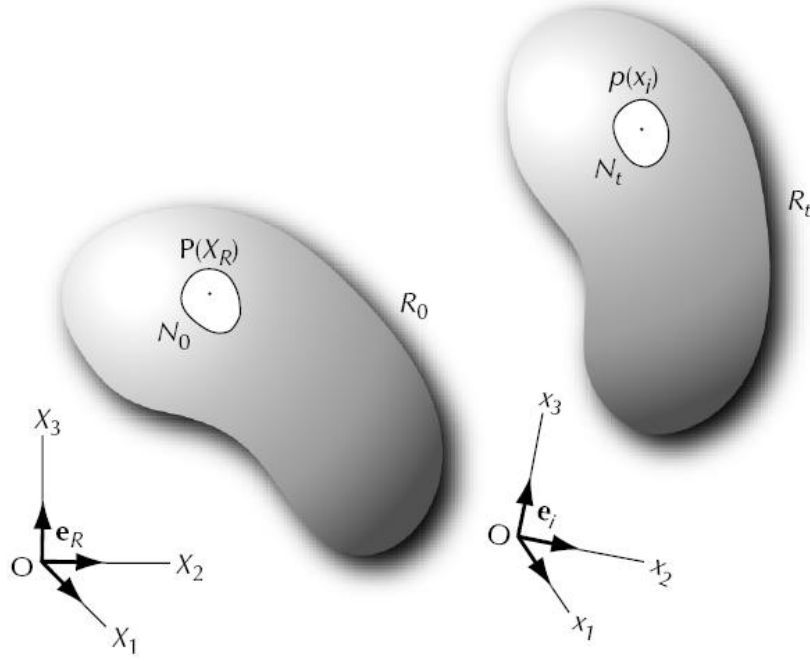


Figure 2.1 : Reference and current configuration [5].

A material (Lagrangian) coordinate system and a spatial (Eulerian) coordinate system, denoted by X_R and x_i respectively, introduced to define the two configurations. The material coordinate X_R is the position of the material point P at the reference configuration and spatial coordinates x_i is the position of the material point p at current configuration. Reference and current configurations are usually selected from the undeformed configuration of a material element for being suitable. However, if it is necessary any other configuration may also be chosen as the reference configuration. After a reference configuration has been selected, using the material coordinates X_R , all the material points of a body can be expressed. In most situation, the origin of the X_R and x_i axes and orientations are not the same as seen in Figure 2.1. Although in many situations two coordinate systems can be selected that they have the same origin and orientation, this selection leads to simplicity.

Motion may be described as a family of deformations continuously varying in time. When a continuum is in motion, physical quantities associated with a material element of the body (such as density ρ , velocity v , etc.) change with location and time. The motion of a continuous body may be described in two ways: material and spatial descriptions. The material description has been used mostly in solid

mechanics and the spatial description in fluid mechanics. Some recent works have used a mixed description.

2.1.2 Conservation laws and constitutive equation

The *conservation laws* of physics and the *constitutive equations* are the fundamental equations of a body undergoing deformation. Conservation of mass, linear momentum, angular momentum, and energy are included in the conservation laws of physics. Before these laws are reduced to local form, they are derived in the global form. The global form considers forces, momentum, or energy acted on the whole deformable body, while the local form refers to a representative infinitesimal element of the body [5].

The conservation laws are applicable to a body of any material, besides the constitutive equations are used to identify the body made of different materials. For example, considering a long bar that is under a tensile force, when conservation equations are written for this bar are the same for any kind of the material of the body. However, it is proven experimentally that different kind materials show different deformation properties to same tensile force. So, the constitutive equations are needed to express accurately the different kind of materials.

Basically constitutive equations are described as a set of equations, which defines the thermo mechanical properties of a material. One constitutive equation or a set of constitutive equations represent the material behavior. It means simple or complex reaction of material. The simplest form of constitutive equation for solids is the most known equation called Hooke's law. Although it is usually suitable for small and linearly elastic deformations at room temperature, the validity of Hooke's law depends on some conditions. Under the conditions that not at constant temperature, large deformation and the plastic region, the constitutive equations needs to be modified or extended to a set of (nonlinear) equations.

Hooke's law is also known as the stress–strain relation, because the equation relates the stress to the strain; for isotropic linearly elastic solid, it is written as:

$$\sigma = 2\mu\varepsilon + \lambda I\varepsilon_{kk} \quad (2.1)$$

Where σ and ε are the engineering stress and strain, respectively, and μ and λ are the Lamé constants.

2.2 Peridynamic Theory

2.2.1 Fundamental derivations of peridynamics

To derive the formulation of the theory in a body, first a body is described. The body has a reference configuration in region R as seen in the Figure 2.2. It is assumed that each pair of particles interact each other by a vector –valued function f . Integration of f function over a unit volume is a force function such as L in the unit of force per reference volume. L is a functional of displacement field u [3].

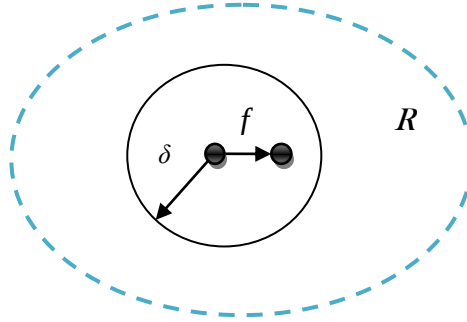


Figure 2.2 : The body in the region R .

According to these definitions the quantity of force function L , at any time t and at any point x in the reference configuration, is described as follow.

$$L_u(x, t) = \int_R f(u(x', t) - u(x, t), x' - x) dV_{x'} \quad \forall x \in R, \quad t \geq 0 \quad (2.2)$$

Without variant of time the equation is written simpler,

$$L_u(x, t) = \int_R f(u' - u, x' - x) dV' \quad \text{on } R \quad (2.3)$$

The peridynamic equation of motion is written as below,

$$\rho \ddot{u} = L_u + b \quad \text{on } R \quad t \geq 0 \quad (2.4)$$

When the condition is equilibrated the peridynamic equilibrium equation is expressed as below,

$$L_u + b = 0 \quad \text{on } R \quad (2.5)$$

Where:

b: Prescribed loading force density in the unit of per unit reference volume described as external force.

f: Pairwise force function.

As seen equation (2.3) there is no spatial derivative.

To simplify the writing of the constitutive equation, some notations are described as η , ξ and $\xi + \eta$.

Relative displacement vector:

$$\eta = u' - u \quad (2.6)$$

Relative position vector:

$$\xi = x' - x \quad (2.7)$$

Relative position of the particle in the deformed configuration:

$$\xi + \eta \quad (2.8)$$

Some specific conditions for equation (2.5) are described equilibrated and pairwise equilibrated as below.

Equilibrated: when $b \equiv u \equiv 0$ is provided in equation (2.5).

Pairwise equilibrated: When $f(0, \xi) = 0 = 0$ is provided for all $\xi \neq 0$ in equation (2.5).

In equation (2.3) the pairwise force function f does not show any memory of the deformation history so such materials in the form $f(u' - u, x' - x)$ are called

peridynamic material without memory. Also the form L_u in the equations involves $f(u' - u, x' - x)$ form is formulated for homogenous material.

Other basic restrictions for f are linear admissibility condition and angular admissibility condition. According to Newton's Third Law the force between two particles due to each other have some magnitude but opposite direction. Therefore the forces must have opposite signs.

$$f(-\eta, -\xi) = -f(\eta, \xi) \quad \forall \eta, \xi \quad (2.9)$$

This condition is called linear admissibility. Also the force between particles must be in the direction of their relative current position.

$$(\xi + \eta) \times f(\eta, \xi) = 0 \quad \forall \eta, \xi \quad (2.10)$$

This condition is called angular admissibility.

From the equation (2.9) and (2.10) the most general form of f can be written including the current relative deformation vector and a scalar valued function F .

$$f(\eta, \xi) = F(\eta, \xi)(\xi + \eta) \quad \forall \eta, \xi \quad (2.11)$$

Other interpretation of equations (2.9) and (2.11) can be made from the mathematical aspect as f must be odd function and F must be even function. Therefore the F function can be explained as follow:

$$F(-\eta, -\xi) = F(\eta, \xi) \quad \forall \eta, \xi \quad (2.12)$$

Other important assumption for f is the definition of horizon. To provide any interaction between particles, both of the two particles must be in a region that the distance between two particles must not exceed some specific value such as δ is called horizon of the material. So, when $|\xi| > \delta$ for any η the f function is equal to zero, $f(\eta, \xi) = 0$.

2.2.2 Material properties in peridynamic

2.2.2.1 Isotropy

For a peridynamic material without memory and its pairwise force function f , it is assumed that there is set of tensor such as $\wp$ and the f function can be written as;

$$f(Q\eta, Q\xi) = Qf(\eta, \xi) \quad \forall \eta, \xi \quad \forall Q \in \wp. \quad (2.13)$$

$\wp$ is called material symmetry group. And $\wp^+$ is the set of proper orthogonal tensors, a peridynamic material is called isotropic if the peridynamic material is described with this symmetry group. Therefore an isotropic material can be rewritten as below;

$$f(Q\eta, Q\xi) = Qf(\eta, \xi) \quad \forall \eta, \xi \quad Q \in \wp^+ \quad (2.14)$$

In addition, isotropic definition for scalar valued function F can rewritten as below;

$$F(Q\eta, Q\xi) = F(\eta, \xi) \quad \forall \eta, \xi \quad Q \in \wp^+ \quad (2.15)$$

The interpretation of these equations is, if the response of a peridynamic material does not depend on the orientation of material, the material has no special direction, the material is called isotropic.

Other explanation for equation (2.15) is that in a isotropic material the value of F does not change when a given pair of vectors ξ and η are rotated without changing the angle between them. So this explains that the value of F is affected only the geometry of the triangle with two sides of lengths $|\xi|$ and $|\eta|$, and the angle between these sides, $(\cos^{-1} \frac{\xi \cdot \eta}{|\xi||\eta|})$.

Because it is more useful and completely equivalent $|\xi + \eta|$ is used instead of $|\eta|$. So the F function can be rewritten in most general form as below:

$$F(\eta, \xi) = I(p, q, r) \quad p = |\xi + \eta| \quad q = |\xi \cdot \eta| \quad r = |\xi| \quad \forall \eta, \xi \quad (2.16)$$

I is described as a scalar valued function. When I is substituted in equation (2.11) it is rewritten equivalently as below;

$$f(\eta, \xi) = I(p, q, r)(\xi + \eta) \quad \forall \eta, \xi \quad (2.17)$$

Due to the reason that p, q and r are invariant according to rigid rotation, if a material is described in equation (2.16) is necessarily isotropic due to equation (2.15).

2.2.2.2 Elasticity

When a peridynamic material obeys the equation below, it is called microelastic.

$$\int_{\Gamma} f(\eta, \xi) \cdot d\eta = 0 \quad \forall \text{ closed curve } \Gamma \quad \forall \xi \neq 0 \quad (2.18)$$

The equation shows that, while particle x' moves along any closed path, the net work done due to the interaction between particles x and x' is equal to zero.

When f is continuously differentiable in η , using Stokes' Theorem, it can be shown that a necessary and sufficient condition for equation (2.18) as below;

$$\nabla_{\eta} \times f(\eta, \xi) = 0 \quad \forall \xi \neq 0 \quad (2.19)$$

$\nabla_{\eta} \times$: The vector curl operator evaluated with respect to the coordinates of η .

In addition another necessary and sufficient condition for a microelastic peridynamic material is the existence of a differentiable, scalar-valued function w , called the pairwise potential function, is defined below.

$$f(\eta, \xi) = \frac{\partial w}{\partial \eta}(\eta, \xi) \quad (2.20)$$

If a peridynamic material is microelastic it must be peridynamic material without memory. Otherwise the material does not have a physical meaning. Because in a material that is not microelastic always produces energy due to the cyclical motion of particles pairs. The material that is not microelastic must be with memory to be physically reasonable.

