ISTANBUL TECHNICAL UNIVERSITY ★ GRADUATE SCHOOL OF SCIENCE ENGINEERING AND TECHNOLOGY

RADICAL, NILRADICAL AND CLASSICAL PRIME SUBMODULES

Ph.D. THESIS

Sibel KILIÇARSLAN CANSU

Department of Mathematical Engineering

Mathematical Engineering Programme

DECEMBER 2013

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Thesis Advisor: Asst. Prof. Recep KORKMAZ

Co-Advisor: Asst. Prof. Erol YILMAZ

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Sibel KILIÇARSLAN CANSU
(509062005)

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Tez Danışmanı: Yrd. Doç. Dr. Recep KORKMAZ

Eş Danışman: Yrd. Doç. Dr. Erol YILMAZ

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Sibel KILIÇARSLAN CANSU, a Ph.D. student of ITU Graduate School of Science Engineering and Technology 509062005 successfully defended the thesis entitled **"RADICAL, NILRADICAL AND CLASSICAL PRIME SUBMODULES"**, which she prepared after fulfilling the requirements specified in the associated legislations, before the jury whose signatures are below.

Thesis Advisor :	**Asst. Prof. Recep KORKMAZ** Istanbul Technical University	....
Co-advisor :	**Asst. Prof. Erol YILMAZ** Abant Izzet Baysal University	....
Jury Members :	**Prof. Dr. Vahap ERDOĞDU** Istanbul Technical University	....
	Assoc. Prof. Sefa Feza ARSLAN Mimar Sinan Fine Art University	....
	Assoc. Prof. Ergün YARANERİ Istanbul Technical University	....
	Assoc. Prof. Gürsel YEŞİLOT Yıldız Technical University	....
	Assoc. Prof. Mustafa ALKAN Akdeniz University	....

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To my family and especially to my precious Fatma Zeynep,

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LIST OF SYMBOLS

$rad_M(N)$	:	Radical of a submodule N of an R-module M
$E_M(N)$	:	Envelope of N in M
$Ass(M/N)$	:	Associated prime ideals of N
$N : \langle f \rangle^\infty$	:	Stable quotient of N by $\langle f \rangle$
$S_p(N)$	:	Saturation of N
$GAss(M/N)$	:	Generalized associated prime ideals of N
$srad_M(N)$	:	Semiradical of a submodule N of an R-module M

RADICAL, NILRADICAL AND CLASSICAL PRIME SUBMODULES

SUMMARY

The set of nilpotent elements of a commutative ring forms an ideal and it is equal to the intersection of all the prime ideals. This is a well-known characterization in commutative rings. R.L.McCasland and M.E.Moore generalized this characterization to modules which gave the concepts prime radical of a module, $rad_M(N)$, which is the intersection of all prime submodules containing N and the envelope of a submodule N, $\langle E_M(N)\rangle$, which is defined to be the set of all elements of the form rm where r is an element of the ring R and m is an element of the R-module M with the condition $r^k m \in N$ where $k \in \mathbb{Z}^+$. The submodule generated by the envelope, $\langle E_M(N)\rangle$, is called (Baer's) lower nilradical.

McCasland and Moore called that N satisfies the radical formula (N s.t.r.f.) if $rad_M(N) = \langle E_M(N)\rangle$; M satisfies the radical formula (s.t.r.f.) if for every submodule N of M, N s.t.r.f.; a ring R satisfies the radical formula (s.t.r.f.) if every R-module satisfies the radical formula. The question of what kind of rings and modules s.t.r.f. has been studied by many authors. Although some methods for computing of radical of a submodule are given by McCasland - Smith and Marcelo - Rodriguez, it seems there is no description for the computation of the lower nilradical of a submodule. One of the main objective of this thesis is determining the lower nilradical of a submodule. We give a formula for computing the lower nilradical of a submodule N if a minimal primary decomposition of N is known.

Chin-Pi Lu proved that if N is a submodule of a finitely generated module M, then the prime radical of N can be written as an intersection of the submodules $S_p(N + pM)$ with p is a prime ideal such that $N : M \subseteq p$, where $S_p(N + pM)$ is the saturation of $N + pM$ which is the set of elements $m \in M$ such that $cm \in N + pM$ for some element $c \in R - p$. A method to find the saturation of the submodule $N + pM$ is given. Also a technique to eliminate the redundant primes in the process of determining the prime radical of a submodule is given.

Also, some properties of semiprime submodules are studied and the relation between the notions of prime radical of a submodule and semiprime radical of a submodule which is the intersection of all semiprimes containing that submodule is examined.

Finally, semiprime submodules which can be written as an intersection of classical primes are investigated. It is shown that if N is a semiprime submodule of a Noetherian R-module and associated prime ideals of N that are determined by the primary decomposition of N form a chain, then N is classical prime submodule. A new definition which we called the simple quasi primary submodule is given. It is shown that if M is a Noetherian R-module and N can be written as an intersection of simple quasi primary submodules with each of them is semiprime, then N can be written as a finite intersection of classical prime submodules. An example which shows that the conjecture of Baziar and Behboodi is false is given.

Throughout R will always denote a commutative ring with identity and M will denote the unitary R-module.

RADİKAL, NİLRADİKAL VE KLASİK ASAL ALT MODÜLLER

ÖZET

R değişmeli bir halka olmak üzere herhangi bir I idealin radikali R'nin I'yı kapsayan bütün asal ideallerinin arakesiti olarak tanımlanır. Bu tanımın $n \in \mathbb{Z}^+$ olmak üzere $r^n \in I$ koşulunu sağlayan bütün $r \in R$ elemanlarının kümesine eşit olduğu çok iyi bilinen bir sonuçtur. Modül teorisinde bu tanımlamaya paralel olarak iki ayrı kavram karşımıza çıkmaktadır. Bunlardan ilki bir modülün asal radikali olarak isimlendirilmiş olup N alt modülünün asal radikali M'nin N'yi kapsayan bütün asal alt modüllerinin ara kesiti olarak tanımlanmıştır, $rad_M(N)$. R halkasını kendisi üzerinde bir modül olarak düşündüğümüz zaman $rad_R(I)$ halkalarda bildiğimiz bir idealin radikaline karşılık gelmektedir. Diğer kavram ise zarf olarak isimlendirilen $E_M(N)$ kümesidir. Bu küme $k \in \mathbb{Z}^+$, $r \in R, m \in M$ olmak üzere $r^k m \in N$ şartını sağlayan tüm rm elemanlarının kümesidir.

Halkalardakinin aksine bu iki kavram modüller üzerinde her zaman birbirine eşit değildir. 1991 senesinde McCasland ve Moore hangi şartlar altında eşitliğin olduğunu araştırmışlardır. Aslında zarf kümesinin tanımına bakıldığında bu küme bir alt modül değildir. Bu sebepten $E_M(N)$ kümesi tarafından üretilen alt modül üzerinde çalışmak daha anlamlıdır. McCasland ve Moore bu makalelerinde N alt modülü için $rad_M(N) = \langle E_M(N) \rangle$ eşitliği var ise N'ye radikal formülünü sağlayan alt modül demişlerdir. Eğer M modülünün her N alt modülü radikal formülünü sağlar ise M radikal formülünü sağlayan modül, eğer her R-modül radikal formülünü sağlar ise R radikal formülünü sağlayan halka olarak tanımlanmıştır. O zamandan beri de bir çok kişi tarafından bu kavramlar üzerinde çalışılmıştır.

Literatüre bakıldığında uzun yıllar boyunca asal radikal ve zarf tarafından üretilen alt modül hesaplanması ile ilgili herhangi bir çalışma bulunmamaktadır. Marcelo ve Rodriguez 2000 senesinde, McCasland ve Smith ise 2008 senesinde bir alt modülün radikalinin hesaplanması ile ilgili makaleler yayınlanmışlardır. Bununla beraber karakterizasyonun zarf kısmı hala belirsiz görünmektedir. Bu tezin temel problemlerinden bir tanesi zarf tarafından üretilen alt modülü belirleyebilmek için herhangi bir yöntem geliştirilip geliştirilemeyeceğidir. N alt modülünün asıl parçalanışı bilindiği takdirde, $E_M(N)$ tarafından üretilen alt modülün hesaplanmasını sağlayan bir formül bu tezde verilmiştir. Bu da Noether M modülü için zarf tarafından üretilen alt modülün her zaman bulunabilmesine imkan sağlar. Ayrıca N alt modülünün zarfı tarafından üretilen alt modül, N'nin kendisine eşit ise N alt modülünün asıl parçalanışındaki izole bileşenlerin asal alt modül olduğu görülmüştür.

2003 yılında Lu, N sonlu üreteçli bir M modülünün alt modülü, p ise $N : M$ kümesini kapsayan asal ideal olmak üzere, N'nin asal radikalinin $S_p(N + pM)$ formundaki asal alt modüllerin kesişimi olarak ifade edilebileceğini gösterdi. Bu tezde $S_p(N)$ kümesinin N'nin asıl parçalanışındaki bazı asıl alt modüllerin kesişimi

olarak yazılabildiği gösterilmiştir. Ayrıca M sonlu üreteçli bir R-modül, N, M'nin alt modülü ve $N : M \subseteq p$ olduğu durumda $S_p(N + pM)$ alt modülünün $N + pM$'in asıl parçalanışındaki p-asal alt modüle eşit olduğu bulunmuştur. R Noether bir halka, M sonlu üreteçli R-modül ve p, $N : M \subseteq p$ şartını sağlayan bir asal ideal ise N alt modülünün asal radikalinin $S_p(N + pM)$ şeklindeki alt modüllerin sonlu ara kesiti olarak yazılabildiği 2008 senesinde McCasland ve Smith tarafından gösterilmiştir. Bu tezde McCasland ve Smith'in teoreminin daha farklı bir ispatı verilmiştir. Daha sonra bu ara kesitte karşımıza çıkan asal alt modüllerin gereksiz olanlarını elemeye yardımcı olacak bir teknik verilmiştir.

Asal alt modül tanımıyla ilintili olarak yarı asal ve klasik asal alt modül tanımları da bu alanda çalışan insanların ilgisini çekmiştir. Klasik asal alt modül ilk olarak 2004 yılında Behboodi ve Koohy tarafından tanımlanmıştır. Tanımlar incelendiği zaman asal alt modülün klasik asal, klasik asal alt modülün de yarı asal olduğu açıkca görülmektedir. Bu çalışmada alt modülün asal radikali tanımına benzer şekilde bir alt modülün yarı asal radikali tanımlanmış, yarı asal radikalin özellikleri incelenmiş, değişmeli bölgeler için herhangi bir modülün yarı asal radikali ile burkulma modülünün yarı asal radikalinin aynı alt modül olduğu görülmüş ve yarı asal radikal ile asal radikal arasındaki ilişki irdelenmiştir. Her serbest R modül için asal radikalin yarı asal radikale eşit olmasının her R modül içinde bu eşitliğin söz konusu olmasını gerektirdiği görülmüştür. Radikal formülüne benzer şekilde yarı radikal formülü tanımlanmış ve yarı radikal formülün hangi şartlar altında sağlandığı belirlenmeye çalışılmıştır. Bunun yanında M_1 R_1-modül, M_2 de R_2-modül olmak üzere $M_1 \times M_2$ modülünün yarı asal alt modülleri belirlenmeye çalışılmış, bu bağlamda $N_1 \times N_2$ alt modülünün yarı asal radikalinin N_1 ve N_2 alt modüllerinin yarı asal radikallerinin çarpımı olarak ortaya çıktığı görülmüştür. Burada N_1, M_1'in; N_2 ise M_2'nin bir alt modülüdür. Ayrıca $M_1 \times M_2$ modülünün yarı radikal formülünü sağlaması ile M_1 ve M_2 modüllerinin yarı radikal formülünü sağlamasının birbirine denk koşullar olduğu gösterilmiştir.

Asal alt modüllerin kesişiminin her zaman yarı asal olmasının yanında klasik asal alt modüllerin kesişiminin de her zaman yarı asal olduğunu görmek zor değildir. Bu durumda asal alt modüllerin kesişimi olarak yazılabilen yarı asalları ve klasik asal alt modüllerin kesişimi olarak yazılabilen yarı asal alt modülleri belirleyebilmek akla ilk gelen soru olacaktır. Sorunun ilk kısmını çözmeye çalışırken Man, her yarı asal alt modülü asal alt modüllerin kesişimi olarak yazılabilen değişmeli Noether halkalar için bir tanımlama vermiştir. Diğer taraftan Behboodi, R değişmeli bir bölge ve $dimR \leq 1$ olduğu takdirde, her yarı asal R alt modülünün klasik asalların kesişimi olarak yazılabildiğini göstermiştir.

Bu tezde Behboodi'nin çalışmasına ek olarak klasik asalların kesişimi olarak yazılabilen yarı asal alt modüller için bir genelleme bulunmaya çalışılmıştır. Bu çalışma esnasında yarı asal alt modüllerin bazı özellikleri incelenmiş, klasik asal alt modüller ve bu alt modüllerin zarfı tarafından üretilen alt modüller arasındaki ilişkiler belirlenmiştir. M Noether R-modül ve N yarı asal alt modül olmak üzere, N ile ilişkili asal idealler bir zincir oluşturuyorsa N alt modülünün klasik asal olması gerektiği gösterilmiştir. Baziar ve Behboodi'nin 2009 yılında yapmış olduğu varsayımın yanlış olduğu gösterilmiş, bu varsayımın şartları zayıflatılarak R Noether bir halka, M sonlu üreteçli bir R-modül ise yarı asal olan her klasik asıl alt modülün klasik asal olması gerektiği ispatlanmıştır. Ayrıca klasik asalların kesişimi olarak yazılabilen yarı asal alt modülleri belirleyebilmek amacıyla basit sözde asıl alt modül tanımı geliştirilmiş ve bu tanım yardımıyla *N* Noether bir *R* modülün yarı asal alt modülü iken *N* her biri yarı asal olan basit sözde asıl alt modüllerin kesişimi olarak yazılabiliyorsa; *N*'nin klasik asal alt modüllerin sonlu kesişimi olarak da yazılabildiği gösterilmiştir.

Bu çalışmada tüm halkalar değişmeli ve birim elemana sahip olup, *M* birimli *R*-modülü temsil edecektir.

1. INTRODUCTION

We begin by recalling that, if I is an ideal in a commutative ring R, then radical of I is the intersection of all prime ideals of R containing I. The radical of an ideal is also characterized by $\sqrt{I} = \{r \in R : r^n \in I \quad for \quad some \quad n \in \mathbb{Z}^+\}$. This notion has been generalized by R.L.McCasland and M.E.Moore to modules over a commutative ring [1]. This generalization brought up the concepts the prime radical of a module and the envelope of a module. If N is a proper submodule of an R-module M, then N is called p-prime (p-primary) if $rm \in N$ for $r \in R, m \in M$ implies that either $m \in N$ or $r \in (N : M) = p$ ($m \in N$ or $r \in \sqrt{N : M} = p$) where $N : M$ is the set of all elements of R that takes M into N.

Hence, the prime radical of N in M, $rad_M(N)$ is defined as the intersection of all prime submodules of M containing N [2]. If no prime submodule of M contains N or there is no prime submodule, then $rad_M(N) = M$. The envelope of N in M, $E_M(N)$, is the collection of all elements $x \in M$ for which there exists $r \in R, m \in M$ such that $x = rm$ and $r^k m \in N$ for some $k \in \mathbb{Z}^+$. The submodule generated by the envelope is called (Baer's) lower nilradical and denoted by $\langle E_M(N) \rangle$. N satisfies the radical formula (N s.t.r.f.) if the radical of N is equal the lower nilradical of N [1] i.e. $rad_M(N) = \langle E_M(N) \rangle$. M satisfies the radical formula (M s.t.r.f.) if for every submodule N of M, N s.t.r.f.. It is said that a ring R satisfies the radical formula (R s.t.r.f.) if every R-module satisfies the radical formula. When we consider the ring R as an R-module, ideals of R will be the submodules of R; and by the characterization of the radical of an ideal every submodule of R satisfies the radical formula.

