

**ARBITRAGE AND THEORY OF VALUATION
IN NON-LINEAR MARKETS**

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**DOĞRUSAL OLMAYAN PİYASALARDA
ARBİTRAJ VE DEĞERLEME TEORİSİ**

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LIST OF SYMBOLS

a.e.	:	Almost everywhere
a.s.	:	Almost surely
$\Rightarrow$	:	Implies
$\Leftrightarrow$, iff	:	If and only if
$\exists$	:	There exist(s)
$\forall$	:	For any
$\in$	:	Belongs to
$\emptyset$	:	The empty set
$\subseteq$	:	Is a subset of
$\subset$	:	Is a proper subset of. (For any two sets A and B , $A \subset B$ iff $A \subseteq B$ and $A \neq B$).
$\cup$	:	Union
$\cap$	:	Intersection
$\text{int}(\cdot)$	:	Topological interior
$\bar{A}$	:	Topological closure of a set A
∂A	:	Topological boundary of a set A defined as $\partial A = \bar{A} \setminus \text{int}(A)$
A^c	:	Complement of the set A (relative to a set X) defined as: $A^c = \{x \in X : x \notin A\}$
$A \setminus B$	:	Difference between the sets A and B defined as: $A \setminus B = \{x \in A : x \notin B\}$
$\sup(A)$	:	The supremum (least upper bound) of a subset A of a partially ordered set.
$\inf(A)$	:	The infimum (greatest lower bound) of a subset A of a partially ordered set.
$\max(A)$	:	The maximal element of a subset A of a partially ordered set.
$\min(A)$	:	The minimal element of a subset A of a partially ordered set.
$\limsup_n x_n$	:	Upper limit of the sequence of real numbers x_n defined as: $\limsup_n x_n = \inf_n (\sup_{k \geq n} x_k)$
$\liminf_n x_n$	:	Lower limit of the sequence of real numbers x_n defined as: $\liminf_n x_n = \sup_n (\inf_{k \geq n} x_k)$
$\psi _M$	:	Restriction of the mapping $\psi : \mathcal{D}(T) \rightarrow Y$ to a subset $M \subset \mathcal{D}(T)$
$\sigma(\mathcal{A})$	:	σ -field generated by a family of sets $\mathcal{A}$

$(\Omega, \mathcal{F})$	:	measurable space
$(\Omega, \mathcal{F}, \mu)$	:	measure space
$\mathcal{L}^2(\Omega, \mathcal{F}, P)$	:	Space of square-integrable random variables on $(\Omega, \mathcal{F}, P)$
$L^2(\Omega, \mathcal{F}, P)$	:	Space of P -equivalence classes of square-integrable random variables on $(\Omega, \mathcal{F}, P)$
$\mathcal{L}^p(\Omega, \mathcal{F}, \mu)$	:	Linear space of p -integrable $\mathcal{F}$ measurable functions with the seminorm $\ f\ _p = \left[\int f ^p d\mu \right]^{1/p}$, $p \geq 1$
$L^p(\Omega, \mathcal{F}, P)$ $p \geq 1$	:	Banach space of P -equivalence classes in $\mathcal{L}^p(\Omega, \mathcal{F}, P)$
$\mathbb{R}^{(\Omega, \mathcal{F})}$	:	Space of $\mathcal{F}$ measurable real valued functions on Ω
$\mathbb{Z}$	:	Integers
$\mathbb{N}$	:	Natural numbers, $\mathbb{N} = \{1, 2, \dots\}$
$\mathbb{Z}_+$	:	Non-negative integers, $\mathbb{Z}_+ = \{0, 1, 2, \dots\}$
$\mathbb{R}$	:	Real numbers
$\overline{\mathbb{R}}$	:	Extended real number system, $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, \infty\}$
$\mathbb{R}_+$	:	Non-negative real numbers, $\mathbb{R}_+ = \{x \in \mathbb{R} : x \geq 0\}$
$\mathbb{R}^N$	:	N -dimensional Euclidean space
$\mathbb{R}_+^N$	:	Positive orthant of $\mathbb{R}^N$ defined as: $\mathbb{R}_+^N = \{x \in \mathbb{R}^N : x_i \geq 0, i = 1, \dots, N\}$
$\mathbb{R}_{++}^N$	:	$\text{int}(\mathbb{R}_+^N)$
$\mathcal{P}(\Omega)$	:	Power set of a set Ω defined as: $\mathcal{P}(\Omega) = \{A : A \subseteq \Omega\}$
A^T	:	Transposed of the matrix A
X'	:	The algebraic conjugate (dual) space of a linear space X .
X^*	:	The topological dual space of a linear space X .
$\langle x, \phi \rangle$	:	$\langle x, \phi \rangle = \phi(x)$, $x \in X$, $\phi \in X'$ where X is a linear space.
$(x y)$	:	Inner product of a pair of vectors x and y
$x \cdot y$	:	Scalar (dot) product defined as $\sum_{i=1}^n x_i y_i$ for any $x, y \in \mathbb{R}^n$
$\max\{x, y\}$	:	Maximum of $x, y \in \mathbb{R}$ defined as: $\max\{x, y\} = \begin{cases} x & \text{if } x \geq y \\ y & \text{if } y > x \end{cases}$
$\min\{x, y\}$	:	Minimum of $x, y \in \mathbb{R}$ defined as: $\min\{x, y\} = \begin{cases} x & \text{if } x \leq y \\ y & \text{if } y < x \end{cases}$
$X \times Y$	:	Cartesian product of X and Y defined as: $X \times Y = \{(x, y) : x \in X, y \in Y\}$
(x, y)	:	Ordered pair
$\{x, y\}$	:	Unordered pair
$\text{span}(\cdot)$	:	Linear hull
$\text{con}(\cdot)$	:	Convex hull
$[\phi = \alpha]$	:	Hyperplane

ARBITRAGE AND THEORY OF VALUATION IN NON-LINEAR MARKETS

SUMMARY

Arbitrage opportunities can be loosely defined as opportunities to make riskless profits on an arbitrarily large scale. Arbitrage concept is the basic technique of analysis in various models in modern finance theory including the Modigliani and Miller's work on the financial structure of the firm and the Black and Scholes Option Pricing Model. In this work, the implications of the absence of arbitrage in a two period security-spot market economy where security pricing operators are non-linear are studied. Non-linear asset pricing is a basic issue in finance that may arise from market frictions like transaction costs. Starting from a formulation of an abstract economy, a review of the established theory of arbitrage is made. Equilibrium concepts for economies in the senses of Arrow-Debreu and Radner are given. The no arbitrage condition is then extended to these economies through the equivalence between Arrow-Debreu and Radner equilibrium allocations. The arbitrage analysis for the Radner economy is generalized to an infinite dimensional case by introducing relevant mathematical techniques. Later, the two period security-spot market economy is modified by allowing the security pricing operators to be non-linear to account for various kind of market frictions. Specifically, this is done by removing the linearity assumption on the asset pricing operator while retaining the linear space assumption for its domain. A geometric visualization of the set of income transfers for such an economy is constructed. It is seen that the assumption that there are no arbitrage opportunities has different implications for frictional markets as characterized by non-linear asset pricing operators and for frictionless markets. Introduction of non-linearity is seen to induce various phenomena that are not seen in canonical security-spot market economies like the presence of limited arbitrage opportunities.

DOĞRUSAL OLMAYAN PİYASALARDA ARBİTRAJ VE DEĞERLEME TEORİSİ

ÖZET

Arbitraj imkanı, kaba olarak, istenilen ölçekte risksiz kazanç sağlama imkanı olarak tanımlanabilir. Arbitraj kavramı başta Modigliani ve Miller'in firmaların finanssal yapıları üzerine olan modeli ve de Black ve Scholes'un Opsiyon Fiyatlandırma Modeli olmak üzere modern finans teorisinde birçok modelin temel analiz metodunu teşkil etmiştir. Bu çalışmada mali varlık fiyatlandırma fonksiyonlarının doğrusal olmadığı iki dönemli bir mali varlık-spot piyasa ekonomisinde arbitraj olmamasının sonuçları araştırılmıştır. Doğrusal olmayan mali varlık fiyatlandırması finasta işlem maliyetleri gibi piyasa sürtünmelerinden ileri gelebilecek temel bir konudur. Soyut bir ekonominin formülasyonu ile başlayarak halihazırdaki arbitraj konusu gözden geçirilmiştir. Arrow-Debreu ve Radner tarzında denge kavramları verilmiştir. Daha sonra Arrow-Debreu ve Radner dengelerindeki paylaşımların eşdeğerliğinden yola çıkarak arbitraj olmaması koşulu bu ekonomilere uyarlanmıştır. Radner ekonomisindeki arbitraj analizi gerekli matematiksel teknikler verilerek sonsuz boyutlu bir mal uzayına genellenmiştir. Daha sonra mali varlık-spot piyasa ekonomisi, muhtelif piyasa sürtünmelerini hesaba katmak maksadıyla mali varlık fiyatlandırma fonksiyonlarının doğrusal olmaması şeklinde değiştirilmiştir. Bu işlem, mali varlık fiyatlandırma fonksiyonlarının tanımlandığı vektör uzayı üzerindeki varsayımlara dokunmayarak, sadece fonksiyonelin doğrusallığı varsayımını kaldırarak yapılmıştır. Bu tür bir ekonomi için geometrik bir grafik canlandırma oluşturulmuştur. Arbitraj olmadığı varsayımının lineer olmayan varlık fiyatlandırma operatörleri ile karakterize edilen sürtünmeli piyasalar ile sürtünmesiz piyasalarda farklı sonuçlar doğurduğu görülmüştür. Doğrusal olmamanın kısıtlı arbitraj gibi standard mali varlık- spot piyasa ekonomilerinde görülmeyen olgulara yol açtığı gözlemlenmiştir.

1. INTRODUCTION

General equilibrium theory is one of the fundamental paradigms of modern economics. Arrow (1953) and Debreu (1959) in the middle of the twentieth century extended this theory by including uncertainty through a redefinition of a commodity. [See Debreu (1959)]. This development set the foundation for further progress in finance within the next fifty years. Apart from the general equilibrium theory in economics, two major contributions have been made to the financial theory in recent decades. Modigliani and Miller's (1958) work on the financial structure of the firm and the Black and Scholes (1973) Option Pricing Model. Both of these models have one thing in common: they make use of the concept of arbitrage as their main tool. Modigliani and Miller (1958) utilized the law of one price or value additivity to show that in a simple world without taxes or transaction costs the value of the firm would be independent of its capital structure. Black and Scholes (1973) in their celebrated *Option Pricing Model* valued options by dynamically replicating them by a portfolio of stocks and bonds. To eliminate arbitrage, the manufacturing cost of the portfolio should be the same as the price of the option. Black-Scholes Option Pricing Theory is now largely considered as one of the greatest success stories of economics as well as finance. The arbitrage concept then has been an important part of the mainstream finance. As opposed to the models like Capital Asset Pricing Model (CAPM) that impose restrictions on *tastes* by specifying the form of the utility functions and on *beliefs* by specifying the probability distributions; arbitrage arguments need no pre-assumption other than non-satiation, that is, individuals prefer more to less.

If a definition is to be made in a broad economic sense; an arbitrage opportunity is a *free lunch* or a transaction that offers *something for nothing*. Thus an arbitrage opportunity is a transaction that guarantees sure profits without making any net investment. In models of economies under uncertainty like the ones employed in finance, an arbitrage opportunity is an investment strategy that guarantees a positive payoff in some contingency with no possibility of a negative payoff and without any

net initial investment. Once present, an arbitrage opportunity can be run at an arbitrary scale by simply increasing the initial investment. Due to this, an arbitrage can be seen as a money pump. The famous notion of the law of one price is also a consequence of the lack of arbitrage opportunities in an economy. Satisfaction of the law of one price is necessary for the lack of arbitrage opportunities but is not sufficient. Therefore, arbitrage is a broader concept than the law of one price.

Although arbitrage arguments can be applied in an isolated manner to derive the pricing relationships, arbitrage concept is closely linked with the general equilibrium theory. In a competitive economy with non-satiated investors, all arbitrage opportunities should be eliminated once an equilibrium is reached. If there was any arbitrage opportunity, agents acting rationally would want to exploit this opportunity driving the economy out of its equilibrium state. Therefore arbitrage is inconsistent with equilibrium. It should be emphasized that, equilibrium prices will be arbitrage-free prices, however, arbitrage-free prices are not necessarily equilibrium prices. There may also be more than one set of arbitrage prices and only one set among these may correspond to the equilibrium prices. It should also be noted that finding the equilibrium prices requires the enumeration of the universe of securities and the agents whereas finding the arbitrage-free prices requires the enumeration of only the particular securities. The restrictions on prices made by arbitrage theory are independent of the unit of value, the universe of agents and the probabilities which agents assign to states, and the universe of securities.

Apart from Beja (1971) who realized the linearity of an asset pricing operator, Ross (1976b) was the first to characterize the absence of arbitrage by the existence of a positive linear pricing operator. This was further studied in Ross (1978). According to his analysis, there is no arbitrage if and only if there is a positive linear pricing operator that values assets, a fact that is now known as the *Fundamental Theorem of Asset Pricing*. This analysis was later extended to a general framework by Kreps (1981) through the use of partially ordered vector spaces and a notion called *viability* of a price process. Harrison and Kreps (1979) incorporated the highly developed martingale theory to show that the absence of arbitrage is equivalent the existence of an *equivalent martingale measure* under which the discounted asset prices are martingales. In general, the existence of a *continuous linear pricing operator* that

values not only marketed assets but all assets in the contingent claims space can be regarded as the basic problems studied in Ross (1978), Clark (1993, 2000). The task that remains is to derive this pricing operator in general spaces. It is well known that truly general results cannot be obtained in spaces other than the finite dimensional spaces. Harrison and Kreps (1979), Chamberlain and Rothschild (1983), Hansen and Richard (1987), Rothschild (1986) study arbitrage and asset pricing theories in an Hilbert space setting. Back and Pliska (1991), Dalang, Morton and Willinger (1990), Delbaen and Schachermayer (1994) and Schachermayer (1992, 1994) basically are in the same lines making their contributions by extending the analysis from finite to infinite dimensional state-spaces, from finite number of periods to infinite number of periods and etc. The generalizations are of course at a cost, to get over technicalities, stronger restrictions on the topological structure of the commodity space and preferences of agents for examples have to be placed.

As opposed to the competitive economies modeled by linear market structures, non-linear economies necessitate different mathematical constructs like non-linear pricing operators. Mathematically, the linear market structure is modeled by price system (M, π) , where M is a marketed subspace of the commodity space which is itself is modeled by a vector space and π is a linear pricing functional on it. Thus the linearity of the market comes from both the vector space structure of the marketed space and the linearity of the pricing functional on it. In this dissertation we make an alteration on this basic model that has a huge impact on the theory of valuation in this economy. In particular, we retain the assumption that the commodity space is a linear space and the marketed space is a subspace of it. Therefore, any linear combination of two choices is still available on the market. However, we remove linearity restriction on the pricing operator. The resulting market structure is non-linear in nature and therefore represents an incompetitive economy.

Historically, the competitive economy or the price taking behavior of agents has been a cornerstone of the general equilibrium theory. Linear pricing, more than just a technical convenience, is the result of price-taking behavior by agents and so is the cornerstone of a competitive economy. Aumann (1964) for example in his path breaking article formally modeled a perfectly competitive economy as an economy consisting of a continuum of traders where the influence of each individual

participant is negligible. In a Walrasian setting, as the influence of an individual in an initially competitive economy is increased, we gradually move to the realm of incompetitive economies. The price taking behavior can no longer be sustained and it is said that monopoly power exists in the markets. From the theoretical point of view there are various reasons to analyze markets where non-linear pricing prevails. Non-linear prices prevail if agents can affect prices by their actions. This can happen either because of the direct actions of agents like the volume pressure exerted on price by a large agent or because other agents believe that the large agent has information that others do not have. There is now a sizable literature on these information effects and the strategic interaction between agents in the stock market for example. Models of Kyle (1985) and Glosten and Milgrom (1985) are two classical examples. Rochet and Vila (1994) and Kyle (1989) are further extensions of these models. See O'Hara (1995) for a survey of various types of market microstructure models. Lindenberg (1979) analyzed the problem of general equilibrium in a securities market with a large investor. This investor behaves so as to maximize his own utility while taking into account the effect of his actions on prices. The resulting model is basically a variation of Capital Asset Pricing Model with a large investor. There is also a now sizable literature available on market manipulation where one agent in stock market engages in activities in which he can turn the effects of his trades into profitable trade for himself. Although various types of market manipulation have been studied, a common type of manipulation is the one that occurs through the volume effects when an investor drives the price of a stock up and then sells at the higher prices to make a profit. See for example Jarrow (1992). See Chatterjea, Cherian, and Jarrow (1993) and Cherian and Jarrow (1995) for overviews. See also Allen and Gorton (1992) and Jarrow (1994) for variations and extensions on these models.

Other reasons for the existence of non-linear prices in an economy are frictions like transaction costs or taxes. There is a large literature studying the affect of transaction costs in models of capital market equilibrium and derivative pricing. There are also various articles on the no arbitrage pricing with taxes [Ross (1987)] and the related issue of tax arbitrage [Dybvig and Ross (1986) and Dammon and Green (1987)].

As a somehow related theme, non-linear pricing of a commodity by its supplier (e.g. volume discounts) actually has been extensively studied from the point of view of the firm and its welfare affects. [See Wilson (1993), Varian (1989) and Philips (1981) for comprehensive surveys]. Here the pricing scheme of the commodity is under the control of the firm and the firm tries to choose the best non-linear price correspondence so as to maximize its profits. There should not be a resale market of course to avoid arbitrage.

Our concern in this dissertation is to study the subject of non-linear pricing, a subject that has implications in terms of pricing of commodities by firms, general equilibrium theory, valuation problem in finance, etc. from the narrower perspective of the concept of arbitrage. This is close in spirit to Jouini and Kallal (1995), Jouini (1997) and Jouini and Kallal (1999). In these series of papers, the implications of no arbitrage and viability as studied by Harrison and Kreps (1979) for linear markets are extended to markets with frictions. An earlier but similar contribution is Prisman (1986). The literature on asset markets with transaction cost is splendid; however Ardalan (1999) and Dermody and Prisman (1993) and Chen (1995) complement each other in some respects.

A coarse plan of the subjects explored to serve our objective are as follows. Section 2.1 establishes the formulation by specifying the commodity space, price system and agents in the economy. Section 2.2 analyzes the concept of arbitrage and the related concepts of viability and free lunches in a broad sense. The concepts of arbitrage and equilibrium have always been linked due to the fact that presence of arbitrage opportunities is not compatible with equilibrium. For this reason it is, almost impossible to ignore equilibrium theories in a study of arbitrage. Section 2.3 is a shallow survey of equilibrium and valuation concepts for different market structures. First the familiar concept of Walrasian equilibrium that is in the heart of the competitive markets is examined. The analysis is extended to a contingent commodity markets economy in the sense of Arrow-Debreu by modeling the information flow and thus incorporating uncertainty into the analysis. This is followed by a sequential market structure in the sense of Radner (1972). The general analysis of arbitrage in Section 2.2 is extended to a security-spot markets economy in Section 2.4. In doing so, the relationship between Arrow-Debreu and Radner

equilibrium allocations is highlighted. This section is divided into two subsections; one being for finite dimensional commodity spaces and the other for infinite dimensional commodity spaces. Although it is easier to work in a finite dimensional setting, infinite dimensional commodity spaces are crucial in the study of economies under uncertainty as an infinite number of states of nature is a bona fide characteristic of real economies. Analysis of arbitrage in an infinite dimensions has its own peculiarities and necessitates the use of relatively more advanced mathematical techniques. Specifically, the no-arbitrage condition requires that the market subspace and the positive cone of the commodity space can be separated by an hyperplane. However, in order to be able to apply classical separating hyperplane theorems to separate two convex sets; one of these sets must have a non-empty interior. In particular, it is well known that the positive cone of the space of equivalence classes of square integrable random variables (L^2 spaces) - the spaces that are most frequently employed in models under uncertainty - have empty interiors and the separating hyperplane theorems cannot be employed. The trick is to separate the strictly preferred or the upper contour set of the agent which has a non-empty interior as preferences are assumed to be continuous from the marketed subspace instead. Finally, Section 2.5 takes up a model of a security-spot market economy assuming that the function defined by security prices on the asset span is non-linear. A geometric visualization of a non-linear security-spot market economy is also given. Finally, the implications of non-linear pricing functions for the pricing of assets are examined.

We now review the general literature on the concept of arbitrage and the theory of value. Due to the importance of the concept of the absence of arbitrage and its implications in the theory of finance, there is now a large body of literature available. Ingersoll (1987), Huang and Litzenberger (1988), Duffie (1988b, 1996a), Dothan (1990), Krouse (1986) and Ohlson (1987) all provide textbook treatments of the subject. Cochrane (2001), and LeRoy and Werner (2001) are two recent additions to this list. Allingham (1991) is a specialized exposition in a discrete time setting. Pliska (1997) is another discrete time treatment starting from a two-period model and later extending the analysis to a multi-period set up parallel to standard models in finance. Naik (1995) is a synopsis of arbitrage again in a finite state setting. Dybvig and Ross (1989) and Varian (1987) aim to give general accounts of the

subject at an introductory level. For the original research articles, reference is made to Ross (1976b,1978) on the absence of arbitrage and its state-pricing implications. Harrison and Pliska (1981) is the classical reference in continuous time setting. In many ways Kreps (1981) and Harrison and Kreps (1979) are two cornerstone research articles for the concept of arbitrage and naturally are the ones that are closest in spirit to our analysis in Section 2.2. Clark (1993, 2000) focus on the extension of the price functional to the whole choice space and the resulting valuation operator.

For equilibrium analysis in infinite-dimensional commodity spaces see for example Bewley (1972), Jones (1995) and Mas-Colell (1995) for a discussion. Being a general survey of the topic Mas-Colell and Zame (1991) is extremely useful. For the problems that may be encountered when working in infinite dimensions see Jones (1983). Duffie (1986a) also includes related material in analyzing equilibrium theory for general choice spaces.

The standard theory of general equilibrium is thoroughly covered in the classical work of Arrow and Hahn (1971). A more modern treatment is Mas-Colell (1985). Eatwell, Milgate, and Newman (1987) takes up various subjects of the general equilibrium theory in an encyclopedic fashion. Hildenbrand and Kirman (1988), Ellickson (1993) are also useful in this regard. At an even more elementary level, standard microeconomic textbooks like Malinvaud (1990) or Kreps (1990) can be consulted. Duffie (1991), Duffie and Sonnenschein (1989), Merton (1982) and Rothschild (1986) are dense surveys of the theory of value and asset pricing.

The theory of incomplete markets has been an active area of research in economics and finance for the last years. As a result of this, it is difficult to isolate the equilibrium theory from the theory of incomplete markets in many of the instances. There is now a large body of literature on the theory of incomplete markets. However, Geanakoplos (1990), Duffie (1995) and Magill and Shafer (1991) are comprehensive surveys on this issue. For equilibria in models with incomplete markets, existence of equilibrium in these models as well as a host of related issues consult, for example, Werner (1985), Duffie and Shafer (1985), Magill and Shafer (1990), Husseini, Lasry and Magill (1990), Geanakoplos and Polemarchakis (1986).

Magill and Quinzii (1998) is a complete textbook treatment of the theory of incomplete markets.

In recent years research in mathematical finance has begun to employ increasingly advanced techniques of functional analysis, measure theoretic probability and stochastic processes. It may be useful give a small sample of the background reading required for the mathematics employed in a study of arbitrage and theory of value. A detailed list actually can be very long and diversified.

Hahn-Banach extension theorem or separating hyperplane theorem is employed in deriving the state prices from the no arbitrage condition. Among the standard references for these theorems are Holmes (1975), Luenberger (1969) and Nikaido (1968). Rockafellar (1997) is the definitive reference on convex sets and functions. As general references on functional analysis and real analysis to get a grasp of function spaces, functionals and etc. see for example the classic Dunford and Schwartz (1957), Royden (1988), Rudin (1973, 1976), Reed and Simon (1980), Aliprantis and Border (1999), Aliprantis and Burkinshaw (1985) and Kolmogorov and Fomin (1975). Kelley (1955), Simmons (1983) and Berge (1997) are good expositions on topology. A more recent treatment is Munkres (2000). Bourbaki's volume on topology (Bourbaki, 1991) is highly recommended. Schaefer (1999), Kelley and Naimoka (1963) and Robertson and Robertson (1973) are classical references on topological vector spaces that literally dominate the modern mathematical economics. Day (1973) is the classic on normed linear spaces. Dixit (1990) is specialized in optimization techniques.

For analytic probability consult Feller (1957, 1971). For measure theoretic probability and stochastic processes consult the classics Meyer (1966), Dellacherie and Meyer (1982), Breiman (1992), Neveu (1965), Billingsley (1995), Chow and Teicher (1997), Shiryaev (1996), Karlin and Taylor (1975, 1981), Lo  ve (1977), Liptser and Shiryaev (1977) and Rogers and Williams (1987). A modern treatment of measure-theoretic probability is Williams (1997).

2. ARBITRAGE AND THEORY OF VALUATION

This chapter gives review of the concept of arbitrage in a competitive economy as modeled by a linear market structure. In Section 2.1, this is done in a broad sense by formulating an abstract economy. Formal definitions of the choice space and the pricing system as building blocks of this economy are given. In Section 2.2 a precise definition of the concept of arbitrage as well as the related notions of viability and free lunches are presented. The special platform chosen to apply the principles laid down is a two period exchange economy under uncertainty. Section 2.3 reviews the equilibrium concepts. Section 2.4 interprets the arbitrage concept previously developed for a two period exchange economy. In the second part of Section 2.4, analysis is extended to an infinite dimensional commodity space assuming that the contingent security payoffs are points of the L^2 space. Finally, Section 2.5 studies the implications of non-linear asset pricing rules from the point of view of arbitrage analysis.