To transform from microelasticity to macroelasticity that consists the whole body not only pairs of particles, the pairwise potential function will be integrated over the whole body.

$$W_u(x) = \frac{1}{2} \int_{\mathfrak{R}} w(u' - u, x' - x) dV' \quad (2.21)$$

$W_u(x)$: The macroelastic energy density.

$$\Phi_u = \int_{\mathfrak{R}} W_u(x) dV \quad (2.22)$$

Φ_u : Total macroelastic energy.

To set a relation between classical theory and peridynamic, the steps are carried out as below.

First it assumed that u depends on time and time derivative of total macroelastic energy is written.

$$\dot{\Phi}_u = \frac{1}{2} \frac{\partial}{\partial t} \int_{\mathfrak{R}} \int_{\mathfrak{R}} w(u' - u, x' - x) dV' dV \quad (2.23)$$

Using chain rule;

$$\dot{\Phi}_u = \frac{1}{2} \frac{\partial}{\partial t} \int_{\mathfrak{R}} \int_{\mathfrak{R}} \frac{\partial w}{\partial \eta} \frac{\partial \eta}{\partial t} = \frac{1}{2} \int_{\mathfrak{R}} \int_{\mathfrak{R}} \frac{\partial w}{\partial \eta} (u' - u, x' - x) (\dot{u}' - \dot{u}) dV' dV \quad (2.24)$$

Using $(x' \leftrightarrow x)$ changes and odd function feature of equation (2.9), when equation (2.4) and (2.20) are substituted into the equation (2.24);

$$\begin{aligned} \dot{\Phi}_u &= - \int_{\mathfrak{R}} \int_{\mathfrak{R}} f(u' - u, x' - x) \dot{u} dV' dV = - \int_{\mathfrak{R}} [\rho \ddot{u} - b] \dot{u} dV \\ &= - \int_{\mathfrak{R}} \left[\frac{\partial}{\partial t} \left(\frac{\rho}{2} \dot{u} \dot{u} \right) - b \dot{u} \right] dV \end{aligned} \quad (2.25)$$

$$\dot{\Phi}_u + \dot{T}_u = \int_{\mathfrak{R}} b \cdot \dot{u} dV \quad (2.26)$$

T_u : The total kinetic energy of the body.

As in the conventional formulation of solid mechanics that if the body is microelastic when the external force does any work on the body, the work is converted either to kinetic energy or to macroelastic energy density. Here the macroelastic energy density can be thought as strain energy of a solid. We can compare the elastic material using this way.

2.2.3 Linearization of the peridynamic theory

For the general peridynamic material assuming that relative displacement vector is so small ($|\eta| \ll 1$) the theory can be linearized. ξ is considered fixed.

$$f(\eta, \xi) = C(\xi)\eta + f(0, \xi) \quad \forall \eta, \xi \quad (2.27)$$

C : the second-order micromodulus tensor.

$$C(\xi) = \frac{\partial f}{\partial \eta}(0, \xi) \quad (2.28)$$

Substituting the equation (2.28) and (2.27) into equation (2.3);

$$Lu = \int_R [C(x' - x)(u' - u) + f(0, x' - x)] dV' \quad \forall x \in R \quad (2.29)$$

Because the L is linear it is used in a different notation ($L_u = Lu$).

If the body has an internal sub region such as R^0 the equation (2.29) can be rewritten as below;

$$Lu = \int_{\mathfrak{R}} C(x' - x)(u' - u) dV' \quad \forall x \in R^0 \quad (2.30)$$

Due to f is odd function, C must be even function according to equation (2.28).

$$C(\xi) = C(-\xi) \quad \forall \xi \quad (2.31)$$

“On physical grounds, one expects that as the distance between a pair of particles gets very large, the interaction between them becomes negligible and consequently that:

$$C(\xi) \rightarrow 0 \quad \text{as } \xi \rightarrow \pm\infty \quad (2.32)$$

In what follows we shall assume that this happens sufficiently fast to ensure the convergence of the various infinite integrals encountered. If the material has a “horizon”, i.e. if there is no interaction between particles that are more than some finite distance ξ_c apart, then $C(\xi) \equiv 0$ for all $\xi \geq \xi_c$ and so Eq. (2.32) holds trivially.

The notion of stress does not play a fundamental role in the peridynamic theory. It is helpful however to introduce such a notion in order to compare the present theory with the classical theory. By adapting Cauchy’s notion of stress in a crystal, one can define the “stress” $\sigma(x, t)$ on the particle x to be,

$$\sigma(x, t) = \int_{-\infty}^x \left(\int_x^{\infty} f(x'', x', t) dx' \right) dx'' \quad (2.33)$$

Where the right-hand side represents the total force that all material particles x'' to the right of x exert on all material particles x' to its left. In the particular case of a homogeneous displacement, $u(x) = \varepsilon x$, $\varepsilon = \text{constant}$; of a linear microelastic material, Eq. (2.33) simplifies to the “stress–strain relation”

$$\sigma = \varepsilon E \quad (2.34)$$

$$E = \int_0^{\infty} \xi^2 C(\xi) d\xi \quad (2.35)$$

Thus E can be identified with the Young's modulus of classical linear elasticity, and Eq. (2.35) relates E to the micromodulus C of the microelastic material. Note that Eq. (2.34) is physically meaningful only in the context of homogeneous deformations.

In summary, the one-dimensional response of a linear microelastic material is characterized by its micromodulus function $C(\xi)$ which is required to obey Eqs. (2.32). C characterizes the “stiffness” of the material in the presence of long-range forces and can be thought of as a nonlocal counterpart of a material's modulus. C involves a “material” length-scale parameter ℓ that captures the effects of the long-range forces. In the limit of short-range forces, i.e. $\ell \rightarrow 0$, one would expect the present theory to converge to classical elasticity theory. In the simplest model of a linear microelastic material, the micromodulus function would involve only two constitutive parameters, viz. E and ℓ : $C = C(\xi; \ell, E)$ for such a material, dimensional analysis of Eq. (2.35) shows that C then necessarily has the form[6].

$$C = C(\xi; \ell, E) = \frac{E}{\ell^3} \hat{C}(\xi / \ell) \quad (2.36)$$

Where $\hat{C}$ does not involve any constitutive parameters. Because of $\hat{C}(\xi) = \hat{C}(-\xi)$ and because of Eq. (2.35),

$$\int_0^\infty \xi^2 \hat{C}(\xi) d\xi = 1 \quad (2.37)$$

Some examples of micromodulus functions:

Consider the following two examples of micromodulus functions,

$$C_1(\xi) = E d_\ell''(\xi) \quad (2.38)$$

$$C_2(\xi) = -2E \frac{d_\ell'(\xi)}{\xi} \quad (2.39)$$

Where each involves the Young's modulus E and a length-scale parameter ℓ , the sequence of functions $d_\ell(\xi)$, parameterized by ℓ , is a generalized function in the sense that it approaches a Dirac-delta distribution as $\ell \rightarrow 0$.

Two specific examples of d_ℓ are;

$$d_\ell^{Gauss}(\xi) = \frac{e^{-(\xi/\ell)^2}}{\ell\sqrt{\pi}} \quad (2.40)$$

$$d_\ell^{Sine}(\xi) = \frac{\sin(\xi/\ell)}{\pi\xi} \quad (2.41)$$

Thus, we consider the following four constitutive models for a linear microelastic material as defined by Eqs. (2.38), (2.39), (2.40) and (2.41):

$$C_1^{Gauss}(\xi) = \frac{2E}{\ell^3\sqrt{\pi}} \left[2(\xi/\ell)^2 - 1 \right] e^{-(\xi/\ell)^2} \quad (2.42)$$

$$C_2^{Gauss}(\xi) = \frac{4E}{\ell^3\sqrt{\pi}} e^{-(\xi/\ell)^2} \quad (2.43)$$

$$C_1^{Sine}(\xi) = \frac{E}{\ell^3\pi} \left[\frac{-2(\xi/\ell)\cos(\xi/\ell) + (2 - (\xi/\ell)^2)\sin(\xi/\ell)}{(\xi/\ell)^3} \right] \quad (2.44)$$

$$C_2^{Sine}(\xi) = \frac{E}{\ell^3\pi} \left[\frac{-2(\xi/\ell)\cos(\xi/\ell) + 2\sin(\xi/\ell)}{(\xi/\ell)^3} \right] \quad (2.45)$$

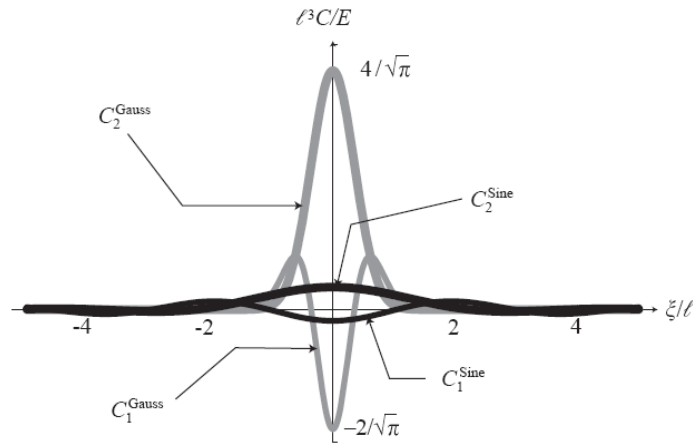


Figure 2.3 : Micromodulus functions [6].

Figure 2.3 shows graphs of these four micromodulus functions. Each $C(\xi)$ satisfies the symmetry requirement, the stability requirement and vanishes as $\xi \rightarrow \pm\infty$. The micromodulus $C_2^{Gauss}(\xi)$ is strictly positive for all ξ , while the other three take on negative values for some ranges of ξ .” [6]

2.2.4 Some examples of peridynamic applications

2.2.4.1 Impact of a sphere on a brittle target

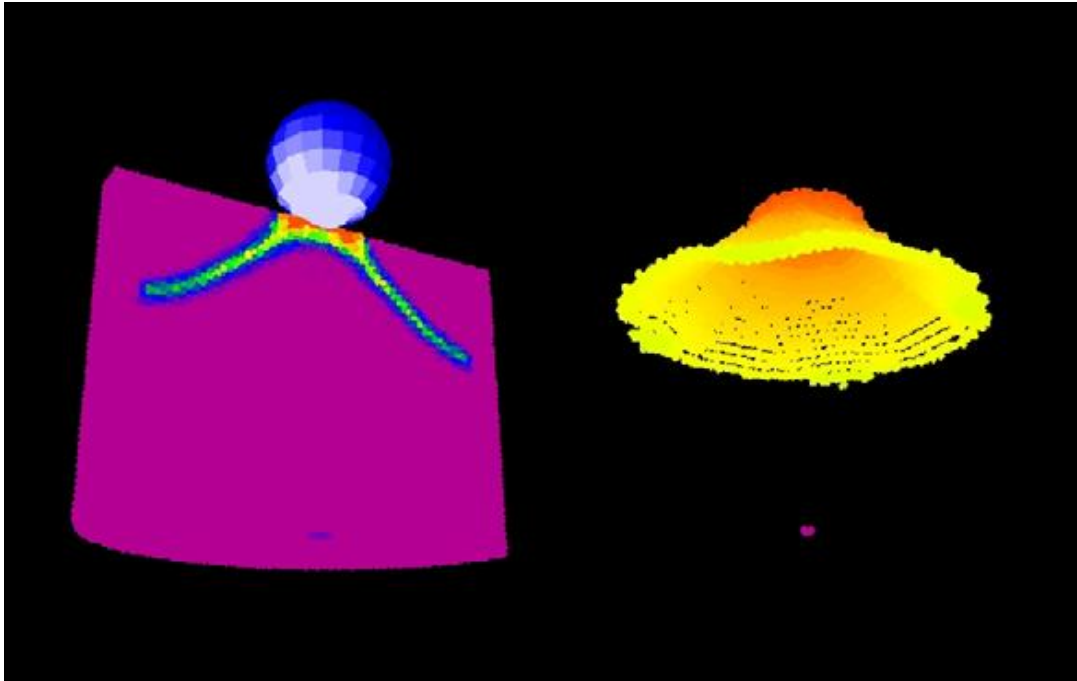


Figure 2.4 : Hertzian cracks on a brittle solid: (left), cross-section showing damage contours and (right) conical crack [7].

