

**RIESZ BASIS AND EIGENVALUE PROBLEMS
FOR ONE AND TWO PARAMETER SELF-ADJOINT
OPERATOR PENCILS**

**Ph.D Thesis by
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**BİR VE İKİ PARAMETRELİ
KENDİNE EŞ OPERATÖR FONKSİYONLAR İÇİN
RIESZ BAZI VE ÖZDEĞER PROBLEMLERİ**

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RIESZ BASIS AND EIGENVALUE PROBLEMS FOR ONE AND TWO PARAMETER SELF-ADJOINT OPERATOR PENCILS

SUMMARY

In the present study, Riesz basis and eigenvalue problems for one or two parameter self-adjoint operator pencils are considered. The study contains six main sections.

In the first section, a general introduction about operator functions, their spectral problems and methods used to solve them is given. The variational method is briefly discussed. Riesz basis problems for self-adjoint operator functions are introduced.

In the second section, the spectral structure of two parameter unbounded operator pencils of waveguide type is studied. Theorems on discreteness of the spectrum for a fixed parameter are proved. Variational principles for real eigenvalues in some parts of the root zones are established. For quadratic operator pencils of waveguide type domains containing the spectrum are described. Conditions in the definition of the pencils of waveguide type arise naturally from physical problems and each of them has a physical meaning. In particular a connection between the energetic stability condition and a perturbation problem for the coefficients is given.

In the third section, the structure of the numerical range and root zones of a class of operator functions, arising from one and two parameter polynomial operator pencils of waveguide type is studied. We construct a general model of such kind of operator pencils. In frame of this model Theorems on distribution of roots and eigenvalues in some parts of root zones are proved. It is shown that, in general the numerical range and root zones are not connected but some connected parts of root zones are determined. It is proved that root zones, under some natural additional conditions which are satisfied for most of waveguide type multi-parameter spectral problems, are non-separated, i.e. they overlap.

In the fourth section, Riesz basis properties for a class of self-adjoint and continuous operator functions are studied. A new approach based on the spectral distribution function is presented.

In the fifth section, spectral theory of two parameter quadratic operator pencils of waveguide type is applied to a partial differential equation and for the resulting operator pencil asymptotics of the spectral distribution function is given.

In the last section, conclusions of the study are given briefly.

BİR VE İKİ PARAMETRELİ KENDİNE EŞ OPERATÖR FONKSİYONLAR İÇİN RIESZ BAZI VE ÖZDEĞER PROBLEMLERİ

ÖZET

Bu çalışmada bir ve iki parametrelî kendine eş operatör fonksiyonların bazı sınıfları için Riesz bazı ve özdeğer problemleri ele alınmıştır. Çalışma altı bölümden oluşmaktadır.

İlk bölümde operatör değerli fonksiyonlar, spektral problemleri ve çözmek için kullanılan yöntemler hakkında genel bir giriş verilmiş ve varyasyon yöntemi kısaca açıklanmıştır. Kendine eş operatör fonksiyonlar için Riesz bazı problemi tanıtılmıştır.

İkinci bölümde, iki parametrelî sınırsız dalga tipi operatör polinomların spektral yapısı incelenmiştir. Bir parametrenin sabit tutulması durumunda spektrumun ayrık olması ile ilgili teoremler ispatlanmıştır. Kök bölgelerinin bazı kısımlarında reel özdeğerler için varyasyon ilkeleri verilmiştir. İkinci derece dalga tipi operatör polinomlar için spektrumu içeren bölgeler tanımlanmıştır. Dalga tipi operatör polinomların tanımındaki koşullar fiziksel problemlerden doğal olarak ortaya çıkar ve her birinin fiziksel bir anlamı vardır. Özel olarak enerji stabilite koşulu ile katsayılar için bir pertürbasyon problemi arasında bir bağlantı verilmiştir.

Üçüncü bölümde, bir ve iki parametrelî dalga tipi operatör polinomlardan ortaya çıkan bir operatör fonksiyonlar sınıfı için sayısal bölgenin ve kök bölgelerinin yapısı incelenmiştir. Bu tür operatör fonksiyonlar için genel bir model oluşturuyoruz. Bu model çerçevesinde kök bölgelerinin bazı kısımlarında köklerin ve özdeğerlerin dağılımıyla ilgili teoremler ispatlanmıştır. Genelde sayısal bölgenin ve kök bölgelerinin bağlantılı olmadığı gösterilmiş ve kök bölgelerinin bazı bağlantılı parçaları belirlenmiştir. Çok parametrelî dalga tipi operatör polinomların çoğu tarafından sağlanan bazı ek koşullar altında kök bölgelerinin ayrık olmadığı, yani üst üste bindiği ispatlanmıştır.

Dördüncü bölümde, kendine eş ve sürekli operatör fonksiyonların bir sınıfı için Riesz bazı özellikleri incelenmiştir. Burada spektral dağılım fonksiyonuna dayanan yeni bir yaklaşım kullanılmıştır.

Beşinci bölümde, ikinci derece iki parametrelî dalga tipi operatör polinomların spektral teorisi bir kısmı türevli diferansiyel denkleme uygulanmış ve denklemden elde edilen operatör polinom için spektral dağılım fonksiyonunun asimptotu verilmiştir.

Son bölümde, çalışmanın sonuçları kısaca değerlendirilmiştir.

1 INTRODUCTION

In this thesis we study Riesz basis and eigenvalue problems for some classes of one and two parameter operator pencils. Spectral problems for operator pencils arise naturally from differential equations and boundary value problems, evolution equations, block matrices, controllable systems and equations depending on one or more parameters. One can find examples on applications of the spectral theory of operator pencils to partial differential equations in domains with singularities on the boundary in [1].

A function $L(\lambda)$, defined on a subset U of real or complex numbers, whose values are operators is called operator pencil or operator function. Notice that the notion of operator function in general is used for nonpolynomial operator pencils. We also will often use this notion. The most important examples of operator pencils are one parameter polynomial operator pencils

$$L(\lambda) = \lambda^n A_n + \lambda^{n-1} A_{n-1} + \cdots + \lambda A_1 + A_0$$

and two parameter operator pencils of the form

$$\begin{aligned} L(\lambda, \mu) = & \lambda^n A_n + \lambda^{n-1} A_{n-1} + \cdots + \lambda A_1 + A_0 + \\ & + \mu^m B_m + \mu^{m-1} B_{m-1} + \cdots + \mu B_1 + B_0, \end{aligned}$$

where $A_0, \dots, A_n$ and $B_0, \dots, B_m$ are bounded or unbounded operators in a Hilbert or Banach space, λ and μ are spectral parameters. In the sequel we assume that H is a separable Hilbert space.

For one parameter operator pencils the spectrum $\sigma(L)$ and the point spectrum or the set of eigenvalues $\sigma_p(L)$ are defined as

$$\begin{aligned} \sigma(L) &= \{\lambda \mid 0 \in \sigma(L(\lambda))\}, \\ \sigma_p(L) &= \{\lambda \mid 0 \in \sigma_p(L(\lambda))\}. \end{aligned}$$

Here $\sigma(L(\lambda))$ refers to the spectrum of the operator $L(\lambda)$ which is the value of the operator pencil L at the point λ . For $\lambda \in \sigma_p(L)$ a nonzero vector x such that

$$L(\lambda)x = 0$$

is said to be an eigenvector. The set of regular points $\rho(L)$ is defined analogously. Note that in the case of $L(\lambda) = A - \lambda I$ these definitions coincide with corresponding definitions for the operator A . For more information about definitions of the spectrum of one or two parameter operator pencils refer to Section 2.

Spectral problems for operator functions can be divided into two groups. The first group consists of determining the spectrum, eigenvalues, spectral distribution function and boundaries for the spectrum, the second group includes problems concerning eigenvectors, their completeness and basis properties.

Here we shortly discuss main methods for solving such kinds of problems. One of these is the method of linearization in which different problems in the theory of operator pencils are reduced to the corresponding problems for a certain linear operator pencil acting in the space H^n . This method is analogous to the reduction of an n -th order differential equation to a system of n first order differential equations and it is applicable only to polynomial operator pencils.

For example, let $A_0, A_1, \dots, A_{n-1}$ be self-adjoint operators on a Hilbert space H . Then the operator pencil L defined by

$$L(\lambda) := \lambda^n I + \sum_{k=0}^{n-1} \lambda^k A_k$$

is called a monic self-adjoint operator polynomial. It is well known ([2]) that the spectrum of the operator

$$A = \begin{bmatrix} 0 & I & \cdots & 0 \\ 0 & 0 & \ddots & \vdots \\ \vdots & & & I \\ -A_0 & -A_1 & \cdots & -A_{n-1} \end{bmatrix}.$$

acting on H^n coincide with the spectrum of the operator pencil L . The operator A is called the companion operator of L and in this case it is self-adjoint in the

indefinite scalar product $[\cdot, \cdot] := (G\cdot, \cdot)$ where $(\cdot, \cdot)$ is the scalar product of Hilbert space H^n and

$$G = \begin{bmatrix} A_1 & \cdots & A_{n-1} & I \\ & & I & 0 \\ \vdots & \ddots & \ddots & \vdots \\ A_{n-1} & I & & \\ I & 0 & \cdots & 0 \end{bmatrix}.$$

Thus by using the linearization method spectral problems for operator pencils are reduced to spectral problems for self-adjoint operators in the indefinite scalar product space $\langle H^n, (G\cdot, \cdot) \rangle$. Such kind of spaces are often called Krein spaces (see [3], [5] and [6]). Spectral problems for nonmonic operator pencils are reduced to spectral problems for linear operator pencils $\tilde{L}(\lambda) = \tilde{A} - \lambda\tilde{B}$ in a Krein space.

Another method is based on the factorization of the operator pencils. The main idea applied here is as follows: Let $A(\lambda)$, $B(\lambda)$ and $C(\lambda)$ holomorphic operator functions defined in a domain U of the complex plane $\mathbb{C}$, whose values are bounded operators and

$$A(\lambda) = B(\lambda)C(\lambda),$$

then there are connections between spectral properties of $A(\lambda)$ and $C(\lambda)$. For example all boundary points of $\sigma(C)$ are in $\sigma(A)$, but in general $\sigma(C) \not\subset \sigma(A)$ (see [2], p. 113). In the decomposition above, if $A(\lambda)$, $B(\lambda)$ and $C(\lambda)$ are operator polynomials and $C(\lambda)$ is monic then $C(\lambda)$ is called a divisor (or right divisor) of $A(\lambda)$. If in addition the spectra of $B(\lambda)$ and $C(\lambda)$ are disjoint, then $C(\lambda)$ is called a spectral divisor (or right spectral divisor) of $A(\lambda)$. In this case $\sigma(A) = \sigma(B) \cup \sigma(C)$, moreover a number $\lambda_0 \in \sigma(C)$ is an eigenvalue of $C(\lambda)$ if and only if it is an eigenvalue of $A(\lambda)$ and the eigenvectors of $C(\lambda)$ and $A(\lambda)$ corresponding to λ_0 coincide. The case of $C(\lambda) = \lambda I - Z$ is of special interest since then the study of some part of the spectrum of the pencil $A(\lambda)$ is reduced to study of the spectral properties of the operator Z .

There are also analytical methods which are based on the estimates of the resolvent.

Finally, we briefly discuss the variational method which is the main subject of

this thesis. It is well known that discrete eigenvalues of a self-adjoint operator A on a Hilbert space H which lie below or above the limit spectrum (or essential spectrum) of A can be characterized by three fundamental variational principles, namely by Rayleigh's principle, by the Poincaré-Ritz minmax principle and by the Courant-Fischer-Weyl principle applied to the Rayleigh quotients $p(x) = \frac{(Ax, x)}{(x, x)}$, $x \in H$, $x \neq 0$.

For example, if the eigenvalues below the minimum of the limit spectrum of A are denoted by

$$\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n \leq \dots,$$

counted according to their multiplicities, then these eigenvalues can be characterized by Rayleigh's principle,

$$\lambda_n = \min_{\substack{x \neq 0, (x, x_i) = 0 \\ i=1, \dots, n-1}} p(x) \quad (1.1)$$

where x_i are eigenvectors corresponding to the eigenvalues λ_i and the minimum is attained at the eigenvector x_n , or by the Poincaré-Ritz principle,

$$\lambda_n = \min_{\substack{L \subset D(A) \\ \dim L = n}} \max_{\substack{x \in L \\ x \neq 0}} p(x), \quad (1.2)$$

or by the Courant-Fischer-Weyl variational principle,

$$\lambda_n = \max_{\substack{L \subset H \\ \dim L = n-1}} \min_{\substack{x \in D(A) \\ x \perp L}} p(x). \quad (1.3)$$

These variational principles were generalized to the λ -nonlinear eigenvalue problems of the form $L(\lambda)x = 0$ by many authors, where the Rayleigh quotient $p(x) = \frac{(Ax, x)}{(x, x)}$ of a linear problem $Ax = \lambda x$ was replaced by the so-called Rayleigh functional p , which is a homogeneous, real-valued, nonlinear functional defined by the equation $(L(p(x))x, x) = 0$, for $x \neq 0$. Note that for a λ -nonlinear eigenvalue problem in general there are several functionals satisfying $(L(p(x))x, x) = 0$ and each of these functionals enables us to describe only a part of the spectrum. Namely, if we define a functional by the equation $(L(p(x))x, x) = 0$, $0 \neq x \in H$ then the part of the spectrum of L in $\overline{W_p}$, where $W_p := \{p(x) \mid x \in D(p)\}$, can be characterized by the functional $p(x)$. Here the set W_p is called the numerical range (or the root zone) and $D(p)$ denotes the domain of $p(x)$ which may be

different from whole space H . For example, there are n Rayleigh functionals $p_1(x), p_2(x), \dots, p_n(x)$ corresponding to

$$L(\lambda) = \lambda^n A_n + \lambda^{n-1} A_{n-1} + \dots + \lambda A_1 + A_0.$$

The main difficulties arising in the spectral theory, especially in the variational theory of the spectrum of operator pencils are explained by the fact that the numerical ranges $W_{p_1}, W_{p_2}, \dots, W_{p_n}$ may be disjoint (the classical case) or overlapping (see [4]). If $A_i = A_i^*$, $A_n \gg 0$, $W_{p_i} \cap W_{p_j} \neq \emptyset$ and contains more than one point then this means that some roots of the equation $(L(\lambda)x, x) = 0$ are complex (see [2], § 31) so some of the functionals $p_i(x)$ are not defined on whole space H but on a conic subset of it. This is the case we are interested in. Thus difficulties arising in λ -nonlinear spectral problems definitely depend on the distribution of roots of the equation $(L(\lambda)x, x) = 0$.

To explain better different cases of the distribution of roots we give here an example. Take

$$L(\lambda) = \lambda^2 A_2 + \lambda A_1 + A_0$$

where

$$A_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}, \quad A_0 = \begin{bmatrix} a & -1 \\ -1 & a \end{bmatrix},$$

a is real parameter and let $H = \mathbb{R}^2$ where $\mathbb{R}$ is the real line. Note that $A_i = A_i^*$, $i = 0, 1, 2$ and $A_2 \gg 0$. If we choose $a = -2$ then all the roots of the equation $(L(\lambda)x, x) = 0$ are real and distinct for all $0 \neq x \in H$. In this case the operator pencil is called hyperbolic (see [2]). Let $p_1(x)$ and $p_2(x)$ denote these roots in nonincreasing order, then p_1 and p_2 are defined on the whole space H and the numerical ranges W_{p_1} and W_{p_2} are disjoint as it is seen in the following figure.

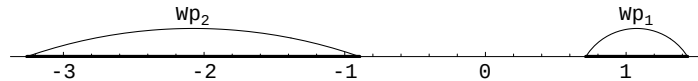


Figure 1.1: The numerical ranges W_{p_1} and W_{p_2} for $a = -2$.

If $a = -1$ then all the roots are real but not necessarily distinct. In this case the operator pencil is called weakly hyperbolic. If we denote the roots as above,

taking the multiplicities into account, again the functionals p_1 and p_2 are defined on whole space H but W_{p_1} and W_{p_2} contains one point in common as can be seen below.

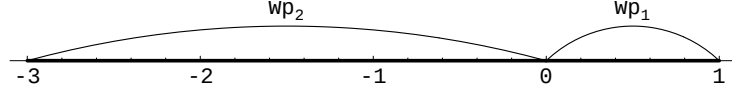


Figure 1.2: The numerical ranges W_{p_1} and W_{p_2} for $a = -1$.

Note that both hyperbolic and weakly hyperbolic cases are classical ones and variational theory of the spectrum is well known in these cases (see [2], [7], [8], [9]).

If $a = 0$ then for some $x \in H$ the equation has two complex roots, consequently the functionals p_1 and p_2 are defined only on a conic subset of H and the numerical ranges W_{p_1} and W_{p_2} overlap as it is seen in the following figure.

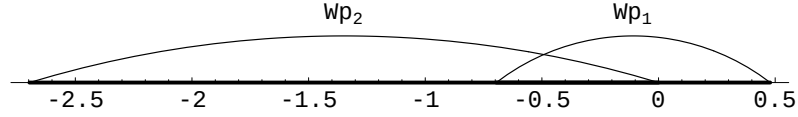


Figure 1.3: The numerical ranges W_{p_1} and W_{p_2} for $a = 0$.

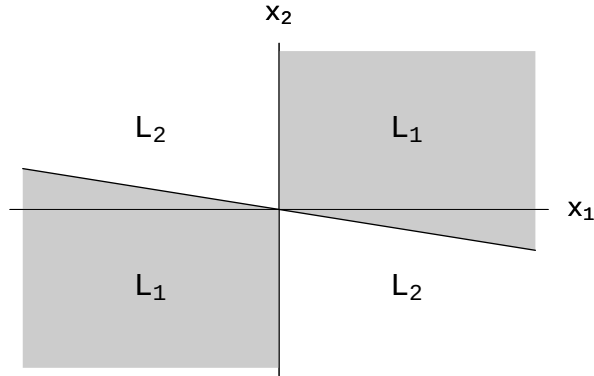


Figure 1.4: The domain of p_1 and p_2 for $a = 0$.

A simple computation shows that if $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in L_1$ then there are two real roots and if $x \in L_2$ there are two complex roots. Thus $D(p_1) = D(p_2) = L_1$.

As we can see in the example above there are two cases in the variational theory of λ -nonlinear spectral problems:

- A) The Rayleigh functional $p(x)$ is defined on the entire space $H \setminus \{0\}$. In this case the λ -nonlinear eigenvalue problem

$$L(\lambda)x = 0$$

is called overdamped. Methods applied in this case depend on whether the operator function $L(\lambda)$, $\lambda \in \Delta = [\alpha, \beta[$ is polynomial, analytic, smooth or nonsmooth. On this subject the classical works of R. J. Duffin, R. Turner, E. H. Rogers, B. Werner [9] should be mentioned. See also the books [2], [7] and the paper [8].

