İSTANBUL TECHNICAL UNIVERSITY ★ INSTITUTE OF SCIENCE AND TECHNOLOGY

APPLICATION OF DTM TO LINEAR-NONLINEAR ENGINEERING PROBLEMS

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CONTENTS

ABBREVIATIONS

DTM : Differential Transform Method
BVP : Boundary Value Problem
B.C. : Boundary Condition
2D : 2 Dimensional
3D : 3 Dimensional

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L	: Beam length
Ω	: The angular velocity of the beam
a	: The radius of hub
A	: Cross section area of the beam
ω_w	: Natural Frequency
p_w	: The force applied per unit length in the *z* direction
E	: Modulus of elasticity
I	: Moment of inertia
η , ζ	: Similarity variables
N	: Number of terms taken
m	: An arbitrary large number
β	: Wedge angle/ π
$U(x)$	: Potential velocity distribution
u_t	: The tangent velocity to the wedge
K_n	: Knudsen Number
γ	: Slip factor defined in section 2.4
σ	: Tangential momentum accomodation coefficient
λ	: Mean free path
A_p	: $\lambda(2-\sigma)/\sigma$
$c(\gamma,\beta)$	: Surface skin friction
ν	: Kinematic viscosity
ρ	: Density
u	: Velocity component in radial direction
v	: Velocity component in circumferential direction
w	: Velocity component in axial direction
F, G, H	: Velocity components defined by Karman
P	: Pressure defined by Karman
P_0	: Value of pressure at $\zeta = 0$
ω	: Slip factor defined as $\left[(2-\sigma)\lambda\Omega^{1/2}\right]/\sigma\nu^{1/2}$
f_1, g_1	: Unknown B.C.'s
ξ	: Benton's independant variable

DTM'NİN LİNEER-NONLİNEER MÜHENDİSLİK PROBLEMLERİNE UYGULANMASI

ÖZET

Bu çalışmada, Diferansiyel Transform Metodu (DTM) dört mühendislik problemine uygulanmıştır. Bunlardan ilki, lineer bir özdeğer problemi olan dönen bir kirişin eğilme modlarının belirlenmesidir. Söz konusu kiriş Euler-Bernoulli kirişi olarak modellenmiştir ve homojen malzeme özelliklerine sahiptir. Sonuçlar beşinci doğal frekansa kadar elde edilmiştir. İkincisi ise düz levha sınır tabakası ile ilgili olarak Blasius Denklemidir. Bu problemdeki bilinmeyen sınır şartı olan yüzey gerilmesi açık formda elde edilebilmiştir. İç ve dış bölgelere DTM uygulanmış ve grafiksel olarak, neden bu iki çözümün denkleştirilmesine ihtiyaç olduğu açıklanmıştır. Üçüncüsü 2B bir köşe etrafındaki akımın Falkner-Skan benzerlik çözümüdür. Sonuncusu ise, serbest bir şekilde dönen sonsuz bir disk üzerindeki 3B akım alanıdır. Ayrıca dönen disk ve köşe problemlerine Navier'in kayma şartı uygulanmıştır. Knudsen sayısının $0.1 < Kn < 0.01$ aralığında geçerli olan bu şart $Kn < 0.01$ için kaymama sınır şartı olarak değiştirilir. Bu iki problem için, kaymanın akım alanı parametrelerine etkisi, detaylı bir şekilde incelenmiştir. Blasius ve Falkner-Skan denklemlerinin çözümünde, iç (yakın) ve dış (uzak) çözümler elde edilmiş ve daha sonra uygun bir değerde denkleştirilmişlerdir. Ayrıca Falkner-Skan denkleminin çözümünde parçalara ayırma ve shooting tekniklerinin birleşimi bir alternatif olarak sunulmuştur. Metodun yakınsaklığının bölge uzunluğu ile değişimine örnek olarak Heimenz akışı (durma noktası akışı) ve kaymama sınır şartı verilmiştir. Hassasiyet gelişimleri ve sonuçlar, ayrıntılı ve daha önceki sonuçlarla kıyas içinde sunulmuştur. Bulunan sonuçların literatür ile uyum içinde olduğu gösterilmiştir.