This example demonstrates the influence on a homogeneous block of brittle material by a rigid sphere. The features of the sphere is mass = 4.16 g, diameter = 10 mm, and impact velocity = 35 m/s directed normal to the target surface. The density of the target material is $\rho = 2200 \text{ kg/m}^3$. The prototype microelastic brittle (PMB) material model is used for the target. The bulk modulus is $k = 14.9 \text{ GPa}$. The target is a cylinder with diameter 34 mm and height 25 mm.

Figure 2.4 shows two the predicted crack views. In the left plot there is a capture of a cross-section of the target after the projectile has rebounded, it shows the damage in the target material and damage is the quantity displacement. The crack occurs is as a

cone-shaped surface that is called as a Hertzian crack. In the right capture the three-dimensional view of this crack is shown.

2.2.4.2 Dynamic growth of a single crack from a defect

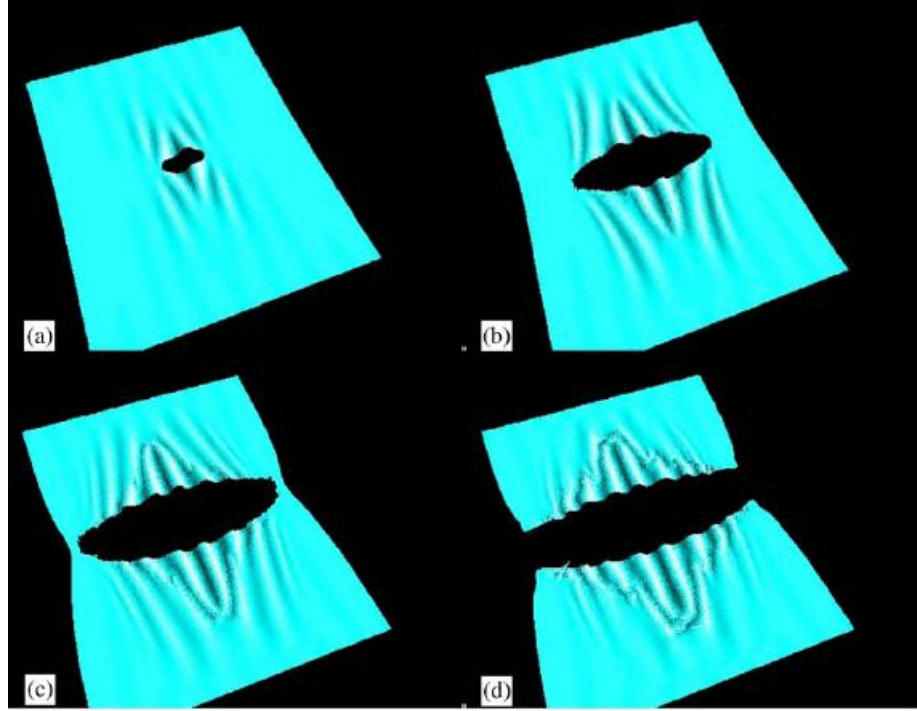


Figure 2.5 : Tensile loading of a membrane containing a slit, showing transition to dynamic fracture [8].

In this example a square elastic membrane with thickness 0.5mm and side length 50mm including a narrow fissure with length 10 mm is examined. The edges of the membrane that is parallel to the slit and the other edges are free. The membrane is also free in normal direction.

Figure 2.5 shows the way that the fissure develops into a dynamic crack at four different times. In view (a) it is shown that, the fissure has not yet started to develop. There is some wrinkles at the fissure edges. The reason of the existence wrinkles is the freedom of the membrane at the slit edges in the normal direction. In the other captures the crack is developing. In the last view (d), the crack has reaches to the free edges of the membrane, and the membrane has separated to two different pieces.

2.2.4.3. Bursting of a balloon

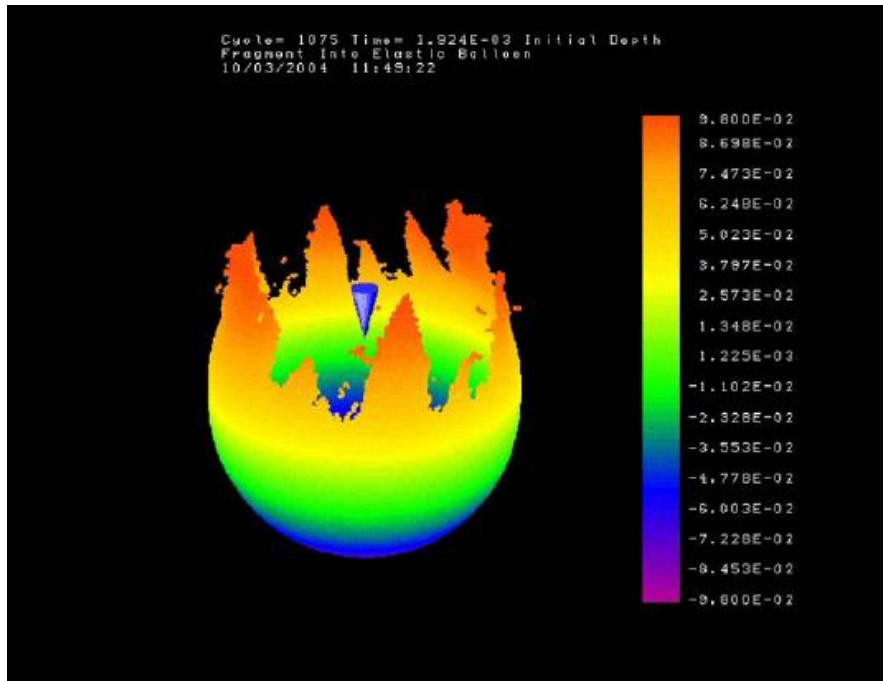


Figure 2.6 : Internally pressurized spherical membrane damaged due to impact of a sharp fragment [8].

In this sample, a spherical membrane is struck by a rigid shard. The spherical membrane has an internal pressure. The density of the unpressurized balloon is $\rho = 1200 \text{ kg/m}^3$. The bulk modulus of the material of the balloon is 800MPa. The sphere has a radius 0.1m and thickness is 0.0015m. The internal pressure is 1.5MPa.

In Figure 2.6 it is shown that the propagation of dynamic fractures from the point of impact around the circumference of the balloon is irregular and the cracks are blunt.

2.2.4.4 Fracture

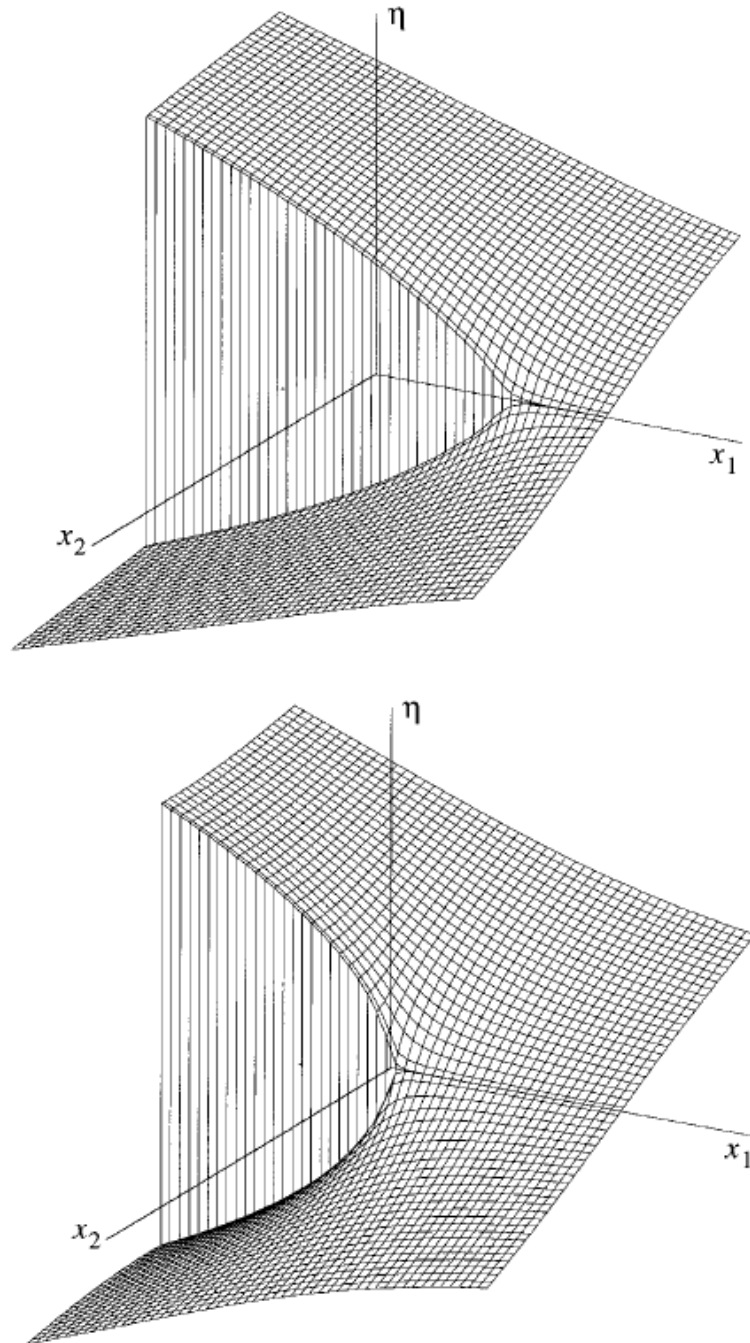


Figure 2.7 : Crack tip behaviour predicted by the peridynamic (up) and the conventional theory (down) [3].

In Figure 2.7 a thin sheet initially parallel to the $x_1 - x_2$ plane is deformed. The vertical surface represents a crack surface. The crack is examined both using peridynamic theory and conventional theory respectively (a) at the top and (b) at the bottom. The basic difference is in the crack tip. An interesting feature of the peridynamic solution is the cusp-like shape of the crack tip. It is different in conventional theory of elasticity. In the elasticity theory the shape of the crack and its tip is parabolic.

2.2.4.5. Other literatures on peridynamic

In addition to cracks, dynamic fracture of a steel material, opening a hole on a thin brittle structure with a sharp object, dynamic bursting of a balloon, fracture of reinforced concrete, fracture of composites and layer separation of multiple layer composites structures have been modeled using peridynamic theory [9].

Analysing an infinite bar under self-equilibrated load with peridynamic theory, it has been observed that there is decaying displacement propagation different than classical results [10]. Also it has been showed that, taking the horizon zero the theory converges to classical continuum theory. [10] Also for different fracture conditions initial value problems have been solved [11, 12] The error has been caused by numerical methods while solving the problem with peridynamic theory has been investigated [13].

In another work detailed analysis of peridynamic theory and compare with classical theory has been carried out [2]. In this work the existing problem has been defined and the remedies, have been applied so far, have been showed. The contributions of peridynamic theory, analytical and numerical solution techniques have been explained. By comparing peridynamic and classical theory, the superiority and shortcomings of the theory, and the case that peridynamic theory converges to classical theory has been analysed in detail. Also with the fracture examples the application area has been extended [2].

In the case of fast transformation, features using on electrical and optical materials have been explained with examples in other works [14, 15].

The wave propagation in anisotropy condition and high rate material errors have been analysed [16, 17]. The force flux and peridynamic stress have been defined

representing the peridynamic state and constitutive equations[18-20]. These definitions have been implemented using finite element method[21]. The conditions in which peridynamic theory converges to classical theory have been showed using peridynamic stress definition[22].

The implementation of peridynamic theory with molecular dynamic code has been carried out and the similarities have been emphasized [23]. The features of the peridynamic theory for multiscale modeling has been studied [24]. Recently, for composites progressive damage prediction and error prediction for brittle materials such as glasses have been studied [25, 26]. Also using peridynamic theory, the material stability and failure analysis have been carried out [27].