- B) The Rayleigh functional $p(x)$ is defined only on a proper conic subset of $H \setminus \{0\}$ and the conditions given in the above mentioned works are partially fulfilled. These conditions are:

- a) $L(\alpha)$ is uniformly positive definite, i. e., $L(\alpha) \gg 0$,
- b) $L(\beta) \ll 0$,
- c) the equation $(L(\lambda)x, x) = 0$ has exactly one simple zero $p(x)$ in $] \alpha, \beta[$ for every $x \in H \setminus \{0\}$.

Such spectral problems are called the nonoverdamped ones and they are the main subject of the thesis.

Notice that recent studies are concentrated on the case B) (see [10], [11], [12], [13], [14], [15], [16], [17], [18], [19], [20], [21]). The operator pencils of waveguide type form one of the main classes of nonoverdamped operator pencils and have very important applications to physical problems (see [10], [11], [17], [22], [23], [24] and [25]). E. M. Barston [13] considered a class of nonoverdamped quadratic eigenvalue problem and gave variational characterizations for real eigenvalues in the finite dimensional case. It seems that H. Voss and B. Werner [19] first established minmax principles for nonlinear nonoverdamped eigenvalue problems in the infinite dimensional case (see also [20] and [21]). In these papers the variational principles are used to locate the eigenvalues of a rational eigenvalue problem arising in the vibrations of interacting fluid-solid structures ([20], [21]).

A wide class of nonoverdamped spectral problems are connected with spectral problems of block operator matrices and λ -rational Sturm-Liouville problems

(see [14], [15], [26] and [27]). More general operator functions of nonoverdamped type defined on the interval $[\alpha, \beta[\subset \mathbb{R}$ with values in the class of bounded self-adjoint operators were studied in [14], where the classical conditions a), b) and c) given above for overdamped spectral problems in general are not satisfied (see also [28]).

It follows from the condition a) that

$$\kappa_\alpha := \dim\{E \mid 0 \neq x \in E, (L(\alpha)x, x) < 0\} = 0.$$

That is why the index shift does not occur in the variational principles for overdamped spectral problems. It was shown in [14] (see Theorem 3.5) that for $\kappa_\alpha > 0$ the classical variational principle (1.3) is replaced by

$$\lambda_n = \min_{\substack{L \\ \dim L = n + \kappa_\alpha}} \sup_{\substack{x \in L \\ x \neq 0}} p(x), \quad n = 1, 2, \dots$$

The same formula together with an analog of the Courant-Fischer-Weyl principle was established in [15] (see p. 293, Theorem 2.1) for operator pencils with values in the class of self-adjoint unbounded operators. In general operator pencils arise from differential operators and therefore they are unbounded operator functions. On the other hand, there are standard techniques to transform some unbounded operator pencils into bounded ones. But in some cases it is better to deal with unbounded pencils, particularly for pencils of waveguide type. For this reason in Section 2 we use methods and results given in the paper [15] to establish variational principles for real eigenvalues.

Although nonoverdamped pencils to be studied in this thesis come from waveguiding systems, there are many examples of polynomial or nonpolynomial operator pencils of nonoverdamped type. Here we give an example from block operator matrices. Let $H = H_1 \oplus H_2$, where H_1 and H_2 are Hilbert spaces. Let $A : H_1 \rightarrow H_1$, $B : H_2 \rightarrow H_1$ and $D : H_2 \rightarrow H_2$ bounded operators and $A = A^*$, $B = B^*$. Consider the block operator matrix

$$\tilde{A} = \begin{pmatrix} A & B \\ B^* & D \end{pmatrix}$$

which is a self-adjoint operator on the space H . The spectrum (the point spectrum) of $\tilde{A}$ outside $\sigma(D)$ coincides with the spectrum (point spectrum) of

the operator pencil

$$L(\lambda) := A - \lambda I - B(D - \lambda I)^{-1}B^*$$

(see [15], [18], [29], [30]). So we can investigate $\sigma(\tilde{A})$ by studying the spectrum of the pencil L which is in general a nonoverdamped spectral problem.

Another problem to be studied in this thesis is the Riesz basis problem for a class of self-adjoint operator functions. Here we give the definition of orthonormal and Riesz bases.

Definition 1.1. A system of vectors $\{u_n\}_{n=1}^{\infty}$ of the Hilbert space H is called an orthonormal basis if the following conditions are satisfied:

- i) for every $x \in H$ there is a unique representation $x = \sum_{n=1}^{\infty} x_n u_n$,
- ii) $(u_i, u_j) = \begin{cases} 1, & i = j, \\ 0, & i \neq j. \end{cases}$

Sometimes the condition ii) is not satisfied but there is an invertible operator A such that $\{Au_n\}_{n=1}^{\infty}$ is orthonormal. In this case the system $\{u_n\}_{n=1}^{\infty}$, satisfying the condition i) is called a Riesz basis.

The most important classical theorem about orthonormal basis in the spectral theory of self-adjoint operators is the following

Theorem 1.1 ([31], [32]). There is an orthonormal basis, consisting of eigenvectors of the self-adjoint compact operators.

This theorem has important applications in the theory of differential and integral equations. Note that if a compact operator is not self-adjoint but it is similar to self-adjoint one then clearly eigenvectors form a Riesz basis. Evidently the notion of Riesz basis comes from operators similar to self-adjoint.

An analog of this theorem in the spectral theory of self-adjoint operator functions is given in Corollary 1.1 of the following

Theorem 1.2 ([2]). Let $[a, b]$ be an interval of the real axis, U be a connected neighborhood of it which is symmetric with respect to $\mathbb{R}$, and $L(\lambda)$ be a

self-adjoint holomorphic operator function on U . If $L(a) \ll 0$, $L(b) \gg 0$ and the function $(L(\lambda)x, x)$, $x \neq 0$ has exactly one root in U , then $L(\lambda)$ admits a factorization

$$L(\lambda) = L_+(\lambda)(\lambda I - Z)$$

where the operator function $L_+(\lambda)$ is holomorphic and invertible in U , Z is a bounded operator and $\sigma(Z) \subset (a, b)$. Further, Z is similar to a self-adjoint operator.

Corollary 1.1 ([2]). Let $\pi(L) = \{\gamma\}$ for some $\gamma \in (a, b)$ then under the conditions of Theorem 1.2 eigenvectors of L corresponding to eigenvalues from the interval (a, b) form a Riesz basis.

In the sequel $\pi(L)$ denotes the limit spectrum of the operator function L , i.e.,

$$\pi(L) = \{\lambda \in (a, b) \mid \exists x_n, \|x_n\| = 1, x_n \xrightarrow{w} 0, L(\lambda)x_n \rightarrow 0\}. \quad (1.4)$$

(Note that it is sometimes denoted by $\sigma_{ess}(L)$ and is called essential spectrum). If $\pi(L) = \{\gamma\} \in (a, b)$ then the operator $Z_\gamma := Z - \gamma I$ is similar to a self-adjoint operator and since $\pi(Z_\gamma) = \{0\}$ it is compact (see [31], p. 210, Theorem 1). Moreover the eigenvectors of Z_γ and Z are the same. Now the Riesz basis property pointed out in Corollary 1.1 follows from Theorem 1.1.

A similar result via a factorization formula for a class of nonanalytic operator functions is given by

Theorem 1.3 ([33]). Let $L \in C^2([a, b], S(H))$ and the conditions:

- i) $L(a) \ll 0$, $L(b) \gg 0$,
- ii) $\int_0^{t_0} \frac{w(t, L'')}{t} dt < +\infty$ for sufficiently small t_0 , where $w(t, L'')$ is the modulus of continuity for L'' ,
- iii) the operator function L satisfies the regularity condition i.e., there exist positive numbers δ and ϵ such that for every $\lambda \in [a, b]$ and $x \in H$, $\|x\| = 1$,
 $|(L(\lambda)x, x)| < \epsilon \Rightarrow (L'(\lambda)x, x) > \delta$

are satisfied. Then L admits a factorization of the form $L(\lambda) = B(\lambda)(\lambda I - Z)$, where $B(\lambda)$ is a continuous and invertible operator function on $[a, b]$, $\sigma(Z) \subset (a, b)$, Z is bounded and is similar to a self-adjoint operator.

By $C^k([a, b], S(H))$ we denote the class of k -times continuously differentiable self-adjoint operator functions defined on the interval $[a, b]$.

A similar factorization theorem was given in the paper of Azizov, Dijkma and Sukhocheva [34]. This paper also contains some sufficient conditions on L under which the closed linear span of all eigenvectors, corresponding to eigenvalues from $[a, b]$ has a Riesz basis consisting of eigenvectors of L (see [35], also Matsaev and Spiegel [36]).

The problem about Riesz basis properties of the eigenvectors of an operator function of the class $C([a, b], S(H))$ satisfying the conditions given in Theorem 1.2 is still open. More precisely, there is a hypothesis that if $L(\lambda)$ is an operator function of the class $C([a, b], S(H))$ such that $L(a) \ll 0$, $L(b) \gg 0$, for all $x \in H \setminus \{0\}$ the function $(L(\lambda)x, x)$ has exactly one zero in (a, b) and $\pi(L) = \{\gamma\} \in (a, b)$ then the eigenvectors of L , corresponding to eigenvalues in (a, b) form a Riesz basis for the Hilbert space H or they are complete in H .

Note that in all of above mentioned papers the basis properties of the eigenvectors of the operator functions are studied using the factorization method which requires some smoothness of the operator function L . In Section 4 we study the same problem for a subclass $C([a, b], S(H))$ using a new variational approach based on the spectral distribution function.

The thesis is organized as follows: In Section 2 the spectral structure of two parameter unbounded operator pencils of waveguide type is studied. Theorems on discreteness of the spectrum for a fixed parameter are proved (Theorems 2.2 and 2.3). Variational principles for real eigenvalues in some parts of the root zones are established (Theorems 2.4 and 2.5). In the case of quadratic pencils domains containing the spectrum are described (see Fig. 2.1–2.3). Conditions in the definition of the pencils of waveguide type arise naturally from physical problems and each of them has a physical meaning. In particular a connection between the energetic stability condition and a perturbation problem for the coefficients is given (Theorem 2.1).

In Section 3 the structure of the numerical range and root zones of a class of operator functions, arising from one or two parameter polynomial operator pencils

of waveguide type is studied. We construct a general model of such kind of operator pencils. In frame of this model Theorems on distribution of roots and eigenvalues in some parts of root zones are proved (Theorems 3.2 and 3.3). It is shown that, in general the numerical range and root zones are not connected (see Fig. 3.3) but some connected parts of root zones are determined (Theorem 3.4). It is proved that root zones, under some natural additional conditions which are satisfied for most of waveguide type multi-parameter spectral problems, are non-separated, i.e. they overlap (Theorem 3.5).

In Section 4 it is shown that the above mentioned hypothesis about eigenvectors of operator functions of the class $C([a, b], S(H))$ holds on a dense subspace.

Finally in Section 5 spectral theory of two parameter quadratic operator pencils of waveguide type is applied to a partial differential equation and for the resulting operator pencil asymptotics of the spectral distribution function is given.

2 EIGENVALUES OF TWO PARAMETER POLYNOMIAL OPERATOR PENCILS OF WAVEGUIDE TYPE

2.1 Introduction

In this section we study spectral properties including variational principles for real eigenvalues of operator pencils in the form

$$L(k, w) := A + \sum_{s=1}^n k^{2s} C_{s-1} + \sum_{s=0}^{n-1} k^{2s+1} B_s + iwD - w^2 I, \quad (2.1)$$

where A, D, B_s and C_s , $s = 0, 1, \dots, n-1$ are symmetric operators in a Hilbert space H and all but C_{n-1} may be unbounded. In addition we assume that all conditions given in Definition 2.1 are satisfied. This class is called operator pencils of waveguide type (w.g.t.) (see [11] and [37]).

Note that such operator pencils arise from waveguiding systems given in a Hilbert

space H by the following dynamical equation (see [17], [23], [24] and [25])

$$V_{tt} + DV_t + \sum_{s=1}^n (-1)^s C_{s-1} \frac{\partial^{2s} V}{\partial x^{2s}} + \sum_{s=0}^{n-1} (-1)^s B_s \frac{\partial^{2s+1} V}{\partial x^{2s+1}} + AV = 0, \quad (2.2)$$

where $V(x, t) : \mathbb{R} \times \mathbb{R} \rightarrow H$ is a smooth vector-valued function.

Indeed looking for special guided waves in the form $V(t, x) = ue^{i(wt-kx)}$, $u \in H$ of the evolution equation (2.2) we obtain a two parameter eigenvalue problem

$$L(k, w)u = 0,$$

where $L(k, w)$ is the operator pencil (2.1), the parameter k is the wavenumber and w is the frequency. Remark that $D = 0$ in most cases of waveguiding systems. The operator D is involved in the systems accounting for energy dissipation (see [25], p. 287). Although some theorems in this thesis will be given for pencils with $D \neq 0$, we assume $D = 0$ for some parts of the results .

Define the following spectral sets:

$$\begin{aligned} \sigma_p(L) &= \{(k, w) \mid 0 \in \sigma_p(L(k, w))\}, \\ \sigma(L) &= \{(k, w) \mid 0 \in \sigma(L(k, w))\} \end{aligned}$$

which are called the point spectrum and the spectrum, respectively. The set of regular points $\rho(L)$ and the limit spectrum $\pi(L)$ are defined analogously. Note that $0 \in \pi(L(k, w))$ means that there exists a sequence x_n such that $\|x_n\| = 1$, $x_n \xrightarrow{w} 0$ and $L(k, w)x_n \rightarrow 0$. Besides, the set of eigenvalues corresponding to a given value of w and the set of eigenfrequencies corresponding to a fixed value of the wavenumber k are respectively defined by

$$\begin{aligned} \sigma_p^w(L) &= \{k \mid (k, w) \in \sigma_p(L)\}, \\ \sigma_p^k(L) &= \{w \mid (k, w) \in \sigma_p(L)\}. \end{aligned}$$

The spectral sets $\sigma_w(L)$, $\sigma_k(L)$, $\pi_w(L)$, $\pi_k(L)$ are defined analogously. The main concern of the section is $\sigma_p^w(L)$, $\sigma_p^k(L)$ and variational characterizations of the eigenvalues in $\sigma_p^w(L)$.

The quadratic operator pencils of w.g.t. has been studied by several authors. In the book [25] one can find information about general structure of the spectrum,

for example discreteness of $\sigma_w(L)$ and $\sigma_k(L)$, domains containing the spectrum $\sigma_w(L)$ and asymptotics of the eigenvalues $w_n(k)$, $k \in \mathbb{R}$. Variational principles for eigenvalues of quadratic operator pencils in some parts of the spectrum have been given in the papers [10] and [11]. In the same class Kostyuchenko and Orazov ([17]) have studied general structure of the spectrum and completeness of root vectors by transforming them to operator pencils of Krein-Langer type (see [17]). They also have studied behavior of the eigenvalues under the variation of the parameter w .

Operator pencils of waveguide type belong to the class of nonoverdamped pencils. In the case of two parameter operator pencils of waveguide type, for fixed $w \in \mathbb{R}$, through the real roots of the equation $(L(k, w)x, x) = 0$ for $x \in H \setminus \{0\}$ we define functionals $p_i(x, w)$, $i = 1, 2, \dots, 2n$. Note that these functionals in general are not defined on whole space H but on some conic proper subsets of the space. We give variational principles in some parts of the spectrum where numerical ranges corresponding to functionals p_i and p_j are similar to the case given in Figure 1.3. For this purpose, as mentioned in the introduction, we use methods and results for operator pencils with unbounded coefficients given in the paper [15]. It seems that this is the first study on the spectral theory of two parameter polynomial operator pencils of w.g.t.

In Subsection 2.2 we give the definition of pencils of waveguide type and study general spectral properties of these pencils. A theorem (see Theorem 2.1) explaining the meaning of the energetic stability condition in Definition 2.1 is proved. For quadratic pencils of waveguide type domains containing $\sigma_w(L)$ and the set of real pairs in $\sigma(L)$ are given in Figures 2.1–2.3.

Variational principles for real eigenvalues in $\sigma_p^w(L)$ are established in Subsection 2.3. A conclusion about the motion of eigenvalues is given at the end of this subsection.

2.2 The Spectral Structure of Pencils of Waveguide Type

In this subsection we define operator pencils of waveguide type and study their spectral structure. Particularly, we prove that the spectra $\sigma_w(L)$ and $\sigma_k(L)$ are

discrete under the conditions given in Definition 2.1. We begin with the following

Definition 2.1. A two parameter operator pencil $L(k, w)$ of the form (2.1) is said to be an operator pencil of waveguide type (w.g.t) iff

(A.I) A is a self-adjoint nonnegative operator satisfying $(A + I)^{-1} \in S_\infty$ and D is a symmetric operator which satisfies $D(A + I)^{-1/2} \in S_\infty$, where S_∞ is the set of compact operators,

(A.II) C_{n-1} is a bounded and positive definite operator:

$$c_1^2(u, u) \leq (C_{n-1}u, u) \leq c_2^2(u, u), \quad u \in H \text{ and } 0 < c_1 \leq c_2,$$

(A.III) The operators B_s , $s = 0, 1, \dots, n-1$ and C_s , $s = 0, 1, \dots, n-2$ are symmetric and $(A+I)^{-1/2}B_s(A+I)^{-1/2} \in S_\infty$ and $(A+I)^{-1/2}C_s(A+I)^{-1/2} \in S_\infty$. Particularly, these conditions mean $D((A + I)^{1/2}) \subset D(B_s)$, $s = 0, 1, \dots, n-1$ and $D((A + I)^{1/2}) \subset D(C_s)$, $s = 0, 1, \dots, n-1$.

(A.IV) There exists a number $\mu \geq 0$ such that for all $k \in \mathbb{R}$ and $u \in D((A + I)^{1/2})$ the following inequality holds:

$$(Au, u) + \sum_{s=1}^n k^{2s}(C_{s-1}u, u) + \sum_{s=0}^{n-1} k^{2s+1}(B_s u, u) \geq \mu^2(u, u).$$

In addition we say that a two parameter pencil of w.g.t satisfies the energetic stability condition if the following condition is fulfilled:

(A.V) There exist real numbers $\zeta \geq 0$ and $c_0 > 0$ such that for all $k \in \mathbb{R}$ and all $u \in D((A + I)^{1/2})$

$$(Au, u) + \sum_{s=1}^n k^{2s}(C_{s-1}u, u) + \sum_{s=0}^{n-1} k^{2s+1}(B_s u, u) \geq (c_0^{2n}k^{2n} + \zeta)(u, u).$$

As mentioned above the conditions (A.I)-(A.V) arise naturally in physical problems and they are satisfied for a wide class of waveguides (see [17], [25] and [37]).

Now we prove a theorem explaining the exact meaning of the energetic stability condition. Namely it means that the energy positivity condition (A.IV) will be satisfied if the operator C_{n-1} is replaced by operators εC_{n-1} for some $\varepsilon \in (0, 1)$.

