

İSTANBUL TECHNICAL UNIVERSITY ★ INSTITUTE OF SCIENCE AND TECHNOLOGY

**ON SOME GENERALIZED EINSTEIN METRIC
CONDITIONS**

**M.Sc. Thesis by
İlker TUNÇ**

Department: Mathematics

Programme: Mathematics Engineering

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**BAZI GENELLEŞTİRİLMİŞ EINSTEIN METRİK
ŞARTLARI**

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LIST OF THE SYMBOLS

$\mathbf{C}$	: Weyl Conformal curvature tensor
$\mathbf{g}$	: Metric tensor
$\mathbf{H}$	: Second fundamental tensor
$\mathbf{R}$	: Riemannian-Christoffel curvature tensor
$\mathbf{S}$	: Ricci tensor
$T_p M$	: Tanget space of M at a point p
$T_p^* M$	: Dual base of the tanget space at a point p
$\otimes$	: Tensor product
∇	: Levi-Civita Connection
$\mathfrak{L}$	: Lie Algebra of vector fields on M
$[\cdot, \cdot]$	: Lie Bracket

ON SOME GENERALIZED EINSTEIN METRIC CONDITIONS

SUMMARY

In this thesis, some family of generalized Einstein metric conditions on semi-Riemannian manifolds are presented. It is proved that every Einstein manifold of dimension ≥ 4 satisfies some pseudosymmetry type curvature conditions. Basing on this fact, we introduce a family of curvature conditions holding on every semi-Riemannian manifold.

It is also proved that on any Einstein manifold the condition

$$R \cdot C - C \cdot R = \frac{\kappa}{n(n-1)} Q(g, C) \quad (1)$$

is satisfied. By using this fact, we investigate any non-Einstein, non-conformally flat manifolds satisfying the condition

$$R \cdot C - C \cdot R = L Q(g, C) \quad (2)$$

and obtained that any manifold satisfying (2) is pseudosymmetric and in addition $C \cdot R = 0$ on that manifold. Then we state two inverse theorems giving sufficient conditions for (2).

Further, we investigate hypersurfaces immersed isometrically in semi-Riemannian space forms. Investigations of Cartan hypersurfaces and Ricci-pseudosymmetric hypersurfaces lead to curvature identities holding on every hypersurface M . Under some assumptions, we show that these identities give rise to new generalized Einstein metric conditions holding on every hypersurface M . We describe an example of a hypersurface having some of these properties.

BAZI GENELLEŐTİRİLMİŐ EİNSTEIN METRİK ŐARTLARI ÖZET

Bu tezde semi-Riemannian manifoldlarda bazı genelleőtirilmiş Einstein metrik őartları sunuldu. Boyutu 4 ten büyük olan her Einstein manifoldunun bazı psödosimetri tip eğrilik őartlarını sağladığı ispatlandı. Bu gerçek kullanılarak herhangi bir semi-Riemannian manifoldunda gerçekleşen eğrilik őartları verildi.

Herhangi bir Einstein manifoldunda

$$R \cdot C - C \cdot R = \frac{\kappa}{n(n-1)} Q(g, C) \quad (3)$$

eğrilik őartının sağlandığı gösterildi. Bu özellik kullanılarak

$$R \cdot C - C \cdot R = L Q(g, C) \quad (4)$$

eğrilik őartını sağlayan herhangi Einstein olmayan, konformal düz olmayan manifoldlar incelendi ve (2) őartını sağlayan bir manifoldun psödosimetrik olduğu ve ayrıca $C \cdot R = 0$ özelliğinin bu manifoldda sağlandığı elde edildi. Daha sonra (2) için gerekli yeter őartları veren iki karőt teorem sunuldu.

İlave olarak, bir semi-Riemannian uzay formunda izometrik daldırılmış hiperyüzeyler incelendi. Cartan hiperyüzeyleri ve Ricci-psödosimetrik hiperyüzeylerin incelenmesi herhangi bir hiperyüzeyde geçerli olan eğrilik özelliklerinin ortaya çıkmasına neden oldu. Bazı kabuller altında bu eğrilik özellikleri kullanılarak her hiperyüzeyde gerçekleşen yeni genel Einstein metrik őartları gösterildi. Son olarak, bu özellikleri sağlayan bir hiperyüzey örneği sunuldu.

1. INTRODUCTION

A semi-Riemannian manifold is said to be locally symmetric if the condition $\nabla R = 0$ is satisfied on that manifold. These manifolds are first studied and classified by E. Cartan in the late twenties.

A semi-Riemannian manifold is said to be semi-symmetric if the condition $R \cdot R = 0$ is satisfied on that manifold. E. Cartan studied semi-symmetric spaces which is a natural generalization of symmetric spaces. Z. Szabó classified semi-symmetric spaces in 1980's. Their semisymmetric or semiparallel submanifolds was studied first by J. Deprez. He also classified those hypersurfaces and surfaces in Euclidean spaces.

The study on totally umbilical submanifolds of semisymmetric manifolds and study of geodesic mappings onto semisymmetric manifolds lead to the notion of pseudosymmetric manifolds. A semi-Riemannian manifold is said to be pseudosymmetric if at every point of M the following condition is satisfied:

The tensors $R \cdot R$ and $Q(g, R)$ are linearly dependent. This condition is equivalent to the relation $R \cdot R = L_R Q(g, R)$ where L_R is a function on the set $\mathcal{U}_R = \{x \in M | R - \frac{\kappa}{n(n-1)} G \neq 0 \text{ at } x\}$. Pseudosymmetric manifolds are a generalization of semisymmetric manifolds.

Notion of pseudosymmetry give rise also to other curvature conditions called conditions of pseudosymmetry type or generalized Einstein metric conditions.

In this thesis, we study generalized Einstein metric conditions on semi-Riemannian manifolds. This thesis is divided into 4 chapters:

Chapter 1, provides the necessary structure for this thesis. We cite pseudosymmetrically related tensors, the basic definitions, some curvature conditions and some results.

In Chapter 2, it is proved that every Einstein manifold satisfies some pseudosymmetry type curvature conditions, and then by considering non-Einstein and non-conformally flat manifolds satisfying that condition, we introduce a family of curvature conditions holding on every semi-Riemannian manifold.

Chapter 3 is concerned with hypersurfaces immersed isometrically in semi-Riemannian space forms. It is known that investigations of Cartan hypersurfaces and Ricci-pseudosymmetric hypersurfaces lead to curvature identities holding on every hypersurface M . Then by using these identities we introduce new generalized Einstein metric conditions holding on every hypersurface M . And finally we describe an example of a hypersurface having some of these properties.

Finally, in Chapter 4, we give results and discussion.

2. PSEUDOSYMMETRIC MANIFOLDS

In this chapter we give the basic definitions, properties and results related with the pseudosymmetric curvature conditions which will be used in the following sections.

2.1. Pseudosymmetrically Related Tensors

Let (M, g) , be an n -dimensional, $n \geq 3$, semi-Riemannian connected manifold of class $\mathcal{C}^\infty$ with Levi-Civita connection ∇ . The Ricci operator $\mathcal{S}$ is denoted by $g(\mathcal{S}X, Y) = S(X, Y)$, where $X, Y \in \Xi(M)$, $\Xi(M)$ being the Lie algebra of vector fields on M .

We define the endomorphisms $X \wedge_A Y$, $\mathcal{R}(X, Y)$ and $\mathcal{C}(X, Y)$ of $\Xi(M)$ by

$$(X \wedge_A Y)Z = A(Y, Z)X - A(X, Z)Y, \quad (2.1)$$

$$\mathcal{R}(X, Y)Z = [\nabla_X, \nabla_Y]Z - \nabla_{[X, Y]}Z, \quad (2.2)$$

$$\begin{aligned} \mathcal{C}(X, Y)Z &= \mathcal{R}(X, Y)Z \\ &\quad - \frac{1}{n-2}(X \wedge_g \mathcal{S}Y + \mathcal{S}X \wedge_g Y - \frac{\kappa}{n-1}X \wedge_g Y)Z, \end{aligned} \quad (2.3)$$

respectively, where $X, Y, Z \in \Xi(M)$, A is a symmetric $(0,2)$ -tensor, κ the scalar curvature and $[X, Y]$ is the Lie bracket of vector fields X and Y . In particular we have $(X \wedge_g Y) = X \wedge Y$.

The Riemannian-Christoffel curvature tensor R , the Weyl conformal curvature tensor C and the $(0,4)$ -tensor G of (M, g) are defined by

$$\begin{aligned} R(X_1, X_2, X_3, X_4) &= g(\mathcal{R}(X_1, X_2)X_3, X_4), \\ C(X_1, X_2, X_3, X_4) &= g(\mathcal{C}(X_1, X_2)X_3, X_4), \\ G(X_1, X_2, X_3, X_4) &= g((X_1 \wedge_g X_2)X_3, X_4), \end{aligned} \quad (2.4)$$

respectively. A tensor $\mathcal{B}$ of type $(1,3)$ on M is said to be generalized curvature tensor if

$$\begin{aligned}
\sum_{X_1, X_2, X_3} \mathcal{B}(X_1, X_2) X_3 &= 0, \\
\mathcal{B}(X_1, X_2) + \mathcal{B}(X_2, X_1) &= 0, \\
B(X_1, X_2, X_3, X_4) &= B(X_3, X_4, X_1, X_2),
\end{aligned} \tag{2.5}$$

where $B(X_1, X_2, X_3, X_4) = g(\mathcal{B}(X_1, X_2) X_3, X_4)$.

For symmetric $(0, 2)$ -tensors E and F we define their Kulkarni-Nomizu product $E \wedge F$ by

$$\begin{aligned}
(E \wedge F)(X_1, X_2, X_3, X_4) &= E(X_1, X_4)F(X_2, X_3) + E(X_2, X_3)F(X_1, X_4) \\
&\quad - E(X_1, X_3)F(X_2, X_4) - E(X_2, X_4)F(X_1, X_3).
\end{aligned}$$

For a $(0, k)$ -tensor field T , $k \geq 1$, a $(0, 2)$ -tensor field A and a generalized curvature tensor $\mathcal{B}$ on (M, g) we define the tensors $B \cdot T$ and $Q(A, T)$ by

$$\begin{aligned}
(B \cdot T)(X_1, \dots, X_k; X, Y) &= -T(\mathcal{B}(X, Y) X_1, X_2, \dots, X_k) \\
&\quad - \dots - T(X_1, \dots, X_{k-1}, \mathcal{B}(X, Y) X_k), \tag{2.6}
\end{aligned}$$

$$\begin{aligned}
Q(A, T)(X_1, \dots, X_k; X, Y) &= -T((X \wedge_A Y) X_1, X_2, \dots, X_k) \\
&\quad - \dots - T(X_1, \dots, X_{k-1}, (X \wedge_A Y) X_k), \tag{2.7}
\end{aligned}$$

respectively, where $X, Y, Z, X_1, X_2, \dots, X_k \in \Xi(M)$. Putting in the above formulas $\mathcal{B} = \mathcal{R}$ or $\mathcal{B} = \mathcal{C}$, $T = R$ or $T = C$ or $T = S$, $A = g$ or $A = S$ we obtain the tensors $R \cdot R$, $R \cdot C$, $C \cdot R$, $R \cdot S$, $C \cdot S$, $Q(g, R)$, $Q(S, R)$, $Q(g, C)$ and $Q(g, S)$.

Let (M, g) be covered by a system of charts $\{W; x^k\}$. We denote by g_{ij} , R_{hijk} , S_{ij} , $G_{hijk} = g_{hk}g_{ij} - g_{hj}g_{ik}$ and

$$\begin{aligned}
C_{hijk} &= R_{hijk} - \frac{1}{n-2}(g_{hk}S_{ij} - g_{hj}S_{ik} + g_{ij}S_{hk} - g_{ik}S_{hj}) \\
&\quad + \frac{\kappa}{(n-1)(n-2)}G_{hijk}, \tag{2.8}
\end{aligned}$$

the local components of the metric tensor g , the Riemann-Christoffel curvature tensor R , the Ricci tensor S , the tensor G and the Weyl tensor C , respectively. Further, we denote by $S_{ij} = S_{ir}S_j^r$ and $S_i^j = g^{jr}S_{ir}$ the local components of the tensor S^2 defined by $S^2(X, Y) = S(\mathcal{S}, Y)$, and of the Ricci operator $\mathcal{S}$, respectively.

2.2. Some curvature conditions

In this section, we present some considerations leading to the definition of a pseudosymmetric manifold.

A semi-Riemannian manifold (M, g) , $n \geq 3$, is said to be an *Einstein manifold* if

$$S = \frac{\kappa}{n}g \quad (2.9)$$

on M . Einstein manifolds form a natural subclass of the class of *quasi-Einstein manifolds*.

A semi-Riemannian manifold (M, g) , $n \geq 3$, is called a quasi-Einstein manifold if at every point $x \in M$ its Ricci tensor S has the form

$$S = \alpha g + \beta w \otimes w, \quad (2.10)$$

where $w \in T_x^*M$ and $\alpha, \beta \in \mathbb{R}$.

The semi-Riemannian manifold (M, g) satisfying the condition $\nabla R = 0$ is said to be *locally symmetric* [14]. Locally symmetric manifolds form a subclass of the class of manifolds characterized by the condition

$$R \cdot R = 0. \quad (2.11)$$

Semi-Riemannian manifolds fulfilling this condition are called *semisymmetric* [15]. They are not locally symmetric in general. Here $R \cdot R$ is a $(0, 6)$ -tensor with components

$$\begin{aligned} (R \cdot R)_{hijklm} &= \nabla_m \nabla_l R_{hijk} - \nabla_l \nabla_m R_{hijk} \\ &= R_{rijk} R_{hlm}^r + R_{hrjk} R_{ilm}^r + R_{hirk} R_{jlm}^r + R_{hijr} R_{klm}^r. \end{aligned} \quad (2.12)$$

A semi-Riemannian manifold is said to be *Ricci-semisymmetric* if on M we have $R \cdot S = 0$. A more general class of manifolds than the class of semisymmetric manifolds is the class of pseudosymmetric manifolds [6].