Prior to [1], the only rings known to s.t.r.f. were fields (every proper subspace of a vector space is prime) and the only modules (other than vector spaces) known to s.t.r.f. were multiplication modules [3]. Later, in 1992 Jenkins and Smith showed that Dedekind domains satisfy the radical formula [4]. In [4], they gave a conjecture that Dedekind domains are the only Noetherian domains which satisfy the radical formula. Then in [5] Man tried to tackle this problem and he showed that if R is a

Noetherian domain of Krull dimension one and $R \oplus R$ s.t.r.f. as an R-module, then R is a Dedekind domain. In 1997, Leung and Man [6] proved that the only Noetherian rings which s.t.r.f. are of dimension at most one and they gave a complete characterization of Noetherian rings which s.t.r.f. Also Sharif, Sharifi and Namazi [7] showed that Artinian rings satisfy the radical formula.

Up to 2001, there was no work which deals with the computation neither the lower nilradical nor the radical of a submodule. One of the main problems of this thesis is computing the lower nilradical of a submodule. In 2001, Smith [8] showed that if N is p-primary submodule, then $\langle E_M(N) \rangle = N + pM$. After two years, Lu gave the definition of saturation [9] which enables him to compute the radical of a submodule. For any submodule N of M and for any prime ideal p of R, saturation of N defined as the set

$$S_p(N) := \{m \in M : cm \in N \, for \, some \; c \in R \setminus p\}.$$

Lu [9] showed that if M is Noetherian R-module, then $rad_M(N) = \bigcap_{i=1}^{n} S_{p_i}(N + p_i M)$ where p_i's are prime ideals containing $N : M$. But it was a question that how can these prime ideals are determined. This question is answered by Smith and McCasland [10] in 2008. They gave a decomposition of the radical of a submodule in a Noetherian module as an intersection of finitely many known prime submodules. To give that decomposition they define generalized associated prime ideals of a submodule.

The notion of classical prime submodule was introduced in [11] and has received the attention of many authors [12], [13], [14]. A proper submodule N of an R-module M is called classical prime if $abm \in N$ implies either $am \in N$ or $bm \in N$ for $a, b \in R$ and $m \in M$. Sometimes weakly prime is used for classical prime. A proper submodule N is semiprime if $r^k m \in N$ implies $rm \in N$ where $r \in R, m \in M$ and $k \in \mathbb{Z}^+$. When we consider the definitions, it is clear that every prime submodule is classical prime and every classical prime submodule is semiprime. An intersection of prime submodules is semiprime. But the converse is not true in general. Man [15] gave a characterization of a commutative Noetherian ring R with the property that every R-semiprime submodule is an intersection of prime submodules. Also an intersection of classical prime submodules is semiprime. Thus it is interesting to characterize the rings over which each semiprime submodule is an intersection classical primes. Behboodi [12] showed that if R is a commutative domain with $dimR \leq 1$, then every semiprime submodule of a module M is an intersection of classical prime submodules. A proper submodule N of M is classical primary if $abm \in N$ where $a, b \in R$ and $m \in M$, then either $bm \in N$ or $a^k m \in N$ for some $k > 1$ [16]. Behboodi and Baziar showed that if M is finitely generated module over a Noetherian ring, then associated prime ideals of a classical primary submodule form a chain [16].

This introduction forms Chapter 1.

Chapter 2 contains one of the main results of this thesis. We give a formula to compute the lower nilradical of a submodule,$\langle E_M(N) \rangle$, if a minimal primary decomposition of the submodule is known.

Chapter 3 has three sections. Section 1 contains some known results about saturation and we will show that the saturation of a submodule is the intersection of some of the primary components of its primary decomposition. Moreover, we also prove that if N is a submodule of a finitely generated module M with $N : M \subseteq p$ where p is a prime ideal, then $S_p(N + pM)$ is equal to the p-prime component of the primary decomposition of $N + pM$. In section 2, definition of generalized associated prime ideal is given and an alternative proof of McCasland - Smith theorem [17] is stated. In the third section, a technique to determine the redundant prime submodules which appears in the computation of the radical is investigated.

Chapter 4 constitute some results about semiprime submodules and semiradical of a module which is defined as the intersection of all semiprime submodules containing that module. In the third section of this chapter we define semiradical formula and investigate the modules which satisfy the semiradical formula. Since semiradical of a module always contained in the prime radical, we investigate the conditions when the equality holds and we call it semiradical equality. We also show that a ring R satisfy the semiradical equality if and only if every free R-module satisfy the semiradical equality. Also, semiprime submodules of the $R = R_1 \times R_2$-module $M_1 \times M_2$ are characterized where each M_i is an R_i-module. We also examine that semiradical of any submodule $N_1 \times N_2$ of $M_1 \times M_2$ is the same as the cartesian product of semiradicals of N_1 and N_2 which also allows us to show that $M_1 \times M_2$ satisfy the semiradical formula if and only if M_i satisfy the semiradical formula for $i = 1, 2$.

In chapter 5 the relation between classical prime submodules and their lower nilradical is investigated. We show that if N is a semiprime submodule of a Noetherian R-module and associated prime ideals of N form a chain, then N is classical prime submodule. Also in this chapter we show that the lower nilradical of a classical primary submodule need not be a classical prime submodule which is the answer of the conjecture of Baziar and Behboodi [16]. We also show that a semiprime submodule N of M is classical prime if it is classical primary when M is Noetherian. We give a new definition which we call simple quasi primary submodule. We also examine that if N is a semiprime submodule of a Noetherian R-module which can be written as an intersection of simple quasi primary submodules, then each of these simple quasi primary submodules is semiprime, and hence N can be written as a finite intersection of classical prime submodules.

Finally, Chapter 6 is the conclusion chapter which contains a brief summary with some suggestions for future study.

The main results of this thesis (namely, the results of Chapter 2 and Chapter 5) will appear in the Bulletin of the Iranian Mathematical Society [18].

Throughout R will always denote a commutative ring with identity and M will denote the unitary R-module.

2. ENVELOPE OF A SUBMODULE

2.1 Primary Decomposition

In this section we recall some basic definitions and well-known results about primary decomposition.

Definition 2.1.1. A proper submodule Q of M is called primary submodule if $rm \in Q$ implies that $m \in Q$ or $r^k M \subseteq Q$ for some $k \in \mathbb{Z}^+$, for all $r \in R, m \in M$.
If $p = \sqrt{N : M}$ is a prime ideal of R, then Q is called p-primary submodule.

For an arbitrary submodule N, a primary decomposition of N in M is a representation of N as an intersection of finitely many primary submodules of M

$$N = Q_1 \cap Q_2 \cap \cdots \cap Q_n$$

with p_i-primary submodules $Q_i \subseteq M$.

We call a primary decomposition minimal precisely when

(a) $p_1, p_2, \ldots, p_n$ are n different prime ideals of R, and

(b) for all $j = 1, 2, \ldots, n$ we have

$$Q_j \not\supseteq \bigcap_{i \neq j} Q_i.$$

If $Q_1, Q_2, \ldots, Q_k$ are p-primary submodules, then $\bigcap_{i=1}^{k} Q_i$ is p-primary. This result provides that we can always refine any primary decomposition to produce a minimal primary decomposition by discarding those Q_i that contain $\bigcap_{i \neq j} Q_i$ and intersecting those Q_i that are p-primary for the same p.

Definition 2.1.2. A prime ideal p of R is called the associated prime of N if $p = (N : m)$ for some $m \in M - N$. The set of all associated primes of N is denoted by $Ass(M/N)$.

The prime ideals in $Ass(M/N)$ that are minimal with respect to inclusion are called the isolated primes of N, the remaining associated prime ideals are the embedded primes of N.

Theorem 2.1.1. *([19], Theorem 3.1) Let R be a Noetherian ring and M be a finitely generated non-zero R-module. Then $Ass(M/N)$ is finite, non-empty set of primes each containing $Ann(M/N) = (N : M)$. The set $Ass(M/N)$ includes all the primes minimal among primes containing $Ann(M/N) = (N : M)$.*

Theorem 2.1.2. *Let R be a Noetherian ring, M be a finitely generated R-module and N be any submodule of M. Suppose $N = Q_1 \cap Q_2 \cap \cdots \cap Q_n$ is a minimal primary decomposition of N with Q_i is p_i-primary. Then*

(a) the p_i's are uniquely determined by N.

(b) if p_i is minimal, then Q_i is uniquely determined by N.

Theorem 2.1.3. *([19], Theorem 3.10) Let R be a Noetherian ring, M be a finitely generated R-module and let $N = Q_1 \cap Q_2 \cap \cdots \cap Q_n$ be a minimal primary decomposition with Q_i is p_i-primary submodules. Then $Ass(M/N) = \{p_1, p_2, \ldots, p_n\}$.*

2.2 Stable Quotient

Definition 2.2.1. Let R be a ring and, M be an R-module and N be a proper submodule of M. If $f \in R$, then

$$N : \langle f \rangle^{\infty} = \{m \in M : f^k m \in N \ for \ some \ k \in \mathbb{Z}^+\}.$$

is called stable quotient of N by $\langle f \rangle$.

Lemma 2.2.1. *([20], Lemma 1) Let P be a primary submodule of M and $f \in R$. Then*

(i) $P : \langle f \rangle^{\infty} = M$ if $f \in \sqrt{P : M}$,

(ii) $P : \langle f \rangle^{\infty} = P$ if $f \notin \sqrt{P : M}$,

More generally if primary decomposition of any submodule N of M is $N = \bigcap_{i=1}^{k} Q_i$ into q_i-primary submodules, then *(iii)* $N:\langle f\rangle^{\infty}=\bigcap_{f\notin q_i} Q_i$.

Proof. (i) If $f\in\sqrt{P:M}$, then for some positive integer k, $f^kM\subseteq P$ which implies that $P:\langle f\rangle^{\infty}=M$.

(ii) Let $f\notin\sqrt{P:M}$ and $n\in P:\langle f\rangle^{\infty}$. Then $f^kn\in P$ for some $k\in\mathbb{Z}^+$. Since P is primary submodule, we get $n\in P$. Thus $P:\langle f\rangle^{\infty}=P$.

(iii) Suppose $N=\bigcap_{i=1}^{k} Q_i$ where $\sqrt{Q_i:M}=q_i$. Then

$$N:\langle f\rangle^{\infty}=(\bigcap_{i=1}^{k} Q_i):\langle f\rangle^{\infty}=\bigcap_{i=1}^{k}(Q_i:\langle f\rangle^{\infty})=\bigcap_{f\notin q_i} Q_i$$

by part (i) and (ii).

□

We can easily show that;

Lemma 2.2.2. *([21], Lemma 1.3.3) Let N be p-primary submodule of an R-module M. Then*

$$N:h=\begin{cases} N, & \text{if } h\notin p;\\ M, & \text{if } h\in(N:M).\end{cases}$$

where $N:h=\{m\in M:h.m\subseteq N\}$.

2.3 Formula for Lower Nilradical

Recall that the envelope of N is the set

$$E_M(N)=\{rm:r\in R, m\in M \quad and \quad r^km\in N \quad for \quad some \quad k\in\mathbb{Z}^+\}.$$

Lemma 2.3.1. *Let $N=Q_1\cap Q_2\cap\cdots\cap Q_k$ be minimal primary decomposition of N where $\sqrt{Q_i:M}=p_i$ for all $i=1,2,\ldots,k$. If $S=\{1,2,\ldots,k\}$ and $\emptyset\neq T\subseteq S$, then*

$$(\bigcap_{i\in T} p_i)(\bigcap_{i\in S-T} Q_i)\subseteq\langle E_M(N)\rangle.$$

Proof. Let $n\in(\bigcap_{i\in T} p_i)(\bigcap_{i\in S-T} Q_i)$. Then there exist elements $r_j\in\bigcap_{i\in T} p_i$ and $m_j\in\bigcap_{i\in S-T} Q_i$ such that

$$n=r_1m_1+r_2m_2+\cdots+r_sm_s$$

for some $s \in \mathbb{Z}^+$. Then $r_j \in \bigcap_{i\in T} p_i$ implies that $r_j^{k_j} M \subseteq \bigcap_{i\in T} Q_i$ with $k_j \in \mathbb{Z}^+$. In particular, $r_j^{k_j} m_j \in \bigcap_{i\in T} Q_i$ for all j.

Also, $m_j \in \bigcap_{i\in S-T} Q_i$ implies that $r_j^{k_j} m_j \in \bigcap_{i\in S-T} Q_i$. Hence $r_j^{k_j} m_j \in \bigcap_{i=1}^{k} Q_i = N$. Thus $r_j m_j \in E_M(N)$ for all j. □

The following theorem gives us a formula for determining the lower nilradical of a submodule.

Theorem 2.3.1. *Let $N = Q_1 \cap Q_2 \cap \cdots \cap Q_k$ be minimal primary decomposition of N where $\sqrt{Q_i : M} = p_i$ for all $i = 1,2,\ldots,k$ and $S = \{1,2,\ldots,k\}$. Then*

$$\langle E_M(N)\rangle = N + (\bigcap_{i=1}^{k} p_i)M + \sum_{T\subset S} (\bigcap_{i\in T} p_i)(\bigcap_{i\in S-T} Q_i)$$

where the summation runs over each non-empty subset T of S.

Proof. Let $m \in \langle E_M(N)\rangle$. Then there exist $m_j \in M, r_j \in R$ such that

$$m = r_1m_1 + r_2m_2 + \cdots + r_tm_t \text{ where } r_j^{k_j} m_j \in N.$$

By the definition of stable quotient, $m_j \in N : \langle r_j\rangle^\infty$ for each j.

For each r_j, we have three cases: $r_j \in p_i$ for all i, $r_j \notin p_i$ for all i which means that $r_j \in R - \bigcup_{i=1}^{k} p_i$ or there is a maximal subset T of S such that $r_j \in \bigcap_{i\in T} p_i$ for all j.

- If $r_j \in \bigcap_{i=1}^{k} p_i$, then $m = r_1m_1 + r_2m_2 + \cdots + r_tm_t \in (\bigcap_{i=1}^{k} p_i)M$.
- If $r_j \in R - \bigcup_{i=1}^{k} p_i$, then consider the set $N : \langle r_j\rangle^\infty$. By Lemma 2.2.1, $N : \langle r_j\rangle^\infty = N$. Hence $m = r_1m_1 + r_2m_2 + \cdots + r_tm_t$ with $m_j \in N : \langle r_j\rangle^\infty$ and thus $m \in N$.
- If $r_j \in \bigcap_{i\in T} p_i$ for some maximal subset T of S, then $N : \langle r_j\rangle^\infty = \bigcap_{i\in S-T} Q_i$. Hence

  $$r_jm_j \in (\bigcap_{i\in T} p_i)(\bigcap_{i\in S-T} Q_i).$$

  Thus $m \in (\bigcap_{i\in T} p_i)(\bigcap_{i\in S-T} Q_i)$.

Thus for all cases, $\langle E_M(N)\rangle \subseteq N+(\bigcap\limits_{i=1}^{k} p_i)M+\sum\limits_{T\subset S}(\bigcap\limits_{i\in T} p_i)(\bigcap\limits_{i\in S-T} P_i)$.

For the other side of the inclusion, Lemma 2.3.1 implies that

$$\sum_{\emptyset\neq T\subsetneq S}(\bigcap_{i\in T} p_i)(\bigcap_{i\in S\setminus T} Q_i)\subseteq \langle E_M(N)\rangle.$$

Also N and $(\bigcap\limits_{i=1}^{k} p_i)M$ are clearly in $\langle E_M(N)\rangle$. □

As a result of this theorem, we can write the following corollary. This is also proved by Patrick F.Smith in [8].

Corollary 2.3.1. *If N is a p-primary submodule, then*

$$\langle E_M(N)\rangle = N+pM.$$

Now we will give an application of Theorem 2.3.1. The computer algebra system **SINGULAR** [22] was used during the computations.

Example 2.3.1. Let $R=\mathbb{Q}[x,y,z]$ and let M be an R-submodule $R\oplus R\oplus R$. Consider the submodule $N=\langle xz\mathbf{e}_3-z\mathbf{e}_1, x^2\mathbf{e}_3, x^2y^3\mathbf{e}_1+x^2y^2z\mathbf{e}_2\rangle$.