Theorem 2.1. Let C_{n-1} be a bounded positive definite operator, i.e. the condition (A.II) is satisfied. Then the condition (A.V) is equivalent to the existence of a real number ε , $0 < \varepsilon < 1$ such that for all $k \in \mathbb{R}$ and all $u \in D((A + I)^{1/2})$

$$(Au, u) + \sum_{s=1}^{n-1} k^{2s} (C_{s-1}u, u) + \sum_{s=0}^{n-1} k^{2s+1} (B_s u, u) + \varepsilon^{2n} k^{2n} (C_{n-1}u, u) \geq \zeta(u, u),$$

where ζ is the same as in the condition (A.V).

Proof. Let us use the notation

$$\begin{aligned} A(k) &:= A + \sum_{s=1}^n k^{2s} C_{s-1} + \sum_{s=0}^{n-1} k^{2s+1} B_s, \\ A_\varepsilon(k) &:= A + \sum_{s=1}^{n-1} k^{2s} C_{s-1} + \sum_{s=0}^{n-1} k^{2s+1} B_s + \varepsilon^{2n} k^{2n} C_{n-1}. \end{aligned}$$

Assume that there exist numbers ε , $0 < \varepsilon < 1$ and ζ such that for all $k \in \mathbb{R}$ and $u \in D((A + I)^{1/2})$

$$(A_\varepsilon(k)u, u) \geq \zeta(u, u).$$

Then it follows that

$$(A(k)u, u) \geq \zeta(u, u) + (1 - \varepsilon^{2n})k^{2n} (C_{n-1}u, u).$$

Using the condition (A.II) we get

$$(A(k)u, u) \geq [\zeta + (1 - \varepsilon^{2n})c_1^2 k^{2n}](u, u).$$

Consequently

$$(A(k)u, u) \geq [\zeta + c_0^{2n} k^{2n}](u, u),$$

where $c_0^{2n} = (1 - \varepsilon^{2n})c_1^2$.

Now let us assume that there exist numbers $c_0 > 0$ and $\zeta \geq 0$ such that for all $k \in \mathbb{R}$ and $u \in D((A + I)^{1/2})$

$$(A(k)u, u) \geq (c_0^{2n} k^{2n} + \zeta)(u, u).$$

Then it follows that

$$(A_\varepsilon(k)u, u) \geq (c_0^{2n} k^{2n} + \zeta)(u, u) - (1 - \varepsilon^{2n})k^{2n} (C_{n-1}u, u).$$

Let $0 < \varepsilon < 1$, then using the condition (A.II) we can write

$$(A_\varepsilon(k)u, u) \geq [\zeta + (c_0^{2n} - (1 - \varepsilon^{2n})c_2^2)k^{2n}](u, u).$$

We can choose ε such that $1 - \frac{c_0^{2n}}{c_2^2} < \varepsilon^{2n} < 1$ then $c_0^{2n} - (1 - \varepsilon^{2n})c_2^2$ will be positive so we get

$$(A_\varepsilon(k)u, u) \geq \zeta(u, u).$$

□

Now we prove a theorem on discreteness of $\sigma_w(L)$. Here we use a standard technique from the spectral theory of operator pencils (see [2] and [32]).

Theorem 2.2. Let $L(k, w)$ be an operator pencil of w.g.t satisfying the energetic stability condition; then for all $w \in \mathbb{R}$ the spectrum $\sigma_w(L)$ is discrete, i.e. $\sigma_w(L)$ consists of isolated eigenvalues of finite multiplicity. Moreover, if $D = 0$ then $\sigma_w(L)$ is discrete for all $w \in \mathbb{C}$.

Proof. Let the eigenvalues of the operator A be $0 \leq \nu_1^2 \leq \nu_2^2 \leq \dots \leq \nu_n^2 \leq \dots$ where the multiplicities are taken into account. From the condition (A.I) we have $(A + I)^{-1} \in S_\infty$ and $A + I$ is a self-adjoint operator. Consequently, there is an orthonormal basis consisting of eigenfunctions φ_n , where $A\varphi_n = \nu_n^2\varphi_n$, $n = 1, 2, \dots$.

We start with the case where $w \neq \pm\nu_n$, $n = 1, 2, \dots$. Then by the condition (A.I) the operator $A - w^2I$ is invertible. For arbitrary $w \in \mathbb{C}$, we define a square root of the operator $A - w^2I$ by specifying it on the elements of the orthonormal basis $\{\varphi_n\}_{n=1}^\infty$ by the equalities

$$(A - w^2I)^{1/2}\varphi_n = \sqrt{\nu_n^2 - w^2}\varphi_n, \quad n = 1, 2, \dots,$$

where the principal branch of the square root of a complex number is chosen. By the condition (A.I), for $w^2 \neq \nu_n^2$ the operator $(A - w^2I)^{-1/2}$ is defined and

$$(A - w^2I)^{-1/2}\varphi_n = \frac{1}{\sqrt{\nu_n^2 - w^2}}\varphi_n, \quad n = 1, 2, \dots$$

Consequently it is a compact operator.

Now we define a new operator pencil

$$\begin{aligned}\tilde{L}(k, w) &:= (A - w^2 I)^{-1/2} L(k, w) (A - w^2 I)^{-1/2} \\ &= I + iw\tilde{D}(w) + \sum_{s=1}^n k^{2s} \tilde{C}_{s-1}(w) + \sum_{s=0}^{n-1} k^{2s+1} \tilde{B}_s(w),\end{aligned}$$

where

$$\begin{aligned}\tilde{B}_s(w) &= (A - w^2 I)^{-1/2} B_s (A - w^2 I)^{-1/2}, \\ \tilde{C}_{s-1}(w) &= (A - w^2 I)^{-1/2} C_{s-1} (A - w^2 I)^{-1/2}, \\ \tilde{D}(w) &= (A - w^2 I)^{-1/2} D (A - w^2 I)^{-1/2}.\end{aligned}$$

Note that $\sigma_w(L) = \sigma_w(\tilde{L})$. We can rewrite $\tilde{B}_s(w)$ and $\tilde{C}_{s-1}(w)$ in the form

$$\begin{aligned}\tilde{B}_s(w) &= K_1 (A + I)^{-1/2} B_s (A + I)^{-1/2} K_2, \\ \tilde{C}_{s-1}(w) &= K_1 (A + I)^{-1/2} C_{s-1} (A + I)^{-1/2} K_2.\end{aligned}$$

where

$$\begin{aligned}K_1 &= (A - w^2 I)^{-1/2} (A + I)^{1/2}, \\ K_2 &= (A + I)^{1/2} (A - w^2 I)^{-1/2}.\end{aligned}$$

It is clear that K_2 is a bounded operator. To prove that K_1 is a bounded operator we use the spectral expansion of K_1 .

Evidently,

$$D[(A + I)^{1/2}] = \left\{ x = \sum_{n=1}^{\infty} c_n \varphi_n \left| \sum_{n=1}^{\infty} |c_n|^2 (\nu_n^2 + 1) < +\infty \right. \right\}$$

and for $x \in D[(A + I)^{1/2}]$ we can write $(A + I)^{1/2} x = \sum_{n=1}^{\infty} c_n \sqrt{\nu_n^2 + 1} \varphi_n$. This is the spectral expansion of $(A + I)^{1/2}$ and it is a discrete version of the formula

$$(A + I)^{1/2} = \int_1^{+\infty} \sqrt{\lambda + 1} dE_A(\lambda),$$

where $E_A(\lambda)$ denotes the spectral measure of the operator A . On the other hand the operator $(A - w^2 I)^{-1/2}$ is bounded, whence

$$(A - w^2 I)^{-1/2} (A + I)^{1/2} x = \sum_{n=1}^{\infty} c_n \sqrt{\nu_n^2 + 1} (A - w^2 I)^{-1/2} \varphi_n = \sum_{n=1}^{\infty} c_n \frac{\sqrt{\nu_n^2 + 1}}{\sqrt{\nu_n^2 - w^2}} \varphi_n.$$

Using the definition $K_1x := (A - w^2)^{-1/2}(A + I)^{1/2}x$ and the fact that $\frac{\sqrt{\nu_n^2 + 1}}{\sqrt{\nu_n^2 - w^2}}$ is a bounded sequence we have

$$\|K_1x\|^2 = \sum_{n=1}^{\infty} |c_n|^2 \left| \frac{\nu_n^2 + 1}{\nu_n^2 - w^2} \right| \leq C \sum_{n=1}^{\infty} |c_n|^2 = C \|x\|^2.$$

Thus $\|K_1x\| \leq \sqrt{C} \|x\|$.

Note that the operator K_1 can be extended to a bounded operator on H since $\overline{D[(A + I)^{1/2}]} = H$ and K_1 is bounded on $D[(A + I)^{1/2}]$.

Consequently, by conditions (A.II)-(A.III), it follows that the operators $\tilde{B}_s(w)$ and $\tilde{C}_{s-1}(w)$ are compact.

If $D = 0$, then

$$\tilde{L}(k, w) = I + \sum_{s=1}^n k^{2s} \tilde{C}_{s-1}(w) + \sum_{s=0}^{n-1} k^{2s+1} \tilde{B}_s(w)$$

has compact coefficients and $\tilde{L}(0, w) = I$ is invertible; then it is known that the spectrum $\sigma_w(\tilde{L})$ consists of eigenvalues of finite multiplicity with infinity as their only possible limit point ([2], Theorem 12.9, or [32], p. 267).

If $D \neq 0$ then in a similar way we can show that $\tilde{D}(w)$ is compact, so

$$\tilde{L}(k, w) = I + iw\tilde{D}(w) + \sum_{s=1}^n k^{2s} \tilde{C}_{s-1}(w) + \sum_{s=0}^{n-1} k^{2s+1} \tilde{B}_s(w)$$

has compact coefficients. If w is real, $\tilde{L}(0, w) = I + iw\tilde{D}(w)$ or $\tilde{L}_\alpha(0, w)$ is invertible where $\tilde{L}_\alpha(k, w)$ is a perturbed pencil, so again the spectrum $\sigma_w(\tilde{L})$ is discrete. In what follows we use this perturbation method.

If $w^2 = \nu_m^2$, then the operator $A - w^2I$ is not invertible. In this case we introduce a new spectral parameter α into the spectral problem for the operator pencil $L(k, w)$ by replacing k by $k + \alpha$, $\alpha \in \mathbb{R}$:

$$L(k + \alpha, w) = A(\alpha) - w^2I + iwD + \sum_{s=1}^{2n} k^s F_s(\alpha)$$

where

$$A(\alpha) = A + \sum_{s=1}^n \alpha^{2s} C_{s-1} + \sum_{s=0}^{n-1} \alpha^{2s+1} B_s$$

and $F_s(\alpha)$'s are finite linear combinations of B_s and C_s . We specify the value of α such that $c_0^{2n}\alpha^{2n} + \zeta - \nu_m^2$ is positive; then the condition (A.V) implies the inequality

$$([A(\alpha) - \nu_m^2 I]u, u) \geq (c_0^{2n}\alpha^{2n} + \zeta - \nu_m^2)(u, u).$$

Hence the operator $A(\alpha) - \nu_m^2 I$ is positive definite, so it is invertible. Since $A(\alpha) = A + (\sum_{s=1}^n \alpha^{2s} C_{s-1} + \sum_{s=0}^{n-1} \alpha^{2s+1} B_s)$ and the second term is compact relatively to the operator A , we conclude that the spectrum of $A(\alpha)$ is also discrete. Therefore a neighborhood $U_\alpha(\nu_m) \subset \mathbb{C}$ of the point $\pm \nu_m$ can be found such that the operator $A(\alpha) - w^2 I$ is invertible for all $w \in U_\alpha(\nu_m)$. We define a pencil

$$\begin{aligned} \tilde{L}_\alpha(k, w) &:= (A(\alpha) - w^2 I)^{-1/2} L(k + \alpha, w) (A(\alpha) - w^2 I)^{-1/2} \\ &= I + iw\tilde{D}(\alpha, w) + \sum_{s=1}^{2n} k^s \tilde{F}_s(\alpha, w), \end{aligned}$$

where

$$\begin{aligned} \tilde{F}_s(\alpha, w) &= (A(\alpha) - w^2 I)^{-1/2} F_s(\alpha) (A(\alpha) - w^2 I)^{-1/2}, \\ \tilde{D}(\alpha, w) &= (A(\alpha) - w^2 I)^{-1/2} D (A(\alpha) - w^2 I)^{-1/2}. \end{aligned}$$

As in the case $w \neq \pm \nu_n$, we can write

$$\tilde{F}_s(\alpha, w) = K_1 (A + I)^{-1/2} F_s(\alpha) (A + I)^{-1/2} K_2,$$

where K_1 and K_2 are bounded operators and using the conditions (A.II)-(A.III) we can show that operators $\tilde{F}_s(\alpha, w)$ are compact. If $D = 0$, then $\tilde{L}_\alpha(0, w) = I$ is invertible so $\sigma_w(\tilde{L}_\alpha)$ is discrete and consequently $\sigma_w(L)$ is discrete.

Finally if $D \neq 0$, then $\tilde{D}(\alpha, w)$ becomes compact by the condition (A.I). Since $w = \pm \nu_m$ is real, $\tilde{L}_\alpha(0, w) = I + iw\tilde{D}(\alpha, w)$ is invertible. Thus $\sigma_w(\tilde{L}_\alpha)$ and consequently $\sigma_w(L)$ is discrete. $\square$

Theorem 2.3. Let $L(k, w)$ satisfy the conditions (A.I)-(A.III). Then for all $k \in \mathbb{C}$ the spectrum $\sigma_k(L)$ is discrete.

Proof. Let us define a new operator pencil

$$\begin{aligned} \tilde{L}(k, w) &= (A + I)^{-1/2} L(k, w) (A + I)^{-1/2} \\ &= I - T_0(k) - wKT_1 - w^2 K^2 \end{aligned}$$

where

$$\begin{aligned}
T_0(k) &= (A + I)^{-1} - \sum_{s=1}^n k^{2s} (A + I)^{-1/2} C_{s-1} (A + I)^{-1/2} \\
&\quad - \sum_{s=0}^{n-1} k^{2s+1} (A + I)^{-1/2} B_s (A + I)^{-1/2}, \\
T_1 &= -iD (A + I)^{-1/2}, \\
K &= (A + I)^{-1/2}.
\end{aligned}$$

Note that $T_1 \in S_\infty$ and $T_0(k) \in S_\infty$ for all $k \in \mathbb{C}$ by the conditions (A.I)-(A.III). K is self-adjoint and $\ker K = \{0\}$. Consequently for fixed k , $\tilde{L}(k, w)$ is a left Keldysh pencil so $\sigma_k(\tilde{L})$ consists of eigenvalues of finite multiplicity with infinity as their only possible limit point ([2], Theorem 17.3). Since $\sigma_k(L) = \sigma_k(\tilde{L})$, we conclude that $\sigma_k(L)$ is discrete. $\square$

Finally, to take a further look at spectral properties, particularly at variational principles for real eigenvalues in $\sigma_p^w(L)$ we determine domains containing $\sigma_w(L)$ and the spectral set

$$\sigma_R(L) := \{(k, w) \mid (k, w) \in \sigma_p(L), k \text{ and } w \in \mathbb{R}\}$$

in special cases (see Figures 2.1-2.3). Let us consider a quadratic pencil of w.g.t $L(k, w) = k^2 C + kB + A - w^2 I$ (with $D = 0$), so the operator A satisfies the condition (A.I), $(A+I)^{-1/2} B (A+I)^{-1/2} \in S_\infty$ and C is a bounded positive definite operator. In addition, the conditions (A.IV)-(A.V) are satisfied for $n = 1$.

Then it was proved (see [25], p. 60, Theorem 11.1) that the following inequalities

$$c_2^2 (\operatorname{Im} k)^2 + (\operatorname{Re} w)^2 - (\operatorname{Im} w)^2 - \mu^2 \geq 0, \quad (2.3)$$

and

$$c_2^2 (\operatorname{Im} k)^2 [c_2^2 (\operatorname{Im} k)^2 + (\operatorname{Re} w)^2 - (\operatorname{Im} w)^2 - \mu^2] \geq (\operatorname{Re} w)^2 (\operatorname{Im} w)^2 \quad (2.4)$$

hold. Here the number μ is the same as in the condition (A.IV). Clearly, for $\operatorname{Im} k \neq 0$ the inequality (2.4) is stronger than (2.3). The following properties of $\sigma(L)$ follow from these inequalities.

Corollary 2.1. a) If $\text{Im } w \neq 0$ then $\text{Im } k \neq 0$.

b) If $\text{Im } w = 0$ and $|w| < \mu$ then $\text{Im } k \neq 0$.

Consequently, real wavenumbers exist only at real frequencies satisfying the inequality $|w| \geq \mu$.

For pencils satisfying the energy stability condition (A.V) we have

$$c_2^2(\text{Im } k)^2 + (\text{Re } w)^2 - (\text{Im } w)^2 - c_0^2(\text{Re } k)^2 - \zeta \geq 0,$$

$$c_2^2(\text{Im } k)^2 [c_2^2(\text{Im } k)^2 + (\text{Re } w)^2 - (\text{Im } w)^2 - c_0^2(\text{Re } k)^2 - \zeta] \geq (\text{Re } w)^2(\text{Im } w)^2, \quad (2.5)$$

where $c_0^2 = (1 - \varepsilon^2)c_1^2$.

Evidently, (2.4) follows from the inequality (2.5) and in the case for real pairs of (w, k) we have

$$w^2 - c_0^2 k^2 \geq \zeta. \quad (2.6)$$

Now using these inequalities we have Figures 2.1–2.3.

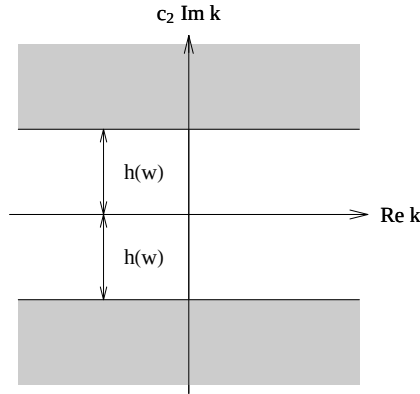


Figure 2.1: The domain containing $\sigma_w(L)$ without the energetic stability condition. $h^2(w) = \mu^2 + (\text{Im } w)^2 - (\text{Re } w)^2$.

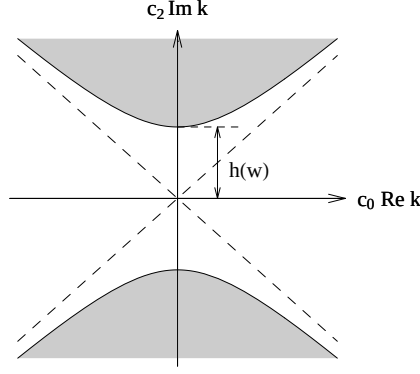


Figure 2.2: The domain containing $\sigma_w(L)$ with the energetic stability condition. $h(w) = \sqrt{\zeta + (\text{Im } w)^2 - (\text{Re } w)^2}$, where $\zeta + (\text{Im } w)^2 - (\text{Re } w)^2 \geq 0$.