A semi-Riemannian manifold (M, g) is said to be *pseudosymmetric* if at every point of M the condition

$$R \cdot R = L_R Q(g, R) \quad (2.13)$$

holds on the set $\mathcal{U}_R = \{x \in M \mid R - \frac{\kappa}{n(n-1)} G \neq 0 \text{ at } x\}$, where L_R is some function on $\mathcal{U}_R$. There exist various examples of pseudosymmetric manifolds which are non semisymmetric.

A semi-Riemannian manifold (M, g) is said to be *Ricci-pseudosymmetric* [7] if at every point of M the condition

$$R \cdot S = L_S Q(g, S) \quad (2.14)$$

holds on the set $\mathcal{U}_S = \{x \in M \mid S - \frac{\kappa}{n} g \neq 0 \text{ at } x\}$, where L_S is some function on $\mathcal{U}_S$.

A semi-Riemannian manifold (M, g) is said to be *Weyl-pseudosymmetric* if at every point of M the condition

$$R \cdot C = L_C Q(g, C) \quad (2.15)$$

holds on the set $\mathcal{U}_C = \{x \in M \mid C \neq 0 \text{ at } x\}$, where L_C is some function on $\mathcal{U}_C$.

(2.13), (2.14), (2.15) or other conditions of this kind are called *curvature conditions of pseudosymmetry type*.

In this thesis, we present a family of curvature conditions of pseudosymmetry type which was found recently:

The condition

$$R \cdot C - C \cdot R = L_C Q(g, C) \quad (2.16)$$

holds on the set $\mathcal{U}_C = \{x \in M \mid C \neq 0 \text{ at } x\}$, where L_C is some function on $\mathcal{U}_C$.

The condition

$$R \cdot C - C \cdot R = L_R Q(g, R) \quad (2.17)$$

holds on the set $\mathcal{U}_R = \{x \in M \mid R - \frac{\kappa}{n(n-1)} G \neq 0 \text{ at } x\}$, where L_R is some function on $\mathcal{U}_R$.

The condition

$$R \cdot C - C \cdot R = L_1 Q(S, R) \quad (2.18)$$

holds on the set $\mathcal{U}_1 = \{x \in M \mid Q(S, R) \neq 0 \text{ at } x\}$, where L_1 is some function on $\mathcal{U}_1$.

The condition

$$R \cdot C - C \cdot R = L_2 Q(S, C) \quad (2.19)$$

holds on the set $\mathcal{U}_2 = \{x \in M \mid Q(S, C) \neq 0 \text{ at } x\}$, where L_2 is some function on $\mathcal{U}_2$.

These conditions are curvature conditions of pseudosymmetry type and also we note that curvature conditions of pseudosymmetry type (2.16)-(2.19) are *generalized Einstein metric conditions*.

2.3. Basic Definitions on Hypersurfaces and Some Results

Let M , $n = \dim M \geq 3$, be a connected hypersurface isometrically immersed in a semi-Riemannian manifold $(N, \tilde{g})$. We denote by g the metric tensor of M , induced from the metric tensor $\tilde{g}$. Further, we denote by $\tilde{\nabla}$ and ∇ the Levi-Civita connections corresponding to the metric tensors $\tilde{g}$ and g , respectively. Let ξ be a local unit normal vector field on M in N and let $\varepsilon = \tilde{g}(\xi, \xi) = \pm 1$. We can present the *Gauss formula* and the *Weingarten formula* of M in N in the following form:

$$\tilde{\nabla}_X Y = \nabla_X Y + \varepsilon H(X, Y) \xi, \quad \tilde{\nabla}_X \xi = -\mathcal{A}(X),$$

respectively, where X, Y are vector fields tangent to M , H is the *second fundamental tensor* of M in N , $\mathcal{A}$ is the *shape operator* of M in N and $H^k(X, Y) = g(\mathcal{A}^k(X), Y)$, $\text{tr}(H^k) = \text{tr}(\mathcal{A}^k)$, $k \geq 1$, $H^1 = H$ and $\mathcal{A}^1 = \mathcal{A}$. We denote by R and $\tilde{R}$ the Riemann-Christoffel curvature tensors of M and N , respectively. We denote by U_H the set consisting of all points $x \in M$ at which the transformation $\mathcal{A}^2$ is not a linear combination of the shape operator $\mathcal{A}$ and the identity transformation Id at x . Note that $\mathcal{U}_H \subset \mathcal{U}_S \cap \mathcal{U}_C$ [12]. The *Gauss equation* of M in N has the form

$$R(X_1, X_2, X_3, X_4) = \tilde{R}(X_1, X_2, X_3, X_4) + \varepsilon \overline{H}(X_1, X_2, X_3, X_4), \quad (2.20)$$

where $X_1, \dots, X_4$ are vector fields tangent to M and $\overline{H} = \frac{1}{2} H \wedge H$. Let the equations $x^r = x^r(y^h)$ be the local parametric expression of M in $(N, \tilde{g})$, where y^r and x^r are the local coordinates of M and N , respectively, and

$h, i, j, k, l, m, p, q \in \{1, \dots, n\}$ and $r, s, t, u \in \{1, \dots, n+1\}$. Now we can write (2.20) in the form

$$R_{hijk} = \tilde{R}_{rstu} B_h^r B_i^s B_j^t B_k^u + \varepsilon \bar{H}_{hijk}, \quad B_h^r = \frac{\partial x^r}{\partial y^k}, \quad (2.21)$$

where $\tilde{R}_{rstu}$, R_{hijk} , $\bar{H}_{hijk} = H_{hk}H_{ij} - H_{hj}H_{ik}$ and H_{hk} are the local components of the tensors $\tilde{R}$, R , $\bar{H}$ and H , respectively. If M is a hypersurface in $N_s^{n+1}(c)$, $n \geq 4$, then (2.21) becomes

$$R_{hijk} = \varepsilon(H_{hk}H_{ij} - H_{hj}H_{ik}) + \frac{\tau}{n(n+1)} G_{hijk}, \quad (2.22)$$

where τ is the scalar curvature of the ambient space and G_{hijk} are the local components of the tensor G . Contracting (2.22) with g^{ij} and g^{hk} , respectively we get

$$S_{hk} = \varepsilon(tr(H)H_{hk} - H_{hk}^2) + \frac{(n-1)\tau}{n(n+1)} g_{hk} \quad (2.23)$$

and

$$\kappa = \varepsilon \left((tr(H))^2 - tr(H^2) \right) + \frac{(n-1)\tau}{n+1}, \quad (2.24)$$

Ricci tensor and the scalar curvature, respectively, where $tr(H) = g^{hk}H_{hk}$, $tr(H^2) = g^{hk}H_{hk}^2$ and S_{hk} are the local components of the Ricci tensor S of M .

At the end of this section we present some results which will be used in the next sections.

LEMMA 2.3.1 [11]. Let (M, g) , $n \geq 4$, be a semi-Riemannian manifold. Let at point $x \in \mathcal{U}_S \cap \mathcal{U}_C$ the following two conditions be satisfied:

$$\begin{aligned} S &= \mu g + \rho a \otimes a, \\ \sum_{X,Y,Z} a(X) \mathcal{B}(Y, Z) &= 0, \end{aligned}$$

for some nonzero covector a , where $B = R - \gamma G$, $\mu, \rho, \gamma \in \mathbb{R}$. Then at x we have :

$$R \cdot R = \frac{\kappa}{n(n-1)} Q(g, R) = Q(S, R) - \frac{(n-2)\kappa}{n(n-1)} Q(g, C), \quad \mu = \frac{\kappa}{n}.$$

LEMMA 2.3.2 [8]. Let (M, g) , $n \geq 3$, be a semi-Riemannian manifold. Let at a point $x \in M$ be given a nonzero symmetric (0,2)-tensor A and a generalized

curvature tensor $\mathcal{B}$ such that at x the following condition is satisfied: $Q(A, \mathcal{B})=0$. Moreover, let V be a vector at x such that the scalar $\rho = a(V)$ is nonzero, where a is a covector defined by $a(X) = A(X, V)$, $X \in T_x M$.

- (i) If $A = \frac{1}{\rho} a \otimes a$ then $\sum_{X,Y,Z} a(X), \mathcal{B}(Y, Z) = 0$ at x , where $X, Y, Z \in T_x M$.
- (ii) If $A - \frac{1}{\rho} a \otimes a$ is nonzero, then $B = \frac{\gamma}{2} A \wedge A$, $\gamma \in \mathbb{R}$, at x . Moreover in both cases, $B \cdot B = Q(\text{Ric}(\mathcal{B}), B)$ at x .

LEMMA 2.3.3 [8]. Let (M, g) , $n \geq 3$, be a semi-Riemannian manifold. Let at a point $x \in M$ be given a symmetric $(0,2)$ -tensor A and two generalized curvature tensors $\mathcal{B}_1$ and $\mathcal{B}_2$ such that $B_1 = \frac{1}{2} A \wedge A$ and $B_2 = g \wedge A$, respectively. Then at x we have

$$Q(A, G) = -Q(g, B_2) \quad \text{and} \quad Q(A, B_2) = -Q(g, B_1).$$

THEOREM 2.3.1 [8]. Let (M, g) , $n \geq 3$, be a semi-Riemannian manifold. If at a point $x \in \mathcal{U}_S \cap \mathcal{U}_C$ its curvature tensor R is of the form

$$R = \frac{\phi}{2} S \wedge S + \mu g \wedge S + \eta G, \quad \phi, \mu, \eta \in \mathbb{R},$$

then at x we have

$$R \cdot R = L_R Q(g, R) = Q(S, R) + (L_R + \frac{\mu}{\phi}) Q(g, C),$$

where $L_R = \frac{\mu}{\phi}((n-2)\mu - 1) - \eta(n-2)$.

LEMMA 2.3.4 [5]. Let E be a symmetric $(0,2)$ -tensor at a point x of a semi-Riemannian manifold (M, g) , $n \geq 3$. If

$$E = \alpha g + \beta u \otimes u, \quad \alpha, \beta \in \mathbb{R}, u \in T_x^* M,$$

then at x we have

$$E^2 = \tilde{\alpha} E + \tilde{\beta} g, \quad \tilde{\alpha}, \tilde{\beta} \in \mathbb{R}.$$

LEMMA 2.3.5 [5]. Let (M, g) , $n \geq 3$, be a semi-Riemannian manifold. Let E be a nonzero symmetric $(0,2)$ -tensor at a point $x \in M$. If at x we have $Q(E - \alpha g, g \wedge E) = 0$, $\alpha \in \mathbb{R}$, then

$$E^2 = \tilde{\alpha} E + \tilde{\beta} g,$$

where $\tilde{\alpha}, \tilde{\beta} \in \mathbb{R}$.

PROPOSITION 2.3.1 [13]. Let M be a hypersurface in $N_s^{n+1}(c)$, $n \geq 4$. The following identities are satisfied on M

$$R \cdot S = Q(H, \text{tr}(H)H^2 - H^3) + \frac{\varepsilon\tau}{n(n+1)}Q(g, \text{tr}(H)H - H^2), \quad (2.25)$$

$$\begin{aligned} R \cdot C &= Q(S, R) - \frac{1}{n-2}g \wedge (R \cdot S) \\ &\quad - \frac{(n-2)\tau}{n(n+1)}Q(g, R) - \frac{\tau}{n(n+1)}Q(S, G). \end{aligned} \quad (2.26)$$

PROPOSITION 2.3.2 [13]. Let M be a hypersurface in $N_s^{n+1}(c)$, $n \geq 4$, satisfying the condition

$$\sum_{(X_1, X_2), (X_3, X_4), (X, Y)} (R \cdot C)(X_1, X_2, X_3, X_4; X, Y) = 0.$$

Then the following identities are satisfied on M

$$H^3 = \text{tr}(H)H^2 + \lambda H - \frac{\mu}{n}g, \quad (2.27)$$

$$R \cdot S = \frac{\tau}{n(n+1)}Q(g, S) - \frac{\mu}{n}Q(g, H), \quad (2.28)$$

$$\begin{aligned} R \cdot C &= Q(S, R) - \frac{\mu}{(n-2)n}Q(H, G) - \frac{(n-2)\tau}{n(n+1)}Q(g, R) \\ &\quad - \frac{(n-3)\tau}{(n-2)n(n+1)}Q(S, G), \end{aligned} \quad (2.29)$$

where λ is some function on $\mathcal{U}_H$ and

$$\mu = \text{tr}(H)\text{tr}(H^2) - \text{tr}(H^3) + \lambda \text{tr}(H).$$

PROPOSITION 2.3.3 [12]. Let M be a hypersurface in $N_s^{n+1}(c)$, $n \geq 4$. Then the following identity is satisfied on M

$$R \cdot R = Q(S, R) - \frac{(n-2)\tau}{n(n+1)}Q(g, C), \quad (2.30)$$

where τ is the scalar curvature of N .

3. SOME FAMILY OF GENERALIZED EINSTEIN METRIC CONDITIONS

In this chapter we prove that every Einstein manifold of dimension ≥ 4 satisfies some pseudosymmetry type curvature conditions. Basing on this fact we introduce a family of curvature conditions. We investigate non-Einstein manifolds satisfying one of these conditions [10].

3.1. Einstein Manifolds Satisfying a Certain Curvature Condition

In this part we obtain some basic identities which will be used in the next results.

