İSTANBUL TECHNICAL UNIVERSITY★ INSTITUTE OF SCIENCE AND TECHNOLOGY

AN ALGEBRAIC APPROACH TO SENSITIVITY ANALYSIS IN LINEAR PROGRAMMING

M.Sc. Thesis by

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Department: INDUSTRIAL ENGINEERING
Program: INDUSTRIAL ENGINEERING

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FOREWORD

This thesis actually had started back in 2004 when I was taking the course END331 Operations Research as an undergraduate student. A homework given to me by the course instructor about the 100% rule had a question which I thought I'd replied correctly. So I was surprised to find out that I hadn't got any marks on that question, which continued to haunt me for weeks. My interest in the subject of sensitivity analysis had started at that point and my answer was the basis for this study.

I'd like to thank my supervisor and instructor Associate Professor Y.İlker Topçu for his help and support in this thesis, if he hadn't given that homework I'd probably never been this much interested in Sensitivity Analysis. I'd especially like to thank Research Assistants Özgür Kabak and Emel Aktaş for inspiring me. I'd also like to thank Research Assistant Senem Kurşun for her most valuable help in formatting this thesis.

My most sincere thanks go to my family and friends for their support and understanding.

MAY 2007 H.Kutay TİNÇ

CONTENTS

ABBREVIATIONS

QM:	Quantitative Methods
OR:	Operations Research
MS:	Microsoft
SA:	Sensitivity Analysis
DM:	Decision Maker
LP:	Linear Programming
N_{obj}:	Normal of the objective function
N_{ci}:	Normal of the *i*th constraint

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FIGURE LIST

SUMMARY

Linear Programming covers an important part of Operations Research. Although it is a basis for advanced study of Operations Research, Sensitivity Analysis in Linear Programming for multiple coefficient changes is limited to the 100% (one-hundred percent) rule, and a more efficient method has not been developed.

In this thesis a new method, which is based on simple mathematical operations, is proposed to eliminate this deficiency. The validity of the method has been shown mathematically and an application with MS Excel © has been done. After explaining the need for this method, the five steps that are used has been shown.

In the first step, the binding constraints of a solved linear programming model are found. This step can be easily realized by checking on the non-basic variables from the program QM ©.

In the second step, the normal functions, which are vectors, of these functions are found.

In the third step, using these normal functions, a hyper-plane function for n-dimension is calculated. This hyper-plane contains the extreme points of all the normal functions.

In the fourth step, the hyper-volume generated by the normal functions is calculated via determinant.

In the fifth step, after being normalized on the hyper-plane, the new objective function, which has been changed for sensitivity analysis, replaces every normal function one at a time in the hyper-volume determinant.

The steps given above are easily realized by MS Excel's © Solver add-on.

As a result, this thesis proposes a method that allows us to analyze the sensitivity of any model, which has been solved, by changing its objective function coefficients in any way.

ÖZET

Doğrusal Programlama, Yöneylem Araştırması konularının önemli bir bölümünü kaplamaktadır. Pek çok açıdan ileri konulara temel oluşturan Doğrusal Programlama'nın, duyarlılık analizi konusu için yapılan çalışmalar özellikle çoklu katsayı değişimleri için 100% (yüzde yüz) kuralı ile sınırlı kalmış, daha iyi bir yöntem geliştirilmemiştir.

Bu tez çalışmasında, varolan bu eksiklik yeni bir matematiksel yaklaşım ile giderilmeye çalışılmış, ve basit matematiksel kuramlara dayalı yeni bir yöntem önerilmiştir. Önerilen yöntemin geçerliliği matematiksel olarak gösterilmiş ve MS Excel© programı ile bir uygulaması yazılmıştır. Yönteme neden ihtiyaç duyulduğu açıklandıktan sonra yöntemi oluşturan beş adım açıklanmıştır.

Birinci adımda optimal sonucu bulunmuş doğrusal programlama modellerine ait aktif (bağlayıcı) kısıtlar bulunuyorlar. Bu adımı örneğin, QM © programındaki çıktılardan temel olmayan değişkenlere bakarak yapmamız mümkündür.

İkinci adımda bu kısıtların, aynı zamanda birer vektör olan, normal doğru denklemleri bulunuyor.

Üçüncü adımda bu denklemlerden yola çıkılarak n-boyut için oluşan hiperdüzlemin denklemi hesaplanıyor. Bu hesaplanan hiperdüzlem tüm normal doğrularının uç noktalarını üstünde bulunduruyor.

Dördüncü adımda normal doğrularının oluşturduğu hiperhacim determinantla hesaplanıyor.

Beşinci adımda duyarlılık analizi için değiştirilmiş olan amaç fonksiyonu, hiper-düzlem üstüne normalize edildikten sonra, hiperhacim determinantında her normal doğrusu ile bir kez değiştirilerek analiz yapılıyor.

Yukarıda verilen adımlar MS Excel© programında çözücü aracı kullanılarak kolay bir şekilde uygulanabiliyor.

Sonuç olarak bu tez çalışması, optimal çözümü bulunmuş olarak verilen her lineer programlama modeli için amaç fonksiyon katsayılarından istediklerimizi istediğimiz kadar değiştirerek duyarlılık analizi yapmamıza izin veren bir yöntem öneriyor.

1. INTRODUCTION

Operations Research (OR) is an interdisciplinary science that uses tools like mathematical modeling, probability and statistics, and algorithms to find solutions for complex decision making problems in real-life.

Operations Research has been in existence as a concept since the World War II. During this sixty year period, many developments have been done, both as theoretically and in applications. Although the concept of Operations Research is widely viewed as a science that is only understood buy those who study it, the mathematical background for the fundamentals of OR is limited with basic algebra.

Due to developments in computer science, decision making process has become much easier with programs like Lindo©, Lingo©, QM©, MS Excel© Solver Add-On. As long as we can model the problem with a clear linear function, the programs mentioned can solve them easily. After solving the model, mostly, a Post Optimality Analysis is required.

The Post-Optimality Analysis, or more known as Sensitivity Analysis (SA), is the part where the Decision Maker (DM) wishes to check his result and see the risks involved. A general definition for SA would be apportioning the output of a model to different types of variations of the input. For Linear Programming (LP), SA is the study of changes in an LP's parameters and how these changes affect the optimal solution.

This change of parameters can occur from the objective function coefficients or from the right-hand side values of the constraints. Handling both changes require an exacting study and are sometimes time-consuming for the DM.

In the following chapters the current method for SA in LPs will be examined and its inadequacy to answer some questions will be shown, then a new method will be introduced after a mathematical background is established for the reader.

The second chapter will familiarize the reader with the 100% rule, the current method that is being used for SA in LPs. Some examples will be given and the reason for the rule's inadequacy will be shown in these examples.

In the third chapter the mathematical background for the new method will be given, where determinants and vectors hold a great deal of importance.