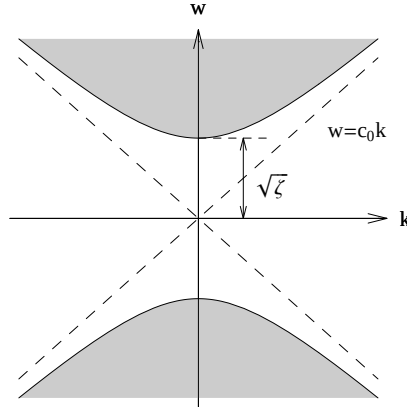


Figure 2.3: The domain containing $\sigma_R(L)$ with the energetic stability condition.

2.3 Variational Principles for Real Eigenvalues

In this subsection we study variational principles for real eigenvalues of pencils of w.g.t in the unbounded form. For quadratic pencils these problems will be investigated in detail. Finally, a theorem about variational principles for real eigenvalues in $\sigma_p^w(L)$ of polynomial pencils of w.g.t is presented at the end of the subsection (Theorem 2.5).

Now we give some preliminary definitions and facts. Fix w and take $n = 1$ in (2.1). In this section write $L_w(k)$ for $L(k, w)$. Consider the following two

parameter operator pencil

$$L_w(k) = A + kB + k^2C - w^2I$$

satisfying the conditions (A.I)-(A.V) in the case $n = 1$. Clearly $\sigma_p^w(L) = \sigma_p^{-w}(L)$ for $w \in \mathbb{R}$. For this reason in what follows we consider only the case $w \geq 0$. Note that a solution of eigenvalue problems in this thesis is understood in the generalized sense, i.e. a function $u \in D((A + I)^{1/2})$ is a solution of $L_w(k)u = 0$ iff

$$[Au, v] + k(Bu, v) + k^2(Cu, v) - w^2(u, v) = 0$$

for every v from a dense subspace of H . Here $[Au, v]$ is the closure of the form (Au, v) in the standard way (see [31] and [38]). For we consider the completion of $D(A)$ with respect to the norm, generated by the scalar product $((A + \alpha)u, u)$ for some $\alpha \in \mathbb{R}$ such that $((A + \alpha)u, u) > 0$, $u \neq 0$. It is known ([31], p. 222, Theorem 1) that $D([Au, v]) = D((A + I)^{1/2})$. On the other hand, B is a well defined operator on $D((A + I)^{1/2})$ by the condition (A.III) and C is a bounded operator, that is why the initial domain $D(A)$ of $L_w(k)$ can be extended to the set

$$\mathcal{D} := D[(A + I)^{1/2}]. \quad (2.7)$$

The equation

$$(L_w(k)x, x) = [Ax, x] + k(Bx, x) + k^2(Cx, x) - w^2(x, x) = 0, \quad x \in \mathcal{D}$$

defines two functionals

$$p_{\pm}(x, w) = \frac{-(Bx, x) \pm \sqrt{d(x, w)}}{2(Cx, x)},$$

known as Rayleigh functionals, where

$$d(x, w) = (Bx, x)^2 - 4[(A - w^2I)x, x](Cx, x).$$

Operator pencils of w.g.t belong to the class of nonoverdamped pencils (see [11], p. 1279 and [22], see also Example 2.1). Consequently, the overdamping condition $d(x, w) \geq 0$ for every $x \in \mathcal{D}$ is not satisfied. Taking into account this fact, the following conic subsets of $\mathcal{D}$ are introduced:

$$G(w) = \{x \in \mathcal{D} \mid d(x, w) > 0\}$$

and

$$G'(w) = \{x \in \mathcal{D} \setminus \{0\} \mid d(x, w) \geq 0\} :$$

which is the domain of the functionals $p_{\pm}(x, w)$. Now define the bounds of the range of $p_{\pm}(x, w)$ on $G(w)$ and $G'(w)$.

$$\begin{aligned} k_-(w) &= \inf_{x \in G(w)} p_-(x, w), & k'_-(w) &= \inf_{x \in G'(w)} p_-(x, w), \\ k_+(w) &= \sup_{x \in G(w)} p_+(x, w), & k'_+(w) &= \sup_{x \in G'(w)} p_+(x, w), \\ \delta_-(w) &= \inf_{x \in G(w)} p_+(x, w), & \delta_+(w) &= \sup_{x \in G(w)} p_-(x, w). \end{aligned}$$

Evidently, $k'_-(w) \leq k_-(w) \leq \delta_-(w) \leq \delta_+(w) \leq k_+(w) \leq k'_+(w)$ (for the inequality $\delta_-(w) \leq \delta_+(w)$ see [11], p. 1279, [10] and [22], Proposition 3.1). In Example 2.1, we define an operator pencil of w.g.t with $\delta_-(w) < \delta_+(w)$ for some $w \in \mathbb{R}$. The most interesting fact is that real eigenvalues are distributed as in Figure 2.4 (see [11]).

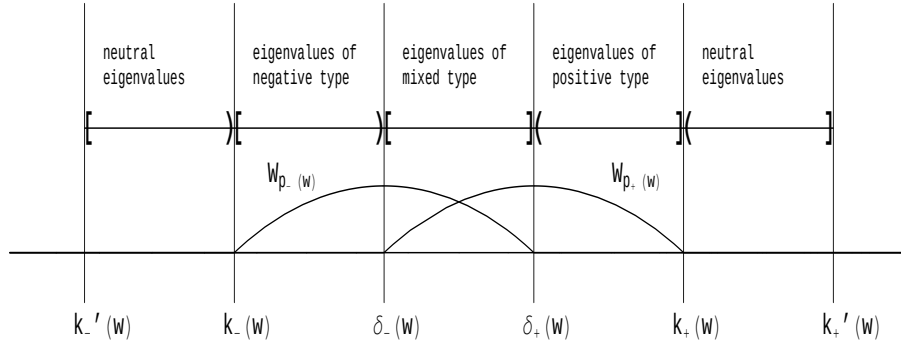


Figure 2.4: The distribution of the real eigenvalues.

Here $W_{p_{\pm}(w)} = \{p_{\pm}(x, w) \mid x \in G(w)\}$. There are only multiple roots of the equation $(L_w(k)x, x) = 0$ on the intervals $[k'_-(w), k_-(w))$ and $(k_+(w), k'_+(w)]$. On the other hand, by the definition of the numbers $k_{\pm}(w)$, $k'_{\pm}(w)$ and $\delta_{\pm}(w)$ we have $L_w(k) \geq 0$ for all $k \geq k_+(w)$ and $k \leq k_-(w)$. Consequently, for these values of k we can write

$$\|L_w(k)x\|^2 \leq \|L_w(k)\| (L_w(k)x, x),$$

which implies that a root of the equation $(L_w(k)x, x) = 0$, $x \in G'(w) \setminus G(w)$ satisfying $k > k_+(w)$ or $k < k_-(w)$ is a neutral eigenvalue of $L_w(k)$. Moreover, it

follows from this fact that the sets $W'_{p_+(w)} \cap (k_+(w), k'_+(w)]$ and $W'_{p_-(w)} \cap [k'_-(w), k_-(w))$ are either a single point or a disconnected set, where $W'_{p_\pm(w)} = \{p_\pm(x, w) \mid x \in G'(w)\}$.

Now we give an example that, in general the spectral sets $\sigma_w(L) \cap (k_+(w), k'_+(w)]$ and $\sigma_w(L) \cap [k'_-(w), k_-(w))$ are nonempty. In this example we also show that $\delta_-(w) < \delta_+(w)$ for some $w \in \mathbb{R}$.

Example 2.1. Let $M(k) = A + kB + k^2C$ be a one parameter operator pencil, $H = \mathbb{R}^2$ and

$$A = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & 7 \\ 7 & 4 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}.$$

The distribution of the real roots $p_\pm(x)$ of the equation $(M(k)x, x) = 0$, $\|x\| = 1$ can be seen in Figure 2.5, where $W_{p_\pm} = \{p_\pm(x) \mid \|x\| = 1\}$. It immediately follows from this that $\sigma_p(M) = \{-5, 0, 1\}$ and 1 is a multiple root.

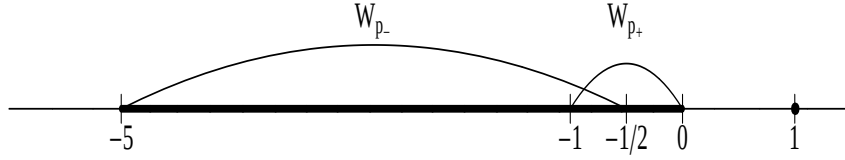


Figure 2.5: The root domain of the pencil $M(k)$.

Now we are going to construct a two parameter pencil $L(k, w)$ of w.g.t such that $L(k, w^*) = M(k)$ for a real number $w^* > 0$. For this purpose define a pencil $L^*(k) := (A + \theta^*I) + kB + k^2C$ and choose θ^* from the energetic stability condition $L^*_\varepsilon(k) = (A + \theta^*I) + kB + \varepsilon^2 k^2 C \geq 0$, $0 < \varepsilon < 1$ (see Theorem 2.1). Since $C > 0$ then $L^*_\varepsilon(k) \geq 0$ is equivalent to the condition

$$d(x, \theta^*) = (Bx, x)^2 - 4\varepsilon^2(Ax, x)(Cx, x) - 4\theta^*\varepsilon^2(Cx, x) \leq 0,$$

for $\|x\| = 1$, whence

$$\theta^* \geq \frac{(Bx, x)^2 - 4\varepsilon^2(Ax, x)(Cx, x)}{4\varepsilon^2(Cx, x)}.$$

Clearly, there exists a vector x such that $(Bx, x)^2 - 4(Ax, x)(Cx, x) > 0$ and $\|x\| = 1$ (see Fig. 2.5) which implies $(Bx, x)^2 - 4\varepsilon^2(Ax, x)(Cx, x) > 0$.

Consequently

$$0 < \max_{x \neq 0} \frac{(Bx, x)^2 - 4\varepsilon^2(Ax, x)(Cx, x)}{4\varepsilon^2(Cx, x)} < +\infty$$

and we can choose

$$\theta^* \geq \max_{x \neq 0} \frac{(Bx, x)^2 - 4\varepsilon^2(Ax, x)(Cx, x)}{4\varepsilon^2(Cx, x)},$$

such that $A + \theta^*I$ is invertible. Thus $L_\varepsilon^*(k) \geq 0$ for all $k \in \mathbb{R}$.

Finally, define a two parameter pencil $L(k, w)$ as

$$L(k, w) := L^*(k) - w^2I = (A + \theta^*I) + kB + k^2C - w^2I.$$

$L(k, w)$ is an operator pencil of w.g.t. Indeed, (A.I) and (A.III) follows from the fact that $A + \theta^*I$ is invertible. By the definition of C , it is positive definite. Thus the condition (A.II) is satisfied. The inequality $L_\varepsilon^*(k) \geq 0$ established above is the energetic stability condition (A.V) and (A.IV) follows from (A.V) with $\mu = 0$.

Now choosing $w^* = \sqrt{\theta^*}$ we have $\sigma_{w^*}(L) = \sigma(M)$ and $W_{p_\pm(w^*)} = W_{p_\pm}$ so

$$\begin{aligned} k'_-(w^*) &= k_-(w^*) = -5, & \delta_-(w^*) &= -1, \\ \delta_+(w^*) &= -\frac{1}{2}, & k_+(w^*) &= 0, & k'_+(w^*) &= 1 \end{aligned}$$

and

$$\sigma(L) \cap (k_+(w^*), k'_+(w^*)) = \{1\}.$$

It turns out that eigenvalues from $[k_-, \delta_-)$ and $(\delta_+, k_+]$ can be characterized by variational principles. Such characterizations for quadratic pencils of w.g.t with bounded coefficients were studied in [10], [11] and [22]. Here pencils of w.g.t are defined by conditions which are quite different from that given in (A.I)-(A.V). To use methods of the papers [10], [11] and [22], we have to transform the pencil $L(k, w)$ to the pencil $\tilde{L}(k, w) := (A + I)^{-1/2}L(k, w)(A + I)^{-1/2}$ and then use the fact that $\sigma(\tilde{L}) = \sigma(L)$.

The reasons for studying variational problems for pencils of w.g.t in the unbounded form are:

- I) The methods given in the above mentioned papers are applicable only in the case $n = 1$. Because these methods are based on the properties of the

fraction $\frac{-(Bx,x) \pm \sqrt{d(x,w)}}{2(Cx,x)}$ and consequently the roots $p_{\pm}(x, w)$ must be given in the open form. Moreover, to deal with the coefficients of the transformed pencil $\tilde{L}(k, w)$ and the new conditions determining pencils of w.g.t with bounded coefficients (see [10], [11] and [22]) is troublesome.

- II) The other motivation for this paper comes from the recently published work of D. Eschwé and M. Langer [15], where the operator functions considered were not restricted to the functions with bounded values and an analog of the variational principles (1.2) and (1.3) with the index shift was established ([15], Theorem 2.1) in an interval $[\alpha, \lambda_e[$, where $\lambda_e = \min \pi(L)$ if $\pi(L) \neq \emptyset$, otherwise λ_e is a number β . Additional conditions guarantee that the spectrum on $[\alpha, \lambda_e[$ is discrete.

Moreover, the conditions given in [15] easily follow from the general spectral properties of pencils of w.g.t given in Subsection 2.2. For this reason we study variational principles for real eigenvalues for operator pencils of w.g.t in the unbounded form by using a theorem from [15] and the results presented in Subsection 2.2.

Let $\Delta \subset [k_-(w), \delta_-(w))$ (or $\Delta \subset (\delta_+(w), k_+(w)]$). Recall that we fix $w \geq 0$. Now comparing the unbounded operator functions given in [15] with the operator pencils of w.g.t and using their spectral structure (see Theorem 2.2 and 2.3, Fig.2.1–2.4), we obtain that the following version of the conditions of Theorem 2.1 from [15] (see p. 291–292) are to be satisfied in order to establish variational principles for eigenvalues from $\Delta \subset [k_-(w), \delta_-(w))$ (or $\Delta \subset (\delta_+(w), k_+(w)]$).

- 1) $L_w(k)$ is an operator function defined on an interval $\Delta = [\alpha, \beta] \subset [k_-(w), \delta_-(w))$ with values in the class of self-adjoint unbounded operators, and the form $l_w(k)[x, y] := (L_w(k)x, y)$ for $k \in \Delta$ and $x, y \in D(L_w(k))$ is closable. The closure is also denoted by $l_w(k)[x, y]$. Moreover there exists a subspace $\mathcal{D} \subset H$, $\overline{\mathcal{D}} = H$ so that

$$D(L_w(k)) \subset \mathcal{D} \subset D(l_w(k)), \text{ for all } k \in \Delta. \quad (2.8)$$

Here $D(L_w(k))$ is the initial domain of $L_w(k)$ (or $(L_w(k)x, y)$) and $D(L_w(k)) = D(A)$ for every k .

2) L_w is continuous in the norm resolvent topology, and the function $l_w(\cdot)[x]$ is continuous for every $x \in \mathcal{D}$.

3) For every $x \in \mathcal{D}$, $x \neq 0$ one of the following conditions is fulfilled:

- a) $l_w(k)[x] > 0$, $\forall k \in \Delta$, or
- b) $l_w(k)[x] < 0$, $\forall k \in \Delta$, or
- c) $\exists k_0 \in \Delta$, $l_w(k_0)[x] = 0$ and then $l_w(k)[x]$ is decreasing at value k_0 ,
where $l_w(k)[x] := l_w(k)[x, x]$.

4) $\kappa_-(\alpha) := \max \dim\{E \mid l_w(\alpha)[x] < 0, 0 \neq x \in E\} < +\infty$.

The Rayleigh functional now is well defined only in the case 3) c), by setting $p(x) = k_0$. For other vectors in $\mathcal{D}$ the functional $p(x)$ is defined as

$$p(x) := \begin{cases} -\infty, & \text{if } l_w(k)[x] < 0 \text{ for all } k \in \Delta, \\ +\infty, & \text{if } l_w(k)[x] > 0 \text{ for all } k \in \Delta. \end{cases}$$

The extended functional $p(x)$ defined in such way is called the generalized Rayleigh functional (see [14]).

In the next lemma we show that the condition 4) is always satisfied for pencils of w.g.t for every $t \in \Delta$.

Lemma 2.1. $\kappa_-(t) < +\infty$ for all $t \in (k_-(w), \delta_-(w))$, and $\kappa_-(t) = 0$ if $t = k_-(w)$.

Proof. It is known (see [15], Lemma 2.5) that $\kappa_-(t) = \dim E_{(-\infty, 0)}(L_w(t))$, where by E is denoted the spectral measure of the self-adjoint operator $L_w(t)$. Evidently, $\lambda < 0$ and $\lambda \in \sigma(L_w(t))$ means $(t, w^2 + \lambda) \in \sigma(L)$. Thus for fixed t and w we have

$$\dim E_{(-\infty, 0)}(L_w(t)) = \text{card} \{\lambda \mid \lambda < 0, (t, w^2 + \lambda) \in \sigma(L)\}.$$

By the condition $t \in \mathbb{R}$, whence $w \in \mathbb{R}$ by Corollary 2.1. It means $w^2 + \lambda \in \mathbb{R}$. Now it follows from the inequality (2.6) (see also Fig. 2.3) and discreteness of the spectrum $\sigma_k(L)$ that there is only a finite number λ satisfying $w^2 + \lambda \geq 0$, $\lambda < 0$ and $w^2 + \lambda \in \sigma_t(L)$.

On the other hand, by the energetic stability condition (A.V) the number $w^2 + \lambda$ must be nonnegative. Thus $\text{card} \{\lambda \mid \lambda < 0, (t, w^2 + \lambda) \in \sigma(L)\} < +\infty$ and consequently $\dim E_{(-\infty, 0)}(L_w(t)) = \kappa_-(t) < +\infty$.

Finally, the relation $\kappa_-(t) = 0$ for $t = k_-(w)$ follows directly from the definition of $k_-(w)$ (see Fig. 2.4). $\square$

Note that if we take $\Delta = [k_-(w), \delta_-(w))$ or $\Delta = (\delta_+(w), k_+(w)]$, then an index shift in the variational principles for eigenvalues of $L_w(k)$ in Δ does not occur, since $\kappa_-(\alpha) = 0$ for $\alpha = k_{\pm}(w)$.

Now we give the main theorem for quadratic pencils of w.g.t.