Using the definition of the tensors $R \cdot C$ and $C \cdot R$ given in (2.4) we get

$$(R \cdot C)_{hijklm} = g^{rs}(C_{rijk}R_{shlm} + C_{hrjk}R_{silm} + C_{hirk}R_{sjlm} + C_{hijr}R_{sklm}), \quad (3.1)$$

$$(C \cdot R)_{hijklm} = g^{rs}(R_{rijk}C_{shlm} + R_{hrjk}C_{silm} + R_{hirk}C_{sjlm} + R_{hijr}C_{sklm}). \quad (3.2)$$

Contracting the tensor $Q(g, R)$ with g^{ij} and g^{hm} , respectively, we get

$$\begin{aligned} g^{ij}Q(g, R)_{hijklm} &= -g^{ij}g_{mh}R_{lijk} - g_{mi}g^{ij}R_{hljk} - g_{mj}g^{ij}R_{hilk} - g_{mk}g^{ij}R_{hijl} \\ &\quad + g_{lh}g^{ij}R_{mikj} + g_{li}g^{ij}R_{hmjk} + g_{lj}g^{ij}R_{himk} + g_{lk}g^{ij}R_{hijm} \\ &= Q(g, S)_{hklm}, \end{aligned} \quad (3.3)$$

$$\begin{aligned} g^{hm}Q(g, R)_{hijklm} &= g^{hm}(-g_{mh}R_{lijk} - g_{mi}R_{hljk} - g_{mj}R_{hilk} - g_{mk}R_{hijl} \\ &\quad + g_{lh}R_{mikj} + g_{li}R_{hmjk} + g_{lj}R_{himk} + g_{lk}R_{hijm}) \\ &= -(n-1)R_{lijk} - g_{lj}S_{ik} + g_{lk}S_{ij}, \end{aligned} \quad (3.4)$$

respectively. Contracting the tensor $Q(g, C)$ with g^{ij} , we have

$$\begin{aligned} g^{ij}Q(g, C)_{hijklm} &= -g_{mh}g^{ij}C_{lijk} - C_{hlmk} - C_{hmlk} - g_{mk}g^{ij}C_{hijl} + g_{lh}g^{ij}C_{mikj} \\ &\quad + g_{li}g^{ij}C_{hmjk} + C_{hmlk} + C_{hlmk} + g^{ij}g_{lk}C_{hijm} \\ &= 0. \end{aligned} \quad (3.5)$$

Contracting the tensor $Q(g, C)$ with g^{hm} and $Q(S, R)$ with g^{ij} and g^{hm} , respectively, we get

$$\begin{aligned} g^{hm}Q(g, C)_{hijklm} &= g^{hm}(-g_{mh}C_{lijk} - g_{mi}C_{hljk} - g_{mj}C_{hilk} - g_{mk}C_{hijl} \\ &\quad + g_{lh}C_{mijk} + g_{li}C_{hmjk} + g_{lj}C_{himk} + g_{lk}C_{hijm}) \\ &= -(n-1)C_{lijk}, \end{aligned} \quad (3.6)$$

$$g^{ij}Q(S, R)_{hijklm} = A_{lkhm} - A_{lhmk} - A_{mkhl} + A_{mhlk}, \quad (3.7)$$

$$\begin{aligned} g^{hm}Q(S, R)_{hijklm} &= A_{lijk} - A_{iljk} - A_{jilk} - A_{kijl} - \kappa R_{lijk} \\ &\quad + S_{kl}S_{ij} - S_{jl}S_{ik}, \end{aligned} \quad (3.8)$$

respectively, where

$$A_{mijk} = S_m^s R_{sijk}. \quad (3.9)$$

Using (2.8) in (3.1) we obtain

$$\begin{aligned} (R \cdot C)_{hijklm} &= g^{rs}(C_{rijk}R_{shlm} + C_{hrjk}R_{silm} + C_{hirk}R_{sjlm} + C_{hijr}R_{sklm}) \\ &= (R \cdot R)_{hijklm} - \frac{1}{(n-2)} \left[R_{hklm}S_{ij} - R_{jhlm}S_{ik} + R_{jilm}S_{hk} \right. \\ &\quad - R_{kilm}S_{hj} - R_{hijl}S_{ik} + R_{ijlm}S_{hk} + R_{hkil}S_{ij} - R_{iklm}S_{hj} \\ &\quad + g_{ij}S_k^s R_{shlm} + g_{hk}S_j^s R_{silm} + g_{hk}S_i^s R_{sjlm} + g_{ij}S_h^s R_{sklm} \\ &\quad \left. - g_{ik}S_j^s R_{shlm} - g_{hj}S_k^s R_{silm} - g_{ik}S_h^s R_{sjlm} - g_{hj}S_i^s R_{sklm} \right] \\ &\quad + \frac{\kappa}{(n-1)(n-2)} \left[R_{hklm}g_{ij} - R_{jhlm}g_{ik} + R_{jilm}g_{hk} - R_{kilm}g_{hj} \right. \\ &\quad \left. + R_{ijlm}g_{hk} - R_{hijl}g_{ik} + R_{hkil}g_{ij} - R_{iklm}g_{hj} \right] \\ &= (R \cdot R)_{hijklm} - \frac{1}{(n-2)} \left[g_{ij}(A_{khlm} + A_{hkml}) + g_{hk}(A_{jilm} + A_{ijlm}) \right. \\ &\quad \left. - g_{ik}(A_{jhlm} + A_{hjlm}) - g_{hj}(A_{kilm} + A_{iklm}) \right]. \end{aligned} \quad (3.10)$$

Applying in the same way (2.8) in (3.2) we get

$$\begin{aligned} (C \cdot R)_{hijklm} &= g^{rs}(R_{rijk}C_{shlm} + R_{hrjk}C_{silm} + R_{hirk}C_{sjlm} + R_{hijr}C_{sklm}) \\ &= (R \cdot R)_{hijklm} \\ &\quad - \frac{1}{n-2}Q(S, R)_{hijklm} + \frac{\kappa}{(n-1)(n-2)}Q(g, R)_{hijklm} \\ &\quad - \frac{1}{n-2}(g_{hl}A_{mijk} - g_{hm}A_{lijk} - g_{il}A_{mhjk} + g_{im}A_{lhjk} \\ &\quad + g_{jl}A_{mkhi} - g_{jm}A_{lkhi} - g_{kl}A_{mjhi} + g_{km}A_{ljhi}). \end{aligned} \quad (3.11)$$

Now combining (3.10) and (3.11) we obtain

$$\begin{aligned}
(n-2)(R \cdot C - C \cdot R)_{hrsklm} &= Q(S, R)_{hrsklm} - \frac{\kappa}{(n-1)} Q(g, R)_{hrsklm} \\
&- g_{rs}(A_{hklm} + A_{khlm}) - g_{hk}(A_{rslm} + A_{srlm}) + g_{rk}(A_{hslm} + A_{shlm}) \\
&+ g_{hs}(A_{rklm} + A_{krlm}) - g_{kl}A_{mshr} - g_{sm}A_{lkhr} + g_{hl}A_{mrsk} \\
&- g_{hm}A_{lrsk} - g_{rl}A_{mhsk} + g_{rm}A_{lhsk} + g_{sl}A_{mkhr} + g_{km}A_{lshr}. \quad (3.12)
\end{aligned}$$

By contraction with g^{rs} and making use of (3.3) and (3.7) yields,

$$\begin{aligned}
(n-2)g^{rs}(R \cdot C - C \cdot R)_{hrsklm} &= g^{rs}Q(S, R)_{hrsklm} - \frac{\kappa}{(n-1)}g^{rs}Q(g, R)_{hrsklm} \\
&- g^{rs}g_{rs}(A_{hklm} + A_{khlm}) - g^{rs}g_{hk}(A_{rslm} + A_{srlm}) \\
&+ g^{rs}g_{rk}(A_{hslm} + A_{shlm}) + g^{rs}g_{hs}(A_{rklm} + A_{krlm}) \\
&+ g^{rs}g_{hl}A_{mrsk} - g^{rs}g_{hm}A_{lrsk} - g^{rs}g_{rl}A_{mhsk} + g^{rs}g_{rm}A_{lhsk} \\
&+ g^{rs}g_{sl}A_{mkhr} - g^{rs}g_{sm}A_{lkhr} - g^{rs}g_{kl}A_{mshr} + g^{rs}g_{km}A_{lshr}.
\end{aligned}$$

This, making use of (3.9), yields

$$\begin{aligned}
(n-2)g^{rs}(R \cdot C - C \cdot R)_{hrsklm} &= A_{lkhm} - A_{lhmk} - A_{mkhl} + A_{mhlk} \\
&- \frac{\kappa}{n-1}(g_{hl}S_{km} + g_{kl}S_{hm} - g_{hm}S_{kl} - g_{km}S_{hl}) \\
&- n(A_{hklm} + A_{khlm}) - g^{rs}g_{hk}(A_{rslm} + A_{srlm}) + g^{rs}g_{hl}A_{mrsk} \\
&- g^{rs}g_{hm}A_{lrsk} - g^{rs}g_{kl}A_{mshr} + g^{rs}g_{km}A_{lshr} + \delta_k^s(A_{hslm} + A_{shlm}) \\
&+ \delta_h^r(A_{rklm} + A_{krlm}) - \delta_l^sA_{mhsk} + \delta_m^sA_{lhsk} + \delta_l^rA_{mkhr} - \delta_m^rA_{lkhr} \\
&= -(n-2)(A_{hklm} + A_{khlm}) + g_{hl} \left(S_{mk}^2 - \frac{\kappa}{(n-1)}S_{km} \right) \\
&+ g_{kl} \left(S_{mh}^2 - \frac{\kappa}{(n-1)}S_{hm} \right) - g_{mk} \left(S_{lh}^2 - \frac{\kappa}{(n-1)}S_{lh} \right) \\
&- g_{hm} \left(S_{lk}^2 - \frac{\kappa}{(n-1)}S_{kl} \right).
\end{aligned}$$

If we define tensor D by

$$D = \frac{1}{n-2}S^2 - \frac{\kappa}{(n-1)(n-2)}S, \quad (3.13)$$

we obtain

$$\begin{aligned}
g^{ij}(R \cdot C - C \cdot R)_{hijklm} &= g_{hl}D_{km} + g_{kl}D_{hm} - g_{hm}D_{kl} - g_{km}D_{hl} \\
&- (A_{hklm} + A_{khlm}). \quad (3.14)
\end{aligned}$$

Contracting (3.12) with g^{hm} and using (3.4) and (3.8) we obtain

$$\begin{aligned}
(n-2)g^{hm}(R \cdot C - C \cdot R)_{hijklm} &= S_{kl}S_{ij} - S_{jl}S_{ik} - nA_{lijk} - 2A_{iljk} \\
&+ g_{kl}(E_{ij} - \frac{\kappa}{n-1}S_{ij}) - g_{jl}(E_{ik} - \frac{\kappa}{n-1}S_{ik}) \\
&+ g_{ik}(E_{jl} - S_{jl}^2) - g_{ij}(E_{kl} - S_{kl}^2) + 2A_{lijk} + 2A_{kilj} + 2A_{jikl} \\
&+ (A_{lijk} + A_{lkij} + A_{jki}) + (A_{iljk} + A_{ijkl} + A_{iklj}).
\end{aligned}$$

Finally we get

$$\begin{aligned}
(n-2)g^{hm}(R \cdot C - C \cdot R)_{hijklm} &= S_{kl}S_{ij} - S_{jl}S_{ik} + g_{kl}(E_{ij} - \frac{\kappa}{n-1}S_{ij}) \\
&- g_{jl}(E_{ik} - \frac{\kappa}{n-1}S_{ik}) - g_{ij}(E_{kl} - S_{kl}^2) + g_{ik}(E_{jl} - S_{jl}^2) \\
&- nA_{lijk} - 2A_{iljk} + 2(A_{lijk} + A_{jilk} + A_{kilj}), \tag{3.15}
\end{aligned}$$

where the tensor E is defined by

$$E_{ij} = S^{rs}R_{rijs}. \tag{3.16}$$

By using the identity (3.12) we prove the following theorem

THEOREM 3.1.1. On any semi-Riemannian Einstein manifold (M, g) , $n \geq 4$, we have

$$R \cdot C - C \cdot R = \frac{\kappa}{(n-1)n} Q(g, R) = \frac{\kappa}{(n-1)n} Q(g, C).$$

PROOF. Using (2.9) in (3.9) we get,

$$A_{mijk} = S_m^s R_{sijk} = \frac{\kappa}{n} g^{st} g_{tm} R_{sijk} = \frac{\kappa}{n} R_{mijk}. \tag{3.17}$$

Now if we use (3.17) in the identity (3.12) we obtain

$$\begin{aligned}
(n-2)(R \cdot C - C \cdot R)_{hijklm} &= \frac{\kappa}{n} Q(g, R)_{hijklm} - \frac{\kappa}{n-1} Q(g, R)_{hijklm} \\
&+ \frac{\kappa}{n} \left[g_{hl}R_{mijk} - g_{hm}R_{lijk} - g_{il}R_{mhjk} + g_{im}R_{lhjk} + g_{jl}R_{mkhi} \right. \\
&- g_{jm}R_{lkhi} - g_{kl}R_{mjhi} + g_{km}R_{ljhi} - g_{ij}(R_{hklm} + R_{khlm}) \\
&\left. - g_{hk}(R_{ijlm} + R_{jilm}) + g_{ik}(R_{hjlm} + R_{jhlm}) + g_{hj}(R_{iklm} + R_{kilm}) \right].
\end{aligned}$$

Then we have

$$R \cdot C - C \cdot R = \frac{\kappa}{(n-1)n} Q(g, R) \tag{3.18}$$

and

$$C = R - \frac{\kappa}{n(n-1)} G. \quad (3.19)$$

Then we obtain $Q(g, C) = Q(g, R)$. Applying this in (3.18) we get our assertion.