As our primary objective is to study valuation by arbitrage, we will keep the theoretical specifications at an absolute minimum that is required for an elaboration of the subject. On the other hand, we will try to present all the required mathematical background as a part of our formulation to provide a complete and unified treatment of the subject.

2.1. Definitions and Assumptions

In this section we present a description of our working environment by introducing an abstract economy encompassing a choice space and a pricing operator on it. Two issues of primary concern are what choices are available to agents and how to value these choices. In our economy, the choices available to agents are given by a commodity space (or choice space) with commodity bundles as generic elements. The method of valuation of the commodity bundles is specified by a price system.

2.1.1. Commodity space

Whatever the initial set of choices available to agents, intuition suggests that it should be possible to combine these choices to create new choices. This suggests imposing an algebraic structure on the commodity space and this is achieved by modeling the commodity space as a linear space. We begin by a formal definition of a linear space for this purpose. Let X be a set and Φ a field. Define the operation *addition* as a mapping $(x, y) \mapsto x + y$ from $X \times X$ into X and the operation *scalar multiplication* as a mapping $(\lambda, x) \mapsto \lambda x$ from $\Phi \times X$ into X . Furthermore, assume that the following algebraic laws are satisfied for all $x, y, z \in X$ and $\alpha, \beta \in \Phi$:

- (1) $x + y = y + x$ (commutative law)
- (2) $(x + y) + z = x + (y + z)$ (associative law)
- (3) there exists an element $0 \in X$ called the **zero element**, such that $x + 0 = x$ for all $x \in X$.
- (4) to every $x \in X$, there corresponds an element in X denoted by $-x$ and called the **negative** of x such that $x + (-x) = 0$.
- (5) $\alpha(x + y) = \alpha x + \alpha y$ (vector distributive law)
- (6) $(\alpha + \beta)x = \alpha x + \beta x$ (scalar distributive law)
- (7) $\alpha(\beta x) = (\alpha \beta)x$ (scalar associative law)
- (8) $1x = x$ (identity law)

Definition. A set X of elements in which the functions *addition* and *scalar multiplication* are defined and for which postulates (1) to (8) above hold is called a

linear space (or a **vector space**) over the field Φ . The elements of the linear space are called **vectors** and the elements of the field Φ are called **scalars**.

In the sequel, the field Φ over which the linear space is defined will always be taken as the real field $\mathbb{R}$. The zero element of a vector space will be symbolized by 0 but the context in which it appears should make it clear to distinguish it from the zero element of $\mathbb{R}$. Once the negative of an element is defined as in (4), $x + (-y)$ is conveniently written as $x - y$.

By using the basic eight axioms above, the following additional elementary properties can be derived.

(1') The zero element 0 of X defined in (3) is unique.

(2') For each $x \in X$; the negative of x , written as $-x$ defined in (4) is unique.

(3') $x + y = x + z \Rightarrow y = z$

(4') $0x = 0$

(5') $\alpha 0 = 0$

(6') $(-1)x = -x$

(7') If $\alpha x = 0$ then either $\alpha = 0$ or $x = 0$ or both.

(8') $\alpha x = \alpha y$ and $\alpha \neq 0 \Rightarrow x = y$

(9') $\alpha x = \beta x$ and $x \neq 0 \Rightarrow \alpha = \beta$

(10') $-0 = 0$

Furthermore, it can be shown by induction and using the commutative, associative and distributive laws that a finite sum $\sum_{i \leq n} x_i$ of elements of X is invariant under different order or groupings of terms.

Typical examples of a linear space are:

- any finite-dimensional Euclidean space $\mathbb{R}^n$
- the set X of infinite sequences with real terms of the form $(x_0, x_1, x_2, \dots)$
- the set of all continuous functions on the interval $[a, b] \quad \forall a, b \in \mathbb{R}$.

Definition. A non-empty subset M of a linear space X is called a **subspace** of X if it is closed under the operations of addition and scalar multiplication, thus:

$$\alpha x + \beta y \in M, \quad x, y \in M, \quad \alpha, \beta \in \mathbb{R} \quad (2.1)$$

It can be shown that a subspace satisfies the linear space properties and hence is a linear space itself. Since any linear space must at least contain its zero element 0 and since a subspace is a non-empty set; it is clear that the set $\{0\}$ is a subspace. Moreover, a linear space is a subspace of itself.

Definition. Given a set of vectors in a linear space X indexed by the set $\mathcal{I} : S = \{x_i \in X : i \in \mathcal{I}\}$; a (finite) **linear combination** of S is the vector $\sum_{i \in \mathcal{I}} \lambda_i x_i$ for some scalars λ_i provided that only a *finite* number of λ_i are non-zero. By definition, a linear space contains every possible linear combination of its elements.

Definition. The **span** (or the **linear hull**) of a set $S \subseteq X$ denoted by $\text{span}(S)$ is the set of all *finite* linear combinations of elements of S . $\text{span}(S)$ is at the same time the smallest subspace of X which contains S . This subspace is said to be **spanned** (or **generated**) by S . By this definition, $\text{span}(\{0\}) = 0$. Also it is assumed that $\text{span}(\emptyset) = 0$.

Definition. A set $S \subseteq X$ is **linearly independent** if S is not $\emptyset$ or $\{0\}$ and no vector in S belongs to the span of the remaining vectors in S .

Definition. A **Hamel basis** of a linear space X is a linearly independent subset S of X with $\text{span}(S) = X$. At the same time, a Hamel basis of X is a linearly independent subset of X that is maximal with respect to set inclusion. Every non-trivial linear space has a Hamel basis. The Hamel basis for $X = \{0\}$ is $\emptyset$; in fact, this is the only linear space that has the empty set as the Hamel basis. Any two bases of X have the same cardinality that is called the **Hamel dimension** of X and denoted by $\dim(X)$. If $X = \{0\}$, then $\dim(X) = 0$. Again, this linear space is the only linear space that has the dimension of zero.

The next step in the specification of the choice space is to establish a sense of closeness of the elements of the commodity space. This is done by equipping the commodity space X with a topology τ . A finite dimensional commodity space admits a unique linear topology. However, there is more than one linear topology in infinite-dimensional commodity spaces. In the sequel, infinite-dimensional commodity spaces in addition to the canonical Euclidean space will be employed. This necessitates an explicit specification of the topological structure of the commodity space. As the elements of the commodity space represent different commodity bundles or choices available to the agents, the topology chosen determines the ability of agents to differentiate between these commodity bundles. Being so, the topological structure of the commodity space has to be justified economically.

Definition. A **topological space** is an ordered pair (X, τ) where X is a set and τ is a family of subsets of X satisfying:

- (1) $\emptyset \in \tau, X \in \tau$
- (2) any union of sets in τ belongs to τ
- (3) any *finite* intersection of sets in τ belongs to τ .

Thus τ is closed under the formation of *finite* intersections and *arbitrary* unions. The family τ is called a **topology** on X . The sets $T \in \tau$ are called τ -**open** (or simply **open**) and their complements $T^c = X \setminus T$ are called τ -**closed**. The sets $\emptyset$ and X are both open and closed. The family of closed sets is closed under *arbitrary* intersections and *finite* unions.

For any nonempty set X , the family $\mathcal{T} = \{\emptyset, X\}$ is called the **trivial topology**. The family of all subsets of X : $\mathcal{G} = \mathcal{P}(X)$ is called the **discrete topology** on X . Consider two topologies τ_1 and τ_2 on X . If $\tau_1 \supseteq \tau_2$; τ_1 is said to be **stronger** or **finer** than τ_2 and τ_2 is said to be **weaker** or **coarser** than τ_1 . It should be noted that two topologies on a given set may not be comparable at all.

For a subset A of X , the **interior** of A denoted by $\text{int}(A)$ is the union of all open sets contained in A . The **closure** of A denoted by $\bar{A}$ is the intersection of all closed sets containing A . If A is open then $\text{int}(A) = A$ and if A is closed then $\bar{A} = A$. The difference $\partial A = \bar{A} \setminus \text{int}(A)$ is called the **boundary** of A . An element $x \in \text{int}(A)$ is called an **interior point** of A . An element $x \in \bar{A}$ is called a **contact point** or **adherent point** of A . A set $A \in X$ is **dense** in X if $\bar{A} = X$. A **neighborhood** of a point $x \in X$ is a set $E \subseteq X$ such that $x \in \text{int}(E)$. Note that according to this definition, a neighborhood of a point can be closed or open. Similarly the neighborhood of a subset A of X is a set $E \subseteq X$ such that $A \subseteq \text{int}(E)$.

Definition. The topology on $\mathbb{R}$ generated by the basis the family of all open intervals of the form $(a, b) = \{x : a < x < b\}$ in $\mathbb{R}$ is called the **standard topology** on $\mathbb{R}$.

Definition. Given a topological space (X, τ) , a **base** for the topology τ is a class $\mathcal{B} \subseteq \tau$ such that every set belonging to τ (every open subset of X) is the union of sets belonging to $\mathcal{B}$.

Definition. If $\mathcal{B}$ is a subset $\mathcal{P}(X)$, the **topology $\mathcal{T}$ generated by $\mathcal{B}$** consists of $\emptyset$, X and all unions of finite intersections of members of $\mathcal{B}$.

Definition. Given two topological spaces (X_1, τ_1) and (Y_2, τ_2) , the **product topology** on $X_1 \times X_2$ is the topology having as the base the collection $\mathcal{B}$ of all the sets of the form $G_1 \cup G_2$ where $G_1 \in \tau_1$ and $G_2 \in \tau_2$.

In the sequel we will basically make use of the finite dimensional Euclidean space and the L^2 when defining the commodity space. Both of these spaces are normed and hence metric spaces. We will further presume norm topologies as the topological structure of these spaces. A quick review of the concepts of metric and normed spaces will facilitate the subsequent analysis.

Definition. A functional $\|\cdot\| : X \rightarrow \Phi$ where X is a linear space over the scalar field Φ is called a **norm** in X if it satisfies the following properties for all $x, y \in X$ and $\lambda \in \Phi$:

- (1) $\|x\| = 0$ iff $x = 0$
- (2) $\|x\| > 0$ if $x \neq 0$
- (3) $\|x + y\| \leq \|x\| + \|y\|$ (the triangle inequality)
- (4) $\|\lambda x\| = |\lambda| \|x\|$

A **normed linear space** is a linear space equipped with a norm.

Definition. A **metric space** is a pair (X, d) , where X is a non-empty set and $d : X \times X \rightarrow \mathbb{R}$ such that for all $x, y, z \in X$:

- (1) $d(x, y) \geq 0$
- (2) $d(x, y) = 0$ iff $x = y$
- (3) $d(x, y) = d(y, x)$ (symmetry)

$$(4) \quad d(x, y) \leq d(x, z) + d(z, y) \quad (\text{the triangle inequality})$$

The elements of X are called points of the metric space and the function d is called the **metric** on X .

Every normed linear space X is a metric space with respect to **the metric induced by the norm** defined as:

$$d(x, y) = \|x - y\|, \quad x, y \in X \quad (2.2)$$

Definition. If (X, d) is a metric space and $x \in X$, the **open ε -ball** centered at x with radius $\varepsilon > 0$ is given by

$$B_d(x, \varepsilon) = \{y : y \in X, d(x, y) < \varepsilon\} \quad (2.3)$$

A subset O of X is open in X if for every $x \in O$ there exists some $\varepsilon > 0$ such that $B_d(x, \varepsilon) \subseteq O$. The family of all open sets with respect to the metric d is called the **metric topology** on X . A topological space whose topology is generated by some metric is called **metrizable**.

Definition. The **norm topology** for a norm $\|\cdot\|$ is the metrizable topology derived from the metric induced by the norm, $d(x, y) = \|x - y\|$.

Definition. A function f from a topological space (X_1, τ_1) to another topological space (X_2, τ_2) is **continuous** if $f^{-1}(G)$ is τ_1 -open for any τ_2 -open set G .

Notice that the continuity of a function depends on the topologies defined on domain and range of the function as well as the function itself.

When the linear space structure and the topological structure are simultaneously imposed on the commodity space X , it becomes a topological vector space. This is made precise in the following definition:

Definition. Let X be a linear space over $\mathbb{R}$ and let τ be a topology on X . The topology τ is called a **linear topology** if the linear space operations (addition and scalar multiplication) are both continuous, that is;

- (i) the mapping from $X \times X$ to X defined by $(x, y) \mapsto x + y$ is continuous;

and

- (ii) the mapping from $\mathbb{R} \times X$ to X defined by $(\lambda, x) \mapsto \lambda x$ is continuous.

Note that here $X \times X$ is given the usual product topology determined by τ . $\mathbb{R} \times X$ is given the usual product topology determined by the standard topology on $\mathbb{R}$ and τ . A linear topology on X is thus compatible with the linear space structure of X .

Definition. A pair (X, τ) where X is a linear space and τ is a linear topology on X is called a **topological vector space** (or a topological linear space).

Definition. A topological space is a **Hausdorff space** (T_2 space) if every pair of distinct points have disjoint open neighborhoods. Formally, a topological space (X, τ) is a Hausdorff space if; for any $x, y \in X$ such that $x \neq y$ there exist disjoint open sets G_x and G_y with $x \in G_x$ and $y \in G_y$.

Definition. A topological vector space is called **locally convex** if there is a base for the topology consisting of convex sets (that is, sets A such that if $x, y \in A$ then $tx + (1-t)y \in A$ for $0 < t < 1$).

After defining the algebraic properties of the commodity space, we need to establish a method to compare different commodity bundles. This is done by endowing the commodity space with an order structure.

Definition. A non-empty subset C of a linear space X is a **cone** if it is closed under *positive* scalar multiplication in the sense that if $x \in C$ then $\alpha x \in C$ for all $\alpha > 0$. Equivalently, a non-empty subset C of a linear space X is a cone if $\alpha C \subseteq C$

for all $\alpha > 0$. According to our definition, a cone may or may not contain the zero element of the linear space.

Definition.

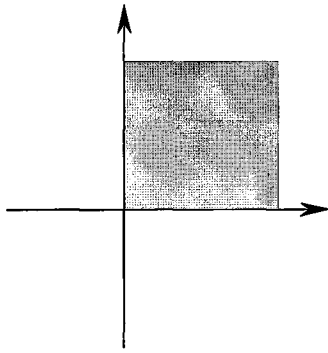
- a) A cone C is **convex** if $C + C \subseteq C$ or equivalently $x, y \in C$ implies $x + y \in C$. Thus, a convex cone is a cone that is at the same time a convex set.
- b) A cone C is **proper** if $C \cap (-C) = \{0\}$ or $C \cap (-C) = \emptyset$.
- c) A cone C is **pointed** if it is a proper cone and $0 \in C$, or equivalently $C \cap (-C) = \{0\}$. Notice that a pointed cone contains no lines.

If C is a convex cone containing 0 , then the smallest subspace containing C is given by $C - C = \{x - y : x \in C, y \in C\}$ and the largest subspace contained within C is given by $C \cap (-C)$. This fact implies that a pointed cone is a cone that contains no subspace other than 0 .

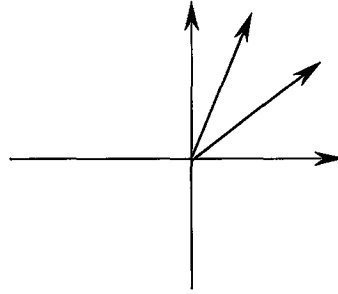
Figure 2.1 exemplifies different types of cones in $\mathbb{R}^2$. All the cones given in this figure contain 0 . The cone shown in (a) is $\mathbb{R}_+^2$; the positive cone of the Euclidean space under the usual ordering. It is a convex and proper cone. As it is proper and contains 0 , it is pointed. The cone in (b) is pointed (hence proper) but not convex. The cone in (c) is a closed hyperplane and at the same time a convex cone. However it is not proper and it is not pointed since $C \cap (-C) \neq \{0\}$. The cone in (d) is a subspace and a convex cone. However it is not proper and it is not pointed. The cone in (e) is convex, proper and pointed. The cone in (f) is not convex, is not proper and is not pointed.

Definition. The **Cartesian product** of the sets X and Y is defined as $X \times Y = \{(x, y) : x \in X, y \in Y\}$.

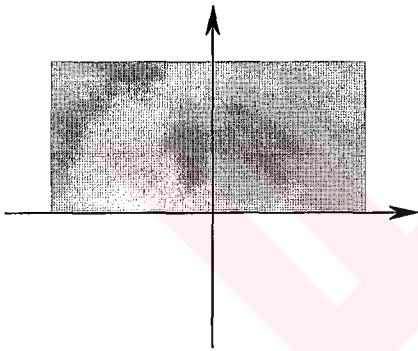
Definition. A **relation** from X to Y is a subset of $X \times Y$.



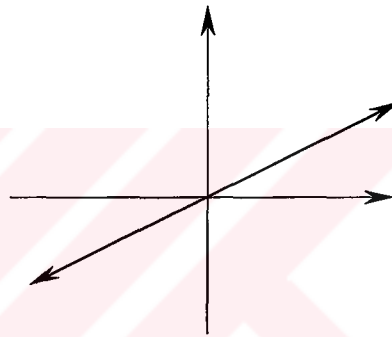
(a)



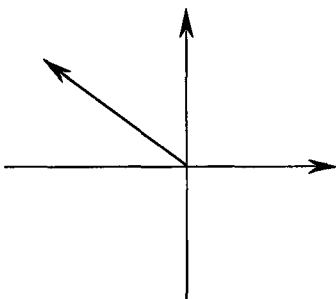
(b)



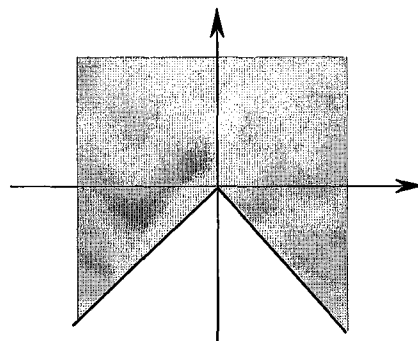
(c)



(d)



(e)



(f)

FIGURE 2.1 CONES IN THE EUCLIDEAN SPACE

A relation from X to X is said to be a relation on X . If R is a relation from X to Y , xRy means $(x, y) \in R$.

Definition. A relation on a set X is;

- (i) **reflexive** if $xRx \quad \forall x \in X$
- (ii) **symmetric** if xRy implies $yRx \quad \forall x, y \in X$
- (iii) **anti-symmetric** if xRy and yRx imply $x = y \quad \forall x, y \in X$
- (iv) **transitive** if xRy and yRz imply $xRz \quad \forall x, y, z \in X$.

Definition. A **partial ordering** on a non-empty set X , is a relation on X that is *reflexive, transitive and anti-symmetric*.

Partial orderings are usually denoted by $\leq$. The expression $x \leq y$ can be equivalently written as $y \geq x$. Also $x < y$ or $y > x$ means that $x \leq y$ but $x \neq y$.

Definition. A **partially ordered set** is a pair $(X, \leq)$ where X is a non-empty set and $\leq$ is a partial ordering on X .

Any non-empty subset of a partially ordered set is also a partially ordered set. There are two alternative methods to impose order structure on a linear space.

Construct 1:

In the first method, a partial order relation is directly imposed on the linear space. Once the partial ordering is specified, the positive cone is defined through this relation.

Definition. An **ordered linear space** is a pair $(X, \leq)$ where X is a linear space over $\mathbb{R}$ and $\leq$ a partial ordering on X such that the following axioms are satisfied:

$$(1) \quad x \leq y \text{ implies } x + z \leq y + z \quad \forall x, y, z \in X$$

$$(2) \quad x \leq y \text{ implies } \lambda x \leq \lambda y \quad \forall \lambda \geq 0$$

These properties require the partial ordering $\leq$ on X to be invariant under translation and positive scalar multiplication and thus compatible with the algebraic structure of X .

Definition. Given an ordered linear space $(X, \leq)$; the subset $X_+ = \{x \in X : 0 \leq x\}$ of X is called the **positive cone** of X . The elements of X_+ are called **positive vectors**.

Notice that under these definitions, X_+ is a *pointed (hence proper) and convex* cone.

Construct 2

Alternatively, a *pointed (hence proper), convex* cone C in a linear space X induces a partial ordering $\leq$ on X defined by;

$$y \leq x \text{ whenever } x - y \in C$$

Then it follows that $(X, \leq)$ is an ordered linear space (thus $\leq$ satisfies the algebraic properties (1) and (2) above) and $X_+ = \{x \in X : 0 \leq x\} = C$.

Remark. It is seen from the two alternative constructs given above that there is a one-to-one correspondence between the partial orderings on a linear space X that make it an ordered linear space and the pointed convex cones of X . Consequently, a linear space X and a pointed convex cone in it completely specify an ordered linear space.

Definition. Let (X, τ) be a topological linear space and at the same time let $(X, \leq)$ be an ordered linear space. Then $(X, \leq)$, along with the topology τ is an **ordered topological linear space** if the positive cone of X defined by $X_+ = \{x \in X : 0 \leq x\}$ is τ -closed in X ; or in other words, if the topology τ is compatible with the ordered linear space structure of X .

Remark: According to this definition, an ordered linear space is an ordered topological linear space if its ordering is continuous as defined by the positive cone being close.

2.1.2. Price System

After defining the commodity space, we focus on the issue of valuation and we define a price system to this end.

Definition. A scalar valued mapping ϕ on a linear space X is called a **functional** on X .

A functional ϕ on X is said to be **additive** if;

$$\phi(x + y) = \phi(x) + \phi(y), \quad x, y \in X \quad (2.4)$$

A functional ϕ on X is said to be **homogeneous** if:

$$\phi(\lambda x) = \lambda \phi(x), \quad x \in X, \quad \lambda \text{ scalar} \quad (2.5)$$

Definition. A functional that is both *additive* and *homogeneous* is said to be **linear**.

Definition. Let X be a linear space over the scalar field $\mathbb{R}$. The set X' of all *linear functionals* on X is called the **algebraic dual (conjugate) space** of X .

For two elements $\phi, \varphi \in X'$; define addition as:

$$(\phi + \varphi)(x) = \phi(x) + \varphi(x), \quad x \in X \quad (2.6)$$

and define scalar multiplication as:

$$(\lambda \phi)(x) = \lambda \phi(x), \quad \alpha \text{ scalar} \quad (2.7)$$

If further a zero element is defined as $0(x) = 0$ and the negative $-\phi$ of the functional ϕ is defined as $(-\phi)(x) = -\phi(x)$, then it is easy to see that X' is a linear space itself.

Definition. Let X be a topological vector space over the scalar field $\mathbb{R}$. The set X^* of all *continuous linear functionals* on X is called the **topological dual space** of X .

Remark. Again it can be verified by using the linear space axioms that X^* is a linear space.

Definition. Given a commodity space X which is a linear space; a **price system** is a pair (M, π) where M is a subspace of X and π is a non-zero, linear functional on M .

Notation. The value of the commodity bundle $m \in M$ at the price π is given by $\langle m, \pi \rangle$. The following notation shall than be used:

$$\pi(m) = \langle m, \pi \rangle, \quad m \in M, \quad \pi \in X' \quad (2.8)$$

There are several properties that a pricing functional defined on a linear subspace of a commodity space may satisfy. Some of these properties are desired because their existence makes economic sense while others are only technical assumptions that facilitate the analysis.

(P1) Homogeneity. This assumption requires that the unit price of each component is independent of the number of units of the good purchased or sold, thus there are no *scale effects*. In symbols;

$$\pi(\alpha m) = \alpha \pi(m), \quad m \in M, \quad \alpha \in \mathbb{R} \quad (2.9)$$

(P2) Additivity. This assumption requires that the price of a bundle of goods is equal to the sum of the prices of its components. In other terms, the price of a bundle is the same whether it is bought or sold alone or in combination with other bundles; thus there are no *scope effects*.

$$\pi(m + m') = \pi(m) + \pi(m'), \quad m, m' \in M \quad (2.10)$$

(P3) Linearity. Properties (P1) and (P2) taken together is called **linearity**. Thus π is linear if:

$$\pi(\alpha m + \beta m') = \alpha \pi(m) + \beta \pi(m'), \quad m, m' \in M, \quad \alpha, \beta \in \mathbb{R} \quad (2.11)$$

The linearity assumption is the basic characterization of *competitive markets*. It implies the absence of monopoly power meaning that agents believe that they can buy and sell as much of a commodity as they want at the market price without affecting the prices. In other words agents are price takers. Linearity at the same time implies a frictionless market, i.e., markets with no frictions such as transaction costs or taxes.

(P4) Preservation of the null bundle. The zero element of the linear space must have a price of zero. In other words the null bundle costs nothing;

$$\pi(0) = 0 \quad (2.12)$$

Recall that homogeneity of a functional requires that

$$\pi(\alpha m) = \alpha \pi(m), \quad m \in M, \quad \alpha \in \mathbb{R} \quad (2.13)$$

For $\alpha = 0$ and $m = 0$, this gives $\phi(00) = 0\phi(0) = 0$, so a homogeneous functional preserves the zero element of the linear space. Consequently, this property is automatically satisfied by a linear (and hence homogeneous) functional.

(P5) Boundedness. A functional f on a normed linear space X is said to be **bounded** if there is a constant $M \geq 0$ such that

$$|f(x)| \leq M \|x\| \text{ for all } x \in X \quad (2.14)$$

It is sensible for a pricing operator to be bounded.