Theorem 2.4. Let $L_w(k)$ be a quadratic operator pencil of w.g.t., i.e. $n = 1$ and $\Delta = [k_-(w), \delta_-(w))$, $k_-(w) < \delta_-(w)$. Then

- a) There is only a finite number of eigenvalues in Δ : arrange them in the nondecreasing order $k_1^-(w) \leq k_2^-(w) \leq \dots \leq k_n^-(w)$ counted according to their multiplicities,
- b) All eigenvalues are of negative type, i.e. $(L'_w(k)x, x) < 0$ for all $x \in \ker L_w(k) \setminus \{0\}$ and $k \in \sigma_w(L) \cap \Delta$,
- c)

$$k_i^-(w) = \min_{\substack{L \subset D \\ \dim L = i}} \max_{\substack{x \in L \\ x \neq 0}} p_-(x, w), \quad i = 1, 2, \dots, n, \quad (2.9)$$

$$k_i^-(w) = \max_{\substack{L \subset H \\ \dim L = i-1}} \inf_{\substack{x \in D, x \neq 0 \\ x \perp L}} p_-(x, w), \quad i = 1, 2, \dots, n. \quad (2.10)$$

Proof. a) By Theorem 2.2, $\sigma_w(L)$ is discrete for all $w \in \mathbb{C}$, i.e. $\sigma_w(L)$ consists of isolated eigenvalues of finite multiplicity. We have to consider real frequencies w , otherwise by Corollary 2.1 an eigenvalue k corresponding to w must be nonreal. Now by the inequality (2.6) the set

$$\sigma_R(L) := \{(k, w) \mid k, w \in \mathbb{R}, (k, w) \in \sigma(L)\}$$

lies in the domain $w^2 - c_0^2 k^2 \geq \zeta$ (see Fig. 2.3). It follows from this fact that if $0 \leq w < \sqrt{\zeta}$ then $\sigma_w(L) = \emptyset$ and if $w \geq \sqrt{\zeta}$ then $\sigma_w(L)$ consists of finite number of eigenvalues.

b) It follows from the definition of the numbers $k_-(w)$ and $\delta_-(w)$ that the function $\varphi_{x,w}(k) := l_w(k)[x]$ is decreasing at value zero on Δ . Consequently, all eigenvalues are of negative type.

c) To establish the variational principles (2.9) and (2.10) according to Theorem 2.1 from [15], we have to check the conditions **1)–4)** on the page 28. By Lemma 2.1 $\kappa_-(\alpha) = 0$ for $\alpha = k_-(w)$, so **4)** is fulfilled.

1) The initial domain of the pencil $L_w(k)$ is $D(A)$. But by (2.7) it is extended to be

$$\mathcal{D} := D[(A + I)^{1/2}].$$

Now clearly the relation $D(L_w(k)) \subset \mathcal{D} \subset D(l_w(k))$ holds for all $k \in \Delta$, and consequently the condition **1)** is satisfied.

2) Denote the resolvent of $L_w(\lambda)$ (for fixed w) by $R_\lambda^w := L_w^{-1}(\lambda)$. Then for μ and λ in $\rho(L)$ we have the generalized resolvent equation

$$R_\mu^w - R_\lambda^w = (\lambda - \mu)R_\mu^w[L'_w(\mu) + (\lambda - \mu)\tilde{\Delta}(\mu, \lambda)]R_\lambda^w, \quad (2.11)$$

where

$$\tilde{\Delta}(\mu, \lambda) = \sum_{k=2}^n \frac{L_w^{(k)}(\mu)}{k!} (\lambda - \mu)^{k-2}.$$

Now the continuity of $L_w(k)$ in the norm resolvent topology ([38]) follows from the formula (2.11) which is obtained from Taylor expansion and can be easily checked. It follows also from the fact that $L_w(k)$ is a generalized holomorphic operator function in the sense of Kato (see [38], p. 366).

3) As mentioned in b), the condition **3)** follows from the definition of the numbers $k_\pm(w)$, $k'_\pm(w)$ and $\delta_\pm(w)$.

Thus the conditions **1)–4)** are satisfied and consequently we have variational principles in the form (2.9) and (2.10). $\square$

Now let us choose a proper subinterval $\Delta = [\alpha, \beta]$ of $[k_-(w), \delta_-(w))$. Clearly, the endpoints $k_\pm(w)$ of the numerical ranges $W_{p_\pm}(w)$ belong to $\sigma_w(L)$ and consequently $k_\pm(w) \in \sigma_p^w(L)$. It means that $\kappa_-(\alpha) \neq 0$ if $k_-(w) < \alpha < \delta_-(w)$. On the other hand, $\kappa_-(\alpha)$ is finite by Lemma 2.1. Consequently we have

$$\begin{aligned} k_i^-(w) &= \min_{\substack{L \subset D \\ \dim L = i + \kappa_-(\alpha)}} \max_{\substack{x \in L \\ x \neq 0}} p_-(x, w), \\ k_i^-(w) &= \max_{\substack{L \subset H \\ \dim L = \kappa_-(\alpha) + i - 1}} \inf_{\substack{x \in D, x \neq 0 \\ x \perp L}} p_-(x, w). \end{aligned}$$

The case $n > 1$. Notice that we consider quadratic pencils of w.g.t in detail in order to show that the conditions **1)–4)**, needed to establish variational principles

follow immediately from the properties of these pencils. So these conditions are not extra ones and they are always fulfilled on some parts of the root domain. In the case $n > 1$ we also have theorems on discreteness of the spectra $\sigma_w(L)$ and $\sigma_k(L)$, but we have no exact formulae for roots of $(L_w(k)x, x) = 0$ and for the domain containing spectra, as in Fig. 2.1–2.4. However we show below that the conditions **1)**, **2)** and **4)** are also fulfilled in the case $n > 1$ and therefore we have to choose a root zone Δ satisfying the condition **3)** for establishing variational principles. Now let us check the conditions **1)**, **2)** and **4)**.

1) By the formula (2.1)

$$L_w(k) := L(k, w) = A + \sum_{s=1}^n k^{2s} C_{s-1} + \sum_{s=0}^{n-1} k^{2s+1} B_s + iwD - w^2 I,$$

for the sake of simplicity suppose $D = 0$. By the condition (A.III) of the pencils of w.g.t C_{n-1} is a bounded operator, the operators B_s , $s = 0, 1, \dots, n-1$ and C_s , $s = 0, 1, \dots, n-2$ are well defined operators on $D((A+I)^{1/2})$. Note that the solution of the eigenvalue problem in the case $n > 1$ is also understood in the generalized sense, i.e., a function $u \in D((A+I)^{1/2})$ is a solution of the eigenvalue problem $L_w(k)u = 0$ iff

$$[Au, v] + \sum_{s=1}^n k^{2s} (C_{s-1}u, v) + \sum_{s=0}^{n-1} k^{2s+1} (B_s u, v) - w^2 (u, v) = 0, \quad (2.12)$$

for every v from a dense subspace of H . Here $[Au, v]$ is the closure of the form (Au, v) . Define

$$(L_w(k)x, x) := (Ax, x) + \sum_{s=1}^n k^{2s} (C_{s-1}x, x) + \sum_{s=0}^{n-1} k^{2s+1} (B_s x, x) - w^2 (x, x),$$

$x \in D(A) = D(A + I)$. By (2.12) the domain of generalized solutions is $D((A+I)^{1/2})$. On the other hand by the condition (A.IV) the form $(L_w(k)x, x)$ is a bounded below form whence $(L_w(k)x, x)$ is closable (see [31], [38]). Denoting the closure of the form $(L_w(k)x, x)$ by $l_w(k)[x]$, we obtain $D((A+I)^{1/2}) \subset D(l_w(k))$. It means the condition **1)** is satisfied with $\mathcal{D} = D((A+I)^{1/2})$ i.e.,

$$D(A) = D(L_w(k)) \subset \mathcal{D} \subset D(l_w(k)), \quad \forall k \in \mathbb{R}.$$

Since the generalized resolvent equation (2.11) holds for any polynomial pencil, the condition **2)** is also satisfied in the case $n > 1$. Note that the condition (A.IV)

and the energetic stability condition (A.V) in the definition of pencils of w.g.t is really a condition for the closure of the form $(L_w(k)x, y)$.

Now we prove a lemma about the condition **4)**. It is an analog of the Lemma 2.1 in the case $n = 1$.

Lemma 2.2. Let $L(k, w)$ be a polynomial operator pencil of w.g.t. Then

$$\kappa_-(k) := \dim E_{(-\infty, 0)}(L_w(k)) < +\infty$$

for every $k, w \in \mathbb{R}$.

Proof. We apply the same arguments as in the Lemma 2.1. We know by Theorem 2.3 that $\sigma_k(L)$ is discrete for every $k \in \mathbb{C}$. Evidently, $\lambda \in \sigma(L_w(k))$ if and only if $(k, w^2 + \lambda) \in \sigma(L)$. Consequently

$$\dim E_{(-\infty, 0)}(L_w(k)) = \text{card } \{\lambda \mid \lambda < 0, (k, w^2 + \lambda) \in \sigma(L)\}.$$

By the condition (A.V)

$$(Au, u) + \sum_{s=1}^n k^{2s} (C_{s-1}u, u) + \sum_{s=0}^{n-1} k^{2s+1} (B_s u, u) \geq 0.$$

It means that if $(k, w^2 + \lambda) \in \sigma(L)$, then $w^2 + \lambda \geq 0$. Hence for $\lambda < 0$, we have $w^2 + \lambda \in [0, w^2)$. Since $\sigma_k(L)$ is discrete, for a fixed k there is only a finite number of λ satisfying $w^2 + \lambda \geq 0$ and $\lambda < 0$. It means

$$\text{card } \{\lambda \mid \lambda < 0, (k, w^2 + \lambda) \in \sigma(L)\} < +\infty.$$

□

Thus we have proved that the conditions **1)**, **2)** and **4)** are satisfied for a two parameter polynomial pencil of waveguide type, consequently the following theorem is true.

Theorem 2.5. Let $L_w(k)$ be a polynomial operator pencil of w.g.t and $w, k \in \mathbb{R}$. Then $\sigma_k(L)$ and $\sigma_w(L)$ are discrete. Moreover, if $L_w(k)$ satisfies the condition **3)** on an interval $[\alpha, \beta] \subset \mathbb{R}$, then for eigenvalues $\{k_i(w)\}_{i=1}^n$ in $[\alpha, \beta]$ arranged in the nondecreasing order, counted according to their multiplicities, the following

variational principles hold:

$$\begin{aligned} k_i(w) &= \min_{\substack{L \subset D \\ \dim L = i + \kappa_-(\alpha)}} \max_{\substack{x \in L \\ x \neq 0}} p(x, w), \\ k_i(w) &= \max_{\substack{L \subset H \\ \dim L = \kappa_-(\alpha) + i - 1}} \inf_{\substack{x \in D, x \neq 0 \\ x \perp L}} p(x, w). \end{aligned}$$

Finally, we give a corollary about the motion of the real eigenvalues in the case $n = 1$. We have

$$p_{\pm}(x, w) = \frac{-(Bx, x) \pm \sqrt{d(x, w)}}{2(Cx, x)},$$

where $d(x, w) = (Bx, x)^2 - 4((A - w^2 I)x, x)(Cx, x)$. We see from the expression of $d(x, w)$ that the functions $p_{\pm}(x, w)$ are monotone in w . Moreover,

$$p_+(x, w) \rightarrow +\infty \quad \text{and} \quad p_-(x, w) \rightarrow -\infty \quad \text{as} \quad w \rightarrow +\infty. \quad (2.13)$$

By $k_i^-(w)$ ($k_i^+(w)$) denote eigenvalues in $[k_-(w), \delta_-(w))$ ($(\delta_+(w), k_+(w)]$) arranged in nondecreasing (nonincreasing) order. The variational principles for $k_i^-(w)$ were given by the formulae (2.9) and (2.10). For $k_i^+(w)$, max and min in (2.9) will be replaced and we have

$$k_i^+(w) = \max_{\substack{L \subset D \\ \dim L = i}} \min_{\substack{x \in L \\ x \neq 0}} p_+(x, w). \quad (2.14)$$

Now the following corollary follows from (2.13) and (2.14).

Corollary 2.2. For a fixed i the eigenvalue $k_i^+(w)$ moves to the right and the eigenvalue $k_i^-(w)$ moves to the left as $w \rightarrow +\infty$.

The motion of the eigenvalues in detail for a class of bounded operator pencils of w.g.t was studied in [17], where the Rellich-Nagy theorem from perturbation theory and the angular operator method given in [39] were essentially used.

3 THE NUMERICAL RANGE OF A CLASS OF SELF-ADJOINT OPERATOR FUNCTIONS

3.1 Introduction

In this section we study the numerical range and the structure of root zones for a class of self-adjoint operator functions, arising from one or two parameter polynomial operator pencils of waveguide type (w.g.t.). These are questions mainly from the variational theory of the spectrum of operator pencils of w.g.t. studied in Section 2. Remember that the main difficulties in the variational theory of the spectrum of multi-parameter operator pencils are based on the fact that their root zones overlap i.e., they are nonoverdamped pencils. For this reason, we construct a general model of self-adjoint operator pencils which contains not only operator pencils of w.g.t., but also a wide class of nonoverdamped pencils. Throughout, we study these problems in the frame of this model (see conditions (B.I)-(B.IV) on page 38).

Note that although variational principles for definite type eigenvalues for quadratic operator pencils of w.g.t. were studied in detail in [11], this section also contains some new results (see Theorem 3.2) about the numerical range and root zones in this case.

In the spectral theory, especially in the variational theory of the spectrum of one or two parameter operator pencils of w.g.t. (see Section 2), we often deal with operator functions whose root zones overlap in an interval $[a, b]$ (see [11], [17], [22], [40]). Namely we have an operator function L and the equation $(L(\lambda)x, x) = 0$ has only two roots $p_-(x)$ and $p_+(x)$ in $[a, b]$ for some x from a cone G' in a Hilbert space H . We recall that a real root λ of the equation $(L(\lambda)x, x) = 0$ is said to be of positive type, of negative type, and neutral if the number $(L'(\lambda)x, x)$ is greater than zero, less than zero, and equal to zero respectively. Similarly a real eigenpair λ, x is said to be of positive type, of negative type, and neutral according to the

sign of number $(L'(\lambda)x, x)$. A real eigenvalue λ is said to be of positive type if every eigenpair formed by it and any eigenvector corresponding to it is of the positive type. The notions of eigenvalue of negative type and neutral eigenvalue are defined similarly. We define also the cone

$$G = \{x \in G' \mid p_-(x) \neq p_+(x)\}$$

and the bounds of the ranges of functionals $p_{\pm}(x)$ on G and G' :

$$\begin{aligned} \delta_- &= \inf_G p_+(x), & \delta_+ &= \sup_G p_-(x), \\ k_- &= \inf_G p_-(x), & k_+ &= \sup_G p_+(x), \\ k'_- &= \inf_{G'} p_-(x), & k'_+ &= \sup_{G'} p_+(x), \end{aligned}$$

In the spectral theory of one or two parameter operator pencils of w.g.t., in general, we have two different models for distribution of roots and curves $(L(\lambda)x, x)$ (see Figures 3.1 and 3.2).

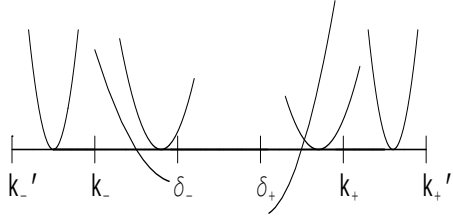


Figure 3.1: Model A

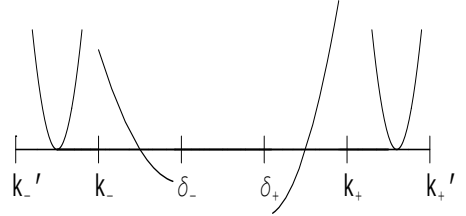


Figure 3.2: Model B

Especially in the solutions of variational problems, distribution of roots as in model B (Figure 3.2) and connectedness of the parts of root zones in the intervals $[k_-, \delta_-]$ and $(\delta_+, k_+]$ are very important.

In this section we aim to give conditions under which such distribution of roots (Figure 3.2) occurs. This is considered in Subsection 3.2.

In addition, examples of one and two parameter operator functions satisfying these conditions are given in Subsection 3.3.

3.2 On the Structure of Root Zones

Let A be a bounded linear operator on the Hilbert space H . The set of all numbers of the form (Ax, x) , where $\|x\| = 1$, is called the numerical domain of A

and denoted by $W(A)$. It is obvious that $W(A)$ is a nonempty subset of $\mathbb{C}$. This set is not closed in general. If $Ax = \lambda x$ ($\|x\| = 1$), then $(Ax, x) = \lambda$, i.e., all the eigenvalues of A are in $W(A)$. The spectrum of A need not be contained in $W(A)$ but is necessarily in $\overline{W(A)}$. We know also that by the Hausdorff theorem $W(A)$ is a convex set.

Let $A(\lambda)$ be an operator function whose values are bounded operators. The set of all roots of all possible functions $(A(\lambda)x, x)$ ($x \neq 0$) is called the numerical range of the operator function $A(\lambda)$ and denoted by $R(A)$. In other words $\lambda_0 \in R(A)$ if there exists a vector x_0 such that $\|x_0\| = 1$ and $(A(\lambda_0)x_0, x_0) = 0$. Obviously, each eigenvalue of $A(\lambda)$ is in the numerical range.

Note that in some resources the notion of ‘numerical domain’ for an operator, defined above, is called ‘numerical range’. That is, the term ‘numerical range’ is used both for operators and operator functions (see [41]). But in order to differentiate between this two notions for operators and operator functions, we prefer to use both terms.

The numerical range of the operator function $A - \lambda I$ coincides with the numerical domain of the operator A , so the concept of the numerical range of an operator function is a natural generalization of the concept of the numerical domain of an operator.

The relation between numerical domain and spectrum of an operator can be generalized to operator functions. If $A(\lambda)$ is an operator function holomorphic in a domain U and there exists a number $z_0 \in U$ such that $0 \notin \overline{W(A(z_0))}$, then $\sigma(A) \subset \overline{R(A)}$ (see [2], p. 139).

In contrast to the numerical domain of an operator, the numerical range of an operator function is nonconvex and even disconnected in general.

In the finite dimensional case, the following theorem exhibits a close relationship between the eigenvalues and numerical range of a monic self-adjoint matrix polynomial, namely, that every real boundary point of $R(L)$ is an eigenvalue of $L(\lambda)$.

Theorem 3.1 ([41], Theorem 10.15). Let $L(\lambda)$ be a monic self-adjoint matrix polynomial, and let $\lambda_0 \in R(L) \cap \overline{(\mathbb{R} \setminus R(L))}$. Then λ_0 is an eigenvalue of $L(\lambda)$.