3. MICROPOLAR PERIDYNAMIC MODEL

3.1 Micropolar Continuum

One of the branches of the classical continuum mechanics can be considered as micropolar continuum. This theory has been proposed first time by The Cosserat Brothers in 1909 [28]. When the size of the sample decreases the strength increases and the classical continuum theory can not estimate this mechanism. Also, the condition that the small samples show high strain gradients and higher strength has been observed experimentally. Cosserat continuum or couple stress theory has the stress and couple stress (represents moment) as below and also is shown in the Figure 3.1 and 3.2.

$$T_i^v = \tau_{ji} v_j \quad (3.1)$$

$$M_i^v = m_{ji} v_j \quad (3.2)$$

Where,

$\vec{T}^v$ the Cauchy stress vector acting on a surface S^v

$\vec{v}$ unit normal

$\vec{M}^v$ is the couple stress vector acting also upon S^v

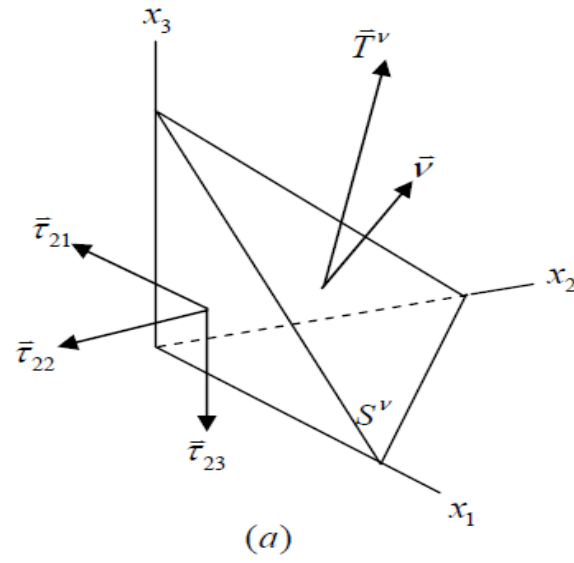


Figure 3.1 : In Cosserat model tetrahedron with Cauchy [28].

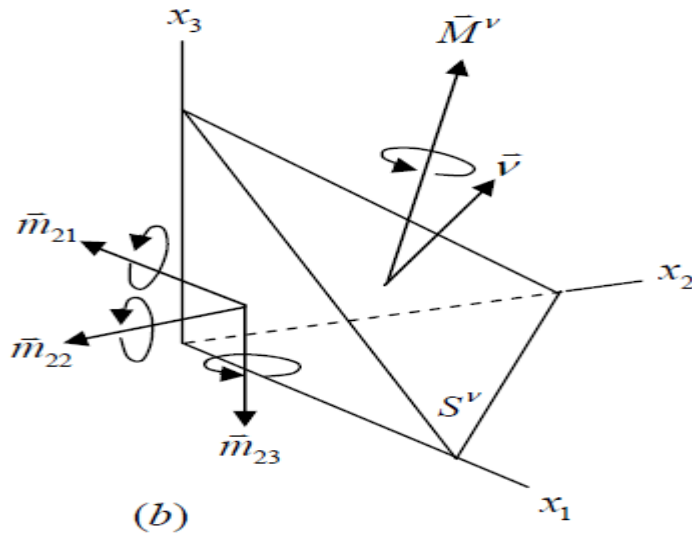


Figure 3.2 : In Cosserat model tetrahedron with couple stresses [28].

On a differential element the differential constitutive equation for static equilibrium condition can be represented as follow:

$$\tau_{ji,j} + b_j = 0 \quad (3.3)$$

$$m_{ji,j} + \varepsilon_{ijk} \tau_{jk} + c_i = 0 \quad (3.4)$$

Where;

b_i the body force

C_i the body couple force

τ_{ji} the stress tensor

m_{ji} the couple stress tensor

On a differential element the differential constitutive equation for dynamic equilibrium condition can be represented as below:

$$\tau_{ji,j} + b_j = \rho \ddot{u}_i \quad (3.5)$$

$$m_{ji,j} + \varepsilon_{ijk} \tau_{jk} + C_i = 0 \quad (3.6)$$

Where;

ρ Material density

$\ddot{u}_i$ Acceleration

3.2 Micropolar Peridynamic Theory

In the following section the fundamental equation of motion for micropolar peridynamic theory will be derived and related mathematical operations are presented explicitly.

In the micropolar peridynamic model, besides peridynamic central forces, the peridynamic moments are also considered, and they interact with particles inside the material horizon. Infinitesimally-small particle i and j interact with each other over a finite distance in the domain R [28]. The horizon definition is same as in section 2. The figure 2.2 explains the interaction and horizon.

The total force in unit volume ΔV_i acting on particle i by particle j is written as below.

$$\vec{F} = \sum_{V_j} \vec{f}_{ij} \Delta V_j \quad (3.7)$$

Where;

$\vec{f}_{ij}$ The pairwise force vector between particle i and j defined in terms of force per unit reference volume squared.

ΔV_j Unit volume of particle j

The total moment in unit volume ΔV_i acting on particle i by particle j is written as below.

$$\vec{M}_i = \sum_{V_j} \vec{m}_{ij} \Delta V_j \quad (3.8)$$

Where;

$\vec{m}_{ij}$ The pairwise moment vector between particle i and j defined in terms of moment per unit reference volume squared.

For the dynamic equilibrium according to Newton's Second Law conservation of linear momentum is given by:

$$\vec{F}_i \Delta V_i + \vec{b}_i \Delta V_i = \rho_i \Delta V_i \ddot{\vec{u}}_i \quad (3.9)$$

$$\vec{M}_i \Delta V_i + \vec{m}_i \Delta V_i = I_i \ddot{\vec{\theta}}_i \quad (3.10)$$

Where;

$\vec{b}_i$ Vector valued external force defined in terms of force per unit reference volume.

$\vec{m}_i$ Vector valued external moment defined in terms of moment per unit reference volume.

ρ_i Material density

$\ddot{\vec{u}}_i$ Acceleration of the particle i

I_i Mass moment of inertia of the particle i

$\ddot{\theta}_i$ Angular acceleration of particle i

Dividing the equation (3.9) and (3.10) by ΔV_i and arranging them;

$$\vec{F}_i + \vec{b}_i = \rho_i \ddot{\vec{u}}_i \quad (3.11)$$

$$\vec{M}_i + \vec{m}_i = I_i \ddot{\theta}_i / \Delta V_i \quad (3.12)$$

The equations can be rewritten after taking the limits while ΔV_i and ΔV_j goes to zero as below;

$$\int_{V_j} \vec{f}_{ij} dV_j + \vec{b}_i = \rho_i \ddot{\vec{u}}_i \quad (3.13)$$

$$\int_{V_j} \vec{m}_{ij} dV_j + \vec{m}_i = 0 \quad (3.14)$$

For static condition the equation become as below;

$$\int_{V_j} \vec{f}_{ij} dV_j + \vec{b}_i = 0 \quad (3.15)$$

$$\int_{V_j} \vec{m}_{ij} dV_j + \vec{m}_i = 0 \quad (3.16)$$

3.3 Linear Elastic Micropolar Peridynamic Model

In linear Micropolar Peridynamic Theory the variables of $\vec{f}_{ij}$ and $\vec{m}_{ij}$ function are relative displacement, relative position of the particles initially, and rotations of the particles. Certainly other features like temperature, humidity or viscosity can be variable of $\vec{f}_{ij}$ and $\vec{m}_{ij}$ functions [28].

$$\int_{V_j} \vec{f}_{ij}(\vec{\eta}_{ij}, \vec{\xi}_{ij}, \vec{\theta}_i, \vec{\theta}_j) dV_j + \vec{b}_i = \rho_i \ddot{\vec{u}}_i \quad (3.17)$$

$$\int_{V_j} \vec{m}_{ij}(\vec{\eta}_{ij}, \vec{\xi}_{ij}, \vec{\theta}_i, \vec{\theta}_j) dV_j + \vec{m}_i = 0 \quad (3.18)$$

Where;

$\vec{\theta}_i$ the rotation of particles i

$\vec{\theta}_j$ the rotation of particles j

$\vec{\xi}_{ij}$ the relative position of particles i and j ,

$\vec{\eta}_{ij}$ the relative displacement of particles i and j .

When the model is in static equilibrium condition;

$$\int_{V_j} \vec{f}_{ij}(\vec{\eta}_{ij}, \vec{\xi}_{ij}, \vec{\theta}_i, \vec{\theta}_j) dV_j + \vec{b}_i = 0 \quad (3.19)$$

$$\int_{V_j} \vec{m}_{ij}(\vec{\eta}_{ij}, \vec{\xi}_{ij}, \vec{\theta}_i, \vec{\theta}_j) dV_j + \vec{m}_i = 0 \quad (3.20)$$

In 2 Dimensional case a link between two particle can be considered as micro beam. The micro beam can be modeled as peridynamic frame element in as shown in the Figure 3.3.

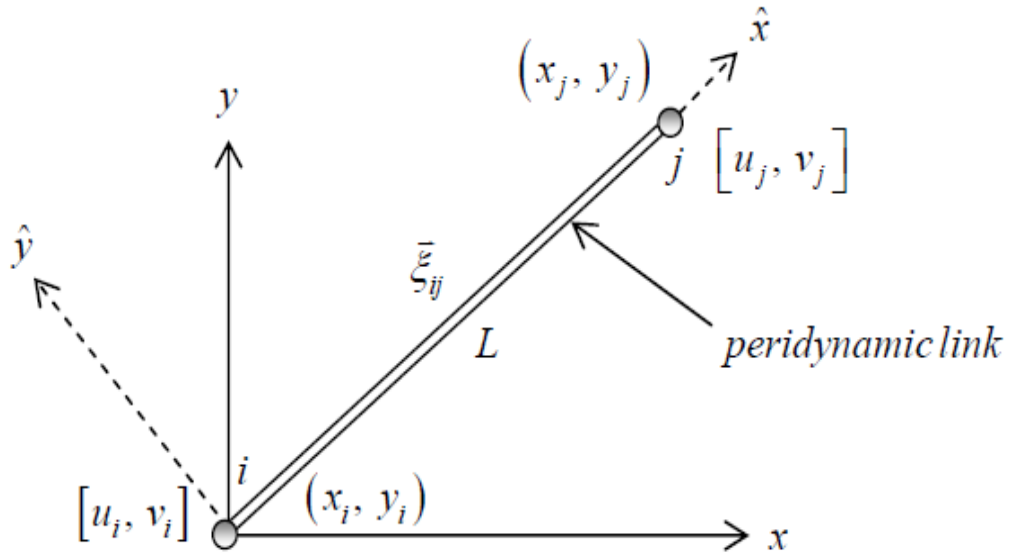


Figure 3.3 : Peridynamic frame element [28].

Where;

E' Young's modulus the element,

L Length of the element

A Cross sectional area of the element. It is radially symmetric.

I Bending moment of inertia of the element

$2I$ Torsional moment of inertia of the element.

Linearized , $\vec{f}_{ij}$ and $\vec{m}_{ij}$ function for a link in local coordinates can be written as below.

$$\begin{bmatrix} \hat{f}_x^i \\ \hat{f}_y^i \\ \hat{m}_z^i \\ \hat{f}_x^j \\ \hat{f}_y^j \\ \hat{m}_z^j \end{bmatrix} = \begin{bmatrix} c/L & 0 & 0 & -c/L & 0 & 0 \\ 0 & 12d/L^3 & 6d/L^2 & 0 & -12d/L^3 & 6d/L^2 \\ 0 & 6d/L^2 & 4d/L & 0 & -6d/L^2 & 2d/L \\ -c/L & 0 & 0 & c/L & 0 & 0 \\ 0 & -12d/L^3 & -6d/L^2 & 0 & 12d/L^3 & -6d/L^2 \\ 0 & 6d/L^2 & 2d/L & 0 & -6d/L^2 & 4d/L \end{bmatrix} \begin{bmatrix} \hat{u}^i \\ \hat{v}^i \\ \hat{\theta}_z^i \\ \hat{u}^j \\ \hat{v}^j \\ \hat{\theta}_z^j \end{bmatrix} \quad (3.21)$$

$$c = E' A \quad (3.22)$$

$$d = E' I \quad (3.23)$$

Where;

$\hat{\eta}_{ij}^x = \hat{u}^j - \hat{u}^i$ Relative displacement in $\hat{x}$ direction

$\hat{\eta}_{ij}^y = \hat{v}^j - \hat{v}^i$ Relative displacement in $\hat{y}$ direction

$\theta_z^i = \hat{\theta}_z^i$ Rotation of particle i on the $\hat{z}$ axes. Positive in counterclockwise

$\theta_z^j = \hat{\theta}_z^j$ Rotation of particle j on the $\hat{z}$ axes. Positive in counterclockwise

$\hat{\xi}_{ij}^x = \hat{x}^j - \hat{x}^i = L$ Relative position of particles i and j in the $\hat{x}$ direction.