APPLICATION OF DTM TO LINEER-NONLINEER ENGINEERING PROBLEMS

SUMMARY

In this study, Differential Transform Method (DTM) is applied to four engineering problems. The first one is a linear eigenvalue problem, which is the determination of the bending modes of a rotating cantilever beam. The beam is modeled as an Euler-Bernoulli beam with homogeneous material properties. The results are evaluated up to the fifth natural frequency. The second one is Blasius Equation, which is related to the flat plate boundary layer. The unknown boundary condition, namely the surface friction, is expressed explicitly in this problem. DTM is applied to both inner and outer regions graphically it is explained why there is a need of the matching of these two solutions. The third one is Falkner Skan’s similarity solution of the flow over a 2D wedge. And the last one is the 3D flow field over an infinite disc rotating freely. Also Navier’s slip condition is applied to the rotating disc and the wedge problems. This condition is valid when Knudsen number is in the range of $0.1 < Kn < 0.01$ and for $Kn < 0.01$ it shall be replaced with the no-slip boundary condition. For these two problems, effect of slip to the flow field parameters is investigated in detail. In the solution of Blasius and Falkner-Skan equations, an inner (close) and an outer (far away) solution are obtained and matched at a suitable value. Also in the solution of the Falkner-Skan Equation, the domain splitting technique combined with the shooting method is introduced as an alternative. The convergence of the method with respect to the domain size is given as an example for Heimenz flow (stagnation point flow) and no slip boundary condition. Accuracy development and results are given in detail with comparison to the already existing ones. The results are in good agreement with open literature.

1. INTRODUCTION

Differential Transform Method is based on Taylor series expansion and can be applied to eigenvalue, initial value or boundary value problems. It was first introduced by Zhou [38] in a study about electrical circuits. Then, numerous studies about DTM are exposed being both as theoretical and application. DTM can be applied to any kind of differential equation or equation system being linear, nonlinear, partial or ordinary. For two and three-dimensional DTM and its applications, the reader can refer to Ayaz [36, 37]. However in this study only one-dimensional transform will be used.

In section 2.1, the method and related theorems will be introduced. Then its application to the determination of the bending modes of a rotating cantilever beam, which is a linear eigenvalue problem, will be discussed in section 2.2. In section 2.3, the well-known flat plate boundary layer similarity solution of Blasius, which is a nonlinear boundary value problem (BVP), will be discussed. In this section a new method for solving infinite domain BVP's will be introduced and named as "Matching of the Inner and the Outer Solutions". After that, in section 2.4, Falkner-Skan's solution of the flow over a wedge will be considered with slip boundary condition. The problem is solved by using the matching technique given in section 2.3 and then, as an alternative, a new method that combines the shooting method with domain splitting is introduced. At last, in section 2.5, the flow over a rotating infinite disc is studied with the use of von Karman's [22] similarity solution and Benton's [34] coordinate transformation. The effect of slip to the flow field is shown in detail.

2. DTM AND ITS APPLICATIONS

In the following sections, firstly the method and the theorems will be given. Then their applications to the recent engineering problems will be discussed.

2.1. Differential Transform Method

The transformation of the nth derivative of a function in one variable is as follows.

$$F(k) = \frac{1}{k!}\left[\frac{d^k f}{dx^k}(x)\right]_{x=x_0} \tag{2.1}$$

And the inverse transformation is defined as

$$f(x) = \sum_{k=0}^{\infty} F(k)(x-x_0)^k \tag{2.2}$$

Theorems that are frequently used in the transformation of the differential equations, which can be evaluated from Eqs. (2.1) and (2.2), are given below [1].

Theorem 1.

If $f(x) = g(x) \pm h(x)$, then $F(k) = G(k) \pm H(k)$

Theorem 2.

If $f(x) = cg(x)$, then $F(k) = cG(k)$. Where, c is a constant

Theorem 3.

If $f(x) = \dfrac{d^n g(x)}{dx^n}$, then $F(k) = \dfrac{(k+n)!}{k!} G(k+n)$

Theorem 4.

If $f(x) = g(x)h(x)$, then $F(k) = \sum_{l=0}^{k} G(l)H(k-l)$

Theorem 5.

If $f(x) = x^n$, then $F(k) = \delta(k-n)$

Where, $\delta(k-n) = \begin{cases} 1 & k = n \\ 0 & k \neq n \end{cases}$

2.2. Flapwise-Bending Vibration Analysis of a Rotating Beam

In this section, the modal analysis of flapwise vibration of a rotating cantilever beam described by the Euler-Bernoulli model is studied. The method is recently applied to vibration analysis of a tapered bar [2].

Chung and Yoo developed a new dynamic modeling method using stretch deformation [3]. They showed that, two of the linear differential equations are coupled through the stretch and chordwise deformations. On the other hand the differential equation related with flapwise deformation is uncoupled.

2.2.1. Equation of Motion

Consider a cantilever beam of a length L, which is fixed at point O of a rigid hub with radius a, as shown in Fig. 2.1. The beam is modeled as the Euler-Bernoulli beam and it has homogeneous, uniform and isotropic material properties. The hub is rotating about the axis of symmetry with a rotating speed Ω.