In the fourth chapter the method itself will be introduced and different types of SA techniques for various coefficient changes will be shown. Excel and QM outputs will be used for the examples in this chapter.

In the final chapter, the method will be reviewed and future study options will be discussed.

2. SENSITIVITY ANALYSIS WITH THE 100% RULE

Linear programming models are usually solved with the Simplex Algorithm [1]. The Simplex algorithm uses basic mathematical operations on a Simplex tableau and finds the optimal solution. When the optimal solution is found, the DM may wish to make a post-optimality analysis. This analysis may include changes of coefficients for one or more variables.

In a given linear programming model, when more than one parameter is changed for sensitivity analysis, the 100% rule is used to determine whether the current basis is still optimal. Once the solution is found, the calculation to see if the changes affect the optimality of the basis is very simple. Every parameter, whether it is an objective function coefficient or a right hand side value of a constraint, has an Allowable Increase and an Allowable Decrease that represent the interval of the values it can take without disrupting the optimality of the current basis. This interval of values only holds for the change of that sole parameter, though. If more than one parameter changes, then the 100% rule should be used.

2.1. The 100% Rule

There are two cases examined in the 100% rule, in the first case, all variables, whose objective function coefficients are changed, have reduced costs greater than zero. In this case, the current basis remains optimal as long as all these objective function coefficients remain in their allowable intervals.

In the second case, some of the objective function coefficients that are changed have non-zero reduced value costs. In this case, the changes in objective function values of these coefficients are compared to their allowable changes and the amount of change is written as a percentage value and all values are added up. If the total percentage does not exceed 100%, the optimality of the current basis does not change, but if the total percentage exceeds 100% then we can not comment on the optimality of the current basis.

This uncertainty is the greatest failing of the 100% rule. Many variations are left unexplored in post-optimality because of this deficiency and sometimes the model is solved again to find out the optimality state.

2.1.1. An Example

Let's consider the following Linear Programming Model, the Dakota Furniture Problem [2] and its given solution with sensitivity analysis (Table 1):

$$\begin{aligned} &Max\ \ 60X_1 + 30X_2 + 20X_3 \\ &ST\ \ \ 8X_1 + 6X_2 + X_3 \le 48 \\ &\qquad 4X_1 + 2X_2 + 1{,}5X_3 \le 20 \\ &\qquad 2X_1 + 1{,}5X_2 + 0{,}5X_3 \le 8 \end{aligned} \tag{2.1}$$

Table 1: Sensitivity Analysis for the Dakota Problem

VARIABLE	VALUE	REDUCED COST	
X_1	2	0	
X_2	0	5	
X_3	8	0	
ROW	SLACK OR SURPLUS	DUAL PRICES	
2	24	0	
3	0	10	
4	0	10	
RANGES IN WHICH THE BASIS IS UNCHANGED			
	OBJ COEFFICIENT RANGES		
VARIABLE	CURRENT	ALLOWABLE	
		INCREASE	DECREASE
X_1	60	20	4
X_2	30	5	INFINITY
X_3	20	2,5	5

It is clear that X_1 and X_3 are basic variables (reduced cost = 0), whereas X2 is a non-basic variable (reduced cost = 5). Now that we know the ranges for these variables we can demonstrate a use of the 100% rule.

Suppose that the coefficient of X_1 is increased by 15 and the coefficient of X_3 is decreased by 1, as they are both basic variables their respective changes will be calculated and summed up to find the total change:

X_1's percentage of change is: $\frac{15}{20} = 75\%$

X_3's percentage of change is: $\frac{1}{5} = 20\%$

Then the total change of percentages $= 75 + 20 = 95\%$, thus the optimality of the current basis holds.

What would happen if the coefficient of X_1 is increased by 18 and the coefficient of X_3 is increased by 1?

X_1's percentage of change is: $\frac{18}{20} = 90\%$

X_3's percentage of change is: $\frac{1}{2{,}5} = 40\%$

Then the total change of percentages $= 90 + 40 = 130\%$, so we can not comment on the optimality of the current basis. This is where the 100% rule fails to investigate, when the total change is complex enough to exceed 100% we cannot decide whether the optimality holds or not. This inadequacy of the 100% rule makes it an inefficient way for post-optimality analysis.

3. VECTORS AND DETERMINANTS – A MATHEMATICAL BACKGROUND

Vectors are directions with magnitude. They can also be described as directed line segments in a space [3]. There can be many examples to vectors such as: Wind velocity at a weather station, the position of a target relative to a naval gun, and the force exerted on a moving electron by a magnetic field [4].

Vectors are shown with directional arrows; a vector on a two dimension would be shown as (Figure 1):

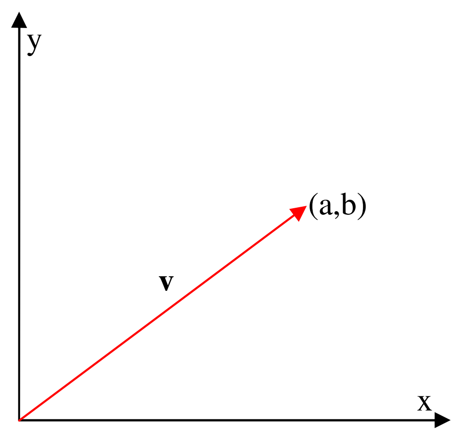

Figure 1: A Vector in 2-D

This vector, denoted as **v** can also be written as $\vec{v} = \mathbf{v} = a\mathbf{i} + b\mathbf{j} = (a,b)$, where **i** and **j** are unit vectors for the axis x and y.

3.1. Magnitude of a Vector

The magnitude of a vector $\mathbf{v} = a\mathbf{i} + b\mathbf{j}$, is equal to its length or the distance of its farthest point from the origin when it is placed on the origin. This length or magnitude can be calculated as:

$$\left|\vec{v}\right| = \sqrt{a^2 + b^2}$$

If the magnitude of a vector is 1 unit, then the vector is called a unit vector.

For example, **u** = 3**i** + 4**j**, has a magnitude of $\sqrt{3^2 + 4^2} = 5$. If we wanted to convert **u** to a unit vector we would divide the coefficients of **u** by its magnitude: **u₁**= $\frac{3}{5}$ **i** + $\frac{4}{5}$ **j**

3.2. Vectors as Normals of Planes and Hyper-Planes

There is only one plane that contains a given point and is perpendicular to a given line [3]. For example, Let $P_0 = (x_0, y_0, z_0)$ is a given point and **N** = a**i** + b**j** + c**k** is a non-zero vector. Then the plane that contains P_0 and perpendicular to **N** is $a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$ and by letting $ax_0 + by_0 + cz_0 = d$ the plane equation becomes $ax + by + cz = d$.

Consider P_0=(1,2,3) and **N** = 2**i** + **j** + 3**k**, the plane derived from the point and vector is: $2(x - 1) + (y - 2) + 3(z - 3) = 0$ which can be written as: $2x + y + 3z = 13$

The vector **N** is called the normal vector of the plane, and can be derived by taking the gradient of the plane, for example the normal vector of $3x + 2y + 3z = 4$ is **N = 3i** + 2**j** + 3**z**, which is also the normal vector for the plane $3x + 2y + 3z = 15$.