Primary decompostion of N is $N=Q_1\cap Q_2\cap Q_3$ where

$$Q_1=\langle \mathbf{e}_3, z\mathbf{e}_1, y\mathbf{e}_1+z\mathbf{e}_2, z^2\mathbf{e}_2\rangle \text{ is } \langle z\rangle-\text{primary},$$
$$Q_2=\langle \mathbf{e}_1, \mathbf{e}_3, y^2\mathbf{e}_2\rangle \text{ is } \langle y\rangle-\text{primary and}$$
$$Q_3=\langle x\mathbf{e}_1, x\mathbf{e}_3-\mathbf{e}_1, x^2\mathbf{e}_2\rangle \text{ is } \langle x\rangle-\text{primary.}$$

By Theorem 2.3.1,

$$\begin{aligned}\langle E_M(N)\rangle = N &+(p_1\cap p_2\cap p_3)M+p_1(Q_2\cap Q_3)+p_2(Q_1\cap Q_3)+p_3(Q_1\cap Q_2)\\ &+(p_1\cap p_2)Q_3+(p_1\cap p_3)Q_2+(p_2\cap p_3)Q_1.\end{aligned}$$

It is clear that $(p_1\cap p_2\cap p_3)M=\langle xyz\mathbf{e}_1, xyz\mathbf{e}_2, xyz\mathbf{e}_3\rangle$. Also we get

$$p_1(Q_2 \cap Q_3) = \langle xz\mathbf{e}_1, xz\mathbf{e}_3 - z\mathbf{e}_1, x^2y^2z\mathbf{e}_2 \rangle$$
$$p_2(Q_1 \cap Q_3) = \langle xyz\mathbf{e}_3 - yz\mathbf{e}_1, x^2y\mathbf{e}_3, x^2y^2\mathbf{e}_1 + x^2yz\mathbf{e}_2 \rangle$$
$$p_3(Q_1 \cap Q_2) = \langle x\mathbf{e}_3, xz\mathbf{e}_1, xy^3\mathbf{e}_1 + xy^2z\mathbf{e}_2 \rangle$$
$$(p_1 \cap p_2)Q_3 = \langle xyz\mathbf{e}_1, xyz\mathbf{e}_3 - yz\mathbf{e}_1, x^2yz\mathbf{e}_2 \rangle$$
$$(p_1 \cap p_3)Q_2 = \langle xz\mathbf{e}_1, xz\mathbf{e}_3, xy^2z\mathbf{e}_2 \rangle$$
$$(p_2 \cap p_3)Q_1 = \langle xy\mathbf{e}_3, xyz\mathbf{e}_1, xy^2\mathbf{e}_1 + xyz\mathbf{e}_2, xyz^2\mathbf{e}_2 \rangle$$

Thus

$$\langle E_M(N) \rangle = \langle z\mathbf{e}_1, x\mathbf{e}_3, xyz\mathbf{e}_2, xy^2\mathbf{e}_1 \rangle.$$

We also have the following result.

Corollary 2.3.2. *If $\langle E_M(N) \rangle = N$ for any submodule of an R-module M, then each isolated component of primary decomposition of N must be prime.*

Proof. Let $N = Q_1 \cap Q_2 \cap \cdots \cap Q_n$ with Q_i's are p_i-primary submodules. Let Q_k be one of the isolated components of N. If Q_k is not a prime submodule, then there exists an element $x \in p_k - (Q_k : M)$ and hence an element $m \in M$ such that $xm \notin Q_k$. Since p_k is an isolated prime, we can find an element $y \in (\bigcap_{j \neq k} p_j) - p_k$. Then

$$xym \in (\bigcap_{j=1}^{n} p_j)M \subseteq \langle E_M(N) \rangle = N \subseteq Q_k.$$

Since Q_k is p_k-primary and $xm \notin Q_k$, $y \in p_k$ which is a contradiction. □

3. COMPUTING RADICAL OF A SUBMODULE

3.1 Saturation of Submodules

Definition 3.1.1. Let R be a ring and N be a submodule of an R-module M. For any prime ideal p of R, the saturation of N defined as

$$S_p(N) := \{m \in M : cm \in N \, for \, some \; c \in R \setminus p\}.$$

It is obvious that $N \subseteq S_p(N)$ and $S_p(N) = \bigcup\limits_{r \in R-p} (N : r)$.

Let us give some well known properties for the saturation of a submodule.

Lemma 3.1.1. *([9], Result 1) Let N be any submodule of an R-module M and p be a prime ideal of R. Then* $S_p(S_p(N)) = S_p(N)$.

Lemma 3.1.2. *([9], Result 2(1)) N is p-prime (p-primary) submodule of M if and only if* $S_p(N) = N$ *and* $N : M = p$ *(* $\sqrt{N : M} = p$*).*

Theorem 3.1.1. *([9], Theorem 2.3) Let M be an R-module, N be a submodule of M and p is a prime ideal of R. Then the following statements are equivalent.*

(i) $S_p(N)$ *is a p-primary submodule of M.*

(ii) $S_p(N) : M$ *is a p-primary ideal of R.*

(iii) $\sqrt{S_p(N) : M} = p$*.*

The following theorem states that saturation of a submodule is the intersection of some of the primary submodules of its primary decomposition.

Theorem 3.1.2. *Let N be a submodule of M and p is a prime ideal of R such that* $N : M \subseteq p$*. If* $N = Q_1 \cap Q_2 \cap \cdots \cap Q_s$ *is a primary decomposition with* p_i*-primary submodules* Q_i*'s, then*

$$S_p(N) = \bigcap_{p_i \subseteq p} Q_i.$$

Proof. Let $r \in R - p$. Then

$$(N:r) = \bigcap_{i=1}^{s}(Q_i:r) = [\bigcap_{p_i \nsubseteq p}(Q_i:r)]\bigcap[\bigcap_{p_i \subseteq p}(Q_i:r)].$$

By Lemma 2.2.2, we get $(N:r) = [\bigcap_{p_i \nsubseteq p}(Q_i:r)]\bigcap[\bigcap_{p_i \subseteq p} Q_i]$. If $p_i \nsubseteq p$, then there exists an element $r_i \in (Q_i:M)$ such that $r_i \notin p$. Say, $r_0 = \prod_{p_i \nsubseteq p} r_i$. Then $r_0 \in (Q_i:M)$ for each i and $r_0 \in R - p$. Then

$$(N:r_0) = M\bigcap[\bigcap_{p_i \subseteq p} Q_i] = \bigcap_{p_i \subseteq p} Q_i \supseteq (N:r).$$

for each $r \in R - p$. Hence $S_p(N) = \bigcup_{r \in R-p}(N:r) = (N:r_0) = \bigcap_{p_i \subseteq p} Q_i$. □

Example 3.1.1. Let $R = \mathbb{Q}[x,y,z]$ and let M be an R-submodule $R \oplus R \oplus R$. Consider the submodule $N = \langle x^2\mathbf{e}_1 + y\mathbf{e}_2 + xy\mathbf{e}_3, z\mathbf{e}_1, y^2z\mathbf{e}_1 + x^2\mathbf{e}_3\rangle$. Then $N:M = \langle x^2yz\rangle$ and primary decomposition of N is $N = Q_1 \cap Q_2 \cap Q_3$ where

$$Q_1 = \langle \mathbf{e}_3, z\mathbf{e}_1, z\mathbf{e}_2, x^2\mathbf{e}_1 + y\mathbf{e}_2\rangle \text{ is } p_1 = \langle z\rangle - \text{primary},$$

$$Q_2 = \langle \mathbf{e}_1, \mathbf{e}_3, y\mathbf{e}_2\rangle \text{ is } p_2 = \langle y\rangle - \text{primary and}$$

$$Q_3 = \langle \mathbf{e}_1, x\mathbf{e}_2, x\mathbf{e}_3 + \mathbf{e}_2\rangle \text{ is } p_3 = \langle x\rangle - \text{primary}.$$

Take $p = \langle z,y\rangle$. Then $N:M \subseteq p$ and $p_1 \subseteq p, p_2 \subseteq p$. Hence by Theorem 3.1.2,

$$S_p(N) = \bigcap_{p_i \subseteq p} Q_i = Q_1 \cap Q_2 = \langle \mathbf{e}_3, z\mathbf{e}_1, yz\mathbf{e}_2, x^2\mathbf{e}_1 + y\mathbf{e}_2\rangle.$$

Proposition 3.1.1. *([9], Proposition 4.1) Let N be a submodule of a finitely generated R-module M and let $p \supseteq N:M$ be a prime ideal of R. Then*

$$(N + pM):M = S_p(N + pM):M = p.$$

Theorem 3.1.3. *([9], Theorem 5.4) Let N be a proper submodule of a finitely generated module M. Then every minimal prime submodule of N must be of the form $S_p(N + pM)$ for some $N:M \subseteq p$.*

The following theorem gives us a method to compute the saturation of $N + pM$.

Theorem 3.1.4. *Let M be finitely generated and let N be a submodule of M such that $N : M \subseteq p$ is a prime ideal of R. Then $S_p(N+pM)$ is p-prime submodule and $S_p(N+pM)$ is equal to p-primary component of the primary decomposition of $N+pM$.*

Proof. Let $N+pM = Q_1 \cap \cdots \cap Q_n$ be primary decomposition of $N+pM$ with $\sqrt{Q_i : M} = p_i$ for $1 \leq i \leq n$. By Proposition 3.1.1, $(N+pM) : M = p$ and hence p is the unique minimal associated prime of $N+pM$. Let $p = p_1$. Hence $p \subsetneq p_i$ for $i = 2, \cdots, n$. By Theorem 3.1.2,

$$S_p(N+pM) = \bigcap_{p_i \subseteq p} Q_i = Q_1.$$

Hence Q_1 is prime. □

Above theorem can be applied if we know the primary decomposition of $N+pM$. If we only know the associated primes of $N+pM$, not the decomposition; then we can compute $S_p(N+pM)$ as follows.

Proposition 3.1.2. *Let R be a Noetherian ring and let N be a submodule of finitely generated R-module M and $p \supseteq N : M$ be a prime ideal of R. If $Ass(M/(N+pM)) = \{p, p_1, \ldots, p_s\}$, then $S_p(N+pM) = (N+pM) : \langle f \rangle^\infty$ for some $f \in (\bigcap_{i=1}^{s} p_i) - p$.*

Proof. Since $N : M \subseteq p$, $(N+pM : M) = p$ and so $p \subseteq p_i$ for each i. Then there exists $f_i \in p_i - p$ for each i. So such an f can be found $f = \prod_{i=1}^{s} f_i \in (\bigcap_{i=1}^{s} p_i) - p$.
Now, if Q_i is the corresponding primary submodule of the decomposition for the prime ideal p_i, by Lemma 2.2.1, $(N+pM) : \langle f \rangle^\infty = Q$ and by Theorem 3.1.4

$$(N+pM) : \langle f \rangle^\infty = Q = S_p(N+pM).$$

□

3.2 Generalized Associated Prime Ideal

In the first part of this section, we will give some known results about saturation and radical of a submodule. Recall that the radical of a submodule N is the intersection of all prime submodules of M containing N and it is denoted by $\text{rad}_M(N)$. In case there is no prime submodule containing N, $\text{rad}_M(N) = M$. The radical of a submodule is studied by some authors [1], [4], [5], [6], [9], [17], [23], [24], [25] and [26].

If M is finitely generated R-module, then $rad_M(N)$ is the intersection of its minimal prime submodules [23]. In [9], Lu showed that to find the radical of any submodule N of M, it is enough to consider saturations of submodules of the form $N + pM$ where p is a prime ideal of R such that $N : M \subseteq p$.

We give a simple proof of the following corollary then its original one.

Corollary 3.2.1. *([9], Corollary 5.5) If N is a submodule of a finitely generated R-module M, then*

$$rad_M(N) = \bigcap_{N:M \subseteq p} S_p(N + pM).$$

Proof. Since M is finitely generated, $rad_M(N)$ is the intersection of all minimal prime submodules of M. By Theorem 3.1.3, minimal prime submodules of N are of the form $S_p(N + pM)$ for some $N : M \subseteq p$. Hence

$$rad_M(N) = \bigcap_{N:M \subseteq p} S_p(N + pM).$$

□

The following corollary states that if M is Noetherian, then $rad_M(N)$ is the finite intersection of saturations $S_p(N + pM)$.

Corollary 3.2.2. *([9], Corollary 5.6) If M is Noetherian R-module and N is proper submodule of M, then there exist finitely many prime ideals $p_i \supseteq N : M$ such that,*

$$rad_M(N) = \bigcap_{i=1}^{n} S_{p_i}(N + p_i M).$$

We now need to determine the followings to compute the radical of any submodule N of an R-module M.

1. $S_p(N + pM)$ where $p \supseteq N : M$.

2. finite number of prime ideals such that $p \supseteq N : M$.

In the first section, we give a method to find the saturation $S_p(N + pM)$. The remaining question is what are the ideals p's where $p \supseteq N : M$? The following example shows that associated prime ideals of N are not necessarily equal to the set of these prime ideals. We use the computer algebra system **SINGULAR** to make the computations.

Example 3.2.1. Let $R = \mathbb{Q}[x,y]$, $M = R \oplus R$ and let $N = \langle x\mathbf{e}_1 + y^3\mathbf{e}_2, x^2\mathbf{e}_1, x\mathbf{e}_2\rangle$.

It is clear that N is $\langle x\rangle$-primary and $\langle x\rangle$ is the only associated prime ideal of N. Then $N + \langle x\rangle M = \langle x\mathbf{e}_1, x\mathbf{e}_2, y^3\mathbf{e}_2\rangle = Q_1 \cap Q_2$ where

$$\begin{aligned} Q_1 &= \langle \mathbf{e}_2, x\mathbf{e}_1\rangle \text{ is } \langle x\rangle-\text{primary}, \\ Q_2 &= \langle x\mathbf{e}_1, x\mathbf{e}_2, y^3\mathbf{e}_1, y^3\mathbf{e}_2\rangle \text{ is } \langle x,y\rangle-\text{primary}. \end{aligned}$$

By Theorem 3.1.4,

$$S_{\langle x\rangle}(N + \langle x\rangle M) = Q_1 = \langle x\mathbf{e}_1, \mathbf{e}_2\rangle.$$

On the other hand, $P = \langle x\mathbf{e}_1, x\mathbf{e}_2, y\mathbf{e}_1, y\mathbf{e}_2\rangle$ is an $\langle x,y\rangle$-prime submodule of M containing N. So $rad_M(N) \subseteq P$ but $S_{\langle x\rangle}(N + \langle x\rangle M) \nsubseteq P$.

This example shows that for finding radical of submodule N, we need some additional prime ideals other than $Ass(M/N)$ containing $N : M$.

For the rest of this section, let N be a proper submodule of a finitely generated module M and p be a prime ideal of R such that $N : M \subseteq p$. For a prime ideal q of R, we write $p \longrightarrow_N q$ if $q \in Ass(M/N + pM)$.
In [17], McCasland and Smith gave the following definition.

Definition 3.2.1. Let N be a proper submodule of a finitely generated module M. A prime ideal p of R is a generalized associated prime ideal of N, if there exist prime ideals $p_0, p_1, \ldots, p_n$ of R such that $p_0 \in Ass(M/N)$ and

$$p_0 \longrightarrow_N \ p_1 \longrightarrow_N \ p_2 \longrightarrow_N \ \cdots \longrightarrow_N \ p_n = p.$$

We will denote the set of generalized associated primes of N by $GAss(M/N)$. If we analize the definition, we get the following properties.

(i) If $p \in GAss(M/N)$, then $N : M \subseteq p$. Also, prime ideals which appear in the definition of generalized associated prime form a chain.

(ii) $Ass(M/N) \subseteq GAss(M/N)$.

Proof. (i) Let $p \in GAss(M/N)$. Then there exist prime ideals $p_0, p_1, \ldots, p_n$ of R such that $p_0 \in Ass(M/N)$ and

$$p_0 \longrightarrow_N p_1 \longrightarrow_N p_2 \longrightarrow_N \cdots \longrightarrow_N p_n = p.$$

Since $p_0 \in Ass(M/N)$, $N : M \subseteq p_0$. Also by the definition of generalized associated prime ideal, $p_i \in Ass(M/N + p_{i-1}M)$ for each i. Hence $(N + p_{i-1}M) : M \subseteq p_i$, and this implies that $N : M \subseteq p_i$.

Since $N : M \subseteq p_{i-1}$, $(N + p_{i-1}M) : M = p_{i-1}$ for each i. That is, we have

$$N : M \subseteq (N + p_{i-1}M) : M = p_{i-1} \subseteq p_i.$$

Hence

$$p_0 \subseteq p_1 \subseteq \cdots \subseteq p_n = p.$$

(ii) If $p \in Ass(M/N)$, then $N : M \subseteq p$. So, $(N + pM) : M = p$ and hence $p \in Ass(M/N + pM)$. Take $n = 1$ and $p_0 = p$, then we have $p \longrightarrow_N p$. This implies that $p \in GAss(M/N)$.