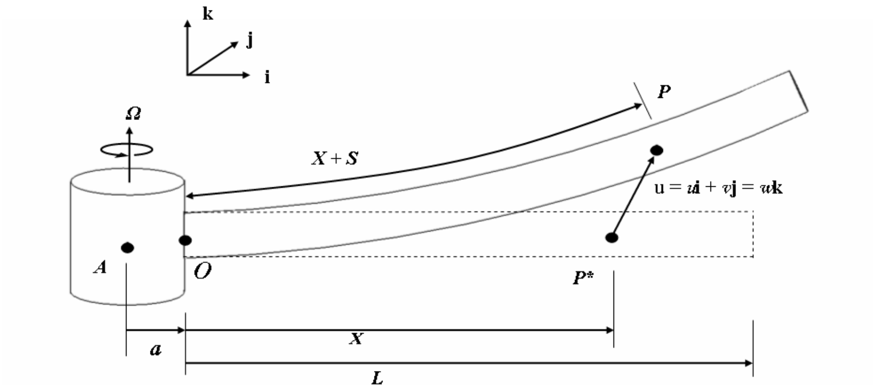

Figure 2.1. Configuration of a rotating cantilever beam

The flapwise bending vibration is governed by the following uncoupled equation of motion [3].

$$\rho \mathrm{A} \frac{\partial^2 w}{\partial t^2} + \mathrm{E\,I} \frac{\partial^4 w}{\partial x^4} - \rho \mathrm{A} \Omega^2 \frac{\partial}{\partial x} \left\{ \left[a\,(\mathrm{L} - \mathrm{x}) + \frac{1}{2}(L^2 - x^2) \right] \frac{\partial w}{\partial x} \right\} = p_w \qquad (2.3)$$

where p_w is the applied force per unit length in the z direction. The boundary conditions are given by

$$\mathrm{w} = \frac{\partial \mathrm{w}}{\partial \mathrm{x}} = 0 \text{ at } x = 0 \qquad (2.4)$$

$$\frac{\partial^2 w}{\partial x^2} = \frac{\partial^3 w}{\partial x^3} = 0 \text{ at } \mathrm{x} = \mathrm{L} \qquad (2.5)$$

2.2.2. Nondimensionalising of the Problem

Following the reference [3], dimensionless parameters are introduced as follows.

$$\tau = \frac{\mathrm{t}}{\mathrm{T}},\ \xi = \frac{\mathrm{x}}{\mathrm{L}},\ \delta = \frac{\mathrm{a}}{\mathrm{L}},\ \gamma = \mathrm{T}\,\Omega \qquad (2.6)$$

where $T = \sqrt{\dfrac{\rho \mathrm{A} \mathrm{L}^4}{\mathrm{E\,I}}}$

The dimensionless natural frequencies are found from the following equation, which is the nondimensionalized form of Eq. (2.3) with the help of Eq. (2.6).

$$\frac{d^4w}{d\xi^4} - \gamma^2\frac{d}{d\xi}\left\{\left[\delta(1-\xi)+\frac{1}{2}\left(1-\xi^2\right)\right]\frac{dw}{d\xi}\right\} - \omega_w^2 w = 0 \tag{2.7}$$

The B.C.'s stated in Eqs. (2.4) and (2.5) become

$$\xi = 0 \rightarrow w = \frac{dw}{d\xi} = 0 \tag{2.8}$$

$$\xi = 1 \rightarrow \frac{d^2w}{d\xi^2} = \frac{d^3w}{d\xi^3} = 0 \tag{2.9}$$

2.2.3. The Solution

After applying Theorems 1-5 to Eqs. (2.7)-(2.9), the following relations are obtained,

$$W(0) = W(1) = 0 \tag{2.10}$$

$$\sum_{k=2}^{\infty} k(k-1)W(k) = \sum_{k=3}^{\infty} k(k-1)(k-2)W(k) = 0 \tag{2.11}$$

$$\begin{aligned} W[k+4] = &-\frac{1}{2(k+1)(k+2)(k+3)(k+4)}\Big\{\left(k(k+1)\gamma^2 - 2\omega_w^2\right)W[k] \\ &+\gamma^2(k+1)\left[2(k+1)\delta W[k+1] - (k+2)(2\delta+1)W[k+2]\right]\Big\} \end{aligned} \tag{2.12}$$