As mentioned above we are mainly interested in problems about the structure of root zones, which we encounter in the variational theory of operator pencils of w.g.t. in the nonoverdamped case. For this reason we prefer to deal, not with conditions on the coefficients, but with conditions that derive from them and are easier to apply in our case. So we construct a general model given by the conditions

(B.I) $L(k) : [a, b] \rightarrow S(H)$, $L \in C^1[a, b]$ and for all $x \neq 0$ from a cone G' in a Hilbert space H the equation $(L(k)x, x) = 0$ has only two roots $p_-(x)$, $p_+(x)$ in $[a, b]$ (multiplicities taken into account and $p_-(x) \leq p_+(x)$) and has no roots in $[a, b]$ for other $x \in H \setminus \{0\}$. Here $S(H)$ denotes the set of bounded self-adjoint operators in H .

(B.II) If $x \in G$ then $(L'(p_-(x))x, x) < 0$ and $(L'(p_+(x))x, x) > 0$, where

$$G = \{x \in G' \mid p_-(x) \neq p_+(x)\}.$$

(B.III) There exist a number $k \in [a, b]$ such that $(L(k)x, x) < 0$ if and only if $x \in G$.

(B.IV) If $\{x_n\} \subset G'$ weakly convergent to $x \in G'$ then

$$\liminf p_-(x_n) \geq p_-(x), \quad \limsup p_+(x_n) \leq p_+(x).$$

Operator pencils of w.g.t. (see examples in Subsections 3.3.1, 3.3.2 and 3.3.3), as well as a wide class of nonoverdamped operator pencils, form a subclass of this model, i.e., they satisfy the conditions (B.I)-(B.IV).

We set

$$W'_{p_{\pm}} := \{p_{\pm}(x) \mid x \in G'\}$$

and

$$W_{p_{\pm}} := \{p_{\pm}(x) \mid x \in G\}$$

which are called root zones of the pencil L .

Lemma 3.1. The functionals $p_{\pm}$ are continuous on G' .

Proof. Let $x_n, x \in G'$ and $x_n \rightarrow x$. We want to show that $p_+(x_n) \rightarrow p_+(x)$. Let $\beta_n = p_+(x_n)$. Since $\beta_n \in [a, b]$ it is bounded so has a convergent subsequence. Let us show it again by β_n and let $\beta_n \rightarrow \beta$. Now we must show that $\beta = p_+(x)$. Since

$$0 = (L(\beta_n)x_n, x_n) \rightarrow (L(\beta)x, x) = 0$$

it follows that $\beta = p_+(x)$ or $\beta = p_-(x)$. If we consider the condition (B.II) then

$$0 \leq (L'(\beta_n)x_n, x_n) \rightarrow (L'(\beta)x, x) \geq 0.$$

Now there are two cases. If $(L'(\beta)x, x) > 0$ then $\beta = p_+(x)$. If $(L'(\beta)x, x) = 0$ then $\beta = p_+(x) = p_-(x)$. Consequently $p_+(x_n) \rightarrow p_+(x)$. $\square$

The following Theorem particularly, shows that for operator pencils of w.g.t. we have distribution of roots as in Figure 3.2.

Theorem 3.2. Let L be an operator function satisfying the conditions (B.I)-(B.IV). Then we have the following properties:

- (i) $k_{\pm} \in \sigma(L)$,
- (ii) If k_+ (k_-) is not a limit point of $\sigma(L)$, every $k \in W'_{p_+} \cap (\delta_+, k_+]$ ($k \in W'_{p_-} \cap [k_-, \delta_-)$) is a root of positive (negative) type. Particularly, all eigenvalues in $(\delta_+, k_+]$ ($[k_-, \delta_-)$) are eigenvalues of positive (negative) type.

Proof. First we prove the property (i) for k_+ . We select a sequence $\{x_n\}$ with the following properties

$$x_n \in G, \quad \|x_n\| = 1, \quad p_+(x_n) \rightarrow k_+, \quad x_n \xrightarrow{w} x. \quad (3.1)$$

Since

$$|(L(k_+)x_n, x_n)| \leq \|L(k_+) - L(p_+(x_n))\|,$$

we have

$$\lim_{n \rightarrow \infty} (L(k_+)x_n, x_n) = 0.$$

From the conditions (B.II)-(B.III) and the definition of k_+ follows that $L(k_+) \geq 0$.

Consequently we have

$$\lim_{n \rightarrow \infty} L(k_+)x_n = 0, \quad L(k_+)x = 0. \quad (3.2)$$

From the existence of a sequence satisfying (3.1) and (3.2) it follows that $k_+ \in \sigma(L)$. In a similar way one shows that $k_- \in \sigma(L)$.

We now prove (ii). We show that roots in $W'_{p_+} \cap (\delta_+, k_+]$ are of positive type. First we establish that k_+ is an eigenvalue and has an eigenvector of positive type. We select a sequence having the properties (3.1) and (3.2). Since k_+ is not a limit point of $\sigma(L)$, the vector x cannot be zero, so from (3.2) it follows that k_+, x is an eigenpair. From the condition (B.IV) we have

$$p_+(x) - p_-(x) \geq \limsup p_+(x_n) - \liminf p_-(x_n).$$

Since $p_+(x_n) \rightarrow k_+$, choosing a subsequence we can write

$$p_+(x) - p_-(x) \geq \lim_{n \rightarrow \infty} p_+(x_n) - \lim_{n \rightarrow \infty} p_-(x_n).$$

We show that the right side of the inequality is strictly positive. Assume that

$$\lim_{n \rightarrow \infty} p_+(x_n) - \lim_{n \rightarrow \infty} p_-(x_n) = 0$$

then

$$k_+ = \lim_{n \rightarrow \infty} p_+(x_n) = \lim_{n \rightarrow \infty} p_-(x_n) \leq \delta_+.$$

So we obtain a contradiction with the fact that $\delta_+ < k_+$. So $p_-(x) < p_+(x)$ and $x \in G$. Since $k_+ \leq p_+(x)$ by the condition (B.IV), we have $k_+ = p_+(x)$ and the pair k_+, x is of positive type by the condition (B.II).

Now we show that k_+ is an eigenvalue of positive type. Let z be an arbitrary (nonzero) eigenvector corresponding to k_+ . If it is of negative type we have a contradiction with the fact that k_+ is the upper bound of p_+ on G . We show that z cannot be neutral. Assuming that $(L'(k_+)z, z) = 0$ we set $z_t = tx + (1-t)z$, $t \in [0, 1]$ where x is the previously found eigenvector of positive type for k_+ . Since $L(k_+)z_t = 0$, we have $z_t \in G'$ and $p_+(z_t) = k_+$ for $t \in [0, 1]$. Let $K = \{z_t \mid t \in [0, 1]\}$ then $K \subset G'$ is a pathwise connected set. Note that the functional p_- is continuous on G' , $p_-(z_0) = p_-(z) = k_+$ and $p_-(z_1) = p_-(x) \leq \delta_+$. Since $p_-(K)$ is connected, for every $k \in (p_-(x), p_-(z))$ there exist a $z_{t_*} \in K$ such that $p_-(z_{t_*}) = k$. If we choose k such that $\delta_+ < k < k_+$ we have $p_-(z_{t_*}) = k < k_+$, $p_+(z_{t_*}) = k_+$ and $z_{t_*} \in G$. Since $p_-(z_{t_*}) = k > \delta_+$ this leads to a contradiction with the fact that δ_+ is the upper bound of p_- on G . We conclude that z cannot be neutral and k_+ is an eigenvalue of positive type.

Now let us prove that if $(L(k_+)z, z) = 0$ then $(L'(k_+)z, z) > 0$. Since $L(k_+) \geq 0$ we can write

$$\|L(k_+)z\|^2 \leq \|L(k_+)\|(L(k_+)z, z)$$

so z, k_+ is an eigenpair and it is of positive type.

Let $\delta_+ < k < k_+$ and z be a corresponding vector such that $(L(k)z, z) = 0$. The vector z cannot be of negative type since from the condition $(L'(k)z, z) < 0$ follows that $z \in G$ and $k = p_-(z)$, contradicting the fact that $\delta_+ < k$. Assume that $(L'(k)z, z) = 0$ therefore $k = p_{\pm}(z)$. We consider an eigenvector x of positive type corresponding to the eigenvalue k_+ and we set $z_{\alpha} = z + \alpha x$. Replacing x by $-x$, we can assume that $\operatorname{Re}(L(k)z, x) \leq 0$. We note that since $\delta_+ < k < k_+$, we have $(L(k)x, x) < 0$ and therefore

$$(L(k)z_{\alpha}, z_{\alpha}) = (L(k)z, z) + 2\alpha \operatorname{Re}(L(k)z, x) + \alpha^2(L(k)x, x) < 0,$$

if $\alpha > 0$. By the condition (B.III) we have $z_{\alpha} \in G$, $\alpha > 0$ and we obtain the contradiction

$$\delta_+ \geq \lim_{\alpha \rightarrow 0^+} p_-(z_{\alpha}) = p_-(z) = k$$

with the fact that $\delta_+ < k$. The case of $W'_{p_-} \cap [k_-, \delta_-)$ is analyzed in an analogous manner. $\square$

Theorem 3.3. Let L be an operator function satisfying the conditions (B.I)-(B.IV). If $k \in W'_{p_+} \cap (k_+, k'_+] (k \in W'_{p_-} \cap [k'_-, k_-))$ then k is a neutral eigenvalue.

Proof. If $k \in W'_{p_+} \cap (k_+, k'_+]$ then there exist $x \in G'$ such that $(L(k)x, x) = 0$. Since $k > k_+$ we have $L(k) \geq 0$. Using the inequality

$$\|L(k)x\|^2 \leq \|L(k)\|(L(k)x, x)$$

we see that $L(k)x = 0$ and k, x is an eigenpair, since $x \in G' \setminus G$ it is a neutral eigenpair. $\square$

As the example below shows the sets $W_{p_{\pm}}$ are not necessarily connected for every operator function L satisfying (B.I)-(B.IV).

Example 3.1. Let $M(k) = A + kB + k^2C$ be a one parameter operator pencil, $H = \mathbb{C}^2$ and

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 4 \\ 4 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

The operator pencil M satisfies (B.I)-(B.IV) and the graphics of sets W_{p_-} and W_{p_+} (see Figure 3.3), obtained by computational methods, are disconnected.

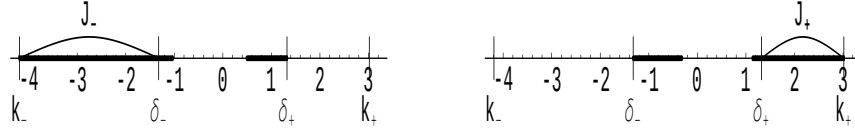


Figure 3.3: Graphics of W_{p_-} and W_{p_+}

Now we show that some parts of $W_{p_{\pm}}$ are connected. For this purpose we define the sets

$$\begin{aligned} G_+ &= \{x \in G \mid p_+(x) > \delta_+\} \\ G_- &= \{x \in G \mid p_-(x) < \delta_-\}. \end{aligned}$$

If G_+ (G_-) is nonempty then some part of $W_{p_{\pm}}$ turn out to be connected, as we can see in Figure 3.3.

We set $J_{\pm} = p_{\pm}(G_{\pm})$. If the set G_+ (G_-) is empty then we consider J_+ (J_-) is empty. In the case $G_{\pm} \neq \emptyset$ we have

$$\begin{aligned} J_+ &= W_{p_+} \cap (\delta_+, k_+] \\ J_- &= W_{p_-} \cap [k_-, \delta_-). \end{aligned}$$

Theorem 3.4. Let L be an operator function satisfying (B.I)-(B.IV). If H is a complex Hilbert space, then the sets $G_{\pm}$ are pathwise connected and $J_{\pm}$ are connected.

Proof. Since the functionals $p_{\pm}$ are continuous on G , the connectedness of $J_{\pm}$ follows from pathwise connectedness of $G_{\pm}$. We show that G_+ is pathwise connected. Let $x, y \in G_+$ then $p_+(x) > \delta_+$, $p_+(y) > \delta_+$, $p_-(x) \leq \delta_+$ and

$p_-(y) \leq \delta_+$. We select $\varepsilon > 0$ such that $p_+(x) > \delta_+ + \varepsilon$, $p_+(y) > \delta_+ + \varepsilon$ and we set $k = \delta_+ + \varepsilon$. Then

$$k \in (p_-(x), p_+(x)) \cap (p_-(y), p_+(y)). \quad (3.3)$$

We select $\lambda = 1$ or -1 so that $\operatorname{Re}[\lambda(L(k)x, y)]$ is nonpositive, and we set $\tilde{x} = \lambda x$, $z_\alpha = \alpha\tilde{x} + (1 - \alpha)y$, $\alpha \in [0, 1]$. Then

$$(L(k)z_\alpha, z_\alpha) = \alpha^2(L(k)x, x) + 2\alpha(1 - \alpha)\operatorname{Re}[\lambda(L(k)x, y)] + (1 - \alpha)^2(L(k)y, y).$$

Note we note that if $x \in G$ from (B.II) and (B.III)

$$(L(t)x, x) < 0 \iff t \in (p_-(x), p_+(x)).$$

Taking $t = k$ from (3.3) we obtain that both $(L(k)x, x)$ and $(L(k)y, y)$ are negative. Therefore $(L(k)z_\alpha, z_\alpha)$ is negative for $\alpha \in [0, 1]$. From this follows that $z_\alpha \in G$ and $k < p_+(z_\alpha)$. Since $k > \delta_+$ we have $z_\alpha \in G_+$ for $\alpha \in [0, 1]$. Thus, $\tilde{x}$ can be joined with y in G_+ by a segment. Since H is a complex space, in the case $\lambda = -1$ the vectors x and $-x$ in G can be joined by $xe^{i\varphi}$, $(0 \leq \varphi \leq \pi)$. For G_- proof is similar. $\square$

Now we show that the root zones, under some additional conditions, are not separated. An example was given above (see Figure 3.3). First we prove the following

Lemma 3.2. The set G is open and $\overline{G} = G'$.

Proof. First we show that G^c is closed. Let $\{x_n\} \subset G^c$ and $x_n \rightarrow x$. Then for every $\alpha \in [a, b]$ we have

$$0 \leq (L(\alpha)x_n, x_n) \rightarrow (L(\alpha)x, x) \geq 0$$

so $x \in G^c$ and G^c is closed.

Now let $\{x_n\} \subset G$ and $x_n \rightarrow x$. Then to every x_n corresponds an $\alpha_n \in [a, b]$ such that $(L(\alpha_n)x_n, x_n) < 0$. Since $\{\alpha_n\} \subset [a, b]$ is bounded has a convergent subsequence. Let us rename it again as $\{\alpha_n\}$ and let $\alpha_n \rightarrow \alpha$. Since

$$0 > (L(\alpha_n)x_n, x_n) \rightarrow (L(\alpha)x, x) \leq 0$$

it follows that $x \in G'$ and $\overline{G} = G'$. $\square$

Theorem 3.5. Let L be an operator function satisfying the conditions (B.I)-(B.IV). If $G \neq \emptyset$ and $G \neq H \setminus \{0\}$ then $\delta_- \leq \delta_+$.

Proof. By the condition $G \neq \emptyset$ there exists $x_1 \in G$. On the other hand it follows from $G \neq H \setminus \{0\}$ that there exists $x_2 \notin G$. Define a path from x_1 to x_2 by $z_t = (1-t)x_1 + tx_2$, $0 \leq t \leq 1$. Since $x_1 \in G$, $x_2 \notin G$ and G is open, there exists a number $t_* \in (0, 1]$ such that $z_t \in G$ for all $t \in [0, t_*)$. From $\overline{G} = G'$ follows that $z_{t_*} \in G'$. Now we can write

$$\begin{aligned} \delta_- &= \inf_{z \in G} p_+(z) \leq \lim_{t \rightarrow t_*-0} p_+(z_t) = p_+(z_{t_*}) = \\ &= p_-(z_{t_*}) = \lim_{t \rightarrow t_*-0} p_-(z_t) \leq \sup_{z \in G} p_-(z) = \delta_+. \end{aligned}$$

□

3.3 Examples

Now we give some examples of classes of operator functions satisfying the conditions (B.I)-(B.IV). Here, in the first two examples, we aim to show where our problem comes from. Note that, even if these two classes are studied extensively, some of the results (see Theorem 3.2) are new for these classes as well.

3.3.1 One parameter pencils of waveguide type

Definition 3.1 ([11], pp. 1278–1279). An operator pencil $L(k) = k^2C + kB + A$, where A , B and C are bounded and symmetric operators in a Hilbert space H is said to be an operator pencil of waveguide type if the following conditions are satisfied

- (A1) $C > 0$;
- (A2) $A = A_1 - A_2$, $A_1 \gg 0$, $A_2 \in S_\infty$;
- (A3) B and C are compact operators;
- (A4) $G \neq \emptyset$, $G \neq H \setminus \{0\}$;
- (A5) $-\infty < k'_-, k'_+ < \infty$.

We set $d(x) = (Bx, x)^2 - 4(Cx, x)(Ax, x)$. The sets G and G' are defined as

$$\begin{aligned} G' &= \{x \in H \setminus \{0\} \mid d(x) \geq 0\}, \\ G &= \{x \in H \mid d(x) > 0\} \end{aligned}$$

and the functionals $p_{\pm}(x)$ have the form

$$p_{\pm}(x) = \frac{-(Bx, x) + \sqrt{d(x)}}{2(Cx, x)}, \quad x \in G'.$$

Now we choose $[a, b] = [k'_-, k'_+]$ and the conditions (B.I)-(B.IV) follow from the conditions (A1)-(A3) [11, p. 1281].

3.3.2 Two parameter quadratic pencils of waveguide type

Definition 3.2 ([22], Definition 2.1). An operator pencil of the form $L(k, w) = A + kB + k^2C - w^2I$ is called weak two parameter pencil of waveguide type if the following conditions are satisfied:

(B1) The operator A is nonnegative and $(A + I)^{-1} \in S_{\infty}$, where S_{∞} is the set of compact operator.

(B2) C is a bounded and positive definite operator.

(B3) B is symmetric and $(A + I)^{-1/2}B(A + I)^{-1/2} \in S_{\infty}$.

Additionally, if the following condition:

(B4) $\exists \varepsilon$ satisfying $0 < \varepsilon < 1$ such that $(Au, u) + k(Bu, u) + \varepsilon^2 k^2(Cu, u) \geq 0$, $\forall k \in \mathbb{R}, u \in D((A + I)^{1/2})$ – the domain of the operator $(A + I)^{1/2}$ – called the energy stability condition is satisfied then we say that we have a weak operator pencil with the energy stability condition.

Here the coefficients A and B may be unbounded. We can transform the pencil $L(k, w)$ to the pencil $\tilde{L}(k, w) := (A + I)^{-1/2}L(k, w)(A + I)^{-1/2}$ which has bounded coefficients. Note that $\sigma(\tilde{L}) = \sigma(L)$. If we write

$$\tilde{L}(k, w) = k^2\tilde{C} + k\tilde{B} + \tilde{A}(w)$$

then

$$\tilde{A}(w) = I - (1 + w^2)(A + I)^{-1}, \quad \tilde{C} = (A + I)^{-1/2}C(A + I)^{-1/2},$$

$$\tilde{B} = (A + I)^{-1/2}C(A + I)^{-1/2}.$$

Now, for fixed $w \in \mathbb{R}$, let us check the conditions (A1)-(A3) and (A5).