□

Theorem 3.2.1. *([17], Theorem 3.5) Let R be a Noetherian ring and let N be a proper submodule of a finitely generated R-module M. Then $GAss(M/N)$ is finite.*

The following theorem is proved by McCasland and Smith [17]. We will give an alternative proof of the theorem.

Theorem 3.2.2. *Let R be a Noetherian ring and let N be a proper submodule of a finitely generated R-module M. Then*

$$rad_M(N) = \bigcap_{p_i \in GAss(M/N)} S_{p_i}(N + p_iM).$$

Proof. By Theorem 3.2.1 , $GAss(M/N)$ is finite. Say, $GAss(M/N) = \{p_1, \ldots, p_s\}$. Let p be a prime ideal of R such that $p \supset N : M$ and $p \notin GAss(M/N)$. Now our aim is to show $S_{p_i}(N + p_iM) \subseteq S_p(N + pM)$ for some i. Then the conclusion of the theorem will be obvious.

Now let us say $K = \{p_i \in GAss(M/N) : p_i \subset p\}$. Since $Ass(M/N) \subseteq GAss(M/N)$, K is not empty. Let p_i be one of the maximal elements of K in terms of the set inclusion order for a fixed i. Then

$$N + p_i M \subset N + pM \subset S_p(N + pM).$$

Consider a minimal primary decomposition

$$N + p_i M = Q_1 \cap N_2 \cap \cdots \cap N_s$$

where Q_1 is p_i-primary and N_j is q_j-primary for $j = 2, 3, \ldots, s$.

Since $(N + p_i M) : M = p_i$, $p_i \subset q_j$ for all j. On the other hand, p_i being a maximal element of K implies that $q_j \not\subset p$. This means that there exists $f_j \in (q_j \setminus p)$ for $j = 2, \ldots, s$. Therefore $f = f_2 f_3 \cdots f_s \in (\bigcap_{j=2}^{s} q_j) \setminus p$.

We claim that $S_{p_i}(N + p_i M) = Q_1 \subset S_p(N + pM)$. Otherwise; there would be $m \in Q_1$ but $m \notin S_p(N + pM)$. Since $f \in q_j$, $f^{n_j} \in (N_j : M)$ for some positive integer n_j. Let $n = \max\{n_2, \ldots, n_s\}$, then $f^n \in (N_2 : M) \cap \cdots \cap (N_s : M) = (N_2 \cap \cdots \cap N_s) : M$. Therefore $f^n m \in N_2 \cap \cdots \cap N_s$. Furthermore, since $m \in Q_1$, $f^n m \in Q_1$. Hence $f^n m \in N + p_i M \subset S_p(N + pM)$. On the other hand, $f^n \notin p$ since p is prime ideal. Then by the definition of saturation, $m \in S_p(S_p(N + pM)) = S_p(N + pM)$ which is a contradiction. □

We will conclude this section with an example. We use **SINGULAR** to make necessary computations.

Example 3.2.2. Let $R = \mathbb{Q}[x, y, z]$ and let M be an R-submodule $R \oplus R \oplus R$. Consider the submodule $N = \langle xz\mathbf{e}_3 - z\mathbf{e}_1, x^2\mathbf{e}_3, x^2y^3\mathbf{e}_1 + x^2y^2z\mathbf{e}_2 \rangle$. Then $N : M = \langle x^2y^2z^2 \rangle$ and primary decomposition of N is $N = Q_1 \cap Q_2 \cap Q_3$ where Q_1 is $\langle z \rangle$-primary, Q_2 is $\langle y \rangle$-primary and Q_3 is $\langle x \rangle$-primary. Then

$$Ass(M/N) = \{\langle z \rangle, \langle y \rangle, \langle x \rangle\}.$$

Now, we have following sets and submodules.

$N + \langle z \rangle M = \langle z\mathbf{e}_1, z\mathbf{e}_2, z\mathbf{e}_3, x^2\mathbf{e}_3, x^2y^3\mathbf{e}_1 \rangle = P_1 \cap P_2 \cap P_3$ where

$$P_1 = \langle \mathbf{e}_1, \mathbf{e}_3, z\mathbf{e}_2 \rangle \text{ is } \langle z \rangle - \text{primary},$$
$$P_2 = \langle \mathbf{e}_3, z\mathbf{e}_1, z\mathbf{e}_2, y^3\mathbf{e}_1, y^3\mathbf{e}_2 \rangle \text{ is } \langle z, y \rangle - \text{primary and}$$
$$P_3 = \langle z\mathbf{e}_1, z\mathbf{e}_2, z\mathbf{e}_3, x^2\mathbf{e}_1, x^2\mathbf{e}_2, x^2\mathbf{e}_3 \rangle \text{ is } \langle z, x \rangle - \text{primary}.$$

By Theorem 3.1.4,

$$S_1 = S_{\langle z \rangle}(N + \langle z \rangle M) = P_1 = \langle \mathbf{e}_1, \mathbf{e}_3, z\mathbf{e}_2 \rangle.$$

Then $Ass(M/N + \langle z \rangle M) = \{\langle z \rangle, \langle z, y \rangle, \langle z, x \rangle\}$.

Since $N + \langle y \rangle M = \langle y\mathbf{e}_1, y\mathbf{e}_2, y\mathbf{e}_3, xz\mathbf{e}_1, xz\mathbf{e}_3 - z\mathbf{e}_1, x^2\mathbf{e}_3 \rangle = P_4 \cap P_5 \cap P_6$ where

$$P_4 = \langle \mathbf{e}_1, \mathbf{e}_3, y\mathbf{e}_2 \rangle \text{ is } \langle y \rangle - \text{ primary},$$
$$P_5 = \langle \mathbf{e}_3, z\mathbf{e}_1, z\mathbf{e}_2, y\mathbf{e}_1, y\mathbf{e}_2 \rangle \text{ is } \langle z, y \rangle - \text{ primary and}$$
$$P_6 = \langle y\mathbf{e}_1, y\mathbf{e}_2, y\mathbf{e}_3, x\mathbf{e}_1, x\mathbf{e}_3 - \mathbf{e}_1, x^2\mathbf{e}_1 \rangle \text{ is } \langle x, y \rangle - \text{ primary}.$$

Then by Theorem 3.1.4, we get

$$S_2 = S_{\langle y \rangle}(N + \langle y \rangle M) = P_4 = \langle \mathbf{e}_1, \mathbf{e}_3, y\mathbf{e}_2 \rangle \text{ and}$$

$$Ass(M/N + \langle y \rangle M) = \{\langle y \rangle, \langle z, y \rangle, \langle x, y \rangle\}.$$

For the associated prime ideal $\langle x \rangle$ of N, we have

$$N + \langle x \rangle M = \langle z\mathbf{e}_1, x\mathbf{e}_1, x\mathbf{e}_2, x\mathbf{e}_3 \rangle = P_7 \cap P_8$$

where

$$P_7 = \langle \mathbf{e}_1, x\mathbf{e}_2, x\mathbf{e}_3 \rangle \text{ is } \langle x \rangle - \text{ primary},$$
$$P_8 = \langle z\mathbf{e}_1, z\mathbf{e}_2, z\mathbf{e}_3, x\mathbf{e}_1, x\mathbf{e}_2, x\mathbf{e}_3 \rangle \text{ is } \langle z, x \rangle - \text{ primary}.$$

Hence,

$$S_3 = S_{\langle x \rangle}(N + \langle x \rangle M) = P_7 = \langle \mathbf{e}_1, x\mathbf{e}_2, x\mathbf{e}_3 \rangle,$$

and $Ass(M/N + \langle x \rangle M) = \{\langle x \rangle, \langle x, z \rangle\}$.

Up to here, we find the saturations of $N + pM$ where p is the associated prime ideal of N. In this process, we get another prime ideals $\langle z, y \rangle, \langle z, x \rangle, \langle x, y \rangle, \langle x, z \rangle$. Now, to get the radical of N, we need to the find saturations of $N + qM$ for each of these prime ideals.

For the prime ideal $\langle z,y\rangle$,

$$N+\langle z,y\rangle M=\langle z\mathbf{e}_1,z\mathbf{e}_2,z\mathbf{e}_3,y\mathbf{e}_1,y\mathbf{e}_2,y\mathbf{e}_3,x^2\mathbf{e}_3\rangle=P_9\cap P_{10}$$

where

$P_9=\langle \mathbf{e}_3,z\mathbf{e}_1,z\mathbf{e}_2,y\mathbf{e}_1,y\mathbf{e}_2\rangle$ is $\langle z,y\rangle-$ primary,

$P_{10}=\langle z\mathbf{e}_1,z\mathbf{e}_2,z\mathbf{e}_3,y\mathbf{e}_1,y\mathbf{e}_2,y\mathbf{e}_3,x^2\mathbf{e}_1,x^2\mathbf{e}_2,x^2\mathbf{e}_3\rangle$ is $\langle z,y,x\rangle-$ primary.

Then

$$S_4=S_{\langle z,y\rangle}(N+\langle z,y\rangle M)=P_9=\langle \mathbf{e}_3,z\mathbf{e}_1,z\mathbf{e}_2,y\mathbf{e}_1,y\mathbf{e}_2\rangle$$

and $Ass(M/N+\langle z,y\rangle M)=\{\langle z,y\rangle,\langle x,y,z\rangle\}$.

For the prime ideal $\langle z,x\rangle$,

$$N+\langle z,x\rangle M=\langle z\mathbf{e}_1,z\mathbf{e}_2,z\mathbf{e}_3,x\mathbf{e}_1,x\mathbf{e}_2,x\mathbf{e}_3\rangle$$

which is $\langle z,x\rangle$-primary. Hence

$$S_5=S_{\langle x,z\rangle}(N+\langle x,z\rangle M)=\langle x\mathbf{e}_1,x\mathbf{e}_2,x\mathbf{e}_3,z\mathbf{e}_1,z\mathbf{e}_2,z\mathbf{e}_3\rangle$$

and $Ass(M/N+\langle x,z\rangle M)=\{\langle x,z\rangle\}$.

For the prime ideal $\langle x,y\rangle$,

$$N+\langle x,y\rangle M=\langle \mathbf{e}_1,y\mathbf{e}_2,y\mathbf{e}_3,x\mathbf{e}_2,x\mathbf{e}_3\rangle=P_{11}\cap P_{12}$$

where

$P_{11}=\langle \mathbf{e}_1,y\mathbf{e}_2,y\mathbf{e}_3,x\mathbf{e}_2,x\mathbf{e}_3\rangle$ is $\langle y,x\rangle-$ primary,

$P_{12}=\langle z\mathbf{e}_1,z\mathbf{e}_2,z\mathbf{e}_3,y\mathbf{e}_1,y\mathbf{e}_2,y\mathbf{e}_3,x\mathbf{e}_1,x\mathbf{e}_2,x\mathbf{e}_3\rangle$ is $\langle x,y,z\rangle-$ primary.

Then

$$S_6=S_{\langle x,y\rangle}(N+\langle x,y\rangle M)=P_{11}=\langle \mathbf{e}_1,y\mathbf{e}_2,y\mathbf{e}_3,x\mathbf{e}_2,x\mathbf{e}_3\rangle$$

and $Ass(M/N+\langle x,y\rangle M)=\{\langle x,y\rangle,\langle x,y,z\rangle\}$.

Since $\langle x,y,z\rangle$ is the associated prime ideal of $N+\langle x,y\rangle M$,

$$N+\langle x,y,z\rangle M=\langle x\mathbf{e}_1,x\mathbf{e}_2,x\mathbf{e}_3,y\mathbf{e}_1,y\mathbf{e}_2,y\mathbf{e}_3,z\mathbf{e}_1,z\mathbf{e}_2,z\mathbf{e}_3\rangle=M$$

and then

$$S_7 = S_{\langle x,y,z\rangle}(N+\langle x,y,z\rangle M) = \langle x\mathbf{e}_1, x\mathbf{e}_2, x\mathbf{e}_3, y\mathbf{e}_1, y\mathbf{e}_2, y\mathbf{e}_3, z\mathbf{e}_1, z\mathbf{e}_2, z\mathbf{e}_3\rangle$$

with $Ass(M/N+\langle x,y,z\rangle M) = \{\langle x,y,z\rangle\}$.

Thus,

$$rad_M(N) = \bigcap_{i=1}^{7} S_i = \langle z\mathbf{e}_1, x\mathbf{e}_3, xy\mathbf{e}_1, xyz\mathbf{e}_2\rangle.$$

3.3 Determining Redundant Primes

In the Example 3.2.2, S_6 is a redundat prime submodule for the radical of N since $S_3 \subseteq S_6$. This implies that some prime submodules which are mentioned in the Theorem 3.2.2 can be redundant.

McCasland and Smith gave a technique to determine the redundant primes [17]. In this section we will give a different technique to eliminate that redundant primes. First of all, we need some technical results for prime submodules similar to Lemma 2.2.1.

Lemma 3.3.1. *Let N be p-prime submodule of an R-module M and let $f \in R$. Then*

$$(i)\ N : \langle f\rangle = \begin{cases} M, & \text{if } f \in p; \\ N, & \text{if } f \notin p. \end{cases}$$

and if I is an ideal of R, then

$$(ii)\ N : I = \begin{cases} M, & \text{if } I \subseteq p; \\ N, & \text{if } I \nsubseteq p. \end{cases}$$

Proof. (i) Let N be a prime submodule of M. If $f \in p = (N : M)$, then $M \subseteq N : \langle f\rangle$. Let $f \notin p$ and $m \in N : \langle f\rangle$. Since N is a prime submodule, $m \in N$.

(ii) Let I be an ideal of R. Then, if $I \subseteq p$, then $IM \subseteq N$ and this implies that $M = N : I$. If $I \nsubseteq p$, then there exists an element $a \in I$ and $a \notin p$. By part (i), $N \subseteq N : I \subseteq N : \langle a\rangle = N$. □

Theorem 3.3.1. *Let $\{(S_i, M, p_i) : i = 1,2,\cdots,n\}$ be a collection of p_i-prime submodules S_i of M with $p_i \neq p_j$ for all $i \neq j$ such that $S = \bigcap S_i$ is a possibly redundant* *prime decomposition of S. Let* $f \in (\bigcap_{p_j \not\subseteq p_i} p_j) - p_i$ *and let* $N_i = (\bigcap_{k=1}^{i} S_k) : \langle f \rangle$. *Then,* S_i *is redundant prime submodule in the decomposition of S if and only if* $N_i = N_i : p_i$. *(in case* $p_j \subseteq p_i$ *for all j take* $f \notin p_i$ *).*

Proof. By Lemma 3.3.1 part (i),

$$N_i = \bigcap_{k=1}^{i} S_k : \langle f \rangle = \bigcap_{f \notin p_k} S_k = \bigcap_{p_k \subseteq p_i} S_k.$$

Again by Lemma 3.3.1 part (ii),

$$N_i : p_i = (\bigcap_{p_k \subseteq p_i} S_k) : p_i = \bigcap_{p_k \subsetneq p_i} S_k$$

Then, $S = \bigcap_{k \neq i} S_k$ if and only if $N_i = N_i : p_i$. □

By using Theorem 3.3.1, we can check whether the prime submodules which we found in the Example 3.2.2 are redundant or not for each step.

In the fourth step of the Example 3.2.2, we get

$$S_1 = \langle \mathbf{e}_1, \mathbf{e}_3, z\mathbf{e}_2 \rangle \text{ is } p_1 = \langle z \rangle - \text{prime,}$$

$$S_2 = \langle \mathbf{e}_1, \mathbf{e}_3, y\mathbf{e}_2 \rangle \text{ is } p_2 = \langle y \rangle - \text{prime,}$$

$$S_3 = \langle \mathbf{e}_1, x\mathbf{e}_2, x\mathbf{e}_3 \rangle \text{ is } p_3 = \langle x \rangle - \text{prime and}$$

$$S_4 = \langle \mathbf{e}_3, z\mathbf{e}_1, z\mathbf{e}_2, y\mathbf{e}_1, y\mathbf{e}_2 \rangle \text{ is } p_4 = \langle z, y \rangle - \text{prime.}$$

Now, since $p_1 \subseteq p_4$ and $p_2 \subseteq p_4$, there is a possibility that S_4 is redundant for the intersection $S = \bigcap_{k=1}^{4} S_k$.