3.3. Determinants

A determinant is a mathematical function that associates a scalar, *s* = det(A), to every $n \times n$ square matrix. Determinants are very useful in the analysis and solution of systems of linear equations. A less commonly known property of the determinant is its ability to calculate area, volume and hyper-volume for vectors. This obscure property of the determinant is this thesis' main focus.

To this purpose the concept of parallelepipeds will be introduced. In three dimensions, a parallelepiped is a prism whose faces are all parallelograms. Let **a**, **b**, and **c** be the basis vectors defining a three-dimensional parallelepiped [5]. Then the parallelepiped has the volume given by the absolute value of the scalar triple-product $\left|\vec{a}.(\vec{b}\times\vec{c})\right|$. The scalar product can also be denoted as a determinant: $\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$. So the volume of the parallelepiped can be calculated by the determinant of its base vectors.

For n-dimensional space, given k vectors ($\mathbf{v_1}$, $\mathbf{v_2}$, …, $\mathbf{v_k}$) , their convex hull is called a parallelepiped. If the number of vectors is equal to the dimension then A = ($\mathbf{v_1}$, $\mathbf{v_2}$, …, $\mathbf{v_k}$) is a square matrix, and the volume of the parallelepiped is given by |Det(A)| [5].

4. THE PARALLELEPIPED METHOD FOR SENSITIVITY ANALYSIS

As we have previously seen, the 100% rule is not efficient for all post-optimality analysis conditions. A more efficient method is proposed in this chapter. The first step is to establish the connection between the objective function, the constraints and vectors.

Each variable in the objective function relates to a dimension for an n-dimension space, and each constraint in a linear model relates to a line (2-D), a plane (for 3-D) or a hyper-plane (n-Dimension).

Although there can be many constraints (including the sign constraints $x_i > 0$), there are only n binding constraints, where n is the number of variables in the objective function. This property of linear models is very important for our method as the normals of the binding constraints will form a $n \times n$ matrix.

To understand the method, we first must take a look at the graphical relation between the objective function and the binding constraints of a linear model.

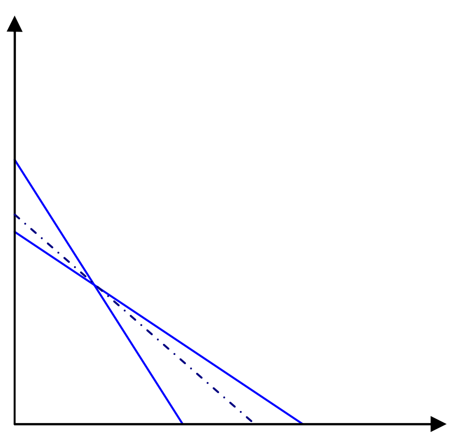

Figure 2: Graphical Illustration of a Linear Model

In Figure 2, the blue lines show the binding constraints, and the dashed line shows the objective function. A property of the binding constraints is that, they contain the optimal point in their lines. Thus all binding constraints and the objective function intersect on the optimal point. In the two dimensional above it is clear that as long as the objective function's slope remains between the slopes of the two binding constraints, optimality holds. But if the slope was increased or decreased too much the model would become like Figure 3:

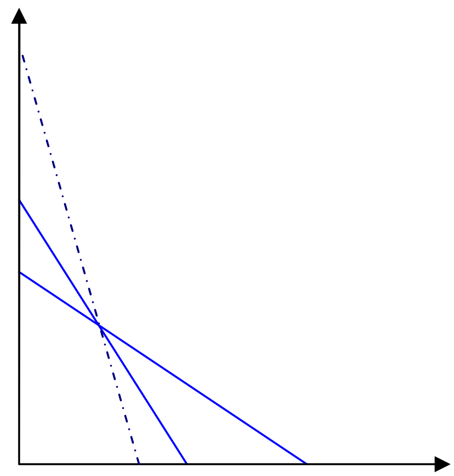

Figure 3: Changed Model

It is clear that the objective function does not leave the feasible region. So the optimality does not hold, actually the graphical optimal solution becomes like Figure 4

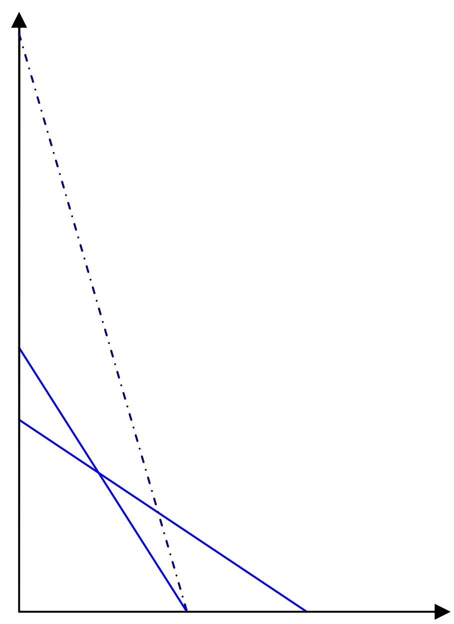

Figure 4: Optimal Solution of the Changed Model

Thus it is seen that for any change in the slope of the objective function a new graphical solution might become plausible. As the slope of a line is perpendicular to its normal we can deduce that the change in the normal of an objective function is subject to some limitations set by the binding constraints of the linear model. Let's consider the first figure and add the normals of the objective function and the binding constraints, the new graphical illustration of the model would be like Figure 5:

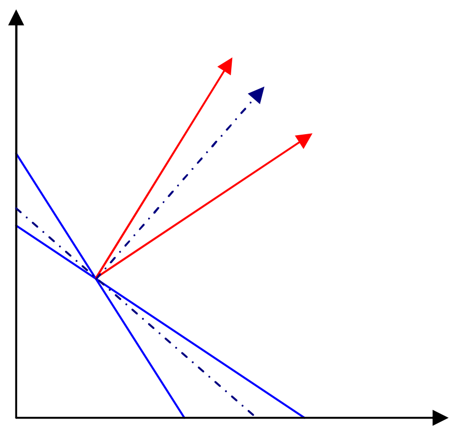

Figure 5: Graphical illustration of normals of constraints and objective function

The red lines represent the normals of the binding constraints and the dashed arrow represents the normal of the objective function. As seen in the figure above, the normal of the objective function (henceforth will be known as $\mathbf{N_{obj}}$) is between the normals of the binding constraints. This property must hold if we are trying to prove the optimality of the current basis.

Naturally the question of how to show that the $\mathbf{N_{obj}}$ stays between the normals of the binding constraints (henceforth will be known as $\mathbf{N_{ci}}$). To this end a method that calculates the volume of the parallelepiped that the N_{ci} span will be used.