(A1) By the condition (B2), $\tilde{C} > 0$.

(A2) $\tilde{A}(w) = I - (1 + w^2)(A + I)^{-1}$. $\tilde{A}_1 = I \gg 0$, $\tilde{A}_2 = (1 + w^2)(A + I)^{-1} \in S_\infty$ by (B1).

(A3) It follows from the conditions (B1)-(B3) that $\tilde{B}$ and $\tilde{C}$ are compact.

(A5) It follows from the conditions (B2) and (B4) that there exists a number $c_0 > 0$ such that for all $k \in \mathbb{R}$ and for all $u \in D((A + I)^{1/2})$

$$(Au, u) + k(Bu, u) + k^2(Cu, u) \geq c_0^2 k^2(u, u).$$

For $v \in H$, $v \neq 0$ let $u = (A + I)^{-1/2}v$ then we have

$$([I - (A + I)^{-1}]v, v) + k(\tilde{B}v, v) + k^2(\tilde{C}v, v) \geq c_0^2 k^2((A + I)^{-1}v, v)$$

and

$$(\tilde{L}(k, w)v, v) \geq (c_0^2 k^2 - w^2)((A + I)^{-1}v, v).$$

If k is a root of the equation $(\tilde{L}(k, w)v, v) = 0$, i.e, $k = p_-(v)$ or $k = p_+(v)$ then

$$0 \geq (c_0^2 k^2 - w^2)((A + I)^{-1}v, v) = (c_0^2 k^2 - w^2) \|(A + I)^{-1/2}v\|^2.$$

Consequently, we have $c_0^2 k^2 \leq w^2$, hence $-\infty < k'_-, k'_+ < \infty$.

Now the conditions (B.I)-(B.IV) follow from Example 3.3.1.

3.3.3 Polynomial operator pencils of waveguide type

For two parameter polynomial operator pencils of waveguide type defined in Section 2 (see Definition 2.1), in the case $D = 0$, the conditions (B.I)-(B.IV) are fulfilled on some parts of the root domain.

4 RIESZ BASES OF EIGENVECTORS FOR A CLASS OF NONANALYTIC OPERATOR FUNCTIONS

4.1 Introduction

In this section we study Riesz basis properties for a class of self-adjoint and continuous operator functions. More detailed history of the problem and some preliminary definitions are given in Section 1 (see pp. 9-11).

Let $L(\alpha)$ be an analytic operator function defined on $[a, b]$ satisfying the conditions

$$(C1) \quad L(a) \ll 0, \quad L(b) \gg 0,$$

$$(C2) \quad \text{for all } x \in H \setminus \{0\} \text{ the function } (L(\alpha)x, x) \text{ has exactly one zero in } (a, b),$$

$$(C3) \quad \pi(L) = \{\gamma\} \in (a, b),$$

where $\pi(L)$ denotes the limit spectrum as defined in (1.4). It was shown that eigenvectors of $L(\alpha)$ corresponding to eigenvalues in (a, b) form a Riesz basis for the Hilbert space H (see [2], Theorem 30.12). The result follows from a representation of the form

$$L(\alpha) = B(\alpha)(\alpha I - Z), \tag{4.1}$$

where $B(\alpha)$ is invertible on $[a, b]$, $\sigma(Z) \subset (a, b)$ and Z is similar to a self-adjoint operator.

A similar result for a class of nonanalytic operator functions was obtained by A. S Markus and V. Matsaev (see [33], [35]). Namely, for an operator function $L(\alpha) \in C^2([a, b], S(H))$ they proved that under the conditions

$$i) \quad L(a) \ll 0, \quad L(b) \gg 0,$$

$$ii) \quad \int_0^{t_0} \frac{w(t, L'')}{t} dt < +\infty \text{ for sufficiently small } t_0, \text{ where } w(t, L'') \text{ is the modulus of continuity for } L'',$$

- iii) the operator function L satisfies the regularity condition i.e., there exist positive numbers δ and ϵ such that for every $\alpha \in [a, b]$ and $x \in H$, $\|x\| = 1$,
 $| (L(\alpha)x, x) | < \epsilon \Rightarrow (L'(\alpha)x, x) > \delta$

the representation (4.1) is still valid. Therefore if we add the condition (C3) then Riesz basis property follows from the representation $L(\alpha) = B(\alpha)(\alpha I - Z)$. A similar factorization theorem under weaker conditions was given in the paper of T. Ya. Azizov, A. Dijksma and L. I. Sukhocheva [34].

Note that in the above mentioned classes the Riesz basis properties of the eigenvalues of the operator functions derive from the existence of the representation (4.1). In the class $C([a, b], S(H))$ main difficulty is that under the conditions (C1)-(C3) such representation is unknown. In this thesis we study this problem using a new variational approach based on the spectral distribution function. We prove that the assertion holds in a dense subspace of the space $C([a, b], S(H))$ under the conditions (C1)-(C3) if the number of eigenvalues either at the right or at the left of $\{\gamma\}$ is finite. We denote this class by $C_\gamma^F([a, b], S(H))$.

4.2 Main Results

In what follows we suppose

Assumption 4.1. $L(a) \ll 0$, $L(b) \gg 0$, for all $x \in H \setminus \{0\}$ the function $(L(\alpha)x, x)$ has exactly one zero in (a, b) and $\pi(L) = \{\gamma\} \in (a, b)$.

Note that all operators in this section are bounded and we say that an operator is invertible if it is boundedly invertible. The main result given in this section is connected with the notion of approximate Riesz basis in the following sense.

Definition 4.1. Let $L \in C([a, b], S(H))$. We say that the eigenvectors of the operator function L form an approximate Riesz basis if there is a sequence $\{L_n\}_{n=1}^\infty \in C([a, b], S(H))$ of operator functions such that $L_n \Rightarrow L$ (uniformly) as $n \rightarrow \infty$, and the eigenvectors of L_n for all n form a Riesz basis for H .

Recall that a collection $\{f_\alpha\}$, $\alpha \in I$ of elements of H is called a Riesz basis of H if there is an invertible, bounded operator G such that $\{Gf_\alpha\}$, $\alpha \in I$ is an

orthonormal basis of H (see [2] and [32]). We study this problem in a separable Hilbert space H and for this reason we choose $I = \mathbb{N}$, where $\mathbb{N}$ denotes the set of natural numbers.

Denote by $PW([a, b], S(H))$ the class of piece-wise linear and continuous operator functions ([8] and [42]) and define the subspace $PW_\gamma^F([a, b], S(H))$ by the same way as the subspace $C_\gamma^F([a, b], S(H))$. We use mainly an approximation method given in [8] and [42]. According to this method a continuous operator function, satisfying the Rayleigh system axioms (see [7] and [42]) can be approximated by piece-wise linear ones from the same class. Note that operator functions studied in this note form a subclass of Rayleigh systems. For this reason we consider first the basis problem for the piece-wise linear operator functions.

Theorem 4.1. Let L be an operator function from $PW([a, b], S(H))$ satisfying Assumption 4.1. If L is differentiable at the point γ or $L \in PW_\gamma^F([a, b], S(H))$ then the eigenvectors, corresponding to eigenvalues in $[a, b]$ form a Riesz basis for H .

Proof. By the conditions of the theorem $L \in PW([a, b], S(H))$. Consequently there are finite number (denoted by k) of points of discontinuities of derivative $L'(\alpha)$. We use the principle of induction and therefore at the beginning we prove this theorem for $k = 1$, i.e. for operator functions of the form

$$L(\alpha) = \begin{cases} \alpha B_+ - A, & \alpha \geq 0, \\ \alpha B_- - A, & \alpha \leq 0, \end{cases} \quad \alpha \in [a, b].$$

Here by shifting the argument we assume that the point of discontinuity of $L'(\alpha)$ is 0. The proof is completely based on a variational approach. Namely we use variational principles for the spectral distribution function $N(\lambda, L)$ to prove this theorem. $N(\lambda, L)$ is the number of eigenvalues of L that strictly larger than λ . For pencils of the form $L(\alpha) = \alpha B - A$, where $B > 0$ or $B < 0$ (the definite case) we use the classical variational principles for $N(\lambda, L)$. If neither $B > 0$ nor $B < 0$ is satisfied then it is an indefinite case and we use a variational principle for pencils in the indefinite case.

Let us first prove the theorem under the definiteness conditions $B_\pm \gg 0$. Since

$\pi(L) = \{\gamma\}$ the spectrum $\sigma(L) \setminus \{\gamma\}$ is discrete (see [42]). We consider two cases: $\gamma = 0$ and $\gamma \neq 0$. Now suppose $\gamma = 0$ and $L \in PW_\gamma^F([a, b], S(H))$. Then either $N(0, L) < +\infty$ or there are finite number of negative eigenvalues. We suppose $N(0, L) = n < +\infty$. Let us construct a self-adjoint operator function of the form

$$F(\alpha) = \begin{cases} \alpha I - B_+^{-\frac{1}{2}} A B_+^{-\frac{1}{2}}, & \alpha \geq 0, \\ \alpha I - B_-^{-\frac{1}{2}} A B_-^{-\frac{1}{2}}, & \alpha \leq 0. \end{cases}$$

We denote by $M_+(L)$ ($M_-(L)$) the closed linear span of eigenvectors, corresponding to positive (nonpositive) eigenvalues of the operator function $L(\alpha)$. Let $H_+(U)$ ($H_-(U)$) be the closed linear span of the eigenvectors, corresponding to positive (nonpositive) eigenvalues of an operator U . We have

$$\sigma(F) = \sigma(L), \quad \sigma_e(F) = \sigma_e(L),$$

and

$$B_+^{\frac{1}{2}} : M_+(L) \rightarrow H_+(T); \quad B_-^{\frac{1}{2}} : M_-(L) \rightarrow H_-(S), \quad (4.2)$$

where $T = B_+^{-\frac{1}{2}} A B_+^{-\frac{1}{2}}$ and $S = B_-^{-\frac{1}{2}} A B_-^{-\frac{1}{2}}$. Denoting by $L_\pm(\alpha) = \alpha B_\pm - A$ we obtain (see [7] and [8]) the following equality for the spectral distribution function of the operator functions $L_\pm(\alpha)$ which plays the key role in the proof of this case.

$$\begin{aligned} \dim H_+(B_-^{-1} A) &= N(0, L_-) \\ &= \max \dim \left\{ E \mid \frac{(Au, u)}{(B_- u, u)} > 0, u \in E \setminus \{0\} \right\} \\ &= \max \dim \{ E \mid (Au, u) > 0 \} \\ &= \max \dim \left\{ E \mid \frac{(Au, u)}{(B_+ u, u)} > 0, u \in E \setminus \{0\} \right\} \\ &= N(0, L_+) = n. \end{aligned} \quad (4.3)$$

We can write

$$H_+(S) = B_-^{\frac{1}{2}} H_+(B_-^{-1} A). \quad (4.4)$$

It follows from (4.3) and (4.4) that $\dim H_+(S) = n$. Consequently, S is a compact and self-adjoint operator. Thus $H = H_-(S) \oplus H_+(S)$ and $\dim H_-^\perp(S) = \dim H_+(S) = n$. Because of the invertibility of the operator $B_-^{-\frac{1}{2}}$ we have

$$\dim [B_-^{-\frac{1}{2}} [H_-(S)]]^\perp = \dim [H_-(S)]^\perp = n. \quad (4.5)$$

Now, writing the formula $H = M_-(L) \oplus M_-(L)^\perp$ in the form

$$H = \left[B_-^{-\frac{1}{2}}[H_-(S)] \right] \oplus \left[B_-^{-\frac{1}{2}}[H_-(S)] \right]^\perp$$

we obtain by (4.5) $\dim M_-(L)^\perp = n$. On the other hand by the condition $\dim M_+(L) = n$ and $M_-(L) \cap M_+(L) = \{0\}$ (see[2], Theorem 32.8). Therefore

$$H = M_-(L) \dot{+} M_+(L). \quad (4.6)$$

Now it follows from (4.2) and (4.6) (see [32]) that the eigenvectors of the operator function $L(\alpha)$ form a Riesz basis of H .

If L is differentiable at the point γ , then $\gamma \neq 0$ and under the conditions of the theorem the spectrum of operator function $L(\alpha)$ in (a, b) consists of the point γ and at most countable number of eigenvalues of finite multiplicity (see [42]). If the set is infinite, then it converges to γ . Thus, here we have the same situation as in the case $\gamma = 0$ and the Riesz basis property in the case follows from the above proved case.

Now let us consider the indefinite case. It means that we do not suppose that the conditions $B_\pm \gg 0$ are satisfied. We have

$$L(\alpha) = \begin{cases} \alpha B_+ - A, & \alpha \geq 0, \\ \alpha B_- - A, & \alpha \leq 0 \end{cases} \quad \alpha \in [a, b].$$

Here $a < 0$, $b > 0$ and, by Assumption 4.1 $L(a) \ll 0$, $L(b) \gg 0$. Recall that we have supposed $N(0, L) < \infty$.

i) if $A \leq 0$ then $\sigma_e^+(L) := \sigma_e(L) \cap (0, +\infty) = \emptyset$. Therefore $\sigma_e(L) = \sigma_e(L_-)$ and the pencil $L_-(\alpha) = \alpha B_- - A$ satisfies Assumption 4.1 on $[a, b]$. Thus the eigenvectors of $L_-(\alpha)$ and consequently the eigenvectors of $L(\alpha)$ form a Riesz basis of H .

ii) if $A \geq 0$ then $\sigma_e^-(L) := (-\infty, 0) \cap \sigma_e(L) = \emptyset$, $\sigma_e(L) = \sigma_e(L_+)$ and we have the same situation as in the case i) if we consider L_+ instead of L_- . In this case since eigenvectors of L (or L_+) form a basis of H it follows from the condition $N(0, L) < +\infty$ that $\gamma = 0$ is an eigenvalue of infinite multiplicity.

iii) Let A be a self-adjoint operator. Define the cones

$$C_-^A = \{x | (Ax, x) < 0\} \quad \text{and} \quad C_+^A = \{x | (Ax, x) > 0\}.$$

It remains to consider the case $C_{\pm}^A \neq \emptyset$. Now using variational principles in indefinite case we have

$$N(0, L) = \max \dim\{E | E \subset C_+^A\}$$

and

$$N(0, L_-) = \max \dim\{E | E \subset C_+^A \cap C_+^{B-}\}.$$

It follows from these formulae that

$$N(0, L_-) \leq N(0, L) = n \quad (4.7)$$

Hence the spectrum of the linear pencil $L_-(\alpha) = \alpha B_- - A$ is discrete and eigenvectors form a Riesz basis of H . We can write

$$H = M_-(L_-) \dot{+} M_+(L_-) \quad (4.8)$$

On the other hand

$$M_+(L) \cap M_-(L_-) = \{0\}. \quad (4.9)$$

Now we obtain from (4.7), (4.8) and (4.9) that

$$N(0, L_-) = N(0, L) = n$$

and

$$H = M_-(L_-) \dot{+} M_+(L) = M_-(L) \dot{+} M_+(L)$$

The general case: We turn now to the inductive step from $n - 1$ to n , so for $k = n - 1$ we assume that all eigenvectors of L , corresponding to eigenvalues from $[a, b]$ form a Riesz basis of H . Let $\{t_k\}_1^n$ be points of discontinuity of the derivative of L . Setting $t_0 = a$ and $t_{n+1} = b$ the piece-wise linear pencil L (it is denoted here by L_n to indicate that L' has n points of discontinuity) can be written in the form

$$L_n(\alpha) = \sum_{k=0}^n \left[\frac{\alpha - t_k}{t_{k+1} - t_k} (A_{k+1} - A_k) + A_k \right] \times \chi_{[t_k, t_{k+1})},$$

where $\chi_{[t_k, t_{k+1})}$ is the characteristic function of the interval $[t_k, t_{k+1})$ and $\{A_k\}_0^{n+1}$ are bounded operators. By shifting the argument we may assume that $t_n = 0$. Despite the fact that the operator function $L_n(\alpha)$ satisfies Assumption 4.1, in

general the function $L_{n-1}(\alpha)$ (it is linearly extended on $[t_n, b]$) does not satisfy it. But it can be extended to $[a, b]$, satisfying the main assumption and having the same number of points of discontinuities of the derivative and such that

$$\sigma_e^-(L_n) = \sigma_e^-(\tilde{L}_{n-1}), \quad N(0, \tilde{L}_{n-1}) = N(0, L_n). \quad (4.10)$$

Now we have

$$M_-(\tilde{L}_{n-1}) = M_-(L_n)$$

and by the assumption for $k = n - 1$

$$H = M_-(\tilde{L}_{n-1}) \dot{+} M_+(\tilde{L}_{n-1}). \quad (4.11)$$

We obtain from (4.10) that

$$\dim M_+(\tilde{L}_{n-1}) = N(0, \tilde{L}_{n-1}) = N(0, L_n) = \dim M_+(L_n) = n.$$

The needed result follows taking into account (4.11) and the fact that $M_-(L_n) \cap M_+(L_n) = \{0\}$. $\square$

Now we are ready to prove an approximate basis property in the class $C_\gamma^F([a, b], S(H))$.

Theorem 4.2. If an operator function L satisfies Assumption 4.1 and $L \in C_\gamma^F([a, b], S(H))$ then the eigenvectors of L form an approximate Riesz basis.

Proof. Since $L \in C([a, b], S(H))$ and $\pi(L) = \{\gamma\}$ there exists a sequence $\{L_n\}_{n=1}^\infty$ such that:

- a) $L_n \Rightarrow L$ and $P_n \Rightarrow P$ (uniformly). Here by $P_n(x)$ and $P(x)$ are denoted roots of the equations $(L_n(\alpha)x, x) = 0$ and $(L(\alpha)x, x) = 0$, respectively.
- b) $L_n \in PW([a, b], S(H))$, having the properties given in Assumption 4.1 (see [8], Theorem 2.2 and Lemma 3.2).