$\hat{\xi}_{ij}^y = \hat{y}^j - \hat{y}^i = 0$ Relative position of particles i and j in the $\hat{y}$ direction.

$L = \|\vec{\xi}_{ij}\|$ Magnitude of relative position vector.

Peridynamic link between particle i and j after deformation has been illustrated in the Figure 3.4.

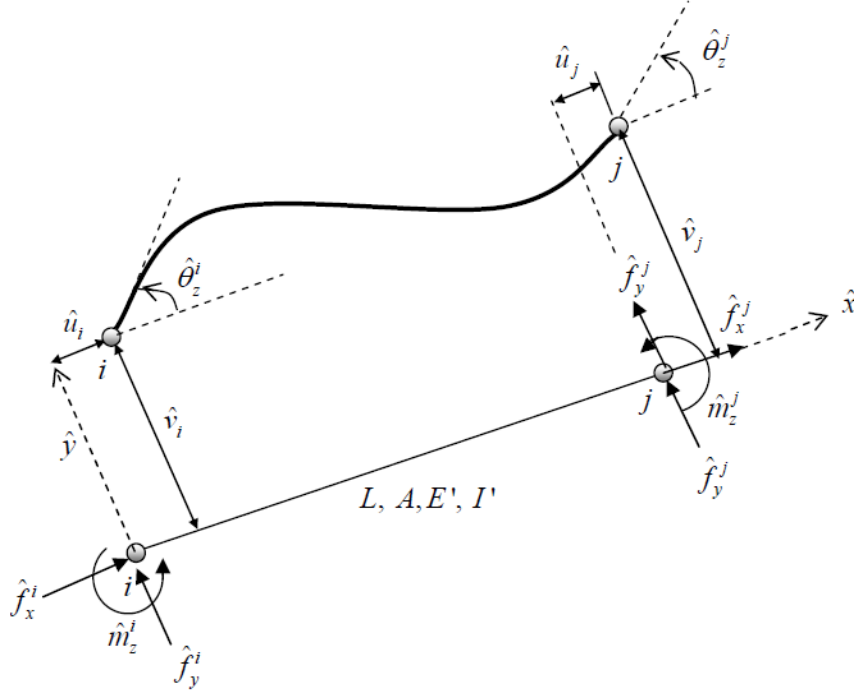


Figure 3.4 : Deformed peridynamic link [28].

To obtain a meaningful comparison between classical theory parameters and micropolar peridynamic theory parameters, the energy stored in the material has been compared. While classical theory calculates strain energy in terms of E and ν , the micropolar peridynamic theory calculates the strain energy in terms of c and d . To obtain the internal strain energy by micropolar peridynamic theory all peridynamic micro beam attached to particle i in a differential volume ΔV_i has been considered.

For principal strains ε_1 and ε_2 the displacement-strain and rotation-displacement relation is defined as following for 2-dimensional plane stress condition.

$$u = \varepsilon_1 x \quad (3.24)$$

$$v = \varepsilon_2 y \quad (3.25)$$

$$\theta_z = \frac{1}{2} \left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} \right) \quad (3.26)$$

Differentiating equ 3.24 and 3.25 and substituting into equ (3.26) we obtain;

$$\frac{\partial u}{\partial y} = 0 \quad (3.27)$$

$$\frac{\partial v}{\partial x} = 0 \quad (3.28)$$

$$\theta_z = 0 \quad (3.29)$$

For one peridynamic micro-beam the strain energy is defined as below;

$$U_0 = \frac{1}{2} [d]^T [k] [d] \quad (3.30)$$

Where;

$[d]$ The displacement vector in equ. (3.21)

$[k]$ The stiffness matrix in equ. (3.21)

Substituting the equ. (3.21) into equ. (3.30) we obtain the strain energy;

$$U = \frac{1}{2} \int_{\mathcal{V}_\delta} \frac{1}{2} \begin{bmatrix} \hat{u} & \hat{v} \end{bmatrix} \begin{bmatrix} c/r & 0 \\ 0 & 12d/r^3 \end{bmatrix} \begin{bmatrix} \hat{u} \\ \hat{v} \end{bmatrix} dV \quad (3.31)$$

Since the link is shared between particle i and j the factor of $\frac{1}{2}$ has been added to equ. (3.31). For more simple expression, the notation changes have been applied as follow;

$$\hat{u} = \hat{u}^i - \hat{u}^j$$

$$\hat{v} = \hat{v}^i - \hat{v}^j$$

$$r = L$$

Since it is more convenient, polar coordinates has been used to obtain the strain energy of all links in a material horizon $\mathcal{S}$. The transformation from local coordinates to global coordinates has been applied to displacement vector as ssen in the Figure 3.5.

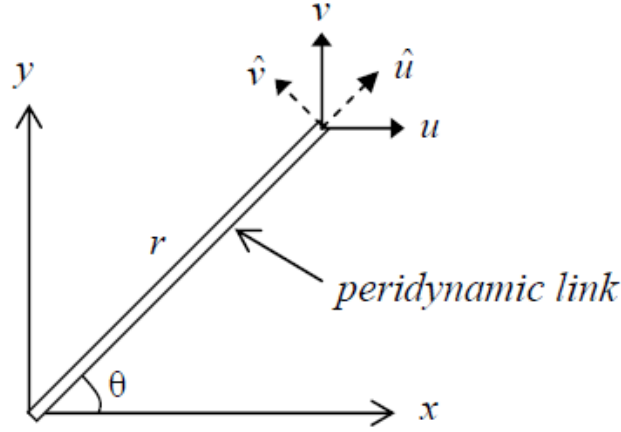


Figure 3.5 : Transformation used for displacement vector [28].

$$\begin{bmatrix} \hat{u} \\ \hat{v} \end{bmatrix} = \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} \quad (3.32)$$

$$u = \varepsilon_1 \cos \theta \quad (3.33)$$

$$v = \varepsilon_2 \sin \theta \quad (3.34)$$

Equation (3.33) and (3.34) express the displacements in polar coordinates.

Substituting all values in to the equation 3.31 and integrating from $r = 0$ to $r = \delta$, $\theta = 0$ to $\theta = 2\pi$, we define strain energy in terms of c , d , δ , ε_1 , ε_2 and t (thickness).

$$U = \frac{1}{2} \int_{r=0}^{r=\delta} \int_{\theta=0}^{\theta=2\pi} U_0 d\theta r dr = \frac{\pi t}{4} \begin{bmatrix} \varepsilon_1 & \varepsilon_2 \end{bmatrix} \begin{bmatrix} \frac{c\delta^3}{4} + 3d\delta & \frac{c\delta^3}{12} - 3d\delta \\ \frac{c\delta^3}{12} - 3d\delta & \frac{c\delta^3}{4} + 3d\delta \end{bmatrix} \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \end{bmatrix} \quad (3.35)$$

For two-dimensional plane stress condition assuming uniform strain field, the energy density for classical theory using principle strains has the following matrix form:

$$U = \frac{1}{2} E \begin{bmatrix} \varepsilon_1 & \varepsilon_2 \end{bmatrix} \begin{bmatrix} \frac{1}{(1-\nu)(1+\nu)} & \frac{\nu}{(1-\nu)(1+\nu)} \\ \frac{\nu}{(1-\nu)(1+\nu)} & \frac{1}{(1-\nu)(1+\nu)} \end{bmatrix} \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \end{bmatrix} \quad (3.36)$$

Where;

E Elasticity Modulus

ν Poisson's ratio

By comparing the equations (3.36) and (3.35), it is possible to write c and d in the terms of Elasticity Modulus E and Poisson's Ratio ν .

$$c = \frac{6E}{\pi\delta^3 t(1-\nu)} \quad (3.37)$$

$$d = \frac{E}{6\pi\delta t} \left[\frac{1-3\nu}{(1-\nu^2)} \right] \quad (3.38)$$

Where;

c has unit in force per unit volume squared

d has unit in force per unit volume per unit length

Choosing appropriate E , ν and δ the structure can be modeled using micropolar peridynamic theory.

4. MATHEMATICAL MODELING

4.1 Micropolar Peridynamic Finite Element Model

To develop an finite element model energy method and displacement based finite element method have been applied[28]. Quadrilateral element has been used to model 2-D plates in plane stress condition. Differential force between particle i and j ;

$$\{df_{ij}\} = [k_{ij}] \{d_{ij}\} dV_i dV_j \quad (4.1)$$

Differential strain energy between particle i and j ;

$$dU_{ij} = \frac{1}{2} [d_{ij}] \{f_{ij}\} = \frac{1}{2} [d_{ij}] [k_{ij}] \{d_{ij}\} dV_i dV_j \quad (4.2)$$

The total strain energy ;

$$U = \frac{1}{2} \int_R \int_R \frac{1}{2} [d_{ij}] [k_{ij}] \{d_{ij}\} dV_i dV_j \quad (4.3)$$

and partitioning Equation 4.3 into num_el finite elements within the domain, the following expression is obtained:

To apply finite element method the total strain energy formula rewritten in terms of num_el .

$$U = \frac{1}{4} \sum_{elemi=1}^{num_el} \left(\int_{elemi} \left(\sum_{elemj=1}^{num_el} \left(\int_{elemj} [d_{ij}] [k_{ij}] \{d_{ij}\} dV_j \right) \right) dV_i \right) \quad (4.4)$$

To simplification the total strain energy formula has been separated into two parts as the total strain energy developed by particles in the different element i and j , and the total strain energy developed by particles in the same element i .

$$U = \sum_{eli=1}^n \left(\sum_{elj=1}^{eli-1} U_{ij} \right) + \sum_{eli=1}^n U_{ii} \quad (4.5)$$

In the element i the displacement field at any point;

$$\{d_i\} = [N^i] \{D_i\} \quad (4.6)$$

Where;

$[N^i]$ Shape matrix used to interpolate $\{d_i\}$ from $\{D_i\}$

$\{D_i\}$ Nodal displacement of element i

Similarly, in the element j the displacement field at any point;

$$\{d_j\} = [N^j] \{D_j\} \quad (4.7)$$

Where;

$[N^j]$ Shape matrix used to interpolate $\{d_j\}$ from $\{D_j\}$

$\{D_j\}$ Nodal displacement of element j

For the pair of elements i and j

$$\{d_{ij}\} = \begin{Bmatrix} \{d_i\} \\ \{d_j\} \end{Bmatrix} = \begin{Bmatrix} [N^i] \{d_i\} \\ [N^j] \{d_j\} \end{Bmatrix} = \begin{bmatrix} [N^i] & [0] \\ [0] & [N^j] \end{bmatrix} \begin{Bmatrix} \{D_i\} \\ \{D_j\} \end{Bmatrix} = [N^{ij}] \{D_{ij}\} \quad (4.8)$$

Substituting into the equation (4.4) we obtain;

$$U = [D_{ij}] \left[\frac{1}{4} \sum_{elemi=1}^{num_el} \left(\int_{elemi} \left(\sum_{elemj=1}^{num_el} \left(\int_{elemj} [N^{ij}]^T [k_{ij}] [N^{ij}] dV_j \right) \right) dV_i \right) \right] \{D_{ij}\} \quad (4.9)$$

In the more compact form the total strain energy stored in the domain R;

$$U = \frac{1}{2} [D] [K] \{D\} \quad (4.10)$$

Where;

$\{D\}$ Nodal displacement vector for all body.

$[K]$ Global stiffness matrix.