Consider that the $\mathbf{N_{ci}}$ span a parallelepiped P with a volume of S. If one of the $\mathbf{N_{ci}}$ was to be replaced with the $\mathbf{N_{obj}}$ to create a fake (The reason that the new parallelepipeds are called fake is that, they are temporary) parallelepiped (P_{fi}), then the volume of the parallelepiped would become smaller, as $\mathbf{N_{obj}}$ lies within P. This property would hold for all the vectors that lie inside P, so for any vector $\mathbf{v_i}$, if $\mathbf{v_i}$ is an element of P (in other words, if the direction of $\mathbf{v_i}$ stays in P), then $\mathbf{v_i}$ is an objective function normal that holds the optimality of the current basis. To give an example to this parallelepiped volume calculation a 2-D illustrative example will be given.

4.1. 2-D Illustrative Example

Let's consider the previous example where the optimal solution is seen in the Figure 6 below:

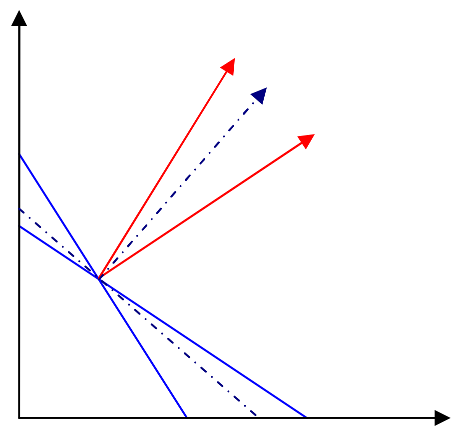

Figure 6: Graphical illustration of the model

To ease the calculations, the functions will be removed from the figure and calculations will be made on the vectors alone, as seen on Figure 7:

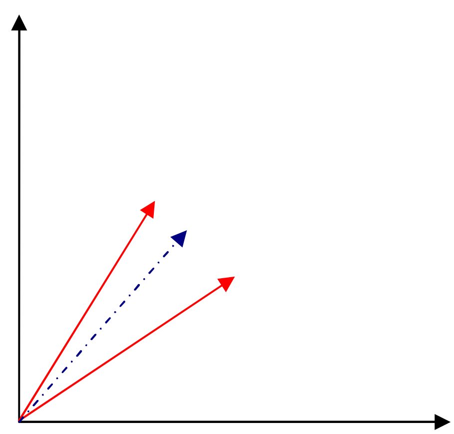

Figure 7: The Normal Vectors

The parallelepiped spanned by the $\mathbf{N}_{ci}$ can be seen in Figure 8:

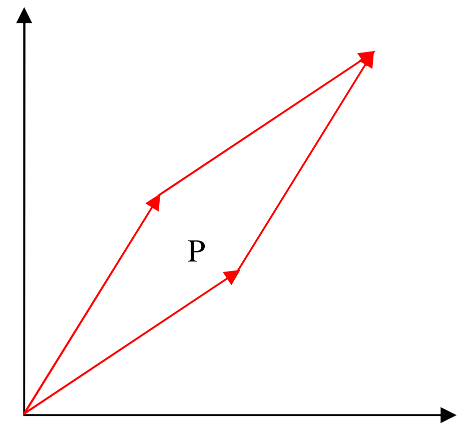

Figure 8: Area of the parallelogram the $\mathbf{N_{ci}}$ span

Replacing both of the $\mathbf{N_{ci}}$ with the $\mathbf{N_{obj}}$ yields Figure 9 and Figure 10:

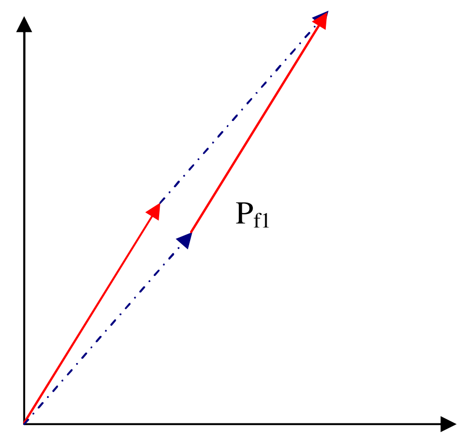

Figure 9: First Fake Area

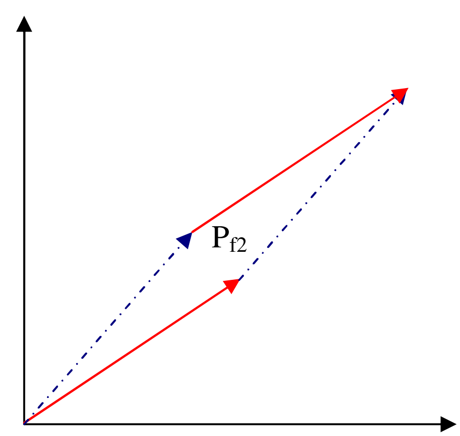

Figure 10: Second Fake Area

It is clear that both these new parallelograms (parallelepipeds in n-Dimensions) have a smaller area (volume for n-Dimensions) than the first one ($A(P)>A(P_{f1})$, $A(P)>A(P_{f2})$). If the $\mathbf{N_{obj}}$ was sloped close to one of the $\mathbf{N_{ci}}$, as in Figure 11, the areas of the above figures would change, where one of them would converge to 0 (Figure 12) and the other would converge to S (Figure 13), the area of the $\mathbf{N_{ci}}$ ($A(P_{f3})\approx 0$, $A(P_{f4})\approx S$).

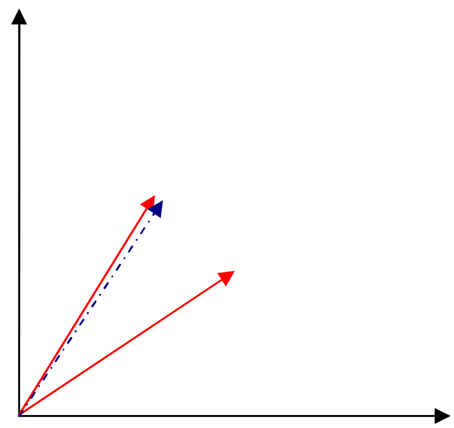

Figure 11: The New Model

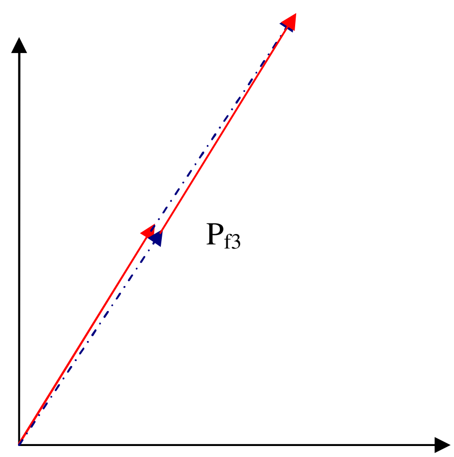

Figure 12: First Fake Area of the New Model

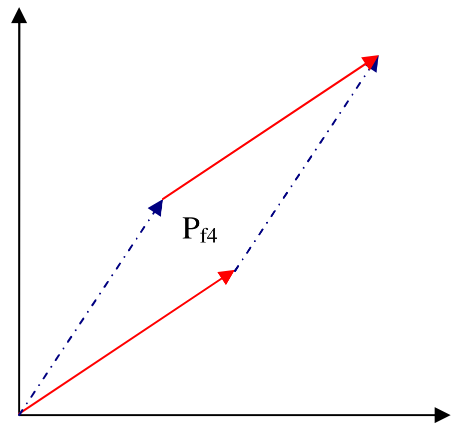

Figure 13: Second fake area of the new model

One important point in calculating the volumes of the parallelepipeds is to make sure that all the vector end points are on one specific plane. Otherwise the calculations can be wrong, for example let's take a look at figure 14:

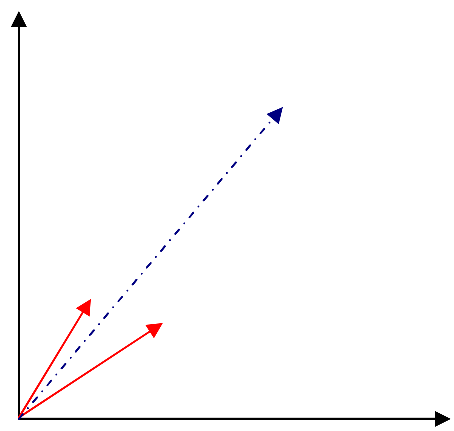

Figure 14: A Non-normalized Objective Function

Although the directions of the vectors are the same as before, their magnitudes have changed, so the areas of the parallelograms change accordingly:

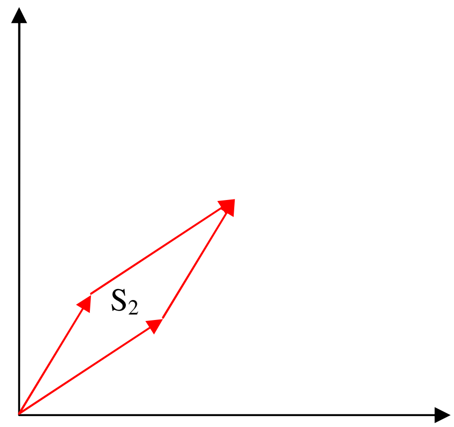

Figure 15: The Area of the Parallelogram

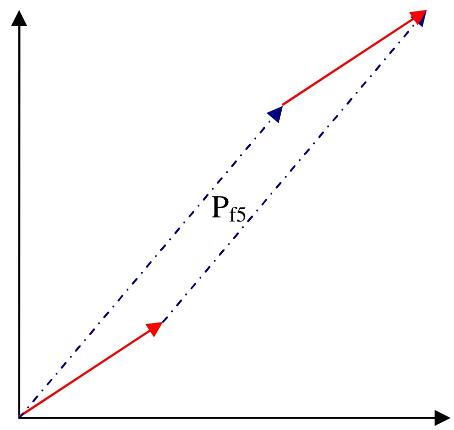

Figure 16: First Fake Area of the Non-normalized Objective Function

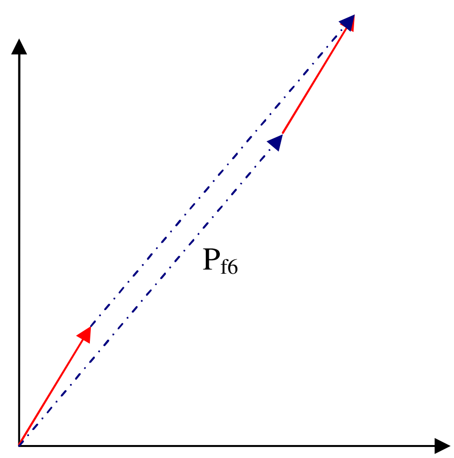

Figure 17: First Fake Area of the Non-normalized Objective Function

It is clear that P_{f5} is larger than S_2, so the vector points should be at the same plane for an accurate calculation.

4.2. Using Determinants for Hyper-Volume Calculations

The calculations for the hyper-volumes of the parallelepipeds can be done by taking the determinant of the matrix that the vectors that span the parallelepiped. Let's consider the Dakota Example [2]:

$$
\begin{array}{lll}
Max & 60X_1 + 30X_2 + 20X_3 & \\
ST & 8X_1 + 6X_2 + X_3 & \leq 48 \\
 & 4X_1 + 2X_2 + 1{,}5X_3 & \leq 20 \\
 & 2X_1 + 1{,}5X_2 + 0{,}5X_3 & \leq 8
\end{array}
\qquad \textbf{(4.1)}
$$

When this linear model is solved, the binding constraints are found as:

C_1: $X_2 \geq 0$

C_2: $4X_1 + 2X_2 + 1{,}5X_3 \leq 48$ **(4.2)**

C_3: $2X_1 + 1{,}5X_2 + 0{,}5X_3 \leq 8$

As this is a maximization question, the binding constraints should be in the form of "greater than", so C_1 becomes: $-X_2 < 0$

The determinants of these vectors are:

$\begin{vmatrix} 0 & -1 & 0 \\ 4 & 2 & 1{,}5 \\ 2 & 1{,}5 & 0{,}5 \end{vmatrix} = -1$, but as we need a positive number we swap the places of two initial vectors as:

$\begin{vmatrix} 2 & 1{,}5 & 0{,}5 \\ 4 & 2 & 1{,}5 \\ 0 & -1 & 0 \end{vmatrix} = 1$, so for any given $\mathbf{v_i}$, the determinant of replacing any $\mathbf{N_{ci}}$ with $\mathbf{v_i}$ should be less than or equal to 1, for the current basis to remain optimal. For example let's assume $\mathbf{v_i} = (3,1,1)$

$$\begin{vmatrix} 3 & 1 & 1 \\ 4 & 2 & 1{,}5 \\ 0 & -1 & 0 \end{vmatrix} = 0{,}5$$

$$\begin{vmatrix} 2 & 1{,}5 & 0{,}5 \\ 3 & 1 & 1 \\ 0 & -1 & 0 \end{vmatrix} = 0{,}5$$

$$\begin{vmatrix} 2 & 1{,}5 & 0{,}5 \\ 4 & 2 & 1{,}5 \\ 3 & 1 & 1 \end{vmatrix} = 0{,}75$$

It is clear that $\mathbf{v_i}$ is in the parallelepiped. Though some doubts may occur, we may check any given vector to see whether it holds the property of being on the plane that the Nci spans. If the given vector needs a coefficient to make it fit the plane, and this coefficient is less than 1, then the calculations done with that vector is greater than they are supposed to be, likewise when the coefficient is less than 1, then the calculations made is less than what they are supposed to be.