Moreover, it follows from the condition $L \in C_\gamma^F([a, b], S(H))$ that $L_n \in PW_\gamma^F([a, b], S(H))$ i.e., $N(\gamma, L_n) < \infty$. Indeed, $N(\gamma, L) < \infty$ and $\|P_n - P\| < \varepsilon$ for all $\varepsilon > 0$ and sufficiently large n . Then denoting

$$\lambda_1 := \min_{\sigma_e(L) \cap [\gamma, b]} \lambda$$

we have $\lambda_1 > \gamma$ and

$$N(\theta, L) = N(\theta - 2\varepsilon, L), \quad (4.12)$$

for $\theta \in (\gamma, \lambda_1)$ and sufficiently small ε . On the other hand using the following formula for the spectral distribution function (see [7])

$$N(\theta, L) = \max \dim \left\{ E \mid P(x) > \theta, x \in E \setminus \{0\} \right\}$$

and choosing ε from the inequality $\|P_n - P\| < \varepsilon$ we obtain from (4.12) that

$$N(\theta - \varepsilon, L_n) = N(\theta, L) = N(\gamma, L) < +\infty.$$

It means that $N(\gamma, L_n) < \infty$. Consequently, the sequence $\{L_n\}_{n=1}^{\infty}$ (we can choose n sufficiently large) satisfies the conditions of Theorem 4.1 and the needed results follow immediately from Theorem 4.1. $\square$

Remark 4.1. The completeness of the eigenfunctions in Theorem 4.1 in the definite case can be proved by using an indefinite scalar product ([3]). Here we illustrate it for a model problem of the form

$$L(\alpha) = \begin{cases} \alpha B_+ - A, & \alpha \geq 0, \\ \alpha B_- - A, & \alpha \leq 0 \end{cases} \quad \alpha \in [a, b].$$

$0 \neq f \perp M := \text{span} \{M_{\pm}(L)\} \iff f \perp M_+(L)$ and $f \perp M_-(L)$. It is clear that $0 \neq f \perp M_-(L)$ implies $B_-^{-\frac{1}{2}} f \in H_+(S)$ and $f \in R(A)$. Define $[x, y] := (A^{-1}x, y)$ ($\ker A \neq 0$). We obtain

$$[f, f] > 0. \quad (4.13)$$

On the other hand using the condition $0 \neq f \perp M_+(L)$ we have $B_+^{-\frac{1}{2}} f \in H_+^{\perp}(T) = H_-(T)$. Finally, since $f \in R(A)$ it is easy to check that

$$[f, f] = (A^{-1}f, f) < 0. \quad (4.14)$$

The inequality (4.14) contradicts the inequality (4.13). Consequently, $f = 0$.

For more information about spectral theory in indefinite inner product spaces see [43], [44], [45], [46] and [47].

5 AN APPLICATION OF TWO PARAMETER OPERATOR POLYNOMIALS TO PDE'S

As mentioned in the introduction, operator pencils have application to a wide range of spectral problems deriving from differential equations and boundary value problems, evolution equations, block matrices, controllable systems and equations depending on one or more parameters.

Although it is not the main aim of the thesis, in this section we give an example to the connection between partial differential equations and operator pencils of waveguide type and for resulting operator pencil we find asymptotics of the spectral distribution function.

Let Ω be a bounded domain in the x_2x_3 -plane and Q be the cylinder $\mathbb{R} \times \Omega$, i.e.

$$Q = \{(x_1, x_2, x_3) \mid x_1 \in \mathbb{R}, (x_2, x_3) \in \Omega\}.$$

We consider the equation

$$\frac{\partial^2 v}{\partial t^2} - \sum_{i,j=1}^3 \frac{\partial}{\partial x_i} \left(a_{ij}(x_2, x_3) \frac{\partial v}{\partial x_j} \right) = 0 \quad (x_1, x_2, x_3) \in Q \quad (5.1)$$

in the cylinder Q with the boundary condition

$$\sum_{i,j=1}^3 \vartheta^i a_{ij} \frac{\partial v}{\partial x_j} \Big|_{\partial Q} = 0, \quad t \in \mathbb{R} \quad (5.2)$$

where $\vartheta = (0, \vartheta^2, \vartheta^3)$ is the normal to the $\partial\Omega$ in the x_2x_3 -plane, $a_{ij} = a_{ji}$ real valued functions, $a_{ij} \in L_\infty(\Omega)$ and

$$\nu \|\xi\|^2 \leq \sum_{i,j=1}^3 a_{ij}(x_2, x_3) \xi_i \bar{\xi}_j \leq \mu \|\xi\|^2, \quad \nu, \mu > 0. \quad (5.3)$$

The condition (5.3) means that the equation (5.1) is an equation of uniformly hyperbolic type. Here we reduce the equation (5.1) with boundary condition (5.2) to an quadratic operator pencil of waveguide type of the form

$$L(k, w) = A + kB + k^2C - w^2I$$

Define a generalized solution of the problem (5.1) and (5.2) as an element $v \in W_2^1(\Omega)$, for any $x \in \mathbb{R}$ and $t \in \mathbb{R}$ by the identity

$$\begin{aligned} & \frac{\partial^2}{\partial t^2} \int_{\Omega} v \bar{\eta} dx_2 dx_3 - \frac{\partial^2}{\partial x^2} \int_{\Omega} a_{11} v \bar{\eta} dx_2 dx_3 \\ & - \int_{\Omega} \left[\sum_{i=2}^3 \frac{\partial}{\partial x_i} \left(a_{i1} \frac{\partial v}{\partial x} \right) + \sum_{j=2}^3 \frac{\partial}{\partial x} \left(a_{1j} \frac{\partial v}{\partial x_j} \right) \right] \bar{\eta} dx_2 dx_3 \\ & - \sum_{i,j=2}^3 \int_{\Omega} \frac{\partial}{\partial x_i} \left(a_{ij} \frac{\partial v}{\partial x_j} \right) \bar{\eta} dx_2 dx_3 = 0, \quad \forall \eta \in W_2^1(\Omega) \end{aligned} \quad (5.4)$$

By integrating by parts and using the boundary value condition (5.2), we can write (5.4) in the form

$$\begin{aligned} & \frac{\partial}{\partial t^2} \int_{\Omega} v \bar{\eta} dx_2 dx_3 - \frac{\partial}{\partial x^2} \int_{\Omega} a_{11} v \bar{\eta} dx_2 dx_3 \\ & - \frac{\partial}{\partial x} \sum_{j=2}^3 \left\{ \int_{\Omega} \left[a_{1j} \frac{\partial v}{\partial x_j} + \frac{\partial}{\partial x_j} (a_{j1} v) \right] \bar{\eta} dx_2 dx_3 - \int_{\partial \Omega} \vartheta^j a_{1j} v \bar{\eta} ds \right\} \\ & + \sum_{i,j=2}^3 \int_{\Omega} a_{ij} \frac{\partial v}{\partial x_i} \frac{\partial \bar{\eta}}{\partial x_j} dx_2 dx_3 = 0. \end{aligned} \quad (5.5)$$

We can treat the operators A , B and C in the following way:

According to the equation (5.5)

$$Cu = a_{11}u. \quad (5.6)$$

The operators B and A are associated with the bilinear forms

$$\mathcal{J}_B[u, \eta] = i \sum_{j=2}^3 \left\{ \int_{\Omega} \left[a_{1j} \frac{\partial u}{\partial x_j} + \frac{\partial}{\partial x_j} (a_{j1} u) \right] \bar{\eta} dx_2 dx_3 - \int_{\partial \Omega} \vartheta^j a_{1j} u \bar{\eta} ds \right\} \quad (5.7)$$

and

$$\mathcal{J}_A[u, \eta] = \sum_{i,j=2}^3 \int_{\Omega} a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial \bar{\eta}}{\partial x_j} dx_2 dx_3, \quad (5.8)$$

respectively. It is known that there exist operators B and A with largest domain satisfying

$$\begin{aligned} \mathcal{J}_B[u, \eta] &= (Bu, \eta), \quad u \in D(B), \eta \in D(\mathcal{J}_B), \\ \mathcal{J}_A[u, \eta] &= (Au, \eta), \quad u \in D(A), \eta \in D(\mathcal{J}_A), \end{aligned}$$

respectively (see [31]). Setting $V(t, x) = v(t, x, x_2, x_3)$, where $x = x_1$ in (5.5) we obtain a evolution equation $V_{tt} - CV_{xx} + iBV_x + AV = 0$ with operator A , B and C defined by (5.6), (5.7) and (5.8). Substituting the function $V(t, x) = ue^{i(wt-kx)}$, $u \in H$ we obtain the operator pencil

$$L(k, w) = A + kB + k^2C - w^2I.$$

Let us show that $L(k, w)$ is an operator pencil of waveguide type, i.e. the operators A , B and C satisfy the conditions in Definition 2.1. For this purpose we give some known results on embedding theorems and bilinear forms. We follow [48] and [31]. Since we consider the boundary value (5.2) problem for the equation (5.1) our energy space is $W_2^1(\Omega)$ (see [48], Chapter IV, p. 168). It is known that under some additional conditions Ω , $W_2^1(\Omega)$ is compactly embedded in $L_2(\Omega)$. Particularly, this is true for domains with the smooth boundary ([48], p. 28, Theorem 6.2). In what follows we suppose that this conditions are satisfied.

Now we are going to give needed results on bilinear forms and associated operators (see [31], Chapter 10). Let $\mathcal{J}[u, v]$ be a densely defined bilinear form in a Hilbert space H . We suppose that for $u \in D(\mathcal{J})$ the inequality $\mathcal{J}[u, u] \geq m(u, u)$ is satisfied, i.e. $\mathcal{J}$ is symmetric and bounded from below. If $m > 0$ then $\mathcal{J}$ is said to be a positive definite form. Let $m > 0$. Taking $[u, v]_{\mathcal{J}} = \mathcal{J}[u, v]$, as a scalar product we transform $D(\mathcal{J})$ into a pre-Hilbert space. The space could turn out to be incomplete, in which case we complete it in the usual way. The completed space is called an energy space and will be denoted by $H_{\mathcal{J}}$. If $\langle D(\mathcal{J}), |u|_{\mathcal{J}} \rangle$ is complete $\mathcal{J}[u, v]$ is called the closed form. It is known that there is one to one correspondence between positive definite forms and self-adjoint, positive definite operators ([31], p. 222, Theorem 2). This correspondence it is given by the formula $\mathcal{J}[u, v] = (A_{\mathcal{J}}^{1/2}u, A_{\mathcal{J}}^{1/2}v)$, $u, v \in D(A_{\mathcal{J}}^{1/2})$. Hence, for positive definite form $\mathcal{J}$ and the associated operator $A_{\mathcal{J}}$, we have $D(\mathcal{J}) = D(A_{\mathcal{J}}^{1/2})$. In the case $m \leq 0$ we consider the forms $\mathcal{J}_{\alpha}[u, v] = \mathcal{J}[u, v] + \alpha(u, v)$, ($\alpha > -m$).

Theorem 5.1 ([31], p. 224, Theorem 5). Let $\mathcal{J}[u, v]$ be a closed, densely defined positive definite form and $A_{\mathcal{J}}$ be the associated positive definite operator. The spectrum of $A_{\mathcal{J}}$ is discrete (equivalently $A_{\mathcal{J}}^{-1}$ is compact) if and only if $H_{\mathcal{J}}$ is embedded compactly in H .

Now let us check the conditions (A.I)-(A.V) beginning from (A.I):

(A.I) The operator A is nonnegative and $(A + I)^{-1} \in S_\infty$.

Proof. The operator A is given by the bilinear form $\mathcal{J}_A[u, \eta] = \sum_{i,j=2}^3 \int_{\Omega} a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial \bar{\eta}}{\partial x_j} dx_2 dx_3$. By the condition, the equation (5.1) is of uniformly hyperbolic type. Then, it follows from (5.3) that $A \geq 0$. The energy space of the operator A is $W_2^1(\Omega)$. Now, from the fact that $W_2^1(\Omega) \hookrightarrow L_2(\Omega)$ is embedded compactly, by Theorem 5.1, we obtain that A has a discrete spectrum. Consequently $(A + I)^{-1} \in S_\infty$. $\square$

The condition (A.II) follows from (5.6).

(A.III) $(A + I)^{-1/2} B (A + I)^{-1/2} \in S_\infty$.

Proof. Let us denote by $\tilde{H}$ the energy space of $A + I$. We have $\tilde{H} = D((A + I)^{1/2})$. The scalar product in $\tilde{H}$ is defined by $\langle u, v \rangle_{\tilde{H}} = ((A + I)^{1/2} u, (A + I)^{1/2} v)$ and the norm in $\tilde{H}$ is denoted by $|u|_{\tilde{H}}$. By $\|\cdot\|$ we denote the usual norm in H . It is known that if the condition (A.I) and the inequality

$$|\mathcal{J}_B[u, \eta]| \leq C (|u|_{\tilde{H}} \|\eta\| + \|u\| |\eta|_{\tilde{H}}) \quad (5.9)$$

are satisfied then $(A + I)^{-1/2} B (A + I)^{-1/2} \in S_\infty$ (see [25], p. 35, Proposition 6.1).

Now the inequality (5.9) follows from (5.7) by Cauchy-Schwarz inequality. $\square$

Finally we check the conditions (A.IV)-(A.V). For this we prove that:

(A.IV)-(A.V) For $u \in W_2^1(\Omega)$ and $k \in \mathbb{R}$,

$$(A(k)u, u) := (Au, u) + k(Bu, u) + k^2(Cu, u) \geq 0. \quad (5.10)$$

Moreover, there exists an ε satisfying $0 < \varepsilon < 1$ such that

$$(Au, u) + k(Bu, u) + \varepsilon^2 k^2 (Cu, u) \geq 0. \quad (5.11)$$

Proof. It follows from the inequality (5.3) that

$$\sum_{i,j=1}^3 \int_{\Omega} a_{ij} \frac{\partial v}{\partial x_i} \frac{\partial \bar{v}}{\partial x_j} dx_2 dx_3 \geq \nu \sum_{j=1}^3 \int_{\Omega} \left| \frac{\partial v}{\partial x_j} \right|^2 dx_2 dx_3.$$

Writing ue^{-ixk} for v we obtain

$$(A(k)u, u) \geq \nu \left[\int_{\Omega} \left(\left| \frac{\partial u}{\partial x_2} \right|^2 + \left| \frac{\partial u}{\partial x_3} \right|^2 \right) dx_2 dx_3 + k^2 \int_{\Omega} |u|^2 dx_2 dx_3 \right].$$

Thus for any $k \in \mathbb{R}$ and $u \in W_2^1(\Omega)$

$$(A(k)u, u) \geq \nu k^2(u, u) \quad (5.12)$$

Now the inequality (5.10) follows from (5.12). The inequality (5.11) follows from (5.12) and the condition (A.II). Indeed by the condition (A.II), $c_1^2(u, u) \leq (Cu, u) \leq c_2^2(u, u)$ where c_1 and c_2 are positive. It is easy to see that ε can be chosen as $\varepsilon^2 = 1 - \nu/c_2^2$. $\square$

For quadratic operator pencil of wave guide type $L(k, w) = A + kB + k^2C - w^2I$ let us denote the spectrum of the operator A by $\{\nu_n^2\}_{n=0}^\infty$ where $0 \leq \nu_0 < \nu_1 < \dots$ then we have the following theorem.

Theorem 5.2 ([25], Theorem 13.1). For $L(k, w)$ let $\sigma_k(L) = \{\pm w_n(k)\}_{n=1}^\infty$ and $k \in \mathbb{R}$ then

$$(i) \quad 0 \leq w_1(k) < w_2(k) < \dots,$$

$$(ii) \quad \lim_{n \rightarrow \infty} \frac{w_n(k)}{\nu_n} = 1.$$

Theorem 5.3. $L(k, w)$ be the operator pencil derived from the boundary value problem (5.1), (5.2) and $a_{ij} \in L_\infty(\Omega) \cap C^1(\Omega)$. Then $w_n(k) \sim c n^{1/2}$ for every $k \in \mathbb{R}$, where c is a constant.

Proof. The operator pencil $L(k, w)$ satisfies the conditions (A.I)-(A.V). Hence $\sigma_k(L)$ is discrete and for any $k \in \mathbb{R}$ by Theorem 5.2 we have

$$\lim_{n \rightarrow \infty} \frac{w_n(k)}{\nu_n} = 1. \quad (5.13)$$

Consequently $w_n(k) \sim \nu_n$ where ν_n^2 's are the eigenvalues of the operator A .

The quadratic form corresponding to the operator A is

$$\mathcal{J}_A[u, u] = \sum_{i,j=2}^3 \int_{\Omega} a_{ij} \frac{\partial u}{\partial x_i} \frac{\overline{\partial u}}{\partial x_j} dx_2 dx_3, \quad u \in W_2^1(\Omega). \quad (5.14)$$

If in condition (5.3) we choose $\xi = (0, \xi_2, \xi_3)$ we have

$$\nu \|\xi\|^2 \leq \sum_{i,j=2}^3 a_{ij}(x_2, x_3) \xi_i \bar{\xi}_j, \quad \nu > 0. \quad (5.15)$$

It follows that the operator A is elliptic on the space $L_2(\Omega)$. Here we can use results given for elliptic operators in [49]. That is $\nu_n^2 \sim c n^{\alpha/m}$ where m is the dimension of the space and α is the order of the equation. In our case $m = \alpha = 2$ so by (5.13) for any $k \in \mathbb{R}$ we have $w_n(k) \sim c n^{1/2}$. $\square$

6 CONCLUSIONS

In the present study, we consider Riesz basis and eigenvalue problems for one and two parameter self-adjoint operator pencils. For two parameter polynomial operator pencils of waveguide type we study the structure of the spectrum and variational principles for real eigenvalues. In the quadratic case results concerning structure of the spectrum are given in [25] and variational principles for the real eigenvalues are proved by Yu. Sh. Abramov in [11]. Although the definition of this class for the nonquadratic case was given by Silbergleit and Kopilevich in [37], this is the first study on spectral properties of polynomial operator pencils of waveguide type. We prove that the spectral sets $\sigma_w(L)$ and $\sigma_k(L)$ are discrete. In quadratic case we give variational principles for real eigenvalues in some parts of $\sigma_w(L)$. Even though in the quadratic case variational principles for bounded operator pencils are known from [11], here we give variational principles for unbounded operator pencils and for this purpose methods and results, from [15], for operator pencils with values in the class of unbounded self-adjoint operators are used. These methods don't require the explicit expression of the Rayleigh functional $p(x)$, so, unlike the method used in [11], are applicable in the polynomial case. Consequently, variational principles for real eigenvalues for operator pencils of waveguide type of arbitrary order in some parts of $\sigma_w(L)$ are given.

We construct a model which includes a wider class of self-adjoint operator functions. This model particularly contains polynomial operator pencils of waveguide type. In the frame of this model the structure of numerical range, root zones and distribution of eigenvalues in root zones are studied. For quadratic

pencils of waveguide type these problems are discussed in [11] but our study contains some new results also in this case.

In this thesis, Riesz basis properties for a class of self-adjoint and continuous operator pencils are also considered. In general, Riesz basis properties for operator pencils are studied using the factorization method and reducing it to the same problem for an operator. Existence of a factorization depends on the smoothness properties of the class. In this study we present a new approach based on the spectral distribution function and for a dense subclass of $C([a, b], S(H))$ we show that, under some conditions, eigenvectors corresponding to eigenvalues in (a, b) form a Riesz basis.

The variational principles given in this study characterize the eigenvalues belonging to intervals where either all eigenvalues are of positive type or all are of negative type. The same questions in a mixed spectral zone were studied only for linear operator pencils of the form $A - \lambda B$ (see [45], [46] and references therein). There are no papers on this subject for λ -nonlinear spectral problems. Particularly, we aim to study some problems in this direction.

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