With respect to Theorem 2.1.1 in the next sections we restrict our considerations to the subset $\mathcal{U} = \mathcal{U}_S \cap \mathcal{U}_C$ of M .

3.2. Manifolds Satisfying $R \cdot C - C \cdot R = L_C Q(g, C)$

Let us write $R \cdot C - C \cdot R = L_C Q(g, C)$ in the form

$$(n-2)(R \cdot C - C \cdot R) = (n-2)L_C Q(g, C). \quad (3.20)$$

Contracting with g^{ij} , in the view of (3.5) and (3.14), we obtain

$$A_{hklm} + A_{khlm} = g_{hl}D_{km} + g_{kl}D_{hm} - g_{hm}D_{kl} - g_{km}D_{hl}. \quad (3.21)$$

Contracting (3.21) with g^{hm} we get

$$\begin{aligned} g^{hm}A_{hklm} + g^{hm}A_{khlm} &= g^{hm}g_{hl}D_{km} + g^{hm}g_{kl}D_{hm} - g^{hm}g_{hm}D_{kl} \\ &\quad - g^{hm}g_{km}D_{hl}, \\ S^{sm}R_{sklm} + S_{lk}^2 &= D_{lk} + g_{kl}tr(D) - nD_{kl} - D_{kl}, \end{aligned}$$

and therefore by using the definition of the tensor E we get

$$E = \frac{-2}{n-2}S^2 + \frac{n\kappa}{(n-1)(n-2)}S + tr(D)g. \quad (3.22)$$

Further, summing cyclically (3.21) in h, l, m we get

$$A_{hklm} + A_{lkmh} + A_{mklh} = 0$$

This together with (3.21) yields

$$-nA_{lijk} - 2A_{iljk} + 2(A_{lij k} + A_{jikl} + A_{kilj}) = -nA_{likj} - 2A_{iljk}.$$

Note that

$$A_{lijk} + A_{iljk} = g_{lj}D_{ik} + g_{ij}D_{lk} - g_{lk}D_{ij} - g_{ik}D_{lj}.$$

Hence

$$\begin{aligned}
-nA_{likj} - 2A_{likj} &= -nA_{likj} - 2(-A_{lijk} + g_{lj}D_{ik} + g_{ij}D_{lk} - g_{lk}D_{ij} - g_{ik}D_{lj}) \\
&= -nA_{lijk} + 2A_{lijk} - 2Q(g, D)_{lijk} \\
&= -(n-2)A_{lijk} - 2Q(g, D)_{lijk} \\
-(n-2)A_{lijk} - 2Q(g, D)_{lijk} &= -nA_{lijk} - 2A_{iljk} + 2(A_{lijk} + A_{jilk} + A_{kilj}).
\end{aligned} \tag{3.23}$$

Contracting (3.20) with g^{hm} and using (3.6) and (3.15) we have

$$\begin{aligned}
g^{hm}(n-2)(R \cdot C - C \cdot R)_{hijklm} &= g^{hm}(n-2)L_C Q(g, C)_{hijklm} \\
&= S_{kl}S_{ij} - S_{jl}S_{ik} \\
&+ g_{kl}\left(E_{ij} - \frac{\kappa}{n-1}S_{ij}\right) - g_{jl}\left(E_{ik} - \frac{\kappa}{n-1}S_{ik}\right) \\
&- g_{ij}(E_{kl} - S_{kl}^2) + g_{ik}(E_{jl} - S_{jl}^2) \\
&- nA_{lijk} - 2A_{iljk} + 2(A_{lijk} + A_{jilk} + A_{kilj}) \\
&= -(n-1)C_{lijk}(n-2)L_C \\
&- S_{jl}S_{ik} + S_{kl}S_{ij} + g_{kl}\left(E_{ij} - \frac{\kappa}{n-1}S_{ij}\right) \\
&- g_{jl}\left(E_{ik} - \frac{\kappa}{n-1}S_{ik}\right) - g_{ij}(E_{kl} - S_{kl}^2) \\
&+ g_{ik}(E_{jl} - S_{jl}^2) - (n-2)A_{lijk} - 2Q(g, D)_{lijk} \\
&= -(n-1)(n-2)L_C C_{lijk}
\end{aligned}$$

Using (3.13) and (3.22), we get

$$\begin{aligned}
E - S^2 + 2D &= -(n-2)D + \text{tr}(D)g, \\
E - \frac{\kappa}{n-1}S + 2D &= \text{tr}(D)g,
\end{aligned}$$

and using this relation we obtain

$$\begin{aligned}
(n-2)A_{lijk} &= S_{kl}S_{ij} - S_{jl}S_{ik} \\
&+ \text{tr}(D)(g_{kl}g_{ij} - g_{jl}g_{ik} + g_{ik}g_{jl} - g_{ij}g_{kl}) \\
&- 2Q(g, D)_{lijk} - g_{ij}\left(\frac{\kappa}{n-1}S_{kl} - S_{kl}^2\right) \\
&+ g_{ik}\left(\frac{\kappa}{n-1}S_{jl} - S_{jl}^2\right) + (n-1)(n-2)L_C C_{lijk} \\
A_{lijk} &= \frac{1}{n-2}(S_{kl}S_{ij} - S_{jl}S_{ik}) \\
&+ g_{ij}D_{kl} - g_{ik}D_{jl} + (n-1)L_C C_{lijk}.
\end{aligned} \tag{3.24}$$

By using (2.8) and (3.13) in (3.24) we get,

$$\begin{aligned}
(n-1)L_C C_{lijk} &= A_{lijk} - \frac{1}{2(n-2)} \bar{S}_{lijk} - g_{ij} D_{kl} + g_{ik} D_{jl} \\
&= S_l^r R_{rijk} - \frac{1}{2(n-2)} \bar{S}_{lijk} - g_{ij} D_{kl} + g_{ik} D_{jl} \\
&= S_l^r (C_{rijk} + \frac{1}{n-2} (g \wedge S)_{rijk}) - \frac{\kappa}{(n-1)(n-2)} G_{rijk} \\
&\quad - \frac{1}{2(n-2)} \bar{S}_{lijk} - g_{ij} D_{kl} + g_{ik} D_{jl} \\
(n-1)L_C C_{lijk} &= S_l^r C_{rijk}. \tag{3.25}
\end{aligned}$$

(3.20) in virtue of (3.12) takes the form

$$\begin{aligned}
L_C Q(g, C)_{hijklm} &= (R.C - C.R)_{hijklm} \\
&= \frac{1}{n-2} \left[Q(S, R)_{hijklm} - \frac{\kappa}{n-1} Q(g, R)_{hijklm} \right. \\
&\quad + g_{ik} (A_{hjlm} + A_{jhl m}) + (g_{hl} A_{mijk} - g_{hm} A_{lijk} - g_{il} A_{mhjk} \\
&\quad + g_{im} A_{lhjk} + g_{hj} (A_{iklm} + A_{kilm}) - g_{hk} (A_{ijlm} + A_{jilm}) \\
&\quad + g_{jl} A_{mkhi} - g_{jm} A_{lkhi} - g_{kl} A_{mjhi} + g_{km} A_{ljhi}) \\
&\quad \left. - g_{ij} (A_{hklm} + A_{khlm}) \right].
\end{aligned}$$

Applying (3.21) and (3.22) on the right hand side of the equation

$$\begin{aligned}
(n-2)L_C Q(g, C)_{hijklm} &= -Q(S, R)_{hijklm} + \frac{\kappa}{n-1} Q(g, R)_{hijklm} \\
&\quad - g_{hl} \left[\frac{1}{n-2} (S_{mk} S_{ij} - S_{mj} S_{ik}) + g_{ij} D_{mk} - g_{ik} D_{mj} + (n-1)L_C C_{mijk} \right] \\
&\quad - g_{hm} \left[\frac{1}{n-2} (S_{lk} S_{ij} - S_{lj} S_{ik}) + g_{ij} D_{lk} - g_{ik} D_{lj} + (n-1)L_C C_{lijk} \right] \\
&\quad - g_{il} \left[\frac{1}{n-2} (S_{mk} S_{hj} - S_{mj} S_{hk}) + g_{hj} D_{mk} - g_{hk} D_{mj} + (n-1)L_C C_{mhjk} \right] \\
&\quad + g_{im} \left[\frac{1}{n-2} (S_{lk} S_{hj} - S_{lj} S_{hk}) + g_{hj} D_{lk} - g_{hk} D_{lj} + (n-1)L_C C_{lhjk} \right] \\
&\quad + g_{jl} \left[\frac{1}{n-2} (S_{mi} S_{kh} - S_{mh} S_{ki}) + g_{kh} D_{mi} - g_{ik} D_{mh} + (n-1)L_C C_{mkhi} \right] \\
&\quad - g_{jm} \left[\frac{1}{n-2} (S_{li} S_{kh} - S_{lh} S_{ik}) + g_{kh} D_{li} - g_{ik} D_{lh} + (n-1)L_C C_{lkhi} \right] \\
&\quad - g_{kl} \left[\frac{1}{n-2} (S_{mi} S_{jh} - S_{mh} S_{ij}) + g_{jh} D_{mi} - g_{ij} D_{mh} + (n-1)L_C C_{mjhi} \right] \\
&\quad + g_{km} \left[\frac{1}{n-2} (S_{li} S_{jh} - S_{lh} S_{ij}) + g_{jh} D_{li} - g_{ij} D_{lh} + (n-1)L_C C_{ljhi} \right]
\end{aligned}$$

$$\begin{aligned}
& + g_{ij}(g_{hl}D_{km} + g_{kl}D_{hm} - g_{hm}D_{kl} - g_{km}D_{hl}) + g_{hk}(g_{il}D_{jm} + g_{jl}D_{im} \\
& - g_{im}D_{jl} - g_{jm}D_{il}) - g_{ik}(g_{hl}D_{jm} + g_{jl}D_{hm} - g_{hm}D_{jl} - g_{jm}D_{hl}) \\
& - g_{jh}(g_{il}D_{km} + g_{kl}D_{im} - g_{im}D_{kl} - g_{km}D_{il}).
\end{aligned}$$

Then we get

$$\begin{aligned}
(n-2)L_C Q(g, C) &= Q(S, R) - \frac{\kappa}{n-1}Q(g, R) \\
&+ \frac{1}{n-2}Q(g, \frac{1}{2}S \wedge S) + (n-1)L_C Q(g, C).
\end{aligned}$$

Hence we obtain

$$Q(S, R) - \frac{\kappa}{n-1}Q(g, R) + \frac{1}{2(n-2)}Q(g, \bar{S}) + L_C Q(g, C) = 0.$$

Applying now Lemma 2.3.3 we have the following

$$Q(g, \frac{1}{2}S \wedge S) = -Q(S, g \wedge S),$$

and therefore we get

$$Q(S, R) - \frac{\kappa}{n-1}Q(g, R) - \frac{1}{n-2}Q(S, g \wedge S) + L_C Q(g, C) = 0. \quad (3.26)$$

Using the definitions of the tensors in (3.26) we get,

$$\begin{aligned}
& - S_{mh}R_{lijk} - S_{mi}R_{hljk} - S_{mj}R_{hilk} - S_{mk}R_{hijl} + S_{lh}R_{mijk} + S_{li}R_{hmjk} \\
& + S_{lj}R_{himk} + S_{lk}R_{hijm} \\
& - \frac{\kappa}{n-1} \left(-g_{mh}R_{lijk} - g_{mi}R_{hljk} - g_{mj}R_{hilk} - g_{mk}R_{hijl} + g_{lh}R_{mijk} \right. \\
& \left. + g_{li}R_{hmjk} + g_{lj}R_{himk} + g_{lk}R_{hijm} \right) \\
& + \frac{1}{n-2} \left(-g_{hk}S_{mh}S_{ij} - g_{ij}S_{mh}S_{lk} + g_{lj}S_{mh}S_{ik} + g_{ik}S_{mh}S_{lj} - g_{hk}S_{mi}S_{lj} \right. \\
& - g_{lj}S_{mi}S_{ij} + g_{hj}S_{mi}S_{lk} + g_{lk}S_{mi}S_{hj} - g_{hk}S_{mj}S_{il} - g_{il}S_{mj}S_{hk} + g_{hl}S_{mj}S_{ik} \\
& + g_{ik}S_{mj}S_{hl} - g_{hl}S_{mk}S_{ij} - g_{ij}S_{mk}S_{hl} + g_{hj}S_{mk}S_{il} + g_{il}S_{mk}S_{hj} + g_{mk}S_{hl}S_{ij} \\
& + g_{ij}S_{lh}S_{mk} - g_{mj}S_{lh}S_{ik} - g_{ik}S_{lh}S_{mj} + g_{hk}S_{li}S_{mj} + g_{mj}S_{li}S_{hk} - g_{hj}S_{li}S_{mk} \\
& - g_{mk}S_{li}S_{hj} + g_{hk}S_{lj}S_{im} + g_{im}S_{lj}S_{hk} - g_{hm}S_{lj}S_{ik} - g_{ik}S_{lj}S_{hm} + g_{hm}S_{lk}S_{ij} \\
& \left. + g_{ij}S_{lk}S_{hm} - g_{hj}S_{lk}S_{im} - g_{im}S_{lk}S_{hj} \right) \\
& + L_C Q(g, C)_{hijklm} \\
& = 0
\end{aligned}$$

and we have

$$\begin{aligned}
& - C_{hmjk}(\frac{\kappa}{n-1}g_{li} - S_{li}) - C_{himk}(\frac{\kappa}{n-1}g_{lj} - S_{lj}) - C_{hijm}(\frac{\kappa}{n-1}g_{lk} - S_{lk}) \\
& + C_{hilk}(\frac{\kappa}{n-1}g_{mj} - S_{mj}) + C_{hijl}(\frac{\kappa}{n-1}g_{mk} - S_{mk}) - C_{mijk}(\frac{\kappa}{n-1}g_{lh} - S_{lh}) \\
& + C_{lijk}(\frac{\kappa}{n-1}g_{mh} - S_{mh}) + C_{hlk}(\frac{\kappa}{n-1}g_{mi} - S_{mi}) \\
& + L_C Q(g, C)_{hijlm} \\
& = 0.
\end{aligned}$$

Finally we obtain

$$\begin{aligned}
Q(S - \frac{\kappa}{n-1}g, C) + L_C Q(g, C) &= 0, \\
Q(S - \left(\frac{\kappa}{n-1} - L_C\right)g, C) &= 0.
\end{aligned} \tag{3.27}$$

We have the following

PROPOSITION 3.2.1. Let (M, g) , $n \geq 4$, be a semi-Riemannian manifold fulfilling the property (2.16). Then on the set $\mathcal{U} \subset M$ we have

$$Q(S - \left(\frac{\kappa}{n-1} - L_C\right)g, C) = 0.$$

According to earlier remark we restrict our consideration to the subset $\mathcal{U}$. If $x \in \mathcal{U}$ then $S - (\frac{\kappa}{n-1} - L_C)g \neq 0$ at x and applying Lemma 2.3.2 we have two cases depending on the rank of the tensor $S - (\frac{\kappa}{n-1} - L_C)g$.