Normally, the sum of $A(P_f)$ gives the area of the first parallelepiped ($\sum A(P_f) = S$), if the sum is greater than S, then the vector's magnitude should be decreased to make it more appropriate, likewise if the sum is less then S then the vector's magnitude should be increases to make it more appropriate.

4.3. The Parallelepiped Method (TPM)

The first step in deciding whether a given objective function satisfies the optimality of the current basis is finding the binding constraints. A constraint is called "binding constraint" if the left-hand side and the right-hand side values of the constraint are equal, in other words when it's slack or excess value equals to 0 [2]. There is a certain restriction to that definition though. For example a given constraint might seem like binding but be totally irrelevant to the feasible region of the model. These constraints only intersect the feasible region at the optimal point and have no impact on the solution itself. So they can be discarded as non-binding constraints for the sake of this method. An example of this situation can be seen in the Figure 18 below:

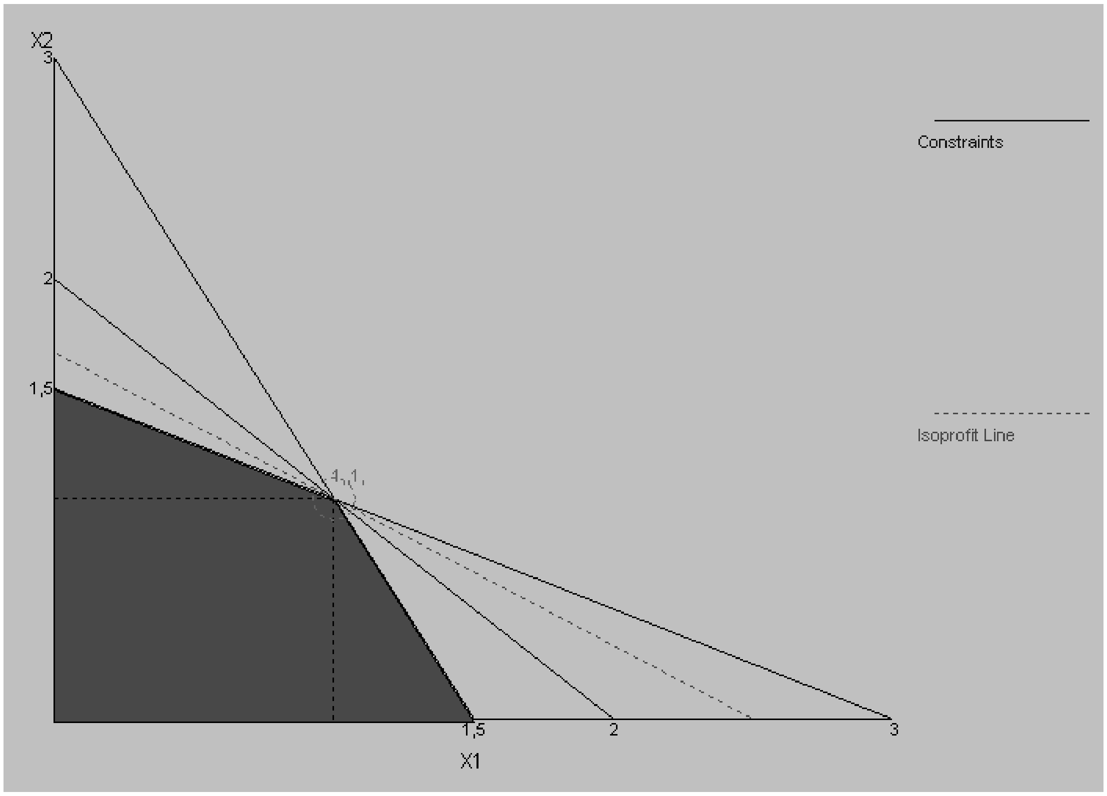

Figure 18: An Example of a Binding Constraint Selection Problem

There are three constraints, and all of them include the optimal point (1, 1) but one of them does not intersect the feasible region on any other point, so it will be discarded.

This example is taken from the QM for Windows version 2.0, the linear model was:

	X_1	X_2		RHS
Maximize	2	3		
Constraint 1	2	1	<=	3
Constraint 2	1	2	<=	3
Constraint 3	1	1	<=	2

The solution itself is given in a more elaborate way in Table 2, and it helps us find the binding constraints that we will be using in the method:

Table 2: Solution Table of the example

Variable	Status	Value
X_1	Basic	1
X_2	Basic	1
slack 1	NONBasic	0
slack 2	NONBasic	0
slack 3	Basic	0

Our third constraint has a slack value of 0, it is a binding constraint according to the definition, but the slack 3 variable is not a NONBasic variable according to QM, and thus it is not useable in our method. By taking further examples it is clear to see that there are only *n* nonbasic variables, where *n* is the number of variables in the objective function. If one of the objective function variable was to become a nonbasic variable, X_1 for example, the binding constraint we use is it's sign constraint ($X_1 > 0$, or $-X_1 < 0$).

The second step is to find the normals of these binding constraints. Taking the gradient of the constraints yield us their normals, so $\mathbf{N_{c1}} = (2, 1)$ and $\mathbf{N_{c2}} = (1, 2)$ are the normal vectors of our binding constraints.

The third step is to find the plane where all the points of the $\mathbf{N_{ci}}$ and $\mathbf{v_i}$ intersect, let's assume this place has an equation of: $ax_1 + bx_2 = c$

Then both (2,1) and (1,2) should satisfy this equation. It is clear that we will have infinitely many solutions for this linear equation, so we choose any eligible one. Assuming a = 1; (2, 1): 2 + b = c and (1, 2): 1 + 2b = c yields b =1 and c =3.

Thus the plane *P* is: $x_1 + x_2 = 3$

The fourth step is the determinant calculation of the $\mathbf{N_{ci}}$:

$\begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = 3 = S$, which is the area of the parallelogram the $\mathbf{N_{ci}}$ span.

The fifth step has two parts; first one is the normalization of the $\mathbf{v_i}$ to the plane *P*. Let's test the original objective function first. Given a coefficient λ, the $\mathbf{N_{obj}} = (2, 3)$ should satisfy the following equation: $\lambda x_1 + \lambda x_2 = 3$, where λ is the normalization coefficient. Replacing x_1 and x_2 with their values yields: $2\lambda + 3\lambda = 3 \Rightarrow \lambda = \frac{3}{5}$.

Then the normalized $\mathbf{N_{obj}}$ is: $(\frac{6}{5},\frac{9}{5})$.

The second part of the step is replacing the normalized $\mathbf{N_{obj}}$ with the $\mathbf{N_{ci}}$ in the determinant one at a time. For two variables, that means taking two determinants:

$P_{f1}=\begin{vmatrix} 6/5 & 9/5 \\ 1 & 2 \end{vmatrix}=\frac{3}{5}$, $P_{f2}=\begin{vmatrix} 2 & 1 \\ 6/5 & 9/5 \end{vmatrix}=\frac{12}{5}$. As both P_{f1} and P_{f2} are less than S, the given objective function is optimal at the given point.