(P6) Continuity. Suppose that X is endowed with a topology. If a sequence of bundles goes to zero, then the sequence of their prices go to zero. Thus if $m_i \rightarrow 0$ then $\pi(m_i) \rightarrow 0$.

Theorem. If $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ are normed spaces and $f : X \rightarrow Y$ is a linear mapping, then the following are equivalent:

- (1) $f(\cdot)$ is bounded
- (2) $f(\cdot)$ is continuous
- (3) $f(\cdot)$ is continuous at 0.

Since we will only consider normed linear spaces to serve as commodity spaces and since $\pi(\cdot)$ is a linear functional as a special case of $f(\cdot)$; the above theorem is readily applicable for $\pi(\cdot)$. Therefore, we will only need to check continuity and boundedness will automatically follow. We will also make use of the fact that in a normed linear space, a linear functional is continuous if and only if it is continuous at 0, so continuity need to be checked at 0 only. In a finite dimensional topological vector space, every linear functional is continuous. Thus if X is a finite dimensional topological vector space, then $X' = X^*$.

(P7) Positivity.

Definition. If $(X, \leq)$ is an ordered linear space with the positive cone X_+ , the **positive dual cone** of X is defined as:

$$X'_+ = \{\phi \in X' : \phi(x) \geq 0, \text{ for } x \in X_+\} \quad (2.15)$$

The elements of X'_+ will be called **positive linear functionals**. It is obvious that if $\phi(\cdot)$ is a positive linear functional on an ordered linear space X , then $\phi(\cdot)$ is monotone; that is, for any $x, y \in X$, $x \leq y$ implies $\phi(x) \leq \phi(y)$.

Definition. Given an ordered linear space $(X, \leq)$ with the positive cone X_+ and the positive dual cone X'_+ ; a linear functional $\phi(\cdot) \in X'_+$ is called **strictly positive** if for any $x \in X_+ \setminus \{0\}$, implies $\phi(x) > 0$.

Assume that the marketed space is an ordered linear space $(M, \leq)$ with the positive cone M_+ .

A pricing operator $\pi : M \rightarrow \mathbb{R}$ is **positive** iff positive choice vectors are assigned positive market values;

$$\pi(m) \geq 0, \quad m \in M_+ \quad (2.16)$$

Positivity is plausible if there is free disposal for example.

When the pricing operator $\pi(\cdot)$ is linear, the positivity property can be written as $\pi \in M'_+$.

A pricing operator is **strictly positive** whenever

$$\pi(m) > 0, \quad m \in M_+ \setminus \{0\} \quad (2.17)$$

(P8) Extension. There exists an extension of π (with the same properties desired for π) to all of the commodity space. In other words there exists a functional ψ on X having the same properties possessed by π and $\psi|_M = \pi$. While this property is very desirable in terms of economic analysis, it is at the same time a very strong condition to place on a pricing operator. We will later see that strictly positive linear functionals for example can be extended to the whole commodity space when the commodity space is finite dimensional. However, the extension of strictly positive linear functionals to the commodity space is not guaranteed when the commodity space is infinite-dimensional.

In the sequel, of the eight properties stated above for a pricing functional, some will be assumed and some will be derived. In particular, linearity will be the main assumption except in Section 2.5 where non-linear markets are analyzed. Homogeneity and additivity will be consequences of linearity. Preservation of the null bundle will follow from homogeneity as mentioned above. Strict positivity of the pricing functional will be derived from the no-arbitrage condition. For finite dimensional spaces, continuity will follow from linearity. For infinite-dimensional spaces, continuity will follow from the no-free lunch condition. As mentioned above, for a normed linear space, boundedness will follow from continuity. Finally, the extension property will be the major task of Section 2.2. In particular, the extension

of the strictly positive, linear and continuous functional $\pi(\cdot)$ on the marketed space M to a strictly positive, linear and continuous functional on the entire linear space X will be the main result obtained in Section 2.2 at the expense of more restrictive assumptions.

2.1.3. Agents

A study of arbitrage in our economy requires no restrictions on preferences of agents other than the condition that agents prefer more to less. In order to make this precise, we first define the preference orderings of agents. Each agent in our economy has a preference ordering defined as a relation on the commodity space X . Recall that a relation R on a set X is a subset of R of $X \times X$ and xRy means $(x, y) \in R$ for $x, y \in X$. Then we make the following definitions:

Definition. A relation R on a set X is:

- (i) **reflexive** if xRx $\forall x \in X$
- (ii) **transitive** if xRy and yRz imply xRz $\forall x, y, z \in X$
- (iii) **complete** if xRy or yRx (or both) $\forall x, y \in X$
- (iv) **asymmetric** if xRy implies $\text{not}(yRx)$ $\forall x, y \in X$
- (v) **negatively transitive** if

$$[\text{not}(xRy) \text{ and } \text{not}(yRz)] \text{ imply } \text{not}(xRz) \quad \forall x, y, z \in X$$

There are two alternative paths to be followed when specifying the comparisons made by the agents. The first approach is to take a preference relation as primitive and derive strict preference and indifference relations from this relation. An equivalent approach is to do this in reverse order; to take a strict preference as

primitive and derive preference and indifference relations from this relation. Both approaches are explained below:

Construct 1

Definition. A **preference** relation on a commodity space symbolized as $\succeq$ is a *complete* and *transitive* relation on the commodity space. It is said that x is *at least as good as* y for $x, y \in X$ and written as $x \succeq y$ if $(x, y) \in \succeq$.

Thus the preference relation $\succeq$ on X is the subset of $X \times X$ defined as:
 $\{(x, y) \in X \times X : x \succeq y\}$

Remark. The completeness of the preference relation $\succeq$ is written as

$$x \succeq y \text{ or } y \succeq x \text{ (or both)} \quad \forall x, y \in X \quad (2.18)$$

Completeness therefore means that every two commodity bundles can be compared.

Transitivity of the preference relation $\succeq$ is written as

$$[x \succeq y \text{ and } y \succeq z] \text{ imply } x \succeq z \quad \forall x, y, z \in X \quad (2.19)$$

Transitivity property is needed to for the consistency of the choices made by the agents. Completeness and transitivity of preference relations provide rationality in choices.

Remark. In addition to completeness and transitivity, a third property: reflexivity is sometimes imposed on a preference relation. Reflexivity in symbols is stated as:

$$x \succeq x \quad \forall x \in X \quad (2.20)$$

It is trivial to prove that reflexivity is a consequence of completeness and thus redundant.

Definition. Given the preference relation $\succeq$, the **strict preference** relation written $\succ$ is defined as:

$$x \succ y \text{ iff } [x \succeq y \text{ and } \text{not}(y \succeq x)] \quad (2.21)$$

The statement $x \succ y$ is read as x is preferred to y .

Definition. Given the preference relation $\succeq$, the **indifference relation** written $\sim$ is defined as :

$$x \sim y \text{ iff } [x \succeq y \text{ and } y \succeq x] \quad (2.22)$$

The statement $x \sim y$ is read as x is indifferent to y .

Proposition. The strict preference relation $\succ$ defined by (2.21) is asymmetric.

Proof. By definition, the preference relation $\succeq$ is complete, thus;

$$x \succeq y \text{ or } y \succeq x \quad (2.23)$$

Using the definition of a strict preference given above, this can be written as;

$$x \succeq y \text{ or } \text{not}(x \succ y) \quad (2.24)$$

But this is logically equivalent to:

$$x \succ y \text{ implies } x \succeq y \quad (2.25)$$

which can be written as;

$$x \succ y \text{ implies } \text{not}(y \succ x) \quad (2.26)$$

This property is nothing but the asymmetry property of $\succ$. ■

Proposition. The strict preference relation $\succ$ defined by (2.21) is negatively transitive.

Proof. By definition, the preference relation $\succeq$ is transitive, thus;

$$[y \succeq x \text{ and } z \succeq y] \text{ imply } z \succeq x \quad (2.27)$$

Using the definition of a strict preference given above, this can be written as;

$$[not(x \succ y) \text{ and } not(y \succ z)] \text{ imply } not(x \succ z) \quad (2.28)$$

This property is nothing but the negative transitivity property of $\succ$. ■

Construct 2

Definition. A **strict preference** relation on a commodity space symbolized as $\succ$ is an *asymmetric* and *negatively transitive* relation on the commodity space. It is said that x is *preferred to* y for $x, y \in X$ and written as $x \succ y$ if $(x, y) \in \succ$.

Definition. Given the strict preference relation $\succ$, the **preference relation** written $\succeq$ is defined as:

$$x \succeq y \text{ if } not(y \succ x) \quad (2.29)$$

The statement $x \succeq y$ is read as x is *at least as good as* y .

Definition. Given the strict preference relation $\succ$, the **indifference relation** written $\sim$ is defined as:

$$x \sim y \text{ if } [not(x \succ y) \text{ and } not(y \succ x)] \quad (2.30)$$

The statement $x \sim y$ is read as x is *indifferent to* y .

$\succ$ is asymmetric means that

$$x \succ y \text{ implies } not(y \succ x) \quad (2.31)$$

$\succ$ is negatively transitive means that

$$[not(x \succ y) \text{ and } not(y \succ z)] \text{ imply } not(x \succ z) \quad (2.32)$$

A logically equivalent statement is:

$$\text{not } [(x \succ y) \text{ or } (y \succ z)] \text{ implies } \text{not}(x \succ z) \quad (2.33)$$

Then by writing the contrapositive form of this proposition we obtain;

$$(x \succ z) \text{ implies } [(x \succ y) \text{ or } (y \succ z)] \quad (2.34)$$

Proposition. The preference relation $\succeq$ defined by (2.29) is complete.

Proof. By definition, the strict preference relation $\succ$ is asymmetric, thus;

$$x \succ y \text{ implies } \text{not}(y \succ x) \quad (2.35)$$

Using the definition of a preference given above, this can be written as;

$$x \succ y \text{ implies } x \succeq y \quad (2.36)$$

But this is logically equivalent to:

$$x \succeq y \text{ or } \text{not}(y \succ x) \quad (2.37)$$

which can be written as;

$$x \succeq y \text{ or } y \succeq x \quad (2.38)$$

This property is nothing but the completeness property of $\succeq$. ■

Proposition. The preference relation $\succeq$ defined by (2.29) is transitive.

Proof. By definition, the strict preference relation $\succ$ is negatively transitive, thus;

$$[\text{not}(x \succ y) \text{ and } \text{not}(y \succ z)] \text{ imply } \text{not}(x \succ z) \quad (2.39)$$

Using the definition of a preference given above, this can be written as;

$$[y \succsim x \text{ and } z \succsim y] \text{ imply } z \succsim x \quad (2.40)$$

This property is nothing but the transitivity property of $\succsim$. ■

Let $\succsim$ be a preference relation defined on a non-empty subset A of an ordered linear space $(X, \leq)$. Let X_+ be the positive cone of X which induces the partial ordering $\leq$ on X . Recall that X_+ is a pointed and convex cone. Let K be the positive cone of X with the origin deleted: $K = X_+ \setminus \{0\}$.

Definition. Let X be a commodity space and $\succsim$ a preference relation on it. The **upper contour set** of bundle x is the set of all bundles that are as good as x ; that is $\{y \in X : y \succsim x\}$. The **lower contour set** of bundle x is the set of all bundles that x is at least as good as; that is $\{y \in X : x \succsim y\}$. The **indifference set** containing point x is the set of all commodity bundles that are indifferent to x , that is $\{y \in X : y \sim x\}$. For a given bundle x ; the indifference set containing x is the intersection of the upper contour set of x and the lower contour set of x .

Definition. A non-empty subset A of an ordered linear space $(X, \leq)$ is **comprehensive upwards** iff

$$x \in A \text{ and } y \geq x \text{ imply } y \in A \quad (2.41)$$

Equivalently, a non-empty subset A of an ordered linear space $(X, \leq)$ with the positive cone X_+ is comprehensive upwards iff:

$$x + x' \in A, \quad x \in A, \quad x' \in X_+ \quad (2.42)$$

Or even more succinctly; a non-empty subset A of an ordered linear space $(X, \leq)$ with the positive cone X_+ is comprehensive upwards iff:

$$A + X_+ = A \quad (2.43)$$

Remark. Being comprehensive upwards for a consumption set means that consuming “more” of a commodity bundle is always feasible, the precise meaning of the word more being conveyed by the order structure of the commodity space.

Example. The positive cone X_+ of an ordered linear space $(X, \leq)$ is a comprehensive upwards set.

Definition (Continuity of preferences). A preference relation $\succeq$ on a topological space (X, τ) is τ -**continuous** iff $\forall x \in X$, the upper contour set of x

$$\{y \in X : y \succeq x\} \quad (2.44)$$

and the lower contour set of x

$$\{y \in X : x \succeq y\} \quad (2.45)$$

are both τ -closed in X .

Since, in a topological space, the complement of a closed set is open, the continuity of preferences can be equivalently stated as follows: A preference relation $\succeq$ on a topological space (X, τ) is τ -continuous iff $\forall x \in X$, the sets

$$\{y \in X : y \succ x\} \quad (2.46)$$

and

$$\{y \in X : x \succ y\} \quad (2.47)$$

are both τ -open in X .

Under the continuity assumption, the indifference set containing a bundle is closed as it is the intersection of the upper and lower contour sets.

Remark. Continuity of preferences is a regularity hypothesis and loosely speaking prevents jumps in the consumer preferences.

Definition (Convexity of preferences). The preference relation $\succsim$ on a linear space X is **convex** if $\forall x \in X$, the upper contour set of x

$$\{y \in X : y \succsim x\} \quad (2.48)$$

is convex.

Equivalently, $\succsim$ is convex if $x, y, z \in X$ with $y \succsim x$ and $z \succsim x$ then

$$\alpha y + (1 - \alpha) z \succsim x \quad \forall \alpha \in [0, 1] \quad (2.49)$$

Definition (Strict monotonicity of preferences). A preference relation $\succsim$ on a non-empty and *comprehensive upward* subset A of an ordered linear space $(X, \leq)$ is **strictly monotonic** iff

$$x, y \in A \text{ and } x > y \text{ implies } x \succ y \quad (2.50)$$

Equivalently; a preference relation $\succsim$ on a non-empty and *comprehensive upward* subset A of an ordered linear space $(X, \leq)$ with the cone $K = X_+ \setminus \{0\}$ is strictly monotonic iff

$$x + k \succ x \quad \forall x \in A \quad \forall k \in K \quad (2.51)$$

Remark. Strictly monotonic preferences mean that more is strictly preferred to less the meaning of “more” being embodied in the partial ordering $\leq$ on X or equivalently the cone $K = X_+ \setminus \{0\}$. Notice that this definition of strict monotonicity is quite general. Consider the case $X = \mathbb{R}^N$ as an example. Take as the ordering $\geq$, the natural partial ordering on $\mathbb{R}^N$ given as;

$$\text{for } x, y \in \mathbb{R}^N \quad x = (x_1, \dots, x_N) \geq y = (y_1, \dots, y_N) \quad \text{iff} \quad x_i \geq y_i \quad \forall n \in \{1, \dots, N\}$$

Then various cases of the concept of monotonicity for $X = \mathbb{R}^N$ can be conveniently dealt with through a different specification of the cone K :

- (i) If $K = \text{int}(\mathbb{R}_+^N)$; then $x, y \in \mathbb{R}^N$ and $x - y \in \text{int}(\mathbb{R}_+^N)$ imply $x \succ y$. This is the assumption sometimes referred to as *monotonicity* in the literature. [See for example Mas-Colell (1985)].
- (ii) If $K = \mathbb{R}_+^N \setminus \{0\}$; then $x, y \in \mathbb{R}^N$ and $x - y \in (\mathbb{R}_+^N \setminus \{0\})$ imply $x \succ y$. This is the situation sometimes referred to as *strong monotonicity* in the literature.
- (iii) If $K = \{x \in \mathbb{R}^N : x_n > 0 \text{ if } n \neq j \text{ and } x_n = 0 \text{ if } n = j\}$; then this means that commodity j is a noxious commodity.

Definition (Non-satiation of preferences). A preference relation $\succeq$ on a set X is **non-satiated** iff $\forall x \in X$ there exists some $y \in X$ such that $y \succ x$.

Definition (Local non-satiation of preferences). A preference relation $\succeq$ on a topological space X is **locally non-satiated** iff $\forall x \in X$ and for any neighborhood V_x of x there exists at least one bundle $y \in V_x$ such that $y \succ x$.

Remark. The local non-satiation assumption, loosely speaking, eliminates thick indifference curves i.e., indifference sets with non-empty interiors.

As a final point it can be shown that for the case $X = \mathbb{R}^N$, strong monotonicity in the sense of (ii) above implies local non-satiation.

2.2. Arbitrage, Viability and Free Lunches

The purpose of this section is to review the concept of arbitrage from the point of view of modern mathematical finance. To this end, arbitrage is defined within the context of a general net trade exchange economy. The advantage of this general formulation is that it can be easily adapted to more specific models like the two period exchange economy that will be taken up in the subsequent sections. The general formulation of arbitrage made is essentially in the lines of Ross (1978), Harrison and Kreps (1979), Kreps (1981), Clark (1993, 2000).

Consider an exchange economy

$$\hat{\mathcal{E}} = (\hat{X}, \{(\hat{X}^i, \hat{\succsim}_i, e^i) : i \in \mathcal{I}\}) \quad (2.52)$$

consisting of a linear space $\hat{X}$, agents indexed by $i \in \mathcal{I}$ each characterized by the triple $(\hat{X}^i, \hat{\succsim}_i, e^i)$ where $\hat{X}^i \subseteq \hat{X}$ is the consumption set for agent i , $\hat{\succsim}_i \in \hat{X}^i \times \hat{X}^i$ is agent i 's preference relation and $e^i \in \hat{X}^i \subseteq \hat{X}$ is agent i 's initial endowment.

This exchange economy $\mathcal{E} = (X, \{(X^i, \succsim_i, e^i) : i \in \mathcal{I}\})$ can be converted into a net trade exchange economy

$$\mathcal{E}' = \{X, (X^i, \succsim_i) : i \in \mathcal{I}\} \quad (2.53)$$

by translating the commodity space and preference relation of each agent as

$$X^i = \hat{X}^i - \{e^i\} \quad \forall i \in \mathcal{I} \quad (2.54)$$

$$x \succsim_i y \text{ iff } x + e^i \hat{\succsim}_i y + e^i \quad \forall x, y \in X^i \quad \forall i \in \mathcal{I} \quad (2.55)$$

The resulting commodity space X_i is also called the **set of net trades** and the bundles in this space which are simply additions to endowments are called **net trades**.

If the consumption set of each agent is taken to be the entire linear space, that is, $X^i = X$; then the economy reduces to:

$$\mathcal{E} = \{(X, \succsim_i) : i \in \mathcal{I}\} \quad (2.56)$$

This economy will be our point of departure to analyze arbitrage and its consequences but we first need to make some assumptions.

Assumption 1. (X, τ) is a *topological linear space*.

Remark. By endowing the choice space with linear space structure, we define the algebraic operations on this space thus, we specify how various choices can be constructed from a given set of choices. Defining the commodity space X as a real linear space embodies the following additional assumptions:

- (A.1.1) Commodity bundles are infinitely divisible.
- (A.1.2) There are no transaction costs.
- (A.1.3) Negative quantities of commodities or short sales are possible.
- (A.1.4) There are no restrictions on the quantity of a commodity bought or sold; e.g., unlimited short selling is allowed.

Assumption 2. $(X, \leq)$ is an *ordered linear space*.

Remark. Note that the positive cone of X defined by $X_+ = \{x \in X : 0 \leq x\}$ is not assumed to be τ -closed in X , and hence $(X, \leq)$, along with the topology τ may not qualify as an ordered topological linear space. Nevertheless, the ordered linear space $(X, \leq)$ has topological structure. Note also that the positive cone is non-empty by the original definition of a cone.

Let the *positive cone with the origin deleted* represented by a cone K . As mentioned before, the order structure of the space X could be specified through this cone. Instead the partial order relation $\leq$ is given and the resulting cone $K = \{x \in X : 0 \leq x\} \setminus \{0\}$ specifies a direction of increasing economic value.

Notation. For an ordered linear space $(X, \leq)$ equipped with a topology, the following notation shall be employed:

$$x \geq 0 \quad \text{iff} \quad x \in X_+ \quad (2.57)$$

$$x > 0 \quad \text{iff} \quad x \in X_+ \setminus \{0\} \quad (2.58)$$

$$x \gg 0 \quad \text{iff} \quad \text{int}(X_+) \quad (2.59)$$

Assumption 3. The topology τ is *Hausdorff*.

Remark. Note that a topological space is Hausdorff if and only if every net converges to at most one point. In the sequel, nets will be used in the definition of the concept of a free lunch. Hausdorff assumption is made to guarantee that convergent nets have unique limits.

Assumption 4. The topology τ is *locally convex*.

Assumption 5. $\forall i \in \mathcal{I}$, $\succsim_i$ is *K strictly increasing*.

Assumption 6. $\forall i \in \mathcal{I}$, $\succsim_i$ is *convex*.

Assumption 7. $\forall i \in \mathcal{I}$, $\succsim_i$ is τ -*continuous*.

For agent $i \in \mathcal{I}$ the preference ordering $\succsim_i$ on X is completely specified by the triple: (X, τ, K) . Let $\mathcal{A} = \mathcal{A}(X, \tau, K)$ denote the set of preference relations (complete and transitive binary relations) on X that are convex, τ -continuous and K -strictly increasing. Thus $\mathcal{A}$ represents the class of agents whose preferences are convex, τ -continuous and K -strictly increasing.

In this abstract net trade exchange economy, there exists markets for an original *non-empty* set M_0 of bundles of goods. Notice that there are no restrictions on the cardinality of M_0 .

Definition. A **marketed bundle** of goods is a commodity bundle that can be bought or sold on the market.

Due to the linear space structure of the net trade space, a marketed bundle is thus a (finite) linear combination of the original set M_0 of marketed bundles of goods. In symbols a marketed bundle is given by:

$$m = \sum_{i=1}^n \theta_i m_i, \quad \{m_1, \dots, m_n\} \subseteq M_0, \quad \theta_1, \dots, \theta_n \text{ scalars} \quad (2.60)$$

Note that, to avoid convergence problems when summing an infinite number of terms, we require that the linear combination is finite. Thus either,

- (i) an infinite number of terms in the linear combination is allowed with the provision that only a finite number of θ_i 's are non-zero

or,

- (ii) only a finite number of terms in the linear combination are allowed

The latter method is employed here.

Definition. The set of commodity bundles that can be constructed out of the original marketed bundles of goods or the set of all marketed bundles denoted M is called the **marketed subspace**.

Thus;

$$\text{span}(M_0) = M \quad (2.61)$$

or,

$$M = \{m \in X : m = \sum_{i=1}^n \lambda_i m_i \text{ for some scalars } \lambda_1, \dots, \lambda_n \text{ and } \{m_1, \dots, m_n\} \subseteq M_0\} \quad (2.62)$$

By definition of span, M is then the smallest subspace of X which contains M_0 .

Assumption 8. $M \cap K \neq \emptyset$.

Remark. This assumption simply says that it is possible to purchase a bundle that increases any agent's level of satisfaction.

Definition. The markets are **incomplete** iff $M \neq X$.

Remark. Incompleteness of markets simply means that there are elements of the choice space X that cannot be constructed out of the marketed bundles.

The relation between M_0 , M and X is given by:

$$M_0 \subseteq (\text{span}(M_0) = M) \subseteq X \quad (2.63)$$

The original non-empty set M_0 of bundles of goods are traded at prices given by the functional $\pi_0 : M_0 \rightarrow \mathbb{R}$.

Definition. Given the set of marketed bundles of goods M , a price correspondence $\pi : M \rightarrow \mathbb{R}$ is defined as:

$$\pi\left(\sum_{i=1}^n \theta_i m_i\right) = \sum_{i=1}^n \theta_i \pi_0(m_i), \quad m_i \in M_0, \quad \theta_1, \dots, \theta_n \text{ scalars} \quad (2.64)$$

Remark. This critical definition says that while new bundles are constructed as linear combinations of the original bundles in M_0 , the prices at which these new bundles can be traded are defined as the linear combinations of the prices of the component bundles in M_0 .

Lemma. $\pi : M \rightarrow \mathbb{R}$ is a *well-defined linear functional*.

Proof. See for example Kreps (1981) or Clark (1993) for a proof that involves a basic arbitrage argument. ■

From this lemma; the linearity of the pricing function which is the standard assumption of competitive markets is seen to be a necessary condition for the absence of arbitrage opportunities.

Consider a marketed bundle $m \in M$ written as a linear combination of two different set of original bundles $\{m_1, \dots, m_n\} \subseteq M_0$ and $\{m'_1, \dots, m'_{n'}\} \subseteq M_0$:

$$m = \sum_{i=1}^n \theta_i m_i = \sum_{i=1}^{n'} \theta'_i m'_i, \quad \{m_1, \dots, m_n\} \subseteq M_0 \text{ and } \{m'_1, \dots, m'_{n'}\} \subseteq M_0 \quad (2.65)$$

As $\pi(\cdot)$ is well-defined, when we calculate the price of m as a linear combination of the prices of the constituent bundles, we must arrive at the same price:

$$\pi(m) = \pi\left(\sum_{i=1}^n \theta_i m_i\right) = \pi\left(\sum_{i=1}^{n'} \theta'_i m'_i\right) = \sum_{i=1}^n \theta_i \pi_0(m_i) = \sum_{i=1}^{n'} \theta'_i \pi_0(m'_i) \quad (2.66)$$

Recall that a price system is a pair (M, π) where M is the set of marketed bundles of goods which is a subspace of the commodity space X and $\pi : M \rightarrow \mathbb{R}$ is a price functional defined by $\pi \mapsto \pi(m)$, $m \in M$.