(i) $S - (\frac{\kappa}{n-1} - L_C)g = \beta a \otimes a$.

In local coordinates, we have

$$a_h C_{ijkl} + a_i C_{jhkl} + a_j C_{hikl} = 0, \tag{3.28}$$

$$Q(Ric(C), C) = 0 \quad \Rightarrow \quad C \cdot C = 0. \tag{3.29}$$

(3.28) implies $a_r a^r = 0$. Now contracting

$S_{ij} = (\frac{\kappa}{n-1} - L_C)g_{ij} + \beta a_i a_j$ with g^{ij} we get

$$L_C = \frac{\kappa}{n(n-1)}. \tag{3.30}$$

Consequently we obtain in sequence:

$$\begin{aligned}
S &= \frac{\kappa}{n}g + \beta a \otimes a, \\
S_{ij}^2 &= \left(\frac{\kappa}{n}g_{ir} + \beta a_{ir} \otimes a_{ir}\right)\left(\frac{\kappa}{n}g_j^r + \beta a_j^r \otimes a_j^r\right) \\
&= \frac{\kappa^2}{n^2}g + 2\frac{\kappa}{n}\beta(a \otimes a).
\end{aligned}$$

Using the above identity in (3.13)

$$\begin{aligned}
D &= \frac{1}{n-2} \left(\frac{\kappa^2}{n^2} g + 2 \frac{\kappa}{n} \beta(a \otimes a) \right) - \frac{\kappa}{(n-1)(n-2)} \left(\frac{\kappa}{n} g + \beta a \otimes a \right) \\
&= \frac{\kappa}{n(n-1)} \left(\frac{\kappa}{n} g + \beta a \otimes a \right) - \frac{\kappa^2}{n^2(n-2)} g \\
&= L_C S - \frac{\kappa^2}{n^2(n-2)} g.
\end{aligned}$$

So we get $Q(g, D) = L_C Q(g, S)$ and using this with (3.21) we get

$$R \cdot S = L_C Q(g, S). \quad (3.31)$$

Applying (2.8) in the identity

$$\begin{aligned}
(C \cdot C)_{hijklm} &= g^{rs} (C_{rijk} C_{shlm} + C_{hrjk} C_{silm} + C_{hirk} C_{sjlm} + C_{hijr} C_{sklm}) \\
&= (C \cdot R)_{hijklm} - \frac{1}{n-2} \left[g_{hk} (C \cdot S)_{ijlm} + g_{ij} (C \cdot S)_{hklm} \right. \\
&\quad \left. - g_{hj} (C \cdot S)_{iklm} - g_{ik} (C \cdot S)_{hjlm} \right] \\
&\quad - \frac{1}{n-2} \left[C_{khlm} S_{ij} - C_{jhlm} S_{ik} + C_{jilm} S_{hk} - C_{kilm} S_{hj} \right. \\
&\quad \left. + C_{ijlm} S_{hk} - C_{hjlm} S_{ik} + C_{hklm} S_{ij} - C_{iklm} S_{hj} \right] \\
&\quad + \frac{\kappa}{(n-1)(n-2)} \left[C_{khlm} g_{ij} - C_{jhlm} g_{ik} + C_{jilm} g_{hk} \right. \\
&\quad \left. - C_{kilm} g_{hj} + C_{ijlm} g_{hk} - C_{hjlm} g_{ik} + C_{hklm} g_{ij} - C_{iklm} g_{hj} \right].
\end{aligned}$$

Then we obtain

$$\begin{aligned}
(C \cdot C)_{hijklm} &= (C \cdot R)_{hijklm} - \frac{1}{n-2} \left(g_{hk} (C \cdot S)_{ijlm} + g_{ij} (C \cdot S)_{hklm} \right. \\
&\quad \left. - g_{hj} (C \cdot S)_{iklm} - g_{ik} (C \cdot S)_{hjlm} \right). \quad (3.32)
\end{aligned}$$

Using (3.25) we get

$$\begin{aligned}
(C \cdot S)_{hijk} &= g^{rs} (S_{ri} C_{shjk} + S_{hr} C_{sijk}) = S_i^s C_{shjk} + S_h^s C_{sijk} \\
&= (n-1) L_C (C_{ihjk} + C_{hijk}) \\
&= 0.
\end{aligned}$$

Using this identity with (3.29) we get

$$C \cdot R = 0.$$

So $R \cdot C = L_C Q(g, C)$ and using (3.31) we have also

$$R \cdot R = L_C Q(g, R).$$

(ii) $\text{rank}(S - (\frac{\kappa}{n-1} - L_C)g) > 1$.

Let us put $\alpha = S - \frac{\kappa}{n-1} - L_C$ and $W = S - \alpha g$. According to Lemma 2.3.2, we have $C = \frac{\gamma}{2}W \wedge W$. Thus using (2.8) we get

$$R = \frac{1}{2}S \wedge S + \left(\frac{1}{n-2} - \alpha\gamma\right)g \wedge S + \left(\gamma\alpha^2 - \frac{\kappa}{(n-1)(n-2)}G\right).$$

Applying now Theorem 2.3.1 we obtain $R \cdot R = L_C Q(g, R)$. In the same manner as in the case (i) we obtain $C \cdot R = 0$. Thus we have proved

THEOREM 3.2.1. Let (M, g) , $n \geq 4$, be a semi-Riemannian manifold fulfilling $R \cdot C - C \cdot R = L_C Q(g, C)$. Then on $\mathcal{U} \in M$ we have $R \cdot R = L_C Q(g, R)$ and $C \cdot R = 0$.

COROLLARY 3.2.1. Let (M, g) , $n \geq 4$, be a semi-Riemannian manifold fulfilling $R \cdot C = C \cdot R$. Then on $\mathcal{U} \in M$ we have $R \cdot R = 0$ and $C \cdot R = 0$.

We have the following inverse statements.

PROPOSITION 3.2.2. Let (M, g) , $n \geq 4$, be a semi-Riemannian manifold satisfying

$$S = \mu g + \beta a \otimes a, \tag{3.33}$$

$$\sum_{X,Y,Z} a(X) \mathcal{C}(Y, Z) = 0. \tag{3.34}$$

Then on the set $\mathcal{U}$ we have $R \cdot R = \frac{\kappa}{n(n-1)} Q(g, R)$, $C \cdot R = 0$ and consequently

$$R \cdot C - C \cdot R = \frac{\kappa}{n(n-1)} Q(g, C).$$

PROOF. (3.33) implies that

$$C = R - \frac{1}{n-2}(2\mu - \frac{\kappa}{n-1})G - \frac{\beta}{n-2}g \wedge (a \otimes a)$$

and if we say

$$B = R - \frac{1}{n-2}(2\mu - \frac{\kappa}{n-1})G$$

then (3.34) will be equivalent to

$$\sum_{X,Y,Z} a(X) \mathcal{B}(Y, Z) = 0$$

because

$$\sum_{X,Y,Z} a(X) \left(g \wedge (a \otimes a) \right) (Y, Z) = 0.$$

Applying Lemma 2.3.1 we get $R \cdot R = \frac{\kappa}{n(n-1)} Q(g, R)$. Using (3.33) and the fact that $a^r a_r = 0$, we easily obtain $C \cdot S = 0$. On the other hand using (3.34) leads to $C \cdot C = 0$. So using (3.32) we get $C \cdot R = 0$.

PROPOSITION 3.2.3. Let (M, g) , $n \geq 4$, be a semi-Riemannian manifold satisfying the relation $C = \frac{\gamma}{2} W \wedge W$, where $W = S - \alpha g$. Then on the set $\mathcal{U} \subset M$ we have $C \cdot R = 0$ and $R \cdot R = L_R Q(g, R)$, where $L_R = \frac{\kappa}{n-1} - \alpha$. Consequently (2.16) holds on $\mathcal{U}$.

PROOF. Contracting the tensor $C_{hijk} = \gamma(W_{hk}W_{ij} - W_{hj}W_{ik})$ with g^{ij} we get

$$\begin{aligned} C_{hijk} &= \gamma(W_{hk}W_{ij} - W_{hj}W_{ik}) \\ g^{ij}C_{hijk} &= \gamma g^{ij}(W_{hk}W_{ij} - W_{hj}W_{ik}) \\ 0 &= \gamma(W_{hk}tr(W) - W_{hk}^2) \\ W^2 &= Wtr(W) \\ C \cdot W &= g^{rs} \left(W_{rj}(W_{sm}W_{il} - W_{sl}W_{im}) + W_{ir}(W_{sm}W_{jl} - W_{sl}W_{jm}) \right) \\ &= W_{mj}^2 W_{il} - W_{lj}^2 W_{im} + W_{im}^2 W_{lj} - W_{il}^2 W_{jm} \\ &= tr(W)(W_{mj}W_{il} - W_{lj}W_{im} + W_{im}W_{lj} - W_{il}W_{jm}) = 0 \\ C \cdot S &= C \cdot (W + \alpha g) = 0 \quad \text{and also} \quad C \cdot C = 0. \end{aligned} \tag{3.35}$$

Using (3.32) we get $C \cdot R = 0$. The curvature tensor R in virtue of (3.35) can be written in the form

$$R = \frac{\gamma}{2}(S \wedge S) - \left(\frac{1}{n-2} - \gamma\alpha \right) (g \wedge S) + \left(\gamma\alpha^2 - \frac{\kappa}{(n-1)(n-2)} \right) G.$$

Using Theorem 2.3.1 we get $R \cdot R = L_R Q(g, R)$.

4. SOME GENERALIZED EINSTEIN METRIC CONDITIONS ON HYPERSURFACES IN SEMI RIEMANNIAN SPACE FORMS

In this chapter we investigate curvature identities holding on every hypersurface M isometrically immersed in a semi-Riemannian space form.

4.1. Some Einstein Metric Conditions on Hypersurfaces

Using the definition of Kulkarni-Nomizu product we prove the following Lemma which will lead to two identities that will be used.

LEMMA 4.1.1. Let E_1, E_2 and F be symmetric $(0,2)$ -tensors at a point x of a semi-Riemannian manifold (M, g) , $n \geq 3$. Then at x we have

$$E_1 \wedge Q(E_2, F) + E_2 \wedge Q(E_1, F) = -Q(F, E_1 \wedge E_2).$$

If $E = E_1 = E_2$ then

$$E \wedge Q(E, F) = -Q(F, \overline{E}). \quad (4.1)$$

PROOF. Let $E_1 = A$ and $E_2 = B$ for simplicity

$$\begin{aligned} E_1 \wedge Q(E_2, F) + E_2 \wedge Q(E_1, F) &= (A \wedge Q(B, F))_{hijklm} + (B \wedge Q(A, F))_{hijklm} \\ &= A_{hk}Q(B, F)_{ijlm} + A_{ij}Q(B, F)_{hklm} - A_{hj}Q(B, F)_{iklm} \\ &\quad - A_{ik}Q(B, F)_{hjlm} + B_{hk}Q(A, F)_{ijlm} + B_{ij}Q(A, F)_{hklm} \\ &\quad - B_{hj}Q(A, F)_{iklm} - B_{ik}Q(A, F)_{hjlm} \\ &= F_{mh}(A \wedge B)_{lij k} + F_{mi}(A \wedge B)_{hljk} + F_{mj}(A \wedge B)_{hil k} \\ &\quad + F_{mk}(A \wedge B)_{hij l} - F_{lh}(A \wedge B)_{mijk} \\ &\quad - F_{li}(A \wedge B)_{hmjk} - F_{lj}(A \wedge B)_{him k} - F_{lk}(A \wedge B)_{hij m} \\ &= -Q(F, A \wedge B)_{hijklm} = -Q(F, E_1 \wedge E_2)_{hijklm}. \end{aligned}$$

Also using the definition of Kulkarni-Nomizu product

$$\begin{aligned}
Q(E, E \wedge F)_{hijklm} &= F_{mh}(E_{ij}E_{lk} - E_{ik}E_{lj}) + F_{mi}(E_{hk}E_{lj} - E_{lk}E_{hj}) \\
&\quad + F_{mj}(E_{hk}E_{li} - E_{lh}E_{ik}) + F_{mk}(E_{lh}E_{ij} - E_{li}E_{hj}) \\
&\quad - F_{lh}(E_{mk}E_{ij} - E_{mj}E_{ik}) - F_{li}(E_{mj}E_{hk} - E_{mk}E_{hj}) \\
&\quad + F_{lj}(E_{mi}E_{hk} - E_{mh}E_{ik}) - F_{lk}(E_{mh}E_{ij} - E_{mi}E_{hj}) \\
&= F_{mh}\bar{E}_{lijk} + F_{mi}\bar{E}_{hljk} + F_{mj}\bar{E}_{hilk} + F_{mk}\bar{E}_{hijl} \\
&\quad - F_{lh}\bar{E}_{mijk} - F_{li}\bar{E}_{hmjk} - F_{lj}\bar{E}_{himk} - F_{lk}\bar{E}_{hijm} \\
&= -Q(F, \bar{E})_{hijklm},
\end{aligned}$$

where $\bar{E} = \frac{1}{2}E \wedge E$ so we get

$$Q(E, E \wedge F) = -Q(F, \bar{E}). \quad (4.2)$$

As a consequence of (2.26) and (2.30) we get

$$E \wedge Q(E, F) = Q(E, E \wedge F). \quad (4.3)$$

PROPOSITION 4.1.1. Let M be a hypersurface in $N_s^{n+1}(c)$, $n \geq 4$.