Consider $\mathbf{v_i}$ = (2, 5), in other words the new objective function is $2x_1 + 5x_2$. Placing the values in $\lambda x_1 + \lambda x_2 = 3$ we see that: $2\lambda + 5\lambda = 3 \Rightarrow \lambda = \frac{3}{7} \Rightarrow \mathbf{v_{in}} = \left(\frac{6}{7},\frac{15}{7}\right)$. By taking both determinants we check the optimality:

$P_{f1}=\begin{vmatrix} 6/7 & 15/7 \\ 1 & 2 \end{vmatrix}=-\frac{3}{7}$, $P_{f2}=\begin{vmatrix} 2 & 1 \\ 6/7 & 15/7 \end{vmatrix}=\frac{24}{7}$. P_{f1} is negative and P_{f2} > S, so the optimality does not hold.

4.4. Excel Application of the Method

The application of the method in Excel is very easy, considering the number of determinants you need to calculate in higher dimensions. Having an excel table of n dimensions gives us the ability to check the optimality for any dimension m for $m \leq n$. For example, let's consider the Dakota Furniture Problem [2]. The QM software gives us the following solution in Table 3:

Table 3: QM Solution of the Dakota Problem

Variable	Status	Value
X_1	Basic	2
X_2	NONBasic	0
X_3	Basic	8
slack 1	Basic	24
slack 2	NONBasic	0
slack 3	NONBasic	0

It is clear that constraints 2 and 3 are binding, and the sign constraints $X_2 > 0$ is binding too. So our three normal vectors are: $(2 \quad 0{,}5 \quad 1{,}5),(4 \quad 2 \quad 1{,}5),(0 \quad -1 \quad 0)$.

An excel table for 8-Dimensions was used to make the calculations (Table 4).

Table 4: The 8-D Excel Table for TPM

	X1	X2	X3	X4	X5	X6	X7	X8
C1	2,0	1,5	0,5					
C2	4,0	2,0	1,5					
C3	0,0	-1,0	0,0					
C4				1,0				
C5					1,0			
C6						1,0		
C7							1,0	
C8								1,0
Plane	7,88	-3,5	-14	3,5	3,5	3,5	3,5	3,5
Objective	60,0	30,0	20,0	0,0	0,0	0,0	0,0	0,0
Normalized	2,4	1,2	0,8	0,0	0,0	0,0	0,0	0,0
OPTIMAL								

The calculations made are only for plane equation, and for every different objective function a new normalized vector is found instantly. The determinant calculations are as follows (Figure 19):

$$\begin{vmatrix} 2 & 1{,}5 & 0{,}5 & 0 & 0 & 0 & 0 & 0 \\ 4 & 2 & 1{,}5 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{vmatrix} = 1 \qquad \begin{vmatrix} 2{,}4 & 1{,}2 & 0{,}8 & 0 & 0 & 0 & 0 & 0 \\ 4 & 2 & 1{,}5 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{vmatrix} = 0{,}4$$

$$\begin{vmatrix} 2 & 1{,}5 & 0{,}5 & 0 & 0 & 0 & 0 & 0 \\ 2{,}4 & 1{,}2 & 0{,}8 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{vmatrix} = 0{,}4 \qquad \begin{vmatrix} 2 & 1{,}5 & 0{,}5 & 0 & 0 & 0 & 0 & 0 \\ 4 & 2 & 1{,}5 & 0 & 0 & 0 & 0 & 0 \\ 2{,}4 & 1{,}2 & 0{,}8 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{vmatrix} = 0{,}2$$

Figure 19: Determinants of the 8-D Excel Table

The determinant on the upper left is the determinant of the binding constraints, and the other three determinants are calculations where one binding constraint is replaced by the objective function. The rest of the determinants always yield 0 as our problem is for only 3-D. Let's change the objective function to see if the optimality conditions hold for the Lindo output of the sensitivity analysis (Table 5):

Table 5: Sensitivity Output of the Dakota Problem

VARIABLE	VALUE	REDUCED COST	
X_1	2	0	
X_2	0	5	
X_3	8	0	
ROW	SLACK OR SURPLUS	DUAL PRICES	
2	24	0	
3	0	10	
4	0	10	
RANGES IN WHICH THE BASIS IS UNCHANGED			
	OBJ COEFFICIENT RANGES		
VARIABLE	CURRENT	ALLOWABLE	
		INCREASE	DECREASE
X_1	60	20	4
X_2	30	5	INFINITY
X_3	20	2,5	5

According to this SA table, X1 can only increase up to 80, and cannot become 81 without affecting the optimality. Testing this with the Parallelepiped Method gives us the result (Table 6) and the following calculations (Figure 20):

Table 6: Modification Analysis for X1 with TPM

	X1	X2	X3	X4	X5	X6	X7	X8
C1	2,0	1,5	0,5					
C2	4,0	2,0	1,5					
C3	0,0	-1,0	0,0					
C4				1,0				
C5					1,0			
C6						1,0		
C7							1,0	
C8								1,0
Plane	7,88	-3,5	-14	3,5	3,5	3,5	3,5	3,5
Objective	80,0	30,0	20,0	0,0	0,0	0,0	0,0	0,0
Normalized	1,1	0,4	0,3	0,0	0,0	0,0	0,0	0,0

OPTIMAL

$$\begin{vmatrix} 2 & 1,5 & 0,5 & 0 & 0 & 0 & 0 & 0 \\ 4 & 2 & 1,5 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{vmatrix} = 1 \qquad \begin{vmatrix} 1,14 & 0,43 & 0,29 & 0 & 0 & 0 & 0 & 0 \\ 4 & 2 & 1,5 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{vmatrix} = 0,57$$

$$\begin{vmatrix} 2 & 1,5 & 0,5 & 0 & 0 & 0 & 0 & 0 \\ 1,14 & 0,43 & 0,29 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{vmatrix} = 0 \qquad \begin{vmatrix} 2 & 1,5 & 0,5 & 0 & 0 & 0 & 0 & 0 \\ 4 & 2 & 1,5 & 0 & 0 & 0 & 0 & 0 \\ 1,14 & 0,43 & 0,29 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{vmatrix} = 0,43$$

Figure 20: Determinants of the Modified Model

The important point here is that one of the determinants is equal to 0, which means that any further change in the same direction would affect the optimality negatively as given in Table 7 and Figure 21:

Table 7: Further Modification Analysis for X1 with TPM

	X1	X2	X3	X4	X5	X6	X7	X8
C1	2,0	1,5	0,5					
C2	4,0	2,0	1,5					
C3	0,0	-1,0	0,0					
C4				1,0				
C5					1,0			
C6						1,0		
C7							1,0	
C8								1,0
Plane	7,88	-3,5	-14	3,5	3,5	3,5	3,5	3,5
Objective	81,0	30,0	20,0	0,0	0,0	0,0	0,0	0,0
Normalized	1,1	0,4	0,3	0,0	0,0	0,0	0,0	0,0