Definition. The set of marketed claims with negative prices

$$F = \{m \in M : \pi(m) \leq 0\} \quad (2.67)$$

is called **set of feasible net trades**.

Definition. An **arbitrage opportunity** is a feasible bundle that is strictly positive.

Symbolically, an arbitrage opportunity is a marketed bundle that satisfies

$$m > 0 \quad \text{and} \quad \pi(m) \leq 0, \quad m \in M \quad (2.68)$$

We are now ready to formally define the notions related to arbitrage.

Definition [No arbitrage, Ross (1978)]. The following are all alternative but equivalent forms of the no arbitrage condition:

(i) A price system (M, π) admits *no arbitrage* opportunities if there exists no $m \in M \cap X_{++}$ with $\pi(m) \leq 0$.

(ii) A price system (M, π) admits *no arbitrage* opportunities if

$$m > 0 \text{ implies } \pi(m) > 0 \quad \forall m \in M \quad (2.69)$$

Thus absence of arbitrage requires that any marketed claim that is strictly positive has a positive price.

(iii) A price system (M, π) admits *no arbitrage* opportunities if

$$X_{++} \cap F = \emptyset \quad (2.70)$$

This says that the strictly positive cone and the set of feasible net trades have empty intersection.

Theorem. A price system (M, π) admits *no arbitrage* opportunities if and only if $\pi : M \rightarrow \mathbb{R}$ is *well defined, strictly positive and linear*.

Proof. See Ross (1978), Kreps (1981), Dybvig and Ross (1989) or Clark (1993, 2000) for example for this well-known result in finance sometimes referred to as the *Fundamental Theorem of Asset Pricing*. ■

In finite-dimensional linear spaces, all linear functionals are continuous. However, this is not necessarily so in infinite-dimensional linear spaces. In finite-dimensional linear spaces, the no arbitrage condition guarantees linearity of the price functional $\pi : M \rightarrow \mathbb{R}$ and continuity follows from linearity. In infinite-dimensional linear spaces, a stronger condition than the no arbitrage condition is needed to ensure that $\pi : M \rightarrow \mathbb{R}$ is continuous also in addition to being strictly positive and linear. [See Clark (2000)]. This condition is first formulated by Kreps (1981).

Definition [Kreps (1981)]. A **free lunch** is a net $\{(m_\alpha, x_\alpha) : \alpha \in a\} \subseteq M \times X$ and a bundle $k \in K$ such that

- (i) $m_\alpha - x_\alpha \in K \cup \{0\} \quad \forall \alpha \in a$
- (ii) $\lim_\alpha x_\alpha = k \quad \text{and} \quad \liminf_\alpha \pi(m_\alpha) \leq 0$

Remark. A free lunch is formally a strictly positive bundle to which a net of bundles dominated by a separate feasible net of bundles converge. As the dominated net of bundles converge to the strictly positive bundle, it is said to approximate the strictly positive bundle. A free lunch thus is a notion of approximate arbitrage.

Definition [No free lunches, Harrison and Kreps (1979)]. A price system (M, π) admits *no free lunches* if and only if there does not exist a free lunch.

The no free lunch condition can be alternatively expressed as in Clark (2000): A price system (M, π) admits no free lunches iff

$$K \cap (\overline{F - K}) = \emptyset \quad (2.71)$$

The no arbitrage condition makes it possible to derive a *strictly positive linear* functional $\pi : M \rightarrow \mathbb{R}$ that values all marketed assets. Furthermore, the stronger condition of no free lunches assures that this functional is continuous also. The next challenge is to be able to value the remaining or non-marketed bundles (those bundles in X that are not in M) by a strictly positive, linear and continuous pricing functional. However, this requires still stronger conditions to impose. There are two methods: In linear spaces with appropriate structure, the no free lunch condition is sufficient. However, to be able to value all bundles (marketed or non-marketed) with a strictly positive, linear and continuous pricing functional in greater generality without imposing any restrictions on the structure of X , the concept of viability due to Kreps (1981) will be employed.

Let Ψ denote the set of all τ -continuous, K -strictly positive and linear functionals on X .

Definition [Ross (1978)]. Given (X, τ, K) , a **valuation operator** for a price system (M, π) is a τ -continuous, K -strictly positive and linear extension of π to X . Thus a valuation operator is a $\psi \in \Psi$ such that $\pi = \psi|_M$.

Definition [Extension property, Kreps (1981)]. A price system (M, π) has the **extension property** for (X, τ, K) if there exists a valuation operator (a τ -continuous, K -strictly positive and linear extension of π to X).

Theorem [Kreps (1981)]. A price system (M, π) is (X, τ, K) -**viable** (or simply **viable**) iff there exist some $\succsim \in \mathcal{A}$ and $m^* \in F$ such that

$$m^* \succsim m \quad \forall m \in F \quad (2.72)$$

Remark. This definition simply says that there is some agent in the class $\mathcal{A}$ whose demand set is non-empty. As we will see in the sequel, in a Walrasian

equilibrium, an allocation is such that each agent's consumption bundle must be budget constrained and strictly supported by an equilibrium price system. In other words, each agent must be able to find a maximal element in his budget set. Viability simply says that there is some agent in the class $\mathcal{A}$ who is able to find such a maximal element. Viability is needed if (M, π) is to be considered as consistent with equilibrium otherwise agents will want to hold unlimited positions of net trades and supply and demand will never balance.

Theorem [Harrison and Kreps (1979)]. A price system (M, π) is *viable* iff it satisfies the *extension property* for (X, τ, K) .

Theorem [Kreps (1981)]. If a price system (M, π) is *viable* then it admits *no free lunches*.

The relation between the concepts formulated can be schematically illustrated in Figure 2.2.

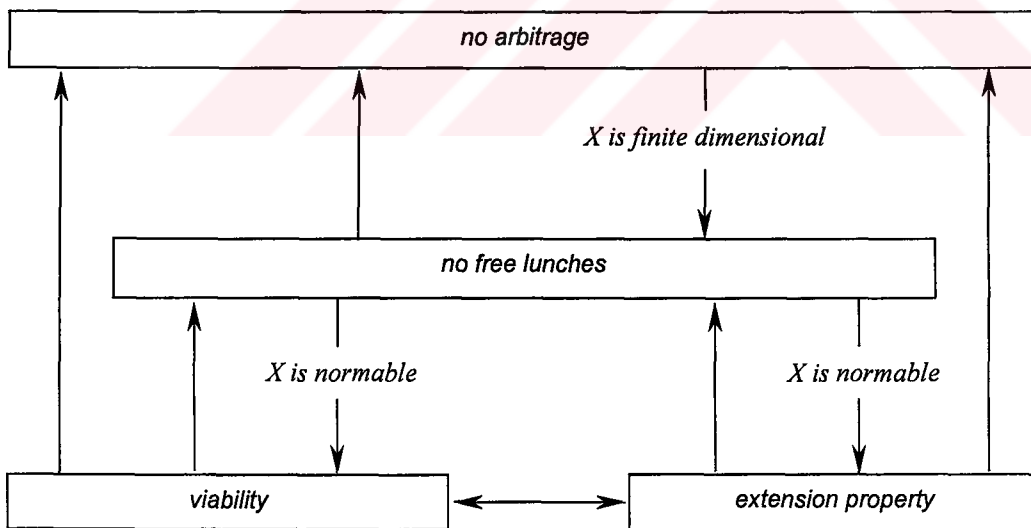


FIGURE 2.2 ARBITRAGE AND RELATED CONCEPTS

2.3. Equilibrium and Theory of Value

The primary focus of the general equilibrium theory is the allocation of resources in an economy when the activities of agents acting independently in their self-interest are compatible with each other. The **Arrow-Debreu-McKenzie Theory** developed in the 1950's can be taken in many ways as a precise model of the modern decentralized competitive economy under certainty. However; a realistic model of an economy requires the incorporation of time and uncertainty into the analysis. This was accomplished by Arrow (1953) and Debreu (1959). The celebrated **Arrow-Debreu Model of Competitive Equilibrium Under Uncertainty** is an extension of the general equilibrium theory to an economy with uncertainty and **complete contingent markets**. The key device that enabled this extension was the definition of a commodity as a good distinguished by the environmental event in which it is delivered in addition to its physical attributes, its location and its date of delivery. With this definition, the price of a commodity is the amount measured in accounting units that has to be paid now for the availability of one unit of that commodity at a specified location, time period and event. Complete contingent market then means that there is a price for each commodity thus defined. In the Arrow-Debreu model equilibrium exists and it is ex ante Pareto optimum. Due to this nature of equilibrium, it is said that markets open once in the Arrow-Debreu model. All transactions are contracted at an initial date via the complete set of futures markets and agents would not undertake further transactions if the markets were to open at a later date due to the Pareto optimality of equilibrium. The Arrow-Debreu model is later extended to the case in which different economic agents have different information.

In an economy under certainty, it is naturally assumed that there is a market and a price for each commodity. When uncertainty and time is introduced into the analysis, this may not be the case. As the assumption of the existence of a complete set of contingent markets in the Arrow-Debreu model is removed, the resulting analysis gives rise to the theory of **incomplete markets**. This theory is developed in a sequence of markets and expectations of agents play a dominant role in the analysis. Different equilibrium constructs differing on the restrictions they place on the expectations of agents have been developed. When no restrictions are placed on the

expectations of agents an equilibrium concept called the **temporary equilibrium** (Grandmont, 1982) is obtained. When the agents base their current decisions on the correct anticipations of future spot prices, we have so called **the equilibrium of plans, prices and price expectations or equilibrium with perfect foresight**, (Radner, 1972). When the equilibrium of plans, prices and price expectations is adapted allowing agents to hold common probability assessments on future events, we arrive at the concept of **rational expectations equilibrium** that is commonly employed in macroeconomics.

In this section we extend the analysis of the previous section to the case where uncertainty prevails. To this end, we formulate a two period exchange economy as a special model of the abstract economy already analyzed. This will provide us a canonical set up to apply the definitions of the previous sections and conduct our analysis. This chapter is a general survey of the theory of value. It starts with the familiar Walrasian theory of value applied to an exchange economy with a finite number of agents and a finite or infinite dimensional commodity space. The essential reason for studying the theory of value is the close relation between arbitrage and equilibrium. It was previously mentioned that the existence of arbitrage opportunities is not consistent with equilibrium. As also mentioned in the previous section, a redefinition of commodity makes it possible to apply the Walrasian equilibrium concept to an economy that incorporates uncertainty. In dealing with uncertainty, two basic *market structures* are considered: contingent commodity markets and security-spot markets.

2.3.1. Walrasian equilibrium

Consider again a linear space X and a price space given by its algebraic dual space X' . As we will content ourselves with the formulation of the basic equilibrium concept and we will not study issues like the existence and uniqueness of equilibrium; we will not assume any structure on X other than that it is a linear space. We first repeat the definition of an exchange economy given in Section 2.2:

Definition. An **exchange economy** is an array

$$\mathcal{E} = (X, \{(X^i, \succsim_i, e^i) : i \in \mathcal{I} = 1, \dots, I\}) \quad (2.73)$$

consisting of a linear space X and a finite set of agents indexed by $i \in \mathcal{I} = 1, \dots, I$ each characterized by the triple $(X^i, \succsim_i, e^i)$ where $X^i \subseteq X$ is the consumption set for agent i , $\succsim_i \in X^i \times X^i$ is agent i 's preference relation and $e^i \in X^i \subseteq X$ is agent i 's initial endowment.

Definition. An **allocation** for the economy $\mathcal{E}$ is an I -tuple $x = (x^1, \dots, x^I)$ where $x^i \in X^i$ for each $i \in \mathcal{I} = 1, \dots, I$.

Definition. An allocation $x = (x^1, \dots, x^I) \in X^1 \times \dots \times X^I$ for the economy $\mathcal{E}$ is **attainable** if

$$\sum_{i=1}^I (x^i - e^i) = 0 \quad (2.74)$$

Definition. The **attainable set** for the economy $\mathcal{E}$ consisting of all attainable allocations for $\mathcal{E}$ is given by;

$$A(\mathcal{E}) = \{x = (x^1, \dots, x^I) \in X^1 \times \dots \times X^I : \sum_{i=1}^I (x^i - e^i) = 0\} \quad (2.75)$$

Definition. Agent i 's **budget set** at price $p \in X'$ is given by

$$B^i(p) = \{x \in X^i : p \cdot x - p \cdot e^i \leq 0\} \quad (2.76)$$

Definition. An allocation $x = (x^1, \dots, x^I) \in X^1 \times \dots \times X^I$ for the economy $\mathcal{E}$ is **budget constrained** by a price $p \in X'$ if

$$x^i \in B^i(p) \quad \forall i \in \mathcal{I} \quad (2.77)$$

Definition. An element of the budget set $x \in B^i(p)$ is **$\succsim_i$ -maximal** in $B^i(p)$ if the set of commodity bundles that are strictly preferred to x and $B^i(p)$ have empty intersection: $\{y \in X^i : y \succ_i x\} \cap B^i(p) = \emptyset$.

In other words; an element $x \in B^i(p)$ is $\succsim_i$ -maximal in $B^i(p)$ if

$$y \succ_i x \text{ implies } p \cdot y - p \cdot e^i > 0 \quad (2.78)$$

As the preference relation $\succsim_i$ is complete, this is equivalent to the statement:

$$x \in B^i(p) \quad \text{and} \quad x \succsim_i y \quad \forall y \in B^i(p) \quad (2.79)$$

Definition. An allocation $x = (x^1, \dots, x^I)$ in the economy $\mathcal{E}$ is **strictly supported** by a non-zero price $\bar{p} \in X' \setminus \{0\}$ if

$$y \succ_i x \text{ implies } p \cdot x - p \cdot e^i > 0 \quad \forall y \in X^i, \quad \forall i \in \mathcal{I} \quad (2.80)$$

Definition. Agent i 's **demand set** at price $p \in X'$, denoted $\varphi^i(\succsim_i, p)$ is the set of all $\succsim_i$ -maximal elements in agent i 's budget set $B^i(p)$, thus;

$$\varphi^i(\succsim_i, p) = \{x \in B^i(p) : x \succsim_i y \text{ for all } y \in B^i(p)\} \quad (2.81)$$

Definition. A **Walrasian equilibrium** (or **competitive equilibrium**) for the economy $\mathcal{E}$ is an $(I+1)$ -tuple $(\bar{p}, \bar{x}^1, \dots, \bar{x}^I)$ consisting of a non-zero price $\bar{p} \in X' \setminus \{0\}$ called the **equilibrium price system** and an *attainable allocation* $(\bar{x}^1, \dots, \bar{x}^I) \in X^1 \times \dots \times X^I$ called the **Walrasian equilibrium allocation** (or **competitive allocation**) such that

$$x^i \in \varphi^i(\succsim_i, \bar{p}) \quad \forall i \in \mathcal{I} = 1, \dots, I \quad (2.82)$$

For purpose of comparison with other equilibrium models that will be given in the sequel, we will make the following alternative definition of Walrasian equilibrium:

Definition. A Walrasian equilibrium for the economy $\mathcal{E}$ is an $(I+1)$ -tuple $(\bar{p}, \bar{x}^1, \dots, \bar{x}^I) \in X' \times X^1 \times \dots \times X^I$ that satisfies:

$$(1) \quad \bar{x}^i \in B^i(\bar{p}) \quad \forall i \in \mathcal{I} = 1, \dots, I$$

$$(2) \quad \bar{x}^i \succsim_i y \quad \forall y \in B^i(\bar{p}), \quad \forall i \in \mathcal{I} = 1, \dots, I$$

$$(3) \quad \sum_{i=1}^I (\bar{x}^i - e^i) = 0$$

According to these definitions, a Walrasian equilibrium is characterized by two properties;

- (i) each agent's consumption bundle is *maximal in his budget set*, and
- (ii) *markets clear*.

A Walrasian equilibrium can be equivalently defined as an *attainable* allocation and a non-zero price vector that *budget constrains* and *strictly supports* this allocation. A market clearing price supporting the Walrasian equilibrium allocation immediately invokes the essential theme of the Walrasian theory of value: *decentralization*.

Definition. An attainable allocation $(x^1, \dots, x^I) \in X^1 \times \dots \times X^I$ for the economy $\mathcal{E}$ is **Pareto optimal** if there is no attainable allocation $(y^1, \dots, y^I) \in X^1 \times \dots \times X^I$ such that;

$$y^i \succsim_i x^i \text{ for all } i \in \mathcal{I} = 1, \dots, I \text{ and } y^i \succ_i x^i \text{ for some } i \in \mathcal{I} = 1, \dots, I \quad (2.83)$$

The relationship between equilibria and optima or more specifically, the equivalence between Pareto optimal allocations and Walrasian equilibrium allocations are revealed in the first and second welfare theorems. The first welfare theorem simply says that if the markets are complete, any Walrasian allocation is Pareto optimal. This means that agents acting in their self-interest arrive at an allocation of resources that cannot be improved upon. Second welfare theorem says that if the markets are complete and agents have convex preferences, any Pareto optimal allocation can be implemented as a Walrasian equilibrium if appropriate lump-sum transfers are made.

Thus the second welfare theorem states that under certain restrictions every Pareto optimal allocation is a competitive allocation subject to transfers.

2.3.2. Contingent Commodity Markets Economy

In this section we formulate a two-period exchange economy under uncertainty. An important part of the specification of the economies under uncertainty is the model of information flow. Specifically, the model of information flow underlying the economy should reflect the fact that agents in this economy have exactly the full knowledge of the past and present prices but nothing more. In this section we make a complete specification of information flow as viewed from a perspective of modern multi-period models. One reason for keeping framework general is that most of the models in finance are multi-period in nature. Although we consider a two period economy for simplicity, most of the results can be extended to multi-periods without much difficulty.

Taken as primitive is the probability space $(\Omega, \mathcal{F}, P)$ where Ω is the sample space, $\mathcal{F}$ is a σ -field of subsets of Ω and P is a **probability measure**. Each generic element ω of Ω which is called a **state**. Each state ω completely specifies the exogenous environment. Subsets of Ω are called **events** and these are the sets that can be assigned probabilities. P is the common probability assessments held by the agents in our economy. For any event $F \in \mathcal{F}$, $P(F)$ indicates the probability of occurrence of this event.

The sample space may have finite or infinite cardinality. If Ω is finite, there are $S \geq 1$ conceivable states of nature indexed by the set $s \in \mathcal{S} = \{1, \dots, S\}$ and the sample space is given by $\omega_s \in \Omega = \{\omega_1, \omega_2, \dots, \omega_S\}$. The information revelation through time is described by a filtration $\mathbb{F}$. This means that the sample space $(\Omega, \mathcal{F})$ is equipped with a filtration $\mathbb{F} = \{\mathcal{F}_0, \mathcal{F}_1, \dots, \mathcal{F}_T\}$: an increasing family of sub- σ -fields of $\mathcal{F}$ with $\mathcal{F}_0 \subseteq \dots \subseteq \mathcal{F}_T$. An element of $\mathcal{F}_t$ for $t \in \mathcal{T} = \{1, \dots, T\}$ is called a date-event. According to this, a date-event is an event whose occurrence or the non-occurrence is conveyed by the σ -field at that date. $\mathbb{F}$ is increasing means that agents do not forget the occurrence of an event once it is revealed. $\mathcal{F}_t$ contains complete information about the history up to the time t , but no information at all

after t . Furthermore, without loss of generality it is assumed that $\mathcal{F}_0 = \{\emptyset, \Omega\}$, the **trivial σ -field** and $\mathcal{F}_T = \mathcal{F} = \mathcal{P}(\Omega)$, the **total σ -field**. This means that at time 0, no information is known and all uncertainty is resolved at time T . Collectively, $\mathcal{B} = (\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t \geq 0}, P)$ is called a **filtered probability space** or a **stochastic basis**. Thus the filtered probability space $\mathcal{B} = (\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t \geq 0}, P)$ provides us a complete description of the exogenously given information structure. All agents in the economy are assumed to have the same information, thus there is no private information.

In our economy, there are two periods indexed by $t \in \mathcal{T} = \{0, 1\}$. Date 0 is called **ex ante** and date 1 is called **ex post**. The filtration then reduces to $\mathbb{F} = \{\mathcal{F}_0, \mathcal{F}_1\}$ where $\mathcal{F}_0 = \{\emptyset, \Omega\}$ and $\mathcal{F}_1 = \mathcal{F} = \mathcal{P}(\Omega)$, the *total σ -field*. The situation is depicted in Figure 2.3.

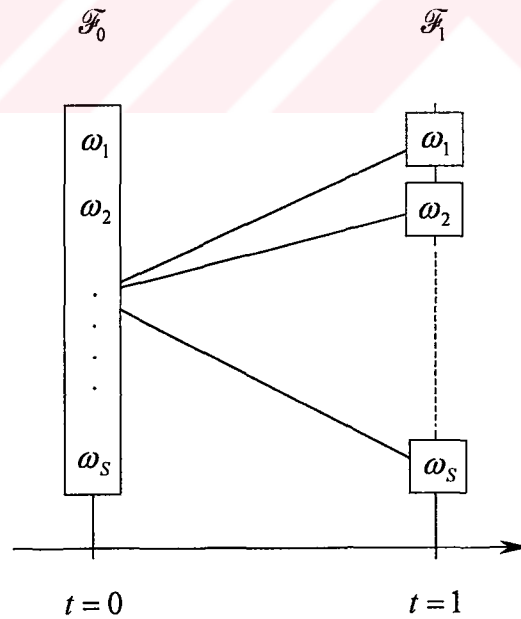


FIGURE 2.3. INFORMATION FLOW

We posit a standard Arrow-Debreu economy. Accordingly, markets for exchanging commodities are opened at date 0 only.

Definition. A **state-contingent commodity** (or an **elementary claim**) for physical good l and state s is a contract to receive *one unit* of the physical good l if state s occurs and nothing otherwise.

The price of a state-contingent commodity for physical good l and state s denoted ψ_{sl} is the number of units of account that have to be paid in order to have one unit of physical good l delivered if state s occurs.

Notation. As we will see in the sequel; given a linear map $\psi : \mathbb{R}^n \rightarrow \mathbb{R}^m$ there exists a unique $m \times n$ matrix A such that $\psi(x) = Ax$ for all $x \in \mathbb{R}^n$. Conversely, any $m \times n$ matrix A corresponds to a unique linear mapping $\psi : \mathbb{R}^n \rightarrow \mathbb{R}^m$. We will use the same symbol for both the linear operator and its associated matrix. Therefore ψ denote the linear operator $\psi(\cdot)$ or the matrix A ; the exact meaning should be obvious from the context. Further, when ψ denotes the matrix; matrix entries will be represented by the same symbol ψ_{ij} . This practice is believed to simplify the notation.

Definition. A **state-contingent commodity vector** is a contract to receive the commodity vector $C_s = (C_s^1, \dots, C_s^L) \in \mathbb{R}^L$ if state s occurs. Thus a state-contingent commodity vector is a bundle of state-contingent commodities. Viewing the date 0 as an extra state, the collection of all state-contingent commodity vectors can be represented by a $(S+1) \times L$ matrix:

$$C = \begin{bmatrix} C_0^1 & \dots & C_0^L \\ C_1^1 & \dots & C_1^L \\ \vdots & \ddots & \vdots \\ C_S^1 & \dots & C_S^L \end{bmatrix} = \begin{bmatrix} C_0 \\ C_1 \\ \vdots \\ C_S \end{bmatrix} = [C^1 \quad \dots \quad C^L] \quad (2.84)$$

There are a finite number of physical goods indexed by $l=1, \dots, L$. The commodity space is given by $X = \mathbb{R}_+^L \times (\mathbb{R}_+^L)^S$ with a typical element as a consumption bundle $x = (x_0, x_1) = (x_0, x_1, \dots, x_S) \in X = \mathbb{R}_+^L \times (\mathbb{R}_+^L)^S$. The bundle $x_1 = (x_1, \dots, x_S) \in (\mathbb{R}_+^L)^S$

is called a **state-contingent consumption plan**. It represents the commodity bundle $x_s \in \mathbb{R}^L$ if the state is s .

There are a finite number of agents indexed by $i = \mathcal{I} = 1, \dots, I$. Each agent is characterized by a pair $(\succsim_i, e^i)$ where $\succsim_i \in \mathbb{R}_+^{L(S+1)} \times \mathbb{R}_+^{L(S+1)}$ is agent i 's preference relation and $e^i = (e_0^i, e_1^i) = (e_0^i, e_1^i, \dots, e_S^i) \in \mathbb{R}_+^L \times (\mathbb{R}_+^L)^S$ is agent i 's initial endowment.

Definition. A process is said to be **adapted** if its values at time t can only depend on the information at that time.

The state contingent consumption at date 1 is a real valued measurable function or a random variable so it is adapted to the filtration $\mathbb{F} = \{\mathcal{F}_0, \mathcal{F}_1\}$.

At date 0 there are markets for $L(S+1)$ contingent commodities. Thus the payments for contingent commodities are made at date 0 before uncertainty unfolds.

Definition. A **contingent commodity markets** (or an Arrow-Debreu contingent claims market) economy is a collection of primitives:

$$\mathcal{E}^{\text{GE}} = ((\Omega, \mathcal{F}, \mathbb{F}, P), \{(X^i = \mathbb{R}_+^{L(S+1)}, \succsim_i, e^i) : i = 1, \dots, I\}, C) \quad (2.85)$$

Sometimes, we will shortly refer to this as:

$$\mathcal{E}^{\text{GE}} = ((\succsim_i, e^i)_{i=1}^I, C) \quad (2.86)$$

Definition. Agent i 's **budget set** at price $\psi \in \mathbb{R}^{L(S+1)}$ is given by:

$$B^i(\psi) = \{x \in X^i : \psi(x - e^i) \leq 0\} \quad (2.87)$$

Application of the Walrasian concept of equilibrium to the contingent-commodity market structure results in the Arrow-Debreu contingent claims market equilibrium.