where $W(k)$ is the differential transformation of $w(\xi)$. By using Eqs. (2.10) and (2.12), $W(k)$'s for $k = 4,5,6,7...$can be evaluated in terms $\delta, \omega_w\ c_2, c_3$ and γ, by the usage of Mathematica 4.0, as follows:

$$W(4) = \frac{1}{24}(1+2\delta)\gamma^2 c_2$$

$$W(5) = \frac{1}{120}\left[-2\delta\gamma^2 c_2 + (3+6\delta)\gamma^2 c_3\right]$$

$$W(6) = \frac{1}{1440}\left\{\left[(-12-12\delta)\gamma^2 + \left(1+4\delta+4\delta^2\right)\gamma^4 + 4\omega_w^2\right]c_2 - 24\delta\gamma^2 c_3\right\}$$

$$W(7) = \frac{1}{10080}\left\{\left(-8\delta-16\delta^2\right)\gamma^4 c_2 + \left[(-72-48\delta)\gamma^2 + \left(3+12\delta+12\delta^2\right)\gamma^4 + 12\omega_w^2\right]c_3\right\}$$

$$W(8)=\frac{1}{161280}\left\{\begin{matrix}\left[\left(8\omega_w^2+16\delta\omega_w^2\right)\gamma^2+\left(-52-136\delta-32\delta^2\right)\gamma^4+\left(1+6\delta+12\delta^2+8\delta^3\right)\gamma^6\right]c_2+\\ \left(-72\delta+144\delta^2\right)\gamma^4 c_3\end{matrix}\right\}$$

$$W(9)=\frac{1}{483840}\left\{\begin{matrix}\left[-16\delta\gamma^2\omega_w^2+\left(80\delta+56\delta^2\right)\gamma^4+\left(-6\delta-24\delta^2-24\delta^3\right)\gamma^6\right]c_2+\\ \left[\left(8\omega_w^2+16\delta\omega_w^2\right)\gamma^2+\left(-84-208\delta\right)\gamma^4+\left(1+6\delta+12\delta^2+8\delta^3\right)\gamma^6\right]c_3\end{matrix}\right\}$$

The coefficients are obtained exactly in rational arithmetic and the constants c_2 and c_3 that appear in $W(k)$'s are as follows.

$$c_2=W(2)=\frac{1}{2!}\left(\frac{d^2w}{d\xi^2}\right)_{\xi=0},c_3=W(3)=\frac{1}{3!}\left(\frac{d^3w}{d\xi^3}\right)_{\xi=0} \qquad (2.13)$$

With the B.C. stated in (2.11), it finally becomes an eigenvalue problem of the form:

$$\begin{bmatrix} f_1(\gamma,\delta,\omega_w) & f_2(\gamma,\delta,\omega_w) \\ f_3(\gamma,\delta,\omega_w) & f_4(\gamma,\delta,\omega_w)\end{bmatrix}\begin{Bmatrix} c_2 \\ c_3\end{Bmatrix}=0 \qquad (2.14)$$

2.2.4. Results

For the solution of the equation system (2.14) to be non trivial, it is necessary that $f_1 f_4 - f_2 f_3 = 0$. For various values of δ and γ, natural frequencies (ω_w) that satisfy this condition are tried to be found. In Table 2.1, for $\delta = 0$, variation of ω_w with respect to γ is given with comparison to the values obtained by Chung et. al. [3].

Table 2.1. Variation of Natural Frequencies (ω_w) with respect to γ

γ	1. N. F.	Ref. [3]	2. N. F.	Ref. [3]	3. N. F.	4. N.F.	5. N.F.
1	3.68165	3.6816	22.18101	22.1810	61.84176	121.05092	200.01157
2	4.13732	4.1373	22.61492	22.6149	62.27318	121.49669	200.46692
3	4.79728	4.7973	23.32026	23.3203	62.98497	122.23555	201.22325
4	5.58500	5.5850	24.27335	24.2733	63.96676	123.26148	202.27671
5	6.44954	6.4495	25.44608	25.4461	65.20504	124.56642	203.62208
6	7.36037	7.3604	26.80908	26.8091	66.68391	126.14045	205.25281
7	8.29964	8.2996	28.33408	28.3341	68.38595	127.97218	207.16198
8	9.25684	9.2568	29.99538	29.9954	70.29296	130.04903	209.33851
9	10.22568	10.2257	31.77051	31.7705	72.38668	132.35763	211.77518
10	11.20231	11.2023	33.64037	33.6404	74.64929	134.8841	214.46095

In Ref. [3], only the first two natural frequencies were given however we are giving three more. As one can see, DTM is pretty successful in determining the eigenvalues.