- (i) $R \cdot S = Q(A, H) + \frac{\tau}{n(n+1)} Q(g, S)$ on M .
- (ii) On M ,

$$\begin{aligned}
R \cdot C &= Q(S, R) - \frac{1}{n-2} g \wedge Q(A, H) \\
&\quad + \frac{\tau}{n(n+1)} \left(\frac{1}{n-2} Q(S, G) - (n-2) Q(g, C) \right), \quad (4.4)
\end{aligned}$$

where

$$A = H^3 - \text{tr}(H)H^2 + \frac{\varepsilon\kappa}{n-1}H. \quad (4.5)$$

PROOF. Using (2.25) and (2.23)

$$\begin{aligned}
R \cdot S &= Q(H, \text{tr}(H)H^2 - H^3) + \frac{\varepsilon\tau}{n(n+1)} Q(g, \text{tr}(H)H - H^2) \\
&= Q(H, \frac{\varepsilon\kappa}{n-1}H - A) + \frac{\varepsilon\tau}{n(n+1)} Q(g, \varepsilon S - \frac{(n-1)\tau}{n(n+1)}g) \\
&= \frac{\varepsilon\kappa}{n-1} Q(H, H) + Q(A, H) + \frac{\varepsilon\tau}{n(n+1)} \left(Q(g, \varepsilon S) + \frac{(n-1)\tau}{n(n+1)} Q(g, g) \right) \\
&= Q(A, H) + \frac{\tau}{n(n+1)} Q(g, S).
\end{aligned}$$

Using (2.26) and (2.8)

$$\begin{aligned}
R \cdot C &= Q(S, R) - \frac{1}{n-2} \left[g \wedge \left(Q(A, H) + \frac{\tau}{n(n+1)} Q(g, S) \right) \right] \\
&\quad - \frac{(n-2)\tau}{n(n+1)} Q \left(g, C + \frac{1}{n-2} g \wedge S - \frac{\kappa}{(n-1)(n-2)} G \right) \\
&\quad - \frac{\tau}{n(n+1)} Q(S, G) \\
&= Q(S, R) - \frac{1}{n-2} g \wedge Q(A, H) - \frac{\tau}{n(n+1)(n-2)} g \wedge Q(g, S) \\
&\quad - \frac{(n-2)\tau}{n(n+1)} Q(g, C) - \frac{\tau}{n(n+1)} Q(g, g \wedge S) \\
&\quad + \frac{\kappa\tau}{n(n+1)(n-1)} Q(g, G) - \frac{\tau}{n(n+1)} Q(S, G) \\
&= Q(S, R) - \frac{1}{n-2} g \wedge Q(A, H) \\
&\quad + \frac{\tau}{n(n+1)} \left(\frac{1}{n-2} Q(S, G) - (n-2) Q(g, C) \right).
\end{aligned}$$

THEOREM 4.1.1. On every hypersurface M in $N_s^{n+1}(c)$, $n \geq 4$, the following identities hold

$$\begin{aligned}
R \cdot C &= Q(S, R) + \frac{1}{n-2} g \wedge Q(H, A) \\
&\quad - \frac{\tau(n-3)}{n(n+1)(n-2)} Q(S, G) - \frac{\tau(n-2)}{n(n+1)} Q(g, R), \quad (4.6)
\end{aligned}$$

$$\begin{aligned}
C \cdot R &= \frac{(n-3)}{(n-2)} Q(S, R) - \frac{(n^2-3n+3)\tau}{n(n+1)(n-2)} Q(g, R) \\
&\quad - \frac{(n-3)\tau}{(n-2)n(n+1)} Q(S, G) + \frac{1}{n-2} H \wedge Q(g, A), \quad (4.7)
\end{aligned}$$

$$\begin{aligned}
R \cdot C - C \cdot R &= \frac{1}{n-2} Q(S, R) + \frac{(n-1)\tau}{(n-2)n(n+1)} Q(g, R) \\
&\quad + \frac{1}{n-2} (g \wedge Q(H, A) - H \wedge Q(g, A)). \quad (4.8)
\end{aligned}$$

PROOF. Applying the relations (2.8) and (4.2) in (4.4) we get

$$\begin{aligned}
R \cdot C &= Q(S, R) - \frac{1}{n-2} g \wedge Q(A, H) \\
&\quad + \frac{\tau}{n(n+1)} \left[\frac{1}{n-2} Q(S, G) - (n-2) Q(g, C) \right] \\
&= Q(S, R) - \frac{1}{n-2} g \wedge Q(A, H) + \frac{\tau}{n(n+1)(n-2)} Q(S, G) \\
&\quad - \frac{\tau(n-2)}{n(n+1)} \left[Q(g, R) - \frac{1}{n-2} g \wedge S + \frac{\kappa}{(n-1)(n-2)} G \right]
\end{aligned}$$

$$\begin{aligned}
&= Q(S, R) - \frac{1}{n-2} g \wedge Q(A, H) + \frac{\tau}{n(n+1)(n-2)} Q(S, G) \\
&\quad - \frac{\tau(n-2)}{n(n+1)} Q(g, R) + \frac{\tau}{n(n+1)} Q(g, g \wedge S) - \frac{\tau\kappa}{n(n-1)(n+1)} Q(g, G) \\
&= Q(S, R) - \frac{1}{n-2} g \wedge Q(A, H) + Q(S, G) \left[\frac{\tau}{n(n+1)(n-2)} - \frac{\tau}{n(n+1)} \right] \\
&\quad - \frac{\tau(n-2)}{n(n+1)} Q(g, R) - \frac{\tau\kappa}{n(n-1)(n+1)} Q(g, G).
\end{aligned}$$

Using the fact that $Q(g, G) = 0$ we obtain (4.6).

Transvecting (2.22) with $H_l^r = g^{rh} H_{hl}$ we have

$$\begin{aligned}
H_l^r R_{rijk} &= \varepsilon(H_{kl}^2 H_{ij} - H_{jl}^2 H_{ik}) + \frac{\tau}{n(n+1)} (g_{ij} H_{kl} - g_{ik} H_{jl}) \\
(R \cdot H)_{iljk} &= g^{rs} (H_{sl} R_{rijk} + H_{is} R_{rljk}) \\
&= \varepsilon(H_{kl}^2 H_{ij} - H_{jl}^2 H_{ik} + H_{ki}^2 H_{lj} - H_{ji}^2 H_{lk}) \\
&\quad + \frac{\tau}{n(n+1)} (g_{ij} H_{kl} - g_{ik} H_{jl} + g_{lj} H_{ki} - g_{lk} H_{ji}) \\
&= \varepsilon Q(H, H^2)_{iljk} + \frac{\tau}{n(n+1)} Q(g, H)_{iljk}
\end{aligned}$$

which implies

$$R \cdot H = \varepsilon Q(H, H^2) + \frac{\tau}{n(n+1)} Q(g, H). \quad (4.9)$$

From (2.22) we also get

$$\begin{aligned}
R - \frac{1}{n-2} \left(g \wedge S - \frac{\kappa}{n-1} G \right) &= \varepsilon \bar{H} - \frac{1}{n-2} g \wedge S \\
&\quad + \left(\frac{\tau}{n(n+1)} + \frac{\kappa}{(n-1)(n-2)} \right) G.
\end{aligned}$$

By making use of (2.23) and (1.1), turns into

$$\begin{aligned}
C &= \varepsilon \bar{H} - \frac{1}{n-2} g \wedge \left(\varepsilon (tr(H)H - H^2) + \frac{\tau(n-1)}{n(n+1)} g \right) \\
&\quad + \left(\frac{\tau}{n(n+1)} + \frac{\kappa}{(n-1)(n-2)} \right) G \\
&= \varepsilon \bar{H} + \frac{\varepsilon}{n-2} g \wedge (H^2 - tr(H)H) \\
&\quad + \frac{1}{n-2} \left(\frac{\kappa}{n-1} - \frac{\tau}{n+1} \right) G.
\end{aligned} \quad (4.10)$$

Let us define the tensor $W = H^2 - tr(H)H$ for simplicity. Using (4.10)

$$(C \cdot H)_{hijk} = g^{rs} H_{ri} C_{shjk} + g^{rs} H_{hr} C_{sijk}$$

$$\begin{aligned}
&= g^{rs} H_{ri} \left[\varepsilon (H_{sk} H_{hj} - H_{sj} H_{hk}) + \frac{\varepsilon}{n-2} (g_{sk} W_{hj} + g_{hj} W_{sk} - g_{sj} W_{hk} - g_{hk} W_{sj}) \right. \\
&\quad \left. + \frac{1}{n-2} \left(\frac{\kappa}{n-1} - \frac{\tau}{n+1} \right) (g_{sk} g_{hj} - g_{sj} g_{hk}) \right] + g^{rs} H_{hr} \left[\varepsilon (H_{sk} H_{ij} - H_{sj} H_{ik}) \right. \\
&\quad \left. + \frac{\varepsilon}{n-2} (g_{sk} W_{ij} + g_{ij} W_{sk} - g_{sj} W_{ik} - g_{ik} W_{sj}) \right. \\
&\quad \left. + \frac{1}{n-2} \left(\frac{\kappa}{n-1} - \frac{\tau}{n+1} \right) (g_{sk} g_{ij} - g_{sj} g_{ik}) \right] \\
&= \varepsilon \frac{(n-3)}{n-2} Q(H, H^2)_{hijk} - \frac{\tau}{(n+1)(n-2)} Q(g, H)_{hijk} \\
&\quad + \frac{\varepsilon}{n-2} \left[-g_{hk} H_{ij}^3 - g_{hk} (-tr(H) H_{ij}^2) - g_{hk} \left(\frac{\kappa(n-2)}{\varepsilon(n-1)(n-2)} H_{ij} \right) \right. \\
&\quad + g_{ij} H_{hk}^3 + g_{ij} (-tr(H) H_{hk}^2) + g_{ij} \left(\frac{\kappa(n-2)}{\varepsilon(n-1)(n-2)} H_{hk} \right) \\
&\quad + g_{hj} H_{ik}^3 - g_{hj} (-tr(H) H_{ik}^2) + g_{hj} \left(\frac{\kappa(n-2)}{\varepsilon(n-1)(n-2)} H_{ik} \right) \\
&\quad \left. - g_{ik} H_{hj}^3 - g_{ik} (-tr(H) H_{hj}^2) - g_{ik} \left(\frac{\kappa(n-2)}{\varepsilon(n-1)(n-2)} H_{hj} \right) \right].
\end{aligned}$$

Thus we get

$$\begin{aligned}
(C \cdot H)_{hijk} &= \varepsilon \frac{(n-3)}{n-2} Q(H, H^2) \\
&\quad - \frac{\tau}{(n+1)(n-2)} Q(g, H) + \frac{\varepsilon}{n-2} Q(g, A), \tag{4.11}
\end{aligned}$$

where

$$A = H^3 - tr(H) H^2 + \frac{\varepsilon \kappa}{(n-1)} H.$$

Using this we get

$$\begin{aligned}
(C \cdot R)_{hijklm} &= g^{rs} (R_{rijk} C_{shlm} + R_{hrjk} C_{silm} + R_{hirk} C_{sjlm} + R_{hijr} C_{sklm}) \\
&= g^{rs} \left[(\varepsilon (H_{rk} H_{ij} - H_{rj} H_{ik}) + \frac{\tau}{n(n+1)} G_{rijk}) C_{shlm} \right. \\
&\quad + (\varepsilon (H_{hk} H_{rj} - H_{hj} H_{rk}) + \frac{\tau}{n(n+1)} G_{hrjk}) C_{silm} \\
&\quad + (\varepsilon (H_{hk} H_{ir} - H_{hr} H_{ik}) + \frac{\tau}{n(n+1)} G_{hirk}) C_{sjlm} \\
&\quad \left. + (\varepsilon (H_{hr} H_{ij} - H_{hj} H_{ir}) + \frac{\tau}{n(n+1)} G_{hijr}) C_{sklm} \right] \\
&= \varepsilon H_{hk} (g^{rs} H_{rj} C_{silm} + g^{rs} H_{ir} C_{sjlm}) + \varepsilon H_{ij} (g^{rs} H_{rk} C_{shlm} \\
&\quad + g^{rs} H_{hr} C_{sklm}) - \varepsilon H_{hj} (g^{rs} H_{rk} C_{silm} \\
&\quad + g^{rs} H_{ir} C_{sklm}) - \varepsilon H_{ik} (g^{rs} H_{rj} C_{shlm} + g^{rs} H_{hr} C_{sjlm}) \\
&\quad + g^{rs} \frac{\tau}{n(n+1)} \left(G_{rijk} C_{shlm} + G_{hrjk} C_{silm} + G_{hirk} C_{sjlm} + G_{hijr} C_{sklm} \right)
\end{aligned}$$

$$\begin{aligned}
&= (\varepsilon H \wedge (C \cdot H))_{hijklm} + \frac{\tau}{n(n+1)} \left(g_{ij}C_{khlm} - g_{ik}C_{jhlm} + g_{hk}C_{jilm} - g_{hj}C_{kilm} \right. \\
&\quad \left. + g_{hk}C_{ijlm} - g_{ik}C_{hjlm} + g_{ij}C_{hkml} - g_{hj}C_{iklm} \right) \\
&= (\varepsilon H \wedge (C \cdot H))_{hijklm}.
\end{aligned} \tag{4.12}$$