Definition. A **contingent commodity market equilibrium** (or **Arrow-Debreu contingent claims market equilibrium**) for the economy $\mathcal{E}^{\text{GE}} = ((\succsim_i, e^i)_{i=1}^I, C)$ is a

collection $(\bar{\psi}, \bar{x}) = (\bar{\psi}, (\bar{x}^i)_{i=1}^I)$ consisting of a linear price functional $\psi : X \rightarrow \mathbb{R}$ and an allocation $(\bar{x}^1, \dots, \bar{x}^I) \in X^1 \times \dots \times X^I$ such that

(1) $\bar{x}^i$ is *budget feasible* for each $i = 1, \dots, I$, that is;

$$\bar{x}^i \in B^i(\bar{\psi}) = \{x \in X^i : \bar{\psi}(x - e^i) \leq 0\}, \quad i = 1, \dots, I$$

(2) $\bar{x}^i$ is $\succeq_i$ -*maximal* in $B^i(\bar{\psi})$ for each $i = 1, \dots, I$, that is;

$$x^i \in B^i(\bar{\psi}) \text{ implies } \bar{x}^i \succeq_i x^i, \quad i = 1, \dots, I$$

(3) Markets clear:

$$\sum_{i=1}^I (\bar{x}^i - e^i) = 0$$

Sometimes we will simply refer to a contingent-commodity markets economy and its associated equilibrium concept contingent-commodity market equilibrium as **Arrow-Debreu economy** and **Arrow-Debreu equilibrium** respectively.

A special case of the Arrow-Debreu economy is the one with incomplete markets; that is $M \neq X$. Recall the relation from Section 2.2:

$$M_0 \subseteq (\text{span}(M_0) = M) \subseteq X \tag{2.88}$$

Notice that, adapted to the terminology of this section, the marketed subspace M is the subspace spanned by the columns of the matrix C . Then the Arrow-Debreu equilibrium concept is adapted to incorporate incomplete markets as given below:

Definition. An **incomplete contingent-commodity market equilibrium** (or **Arrow-Debreu contingent claims market equilibrium**) for the economy $\mathcal{E}^{\text{GE}}$ is a

collection $(\bar{\pi}, \bar{x}) = (\bar{\pi}, (\bar{x}^i)_{i=1}^I)$ consisting of a linear price functional $\pi : M \rightarrow \mathbb{R}$ and an allocation $(\bar{x}^1, \dots, \bar{x}^I) \in X^1 \times \dots \times X^I$ such that

(1) $\bar{x}^i$ is *budget feasible* for each $i = 1, \dots, I$, that is;

$$\bar{x}^i \in B^i(\bar{\pi}) = \{x \in X^i \cap M : \bar{\pi}(x - e^i) \leq 0\}, \quad i = 1, \dots, I$$

(2) $\bar{x}^i$ is $\succeq_i$ -*maximal* in $B^i(\bar{\pi})$ for each $i = 1, \dots, I$, that is;

$$x^i \in B^i(\bar{\pi}) \text{ implies } \bar{x}^i \succeq_i x^i, \quad i = 1, \dots, I$$

(3) Markets clear:

$$\sum_{i=1}^I (\bar{x}^i - e^i) = 0$$

2.3.3. Security-Spot Markets Economy

Again we take up a similar two-period exchange economy under uncertainty. Uncertainty is again resolved according to the information flow modeled in the previous section. However, the contingent commodity market structure is enriched by incorporating financial markets into the analysis to obtain a different market structure: security-spot markets. Securities or assets are traded in financial markets. Securities are claims to commodity bundles in the second period that are contingent on the state of nature. In order to analyze this market structure we first precisely define the asset structure.

Definition. A **real security** (or a **real asset**) A^k is a contract that promises to pay the vector of goods

$$A_s^k = \begin{bmatrix} a_{1s} \\ \vdots \\ a_{Ls} \end{bmatrix} \in \mathbb{R}^L \quad (2.89)$$

in each state $s = 1, \dots, S$ at date 1.

Thus a real security A^k can be represented by the following vector:

$$A^k = \begin{bmatrix} A_1^k \\ \vdots \\ A_S^k \end{bmatrix} = \begin{bmatrix} a_{11} \\ \vdots \\ a_{L1} \\ \vdots \\ a_{1S} \\ \vdots \\ a_{LS} \end{bmatrix} \in \mathbb{R}^{LS} \quad (2.90)$$

There are a finite number of real securities indexed by $k \in \{1, \dots, K\}$. The K securities can be collected in a $LS \times K$ matrix A called the **asset structure**:

$$A = \begin{bmatrix} A^1 & \dots & A^K \end{bmatrix} \quad (2.91)$$

Security prices, measured in units of account at date 0, are given by a vector $q = (q_1, \dots, q_K) \in \mathbb{R}^K$.

For $p_1 = (p_1, \dots, p_S) \in (\mathbb{R}^L)^S$ and $x_1 = (x_1, \dots, x_S) \in (\mathbb{R}^L)^S$ define $p_1 \square x_1$ as:

$$p_1 \square x_1 = \begin{bmatrix} p_1 \cdot x_1 \\ \vdots \\ p_S \cdot x_S \end{bmatrix} \in \mathbb{R}^S \quad (2.92)$$

Further let $p_1 \square A$ denote the $S \times K$ matrix whose j th column is given by $p_1 \square A^k \in \mathbb{R}^S$. Then we define the matrix $S \times K$ matrix called the **market structure** (or **securities-states tableau**) as:

$$V(p_1, A) = \begin{bmatrix} V_1^1 & \cdots & V_1^K \\ \vdots & \ddots & \vdots \\ V_S^1 & \cdots & V_S^K \end{bmatrix} = p_1 \square A = \begin{bmatrix} p_1 A_1^1 & \cdots & p_1 A_1^K \\ \vdots & \ddots & \vdots \\ p_S A_S^1 & \cdots & p_S A_S^K \end{bmatrix} \quad (2.93)$$

Notice that a typical entry V_s^k of this matrix denotes the payoff of the security k in state s . The market structure can also be written as:

$$V(p_1, A) = \begin{bmatrix} V_1^1 & \cdots & V_1^K \\ \vdots & \ddots & \vdots \\ V_S^1 & \cdots & V_S^K \end{bmatrix} = \begin{bmatrix} V^1 & \cdots & V^K \end{bmatrix} = \begin{bmatrix} V_1 \\ \vdots \\ V_S \end{bmatrix} \quad (2.94)$$

Then each column V^k , $k = 1, 2, \dots, K$ is a random variable denoting the payoffs on security k in distinct states.

There are a finite number of physical goods indexed by $l = 1, \dots, L$. The commodity space is given by $X = \mathbb{R}_+^L \times (\mathbb{R}_+^L)^S$ with a typical element as a consumption bundle $x = (x_0, x_1) = (x_0, x_1, \dots, x_S) \in X = \mathbb{R}_+^L \times (\mathbb{R}_+^L)^S$. The bundle $x_1 = (x_1, \dots, x_S) \in (\mathbb{R}_+^L)^S$ represents the commodity bundle $x_s \in \mathbb{R}_+^L$ if the state is s . There are a finite number of agents indexed by $i = \mathcal{I} = 1, \dots, I$. Each agent is characterized by a pair $(\succsim_i, e^i)$ where $\succsim_i \in \mathbb{R}_+^{L(S+1)} \times \mathbb{R}_+^{L(S+1)}$ is agent i 's preference relation and $e^i = (e_0^i, e_1^i) = (e_0^i, e_1^i, \dots, e_S^i) \in \mathbb{R}_+^L \times (\mathbb{R}_+^L)^S$ is agent i 's initial endowment.

At date 0 ; there are

(i) spot markets for L commodities and

(ii) future markets for K securities

and at date 1, in each state of nature; there are spot markets for L commodities. The spot prices are given by $p = (p_0, p_1) = (p_0, p_1, \dots, p_S) \in \mathbb{R}_{++}^L \times (\mathbb{R}_{++}^L)^S$.

Definition. An admissible portfolio is a vector $\theta \in \mathbb{R}^K$.

In security-spot markets, wealth is transferred between date-event pairs using securities and then allocated between goods in the spot markets. Buying a security k ($\theta_k > 0$) in this context means a transfer of wealth from the present to some future state; short selling a security ($\theta_k < 0$) means a transfer of wealth from a date 1 state to date 0.

Definition. A two period security-spot market economy is a collection of primitives:

$$\mathcal{E}^{\text{GEI}} = ((\Omega, \mathcal{F}, \mathbb{F}, P), \{(X^i = \mathbb{R}_+^{L(S+1)}, \bar{z}_i, e^i) : i = 1, \dots, I\}, A) \quad (2.95)$$

Sometimes, we will shortly write this as:

$$\mathcal{E}^{\text{GEI}} = ((\bar{z}_i, e^i)_{i=1}^I, A) \quad (2.96)$$

We will sometimes call the security-spot market economy simply a **Radner economy**.

Definition. A **plan** for agent i is a pair $(x^i, \theta^i) \in X^i \times \mathbb{R}^K$ where $x^i \in X^i = \mathbb{R}_+^{L(S+1)}$ is a consumption choice and $\theta \in \mathbb{R}^K$ is an admissible portfolio choice.

Introduction of spot markets leads to a system of $S+1$ budget constraints, one for each date-event. First consider the following budget set for agent i :

$$B'_\theta(p, q) = \{x \in X^i : p_0 \cdot (x_0 - e_0^i) + q \cdot \theta \leq 0$$

$$\text{and } \forall s = 1, \dots, S, p_s \cdot (x_s - e_s^i) - V_s(p_1, A) \cdot \theta \leq 0\} \quad (2.97)$$

Thus in a Radner economy each consumer faces a sequence of budget constraints instead of single budget constraint of an Arrow-Debreu economy. A separate budget constraint corresponds to each market that opens. Notice that this budget set gives the budget feasible consumption for agent i at a portfolio choice θ . Now consider the alternative budget set of plans for agent i :

$$B'_{(x, \theta)}(p, q) = \{(x, \theta) \in X^i \times \mathbb{R}^K : p_0 \cdot (x_0 - e_0^i) + q \cdot \theta \leq 0$$

$$\text{and } \forall s = 1, \dots, S, p_s \cdot (x_s - e_s^i) - V_s(p_1, A) \cdot \theta \leq 0\} \quad (2.98)$$

These two budget sets help in arriving the final form of the budget set for agent i :

Definition. Agent i 's **budget set** at prices (p, q) is given by:

$$B^i(p, q) = \{x \in X^i : (x, \theta) \in B'_{(x, \theta)}(p, q) \text{ for some } \theta \in \mathbb{R}^K\} = \bigcup_{\theta \in \mathbb{R}^K} B'_{\theta}(p, q) \quad (2.99)$$

Definition. A plan (x^i, θ^i) is **budget feasible** for agent i at prices (p, q) whenever $(x^i, \theta^i) \in B^i(p, q)$.

Definition. Given the pair (p, q) , a budget feasible plan (x^i, θ^i) for agent i is **optimal** if there is no budget feasible plan $(\hat{x}^i, \hat{\theta}^i)$ for agent i such that $\hat{x}^i \succ x^i$. As the preference relation is complete, this is equivalent to the following; given the pair (p, q) , a plan $(x^i, \theta^i) \in B^i(p, q)$ is optimal if $(\hat{x}^i, \hat{\theta}^i) \in B^i(p, q)$ implies $x^i \succeq_i \hat{x}^i$.

Application of the Walrasian concept of equilibrium to the second market structure; security-spot markets results in the Radner equilibrium concept.

Definition. A **Radner equilibrium** (or **equilibrium of plans, prices and price expectations**) for the economy $\mathcal{E}^{\text{GEI}} = ((\succeq_i, e^i)_{i=1}^I, A)$ is a collection $(\bar{p}, \bar{q}, (\bar{x}^i, \bar{\theta}^i)_{i=1}^I)$ such that;

(1) $(\bar{x}^i, \bar{\theta}^i)$ is *budget feasible* at prices $(\bar{p}, \bar{q})$ for each $i = 1, \dots, I$;

$$(\bar{x}^i, \bar{\theta}^i) \in B^i(\bar{p}, \bar{q}), \quad i = 1, \dots, I$$

(2) $(\bar{x}^i, \bar{\theta}^i)$ is *optimal* at prices $(\bar{p}, \bar{q})$ for each $i = 1, \dots, I$:

$$(\bar{x}^i, \bar{\theta}^i) \in B^i(\bar{p}, \bar{q}) \text{ implies } \bar{x}^i \succeq_i x^i, \quad i = 1, \dots, I$$

(3) Spot markets clear:

$$\sum_{i=1}^I (\bar{x}^i - e^i) = 0$$

(4) Security markets clear:

$$\sum_{i=1}^I \bar{\theta}^i = 0$$

For the purpose of comparison with the Walrasian equilibrium; we can make the following alternative definition of the Radner equilibrium via the different type of budget sets defined above.

Definition. A Radner equilibrium for the economy $\mathcal{E}^{\text{GEI}} = ((\succsim_i, e^i)_{i=1}^I, A)$ is a collection $(\bar{p}, \bar{q}, (\bar{x}^i, \bar{\theta}^i)_{i=1}^I)$ such that;

- (1) $\bar{x}^i \in B_{\bar{\theta}^i}^i(\bar{p}, \bar{q}), \quad i = 1, \dots, I$
- (2) $x^i \in B^i(\bar{p}, \bar{q})$ implies $\bar{x}^i \succsim_i x^i, \quad i = 1, \dots, I$
- (3) $\sum_{i=1}^I (\bar{x}^i - e^i) = 0$
- (4) $\sum_{i=1}^I \bar{\theta}^i = 0$

2.4. Arbitrage in a Two Period Security-Spot Markets Economy

The purpose of this chapter is to translate the implications of the abstract analysis of arbitrage given in Section 2.2 to a two-period security-spot market economy. The consequences of the no-arbitrage condition are first analyzed in a finite state-space

setting. Recall that when all possible configurations for the economy at period 1 can be described in terms of a finite number of conceivable states of nature indexed by the set $s \in \mathcal{S} = \{1, \dots, S\}$; and there are L spot commodities, the commodity space is given by the finite dimensional Euclidean space $X = \mathbb{R}^L \times (\mathbb{R}^L)^S$. There are basically two alternative approaches to analyze arbitrage in a two period exchange economy under uncertainty. The first approach is to adapt the abstract definition of arbitrage and the consequences of the no-arbitrage condition to the specific case of the finite dimensional commodity space. The second approach starts with a definition of arbitrage-free security prices as in a security-spot markets economy, net trades are attained indirectly through an investment in securities. The results are extended to an infinite dimensional setting in Section 2.4.2. The analysis of the no arbitrage condition and its consequences in a competitive market prepares a basis for the analysis in the subsequent section.

2.4.1. Arbitrage in a Finite State-Space Setting

Now we are ready to state how the abstract analysis of arbitrage given in Section 2.2 translates to our two period exchange economy under uncertainty. We start by a review of the algebraic, topological and order structure of the finite dimensional Euclidean space $\mathbb{R}^N$ under the format of the Section 2.1. First notice that the finite dimensional Euclidean space $\mathbb{R}^N$ is a linear space. The topological structure of $X = \mathbb{R}^n$ is also easily specified by the following definitions:

Theorem. $\mathbb{R}^n$ is a normed linear space under the **Euclidean norm**:

$$\|x\| = \left[\sum_{i=1}^n x_i^2 \right]^{\frac{1}{2}} \quad (2.100)$$

Theorem. $\mathbb{R}^n$ is a metric space under the metric induced by this norm or the **Euclidean metric** given by:

$$d(x, y) = \left[\sum_{i=1}^n (x_i - y_i)^2 \right]^{\frac{1}{2}} \quad (2.101)$$

This metric induces the **Euclidean topology** on $\mathbb{R}^n$. This topology coincides with the product topology derived from the standard topology on $\mathbb{R}$. It is a fact that a finite dimensional vector space admits only one Hausdorff linear topology. [See for example Aliprantis (1999)]. Consequently, the Euclidean topology is the unique Hausdorff linear topology on $\mathbb{R}^n$. It can also be verified that the Euclidean topology is a locally convex topology.

The order structure of the commodity space $X = \mathbb{R}^n$ is specified by declaring the usual positive cone or the positive orthant of the Euclidean space as the positive cone of X thus:

$$X_+ = \mathbb{R}_+^n = \{x = (x_1, \dots, x_n) \in \mathbb{R}^n : x_i \geq 0 \quad \forall i \in \{1, \dots, n\}\} \quad (2.102)$$

The partial ordering induced by this cone is the natural partial ordering on $\mathbb{R}^n$ given as;

$$\text{for } x, y \in \mathbb{R}^n \quad x = (x_1, \dots, x_n) \geq y = (y_1, \dots, y_n) \quad \text{iff} \quad x_i \geq y_i \quad \forall i \in \{1, \dots, n\}$$

It is easy to see that for $n > 1$, the binary relation $\geq$ on $\mathbb{R}^n$ is reflexive, transitive and anti-symmetric, thus $\geq$ partially orders $\mathbb{R}^n$. Moreover, $\geq$ satisfies the ordered linear space axioms so $(\mathbb{R}^n, \leq)$ is an ordered linear space.

For convenience the relations $>$ and $\gg$ on $\mathbb{R}^n$ are defined as; for $x, y \in \mathbb{R}^n$

$$x = (x_1, \dots, x_n) > y = (y_1, \dots, y_n) \quad \text{iff} \quad x_i \geq y_i \quad \forall i \in \{1, \dots, n\} \quad \text{and} \quad x \neq y \quad (2.103)$$

$$x = (x_1, \dots, x_n) \gg y = (y_1, \dots, y_n) \quad \text{iff} \quad x_i > y_i \quad \forall i \in \{1, \dots, n\} \quad (2.104)$$

In the special case that $y = 0$, we have:

$$x \geq 0 \quad \text{iff} \quad x \in \mathbb{R}_+^n \quad (2.105)$$

$$x > 0 \quad \text{iff} \quad x \in \mathbb{R}_+^n \setminus \{0\} \quad (2.106)$$

$$x \gg 0 \quad \text{iff} \quad x \in \text{int}(\mathbb{R}_+^n) \quad (2.107)$$

In order to define the strictly increasing preferences we take $K = \mathbb{R}_+^n \setminus \{0\}$ as the cone with the origin deleted. Under these facts it is seen that assumptions (1) to (4) of Section 2.2 are satisfied for the finite dimensional Euclidean space.

Definition. Given a linear space X , an **inner product** is a function $(\cdot | \cdot) : X \times X \rightarrow \mathbb{R}$ satisfying the following properties:

$$(1) \quad (x | y) = (y | x) \quad \forall x, y \in X$$

$$(2) \quad (x | x) = 0 \text{ iff } x = 0 \quad \forall x \in X$$

$$(3) \quad (x | x) \geq 0 \quad \forall x \in X$$

$$(4) \quad (\alpha x + \beta y | z) = \alpha(x | z) + \beta(y | z) \quad \forall \alpha, \beta \in \mathbb{R}, \quad \forall x, y, z \in X$$

Definition. A linear space with an inner product is called an **inner product space** (or **pre-Hilbert space**).

Theorem. On an inner product space X the function

$$\|x\| = \sqrt{(x | x)}, \quad x \in X \quad (2.108)$$

defines a *norm*. Thus the inner product space X with the norm $\|x\| = \sqrt{(x | x)}$ is a *normed linear space*.

Definition. A **Hilbert space** is an inner product space which is complete with respect to the norm $\|x\| = \sqrt{(x | x)}$.

Remark. The n -dimensional Euclidean space $\mathbb{R}^n$, with the inner product defined as the standard dot product;

$$(x | y) = \sum_{i=1}^n x_i y_i, \quad x, y \in \mathbb{R}^n \quad (2.109)$$

is an inner product space. Furthermore, it is a Hilbert space. The norm defined with this inner product as

$$\|x\| = \sqrt{(x | x)} = \left(\sum_{i=1}^n |x_i|^2 \right)^{1/2}, \quad x \in \mathbb{R}^n \quad (2.110)$$

is called the **Euclidean norm** for $\mathbb{R}^n$.

Theorem (Riesz Representation Theorem). If ϕ is a continuous linear functional on a Hilbert space X , there is a unique element $y \in X$ such that

$$\phi(x) = (x | y) \quad \forall x \in X \quad (2.111)$$

For the n -dimensional Euclidean space $\mathbb{R}^n$, the Riesz Representation Theorem can be written as follows.

Theorem (Riesz Representation Theorem for $\mathbb{R}^n$). If ϕ is a continuous linear functional on $\mathbb{R}^n$, there is a unique element $y \in \mathbb{R}^n$ such that

$$\phi(x) = \sum_{i=1}^n x_i y_i \quad \forall x \in \mathbb{R}^n \quad (2.112)$$

Now we state the following well-known results from linear algebra without proof.

Theorem. Given a linear map $T_A : \mathbb{R}^n \rightarrow \mathbb{R}^m$ there exists a unique $m \times n$ matrix A such that $T_A(x) = Ax$ for all $x \in \mathbb{R}^n$.

Theorem. Any $m \times n$ matrix A corresponds to a unique linear mapping $T_A : \mathbb{R}^n \rightarrow \mathbb{R}^m$.

The direct consequence of these theorems is that there is one-to-one correspondence between linear functions from $\mathbb{R}^n$ to $\mathbb{R}^m$ and $m \times n$ matrices.

Kreps' viability criterion applies to market structures where *marketed* bundles of goods $m \in M$ are bought or sold in a common market at prices $\pi(m)$. Accordingly, it readily applies to a contingent commodity markets economy in which all trades take place at date 0 and there is a single budget constraint. Recall that for finite dimensional commodity spaces, viability is a necessary and sufficient condition for arbitrage, hence we make no distinction between these two. In order to translate the no arbitrage condition to a security-spot market economy, a different characterization of Radner equilibrium is needed. We first convert all economies to net trade economies for ease of notation. In a net trade exchange economy the consumption vectors are net trade vectors or additions to endowments and agents' preferences are defined over these net trades. Consider the two different types of market structures considered in Section 3.1. namely, an Arrow-Debreu economy

$$\hat{\mathcal{E}}^{\text{GE}} = ((X^i = \mathbb{R}_+^{L(S+1)}, \succsim_i, e^i)_{i=1}^I, C) \quad (2.113)$$

and a Radner economy

$$\hat{\mathcal{E}}^{\text{GEI}} = ((X^i = \mathbb{R}_+^{L(S+1)}, \succsim_i, e^i)_{i=1}^I, A) \quad (2.114)$$

Without loss of generality, these economies can be converted into net trade economies $\mathcal{E}^{\text{GE}} = ((\succsim_i)_{i=1}^I, C)$ and $\mathcal{E}^{\text{GEI}} = ((\succsim_i)_{i=1}^I, A)$ respectively by translating the preference relation of each agent as:

$$x \succsim_i y \text{ iff } x + e^i \hat{\succsim}_i y + e^i \quad \forall x, y \in X^i \quad \forall i \in \mathcal{I} \quad (2.115)$$

Now we rewrite the different types equilibrium concepts defined in Section 2.3 for these net trade economies:

First an *Arrow-Debreu complete market equilibrium* for the economy $\mathcal{E}^{\text{GE}} = ((\succsim_i)_{i=1}^I, C)$ is a collection $(\bar{\psi}, (\bar{x}^i)_{i=1}^I)$ where $\bar{\psi} : X \rightarrow \mathbb{R}$ is a linear functional and $(\bar{x}^i)_{i=1}^I$ is an allocation such that

$$(1) \quad \bar{x}^i \in B^i(\bar{\psi}) = \{x \in X^i : \bar{\psi}(x) \leq 0\}, \quad i=1, \dots, I$$

$$(2) \quad \bar{x}^i \text{ is } \succeq_i\text{-maximal in } B^i(\bar{\psi}) = \{x \in X^i : \bar{\psi}(x) \leq 0\} \text{ for each } i=1, \dots, I$$

$$(3) \quad \sum_{i=1}^I \bar{x}^i = 0$$

An *Arrow-Debreu equilibrium* for $\mathcal{E}^{\text{GE}} = ((\succeq_i)_{i=1}^I, C)$ is a collection $(\bar{\pi}, (\bar{x}^i)_{i=1}^I)$

where $\bar{\pi} : M \rightarrow \mathbb{R}$ is a linear functional and $(\bar{x}^i)_{i=1}^I$ is an allocation such that

$$(1) \quad \bar{x}^i \in B^i(\bar{\pi}) = \{x \in X^i \cap M : \bar{\pi}(x) \leq 0\}, \quad i=1, \dots, I$$

$$(2) \quad \bar{x}^i \text{ is } \succeq_i\text{-maximal in } B^i(\bar{\pi}) = \{x \in X^i \cap M : \bar{\pi}(x) \leq 0\} \text{ for each } i=1, \dots, I$$

$$(3) \quad \sum_{i=1}^I \bar{x}^i = 0$$

And finally a *Radner equilibrium* for the economy $\mathcal{E}^{\text{GEI}} = ((\succeq_i)_{i=1}^I, A)$ is a collection