NOT OPTIMAL

$$\begin{vmatrix} 2 & 1,5 & 0,5 & 0 & 0 & 0 & 0 & 0 \\ 4 & 2 & 1,5 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{vmatrix} = 1 \qquad \begin{vmatrix} 1,12 & 0,42 & 0,28 & 0 & 0 & 0 & 0 & 0 \\ 4 & 2 & 1,5 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{vmatrix} = 0,57$$

$$\begin{vmatrix} 2 & 1,5 & 0,5 & 0 & 0 & 0 & 0 & 0 \\ 1,12 & 0,42 & 0,28 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{vmatrix} = -0,007 \qquad \begin{vmatrix} 2 & 1,5 & 0,5 & 0 & 0 & 0 & 0 & 0 \\ 4 & 2 & 1,5 & 0 & 0 & 0 & 0 & 0 \\ 1,12 & 0,42 & 0,28 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{vmatrix} = 0,43$$

Figure 21: Determinant Calculations for Further Modification

The third determinant is negative, thus the optimality no longer holds. This process can be applied to all the variables and all of them give the same result, but the most important part is that, we can apply this to multiple coefficient changes in the objective function.

Consider the example in section 2.1.1 where the objective function variables were changed as follows: the coefficient of x_1 was increased by 18 and the coefficient of x_3 was increased by 1. The 100% rule was inefficient as the total change exceeded 100%. The Parallelepiped Method gives us Table 8 and Figure 21:

Table 8: Multiple Coefficient Changes with TPM

	X1	X2	X3	X4	X5	X6	X7	X8
C1	2,0	1,5	0,5					
C2	4,0	2,0	1,5					
C3	0,0	-1,0	0,0					
C4				1,0				
C5					1,0			
C6						1,0		
C7							1,0	
C8								1,0
Plane	7,88	-3,5	-14	3,5	3,5	3,5	3,5	3,5
Objective	78,0	30,0	21,0	0,0	0,0	0,0	0,0	0,0
Normalized	1,3	0,5	0,3	0,0	0,0	0,0	0,0	0,0
OPTIMAL								

$$\begin{vmatrix} 2 & 1,5 & 0,5 & 0 & 0 & 0 & 0 & 0 \\ 4 & 2 & 1,5 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{vmatrix} = 1 \qquad \begin{vmatrix} 1,27 & 0,49 & 0,34 & 0 & 0 & 0 & 0 & 0 \\ 4 & 2 & 1,5 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{vmatrix} = 0,537$$

$$\begin{vmatrix} 2 & 1,5 & 0,5 & 0 & 0 & 0 & 0 & 0 \\ 1,27 & 0,49 & 0,34 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{vmatrix} = 0,049 \qquad \begin{vmatrix} 2 & 1,5 & 0,5 & 0 & 0 & 0 & 0 & 0 \\ 4 & 2 & 1,5 & 0 & 0 & 0 & 0 & 0 \\ 1,27 & 0,49 & 0,34 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{vmatrix} = 0,415$$

Figure 22: Determinant Calculations for Multiple Coefficient Changes

So the current basis $(78,30,21)$ is optimal. Many other examples may be given for this problem, all easily calculated by The Parallelepiped Method.

4.5. Application of the Method to Right-Hand Side Values

In a linear model, the right-hand side values represent the maximum or minimum usage of the constraints. They are susceptible to change as much as the objective function coefficients, so their sensitivity analysis is as much important. Using the parallelepiped method, the easiest way to examine their sensitivity is by using the linear model's dual.

In the dual, the right-hand side values become the objective function coefficients and the selection of the binding constraints are done according to the dual. Taking the Dakota example's dual we have the following model:

$$
\begin{aligned}
Min \quad & 48Y_1 + 20Y_2 + 8Y_3 & \\
ST \quad & 8Y_1 + 4Y_2 + 2Y_3 & \geq 60 \\
& 6Y_1 + 2Y_2 + 1{,}5Y_3 & \geq 30 \\
& Y_1 + 1{,}5Y_2 + 0{,}5Y_3 & \geq 20
\end{aligned}
\qquad \textbf{(2.1)}
$$

The complementary slackness theorem states that all basic variables become nonbasic for the dual of a problem, and also due to the nature of the dual problem, the objective function coefficients of the primal problem are the right hand side values of the dual problem. Taking the solution table of the primal model we will now select the basic variables as the non-binding constraint (Table 9).

Table 9: Solution table of the primal model

Variable	Status	Value
X_1	Basic	2
X_2	NONBasic	0
X_3	Basic	8
slack 1	Basic	24
slack 2	NONBasic	0
slack 3	NONBasic	0

So the variables of the dual model are as in Table 10:

Table 10: Variables of the Dual Model

Variable	Status
Y_1	NONBasic
Y_2	Basic
Y_3	Basic
slack 1	NONBasic
slack 2	Basic
slack 3	NONBasic

Thus our binding constraints are:

Y1 $\geq$ 0,

$8Y1 + 4Y2 + 2Y3 \geq 60,$

$Y1 + 1{,}5Y2 + 0{,}5Y3 \geq 20.$

Further calculations can be done the same way before; the starting table is in Table 11:

Table 11: Parallelepiped 8-Dimension Dual Dakota Table

	X1	X2	X3	X4	X5	X6	X7	X8
C1	-1,0	0,0	0,0					
C2	-8,0	-4,0	-2,0					
C3	-1,0	-1,5	-0,5					
C4				1,0				
C5					1,0			
C6						1,0		
C7							1,0	
C8								1,0
Plane	-3,5	-12	36,8	3,5	3,5	3,5	3,5	3,5
Objective	48,0	20,0	8,0	0,0	0,0	0,0	0,0	0,0
Normalized	-1,4	-0,6	-0,2	0,0	0,0	0,0	0,0	0,0

OPTIMAL

5. RESULTS AND SUGGESTIONS

In this thesis, an algebraic method for sensitivity analysis in linear programming has been proposed. This new method uses basic mathematical operations and it is very easily applied via MS Excel©. The method is called The Parallelepiped Method because of its tendency to use determinants to calculate parallelepiped volumes.

The method uses the normals of the binding constraints as a basis for its optimality search, and the volume these normals create a threshold that should not be passed, in order to stay optimal.

The parallelepiped method that is proposed in this thesis not only improves the sensitivity analysis capabilities for linear programming, but also gives a new basis for further research. Using a similar method for non-linear programming may provide us new study options, as well as new challenges.

Some obvious research areas would be finding areas of optimality similar to parametric programming [6], changing both objective function coefficients and right hand side values similar to tolerance approach [7]. Further research on these will be conducted by the author.

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AUTOBIOGRAPHY

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