If we take $f = x \in p_3 - p_4$, then

$$N_4 = S : \langle f \rangle = (\bigcap_{k=1}^{4} S_k) : \langle x \rangle = \bigcap_{x \notin p_k} S_k = S_1 \cap S_2 \cap S_4 = \langle \mathbf{e}_3, z\mathbf{e}_1, y\mathbf{e}_1, yz\mathbf{e}_2 \rangle, \text{ and}$$

$$N_4 : p_4 = N_4 : \langle z, y \rangle = S_1 \cap S_2 = \langle \mathbf{e}_1, \mathbf{e}_3, yz\mathbf{e}_2 \rangle.$$

Since $N_4 : p_4 \neq N_4$, S_4 is needed in the intersection and thus $S = \bigcap_{k=1}^{4} S_k$.

Now, consider the submodule $S_5 = \langle x\mathbf{e}_1, x\mathbf{e}_2, x\mathbf{e}_3, z\mathbf{e}_1, z\mathbf{e}_2, z\mathbf{e}_3 \rangle$ with $p_5 = \langle z, x \rangle$. Since $p_1 \subseteq p_5$ and $p_3 \subseteq p_5$, take $f = y \in (p_2 \cap p_4) - p_5$. Then

$$N_5 = S:\langle f\rangle = (\bigcap_{k=1}^{5} S_k):\langle y\rangle = \bigcap_{y\notin p_k} S_k = S_1\cap S_3\cap S_5 = \langle z\mathbf{e}_1, x\mathbf{e}_1, x\mathbf{e}_3, xz\mathbf{e}_2\rangle,$$

$$N_5 : p_5 = N_5:\langle z,x\rangle = S_1\cap S_3 = \langle \mathbf{e}_1, x\mathbf{e}_3, xz\mathbf{e}_2\rangle.$$

Since $N_5 : p_5 \neq N_5$, S_5 is also not a redundant prime. Hence $S = \bigcap_{k=1}^{5} S_k$ in this step. But if we consider S_6 and $p_6 = \langle x,y\rangle$, we have $p_2 \subseteq p_6$ and $p_3 \subseteq p_6$. Take $f = z \in (p_1\cap p_4\cap p_5) - p_6$. Then

$$N_6 = S:\langle f\rangle = (\bigcap_{k=1}^{6} S_k):\langle z\rangle = \bigcap_{z\notin p_k} S_k = S_2\cap S_3\cap S_6 = \langle \mathbf{e}_1, x\mathbf{e}_3, xy\mathbf{e}_2\rangle,$$

$$N_6 : p_6 = N_6:\langle z,y\rangle = S_2\cap S_3 = \langle \mathbf{e}_1, x\mathbf{e}_3, xy\mathbf{e}_2\rangle.$$

Since $N_6 : p_6 = N_6$, S_6 is a redundant prime. Hence $S = \bigcap_{k=1}^{5} S_k$.

Also, for the last step we get $S_7 = \langle x\mathbf{e}_1, x\mathbf{e}_2, x\mathbf{e}_3, y\mathbf{e}_1, y\mathbf{e}_2, y\mathbf{e}_3, z\mathbf{e}_1, z\mathbf{e}_2, z\mathbf{e}_3\rangle$ and $p_7 = \langle x,y,z\rangle$. It is obvious that

$$(\bigcap_{k=1}^{5} S_k)\cap S_7 = \bigcap_{k=1}^{5} S_k.$$

Also by using Theorem 3.3.1, we can say that S_7 is redundant. Since $p_i \subseteq p_7$ for all $i = 1,\cdots,6$, take $f = 1 \notin p_7$. Then

$$N_7 = ((\bigcap_{k=1}^{5} S_k)\cap S_7):\langle f\rangle = ((\bigcap_{k=1}^{5} S_k)\cap S_7):\langle 1\rangle = \bigcap_{k=1}^{5} S_k \text{ and}$$

$$N_7 : p_7 = N_7:\langle x,y,z\rangle = \bigcap_{k=1}^{5} S_k.$$

Thus S_7 is also a redundant prime and therefore

$$rad_M(N) = \bigcap_{i=1}^{5} S_i = \langle z\mathbf{e}_1, x\mathbf{e}_3, xy\mathbf{e}_1, xyz\mathbf{e}_2\rangle.$$

Hence for this example, we only need the prime submodules $S_1,\cdots,S_5$.

4. PRIME AND SEMIPRIME SUBMODULES

4.1 Semiprime Submodules

In this section, we will define semiprime submodule and give some properties of semiprimes.

Definition 4.1.1. A proper submodule N of an R-module M is called *a semiprime submodule* if whenever $r^k m \in N$ for some $r \in R$, $m \in M$ and $k \in \mathbb{N}$, then $rm \in N$.

The following lemma shows that semiprime submodules can be defined in terms of their envelopes.

Lemma 4.1.1. *A proper submodule N is semiprime if and only if $\langle E_M(N)\rangle = N$.*

Proof. Suppose that N is semiprime. Let $x \in \langle E_M(N)\rangle$. Then there exist elements $r_i \in R$, $m_i \in M$ $(1 \le i \le k)$ such that

$$x = r_1 m_1 + \cdots + r_k m_k \qquad with \quad r_i^{t_i} m_i \in N$$

for some $t_i \in \mathbb{Z}^+$. Since N is semiprime, $r_i m_i \in N$ for all i. Hence $x \in N$ and $\langle E_M(N)\rangle = N$.

Conversely suppose that $\langle E_M(N)\rangle = N$. Let $r^k m \in N$ for some $r \in R, m \in M$ and natural number k. By the definition of envelope, $rm \in E_M(N) \subseteq \langle E_M(N)\rangle = N$. Hence N is semiprime. □

It is easy to see that every prime submodule is semiprime. But the converse is not true. For example, if $R = \mathbb{Q}[x,y,z]$, $M = R^3$ and $N = \langle z\mathbf{e}_1, y\mathbf{e}_1, xy\mathbf{e}_2, xy\mathbf{e}_3, xz\mathbf{e}_2 + x^2 z\mathbf{e}_3\rangle$, then by Theorem 2.3.1, $\langle E_M(N)\rangle = N$. Hence N is a semiprime submodule of M with $N : M = \langle xy\rangle$. On the other hand N is not a prime submodule; if we take $r = z$ and $m = (0,x,x^2)$, then $rm = z(0,x,x^2) = (0,xz,x^2z) \in N$ but $r = z \notin N : M$ and $m = (0,x,x^2) \notin N$.

If N is prime submodule, it is well-known that $N : M$ is a prime ideal. When N is semiprime, we have the following.

Lemma 4.1.2. *If N is a semiprime submodule of an R-module M, then $N : M$ is a semiprime ideal.*

Proof. Recall that for a commutative ring R, an ideal I is semiprime if $\sqrt{I} = I$. Now, let $x \in \sqrt{N : M}$. Then $xM \subseteq N$ since N is semiprime. Hence $\sqrt{N : M} = N : M$. This implies that $N : M$ is a semiprime ideal. □

Lemma 4.1.3. *Let N be a primary submodule. Then N is semiprime submodule iff $N : M$ is a semiprime ideal.*

Proof. Suppose $N : M$ is semiprime ideal. Let $r^k m \in N$ where $r \in R, m \in M - N$ and $k \in \mathbb{Z}^+$. Since N is primary and $N : M$ is semiprime, $r \in \sqrt{N : M} = N : M$. Hence $rm \in N$. Otherside is clear by the above lemma. □

Lemma 4.1.4. *Let M be an R-module and N be a proper submodule of M. Then N is prime submodule of M if and only if N is primary and semiprime.*

Proof. Assume that N is primary and semiprime. Let $am \in N$ for $a \in R, m \in M$. Since N is primary, either $m \in N$ or $a \in \sqrt{N : M}$. By Lemma 4.1.2, $m \in N$ or $a \in N : M$. Hence N is prime submodule. The converse is clear. □

Proposition 4.1.1. *A finite intersection of semiprime submodules is also semiprime.*

Proof. Let $N = N_1 \cap N_2 \cap \cdots \cap N_s$ where each N_i is semiprime. If $x \in \langle E_M(N) \rangle$, then

$$x = r_1 m_1 + r_2 m_2 + \cdots r_t m_t$$

where $r_i^{k_i} m_i \in N$ for some $k_i \in \mathbb{N}$. Therefore for each i and j, $r_i^{k_i} m_i \in N_j$. Since each N_j is semiprime, $r_i m_i \in N_j$ for $j = 1, \ldots, s$. Hence $x \in N$. By Lemma 4.1.1 , N is semiprime.

□

Lemma 4.1.5. *Let M be an R-module. Assume that N and K are submodules of M such that $K \subseteq N$ with $N \neq M$. Then, if K and N/K are semiprime submodules then N is also semiprime.*

Proof. Let $r^t m \in N$ for $r \in R, m \in M$ and $t \in \mathbb{Z}^+$. Then $r^t(m+K) = r^t m + K \in N/K$. If $r^t m \in K$, then $rm \in K \subseteq N$ since K is semiprime. Now, we may assume that $r^t m \notin K$. Then $r^t(m+K) \in N/K$ and N/K is semiprime implies that $r(m+K) = rm + K \in N/K$. Hence $rm \in N$. □

Lemma 4.1.6. *Let $M = K \oplus L$ be the direct sum of submodules K, L and N be semiprime submodule of K. Then $N \oplus L$ is a semiprime submodule of M.*

Proof. Let $r \in R, m \in M$ and $r^t m \in N \oplus L$ for some $t \in \mathbb{Z}^+$. Then there exist elements $n \in N, l \in L$ such that $r^t m = n + l$. Since $M = K \oplus L$, there exists an element $k \in K$ such that $r^t k \in N$. Since N is semiprime, $rk \in N$. Hence $rm \in N \oplus L$. □

Lemma 4.1.7. *([1], Result 1.1) Let M, M' be R-modules with $\phi : M \to M'$ an R-module epimorphism and N be a submodule of M such that $Ker\phi \subseteq N$. Then there exists a one-to-one order preserving correspondence between the proper submodules of M containing N and the proper submodules of M' containing $\phi(N)$. Furthermore, for any submodule N' of M' there exists a submodule L of M such that $Ker\phi \subseteq L$ and $\phi(L) = N'$.*

The following lemma gives the relationship between semiprime submodules of a module M and semiprime submodules of its homomorphic image.

Lemma 4.1.8. *Let M, M' be R-modules with $\phi : M \to M'$ an R-module epimorphism and N be a submodule of M such that $Ker\phi \subseteq N$. Then*

(i) If P is a semiprime submodule of M containing N, then $\phi(P)$ is a semiprime submodule of M' containing $\phi(N)$.

(ii) If P' is a semiprime submodule of M' containing $\phi(N)$, then $\phi^{-1}(P')$ is a semiprime submodule of M containing N.

Proof. (i) Assume that P is a semiprime submodule of M containing N. Then $\phi(N) \subseteq \phi(P)$ and $\phi(P) \neq \phi(M) = M'$ by Lemma 4.1.7

Let $r \in R$, $m' \in M'$ and $r^t m' \in \phi(P)$ for some $t \in \mathbb{Z}^+$. Since ϕ is an epimorphism, there exists an element $m \in M$ such that $\phi(m) = m'$. Then $r^t m' = r^t \phi(m) \in \phi(P)$ implies that $r^t m \in P$. Since P is semiprime submodule of M, $rm \in P$. Thus $rm' = r\phi(m) = \phi(rm) \in \phi(P)$.

(ii) Let P' be a semiprime submodule of M' containing $\phi(N)$. By Lemma 4.1.7 , there exists a submodule L of M such that $Ker\phi \subseteq L$ and $\phi(L) = P'$. Then $N \subseteq L = \phi^{-1}(P')$. Let $r^t m \in \phi^{-1}(P')$ for $r \in R, m \in M$ and $t \in \mathbb{Z}^+$. Then $\phi(r^t m) = r^t \phi(m) \in P'$. Since P' is semiprime submodule of M', $r\phi(m) = \phi(rm) \in P'$. Hence $rm \in \phi^{-1}(P')$.

□

Let K and N be any submodules of an R-module M where $N \subseteq K$. If we consider the canonical epimorphism $\phi : M \to M/N$, then by Lemma 4.1.8 it is clear that K is a semiprime submodule of M if and only if K/N is semiprime submodule of M/N.

4.2 Semiprime Radical

If N is a proper submodule of an R-module M, then the prime radical of N, $rad_M(N)$, is the intersection of all prime submodules containing N. The prime radical of submodules has been studied by some authors ([4], [9], [17], [23], [24], [27]).

The semiradical of N, denoted by $srad_M(N)$, is defined as the intersection of all semiprime submodules of M containing N. If there is no semiprime submodule containing N, then $srad_M(N) = M$. We shall denote the semiradical of M by $srad_M(0)$. Since $rad_M(N)$ is semiprime, we have

$$N \subseteq \langle E_M(N) \rangle \subseteq srad_M(N) \subseteq rad_M(N)$$

where

$$E_M(N) = \{rm : r \in R, m \in M \quad and \quad r^k m \in N \quad for \quad some \quad k \in \mathbb{Z}^+\}.$$

Proposition 4.1.1 implies that $srad_M(N)$ is the minimal semiprime submodule of M containing N. In this section we will give generalization of [4] and [12] to semiprime radical. The following two lemmas are generalization of [4] Lemma 4 and Lemma 6.

Lemma 4.2.1. *Let R be a ring, M be an R-module and N, K be submodules of M with $K \subseteq N$. Then $srad_N(K) \subseteq srad_M(K)$.*

Proof. Let P be any semiprime submodule of M with $K \subseteq P$. If $N \subseteq P$, then $srad_N(K) \subseteq P$. If $N \nsubseteq P$, then $N \cap P$ is a semiprime submodule of N. Hence $srad_N(K) \subseteq N \cap P \subseteq P$. Thus in any case $srad_N(K) \subseteq srad_M(K)$. □

Lemma 4.2.2. *Let M be the direct sum of the R-modules M_i, $i \in I$. Let $N = \oplus N_i$ be a submodule of M such that N_i is a submodule of M_i for all $i \in I$. Then $srad_M(N) = \oplus srad_{M_i}(N_i)$.*

Proof. By Lemma 4.2.1, $srad_{M_i}(N_i) \subseteq srad_M(N)$ for all $i \in I$. Let $m \in srad_M(N)$ and $m \notin \bigoplus_i srad_{M_i}(N_i)$. Then there exists $j \in I$ such that $\pi_j(m) \notin srad_{M_j}(N_j)$ where $\pi_j : M \to M_j$ denotes the canonical projection. There exists a semiprime submodule P_j of M_j such that $\pi_j(m) \notin P_j$. By Lemma 4.1.6, $K = P_j \bigoplus (\bigoplus_{i \neq j} M_i)$ is semiprime submodule of M containing N. Since $\pi_j(m) \notin P_j$, $m \notin K$. Then $m \notin srad_M(N)$. Therefore $srad_M(N) = \bigoplus_i srad_{M_i}(N_i)$. □

Lemma 4.2.3. *([4], Corollary 2) Let R be a domain and M be a non-torsion module. Then*

1. *the torsion submodule $T(M)$ of M is prime, and*

2. *$PM = M$ or PM is prime submodule of M, for each maximal ideal P of R.*

Since every prime submodule is semiprime, $T(M)$ and PM are semiprime submodules of a module M over a domain where P is maximal ideal of R.

The general form of [12], Proposition 1.3 is

Proposition 4.2.1. *Let R be a domain and M be an R-module with torsion submodule $T(M)$. If N is a submodule of $T(M)$, then N is semiprime submodule of $T(M)$ if and only if N is semiprime submodule of M.*

Proof. Suppose N is semiprime submodule of $T(M)$. Let $0 \neq r \in R, m \in M$ with $r^k m \in N$ for some $k \in \mathbb{Z}^+$. By Lemma 4.2.3, $rm \in T(M)$. Then there exists nonzero $s \in R$ such that $s(rm) = 0$. Since $sr \neq 0$, we have $m \in T(M)$ which implies that $rm \in N$. Thus, N is a semiprime submodule of M. The converse is clear. □

Now we can show that for domains, the study of semiprime radicals of any modules reduces to torsion modules.

Corollary 4.2.1. *Let R be a domain and M be an R-module with torsion submodule $T(M)$. Then $srad_M(0) = srad_{T(M)}(0)$.*

Proof. Since $T(M)$ is a submodule of M, by Lemma 4.2.1 $srad_{T(M)}(0) \subseteq srad_M(0)$. Now, suppose $srad_{T(M)}(0) = \bigcap N$ where N is a semiprime submodule of $T(M)$. By Proposition 4.2.1, N is also semiprime submodule of M. Hence $srad_M(0) \subseteq srad_{T(M)}(0)$.