$(\bar{p}, \bar{q}, (\bar{x}^i, \bar{\theta}^i)_{i=1}^I)$ such that;

$$(1) \quad \bar{x}^i \in B_{\bar{\theta}^i}^i(\bar{p}, \bar{q}), \quad i=1, \dots, I$$

$$(2) \quad \bar{x}^i \text{ is } \succeq_i\text{-maximal in } B^i(\bar{p}, \bar{q}) = \bigcup_{\theta \in \mathbb{R}^K} B_{\theta}^i(\bar{p}, \bar{q}) \text{ for each } i=1, \dots, I$$

$$(3) \quad \sum_{i=1}^I \bar{x}^i = 0$$

$$(4) \quad \sum_{i=1}^I \bar{\theta}^i = 0$$

where the Radner budget constraints are given by:

$$B_\theta(p, q) = \{x \in X : p_0 \cdot x_0 + q \cdot \theta \leq 0\}$$

$$\text{and } \forall s = 1, \dots, S, \quad p_s \cdot x_s - V_s(p_1, A) \cdot \theta \leq 0\} \quad (2.116)$$

Recall that the market structure was given as:

$$V(p_1, A) = \begin{bmatrix} V_1^1 & \dots & V_1^K \\ \vdots & \ddots & \vdots \\ V_S^1 & \dots & V_S^K \end{bmatrix} = [V^1 \quad \dots \quad V^K] = \begin{bmatrix} V_1 \\ \vdots \\ V_S \end{bmatrix} \quad (2.117)$$

Thus the payoffs of the available set of securities are given by the set of random variables;

$$\{V^1, \dots, V^K\}, \quad V^k \in \mathbb{R}^S, \quad k = 1, \dots, K \quad (2.118)$$

These securities can be combined to form a portfolio

$$V\theta = \sum_{k=1}^K \theta_k V^k = \theta_1 \begin{bmatrix} V_1^1 \\ \vdots \\ V_S^1 \end{bmatrix} + \dots + \theta_K \begin{bmatrix} V_1^K \\ \vdots \\ V_S^K \end{bmatrix}, \quad \theta \in \mathbb{R}^K \quad (2.119)$$

Definition. $\text{span}[V]$ is defined as the set of all possible linear combinations or portfolios of the securities $\{V^1, \dots, V^K\}$ or the smallest subspace of X that contains $\{V^1, \dots, V^K\}$. Namely;

$$\text{span}[V] = \text{span}(\{V^1, \dots, V^K\}) = \{\mathcal{G} \in \mathbb{R}^S : \mathcal{G} = \sum_{k=1}^K \theta_k V^k, \quad \theta_k \in \mathbb{R}\} \quad (2.120)$$

or

$$\text{span}[V] = \{\mathcal{G} \in \mathbb{R}^S : \mathcal{G} = V\theta, \quad \theta \in \mathbb{R}^K\} \quad (2.121)$$

Define a new $(S+1) \times K$ matrix W by stacking the vector of security prices $q = (q_1, \dots, q_K) \in \mathbb{R}^K$ on the market structure $V(p_1)$;

$$W(q, V(p_1)) = \begin{bmatrix} -q \\ V(p_1) \end{bmatrix} = \begin{bmatrix} -q_1 & \dots & -q_K \\ V_1^1 & \dots & V_1^K \\ \vdots & \ddots & \vdots \\ V_S^1 & \dots & V_S^K \end{bmatrix} = \begin{bmatrix} W^1 & \dots & W^K \end{bmatrix} = \begin{bmatrix} W_1 \\ \vdots \\ W_S \end{bmatrix} \quad (2.122)$$

Definition. $\text{span}[W]$ which is defined as the smallest subspace of X that contains $\{W^1, \dots, W^K\}$ is called **the subspace of income transfers**. More precisely;

$$\text{span}[W] = \text{span}(\{W^1, \dots, W^K\}) = \{\tau \in \mathbb{R}^{S+1} : \tau = W\theta, \theta \in \mathbb{R}^K\} \quad (2.123)$$

Each typical element $\tau = (\tau_0, \tau_1, \dots, \tau_S) \in \text{span}[W]$ is then a **vector of income transfers**. According to the Radner budget constraints, the net trades are obtained indirectly through investment in a portfolio of securities.

For $p = (p_0, p_1, \dots, p_S) \in \mathbb{R}^{L(S+1)}$ and $x = (x_0, x_1, \dots, x_S) \in \mathbb{R}^{L(S+1)}$ define $p \square x$ as:

$$p \square x = \begin{bmatrix} p_0 \cdot x_0 \\ p_1 \cdot x_1 \\ \vdots \\ p_S \cdot x_S \end{bmatrix} \in \mathbb{R}^{S+1} \quad (2.124)$$

and similarly

$$p_1 \square x = \begin{bmatrix} p_1 \cdot x_1 \\ \vdots \\ p_S \cdot x_S \end{bmatrix} \in \mathbb{R}^S \quad (2.125)$$

Then the Radner budget constraints can be written in matrix notation as:

$$p_0 \cdot x_0 = -q \cdot \theta \quad (2.126)$$

$$p_1 \square x_1 = V(p_1, A)\theta \quad (2.127)$$

Or introducing the vector of income transfers:

$$p_0 \cdot x_0 = \tau_0 \quad (2.128)$$

$$p_1 \square x_1 = \tau_1 \quad (2.129)$$

These can even be written more compactly as;

$$p \square x = W(q, V)\theta \quad (2.130)$$

or equivalently as;

$$p \square x = \tau \quad (2.131)$$

For the purpose of comparing an Arrow-Debreu equilibrium allocation with a Radner equilibrium allocation we just briefly explain the concept of an effective equilibrium as given in Duffie (1992) for example. Consider the following alternative budget set:

$$\hat{B}^i(\varphi) = \{x \in X^i : \varphi(x) \leq 0 \text{ and } \varphi_1 \square x_1 \in \text{span}[V(\varphi_1, A)]\} \quad (2.132)$$

Definition. An **effective equilibrium** for the (net trade) economy $\mathcal{E}^{\text{GEI}} = ((\succsim_i)_{i=1}^I, A)$ is a collection $(\bar{\varphi}, (\bar{x}^i)_{i=1}^I)$ such that;

$$(1) \quad \bar{x}^i \in \hat{B}^i(\bar{\varphi}), \quad i = 1, \dots, I$$

$$(2) \quad \bar{x}^i \text{ is } \succsim_i\text{-maximal in } \hat{B}^i(\bar{\varphi}) \text{ for each } i = 1, \dots, I$$

$$(3) \quad \sum_{i=1}^I \bar{x}^i = 0$$

Theorem [Duffie, 1995]: For the economy $\mathcal{E}^{\text{GEI}}$, $(x^1, \dots, x^I) \in X_+^I$ is an *effective allocation* iff it is a *Radner equilibrium allocation*.

This is a key result that simply establishes the equivalence between Arrow-Debreu and Radner equilibrium allocations. The analysis given below to establish the link between the no-arbitrage condition of Section 2.2 and the no-arbitrage condition for a Radner economy is essentially parallel to this theorem. The trick is to find an Arrow-Debreu equilibrium corresponding to a given Radner equilibrium. The derivation employed here is essentially parallel to Kreps (1982, 1987).

Now given a spot price, security price pair $(p, q) \in \mathbb{R}_{++}^{(S+1)L} \times \mathbb{R}^K$, define a subset of X as

$$M^*(p, q) = \{x \in X : p_0 \cdot x_0 = -q \cdot \theta \text{ and } p_1 \square x_1 = V_s(p_1, A) \cdot \theta \text{ for some } \theta \in \mathbb{R}^K\} \quad (2.133)$$

Thus $M^*(p, q)$ is the set of all commodity bundles that satisfy the Radner budget constraints with equality for a given pair (p, q) . $M^*(p, q)$ can also be written as

$$M^*(p, q) = \{x \in X : p \square x = \text{span}[W]\} \quad (2.134)$$

Now define the subset X° of X as:

$$X^\circ = \{x = (x_0, x_1) \in X : x_1 = 0\} \quad (2.135)$$

Notice that X° is a subspace of X with dimension L , that is $X^\circ = \mathbb{R}^L$.

Further define

$$M(p, q) = M^*(p, q) + X^\circ \quad (2.136)$$

Define a linear functional $\pi : M(p, q) \rightarrow \mathbb{R}$ as:

$$[\pi(p, q)](m) = p_0 \cdot x^\circ \quad (2.137)$$

for $m = m^* + x^\circ$ where $m^* \in M^*(p, q)$ and $x^\circ \in X^\circ$

Now imagine an Arrow-Debreu economy with the marketed bundles $m \in M$ and the linear pricing functional $\pi(\cdot) : M \rightarrow \mathbb{R}$ as defined above. Hence (M, π) constitute a price system.

Proposition. $M^*(p, q)$ is a subspace of X .

Proof. We want to prove that $\alpha m + \beta m' \in M^*(p, q)$ for each $m, m' \in M^*$ and for each $\alpha, \beta \in \mathbb{R}$. If $m, m' \in M^*(p, q)$ then they satisfy the Radner budget constraints:

$$p \square m = W \theta \quad (2.138)$$

$$p \square m' = W \theta' \quad (2.139)$$

Multiplying (2.138) and (2.139) by scalars α and β respectively, making use of the linearity of matrix operations and adding side by side:

$$p \square (\alpha m + \beta m') = W (\alpha \theta + \beta \theta') \quad (2.140)$$

Consequently $\alpha m + \beta m' \in M^*(p, q)$. ■

Proposition. $M^*(p, q)$ is a *proper* subset (subspace) of X .

Proof. A trivial arbitrage argument is used to prove that $M^*(p, q) \subset X$. Suppose on the contrary that $M^*(p, q) = X$. This means that agents are not in any way constrained by the Radner budget constraints when choosing their net trade vectors. As it is assumed that agents' preferences are strictly increasing, insatiable agents would take unlimited positions in the positive cone of X or $\mathbb{R}_+^{L(S+1)} \setminus \{0\}$ and then supply and demand would never balance. ■

It is also informative to observe that $M^*(p, q) \subset \mathbb{R}^{L(S+1)}$ because $\text{span}[W] \subset \mathbb{R}^{S+1}$. Therefore, the maximum dimension $\text{span}[W]$ can have is S . Then, the maximum dimension $M^*(p, q)$ can have is LS .

Notice that due to the fact that both $M^*(p, q)$ and X° are subspaces of X and that the sum of two linear spaces is a linear space, $M(p, q)$ is a *subspace* of X . As the maximum dimension $M^*(p, q)$ can have is LS and $X^\circ = \mathbb{R}^L$; the possibility that $M(p, q) = X = \mathbb{R}^{L(S+1)}$ is not excluded at all. Actually this is the case if markets are complete.

Now from Section 2.2. we know that the price system (M, π) admits no arbitrage opportunities if and only if the functional $\pi(\cdot) : M \rightarrow \mathbb{R}$ is

- i) well-defined
- ii) strictly positive
- iii) linear
- iv) continuous

If $\pi(\cdot)$ is linear, then for $m = m^* + x^\circ$;

$$\pi(m) = \pi(m^* + x^\circ) = \pi(m^*) + \pi(x^\circ), \quad m^* \in M^*(p, q), \quad x^\circ \in X^\circ \quad (2.141)$$

But $\pi(\cdot)$ was defined as

$$[\pi(p, q)](m) = p_0 \cdot x^\circ, \quad m \in (M^* + X^\circ) \quad (2.142)$$

Thus $\pi(\cdot)$ is functional that assigns a zero value to a bundle $m^* \in M^*$;

$$\pi(m^*) = 0, \quad m^* \in M^*(p, q) \quad (2.143)$$

Notice that m^* is a bundle that satisfies the Radner budget constraints. Being so, $m^* \in M^*$ can be viewed as a spot market consumption vector that is consistent with a particular portfolio choice.

Again writing the Radner budget constraints with the vector of income transfers on the right hand sides as:

$$p_s \cdot m_s^* = \tau_s, \quad s = 0, 1, \dots, S \quad (2.144)$$

If each equation $s = 0, 1, \dots, S$ is rescaled by a *strictly positive* number β_s , the constraints will be unchanged:

$$\beta_s p_s \cdot m_s^* = \beta_s \tau_s, \quad s = 0, 1, \dots, S \quad (2.145)$$

If the equations are summed up, the Radner budget constraints collapse into a single constraint:

$$\beta_0 p_0 \cdot m_0^* + \beta_1 p_1 \cdot m_1^* + \dots + \beta_S p_S \cdot m_S^* = \beta_0 \tau_0 + \beta_1 \tau_1 + \dots + \beta_S \tau_S \quad (2.146)$$

To be consistent with $\pi(m^*) = 0$, the following should hold;

$$\beta \cdot \tau = 0 \quad (2.147)$$

or in other words there exists $\beta \in \text{int}(\mathbb{R}_+^{S+1})$ with $\beta W = 0$

Alternatively; the no arbitrage condition and its implications in a Radner economy can be directly explored via the market structure and the security prices.

Notation. Let $M^{S,K}$ denote the space of real-valued $S \times K$ matrices.

Definition. The pair $(q, V) \in \mathbb{R}^K \times M^{S,K}$ is said to be *arbitrage-free* if there does not exists a portfolio $\theta \in \mathbb{R}^K$ such that

$$q \cdot \theta \leq 0 \text{ and } V \theta > 0 \quad (2.148)$$

or

$$q \cdot \theta < 0 \text{ and } V \theta \geq 0 \quad (2.149)$$

If $(q, V) \in \mathbb{R}^K \times M^{S,K}$ is arbitrage-free, it is also said that q is a *no arbitrage security price vector relative to V* . This definition simply says that q is a no-arbitrage security price vector relative to V if there is no financial investment strategy that costs nothing at date 0 and generates profits at date 1 with no risk of loss.

Again stacking the vector of security prices $q = (q_1, \dots, q_K) \in \mathbb{R}^K$ on the market structure $V(p_1)$ we have

$$W(q, V) = \begin{bmatrix} -q \\ V(p_1) \end{bmatrix} \quad (2.150)$$

Then the above definition can be equivalently stated as: $W(q, V) \in M^{S+1,K}$ is said to be arbitrage-free if there does not exist a portfolio $\theta \in \mathbb{R}^K$ such that $W(q, V)\theta > 0$.

Then the $W(q, V) > 0$ can be equivalently written as:

$$\text{span}[W] \cap (\mathbb{R}_+^{S+1} \setminus 0) = \emptyset \quad (2.151)$$

As $\text{span}[W]$ is a linear subspace by definition, it contains the zero vector, hence the above expression can be further simplified as:

$$\text{span}[W] \cap \mathbb{R}_+^{S+1} = \{0\} \quad (2.152)$$

In other words, there is no arbitrage if the *subspace of income transfers* $\text{span}[W]$ and the *space of free lunches* $\mathbb{R}_+^{S+1}$ are separated.

Lemma. $W(q, V) \in M^{S+1,K}$ is arbitrage-free iff there exist a vector $\beta \in \text{int}(\mathbb{R}_+^{S+1})$ such that $\beta W = 0$.

Proof. Consider the following corollary of the separating hyperplane theorem also known as the Theorem of the Alternative which can be found in Gale (1960) for example:

Corollary (Stiemke's Lemma). Given a $m \times n$ real matrix A , one and only one of the following is true:

(1) there exists $x \in \text{int}(\mathbb{R}_+^m)$ with $x A = 0$

(2) there exists $y \in \mathbb{R}^n$ with $y A > 0$

Now, given the $(S+1) \times K$ real matrix W , according to this corollary one and only one of the following is true:

(1) there exists $\beta \in \text{int}(\mathbb{R}_+^{S+1})$ with $\beta W = 0$

(2) there exists $\theta \in \mathbb{R}^K$ with $W \theta > 0$

As the second condition implies the existence of arbitrage opportunities by definition, no-arbitrage condition requires that the first statement holds. ■

Define a new vector $\hat{\beta} = [1 \quad \hat{\beta}_1 \quad \dots \quad \hat{\beta}_S] \in \mathbb{R}_{++}^{S+1}$ by normalizing $\beta \in \mathbb{R}_{++}^{S+1}$ such that:

$$\hat{\beta} = \frac{1}{\beta_0} \beta \quad (2.153)$$

Definition: The vector $\mu = (\mu_1, \dots, \mu_S) \in \mathbb{R}_{++}^S$ defined as

$$\mu = [\hat{\beta}_1 \quad \dots \quad \hat{\beta}_S] = [\mu_1 \quad \dots \quad \mu_S] \quad (2.154)$$

is called the **state price vector** with each component μ_s being called a **state price** belonging to state s .

Notice that $\beta W(q, V) = 0$ means that $q = \mu V$. Thus the current prices of securities are given by:

$$q_k = \sum_{s=1}^S \mu_s V_s^k, \quad k=1, \dots, K \quad (2.155)$$

or written in matrix notation:

$$q = \mu V = \begin{bmatrix} \mu_1 V_1^1 + \dots + \mu_S V_S^1 \\ \vdots \\ \mu_1 V_1^K + \dots + \mu_S V_S^K \end{bmatrix} = \begin{bmatrix} q_1 \\ \vdots \\ q_K \end{bmatrix} \quad (2.156)$$

This says that the current price of each security is equal to the weighted average of its payoffs in conceivable states of nature the weights being the (strictly positive) state prices. A state price can also be viewed as the marginal cost of obtaining an additional unit of account in a particular state. To further emphasize that security valuation is linear, the following is useful. By the Riesz Representation Theorem for Euclidean space, the vector of state prices $\mu \in \mathbb{R}_{++}^S$ represents a continuous functional $\varphi(\cdot) : \mathbb{R}^S \rightarrow \mathbb{R}$. Then any security $V^k \in \mathbb{R}^S$ which can also be viewed as a random variable is valued by $\varphi(\cdot)$ as:

$$q_k = \sum_{s=1}^S \mu_s V_s^k = \varphi(V_k), \quad k=1, \dots, K \quad (2.157)$$

Sometimes, in order to ensure positive security prices, securities are assumed to have **limited liability** meaning that securities have positive payoffs in every state. Precisely, it is assumed that for any security $k=1, \dots, K$; $V^k \in \mathbb{R}_+^S \setminus \{0\}$. The limited liability assumption coupled by the existence of a positive linear pricing functional then assures that every security has a positive price: $q_k > 0$ for each $k=1, \dots, K$ or equivalently $q \in \text{int}(\mathbb{R}^K)$.

Any income transfer $\tau_1 \in \mathbb{R}^S$ which can also be viewed as random variable or a contingent income stream constructed by holding a portfolio $\theta_k \in \mathbb{R}^K$ of securities as

$$\tau_1 = \sum_{k=1}^K \theta_k V^k \quad (2.158)$$

then will be valued as simply applying the linear valuation operator $\varphi(\cdot)$ as:

$$\varphi(\tau_1) = \varphi\left(\sum_{k=1}^K \theta_k V^k\right) = \sum_{k=1}^K \varphi(V^k) \theta_k = \sum_{k=1}^K q_k \theta_k \quad (2.159)$$

If the state prices are further normalized so as their sum over the possible states is equal to 1 as:

$$\hat{\mu} = \frac{1}{\sum_{s=1}^S \mu_s} \mu \quad (2.160)$$

with

$$\sum_{s=1}^S \hat{\mu}_s = 1 \quad (2.161)$$

The vector $\hat{\mu} = [\hat{\mu}_1 \ \cdots \ \hat{\mu}_S] \in \mathbb{R}_{++}^S$ such obtained can be understood as vector of probabilities as its components sum up to unity. This measure is called equivalent martingale measure and will be denoted by Q .

$$\frac{q_k}{\sum_{s=1}^S \mu_s} = \sum_{s=1}^S \hat{\mu}_s V_s^k = \mathbb{E}_Q[V^k] \quad (2.162)$$

where $\mathbb{E}_Q[\cdot]$ is an expectations operator under the probability measure Q .

Let

$$\sum_{s=1}^S \mu_s = \frac{1}{(1+r)} \quad (2.163)$$

where r is the riskless rate.

Then for any security $k = 1, \dots, K$;

$$q_k = \frac{1}{(1+r)} \mathbb{E}_Q[V^k] \quad (2.164)$$

Q is called a **risk-neutral probability measure** or a **martingale measure**.

In section 2 the completeness of markets was defined based on the marketed subspace M ; the set of commodity bundles that can be constructed out of the original marketed bundles. The commodity space is

$$\text{span}(M_0) = M = X = \mathbb{R}^{(S+1)L} \quad (2.165)$$

Since in a security-spot markets economy, net trades are attained indirectly through an investment in securities. For this reason it is natural to base the definition of market completeness also on securities.

In our case, $\text{rank } W = \text{rank } V$ and $\text{span}[W(q, V(p_1))] = \text{span}[V(p_1)]$. This means completeness of the market can be understood from the matrix $V(p_1)$. We can now define the completeness of security markets parallel to our previous definition.

Definition. The security markets are said to be **complete** iff $\text{span}[p_1 \square A] = \text{span}[V(p_1, A)] = \mathbb{R}^S$ or otherwise they are said to be **incomplete**.

Definition. Two vectors x, y in an inner-product space are said to be **orthogonal** if $(x | y) = 0$. A vector x is orthogonal to a set S if $(x | y) = 0$ for all $s \in S$.

Definition. Let S be a subset of an inner product space X . The set of all vectors orthogonal to S is called the **orthogonal complement** of S and denoted $S^\perp$. Thus

$$S^\perp = \{x \in X : (x | s) = 0 \quad \forall s \in S\} \quad (2.166)$$

Remark. It can be shown that for any set S in X , $S^\perp$ is a *closed subspace* of X .

Definition. A linear space X is said to be the **direct sum** of its two subspaces S and E denoted $X = S \oplus E$, if every vector $x \in X$ has a unique representation of the form

$$x = s + e, \quad s \in S, \quad e \in E \quad (2.167)$$

Definition. If M is a subspace of a linear space X , the dimension of the quotient space X/M is called the **codimension** of M in X .

Theorem. If S is a closed linear subspace of a Hilbert space H , then $H = S \oplus S^\perp$.

$$\text{span}[W] \oplus (\text{span}[W])^\perp = \mathbb{R}^{S+1} \quad (2.168)$$

$$\dim(\text{span}[W]) + \dim((\text{span}[W])^\perp) = S + 1 \quad (2.169)$$

If $W(q, V) \in M^{S+1, K}$ is arbitrage-free there exist a price vector $\beta \in \mathbb{R}_{++}^{S+1}$ such that $\beta W = 0$. This means that

$$\dim((\text{span}[W])^\perp) \geq 1 \quad (2.170)$$

If markets are complete, the columns of the market structure matrix W span the S dimensional Euclidean space; thus

$$\dim(\text{span}[W]) = S \quad (2.171)$$

Then from (2.169) above

$$\dim((\text{span}[W])^\perp) = 1 \quad (2.172)$$

This means that in case of complete markets, $(\text{span}[W])^\perp$ is one-dimensional subspace of $\beta \in \mathbb{R}_{++}^{S+1}$ or a line.

If the markets are incomplete,

$$\dim(\text{span}[W]) = K < S \quad (2.173)$$

and

$$\dim((\text{span}[W])^\perp) = S - K + 1 > 1 \quad (2.174)$$

If the market is complete, then the state-price vector is unique.

Before closing this section, it is instructive to consider a special case of the model analyzed. In a Radner economy, the preferences of agents over bundles induce preferences on net trade vectors. In models in finance especially, it is common to assume that there is only a single consumption good ($L=1$) and this consumption good is taken as a numeraire in each date-event. Thus

$$p_0 = 1 \quad \text{and} \quad p_s = 1 \quad \text{for} \quad s = 1, \dots, S \quad (2.175)$$

The effect of taking the single commodity as a numeraire throughout is the same as normalizing the security prices at time 0 ;

$$\hat{q}_k = \frac{q_k}{p_0} \quad k = 1, \dots, K \quad (2.176)$$

and security payoffs at time 1 in each state s ;

$$\hat{V}(p_1, A) = \begin{bmatrix} \hat{V}_1^1 & \cdots & \hat{V}_1^K \\ \vdots & \ddots & \vdots \\ \hat{V}_S^1 & \cdots & \hat{V}_S^K \end{bmatrix} = \begin{bmatrix} V_1^1/p_1 & \cdots & V_1^K/p_1 \\ \vdots & \ddots & \vdots \\ V_S^1/p_S & \cdots & V_S^K/p_S \end{bmatrix} \quad (2.177)$$

For a given spot price vector $p = (p_0, p_1) \in \mathbb{R}_{++}^L \times \mathbb{R}_{++}^{LS}$ the vector of income transfers are given by:

$$\tau_0 = p_0 \cdot x_0 \quad p_0 \in \mathbb{R}^L \quad (2.178)$$

$$\tau_s = p_s \cdot x_s, \quad s = 1, \dots, S \quad (2.179)$$

For the one good model $p = (p_0, p_1) \in \mathbb{R}_{++} \times \mathbb{R}_{++}^S$ and

$$\tau_0 = p_0 x_0 \quad (2.180)$$

$$\tau_s = p_s x_s, \quad s = 1, \dots, S \quad (2.181)$$

After normalization

$$\hat{\tau}_0 = x_0 \quad (2.182)$$

$$\hat{\tau}_s = x_s, \quad s = 1, \dots, S \quad (2.183)$$

Therefore when income transfers are measured in units of the consumption good, the distinction between *income transfers* and *quantities* disappears. For simplicity of notation, the angle bracketed notation will be suppressed, keeping in mind that the single consumption good is used as a numeraire throughout.