Rewriting the $[K]$ in more explicit manner;

$$[K] = \frac{1}{2} \sum_{elemi=1}^{num_el} \left(\int \left(\sum_{elemj=1}^{num_el} \left(\int [N^{ij}]^T [k_{ij}] [N^{ij}] dV_j \right) \right) dV_i \right) \quad (4.11)$$

$[K]$ can be shown in forms of contribution of different elements, and same element.

$$[K] = \sum_{eli=1}^{num_el} \left(\sum_{elj=1}^{eli-1} [K_{ij}] \right) + \sum_{eli=1}^n [K_{ii}] \quad (4.12)$$

Where;

$$[K_{ij}] = \int \left(\int_{elemj} [N^{ij}]^T [k_{ij}] [N^{ij}] dV_j \right) dV_i \quad (4.13)$$

$$[K_{ii}] = \frac{1}{2} \int \left(\int_{elemi} [N^{ii}]^T [k_{ij}] [N^{ii}] dV_j \right) dV_i \quad (4.14)$$

Dividing the element i into m_i subelements of volume ΔV_i and the element j into m_j subelements of volume ΔV_j , the integral has been solved numerically.

$$[K_{ij}] = \sum_{i=1}^{m_i} \left(\sum_{j=1}^{m_j} ([N^{ij}]^T [k_{ij}] [N^{ij}] \Delta V_j) \Delta V_i \right) \quad (4.15)$$

$$[K_{ii}] = \sum_{i=1}^{m_i} \left(\sum_{j=1}^{i-1} ([N^{ii}]^T [k_{ij}] [N^{ii}] \Delta V_j) \Delta V_i \right) \quad (4.16)$$

Where;

$[N_{ii}]$ Shape matrix for the single element i .

$$[N_{ij}] = \begin{bmatrix} N^i(\xi_i) \\ N^i(\xi_j) \end{bmatrix} \quad (4.17)$$

$[N_{ij}]$ Shape matrix for the element pair i and j .

$$[N_{ij}] = \begin{bmatrix} [N^i] & [0] \\ [0] & [N^j] \end{bmatrix} \quad (4.18)$$

In the equation (4.16) the factor of $\frac{1}{2}$ has been removed after the upper limit on the inner summation has been changed to $i-1$.

The application of the suface and consantrated forces and the total potential energy can be represented as below;

$$\pi = U + \pi_1 + \pi_2 + \pi_3 \quad (4.19)$$

Where;

$$U = [D_{ij}] \left[\frac{1}{4} \sum_{elemi=1}^{num_el} \left(\int_{elemi} \left(\sum_{elemj=1}^{num_el} \left(\int_{elemj} [N^{ij}]^T [k_{ij}] [N^{ij}] dV_j \right) \right) dV_i \right) \right] \{D_{ij}\} \quad (4.20)$$

$$\pi_1 = -[D_{ij}] \left[\sum_{elemi=1}^{num_el} \left(\int_{elemi} [N^{ij}]^T [b_i] dV_i \right) \right] \quad (4.21)$$

$$\pi_2 = -[D_{ij}] \left[\sum_{elemi=1}^{num_el} \left(\int_{elemi} [N^{ij}]^T [t_i] dS_i \right) \right] \quad (4.22)$$

$$\pi_3 = -[D_{ij}] \left[\sum_{elemi=1}^{num_el} \{P_i\} \right] \quad (4.23)$$

$[b_i]$ Body force vector.

$[t_i]$ Surface force vector.

$\{P_i\}$ External force vector.

As a rule of the Finite Element Methods the equation (4.19) has been derived to find the minimum potential energy expression.

$$\begin{aligned}
\frac{\partial \pi}{\partial \{D_{ij}\}} = & \left[\frac{1}{2} \sum_{elemi=1}^{num_el} \left(\int_{elemi} \left(\sum_{elemj=1}^{num_el} \left(\int_{elemj} [N^{ij}]^T [k_{ij}] [N^{ij}] dV_j \right) \right) dV_i \right) \{D_{ij}\} \right. \\
& - \left[\sum_{elemi=1}^{num_el} \left(\int_{elemi} [N^{ij}]^T [b_i] dV_i \right) \right] - \left[\sum_{elemi=1}^{num_el} \left(\int_{elemi} [N^{ij}]^T [t_i] dS_i \right) \right] \\
& \left. - \left[\sum_{elemi=1}^{num_el} \{P_i\} \right] = 0 \right.
\end{aligned} \tag{4.24}$$

When the body force and surface force are equal to zero;

$$\begin{aligned}
& \left[\frac{1}{2} \sum_{elemi=1}^{num_el} \left(\int_{elemi} \left(\sum_{elemj=1}^{num_el} \left(\int_{elemj} [N^{ij}]^T [k_{ij}] [N^{ij}] dV_j \right) \right) dV_i \right) \{D_{ij}\} \right. \\
& \left. = \left[\sum_{elemi=1}^{num_el} \{P_i\} \right] \right.
\end{aligned} \tag{4.25}$$

In other form;

$$[K]\{D\} = [P] \tag{4.26}$$

4.2 Development of Computer Model for Stiffness Matrix

To apply micropolar peridynamic theory to displacement based finite element model, linear quadrilateral element has been used. The finite element is shown in the Figure 4.1 in natural coordinates.

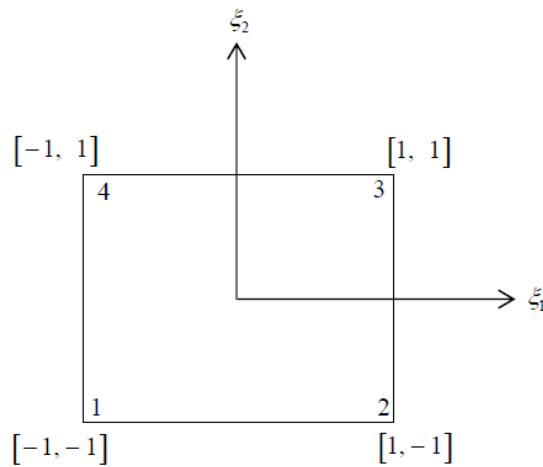


Figure 4.1 : Quadrilateral finite element in natural coordinates [28].

Displacement vector as interpolation of nodal displacement vector by shape matrix is written in terms natural coordinates for element i .

$$\{d_i\} = \begin{bmatrix} N_1(\xi_1^i, \xi_2^i) & 0 & N_2(\xi_1^i, \xi_2^i) & 0 & N_3(\xi_1^i, \xi_2^i) & 0 & N_4(\xi_1^i, \xi_2^i) & 0 \\ 0 & N_1(\xi_1^i, \xi_2^i) & 0 & N_2(\xi_1^i, \xi_2^i) & 0 & N_3(\xi_1^i, \xi_2^i) & 0 & N_4(\xi_1^i, \xi_2^i) \\ \frac{-N_{1,y}(\xi_1^i, \xi_2^i)}{2} & \frac{-N_{1,x}(\xi_1^i, \xi_2^i)}{2} & \frac{-N_{2,y}(\xi_1^i, \xi_2^i)}{2} & \frac{-N_{2,x}(\xi_1^i, \xi_2^i)}{2} & \frac{-N_{3,y}(\xi_1^i, \xi_2^i)}{2} & \frac{-N_{3,x}(\xi_1^i, \xi_2^i)}{2} & \frac{-N_{4,y}(\xi_1^i, \xi_2^i)}{2} & \frac{-N_{4,x}(\xi_1^i, \xi_2^i)}{2} \end{bmatrix} \times \begin{Bmatrix} u_1^i \\ v_1^i \\ u_2^i \\ v_2^i \\ u_3^i \\ v_3^i \\ u_4^i \\ v_4^i \end{Bmatrix} = \begin{Bmatrix} u_i \\ v_i \\ \theta_i \end{Bmatrix} = [N^i(\xi_1^i, \xi_2^i)] \{D_i\} \quad (4.27)$$

Similarly for element j ;

$$\{d_j\} = \begin{bmatrix} N_1(\xi_1^j, \xi_2^j) & 0 & N_2(\xi_1^j, \xi_2^j) & 0 & N_3(\xi_1^j, \xi_2^j) & 0 & N_4(\xi_1^j, \xi_2^j) & 0 \\ 0 & N_1(\xi_1^j, \xi_2^j) & 0 & N_2(\xi_1^j, \xi_2^j) & 0 & N_3(\xi_1^j, \xi_2^j) & 0 & N_4(\xi_1^j, \xi_2^j) \\ \frac{-N_{1,y}(\xi_1^j, \xi_2^j)}{2} & \frac{-N_{1,x}(\xi_1^j, \xi_2^j)}{2} & \frac{-N_{2,y}(\xi_1^j, \xi_2^j)}{2} & \frac{-N_{2,x}(\xi_1^j, \xi_2^j)}{2} & \frac{-N_{3,y}(\xi_1^j, \xi_2^j)}{2} & \frac{-N_{3,x}(\xi_1^j, \xi_2^j)}{2} & \frac{-N_{4,y}(\xi_1^j, \xi_2^j)}{2} & \frac{-N_{4,x}(\xi_1^j, \xi_2^j)}{2} \end{bmatrix} \times \begin{Bmatrix} u_1^j \\ v_1^j \\ u_2^j \\ v_2^j \\ u_3^j \\ v_3^j \\ u_4^j \\ v_4^j \end{Bmatrix} = \begin{Bmatrix} u_j \\ v_j \\ \theta_j \end{Bmatrix} = [N^j(\xi_1^j, \xi_2^j)] \{D_j\} \quad (4.28)$$

The Jacobian matrix should be defined to make coordinate transformation from natural coordinates to global coordinates as below;

$$x = \frac{1}{4} \left[(1-\xi_1)(1-\xi_2)x_1 + (1+\xi_1)(1-\xi_2)x_2 + (1+\xi_1)(1+\xi_2)x_3 + (1-\xi_1)(1+\xi_2)x_4 \right] \quad (4.26)$$

$$y = \frac{1}{4} \left[(1-\xi_1)(1-\xi_2)y_1 + (1+\xi_1)(1-\xi_2)y_2 + (1+\xi_1)(1+\xi_2)y_3 + (1-\xi_1)(1+\xi_2)y_4 \right] \quad (4.26)$$

$$J = \frac{1}{4} \begin{bmatrix} -(1-\xi_2)x_1 + (1-\xi_2)x_2 + (1+\xi_2)x_3 - (1+\xi_2)x_4 & -(1-\xi_2)y_1 + (1-\xi_2)y_2 + (1+\xi_2)y_3 - (1+\xi_2)y_4 \\ -(1-\xi_1)x_1 - (1+\xi_1)x_2 + (1+\xi_1)x_3 + (1-\xi_1)x_4 & -(1-\xi_1)y_1 - (1+\xi_1)y_2 + (1+\xi_1)y_3 + (1-\xi_1)y_4 \end{bmatrix} \quad (4.26)$$

Coordinates transformation of derivative of shape matrix;

$$\frac{\partial N}{\partial x} = \frac{1}{\det(J)} \left[\frac{\partial y}{\partial \xi_2} \frac{\partial N}{\partial \xi_1} - \frac{\partial y}{\partial \xi_1} \frac{\partial N}{\partial \xi_2} \right] \quad (4.32)$$

$$\frac{\partial N}{\partial y} = \frac{1}{\det(J)} \left[\frac{\partial x}{\partial \xi_2} \frac{\partial N}{\partial \xi_1} - \frac{\partial x}{\partial \xi_1} \frac{\partial N}{\partial \xi_2} \right] \quad (4.33)$$

In a more compact form the equation (4.27) and (4.28);

$$\{d_i\} = \begin{Bmatrix} u_i \\ v_i \\ \theta_i \end{Bmatrix} = \begin{bmatrix} N_1^i & 0 & N_2^i & 0 & N_3^i & 0 & N_4^i & 0 \\ 0 & N_1^i & 0 & N_2^i & 0 & N_3^i & 0 & N_4^i \\ N_{12}^i & N_{11}^i & N_{22}^i & N_{21}^i & N_{32}^i & N_{31}^i & N_{42}^i & N_{41}^i \end{bmatrix} \begin{Bmatrix} u_1^i \\ v_1^i \\ u_2^i \\ v_2^i \\ u_3^i \\ v_3^i \\ u_4^i \\ v_4^i \end{Bmatrix} = [N^i(\xi_1^i, \xi_2^i)] \{D_i\} \quad (4.34)$$

$$\{d_j\} = \begin{Bmatrix} u_j \\ v_j \\ \theta_j \end{Bmatrix} = \begin{bmatrix} N_1^j & 0 & N_2^j & 0 & N_3^j & 0 & N_4^j & 0 \\ 0 & N_1^j & 0 & N_2^j & 0 & N_3^j & 0 & N_4^j \\ N_{12}^j & N_{11}^j & N_{22}^j & N_{21}^j & N_{32}^j & N_{31}^j & N_{42}^j & N_{41}^j \end{bmatrix} \begin{Bmatrix} u_1^j \\ v_1^j \\ u_2^j \\ v_2^j \\ u_3^j \\ v_3^j \\ u_4^j \\ v_4^j \end{Bmatrix} = [N^j(\xi_1^j, \xi_2^j)] \{D_j\} \quad (4.35)$$