(4.11) in view of (4.9) leads to,

$$C \cdot H = \frac{n-3}{n-2}(R \cdot H) + \frac{\varepsilon}{n-2}Q(g, A) - \frac{(2n-3)\tau}{(n-2)n(n+1)}Q(g, H). \tag{4.13}$$

Using (4.13) in (4.12) we get

$$\begin{aligned}
C \cdot R &= \varepsilon H \wedge \left[\frac{n-3}{n-2}(R \cdot H) + \frac{\varepsilon}{n-2}Q(g, A) - \frac{(2n-3)\tau}{(n-2)n(n+1)}Q(g, H) \right] \\
&= \varepsilon \frac{n-3}{n-2}H \wedge (R \cdot H) + \frac{1}{n-2}H \wedge Q(g, A) \\
&\quad - \frac{(2n-3)\varepsilon\tau}{(n-2)n(n+1)}H \wedge Q(g, H).
\end{aligned}$$

Observe that

$$\begin{aligned}
(H \wedge (R \cdot H))_{hijklm} &= H_{hk}(R \cdot H)_{ijlm} + H_{ij}(R \cdot H)_{hklm} - H_{hj}(R \cdot H)_{iklm} \\
&\quad - H_{ik}(R \cdot H)_{hjlm} \\
&= g^{rs}R_{sil m}(H_{hk}H_{rj} - H_{hj}H_{rk}) + g^{rs}R_{sjlm}(H_{hk}H_{ir} - H_{ik}H_{hr}) \\
&\quad + g^{rs}R_{shlm}(H_{ij}H_{rk} - H_{ik}H_{rj}) + g^{rs}R_{sklm}(H_{ij}H_{hr} - H_{hj}H_{ir}) \\
&= g^{rs}(\bar{H}_{rijk}R_{shlm} + \bar{H}_{hrjk}R_{sil m} + \bar{H}_{hirk}R_{sjlm} + \bar{H}_{hijr}R_{sklm}) \\
&= (R \cdot \bar{H})_{hijklm}.
\end{aligned}$$

Also

$$\begin{aligned}
R \cdot R &= \varepsilon R \cdot \bar{H} + \frac{\tau}{n(n+1)}(R \cdot G) \Rightarrow R \cdot R = \varepsilon R \cdot \bar{H}, \\
Q(g, R) &= Q(g, \varepsilon \bar{H}) + \frac{\tau}{n(n+1)}Q(g, G) \Rightarrow Q(g, R) = \varepsilon Q(g, \bar{H}).
\end{aligned}$$

Using these relations(2.8),(4.3) and (2.30) we get

$$\begin{aligned}
C \cdot R &= \frac{n-3}{n-2}(R \cdot R) + \frac{1}{n-2}H \wedge Q(g, A) - \frac{(2n-3)\tau\varepsilon}{(n-2)n(n+1)}H \wedge Q(g, H) \\
&= \frac{n-3}{n-2} \left[Q(S, R) - \frac{(n-2)\tau}{n(n+1)}Q(g, C) \right] + \frac{1}{n-2}H \wedge Q(g, A) \\
&\quad - \frac{(2n-3)\tau}{(n-2)n(n+1)}Q(g, R)
\end{aligned}$$

By making use of (1.1) and (4.2) we get (4.7).

$$\begin{aligned}
C \cdot R &= \frac{n-3}{n-2} \left[Q(S, R) - \frac{(n-2)\tau}{n(n+1)} Q(g, R - \frac{1}{n-2} g \wedge S + \frac{\kappa}{(n-2)(n-1)} G) \right] \\
&\quad + \frac{1}{n-2} H \wedge Q(g, A) - \frac{(2n-3)\tau}{(n-2)n(n+1)} Q(g, R) \\
&= \frac{(n-3)}{(n-2)} Q(S, R) - \frac{(n^2-3n+3)\tau}{n(n+1)(n-2)} Q(g, R) \\
&\quad - \frac{(n-3)\tau}{(n-2)n(n+1)} Q(S, G) + \frac{1}{n-2} H \wedge Q(g, A).
\end{aligned}$$

Further using (4.12) together with (4.4) yields

$$\begin{aligned}
R \cdot C - C \cdot R &= \frac{1}{n-2} Q(S, R) - \frac{\tau}{n(n+1)} Q(g, C) \\
&\quad + \frac{1}{n-2} (g \wedge Q(H, A) - H \wedge Q(g, A)) \\
&\quad + \frac{\tau}{n(n+1)(n-2)} Q(S, G) + \frac{(2n-3)\tau}{(n-2)n(n+1)} Q(g, R).
\end{aligned}$$

Applying (1.1) and (4.2) we get (4.8).

THEOREM 4.1.2. Let M be a hypersurface in $N_s^{n+1}(c)$, $n \geq 4$. If

$$A = \left(\lambda + \frac{\varepsilon\kappa}{n-1} \right) H + \varrho g,$$

where

$$\varrho = \frac{1}{n} \left(\text{tr}(A) - \left(\lambda + \frac{\varepsilon\kappa}{n-1} \right) \text{tr}(H) \right) \quad (4.14)$$

is satisfied on $U_H \subset M$ then on this set we have

$$\begin{aligned}
R \cdot C - C \cdot R &= \frac{1}{n-2} Q(S, R) + \frac{(n-1)\tau}{(n-2)n(n+1)} Q(g, R) \\
&\quad + \frac{1}{n-2} \left(\varrho Q(H, G) - \varepsilon \left(\lambda + \frac{\varepsilon\kappa}{n-1} \right) Q(g, R) \right). \quad (4.15)
\end{aligned}$$

PROOF. Using (4.14) and (4.1) in the relation (4.8) we get

$$\begin{aligned}
R \cdot C - C \cdot R &= \frac{1}{n-2} Q(S, R) + \frac{(n-1)\tau}{(n-2)n(n+1)} Q(g, R) \\
&\quad + \frac{1}{n-2} \left(g \wedge Q(H, \left(\lambda + \frac{\varepsilon\kappa}{n-1} \right) H + \varrho g) \right. \\
&\quad \left. - H \wedge Q(g, \left(\lambda + \frac{\varepsilon\kappa}{n-1} \right) H + \varrho g) \right) \\
&= \frac{1}{n-2} Q(S, R) + \frac{(n-1)\tau}{(n-2)n(n+1)} Q(g, R) \\
&\quad + \frac{1}{n-2} \left(\varrho Q(H, G) - \left(\lambda + \frac{\varepsilon\kappa}{n-1} \right) Q(g, \varepsilon(R - \frac{\tau}{n(n+1)} G)) \right).
\end{aligned}$$

Hence we proved the theorem.

4.2. Hypersurfaces with $A = \lambda H + \varrho g$

PROPOSITION 4.2.1. Let (M, g) , $n \geq 4$, be a semi-Riemannian manifold. The following equalities are equivalent on M .

$$\sum_{(X_1, X_2), (X_3, X_4), (X, Y)} (R \cdot C)(X_1, X_2, X_3, X_4; X, Y) = 0, \quad (4.16)$$

$$\sum_{(X_1, X_2), (X_3, X_4), (X, Y)} (R \cdot C - C \cdot R)(X_1, X_2, X_3, X_4; X, Y) = 0, \quad (4.17)$$

$$\sum_{(X_1, X_2), (X_3, X_4), (X, Y)} (C \cdot R)(X_1, X_2, X_3, X_4; X, Y) = 0. \quad (4.18)$$

PROOF. First we will give well known Patterson and Walker identities [1] which will be used

$$Q(E, B)_{hijklm} + Q(E, B)_{jklmhi} + Q(E, B)_{lmhijk} = 0, \quad (4.19)$$

$$(R \cdot R)_{hijklm} + (R \cdot R)_{jklmhi} + (R \cdot R)_{lmhijk} = 0. \quad (4.20)$$

Using the identity (2.11) we get

$$\begin{aligned} (C \cdot R)_{hijklm} &= (R \cdot R)_{hijklm} - \frac{1}{n-2} Q(S, R)_{hijklm} \\ &\quad + \frac{\kappa}{(n-1)(n-2)} Q(g, R)_{hijklm} - \frac{1}{n-2} (g_{hl} V_{mijk} - g_{hm} V_{lij} \\ &\quad - g_{il} V_{mhjk} + g_{im} V_{lhjk} + g_{jl} V_{mkhi} - g_{jm} V_{lkhi} - g_{kl} V_{mjhi} + g_{km} V_{ljhi}), \end{aligned}$$

where

$$V_{mijk} = S_m^s R_{sijk}. \quad (4.21)$$

Now symmetrizing with respect to the pairs (h, i) , (j, k) , (l, m) and using the above identities

$$\begin{aligned} &(C \cdot R)_{lmhijk} + (C \cdot R)_{jklmhi} + (C \cdot R)_{hijklm} \\ &= (R \cdot R)_{hijklm} + (R \cdot R)_{jklmhi} + (R \cdot R)_{lmhijk} \\ &\quad - \frac{1}{n-2} \left(Q(S, R)_{jklmhi} + Q(S, R)_{lmhijk} + Q(S, R)_{hijklm} \right) \\ &\quad + \frac{\kappa}{(n-1)(n-2)} \left(Q(g, R)_{hijklm} + Q(g, R)_{jklmhi} + Q(g, R)_{lmhijk} \right) \\ &\quad - \frac{1}{n-2} \left[g_{hl} (V_{mijk} + V_{imjk}) - g_{hm} (V_{lij} + V_{iljk}) - g_{il} (V_{mhjk} + V_{hmjk}) \right. \\ &\quad + g_{im} (V_{lhjk} + V_{hljk}) + g_{jl} (V_{mkhi} + V_{kmhi}) - g_{jm} (V_{lkhi} + V_{klhi}) \\ &\quad - g_{kl} (V_{lkhi} + V_{jmhi}) + g_{km} (V_{ljhi} + V_{jlhi}) + g_{jh} (V_{iklm} + V_{kil m}) \\ &\quad \left. - g_{ij} (V_{hklm} + V_{khlm}) - g_{kh} (V_{ijlm} + V_{jilm}) + g_{ik} (V_{hjlm} + V_{jhlm}) \right] = P_{hijklm}. \end{aligned}$$

In the same way we get

$$(R \cdot C)_{lmhijk} + (R \cdot C)_{jklmhi} + (R \cdot C)_{hijklm} = -P_{hijklm}.$$

Using the last two relations we obtain

$$\begin{aligned} (R \cdot C - C \cdot R)_{lmhijk} &+ (R \cdot C - C \cdot R)_{jklmhi} \\ &+ (R \cdot C - C \cdot R)_{hijklm} = -P_{hijklm}. \end{aligned}$$

THEOREM 4.2.1. If on the subset $\mathcal{U}_H$ in a hypersurface M of $N_s^{n+1}(c)$, $n \geq 4$, one of the conditions (4.17) , (4.16) or (4.18) is satisfied then (4.14) holds on $\mathcal{U}_H$.

PROOF. Using Proposition 2.3.2 and (4.5), we have

$$\begin{aligned} A &= H^3 - tr(H)H^2 + \left(\frac{\varepsilon\kappa}{n-1}\right)H \\ &= \lambda H - \frac{\mu}{n}g + \left(\frac{\varepsilon\kappa}{n-1}\right)H \\ &= \left(\lambda + \frac{\varepsilon\kappa}{n-1}\right)H + \frac{1}{n}\left(tr(H^3) - tr(H)tr(H^2) - \lambda tr(H)\right)g \\ &= \left(\lambda + \frac{\varepsilon\kappa}{n-1}\right)H + \frac{1}{n}\left(tr(A) - \frac{\varepsilon\kappa}{n-1}tr(H) - \lambda tr(H)\right)g. \end{aligned}$$

Thus we get (4.14).

Using (4.19), (4.20) and Proposition 3.2.1 we immediately get

COROLLARY 4.2.1. If on the subset $\mathcal{U}_H$ in a hypersurface M of $N_s^{n+1}(c)$, $n \geq 4$, one of the tensors $R \cdot C$, $C \cdot R$ or $R \cdot C - C \cdot R$ is a linear combination of $R \cdot R$ and of a finite sum of tensors of the form $Q(E, B)$ where E is a symmetric (0,2)-tensor and B a generalized curvature tensor , then (4.14) holds on $\mathcal{U}_H$.

Using the Theorem 3.2.1 and the Corollary 3.2.1 we prove

THEOREM 4.2.2. Let M be a hypersurface in $N_s^{n+1}(c)$, $n \geq 4$. If at every point of M the following two tensors are linearly dependent:

- (i) $R \cdot C - C \cdot R$ and $Q(g, C)$, or
- (ii) $R \cdot C - C \cdot R$ and $Q(g, R)$, or
- (iii) $R \cdot C - C \cdot R$ and $Q(S, R)$, or
- (iv) $R \cdot C - C \cdot R$ and $Q(S, C)$,

then (4.14) and (4.15) hold on $\mathcal{U}_H \subset M$.