Once the distinction between *income transfers* and *quantities* disappears, the set of initially available list of securities can be represented by M_0 and their span by M in line with the notation in Section 2.2. Thus; the price-payoff pairs of the initially available set of securities are collectively given by

$$M_0 = \{W^1, \dots, W^K\} = \{W^k = (-q_k, V^k) \in \mathbb{R} \times \mathbb{R}^S : k = 1, \dots, K\} \quad (2.184)$$

and

$$M = \text{span}(M_0) = \text{span}[W] \quad (2.185)$$

Further to this, the initial set of contingent claims and their span are given by;

$$M'_0 = \{V^1, \dots, V^K\} = \{V^k \in \mathbb{R}^S : k = 1, \dots, K\} \quad (2.186)$$

and

$$M' = \text{span}(M'_0) = \text{span}[V] \quad (2.187)$$

Further notice that if Y is a linear space over $\mathbb{R}$, then $(\mathbb{R} \times Y)'$ is isomorphic to $\mathbb{R} \times Y'$ meaning that for any $\psi \in (\mathbb{R} \times Y)'$ there is a $(s, \psi) \in \mathbb{R} \times Y'$ such that

$$\psi(r, y) = \gamma r + \phi(y), \quad r \in \mathbb{R}, \quad y \in Y \quad (2.188)$$

[See for example Holmes (1975), page 22 or Schaefer (1999), page 69, exercise 6].

According to this, in a one good model the linear functional $\pi(\cdot)$ on M can be represented as

$$\pi(r, y) = p_0 r + \phi(y), \quad r \in \mathbb{R}, \quad y \in \mathbb{R}^S \quad (2.189)$$

When the single good is used as a numeraire, this can be written as;

$$\hat{\pi}(r, y) = r + \phi(y), \quad r \in \mathbb{R}, \quad y \in \mathbb{R}^S \quad (2.190)$$

Suppressing the angle bracketed notation:

$$\pi(r, y) = r + \phi(y), \quad r \in \mathbb{R}, \quad y \in \mathbb{R}^S \quad (2.191)$$

Therefore in a one-good model, any bundle $(r, y) \in \mathbb{R} \times M'$ can be purchased in units of date zero consumption at a price of $r + \phi(y)$. We will employ this special one-good, normalized price system (M', ϕ) in the sequel.

2.4.2. Arbitrage in an Infinite Dimensional Commodity Space

In this section we take up the analysis of arbitrage in a Radner economy with an infinite dimensional commodity space. In particular, the commodity space is infinite dimensional when the sample space Ω is infinite. In this section, it is assumed that there is only a single consumption good ($L = 1$) and this consumption good is taken as a numeraire in each date-event. We closely follow the analysis of Harrison and Kreps (1979) in what follows.

As mentioned in the previous section, when income transfers are measured in units of the consumption good, the distinction between *income transfers* and *quantities* disappears. Therefore, we can conduct the analysis taking as a net trade vector (x_0, x_1) a vector of income transfers (τ_0, τ_1) . Also the preferences of agents over net trade vectors will be the ones induced by the preferences over the net trades. In this one commodity, infinite-dimensional net trade economy, the net trade space is given by $X = \mathbb{R} \times Y$ where $\mathbb{R}$ is the real line and Y is a space of random variables on $(\Omega, \mathcal{F})$. Y is typically taken as $Y \subseteq \mathbb{R}^{(\Omega, \mathcal{F})}$ where $\mathbb{R}^{(\Omega, \mathcal{F})}$ is the space of $\mathcal{F}$ measurable real valued functions on Ω . Appropriate restrictions are imposed on Y as a subspace of the $\mathcal{F}$ measurable random variables on Ω . In models of economies under uncertainty and especially in models built in finance; the canonical choice space employed is the space $L^2(\Omega, \mathcal{F}, P)$ of equivalence classes of square integrable random variables on $(\Omega, \mathcal{F}, P)$. We will shortly write $L^2(P)$ for $L^2(\Omega, \mathcal{F}, P)$. The major reason for this choice is that elements of this space have finite variances which is a very desirable property for financial variables. Furthermore by using the Cauchy-Schwarz inequality it can be shown that elements of $L^2(P)$ have finite expectations also. Therefore we customarily take Y as $L^2(P)$, the space of equivalence classes of square integrable random variables on $(\Omega, \mathcal{F}, P)$. Then the net trade space is given by $X = \mathbb{R} \times Y = \mathbb{R} \times L^2(\Omega, \mathcal{F}, P)$ and each generic element $(r, y) \in \mathbb{R} \times L^2(\Omega, \mathcal{F}, P)$ denotes r units of consumption at time 0 and $y(\omega)$ units of consumption at time 1 in state ω .

We will employ the price system given in the previous section for the one-good model.

$$M'_0 = \{V^1, \dots, V^K\} = \{V^k \in Y : k = 1, \dots, K\} \quad (2.186)$$

and

$$M' = \text{span}(M'_0) = \text{span}[V] \quad (2.187)$$

The price system is then given as a pair (M', ϕ) where M' is the subspace of Y and ϕ is a linear functional on M' . Any bundle $(r, y) \in \mathbb{R} \times M'$ can then be purchased in units of date zero consumption at a price of $r + \phi(y)$.

Before proceeding to an analysis of arbitrage with these random variables we will take a quick tour of the more general class of function spaces in which $L^2(\Omega, \mathcal{F}, \mu)$ is a member.

Definition. A *reflexive, symmetric and transitive* relation on a set X is said to be an **equivalence relation** on X . An equivalence relation is usually denoted by the symbol $\sim$.

Definition. If $\sim$ is an equivalence relation on X , then the **equivalence class** of an element $x \in X$, denoted $[x]$ is the set $[x] = \{y \in X : y \sim x\}$.

The $\sim$ -equivalence classes form a partition of X .

Definition. A property is said to hold **almost everywhere** (abbreviated as **a.e.**) if it holds except at most on a set of measure zero.

Let $(\Omega, \mathcal{F}, \mu)$ be a measure space and $1 < p < \infty$. $\mathcal{L}^p(\Omega, \mathcal{F}, \mu)$ is the class of all $\mathcal{F}$ -measurable functions on Ω for which $|f|^p$ is integrable, that is;

$$\int_{\Omega} |f|^p d\mu < \infty \quad (2.192)$$

Define an equivalence relation on $\mathcal{L}^p(\Omega, \mathcal{F}, \mu)$ such that; for any $f, g \in \mathcal{L}^p(\Omega, \mathcal{F}, \mu)$, $f \sim g$ if and only if $f = g$ μ -almost everywhere. The equivalence class of a function $f \in \mathcal{L}^p(\Omega, \mathcal{F}, \mu)$ denoted by $[f]$ is then the set of all functions in $\mathcal{L}^p(\Omega, \mathcal{F}, \mu)$ which are μ -equivalent to f . The new space such obtained is called the space of μ -equivalence classes of $\mathcal{F}$ -measurable functions on Ω for which $|f|^p$ is integrable and will be denoted $L^p(\Omega, \mathcal{F}, \mu)$.

Definition. L^p -norm is defined as:

$$\|f\|_p = \left[\int |f|^p d\mu \right]^{1/p}, \quad f \in L^p(\Omega, \mathcal{F}, \mu) \quad (2.193)$$

Notice that $\|f\|_p = 0$ imply that $f = 0$ for almost everywhere or equivalently $f(\omega) = 0$ for almost all $\omega \in \Omega$.

As the elements of $L^p(\Omega, \mathcal{F}, \mu)$ are functions on Ω , algebraic operations on these elements are defined pointwise. This means that addition and scalar multiplication, for example, are defined as:

$$(f + g)(\omega) = f(\omega) + g(\omega), \quad f, g \in L^p(\Omega, \mathcal{F}, \mu) \quad (2.194)$$

$$\lambda f(\omega) = (\lambda f)(\omega), \quad f \in L^p(\Omega, \mathcal{F}, \mu), \quad \lambda \in \mathbb{R} \quad (2.195)$$

Note that as $L^p(\Omega, \mathcal{F}, \mu)$ have equivalence classes of functions as elements, the algebraic operations are defined for equivalence classes also, i.e., in a more precise notation, the sum of two equivalence classes $[f]$ and $[g]$ is the equivalence class $[f + g]$.

According to Minkowski's Inequality, for any $f, g \in L^p(\mu)$ and $\lambda \in \mathbb{R}$; $f + g \in L^p(\mu)$. Furthermore and $\lambda f \in L^p(\mu)$ for all $f \in L^p(\mu)$ and $\lambda \in \mathbb{R}$. It is elementary to prove that $L^p(\mu)$ ($1 < p < \infty$) satisfies the other linear space axioms. As the zero element of $L^p(\mu)$, we take the equivalence class of functions vanishing μ -almost everywhere.

Recall that a functional $\|\cdot\| : X \rightarrow \Phi$ where X is a linear space over the scalar field Φ is called a **norm** in X if it satisfies the following properties for all $x, y \in X$ and $\lambda \in \Phi$:

$$(1) \quad \|x\| = 0 \quad \text{iff} \quad x = 0$$

$$(2) \quad \|x\| > 0 \quad \text{if} \quad x \neq 0$$

$$(3) \quad \|x + y\| \leq \|x\| + \|y\| \quad (\text{the triangle inequality})$$

$$(4) \quad \|\lambda x\| = |\lambda| \|x\|$$

It can be shown that the L^p -norm defined above as:

$$\|f\|_p = \left[\int |f|^p d\mu \right]^{1/p} \quad \forall f \in L^p(\mu) \quad (2.196)$$

satisfies all of these properties thus $L^p(\mu)$ ($1 < p < \infty$) is a normed linear space under the L^p -norm. The metric induced by this norm is given as:

$$d_p(f, g) = \|f - g\|_p \quad \forall f, g \in L^p(\Omega, \mathcal{F}, \mu) \quad (2.197)$$

It can also be shown that $L^p(\mu)$ ($1 < p < \infty$) is complete under the L^p -norm. Since a complete linear space is called a Banach space, $L^p(\mu)$ ($1 < p < \infty$) is a Banach space under the L^p -norm. In the sequel, we will make use of the properties of the $L^p(\mu)$ ($1 < p < \infty$) due to being a normed linear space and we will not make use of the completeness property.

Having completed our brief survey of the space $L^p(\mu)$ ($1 < p < \infty$), we can now be a bit more specific. Firstly as our primary objective is to model random variables, we will replace the measure μ with a probability measure P obtaining the space $L^p(P)$ ($1 < p < \infty$). Recall that a probability measure is a measure with the property $P(\Omega) = 1$. We will even further narrow this space by taking $p = 2$ for the reason stated in the first paragraph of this section.

Having decided to take $L^2(P)$ as the space of contingent claims to consumption at date 1; we briefly review the algebraic, topological and order structure of $X = \mathbb{R} \times L^2(P)$ to see if it satisfies the assumptions (1) to (4) of Section 2.2 given for a commodity space. First it can be shown that the Cartesian product of two linear spaces is a linear space itself. [See Luenberger (1969) for example]. Then the product space $X = \mathbb{R} \times L^2(P)$ is a linear space itself.

For the order structure of $L^2(P)$; it can be proved that $L^2(P)$ is an ordered linear space under the almost everywhere pointwise ordering. The almost everywhere pointwise ordering is given as;

for $V, V' \in L^2(P)$, $V \geq V'$ iff $V(\omega) \geq V'(\omega)$ for P -almost every ω . (a.s.)

Notation. As $L^2(P)$ is an ordered vector space, the following relational notation shall be employed for random variables. For two random variables $V, V' \in L^2(P)$;

$$V = V' \text{ } P\text{-a.s. iff } P(V = V') = P(\{\omega \in \Omega : V(\omega) = V'(\omega)\}) = 1 \quad (2.198)$$

$$V \geq V' \text{ } P\text{-a.s. iff } P(V \geq V') = P(\{\omega \in \Omega : V(\omega) \geq V'(\omega)\}) = 1 \quad (2.199)$$

$$V > V' \text{ } P\text{-a.s. iff } V \geq V' \text{ } P\text{-a.s. and } P(V > V') > 0 \quad (2.200)$$

For the special case $V' = 0$;

$$V \geq 0 \quad \text{iff} \quad V \in L_+^2(P) \quad (2.201)$$

$$V > 0 \quad \text{iff} \quad V \in L_+^2(P) \setminus \{0\} \quad (2.202)$$

The order structure of $X = \mathbb{R} \times L^2(P)$ is specified by taking as the positive cone of X ;

$$X_+ = \mathbb{R}_+ \times Y_+ \quad (2.203)$$

where

$$Y_+ = \{y \in L^2(P) : y(\omega) \geq 0 \text{ a.s.}\} = \{y \in L^2(P) : P(y(\omega) \geq 0) = 1\} \quad (2.204)$$

Thus;

$$X_+ = \{(r, Y) \in \mathbb{R} \times L^2(P) : r \geq 0, Y \geq 0 \text{ a.s.}\} \quad (2.205)$$

We naturally let K be the positive cone of X with the origin deleted:
 $K = X_+ \setminus \{0\} = \mathbb{R}_+ \times L_+^2 \setminus \{(0, 0)\}.$

The topology τ on X is taken as the product topology on $\mathbb{R} \times L^2(P)$ derived from the Euclidean topology on $\mathbb{R}$ and the $L^2(P)$ norm topology on $L^2(P)$. It is well established that this topology is Hausdorff and locally convex. It can be further shown that $X = \mathbb{R} \times L^2(P)$ is topological vector space. We now consider assumptions (5) to (7) of Section 2.2.

The preferences of agents defined on the space of net trades X are assumed to be complete and transitive. In addition, they are assumed to be continuous, strictly increasing and convex as made precise below:

Continuity. The preferences are τ continuous; that is, for all $(r, y) \in \mathbb{R} \times L^2$, the sets

$$\{(r', y') \in \mathbb{R} \times L^2 : (r, y) \succeq (r', y')\} \quad (2.206)$$

and

$$\{(r', y') \in \mathbb{R} \times L^2 : (r', y') \succeq (r, y)\} \quad (2.207)$$

are both Euclidean $\times L^2$ closed in X .

Convexity. The preference relations $\succeq$ are convex; that is, for all $(r, y) \in \mathbb{R} \times L^2$, the upper contour set of (r, y)

$$\{(r', y') \in \mathbb{R} \times L^2(P) : (r', y') \succeq (r, y)\} \quad (2.208)$$

is convex.

Monotonicity. The preference relations $\succsim$ are strictly increasing; that is, for all $(r, y) \in \mathbb{R} \times L^2$, for all $r' > 0$ and all $y' \in Y_+ \setminus \{0\}$

$$(r + r', y) \succ (r, y) \quad \text{and} \quad (r, y + y') \succ (r, y) \quad (2.209)$$

We first note some definitions required for the separation theorem that will be employed.

Definition. Let $x, y \in X$ with $x \neq y$ and α a scalar. The **line segment** joining x and y is the set of points; $[x, y] = \{\alpha x + (1 - \alpha)y : 0 \leq \alpha \leq 1\}$. $(x, y) = [x, y] \setminus \{y\}$ and $(x, y) = [x, y] \setminus \{x\}$ are likewise defined.

Definition. A set $A \in X$ is **convex** if $[x, y] \subseteq A$ for any pair of points $x, y \in A$.

Definition. For a non-zero linear functional ϕ on a linear space X and a scalar α ; an **hyperplane** is defined as the set:

$$[\phi = \alpha] = \{x \in X : \phi(x) = \alpha\} \quad (2.210)$$

A hyperplane $[\phi = \alpha]$ determines two **strict half spaces**;

$$[\phi < \alpha] = \{x \in X : \phi(x) < \alpha\} \quad (2.211)$$

and

$$[\phi > \alpha] = \{x \in X : \phi(x) > \alpha\} \quad (2.212)$$

Similarly, $[f = \alpha]$ determines two **weak half spaces**;

$$[\phi \leq \alpha] = \{x \in X : \phi(x) \leq \alpha\} \quad (2.213)$$

and

$$[\phi \geq \alpha] = \{x \in X : \phi(x) \geq \alpha\} \quad (2.214)$$

Definition. Two subsets of X are said to be **separated** if they lie on opposite weak half-spaces determined by an hyperplane. In other words; two subsets A, B of X are said to be separated by an hyperplane $[\phi = \alpha]$ if either

$$A \subseteq [\phi \leq \alpha] \text{ and } B \subseteq [\phi \geq \alpha] \quad (2.215)$$

or if

$$B \subseteq [\phi \leq \alpha] \text{ and } A \subseteq [\phi \geq \alpha] \quad (2.216)$$

Equivalently, two subsets A, B of X are said to be separated by an hyperplane $[\phi = \alpha]$ if either

$$\phi(x) \leq \alpha \leq \phi(y) \quad \forall x \in A, \quad \forall y \in B \quad (2.217)$$

or if

$$\phi(y) \leq \alpha \leq \phi(x) \quad \forall x \in A, \quad \forall y \in B \quad (2.218)$$

As before securities $V^k(\omega) \in L^2(\Omega, \mathcal{F}, P)$ for $k=1, \dots, K$ are claims to state contingent consumption at date 1 and thus they are random variables. The price-payoff pairs of the initially available set of securities are collectively given by

$$M_0 = \{W^1, \dots, W^K\} = \{W^k = (-q_k, V^k) \in \mathbb{R} \times L^2(P) : k=1, \dots, K\} \quad (2.219)$$

and the market subspace is given as:

$$M = \text{span}(M_0) = \{(-q \cdot \theta, \sum_{k=1}^K \theta_k V^k) \in \mathbb{R} \times L^2(P) : \theta \in \mathbb{R}^K\} \quad (2.220)$$

It is well known that the positive cone of the L^2 space has an empty interior. Hence standard separation theorems like the one given below cannot be applied to find a continuous linear functional that separates the positive cone from the marketed subspace. The trick is to separate the strictly preferred set from the marketed set

instead. The strictly preferred set has a non-empty interior as preferences are continuous. Actually in separating the strictly preferred set from the marketed subspace; we are assuming that the price system is viable. Hence our approach is a more transparent analysis of the implication of viability that was given without proof in Section 2.2.

$$B = \{x \in X : x \succ 0\} = \{(r, y) \in \mathbb{R} \times L^2 : (r, y) \succ (0, 0)\} \quad (2.221)$$

$$H = \{m \in M : \phi(m) \leq 0\} = \{(r, m) \in R \times M : r + \phi(m) \leq 0\} \quad (2.222)$$

Note that H is a subspace of the linear space X and hence convex. B is convex as preferences are convex. Then two sets can be separated by applying the following separation theorem for example:

Theorem [Separating hyperplane theorem, Holmes (1975)]. Let A and B be convex subsets of the linear topological space, X and assume that $\text{int}(A) \neq \emptyset$. Then A and B can be separated by a closed hyperplane iff $\text{int}(A) \cap B = \emptyset$.

We could alternatively note that the space $L^2(P)$ is normable. Now assume that (M', ϕ) admits no arbitrage opportunities. Then viability requires that $\phi(\cdot) : M \rightarrow \mathbb{R}$ is *well-defined, linear and strictly positive*. Referring to Figure 2.2, in order to extend π to a linear, strictly positive, continuous functional $\psi(\cdot) : X \rightarrow \mathbb{R}$, we need to assume either

- (i) (M', ϕ) is viable, or
- (ii) (M', ϕ) admits no free lunches

It is seen that as $L^2(P)$ is normable, extension of the price functional to the whole space can be made with the no-free lunches condition which is less restrictive than viability. As an alternative approach we could initially assume that (M, π) admits no free lunches. Then $\phi(\cdot) : M' \rightarrow \mathbb{R}$ is *well-defined, linear, strictly positive* and

continuous. Knowing that $L^2(P)$ is normable, a valuation operator $\psi(\cdot) : X \rightarrow \mathbb{R}$ is obtained. Now it is well-known that $L^2(\Omega, \mathcal{F}, P)$ is Hilbert space.

Theorem (Riesz Representation Theorem for $L^2(\Omega, \mathcal{F}, P)$). If ϕ is a *continuous linear* functional on $L^2(\Omega, \mathcal{F}, P)$, there is a *unique* element $\xi \in L^2(\Omega, \mathcal{F}, P)$ such that

$$\phi(x) = \mathbb{E}_P(\xi x) \quad \forall x \in L^2(\Omega, \mathcal{F}, P) \quad (2.223)$$

By employing this Riesz Representation Theorem for $L^2(\Omega, \mathcal{F}, P)$ and assuming that security $k=1$ is a riskless security in the sense that $V^1(\omega) = 1$ for all $\omega \in \Omega$ we arrive at the familiar valuation formula:

$$q_k = \frac{1}{(1+r)} \mathbb{E}_Q[V^k], \quad k = 1, \dots, K \quad (2.224)$$

where Q is an equivalent martingale measure.

Parallel to the result for the finite dimensional Euclidean space it is seen that the arbitrage-free price of any random payoff $V^k \in L^2(P)$ is simply its expected payoff under Q discounted at the riskless rate.

2.5. Non-linear Asset Markets and Arbitrage

In Section 2 we have analyzed the valuation of assets by arbitrage in a competitive economy as characterized by a linear market structure. The purpose of this section is to qualitatively analyze the consequences of non-linear pricing within the framework of a security-spot market economy analyzed in Sections 2.3.3. and 2.4.1. We will be engaged in the consequences of non-linear pricing from the perspective of valuation by arbitrage.

Without loss of generality, we assume a single consumption good ($L=1$) security-spot markets economy with two periods and a finite number of states; all possible configurations for the economy at date 1 can be described in terms of a finite number

of conceivable states of nature indexed by the set $s \in \mathcal{S} = \{1, \dots, S\}$. In this economy, the net trade space is given by $X = \mathbb{R} \times \mathbb{R}^S$.

In the linear price system (M, ϕ) of Section 2.4.1 any bundle $(r, y) \in \mathbb{R} \times M$ can be purchased in units of date zero consumption at a price of $r + \phi(y)$. We again take assets or securities are claims to date 1 consumption. A typical contingent claim $V \in \mathbb{R}^S$ is a random variable. V_s^k represents payoff of security k in units of the consumption good at date 1 in state s . Then any security $V^k \in \mathbb{R}^S$ which can also be viewed as a random variable is valued by $\phi(\cdot)$ as:

$$q_k = \phi(V^k), \quad V^k \in \mathbb{R}^S \quad (2.225)$$

Further any contingent claim constructed by holding a portfolio $\theta \in \mathbb{R}^K$ of securities as

$$\sum_{k=1}^K \theta_k V^k \quad (2.226)$$

then will be valued as simply applying the linear valuation operator $\phi(\cdot)$ as:

$$\phi(m) = \phi\left(\sum_{k=1}^K \theta_k V^k\right) = \sum_{k=1}^K \phi(V^k) \theta_k = \sum_{k=1}^K q_k \theta_k \quad (2.227)$$

Note that, to avoid convergence problems when summing infinite number of terms, only finite number of terms in the linear combination are allowed. The situation for the linear pricing is schematically illustrated in Figure 2.4.

In this figure a Radner economy with $S=1$ states and two assets are shown. According to this price-contingent claim pairs (q_k, V^k) for $k=1, 2$ are vectors in $\mathbb{R}^3$. The span of these two vectors is given by $\text{span}[W]$ in line with the notation of Section 2.4. Notice that $\text{span}[W]$ is a subspace of $\mathbb{R}^3$, or more specifically it is a plane that passes through the origin. Notice that this is a case of complete markets. In accordance with the no arbitrage condition, $\text{span}[W]$ intersects the positive cone $\mathbb{R}_+^3$.

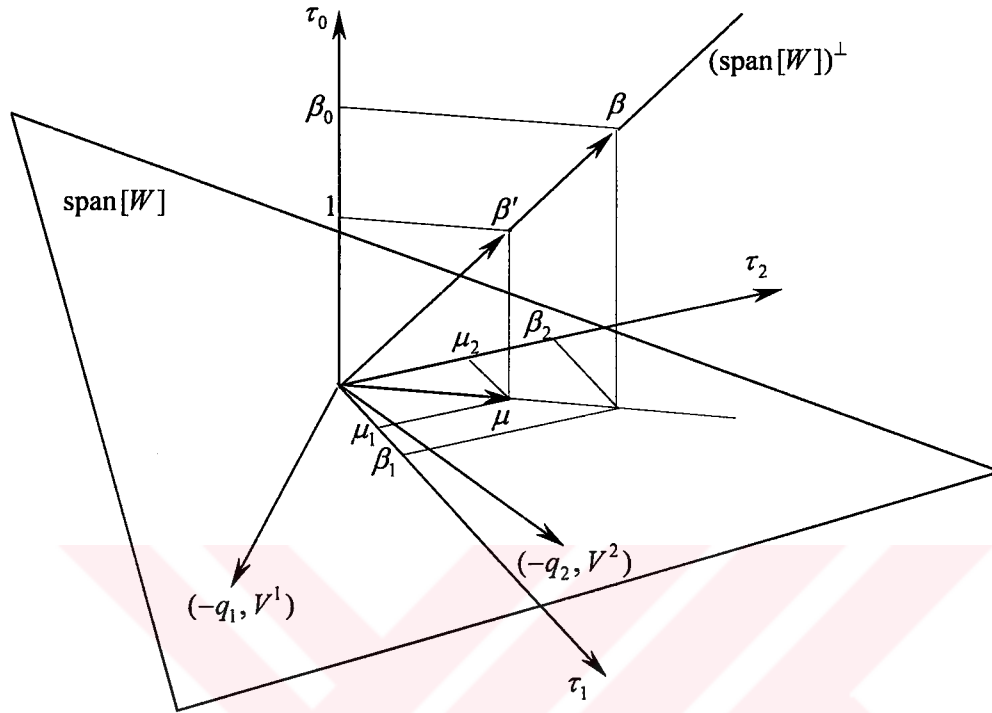


FIGURE 2.4 MARKETS WITH LINEAR ASSET PRICING FUNCTIONALS

at only 0. As the no arbitrage condition is satisfied, there is a strictly positive vector $\beta \in \mathbb{R}^3$ lying in the orthogonal space $\text{span}[W]^\perp$. As markets are complete, this vector is unique. The state price vector is also shown in this figure.