The elements of the [N] matrix in terms of natural coordinates and Jacobian Matrix;

$$N_1 = \frac{(1-\xi_1)(1-\xi_2)}{4} \quad (4.36)$$

$$N_2 = \frac{(1+\xi_1)(1-\xi_2)}{4} \quad (4.37)$$

$$N_3 = \frac{(1+\xi_1)(1+\xi_2)}{4} \quad (4.38)$$

$$N_4 = \frac{(1-\xi_1)(1+\xi_2)}{4} \quad (4.39)$$

$$N_{11} = \frac{-(1-\xi_2)J_{11}^{-1} - (1-\xi_1)J_{12}^{-1}}{8} \quad (4.40)$$

$$N_{12} = \frac{(1-\xi_1)J_{22}^{-1} + (1-\xi_2)J_{21}^{-1}}{8} \quad (4.41)$$

$$N_{21} = \frac{(1-\xi_2)J_{11}^{-1} - (1+\xi_1)J_{12}^{-1}}{8} \quad (4.42)$$

$$N_{22} = \frac{(1+\xi_1)J_{22}^{-1} - (1-\xi_2)J_{21}^{-1}}{8} \quad (4.43)$$

$$N_{31} = \frac{(1+\xi_2)J_{11}^{-1} + (1+\xi_1)J_{12}^{-1}}{8} \quad (4.44)$$

$$N_{32} = \frac{-(1+\xi_1)J_{22}^{-1} - (1+\xi_2)J_{21}^{-1}}{8} \quad (4.45)$$

$$N_{41} = \frac{(1-\xi_2)J_{11}^{-1} - (1-\xi_1)J_{12}^{-1}}{8} \quad (4.46)$$

$$N_{42} = \frac{-(1-\xi_1)J_{22}^{-1} + (1-\xi_2)J_{21}^{-1}}{8} \quad (4.47)$$

While developing computer code elements have been partitioned into material points and every material point has peridynamic links to other material points as in the Figure 4.2.

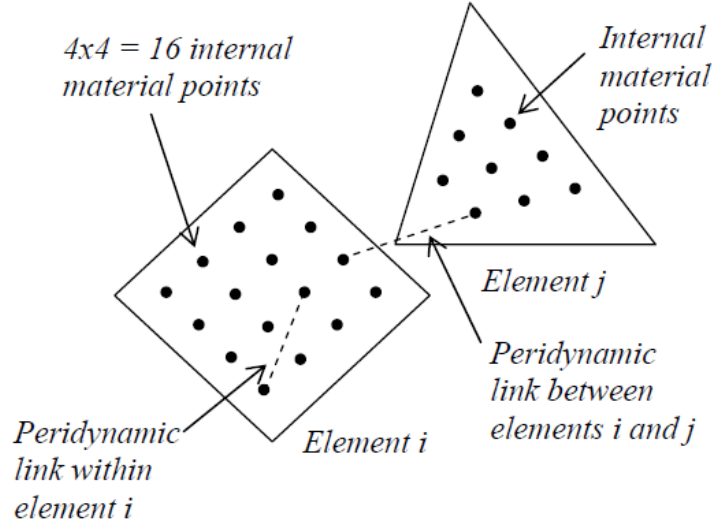


Figure 4.2 : Peridynamic links in the elements [28].

Contribution of the all links evaluated separately and summed into the global stiffness matrix. To obtain the stiffness matrix for the particles, first the L has been calculated from the global coordinates of the material points. If the length is smaller than defined material horizon the stiffness matrix in local coordinates has been calculated using $[\hat{k}_{ij}]$ matrix in the equation 3.21. To evaluate the stiffness matrix in global coordinates $[T]$ transformation matrix has been used.

$$[T] = \begin{bmatrix} ct & st & 0 & 0 & 0 & 0 \\ -st & ct & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & ct & st & 0 \\ 0 & 0 & 0 & -st & ct & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad (4.48)$$

Where;

$$ct = \cos \theta$$

$$st = \sin \theta$$

Then the stiffness matrix in global coordinates of a peridynamic link is written as;

$$[k_{ij}] = [T]^T [\hat{k}_{ij}] [T] \quad (4.49)$$

Substituting into the equation (4.15) and (4.16) the element stiffness matrix has been calculated.

5. APPLICATIONS OF MICROPOLAR PERIDYNAMIC MODEL

Cantilever plates are modeled using micropolar peridynamic model. The model equations have been solved by developing a computer code which was written using software package MATHEMATICA.

The input parameters for the program are E , ν , δ , a , b , t (thickness), p (particle number in one element), boundary conditions and Q (force applied externally). The outputs of the program are K and D .

5.1 For Various Dimensions

Seven cantilever plates with various length and width have been analyzed using the code. For all models t is 1 mm, and concentrated load 500N has been applied to free end in the $-y$ direction. Aluminum 2011 T6 is selected as the material of the structures, $E=70000\text{Mpa}$ and $\nu=0.33$ [29]. Horizon is equal to length of the model. The ratios of length to width are chosen from the range 1-7 as seen in Table 5.1

Table 5.1: Dimensions of the Samples.

Models	Length (mm)	Width (mm)
Model-1	100	100
Model-2	200	100
Model-3	300	100
Model-4	400	100
Model-5	500	100
Model-6	600	100
Model-7	700	100

For analytical result the equation (5.1) is used and the model for the equation is as in Figure 5.1 [30].

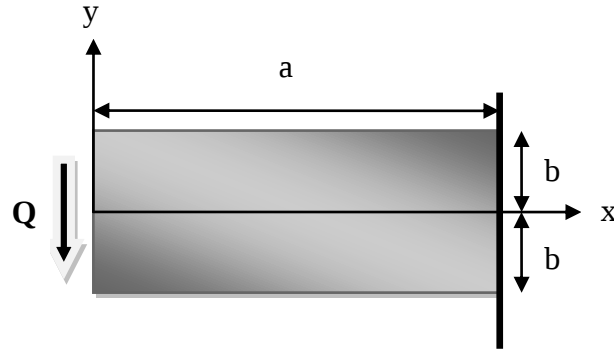


Figure 5.1 : Cantilever plate with an end load.

$$u_y = -\frac{Qa^3}{2Eb^3} \left(1 + \frac{(12+11\nu)}{5} \frac{b^2}{a^2} \right) \quad (5.1)$$

The analysis is carried out for the FEM by using software package ANSYS.

5.2 For Various Horizons

The seven cantilever plates investigated in section 5.1 have been analyzed for four different horizon selections as seen Table 5.2.

Table 5.2: Horizon selection of models

Models	Horizons (mm)			
	25	50	75	100
Model-1	25	50	75	100
Model-2	50	100	150	200
Model-3	75	150	225	300
Model-4	100	200	300	400
Model-5	125	250	375	500
Model-6	150	300	450	600
Model-7	175	350	525	700

6. RESULTS

The maximum displacements obtained using Micropolar Peridynamic approach (MPD) with horizon lengths are equal to length of the models, classical approach and ANSYS, and errors with respect to the results of the classical theory and Ansys are given in Table 6.1 and following figures.

Table 6.1: Maximum displacements

	MPD (mm)	Class. (mm)	Class.Er (%)	Ansys (mm)	AnsysEr.(%)
Model-1	-0,17	-0,05	2,34	-0,07	1,43
Model-2	-0,52	-0,27	0,90	-0,29	0,79
Model-3	-0,98	-0,84	0,17	-0,85	0,15
Model-4	-1,53	-1,92	0,20	-1,93	0,21
Model-5	-2,12	-3,68	0,42	-3,68	0,42
Model-6	-2,85	-6,31	0,55	-6,29	0,55
Model-7	-3,52	-9,96	0,65	-9,93	0,65

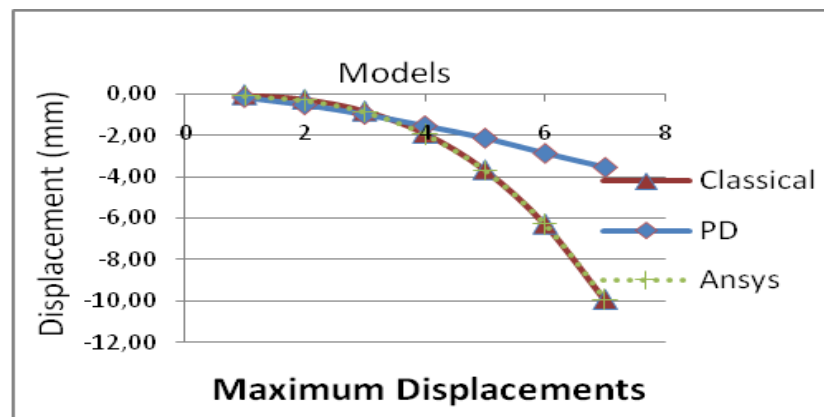


Figure 6.1: Maximum Displacements.

Figure 6.1 shows maximum displacements calculated using Micropolar Peridynamic theory, ANSYS and Classical theory

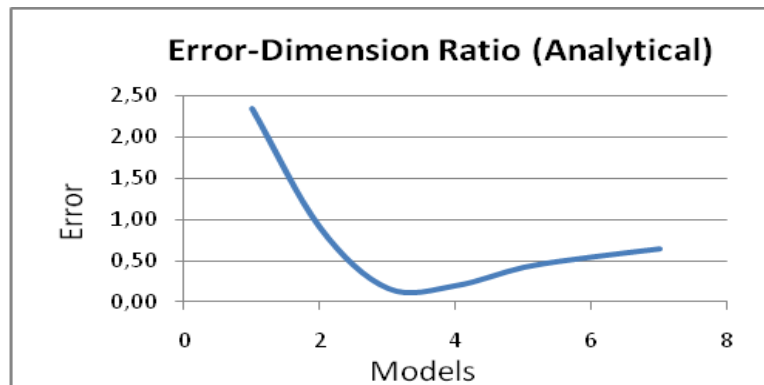


Figure 6.2 : Error with respect the results using the classical theory.

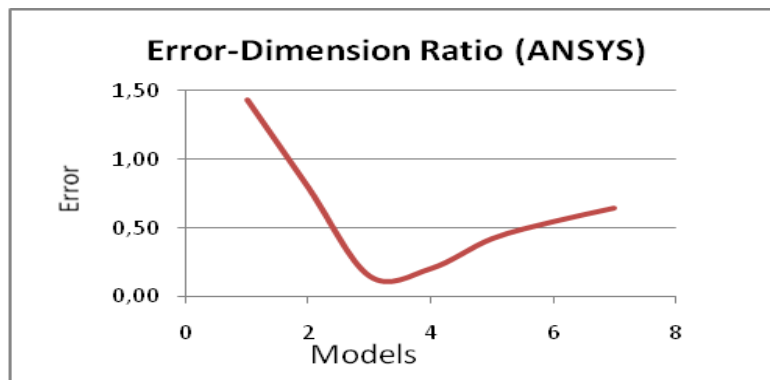


Figure 6.3 : Error with respect to the results from Ansys.

Figure 6.2 and Figure 6.3 show the error of the micropolar peridynamic theory with respect to the results obtained by using the classical theory and the software package ANSYS respectively.