□

We also have the following corollary which is the generalization of [12], Lemma 1.7.

Corollary 4.2.2. *Let R be a domain and M be a left R-module with torsion submodule $T(M)$. Then*

$$srad_M(0) \subseteq \bigcap\{PT(M) : P \text{ is a maximal ideal of } R\}.$$

Proof. By Corollary 4.2.1 and Lemma 4.2.3. □

4.3 Modules Which Satisfy The Semiradical Formula

Note that any submodule N of a module M satisfies the radical formula (s.t.r.f) if $rad_M(N) = \langle E_M(N)\rangle$. It is said that M satisfies the radical formula if for every submodule N of M, $rad_M(N) = \langle E_M(N)\rangle$. A ring R satisfies the radical formula, if every R-module s.t.r.f.. Modules which satisfy the radical formula was studied in [1], [4], [5], [6], [7], [23], [24] and [25].

In the same manner, we say that M satisfies the semiradical formula (s.t.s.r.f.) if for any submodule N of M, $srad_M(N) = \langle E_M(N)\rangle$. Since intersection of semiprime submodules is semiprime, $srad_M(N)$ is the unique smallest semiprime submodule of M containing N.

We know that for an ideal I of R, we have $\sqrt{\sqrt{I}} = \sqrt{I}$; but the envelope of a submodule does not satisfy an equation similiar to this one as the following example shows.

Example 4.3.1. Let $R = \mathbb{Q}[x,y,z]$ and let M be an R-submodule $R \oplus R$. Consider the submodule $N = \langle z^2\mathbf{e}_1, z^2\mathbf{e}_2, yz\mathbf{e}_2, y^2\mathbf{e}_1 + z\mathbf{e}_2, y^2\mathbf{e}_2, y\mathbf{e}_1 + x^3\mathbf{e}_2\rangle$. N is $p = \langle z, y\rangle$-primary, so by Theorem 2.3.1,

$$\langle E_M(N)\rangle = N + \langle z,y\rangle M = \langle z\mathbf{e}_1, z\mathbf{e}_2, y\mathbf{e}_1, y\mathbf{e}_2, x^3\mathbf{e}_2\rangle.$$

Primary decompostion is $\langle E_M(N)\rangle = Q_1 \cap Q_2$ where

$$Q_1 = \langle \mathbf{e}_2, z\mathbf{e}_1, y\mathbf{e}_1\rangle \text{ is } \langle z,y\rangle - \text{primary},$$
$$Q_2 = \langle z\mathbf{e}_1, z\mathbf{e}_2, y\mathbf{e}_1, y\mathbf{e}_2, x^3\mathbf{e}_1, x^3\mathbf{e}_2\rangle \text{ is } \langle x,y,z\rangle - \text{primary}.$$

Hence,

$$\langle E_M(\langle E_M(N)\rangle)\rangle = \langle z\mathbf{e}_1, z\mathbf{e}_2, y\mathbf{e}_1, y\mathbf{e}_2, x\mathbf{e}_2\rangle \neq \langle E_M(N)\rangle.$$

In [28], Azizi and Nikseresht defined the kth envelope of N recursively by $E_0(N) = N, E_1(N) = E_M(N), E_2(N) = E_M(\langle E_M(N)\rangle)$ and $E_k(N) = E_M(\langle E_{k-1}(N)\rangle)$ for every submodule N of M. It is easy to show that

$$N = \langle E_0(N)\rangle \subseteq \langle E_1(N)\rangle \subseteq \langle E_2(N)\rangle \subseteq \cdots\cdots \subseteq \langle E_\infty(N)\rangle \subseteq srad_M(N) \subseteq rad_M(N)$$

where $\langle E_\infty(N)\rangle = \bigcup_{k=0}^{\infty} \langle E_k(N)\rangle$.

It is clear that $\langle E_\infty(N)\rangle$ is semiprime and thus $\langle E_\infty(N)\rangle = srad_M(N)$. Therefore we have the following equivalent conditions.

Theorem 4.3.1. *The following statements are equivalent.*

(i) A module M satisfies the semiradical formula;

(ii) $\langle E_i(N)\rangle = \langle E_j(N)\rangle$ for all i, j;

(iii) $\langle E_M(N)\rangle = \langle E_2(N)\rangle$ for all submodules N of M.

Proof. $(i) \Rightarrow (ii)$ and $(ii) \Rightarrow (iii)$ are clear.

$(iii) \Rightarrow (i)$ $\langle E_M(N)\rangle = \langle E_2(N)\rangle$ implies that $\langle E_M(N)\rangle$ is semiprime submodule and hence M s.t.s.r.f. □

By Theorem 4.3.1, we can conclude that a module M s.t.s.r.f. if and only if $\langle E_M(N)\rangle$ is either M or a semiprime submodule of M for every submodule N of M; R s.t.s.r.f. if and only if either $\langle E_M(0)\rangle = M$ or $\langle E_M(0)\rangle$ is semiprime submodule of M for every non-zero R-module M. In this section, we will investigate the equality $srad_M(N) = \langle E_M(N)\rangle$.

Lemma 4.3.1. *Let M, M' be R-modules with $\phi : M \to M'$ an R-module epimorphism and N be a submodule of M such that $Ker\phi \subseteq N$. Then $\phi(srad_M(N)) = srad_{M'}(\phi(N))$.*

Proof. Let $x \in \phi(srad_M(N))$. Then $x = \phi(y)$ for some $y \in srad_M(N) = \bigcap_i P_i$ where each P_i is semiprime submodule of M containing N. So, $y \in P_i$ for each i. Hence $\phi(y) \in \phi(P_i)$ where $\phi(P_i)$ is a semiprime submodule of M' containing $\phi(N)$ by Lemma 4.1.8. If we take any semiprime submodule Q_j of M' containing $\phi(N)$, then $\phi^{-1}(Q_j)$ is semiprime submodule of M containing N by Lemma 4.1.8. Hence $y \in \phi^{-1}(Q_j)$ for all j, and hence $x = \phi(y) \in Q_j$. Thus $x \in srad_{M'}(\phi(N))$.
Now, let $m' \in srad_{M'}(\phi(N)) = \bigcap_j Q_j$ where Q_j is semiprime submodule of M' containing $\phi(N)$ for each j. Then $m' \in Q_j$ for all j. Since ϕ is an epimorphism, there exists $m \in M$ such that $m' = \phi(m) \in Q_j$. Hence $m \in \phi^{-1}(Q_j)$. By Lemma 4.1.8, $\phi^{-1}(Q_j)$ is semiprime submodule of M containing N. Thus $m \in srad_M(N)$ and $m' = \phi(m) \in \phi(srad_M(N))$. □

Lemma 4.3.2. *Let N be a submodule of a module M. Then $srad_{M/N}(0) = srad_M(N)/N$.*

Proof. Consider the canonical epimorphism $\pi : M \to M/N$. Since $Ker\pi = N$, we can apply Lemma 4.3.1. Then $\pi(srad_M(N)) = srad_{M/N}(\pi(N)) = srad_{M/N}(0)$. Let $srad_M(N) = \bigcap_i P_i$. Then we have;

$$srad_{M/N}(0) = \phi(\bigcap_i P_i) = \bigcap_i (P_i/N) = (\bigcap_i P_i)/N = srad_M(N)/N.$$

□

Corollary 4.3.1. *Let N be a submodule of a module M and N' be a submodule of a module M' such that $M/N \cong M'/N'$. Then $srad_M(N) = \langle E_M(N) \rangle$ if and only if $srad_{M'}(N') = \langle E_{M'}(N') \rangle$.*

Proof. It is clear by the definition of envelope that $\langle E_{M/N}(0)\rangle = \langle E_M(N)\rangle/N$, also by Lemma 4.3.2, we have

$$\begin{aligned} srad_M(N) = \langle E_M(N)\rangle &\Leftrightarrow srad_M(N)/N = \langle E_M(N)\rangle/N \\ &\Leftrightarrow srad_{M/N}(0) = \langle E_{M/N}(0)\rangle \\ &\Leftrightarrow srad_{M'/N'}(0) = \langle E_{M'/N'}(0)\rangle \\ &\Leftrightarrow srad_{M'}(N')/N' = \langle E_{M'}(N')\rangle/N' \\ &\Leftrightarrow srad_{M'}(N') = \langle E_{M'}(N'). \end{aligned}$$

□

Corollary 4.3.2. *Let N, L be submodules of M such that $M = N + L$ and $srad_L(N \cap L) = \langle E_L(N \cap L)\rangle$. Then $srad_M(N) = \langle E_M(N)\rangle$.*

Proof. Note that $M/N = (N+L)/N \cong L/N \cap L$. Apply Corollary 4.3.1. □

Lemma 4.3.3. *Let M, M', ϕ and N be as in Lemma 4.3.1. Then if $srad_M(N) = \langle E_M(N)\rangle$, then $srad_{M'}(\phi(N)) = \langle E_{M'}(\phi(N))\rangle$.*

Proof. Assume that $srad_M(N) = \langle E_M(N)\rangle$. By Lemma 4.3.1,

$$\phi(srad_M(N)) = srad_{M'}(\phi(N)).$$

Now, it is enough to show that

$$\phi(\langle E_M(N)\rangle) = \langle E_{M'}(\phi(N))\rangle.$$

Let $x \in \phi(\langle E_M(N)\rangle)$. Then $x = \phi(y)$ where $y \in \langle E_M(N)\rangle$. So there exist elements $r_i \in R, m_i \in E_M(N)$ such that

$$y = r_1 m_1 + r_2 m_2 + \cdots + r_l m_l$$

and for each i, $m_i = s_i n_i$ with $s_i \in R, n_i \in M$ and $s_i^{t_i} n_i \in N$. Then

$$y = r_1 s_1 n_1 + r_2 s_2 n_2 + \cdots + r_l s_l n_l$$

and so

$$\begin{aligned} x = \phi(y) &= \phi(r_1 s_1 n_1 + r_2 s_2 n_2 + \cdots + r_l s_l n_l) \\ &= r_1 \phi(s_1 n_1) + r_2 \phi(s_2 n_2) + \cdots + r_l \phi(s_l n_l) \end{aligned}$$

where $\phi(s_i^{t_i}n_i) = s_i^{t_i}\phi(n_i) \in \phi(N)$ implies that $s_i\phi(n_i) \in E_{M'}(\phi(N))$. Thus

$$x = r_1s_1\phi(n_1) + r_2s_2\phi(n_2) + \cdots + r_ls_l\phi(n_l) \in \langle E_{M'}(\phi(N))\rangle.$$

Now, let $x' \in \langle E_{M'}(\phi(N))\rangle$. Then

$$x' = a_1m_1' + a_2m_2' + \cdots + a_km_k'$$

with $a_i \in R$, $m_i' \in E_{M'}(\phi(N))$. For each i, $m_i' = b_iy_i'$ where $b_i \in R$, $y_i' \in M'$ and $b_i^{k_i}y_i' \in \phi(N)$ for some positive integer k_i. Hence

$$\begin{aligned} x' &= a_1m_1' + a_2m_2' + \cdots + a_km_k' \\ &= a_1b_1y_1' + a_2b_2y_2' + \cdots + a_kb_ky_k' \\ &= a_1b_1\phi(y_1) + a_2b_2\phi(y_2) + \cdots + a_kb_k\phi(y_k) \\ &= \phi(a_1b_1y_1 + a_2b_2y_2 + \cdots + a_kb_ky_k) \end{aligned}$$

where $y_i \in M$ for each i. Also $b_i^{k_i}y_i' = b_i^{k_i}\phi(y_i) = \phi(b_i^{k_i}y_i) \in \phi(N)$ implies that $b_i^{k_i}y_i \in N$. Hence $b_iy_i \in E_M(N)$ and $a_1b_1y_1 + \cdots a_kb_ky_k \in \langle E_M(N)\rangle$. Thus

$$x' = \phi(a_1b_1y_1 + a_2b_2y_2 + \cdots + a_kb_ky_k) \in \phi(\langle E_M(N)\rangle).$$

□

Lemma 4.3.4. *Let M be an R-module which s.t.s.r.f.. Then every homomorphic image of M s.t.s.r.f. as an R-module.*

Proof. Suppose M s.t.s.r.f.. Let N be any submodule of M. Consider the canonical epimorphism $\pi : M \to M/N$. Let K be any submodule of M containing N. Then $srad_M(K) = \langle E_M(K)\rangle$. By Lemma 4.3.3,

$$srad_{M/N}(\pi(K)) = \langle E_{M/N}(\pi(K))\rangle.$$

Hence

$$srad_{M/N}(K/N) = \langle E_{M/N}(K/N)\rangle$$

and thus M/N s.t.s.r.f.. □

Lemma 4.3.5. *Let M be the direct sum of the R-modules M_i, $i \in I$. Let $N = \oplus N_i$ be a submodule of M such that N_i is a submodule of M_i for all $i \in I$. Then $srad_M(N) = \langle E_M(N)\rangle$ if and only if $srad_{M_i}(N_i) = \langle E_{M_i}(N_i)\rangle$ for each i.*

Proof. Assume $srad_{M_i}(N_i) = \langle E_{M_i}(N_i)\rangle$ for each i. By Lemma 4.2.2,

$$srad_M(N) = \bigoplus_{i\in I} srad_{M_i}(N_i) = \bigoplus_{i\in I}\langle E_{M_i}(N_i)\rangle.$$

The result follows by [29], Lemma 2.3.

Conversely, since N_i is a submodule of M_i and $N = \oplus N_i$, by Lemma 4.2.2

$$srad_{M_i}(N_i) \subseteq srad_M(N) = \langle E_M(N)\rangle.$$

If $m \in srad_{M_i}(N_i)$, then by the definition of envelope it is easy to show that $m \in \langle E_{M_i}(N_i)\rangle$. □

By Lemma 4.2.2 and the above lemma, we have the following result.

Corollary 4.3.3. *Let R be any ring and M be any projective R-module. Then* $srad_M(0) = \langle E_M(0)\rangle$.

Proof. Since M is projective, there exists a free R-module F such that M is the direct summand of F. Then there exist an index set Λ and cyclic submodules F_λ of F such that $F = \bigoplus_{\Lambda} F_\lambda$ where $\lambda \in \Lambda$ by [30]. By Lemma 4.2.2, $srad_F(0) = \bigoplus_{\Lambda} srad_{F_\lambda}(0)$ and since every cyclic module s.t.r.f., $srad_{F_\lambda}(0) = \langle E_{F_\lambda}(0)\rangle$ for all $\lambda \in \Lambda$. Hence $srad_F(0) = \langle E_F(0)\rangle$ by Lemma 4.3.5 and thus $srad_M(0) = \langle E_M(0)\rangle$. □

Since every prime submodule is semiprime, this result can also be obtained from [4], Corollary 8.

Lemma 4.3.6. *Let* $\varphi : S \to R$ *be a module epimorphism and M be an R-module. Then*

(i) a submodule N of M is semiprime as an R-module if and only if it is semiprime as an S-module.

(ii) for any submodule N of M, $E_M({}_SN) = E_M({}_RN)$.

Proof. Since $\varphi : S \to R$ is an epimorphism, an R-module M has an S-module structure as

$$\begin{aligned} f : S\times M &\to M \\ (s,m) &= s\cdot m = \varphi(s)m \end{aligned}$$

(i) Let N be a semiprime submodule of M as an R-module and let $s \in S, m \in M$ with $s^k \cdot m \in N$. Then $s^k \cdot m = \varphi(s^k)m = \varphi(s)^k m \in N$. Since N is semiprime as an R-module, $\varphi(s)m = s \cdot m \in N$.

Conversely, let N be a semiprime submodule of M as an S-module and let $r \in R, m \in M$ with $r^k m \in N$. Since φ is an epimorphism, $r^k m = \varphi(s)^k m = s^k \cdot m \in N$, and since N is semiprime as an S-module, $s \cdot m = \varphi(s)m = rm \in N$. Thus N is semiprime submodule of M as an R-module.

(ii) Let $x \in E_M({}_S N)$. Then $x = s \cdot m$ with $s^k \cdot m \in N$ and $x = s \cdot m = \varphi(s)m$ with $s^k \cdot m = \varphi(s)^k m \in N$. This means that $\varphi(s)m = x \in E_M({}_R N)$. Conversely, let $y \in E_M({}_R N)$. Then $y = rm$ where $r^t m \in N$. Since φ is an epimorphism, $y = rm = \varphi(s)m = s \cdot m$ with $r^t m = \varphi(s)^t m = s^t \cdot m \in N$. Then $y = s \cdot m \in E_M({}_S N)$.