PROOF. It is known that $\mathcal{U}_H \subset \mathcal{U}_S \cap \mathcal{U}_C$ so on $\mathcal{U}_S \cap \mathcal{U}_C$ (i) and (ii) can be written as

$$R \cdot C - C \cdot R = L_C Q(g, C) \quad \text{and} \quad R \cdot C - C \cdot R = L_S Q(g, R),$$

where L_C and L_S are functions defined on $\mathcal{U}_C$ and $\mathcal{U}_S$, respectively. Then using Corollary 3.2.1 we get (4.14).

Consider case (iii) and assume that $Q(S, R) = 0$ at $x \in \mathcal{U}_H$. Then using (2.30)

$$R \cdot R = -\frac{(n-2)\tau}{n(n+1)} Q(g, C).$$

Also using (3.7)

$$\begin{aligned} g^{rs} Q(S, R)_{hrsklm} &= A_{lkhm} - A_{lhmk} - A_{mkhl} + A_{mhlk} \\ 0 &= A_{lkhm} + A_{lhmk} + A_{mhlk} + A_{mkhl} \\ R \cdot S &= 0. \end{aligned}$$

Thus using these in (3.10) we have

$$\begin{aligned} R \cdot C &= R \cdot R - \frac{1}{n-2} g \wedge (R \cdot S), \\ R \cdot C &= -\frac{(n-2)\tau}{n(n+1)} Q(g, C). \end{aligned}$$

Applying Corollary 3.2.1 we get (4.14). Clearly if $Q(S, R)$ is nonzero then Corollary 3.2.1 implies (4.14).

Finally for the case (iv) Let us assume that $Q(S, C)$ is nonzero at $x \in \mathcal{U}_H$. Then it is obvious that Corollary 3.2.1 implies (4.14). Assume now $Q(S, C) = 0$ at x . Then we have $R \cdot R = \frac{\kappa}{n-1} Q(g, R)$. This yields $R \cdot S = \frac{\kappa}{n-1} Q(g, S)$. Using (2.8) and (4.3) we find

$$\begin{aligned} R \cdot C &= R \cdot R - \frac{1}{n-2} g \wedge (R \cdot S) \\ &= \frac{\kappa}{n-1} Q(g, R) - \frac{\kappa}{(n-2)(n-1)} g \wedge Q(g, S) \\ &= \frac{\kappa}{n-1} Q(g, R) - \frac{\kappa}{(n-2)(n-1)} Q(g, g \wedge S) \\ &= \frac{\kappa}{n-1} \left(Q(g, R) - \frac{\kappa}{n-2} Q(g, g \wedge S) \right) \\ &= \frac{\kappa}{n-1} Q(g, C). \end{aligned}$$

In the view of Corollary 3.2.1 we obtain (4.14) on $\mathcal{U}_H$. From Theorem 3.1.2 it follows that (4.15) holds on $\mathcal{U}_H$. This completes the proof.

PROPOSITION 4.2.2. Let (M, g) , $n \geq 4$, be a semi-Riemannian manifold. If at a point $x \in \mathcal{U}_S \cap \mathcal{U}_C$ its curvature tensor R is of the form

$$R = \phi \bar{S} + \mu g \wedge S + \eta G, \quad \phi, \mu, \eta \in \mathbb{R}, \quad (4.22)$$

then at x we have

$$\begin{aligned} R \cdot C - C \cdot R &= \left(\frac{1}{\phi} \left(\mu - \frac{1}{n-2} \right) + \frac{\kappa}{n-1} \right) Q(g, R) \\ &\quad + \left(\frac{\mu}{\phi} \left(\mu - \frac{1}{n-2} \right) - \eta \right) Q(S, G). \end{aligned} \quad (4.23)$$

PROOF. Contracting (4.22) with g^{ij} we get

$$\begin{aligned} R_{hijk} &= \phi(S_{hk}S_{ij} - S_{hj}S_{ik}) + \mu(g_{hk}S_{ij} + g_{ij}S_{hk} - g_{hj}S_{ik} - g_{ik}S_{hj}) \\ &\quad + \eta(g_{hk}g_{ij} - g_{hj}g_{ik}), \\ S_{hk} &= \phi(S_{hk}\kappa - S_{hk}^2) + \mu(g_{hk}\kappa + S_{hk}n - S_{hk} - S_{hk}) + \eta(g_{hk}n - g_{hk}), \\ S_{hk}^2 &= \left(\frac{\phi\kappa + \mu(n-2) - 1}{\phi} \right) S_{hk} + \left(\frac{\mu\kappa + \eta(n-1)}{\phi} \right) g_{hk}, \\ S^2 &= \bar{\alpha}S + \bar{\beta}g \end{aligned} \quad (4.24)$$

where

$$\bar{\alpha} = \kappa + \frac{\mu(n-2) - 1}{\phi} \quad \text{and} \quad \bar{\beta} = \frac{\mu\kappa + \eta(n-1)}{\phi}.$$

Using the relation (4.24) in (4.21) we have

$$\begin{aligned} V_{mijk} &= S_m^h R_{hijk} \\ &= \phi(S_{mk}^2 S_{ij} - S_{mj}^2 S_{ik}) + \mu(S_{mk}S_{ij} \\ &\quad + g_{ij}S_{mk}^2 - S_{mj}S_{ik} - g_{ik}S_{mj}^2) + \eta(S_{mk}g_{ij} - S_{mj}g_{ik}) \\ &= \phi \left((\bar{\alpha}S_{mk} + \bar{\beta}g_{mk})S_{ij} - (\bar{\alpha}S_{mj} + \bar{\beta}g_{mj})S_{ik} \right) \\ &\quad + \mu \left(S_{mk}S_{ik} + g_{ij}(\bar{\alpha}S_{mk} + \bar{\beta}g_{mk}) - S_{mj}S_{ik} - g_{ik}(\bar{\alpha}S_{mj} + \bar{\beta}g_{mj}) \right) \\ &\quad + \eta(S_{mk}g_{ij} - S_{mj}g_{ik}) \\ &= (\phi\bar{\alpha} + \mu)(S_{mk}S_{ij} - S_{mj}S_{ik}) + (\mu\bar{\alpha} + \eta)(S_{mk}g_{ij} - S_{mj}g_{ik}) \\ &\quad + \phi\bar{\beta}(S_{mk}g_{ij} - S_{mj}g_{ik}) + \mu\bar{\beta}G_{mijk} \\ &= (\alpha + \mu)(S_{mk}S_{ij} - S_{mj}S_{ik}) + \left(\frac{\alpha\mu}{\phi} + \eta \right) (S_{mk}g_{ij} - S_{mj}g_{ik}) \\ &\quad + \beta(g_{mk}S_{ij} - g_{mj}S_{ik}) + \frac{\beta\mu}{\phi} G_{mijk}, \end{aligned} \quad (4.25)$$

where

$$\alpha = \phi\kappa - 1 + (n-2)\mu, \quad \beta = \mu\kappa + (n-1)\eta.$$

Using (4.25) we get

$$\begin{aligned} (R \cdot S)_{mijk} &= g^{rs}(S_{ri}R_{smjk} + S_{mr}R_{sijk}) \\ &= S_i^s R_{smjk} + S_m^s R_{sijk} \\ &= V_{imjk} + V_{mijk} \\ R \cdot S &= (n-2)\left(\frac{\mu}{\phi}\left(\mu - \frac{1}{n-2}\right) - \eta\right)Q(g, S). \end{aligned} \quad (4.26)$$

Using (2.12) we get

$$\begin{aligned} (n-2)(R \cdot C - C \cdot R)_{hijklm} &= Q(S, R)_{hijklm} - \frac{\kappa}{(n-1)}Q(g, R)_{hijklm} \\ &+ g_{hl}V_{mijk} - g_{hm}V_{lij} - g_{il}V_{mhjk} + g_{im}V_{lhjk} + g_{jl}V_{mkhi} - g_{jm}V_{lkhi} \\ &- g_{kl}V_{mjhi} + g_{km}V_{ljhi} - g_{ij}(R \cdot S)_{hklm} - g_{hk}(R \cdot S)_{ijlm} \\ &+ g_{ik}(R \cdot S)_{hjlm} + g_{hj}(R \cdot S)_{iklm} \\ &= Q(S, R) - \frac{\kappa}{(n-1)}Q(g, R) + (\alpha + \mu)Q(g, \bar{S}) \\ &- (n-2)\left(\frac{\mu}{\phi}\left(\mu - \frac{1}{n-2}\right) - \eta\right)(g \wedge Q(g, S)) \\ &- \left(\frac{\alpha\mu}{\phi} + \eta\right)Q(S, G). \end{aligned} \quad (4.27)$$

By using (4.22) we obtain

$$\begin{aligned} Q(g, R) &= \phi Q(g, \bar{S}) + \mu Q(g, g \wedge S) + \eta Q(g, G), \\ Q(S, R) &= -\frac{\mu}{\phi}Q(g, R) + \left(\eta - \frac{\mu^2}{\phi}\right)Q(S, G). \end{aligned}$$

If we substitute all these relations into the (4.27) we get (4.23).

4.3. Hypersurfaces with $H^3 = \text{tr}(H)H^2 - \frac{\varepsilon\kappa}{n-1}H$.

We now present an example of a hypersurface satisfying the condition (4.14).

EXAMPLE 4.3.1.[4] Let M be a hypersurface in a Euclidean space E^{n+1} , $n \geq 4$, having three principal curvatures: $0, \sqrt{\gamma}, -\sqrt{\gamma}$ with multiplicities $\frac{n+2p}{3}, \frac{n-p}{3}$ and $\frac{n-p}{3}$, respectively, where $n-p = 3, 6, 12$ or $24, p \geq 1$, and γ is a positive function on M . The hypersurface M is a non-quasi-Einstein

Ricci-semisymmetric manifold. Moreover, if $n - p = 6, 12$ or 24 then M is a non-semisymmetric manifold. It is easy to check that on M we have:

$$\begin{aligned} \text{tr}(H) &= 0, \quad S = -H^2, \quad \kappa = -\frac{2(n-p)\gamma}{3}, \\ H^3 &= \text{tr}(H)H^2 + \gamma H = -\frac{3\kappa}{2(n-p)}H. \end{aligned}$$

Now the relation $H^3 = \lambda H$, where $\lambda = -\frac{3\kappa}{2(n-p)}$, yields (4.14).

THEOREM 4.3.1. Let M be a hypersurface in $N_s^{n+1}(c)$, $n \geq 4$. On $U_H \subset M$ the condition $A = 0$ is equivalent to

$$R \cdot C - C \cdot R = \frac{1}{n-2} Q(S, R) + \frac{(n-1)\tau}{(n-2)n(n+1)} Q(g, R). \quad (4.28)$$

PROOF. Clearly, $A = 0$ by (4.8) implies (4.28). Now assume that (4.28) holds on U_H . Then (4.8) reduces to

$$g \wedge Q(H, A) - H \wedge Q(g, A) = 0.$$

In the virtue of (4.3) and (4.14), we get

$$\begin{aligned} \varrho g \wedge Q(H, g) &= \left(\lambda + \frac{\varepsilon \kappa}{n-1} \right) H \wedge Q(g, H) \\ &= -\varrho g \wedge Q(g, H) + \left(\lambda + \frac{\varepsilon \kappa}{n-1} \right) H \wedge Q(H, g) \\ &= -\varrho g \wedge Q(g, H) + \left(\lambda + \frac{\varepsilon \kappa}{n-1} \right) H \wedge Q(H, g) \\ &= 0. \end{aligned}$$

Thus we have

$$Q \left(\left(\lambda + \frac{\varepsilon \kappa}{n-1} \right) H - \varrho g, g \wedge H \right) = 0. \quad (4.29)$$

We prove that

$$A = \left(\lambda + \frac{\varepsilon \kappa}{n-1} \right) H + \varrho g = 0.$$

First we assert that

$$\left(\lambda + \frac{\varepsilon \kappa}{n-1} \right) = 0.$$

Suppose not; then (4.29) can be written in the form $Q(H - \alpha g, g \wedge H) = 0$, $\alpha \in \mathbb{R}$. Applying Lemma 2.3.5 we have a contradiction because $x \in U_H$ so we have

$$\varrho Q(g, g \wedge H) = 0.$$

Suppose that $\varrho \neq 0$. Then using (4.1) , $Q(H, G) = 0$. Applying Lemma 2.3.2 we get a contradiction. Therefore we have $\varrho = 0$ and $A = 0$.

COROLLARY 4.3.1. Let M be a hypersurface in $\mathbb{E}_s^{n+1}$, $n \geq 4$. On $U_H \subset M$ the condition $A = 0$ is equivalent to

$$R \cdot C - C \cdot R = \frac{1}{n-2} Q(S, R).$$

5. RESULTS AND DISCUSSION

Every semi-Riemannian Einstein manifold satisfies the condition $R \cdot C - C \cdot R = \frac{\kappa}{n(n-1)}Q(g, C)$. Any non-Einstein, non-conformally flat manifold satisfying the condition $R \cdot C - C \cdot R = LQ(g, C)$ is pseudosymmetric and in addition the condition $C \cdot R = 0$ is satisfied on that manifold.

Using the curvature identities holding for any hypersurface immersed isometrically in a semi-Riemannian space form, under some assumptions other generalized Einstein metric conditions can be obtained.

In the future, we aim to investigate new curvature conditions on hypersurfaces in semi-Riemannian space forms.

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