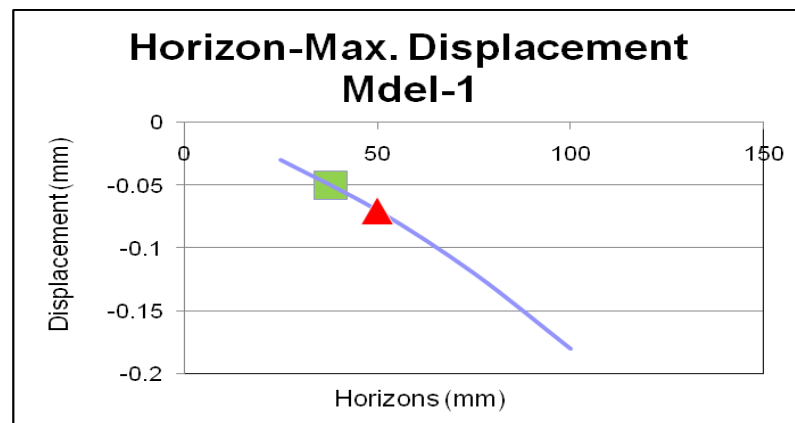
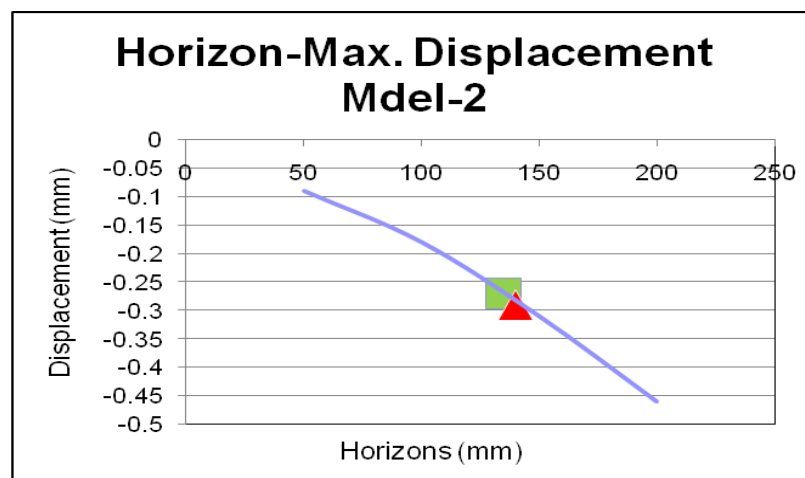
The maximum displacements obtained using Micropolar Peridynamic approach (MPD) for four different horizon selections for each model, classical approach and ANSYS results are obtained. Results for each model are shown graphically.

In table 6.2 the maximum displacements of each model for four different horizon selections have been shown.

Table 6.2: Maximum displacements of MPD for various horizons

Model-1		Model-2		Model-3		Model-4		Model-5		Model-6		Model-7	
Hor.	M.Dis	Hor.	M.Dis	Hor.	M.Dis	Hor.	M.Dis	Hor.	M.Dis	Hor.	M.Dis	Hor.	M.Dis
25	-0,03	50	-0,09	75	-0,12	100	-0,16	125	-0,22	150	-0,29	175	-0,36
50	-0,07	100	-0,18	150	-0,3	200	-0,47	250	-0,69	300	-0,93	350	-1,17
75	-0,12	150	-0,31	225	-0,58	300	-0,93	375	-1,32	450	-1,81	525	-2,15
100	-0,18	200	-0,46	300	-0,94	400	-1,46	500	-2,17	600	2,69	700	-3,02

While in each figure continuous curve represents MPD results, the symbols square and triangle show the Classical and ANSYS results respectively. The continuous curve is obtained for various horizon selections.

**Figure 6.4 :** Maximum displacements versus horizons for Model-1.**Figure 6.5 :** Maximum displacements versus horizons for Model-2.

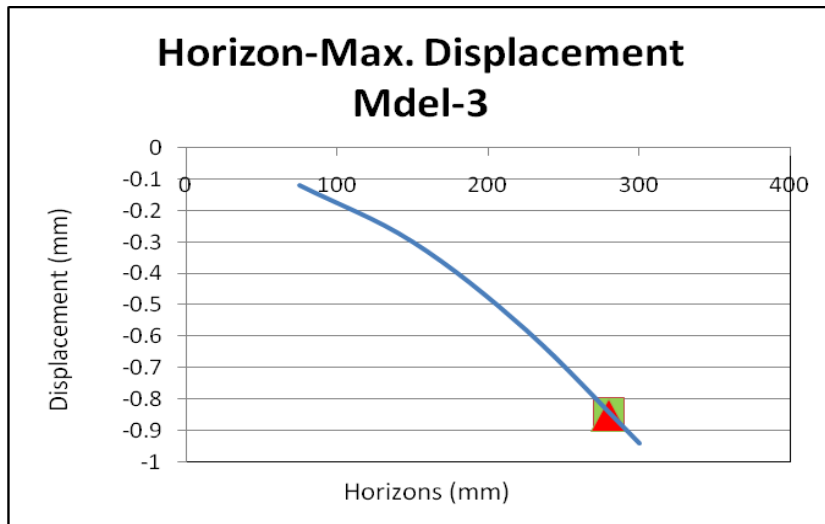


Figure 6.6 : Maximum displacements versus horizons for Model-3.

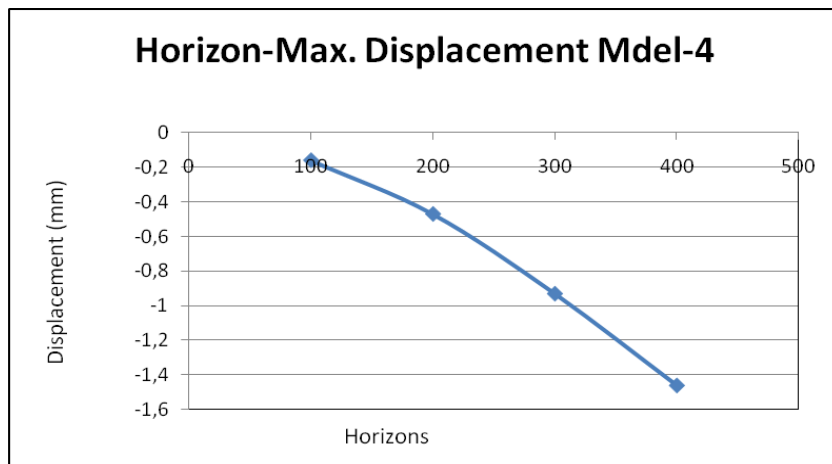


Figure 6.7 : Maximum displacements versus horizons for Model-4.

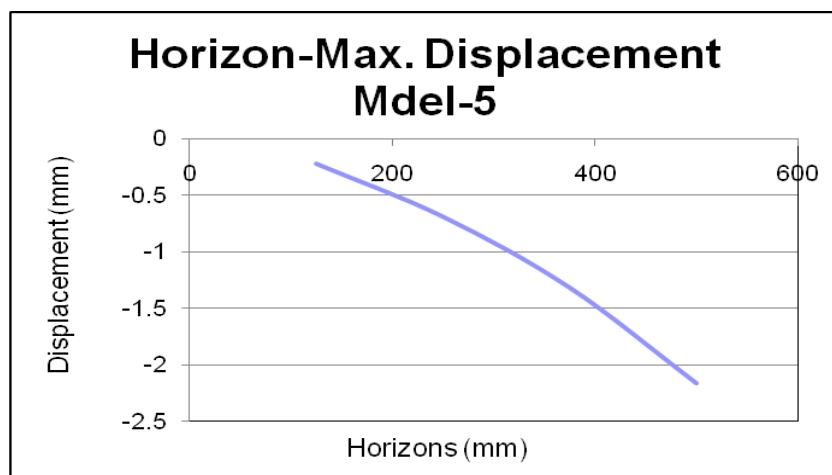


Figure 6.8 : Maximum displacements versus horizons for Model-5.

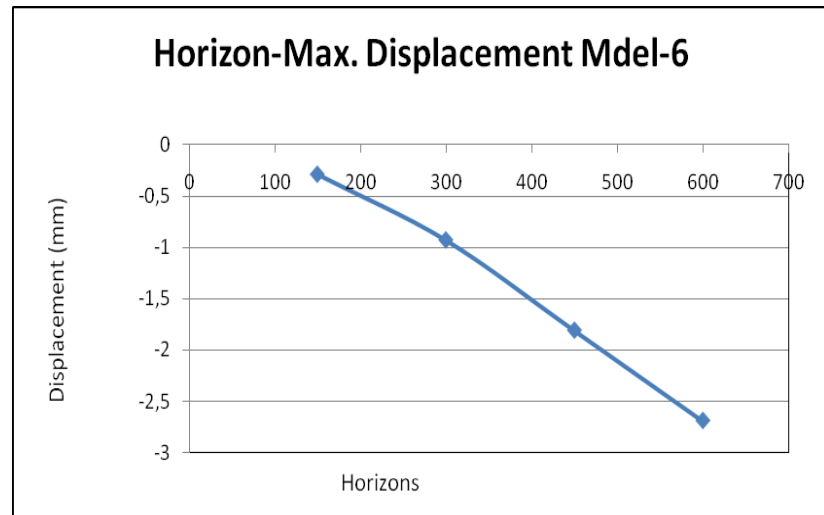


Figure 6.9 : Maximum displacements versus horizons for Model-6.

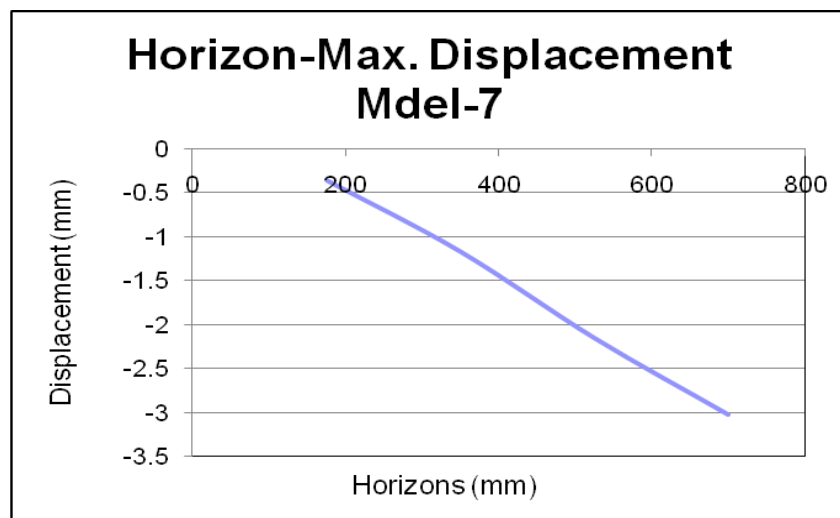


Figure 6.10 : Maximum displacements versus horizons for Model-7.

By close inspection of figures reveals out that, maximum displacements strongly depend on the horizon selection. For the first three models, for some certain horizon selection, there are points at which classical and ANSYS results coincide. However, for the models dimension ratios are bigger than three, the results of classical theory do not coincide with the results of MPD for any horizon selection.

7. CONCLUSION

In this study MPD, classical-analytical and ANSYS analysis of seven the plates with various length and width have been carried out. Each plate has been analyzed for four different horizon selections. It has been observed from Table 6.2 that the displacement fields of plates obtained by MPD are strongly affected by horizon selections. In model-1, model-2 and model-3 for some horizon selection the MPD gives the same results with Classical and ANSYS solutions. However, for other models, the results for the maximum displacements never have the same values.

In addition, it is shown from Table 6.1 that the displacement fields of the plates are strongly affected by dimensional changes of the plates. In Figure 6.2 and Figure 6.3 show that in Model-3 the error takes its minimum value for both analytical solutions and ANSYS results but then the error increases for both results. The graphical representation of error for maximum displacement show increasing trend. However, it is naturally expected that the micropolar peridynamic and classical theories should give the same results, at least to a certain extend. This shows that The MPD might have a shortcoming of validity on every dimension ratios and horizon selection. Even if appropriate horizon could be found for some dimension ratios, there would be no explanation about which horizon could be selected.

Apparently, there is a problem which is related both to dimension ratios and horizon length selection. This might be solved by checking the mathematical model and physical basis on which the theory has been developed.

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