□

In the following theorem, we will give some equivalent conditions for a ring to satisfy the semiradical formula.

Theorem 4.3.2. *Let R be a ring. Then R s.t.s.r.f. provided that any one of the following is satisfied.*

(i) for every free R-module F, F s.t.s.r.f.

(ii) for every R-module M, $srad_M(0) = \langle E_M(0) \rangle$.

(iii) R is a ring homomorphic image of S, where S s.t.s.r.f..

Proof. (i) Since every module M is the homomorphic image of a free R-module, by Lemma 4.3.4 it is clear.

(ii) Clear by the remark after Theorem 4.3.1.

(iii) Let $\Phi : S \to R$ be an epimorphism. Suppose S s.t.s.r.f.. Since Φ is an epimorphism, every S-submodule of M is also an R-submodule and vice versa. Let N be any submodule of M. Then by Lemma 4.3.6,

$$srad_M({}_R N) = srad_M({}_S N) = \langle E_M({}_S N) \rangle = \langle E_M({}_R N) \rangle.$$

Hence R s.t.s.r.f.. □

4.4 Semiradical Equality

When we consider the chain

$$N = \langle E_0(N)\rangle \subseteq \langle E_1(N)\rangle \subseteq \langle E_2(N)\rangle \subseteq \cdots\cdots \subseteq \langle E_\infty(N)\rangle = srad_M(N) \subseteq rad_M(N),$$

it seems meaningfull to focus on the submodules $srad_M(N)$ and $rad_M(N)$ and investigate the conditions where the equality $srad_M(N) = rad_M(N)$ occurs.

Lemma 4.4.1. *Let M be an R-module. Then every semiprime submodule is an intersection of prime submodules if and only if* $srad_M(N) = rad_M(N)$ *for any submodule N of M.*

Proof. (⇒) Obvious since $srad_M(N)$ is a semiprime submodule.

(⇐) Let K be a semiprime submodule of M. Then $K = srad_M(K) = rad_M(K)$. Hence K is an intersection of prime submodules. □

Lemma 4.4.2. *Let N be a submodule of an R-module M such that M/N is projective. Then* $srad_M(N) = rad_M(N)$.

Proof. Since M/N is projective, $rad_{M/N}(0) = \langle E_{M/N}(0)\rangle$ by [4] Lemma 8. Then we have,$rad_M(N) = \langle E_M(N)\rangle$ which implies that $srad_M(N) = rad_M(N)$. □

Corollary 4.4.1. *Let N be a submodule of an R-module M such that M/N is projective. Then* $srad_M(N) = radRM + N$.

Proof. Clear by [29], Theorem 2.7 and the above lemma. □

We say that a module M satisfy the semiradical equality if for every submodule N of M, $srad_M(N) = rad_M(N)$.

It is said that a ring R satisfy the semiradical equality if every R-module satisfy the semiradical equality. Since arithmetical rings satisfy the radical formula, an arithmetical ring satisfy the semiradical equality.

Proposition 4.4.1. *The followings are equivalent.*

(i) The ring R satisfy the semiradical equality.

(ii) for any ideal I of R, the ring R/I satisfy the semiradical equality.

(iii) for any non-maximal semiprime ideal P of R, the ring R/P satisfy the semiradical equality.

Proof. $(i \Rightarrow ii)$ Let M be an R/I-module. By Lemma 4.4.1, it is enough to show that every semiprime R/I-module is an intersection of prime submodules. Let K be a semiprime submodule of an R/I-module M. Then K is a semiprime submodule of M as an R-module. So, $K = srad_{_RM}(K) = rad_{_RM}(K)$.

It is easy to see that every submodule of M is a prime R-submodule if and only if it is a prime R/I-submodule. Hence $rad_{_RM}(K) = rad_{_{R/I}M}(K)$ and thus $K = rad_{_{R/I}M}(K)$.

$(iii \Rightarrow i)$ Let N be a semiprime submodule of an R-module M with $N : M = P$ where P is a non-maximal semiprime ideal. Consider M/N as an R/P-module. Then by our assumption, $srad_{M/N}(0) = rad_{M/N}(0)$. Hence $srad_M(N) = rad_M(N)$.

□

Corollary 4.4.2. *If for any non-maximal prime ideal P of R R/P is a Prüfer domain, then R satisfy the semiradical equality.*

Lemma 4.4.3. *A ring R satisfy the semiradical equality if and only if every free R-module satisfy the semiradical equality.*

Proof. Let M be an R-module. Then there exists a free R-module F such that $M \cong F/K$. By our assumption, for any submodule N of M

$$\begin{aligned} srad_{F/K}(N/K) &= srad_F(N)/K \\ &= rad_F(N)/K \\ &= rad_{F/K}(N/K). \end{aligned}$$

Hence M satisfy the semiradical equality. □

4.5 Semiprime Submodules of Cartesian Product of Modules

Let $R = R_1 \times R_2$ where each R_i is a commutative ring with nonzero identity. Let M_i be an R_i-module for $i = 1,2$ and $M = M_1 \times M_2$ be the R-module with action $(r_1,r_2)(m_1,m_2) = (r_1m_1,r_2m_2)$ where $r_i \in R_i, m_i \in M_i$. These notations are fixed for this section.

Note that since our action is $(r_1,r_2)(m_1,m_2) = (r_1m_1,r_2m_2)$ where $r_i \in R_i, m_i \in M_i$, every submodule of $M_1 \times M_2$ is of the form $N_1 \times N_2$ with N_1 is a submodule of M_1 and N_2 is a submodule of M_2.

Proposition 4.5.1. *Let R and M be as above. Then*

(i) If N_1 is semiprime submodule of M_1, then $N_1 \times M_2$ is semiprime submodule of $M_1 \times M_2$.

(ii) If N_2 is semiprime submodule of M_2, then $M_1 \times N_2$ is semiprime submodule of $M_1 \times M_2$.

Proof. (i) Let $r = (r_1,r_2) \in R$, $m = (m_1,m_2) \in M$ and $r^k m \in N_1 \times M_2$ for some $k \in \mathbb{Z}^+$. Since N_1 is semiprime submodule of M_1, $r_1m_1 \in N_1$. Then $(r_1m_1,r_2m_2) = rm \in N_1 \times M_2$ which implies that $N_1 \times M_2$ is semiprime submodule of M.

(ii) Similiar to case (i). □

Lemma 4.5.1. *Let R and M be as above. Then $Q_1 \times Q_2$ is a semiprime submodule of M if and only if Q_i is semiprime submodule of M_i for all $i = 1,2$.*

Proof. Let $(r_1,r_2)^k(m_1,m_2) \in Q_1 \times Q_2$ where $m_i \in M_i$, $r_i \in R_i$ and $k \in \mathbb{Z}^+$. Since Q_1 and Q_2 are semiprime, $r_im_i \in Q_i$ for $i = 1,2$ which implies that $Q_1 \times Q_2$ is semiprime. Now assume that $Q_1 \times Q_2$ is semiprime submodule of $M_1 \times M_2$. Let $r_1 \in R_1$, $m_1 \in M_1$ with $r_1^k m_1 \in Q_1$. Then $(r_1,1)^k(m_1,0) \in Q_1 \times Q_2$. Since $Q_1 \times Q_2$ is semiprime, $(r_1,1)(m_1,0) = (r_1m_1,0) \in Q_1 \times Q_2$ implies that Q_1 is semiprime submodule of M_1. Similarly it can be shown that Q_2 is semiprime submodule of M_2. □

Lemma 4.5.2. *Let $N = N_1 \times N_2$ be a submodule of M where N_i is a submodule of M_i for $i = 1,2$. Then $N : M = (N_1 : M_1) \times (N_2 : M_2)$*

Proof. Let $x = (x_1,x_2) \in (N : M)$. Then $xM \subseteq N$ which means that

$$(x_1,x_2)(m_1,m_2) = (x_1m_1,x_2m_2) \in N_1 \times N_2$$

for all $m_1 \in M_1$ and $m_2 \in M_2$. So, $x_1m_1 \in N_1$ and $x_2m_2 \in N_2$. Hence

$$x_1 \in (N_1 : M_1), \quad x_2 \in (N_2 : M_2)$$

and thus $x = (x_1, x_2) \in (N_1 : M_1) \times (N_2 : M_2)$.

Conversely, let $y = (y_1, y_2) \in (N_1 : M_1) \times (N_2 : M_2)$. Then $y_1 M_1 \subseteq N_1$ and $y_2 M_2 \subseteq N_2$. Hence for all $m_1 \in M_1, m_2 \in M_2$,

$$(y_1, y_2)(m_1, m_2) = (y_1 m_1, y_2 m_2) \in N_1 \times N_2.$$

This implies that $y \in (N_1 \times N_2) : (M_1 \times M_2) = (N : M)$. □

Let N be a semiprime submodule of an R-module M. If $p = \sqrt{N : M}$ is a prime ideal, then N is called p -semiprime submodule.

Lemma 4.5.3. *Let $N = N_1 \times N_2$ be a submodule of M. Then*

(i) N is $p \times R_2$ semiprime submodule of M iff N_1 is p-semiprime submodule of M_1 and $N_2 = M_2$.

(ii) N is $R_1 \times p$ semiprime submodule of M iff N_2 is p-semiprime submodule of M_2 and $N_1 = M_1$.

Proof. (i) Suppose $N = N_1 \times N_2$ is semiprime submodule of $M_1 \times M_2$. By Lemma 4.5.1, N_1 is semiprime submodule of M_1.

Since $N : M = p \times R_2$, N_1 is p-semiprime and $N_2 : M_2 = R_2$ implies that $N_2 = M_2$.

Other side is clear by Proposition 4.5.1 and the Lemma 4.5.2.

(ii) Similiar to case (i). □

Proposition 4.5.2. *Let $N = N_1 \times N_2$ be a submodule of M. Then*

$$srad_M(N) = srad_{M_1}(N_1) \times srad_{M_2}(N_2)$$

Proof. Let $Q_1 \times Q_2$ be a semiprime submodule of M containing $N_1 \times N_2$. By Lemma 4.5.1, Q_i is semiprime submodule of M_i containing N_i for $i = 1, 2$. Then

$$srad_{M_1}(N_1) \times srad_{M_2}(N_2) \subseteq srad_M(N_1 \times N_2)$$

since $srad_{M_1}(N_1) \times srad_{M_2}(N_2) \subseteq Q_1 \times Q_2$.

Since $srad_{M_i}(N_i)$ is the minimal semiprime submodule of M_i containing N_i, Lemma 4.5.1 implies that $srad_{M_1}(N_1) \times srad_{M_2}(N_2)$ is a semiprime submodule of $M_1 \times M_2$ which contains $N_1 \times N_2$. Hence

$$srad_M(N) \subseteq srad_{M_1}(N_1) \times srad_{M_2}(N_2)$$

□

Corollary 4.5.1. *Let $N = N_1 \times N_2$ be a submodule of M. Then*

(i) $srad_M(N_1 \times M_2) = srad_{M_1}(N_1) \times M_2$

(ii) $srad_M(M_1 \times N_2) = M_1 \times srad_{M_2}(N_2)$

Proof. Clear by the Proposition 4.5.2.

□

Proposition 4.5.3. *([26], Proposition 2.12) Let $N = N_1 \times N_2$ be a submodule of M. Then $\langle E_M(N)\rangle = \langle E_{M_1}(N_1)\rangle \times \langle E_{M_2}(N_2)\rangle$.*

Theorem 4.5.1. *M s.t.s.r.f. if and only if M_i s.t.s.r.f. for all $i = 1,2$.*

Proof. Assume M s.t.s.r.f.. Take a submodule N_1 of M_1. Then $N_1 \times M_2$ s.t.s.r.f., so that $srad_{M_1}(N_1) \times M_2 = \langle E_{M_1}(N_1)\rangle \times \langle E_{M_2}(M_2)\rangle$. Now, let $x \in srad_{M_1}(N_1)$. Then $(x,m) \in srad_{M_1}(N_1) \times M_2$ and hence $x \in \langle E_{M_1}(N_1)\rangle$. Similiarly it can be shown that $srad_{M_2}(N_2) = \langle E_{M_2}(N_2)\rangle$.

Conversely assume that M_1 and M_2 s.t.s.r.f.. Take any submodule $N_1 \times N_2$ of $M_1 \times M_2$. Then

$$\begin{aligned} srad_M(N_1 \times N_2) &= srad_{M_1}(N_1) \times srad_{M_2}(N_2) \\ &= \langle E_{M_1}(N_1)\rangle \times \langle E_{M_2}(N_2)\rangle \\ &= \langle E_M(N_1 \times N_2)\rangle \end{aligned}$$

Thus, $M = M_1 \times M_2$ s.t.s.r.f.. □

5. CLASSICAL PRIME SUBMODULES

5.1 Classical Prime Submodules

Classical prime submodules was introduced in [11]. A proper submodule N of an R-module M is called a classical prime submodule if for each $m \in M$ and $a, b \in R$; $abm \in N$ implies that $am \in N$ or $bm \in N$.

In the same manner classical primary submodule is defined [16]. There are two different definitions for this concept. When the main module is Noetherian, these definitions are coincide [31]. A proper submodule N of an R-module M is a classical primary submodule if $abm \in N$ where $a, b \in R$ and $m \in M$, implies that either $bm \in N$ or $a^k m \in N$ for some $k \geq 1$.

Sometimes weakly prime and weakly primary are used for classical prime and classical primary submodules.

It is clear from the definition that every prime submodule is classical prime and every primary submodule is classical primary but the converse need not be true [[16], Example 1.2].

Lemma 5.1.1. *If N is a classical prime submodule, then $\langle E_M(N) \rangle = N$.*

Proof. Let $x \in \langle E_M(N) \rangle$. Then there exist elements $r_i \in R, m_i \in M$ such that

$$x = r_1 m_1 + r_2 m_2 + \cdots + r_t m_t \quad \text{where } 1 \leq i \leq n \text{ and } \quad r_i^{k_i} m_i \in N.$$

Since N is classical prime, $r_i^{k_i} m_i \in N$ implies that $r_i m_i \in N$ or $r_i^{k_i - 1} m_i \in N$. If $r_i m_i \in N$, then $x = r_1 m_1 + \cdots + r_t m_t \in N$. If $r_i^{k_i - 1} m_i \in N$, then $r_i m_i \in N$ or $r_i^{k_i - 2} m_i \in N$. By the same process, $r_i m_i \in N$ for all cases. Hence $x \in N$, which means that $\langle E_M(N) \rangle \subseteq N$. Other side of the inclusion is obvious. □

The above lemma implies that every classical prime submodule is semiprime by Lemma 4.1.1.

Behboodi and Baziar gave the following proposition which gives a relationship between associated primes of a classical primary submodule.

Proposition 5.1.1. *([16], Proposition 3.1) Let M be a finitely generated module over a Noetherian ring. If N is classical primary submodule of M and $N = Q_1 \cap Q_2 \cap \ldots \cap Q_s$ is its minimal primary decomposition with $\sqrt{Q_i : M} = pi$ for each i, then $p_1 \subset p_2 \subset \cdots \subset p_s$.*

In the next statement we show that the converse of the above proposition is also true for any submodule N of a Noetherian module M satisfying the condition $\langle E_M(N)\rangle = N$.

Theorem 5.1.1. *Let R be a Noetherian ring and M be a finitely generated R-module. Suppose that a submodule N of M has a primary decomposition $N = Q_1 \cap Q_2 \cap \cdots \cap Q_s$ where each Q_i is p_i-primary with $p_1 \subset p_2 \subset \cdots \subset p_s$. If $\langle E_M(N)\rangle = N$, then N is classical prime submodule.*

Proof. Since $p_1 \subset p_2 \subset \cdots \subset p_s$, by Theorem 2.3.1 we have

$$N = \langle E_M(N)\rangle = N + p_1M + \sum_{i=2}^{s} p_i\Big(\bigcap_{j=1}^{i-1} Q_j\Big).$$

Let $abm \in N$ with $a, b \in R$ and $m \in M$. Let i be the first index for which $m \notin Q_i$. Since Q_i is p_i-primary, $ab \in p_i$ and so either $a \in p_i$ or $b \in p_i$. If $i = 1$, then since $p_1M \subset \langle E_M(N)\rangle = N$, either $am \in N$ or $bm \in N$. Let $i > 1$. Since $p_i(\bigcap_{j=1}^{i-1} Q_j) \subset \langle E_M(N)\rangle = N$, either $am \in N$ or $bm \in N$. Hence N is a classical prime submodule. □

The classical quasi-primary submodules are introduced in [31].