As mentioned in the introduction, linearity represents the price-taking behavior which is at the heart of the Walrasian theory of value. We retain the linear space assumption for the commodity space; any linear combination of two choices is still available. We remove the linearity assumption for the pricing functional. Once the linearity assumption is removed, markets are not considered as competitive anymore. Non-linearity in an economy may be an indication of various phenomenon like

monopoly power in asset markets, market manipulation, information effects, market frictions like transaction costs, taxes and etc. The market manipulation could be performed by an investor through a direct or an indirect mechanism. In the direct market manipulation, a large investor could affect the prices of assets by involving in a large volume of trades. In the indirect mechanism, the manipulating investor could cause the prices to rise or fall as the other investors duplicate his or her trades. The other investors follow suit because they believe that the manipulator is well informed. Here, we do not directly attempt to model the mechanism that causes non-linear asset pricing operators. Rather we assume that non-linearity is exogenous and look and study the implications in terms of arbitrage opportunities.

We now proceed to our formulation of non-linear markets. Assume that an initial set of contingent claims V^k indexed by $k \in \mathcal{K} = \{1, \dots, K\}$ is given. In a Radner economy with non-linear asset pricing functionals, a contingent claim $V^k \in \mathbb{R}^S$ is valued by a general **asset pricing operator** $\varphi : \mathbb{R}^S \rightarrow \mathbb{R}$ defined as $V \rightarrow \varphi(V)$. Thus the operator $\varphi(\cdot)$ assigns to each claim $V^k \in \mathbb{R}^S$ a price measured in units of date zero consumption. $\varphi(\cdot)$ can be axiomatized to have the following properties:

(1) **Linearity.** A function $\varphi : \mathbb{R}^S \rightarrow \mathbb{R}$ is linear if

$$\varphi(\alpha V + \beta V') = \alpha \varphi(V) + \beta \varphi(V'), \quad V, V' \in \mathbb{R}^S, \quad \alpha, \beta \in \mathbb{R} \quad (2.228)$$

(2) **Monotonicity.** A function $\varphi : \mathbb{R}^S \rightarrow \mathbb{R}$ is monotonous if

$$\varphi(V') \leq \varphi(V) \quad V' \leq V \in \mathbb{R}^S \quad V, V' \in \mathbb{R}^S \quad (2.229)$$

(3) **Positive homogeneity.** A function $\varphi : \mathbb{R}^S \rightarrow \mathbb{R}$ is positively homogeneous if

$$\varphi(\lambda V) = \lambda \varphi(V) \quad V \in \mathbb{R}^S, \quad \lambda \in \mathbb{R}_+ \quad (2.30)$$

Remark. As noted in Jouini (2000), positive homogeneity is a reasonable assumption when the effect of buying multiples of an asset on the asset price is considered. When trading larger quantities, better prices can be obtained (increasing returns to scale). On the other hand trading larger quantities suppresses the best bid and ask offers widening the bid-ask spread (decreasing returns to scale). Positive homogeneity is reasonable if these two opposing effects are assumed offset each other.

Also note that if $\varphi(\cdot)$ is positively homogeneous then, for $\lambda = 0$, $\varphi(0V) = 0\varphi(V) = 0$, thus $\varphi(0) = 0$. When combined with the monotonicity condition we obtain

$$\varphi(V) \geq 0 \quad V \geq 0 \quad V \in \mathbb{R}^S \quad (2.231)$$

which is a no arbitrage condition.

(4) **Subadditivity.** A function $\varphi : \mathbb{R}^S \rightarrow \mathbb{R}$ is subadditive if

$$\varphi(V + V') \leq \varphi(V) + \varphi(V') \quad V, V' \in \mathbb{R}^S \quad (2.32)$$

Remark. Subadditivity means that it is less expensive to buy the sum $V + V'$ of two contingent claims than to buy the claims V and V' separately and add up the prices.

(5) **Convexity.** A functional $\varphi : \mathbb{R}^S \rightarrow \mathbb{R}$ is convex if for all $x, y \in \mathbb{R}^S$ and all $0 \leq \lambda \leq 1$;

$$\varphi(\lambda x + (1 - \lambda)y) \leq \lambda \varphi(x) + (1 - \lambda)\varphi(y) \quad (2.233)$$

(6) **Sublinearity.** A function $\varphi : \mathbb{R}^S \rightarrow \mathbb{R}$ is sublinear if it is *subadditive* and *positively homogeneous*, thus:

$$\varphi(V + V') \leq \varphi(V) + \varphi(V') \quad V, V' \in \mathbb{R}^S \quad (2.234)$$

and;

$$\varphi(\lambda V) = \lambda \varphi(V) \quad V \in \mathbb{R}^S, \quad \lambda \in \mathbb{R}_+ \quad (2.235)$$

Remark. A *sublinear* function is *convex*.

If $\varphi(\cdot)$ is sublinear then

$$\varphi(V) + \varphi(-V) \geq \varphi(V - V) \quad V \in \mathbb{R}^S \quad (2.236)$$

$$\varphi(V) + \varphi(-V) \geq \varphi(0) \quad V \in \mathbb{R}^S \quad (2.237)$$

$$\varphi(V) + \varphi(-V) \geq 0 \quad V \in \mathbb{R}^S \quad (2.238)$$

$$\varphi(V) \geq -\varphi(-V) \quad V \in \mathbb{R}^S \quad (2.239)$$

Therefore the price at which a contingent claim $V \in \mathbb{R}^S$ can be bought is larger than or equal to the price at which it can be sold.

We will model the non-linearity by a sublinear pricing operator is $\varphi(\cdot)$. As noted in Jouini and Kallal (1995), sublinear pricing operators are the reduced forms of multiperiod securities price models encompassing a larger class of market frictions that includes bid-ask spreads, short selling constraints and costs, borrowing constraints and costs and taxes. In those multiperiod models, the pricing operator is the result of a dynamic hedging process. In particular, a pricing operator assigns to each contingent claim a value that is equal to the minimum cost necessary to dominate that contingent claim and is accordingly sublinear and hence convex.

In the presence of bid-ask spreads or transaction costs, asset pricing operators will be non-linear. Assume that each security V^k , $k \in \mathcal{K} = \{1, \dots, K\}$ can be bought for its

ask price q_k^a and can be sold for its bid price q_k^b . The difference between the bid and ask price is called the bid-ask spread. A trivial no arbitrage argument is sufficient to prove that $q_k^a \geq q_k^b$. Let θ_k^a and θ_k^b be the numbers of units of security k is longed or shorted respectively. In the presence of bid-ask spreads the asset valuation operator can be defined as:

$$\varphi(V) = \min_{(\theta^a, \theta^b) \in \mathbb{R}^K \times \mathbb{R}^K} \left\{ \sum_{k=1}^K q_k^a \theta_k^a - \sum_{k=1}^K q_k^b \theta_k^b : \sum_{k=1}^K (\theta_k^a - \theta_k^b) V^k = V, \theta^a \geq 0, \theta^b \geq 0 \right\} \quad (2.240)$$

This operator assigns to each contingent claim a value that is equal to the minimum cost necessary to duplicate or perfectly hedge that contingent claim. As noted in LeRoy and Werner (2001) it is easy to see that this asset valuation operator is sublinear, thus

$$\varphi(V + V') \leq \varphi(V) + \varphi(V') \quad V, V' \in \mathbb{R}^S \quad (2.241)$$

and

$$\varphi(\lambda V) = \lambda \varphi(V) \quad V \in \mathbb{R}^S, \quad \lambda \in \mathbb{R}_+ \quad (2.242)$$

The reason for this can be understood by noticing that the sum of two strategies that hedge the claims V and V' hedges the claim $V + V'$ but some orders to buy and sell the same security might cancel out resulting in savings in transaction costs.

Rewrite the definitions of a line segment and a convex set given in Section 2.4.2 for $\mathbb{R}^n$:

Definition. Let $x, y \in \mathbb{R}^n$ with $x \neq y$ and λ a scalar. The line segment joining x and y is the set of points; $[x, y] = \{\lambda x + (1 - \lambda)y : 0 \leq \lambda \leq 1\}$.

Definition. A set $A \in \mathbb{R}^n$ is convex if $[x, y] \subseteq A$ for any pair of points $x, y \in A$.

We first note a result as an indication of what may be encountered in non-linear asset markets.

Theorem (Jouini and Kallal, 1999). Assume that standard assumptions about preferences hold. The price system (M, φ) , where the set of marketed contingent claims M is a convex subset of X and φ is a convex pricing rule defined on M , is viable if and only if there exists a *strictly positive* and *continuous linear* functional ψ defined on X such that

$$\psi|_M \leq \varphi \quad (2.243)$$

Proof. See Jouini and Kallal (1999). ■

This result may be regarded as the non-linear counterpart of the Theorem 1 of Harrison and Kreps (1979). In this theorem the pricing operator is assumed to be convex.

We will call a pair $(-\varphi_k(\theta_k V^k), \theta_k V^k) \in \mathbb{R} \times \mathbb{R}^S$ a price-contingent claim pair. This is a pair obtained by holding $\theta_k \in \mathbb{R}$ of the contingent claim V^k . Figure 2.5 illustrates two assets with their corresponding sets G_1 and G_2 for the case $S = 2$. Now define

$$G_k = \{(-\varphi_k(\theta_k V^k), \theta_k V^k) \in \mathbb{R} \times \mathbb{R}^S : \theta_k \in \mathbb{R}\}, \quad k = 1, \dots, K \quad (2.244)$$

Thus G_k for each $k \in \mathcal{K}$ is the set of price-contingent claim pairs that can be obtained by holding different amounts of a given security

Definition. The sum of two sets A, B in $\mathbb{R}^n$ is defined as:

$$A + B \equiv \{x : x = a + b, a \in A, b \in B\} \quad (2.245)$$

Now define the set G as the set of all price-contingent claim pairs $(r, V) \in \mathbb{R} \times \mathbb{R}^S$ that can be obtained as the sum of two or more points each from a different set G_k .

$$G = \{g \in \mathbb{R}^{S+1} : g = \sum_{k \in \mathcal{K}} \lambda_k g_k \text{ for some } \{\lambda_k\}_{k \in \mathcal{K}} \text{ with } \lambda_k \text{ for } k \in \mathcal{K} \text{ is either 0 or 1}\} \quad (2.246)$$

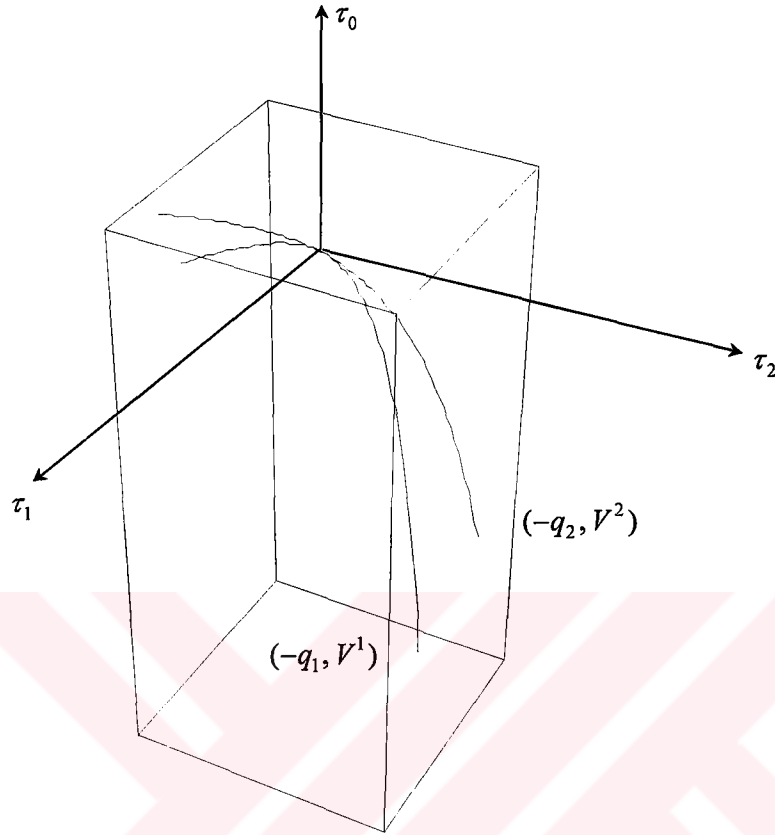


FIGURE 2.5 MARKETS WITH SUBLINEAR PRICING OPERATORS

Thus G in this non-linear economy somehow corresponds to the smallest subspace of $\mathbb{R} \times \mathbb{R}^S$ that contains $\{(-q_k, V^k)\}_{k \in \mathcal{K}}$ or $\text{span}(\{(-q_k, V^k)\}_{k \in \mathcal{K}})$ defined for the linear economy.

Definition. The scalar multiple of a set $A \in \mathbb{R}^n$ is given by, B in a linear space X is defined as:

$$\lambda A = \{\lambda x : x \in A\} \quad (2.247)$$

Theorem. The sum of two *convex* sets C_1 and C_2 in $\mathbb{R}^n$ is *convex*.

Proof. See Rockafellar (1997). ■

Theorem. If $C \in \mathbb{R}^n$ is *convex* and $\lambda \geq 0$ then the following set is *convex*.

$$\lambda C = \{\lambda x : x \in C\} \quad (2.248)$$

Proof. See Rockafellar (1997). ■

These results for example suggest that if G_k for each $k \in \mathcal{K}$ are *convex*, then G will be *convex*.

Figure 2.6 illustrates an example of the type of set G that may be obtained in an economy with non-linear security pricing functionals.

This figure is for the case $S = 2$, and for a set G obtained as the sum of two sets G_1 and G_2 . Only $G \cap \mathbb{R}_+^3$ is shown in the figure because this is the part that represents limited arbitrage opportunities.

For the case where G_k for $k \in \mathcal{K}$ are *convex* sets, the below theorem may help to draw conclusion on the characteristics of the set G . We first define a *convex symmetric* set in $\mathbb{R}^n$.

Definition. A *convex* set A in $\mathbb{R}^n$ is **symmetric** if $-A = A$

Consider the following theorem given in Rockafellar (1997), page 19:

Theorem. Given two *convex* sets A and B in $\mathbb{R}^n$, each vector $x \in A + B$ can be expressed uniquely in the form of $x = a + b$, $a \in A$, $b \in B$ if and only if the *symmetric convex* sets $A - A$ and $B - B$ have only the zero vector of $\mathbb{R}^n$ in common.

A specific case is when the security pricing functionals $\varphi_k(\cdot)$ are *linear*. It was previously shown that no-arbitrage condition requires that there is a unique $\varphi(\cdot)$ that is well-defined and *linear*. Therefore; in arbitrage-free linear markets, each set

G_k for $k \in \mathcal{K}$ is a subspace or more specifically a line passing through the origin. Then for $k \in \mathcal{K}$, $G_k - G_k$ will be a line passing through the origin also. This means that in linear markets each point in $\text{span}(\{(-q_k, V^k)\}_{k \in \mathcal{K}})$ is uniquely determined. This cannot be taken for granted for non-linear markets.

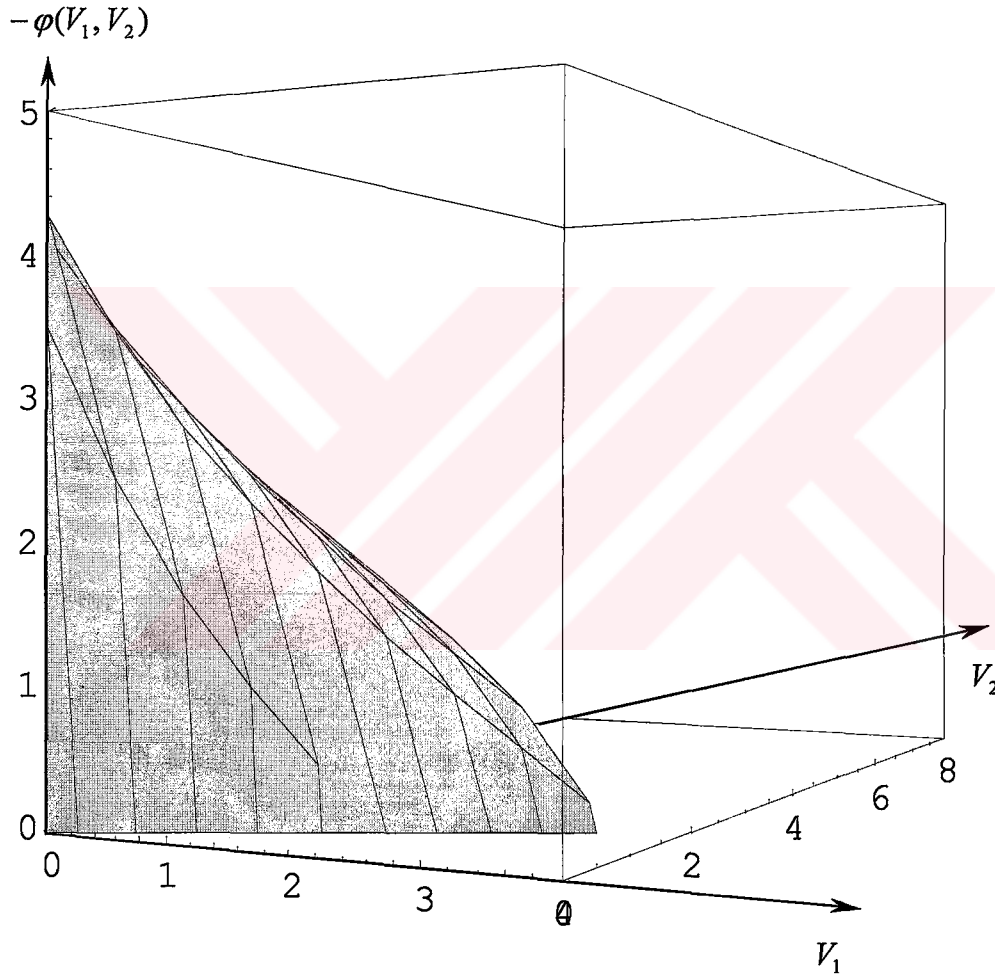


FIGURE 2.6 LIMITED ARBITRAGE OPPORTUNITIES

3. CONCLUSION AND RECOMMENDATIONS

We analyzed the implications of the no arbitrage condition from the perspective of modern finance. Arbitrage opportunities can be loosely defined as opportunities to make riskless profits on an arbitrarily large scale. Initially, arbitrage is analyzed in an abstract net trade exchange economy in the spirit of Kreps (1981). The results are extended to economies that incorporate uncertainty. The market structures employed were contingent claims markets and security-spot markets. We then focused on the issue of valuation of contingent claims. It is seen that in the absence of arbitrage, contingent claims can be priced by taking the expected value of their normalized payoff with respect to an equivalent martingale measure. If this value is unique, the claim is said to be priced by arbitrage and it can be perfectly hedged or duplicated by trading. Later, the implications of the no-arbitrage condition in markets with frictions are analyzed.

Non-linear markets are modeled by removing the linearity assumption on the asset pricing operator while retaining the linear space assumption for its domain. It is assumed that the set of marketed claims is a linear space and the pricing operator at which these claims are available is sublinear and hence convex. Employing sublinear pricing operators is justified because they are the reduced forms of multiperiod securities price models encompassing a larger class of market frictions that includes bid-ask spreads, short selling constraints and costs, borrowing constraints and costs and taxes. In those multiperiod models, the pricing operator is the result of a dynamic hedging process. In particular, a pricing operator assigns to each contingent claim a value that is equal to the minimum cost necessary to dominate that contingent claim and accordingly it is sublinear and hence convex.

In real financial markets, securities have bid and ask prices; the ask price being greater than the bid price to avoid arbitrage. Bid and ask prices are quoted by a specialist who matches buying and selling orders on each security. Agents buy securities from the specialist at ask prices and sell securities to the specialist at bid

prices. Bid-ask spreads are forms of transaction costs that are well characterized by the sublinearity assumption.

The assumption that there are no arbitrage opportunities has different implications for frictionless markets and for markets with frictions. In the absence of arbitrage, a number of interesting phenomena not observed in linear asset pricing systems appeared in non-linear systems. Obviously, in non-linear markets, it is not possible to derive a linear pricing rule or an equivalent martingale measure.

In frictionless markets as analyzed in Section 2.4, arbitrage free asset prices can be represented by the mathematical expectation of their discounted payoff with respect to a probability measure. The analysis of arbitrage in non-linear markets yields valuation bounds for assets as opposed to the arbitrage-free values as determined in a frictionless market. In frictional markets, arbitrage bounds can be computed, for arbitrary contingent claims, taking the expected value of their normalized payoff with respect to all the measures that characterize the absence of arbitrage opportunities. The arbitrage upper bound for a given contingent claim is equal to the minimum amount it costs to hedge it, taking the market frictions into account. The tightness of these arbitrage bounds depends on the specific assumptions about the pricing operators. In frictional markets, results may be derived by placing certain restrictions on the form of the pricing operators like the theorem of Jouini and Kallal (1999) given Section 2.5.

In linear markets, arbitrage opportunities are like money pumps. Once an arbitrage opportunity is present, non-satiable investors will take unlimited positions to obtain infinite gains from this opportunity. This is due to the linearity property of the pricing functionals. In non-linear asset markets, limited arbitrage opportunities may be present. This means that even when an arbitrage opportunity is present, this opportunity may not allow the investors to obtain unlimited gains. It may be wiped away by the non-linearity property of the pricing functional. It is seen that unlimited arbitrage can be operated on any scale desired, whereas limited arbitrage cannot.

Valuation of assets under the assumptions of no arbitrage has several practical applications. Determining the market price of financial innovations or the market prices of non-traded assets is one application. Black and Scholes Option Pricing

Model is the typical example for this. The existence of a definitive formula to value options has greatly enhanced the trading volume in the options market. Determining arbitrage-free prices are also critical for agents in detecting mispricings and benefiting from these mispricings. In real markets, market frictions such as transaction costs or short selling constraints are important facts of life for investors. The implicit assumption when ignoring transaction costs is that these costs are sufficiently small, so that they do not seriously affect the empirical results. However the effects of transaction costs may be exaggerated through frequent trading. Valuation of assets can be improved by incorporating non-linearities into the models.

Weakening the linearity assumption has implications for empirical analysis also. The fundamental assumption of linearity is not very well supported by empirical evidence as frictions play an important role in the functioning of markets. Incorporating these frictions into the model improves the fit between the observed financial data and the theoretical asset pricing models. It may also help to resolve some of the familiar anomalies in the finance literature. A different set of conclusions are drawn from observed securities prices when the underlying pricing operators are assumed to be non-linear as compared to the canonical assumption of linearity.

In the absence of arbitrage opportunities, there exists a stochastic discount factor that relates payoffs to market prices for all assets in the economy. As noted in Campbell (2000) the conditions for the existence of a stochastic discount factor are so general, they place almost no restrictions on financial data. The results obtained from an analysis of arbitrage in non-linear markets will make significant contributions to the theory of asset pricing which is mainly involved with the trade of between risk and return.

One issue that needs to be further analyzed is the link between the existence of limited arbitrage opportunities and the stability of markets. In normal functioning of financial markets, agents take positions in assets to take advantage of temporary arbitrage opportunities while driving prices into their arbitrage free values. As Hart (1977) noted, there is controversy among economists and noneconomists about this type of arbitrage activity. Most noneconomists find arbitrage activity socially undesirable as it introduces disturbance to the economy. Moreover, agents involved in such activity make profits at the expense of other smaller traders. In contrast,

economists usually view arbitrage activity as stabilizing since they believe that agents involved in this type of activity accelerate the transition between equilibrium states. In view of this, a fertile area of research is to investigate the aggregate effect of agents' arbitrage activity in the presence of frictions and limited arbitrage opportunities. A study of this type may illuminate whether arbitrage activity in actual markets has a destabilizing influence or not as real markets can be more realistically modeled by non-linear pricing operators than by their linear counterparts. It may also shed light on the role of arbitrage activity or portfolio insurance in financial crises like the 1987 crash.

As stated before arbitrage opportunities are inconsistent with equilibrium when the markets are assumed to be frictionless. However in non-linear markets, existence of arbitrage opportunities may be consistent with equilibrium. It seems that endogenizing these limited arbitrage opportunities in the theoretical model would represent a major development in modeling. Such a model may also explain the dynamics of the pricing operators in non-linear markets. In actual financial markets it is likely that some limited arbitrage opportunities may persist while others disappear in a dynamic fashion. Formulating the dynamics of the arbitrage trades within the model would greatly enhance our understanding of the dynamic economies.

One apparent shortcoming in our analysis is the sublinearity assumption for the asset pricing operator. Sublinear functions are subsets of convex functions. Assuming a reduced model of sublinear functions may seem prohibitively restrictive at first glance but noticing that the majority of the non-linear phenomena in real markets can be modeled by sublinear functions. However, this does not reduce the importance of a study of the relationship between different forms of non-linearity and asset prices. For example when transaction costs increase less than proportionally with transaction size, the pricing rule will be convex but not sublinear. Therefore, current analysis can be extended by encompassing other types of market frictions in the forms of different restrictions imposed on the pricing operator.

Drawing on the idea of the absence of arbitrage while generalizing it by incorporating real life phenomena as characterized by non-linear pricing operators will contribute to our overall understanding of price formation in financial markets. This type of analysis will have normative implications also, valuation of financial

innovations and non-traded assets being the primary examples. Introduction of financial innovations and opening of new markets will serve to achieve better allocation of risk sharing. We conjecture that elaboration of the arbitrage analysis in this regard will be one of the most productive research tasks of asset pricing and financial economics for some time to come.



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