Definition 5.1.1. A proper submodule N of a Noetherian module M is called classical quasi-primary if $abm \in N$ where $a, b \in R$ and $m \in M$ implies that either $a^k m \in N$ or $b^k m \in N$ for some $k \in \mathbb{N}$.

In [31], it is shown that if N is classical quasi-primary, then the converse of Proposition 5.1.1 is also satisfied.

Proposition 5.1.2. *([31], Proposition 3.4) Let M be a Noetherian R-module and N be a proper submodule of M. Suppose that $N = Q_1 \cap Q_2 \cap \cdots \cap Q_s$ is the minimal primary decomposition where each Q_i is p_i-primary submodule. Then N is classical quasi-primary if and only if $\{p_1, p_2, \ldots, p_s\}$ is a chain of prime ideals.*

It is clear from the definition that every classical primary submodule is classical quasi-primary, but the following example shows that the converse is not true in general.

Example 5.1.1. Let $R = \mathbb{Q}[x,y,z]$ and let $M = R \oplus R \oplus R$. Consider the submodule $N = \langle y\mathbf{e}_1, xy\mathbf{e}_3, x^2\mathbf{e}_2, +xy^2\mathbf{e}_2 + (x^3+y^2z)\mathbf{e}_3, xy^3\mathbf{e}_2 + y^3z\mathbf{e}_3, x^4\mathbf{e}_1\rangle$ with $N : M = \langle x^2y, x^4\rangle$.

Primary decompostion is $N = Q_1 \cap Q_2$ where

$Q_1 = \langle \mathbf{e}_1, x\mathbf{e}_2 + z\mathbf{e}_3\rangle$ is $\langle x\rangle$ − primary,

$Q_2 = \langle y\mathbf{e}_1, xy\mathbf{e}_3, x^2\mathbf{e}_2, y^3\mathbf{e}_2, y^3\mathbf{e}_3, x^3\mathbf{e}_3 + xy^2\mathbf{e}_2 + y^2z\mathbf{e}_3, x^4\mathbf{e}_1\rangle$ is $\langle x,y\rangle$ − primary.

Since $\langle x\rangle \subseteq \langle x,y\rangle$, N is classical quasi-primay by Proposition 5.1.2 with

$$\langle E_M(N)\rangle = \langle y\mathbf{e}_1, x\mathbf{e}_1, x\mathbf{e}_2, x\mathbf{e}_3, yz\mathbf{e}_3\rangle.$$

On the other hand, N is not classical primary. If we take $a = x^3, b = y$ and $m = (0,0,1)$, then $abm = x^3y(0,0,1) = (0,0,x^3y) \in N$ but

$$x^3(0,0,1) \notin N \quad and \quad y^k(0,0,1) \notin N,$$
$$y(0,0,1) \notin N \quad and \quad (x^3)^k(0,0,1) \notin N \quad for \quad some \quad k \geq 1.$$

By using the above proposition, we can show the following theorem.

Theorem 5.1.2. *Let M be a Noetherian R-module. If N is a classical quasi-primary submodule with $\langle E_M(N)\rangle = N$, then N is classical prime submodule.*

Proof. Suppose that $N = Q_1 \cap Q_2 \cap \cdots \cap Q_s$ is classical quasi primary submodule where each Q_i is p_i-primary submodule. By Proposition 5.1.2, $p_1 \subset p_2 \subset \cdots \subset p_s$. Hence by Theorem 5.1.1, N is classical prime.

□

In [16] Theorem 1.9, Baziar and Behboodi showed that if R is domain with $dimR \leq 1$, then for every classical primary submodule Q of M $\langle E_M(Q)\rangle$ is a classical prime submodule. As a result of this theorem, they proposed the following conjecture. Notice that they use the notation $\sqrt[nil]{Q}$ for $\langle E_M(Q)\rangle$.

Conjecture 5.1.1. Let R be a ring and M be an R-module. Then for every classical primary submodule Q of M, $\langle E_M(Q)\rangle$ is a classical prime submodule.

The following example shows that this conjecture is false.

Example 5.1.2. Let $R = \mathbb{Q}[x,y]$ and let $M = R \oplus R$. Consider the submodule $N = \langle x\mathbf{e}_1 + y^3\mathbf{e}_2, x^2\mathbf{e}_1, x\mathbf{e}_2\rangle$. One can easily see that $N : M = \langle x^2\rangle$ and N is $\langle x\rangle$-primary submodule. Hence

$$\langle E_M(N)\rangle = N + \langle x\rangle M = \langle x\mathbf{e}_1, x\mathbf{e}_2, y^3\mathbf{e}_2\rangle$$

Then $\langle E_M(N)\rangle$ is not classical prime submodule since $y^2(0,y) = (0,y^3) \in \langle E_M(N)\rangle$ but $y(0,y) = (0,y^2) \notin \langle E_M(N)\rangle$.

If we weaken the conditions of conjecture as follows, then we obtain the following result.

Corollary 5.1.1. *Let R be a Noetherian ring and M be a finitely generated R-module, N be classical primary submodule of M. Then N is semiprime if and only if N is classical prime.*

Proof. Suppose N is semiprime and $N = Q_1 \cap Q_2 \cap \cdots \cap Q_k$ is primary decomposition of N with $\sqrt{Q_i : M} = p_i$ $(1 \le i \le k)$. By Proposition 5.1.1, $p_1 \subset p_2 \subset \cdots \subset p_k$. Then Theorem 5.1.1 implies that N is classical prime submodule. □

We also have the following result.

Corollary 5.1.2. *Let $N = Q_1 \cap Q_2$ be a submodule of M where Q_i is p_i-primary. If $\langle E_M(N)\rangle = N$, then either Q_1 and Q_2 are both prime or N is classical prime.*

Proof. We have two cases: $p_1 \not\subseteq p_2$ or $p_1 \subseteq p_2$. If $p_1 \not\subseteq p_2$, then both p_1 and p_2 are isolated primes. By Corollary 2.3.2, Q_1 and Q_2 are prime submodules. If $p_1 \subseteq p_2$, then Theorem 5.1.2 implies that N is classical prime.

□

5.2 Semiprime Submodules which are Intersection of Classical Primes

Any intersection of classical prime submodule of an R-module is semiprime. But the converse is not true in general. So, it is interesting to characterize modules over which every semiprime submodule is an intersection of classical prime submodules. In [15], commutative rings over which semiprime submodules are intersection of prime submodules are characterized. In 2006 Behboodi showed the following theorem [12].

Theorem 5.2.1. *([12], Theorem 1.10) Let R be a commutative domain with dimR* ≤ 1*, and let M be an R-module. Then every semiprime submodule of M is an intersection of classical prime submodules.*

In this section we will give some additional definitions and conditions to characterize the semiprime submodules which can be written as an intersection of classical primes.

Definition 5.2.1. A submodule N is called a quasi-p-primary in M, if N has a unique isolated prime p (and possibly embedded primes).

Now, let us write a new definiton.

Definition 5.2.2. A quasi-p-primary submodule N is called simple quasi-p-primary if for any distinct associated primes p_i, p_j and p_k of N, $p_i \subset p_k$ and $p_j \subset p_k$ implies either $p_i \subset p_j$ or $p_j \subset p_i$.

Definition 5.2.3. A Hasse diagram is a graphical rendering of a partially ordered set displayed via the cover relation of the partially ordered set with an implied upward orientation. A point is drawn for each element of the poset, and line segments are drawn between these points according to the following two rules:

(i) If $x < y$ in the poset, then the point corresponding to x appears lower in the drawing than the point corresponding to y.

(ii) The line segment between the points corresponding to any two elements x and y of the poset is included in the drawing if and only if x covers y or y covers x.

In the language of the graph theory, we can say N is a simple quasi-primary submodule, if Hasse diagram of associated primes of N with respect to set inclusion form a rooted tree.

Lemma 5.2.1. *Let M be a Noetherian R-module. If N is a simple quasi-p_1-primary semiprime submodule, then N can be expressed as an intersection of finitely many classical prime submodules containing N.*

Proof. Let $Ass(M/N) = \{p_1, \ldots, p_s\}$ and $S = \{1, \ldots, s\}$. If N contains only one maximal associated prime with respect to inclusion, then its associated primes form a chain $p_1 \subset \cdots \subset p_s$. Hence N is classical prime by Theorem 5.1.2.

Suppose that N has more than one maximal associated prime. For each maximal p_j, we have a unique chain of associated primes $p_1 = p_{j_1} \subset p_{j_2} \subset \cdots \subset p_{j_t} = p_j$. Let $N_j = Q_{j_1} \cap Q_{j_2} \cdots \cap Q_{j_t}$ where $Q_{j_1} = Q_1$ and $Q_{j_t} = Q_j$. From Theorem 2.3.1,

$$\langle E_M(N) \rangle = N + p_1 M + \sum_{T \subset S} (\bigcap_{i \in T} p_i)(\bigcap_{i \in S \setminus T} Q_i)$$

and

$$\langle E_M(N_j) \rangle = N_j + p_1 M + \sum_{r=2}^{t} p_{j_r}(\bigcap_{k=1}^{r-1} Q_{j_k}).$$

Our aim to show that $\langle E_M(N_j) \rangle = N_j$.

Clearly $p_1 M \subset \langle E_M(N) \rangle = N \subset N_j$. Let $B = Ass(M/N) \setminus Ass(M/N_j)$. Take $x \in p_{j_r}$ and $m \in \bigcap_{k=1}^{r-1} Q_{j_k}$. Since p_j is a maximal prime and associated primes pairwise distinct, there exists $y \in (\bigcap_{p \in B} p) \setminus p_j$. Hence

$$yxm \in (p_{j_r} \cap (\bigcap_{p \in B} p))(\bigcap_{k=1}^{r-1} Q_{j_k}) \subset \langle E_M(N) \rangle = N \subset N_j \subset Q_{j_k}.$$

Since each Q_{j_k} is p_{j_k}-primary and $y \notin p_{j_k}$, $xm \in Q_{j_k}$. Hence $xm \in N_j$. This implies $\langle E_M(N_j) \rangle = N_j$ and N_j is classical prime by Theorem 5.1.2. Since $N = \cap N_j$, N is intersection of finitely many classical prime submodules. □

The following proposition is crucial for computing the primary decomposition.

Proposition 5.2.1. *([20],Proposition 2) Assume that $L = \{p_1, \ldots, p_k\}$ are the isolated primes of N. For $i, j = 1, \ldots, k$ take $f_i \in R$ such that $f_i \in p_j$ if $i \neq j$, but $f_i \notin p_i$, $N_i = N : \langle f_i \rangle^\infty$ and take integers e_i such that $f_i^{e_i} N_i \subset N$. Then:*

(i) N_i is a quasi-p_i-primary submodule in M.

(ii) The sets $A_i = Ass(M/N_i) = \{p \in Ass(M/N) : f_i \notin p\}$ are pairwise disjoint.

(iii) For $J := \langle f_1^{e_1}, f_2^{e_2}, \ldots, f_k^{e_k} \rangle$ *we have*

$$N = (\bigcap N_i) \cap (N + JM)$$

This is a decomposition of N *into quasi-primary components* N_i *and a component* $N' := N + JM \subset M$ *of lower (relative) dimension.*

Theorem 5.2.2. *Assume that* $L = \{p_1, \ldots, p_k\}$ *are the isolated primes of a semiprime submodule* N *and define* N_i*'s as in the previous proposition. If* $N = N_1 \cap N_2 \cap \cdots N_k$ *and each* N_i *is simple quasi-*p_i*-primary, then* $\langle E_M(N_i) \rangle = N_i$ *for* $i = 1, \ldots, k$*. Hence* N *can be written as a finite intersection of classical prime submodules.*

Proof. For a fixed i, let $Ass(M/N_i) = \{p_{i_1} = p_i, p_{i_2}, \ldots, p_{i_{s_i}}\}$ and $p_i \subseteq p_{i_k}$ for every k and let $N_i = Q_{i_1} \cap \cdots \cap Q_{i_{s_i}}$ where each Q_{i_k} is p_{i_k}-primary. By the Theorem 2.3.1,

$$\langle E_M(N) \rangle = N + (\bigcap_{i=1}^{k} p_i)M + \sum_{\emptyset \neq T \subsetneq S} (\bigcap_{j \in T} p_{i_j})(\bigcap_{j \in S \setminus T} Q_{i_j})$$

and

$$\langle E_M(N_i) \rangle = N_i + p_i M + \sum_{\emptyset \neq T \subsetneq S_i} (\bigcap_{r \in T} p_{i_r})(\bigcap_{r \in S_i \setminus T} Q_{i_r})$$

where $S_i = \{i_1, i_2, \ldots, i_{s_i}\}$ and $S = \bigcup_{i=1}^{k} S_i$.

Let $x \in p_i$ and $m \in M$. Take $y \in \left(\bigcap_{j \neq i} p_j\right) \setminus \left(\bigcup_{t=2}^{s_i} p_{i_t}\right)$. Then

$$yxm \in (\bigcap_{j=1}^{k} p_j)M \subseteq \langle E_M(N) \rangle \subseteq Q_{i_t}$$

for $t = 1, \ldots, s_i$. Since Q_{i_t} is primary and $y \notin p_{i_t}$, $xm \in Q_{i_t}$. Hence $xm \in N_i$.

Now let $x \in \bigcap_{r \in T} p_{i_r}, m \in \bigcap_{r \in S_i \setminus T} Q_{i_r}$ for some $T \subsetneq S_i$. Take

$$y \in (\bigcap_{j \neq i} p_j) \setminus (\bigcup_{t=2}^{s_i} p_{i_t}).$$

Then

$$yxm \in \left[(\bigcap_{j \neq i} p_j) \cap (\bigcap_{r \in T} p_{i_t})\right](\bigcap_{r \in S_i \setminus T} Q_{i_r}).$$

Since

$$\bigcap_{j \neq i} p_j = \bigcap_{j \neq i} \bigcap_{t=1}^{s_j} p_{j_t},$$

$$\Big[\big(\bigcap_{j\neq i} p_j\big) \cap \big(\bigcap_{r\in T} p_{i_t}\big)\Big]\big(\bigcap_{r\in S_i\setminus T} Q_{i_r}\big) \subseteq \langle E_M(N)\rangle \subseteq N_i.$$

Thus $yxm \in Q_{i_t}$ for $t = 1,\ldots,s_i$. Since Q_{i_t} is primary and $y \notin p_{i_t}$, $xm \in Q_{i_t}$ and hence $xm \in N_i$. Therefore $\langle E_M(N_i)\rangle = N_i$ and hence N can be written as an intersection of classical prime submodules by Lemma 5.2.1. □

6. CONCLUSIONS AND RECOMMENDATIONS

In this thesis, we obtained a formula for finding the lower nilradical of a submodule when the underlying module is Noetherian. Computing the radical of a submodule is also another problem in this area. This problem is solved for Noetherian modules by [17]. Here we gave an alternative proof for that computation and also we got a better technique to determining the prime submodules which are redundant or not for the prime radical.

Since every prime submodule is semiprime, the certain question one would like to think of is what is the intersection of all semiprime submodules containing a specific module. The answer of this question gave us semiprime radical and this suggested that one should consider the modules which satisfy the semiradical formula, and we did so. We also observed the semiprime submodules which can be written as an intersection of classical primes since the converse always holds.

The obvious question one would like to consider is that is it possible to find a formula for determining lower nilradical even in the non-Noetherian case. This would be one of the questions that we would like to answer in our future study. The related question that we would like to answer is whether the definition of semiradical formula can be generalized or not.

Another related question for our future study would be what conditions can be added to simplify semiprimes which are intersection of classical primes.

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CURRICULUM VITAE

Name Surname: Sibel KILIÇARSLAN CANSU

Place and Date of Birth: İstanbul / 05.11.1979

Address: ITU, Ayazağa Campus, Faculty of Science and Letters

E-Mail: sibelkcansu@gmail.com, kilicarslans@itu.edu.tr

B.Sc.: Abant Izzet Baysal University 1997-2001

M.Sc.: Abant Izzet Baysal University 2001-2004

PUBLICATIONS ON THE THESIS:

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- **Kılıçarslan Cansu S.** and Yılmaz E., 2010: Alt Modüllerin Zarfları, *23.Ulusal Matematik Sempozyumu*, August 4-7, 2010, Kayseri, Turkey.

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