To evaluate up to fourth natural frequency to five-digit precision, it was necessary to take 42 terms. It is observed that when γ increases, to achieve the prescribed accuracy; one needs to increase the number of terms used in calculations. As an example, for $\gamma = 1$ and $\delta = 0$, the convergence of these eigenvalues are given below in Fig. 2.2.

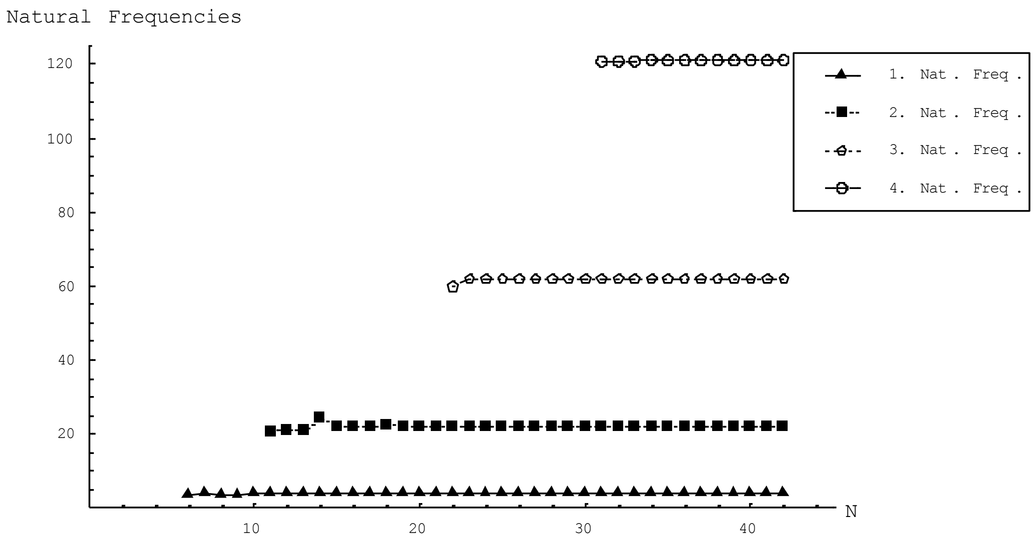

Figure 2.2. Convergence of the Bending Modes

Here *N* represents the number of $W(k)$'s calculated. For the case where $\delta = 0.1$, variation of the natural frequencies with respect to γ is sketched below in Fig. 2.3.

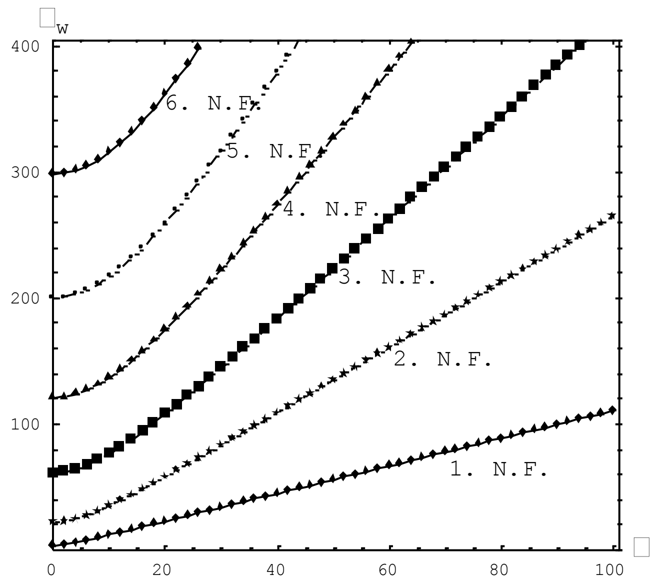

Figure 2.3. Variation of the Natural Frequencies with respect to γ

As γ increases, ω_w starts to vary linearly. In a mathematical point of view, this event can be explained as follows; for the limit case $\gamma \rightarrow \infty$, Eq. (2.7) can be written in the following form:

$$\frac{d}{d\xi}\left\{\left[\delta(1-\xi)+\frac{1}{2}(1-\xi^2)\right]\frac{dw}{d\xi}\right\}-\vartheta^2 w=0 \tag{2.15}$$

Where $\vartheta = \omega_w / \gamma$. From the nth eigenvalue of Eq. (2.15), the following result can be deduced:

$$(\omega_w)_n = \gamma \vartheta_n \tag{2.16}$$

Eq. (2.16) shows that nth natural frequency of Eq. (2.7) varies linearly with γ having the slope ϑ_n, when Eq. (2.15) is valid in the limit case of $\gamma \rightarrow \infty$.

2.3. Blasius Equation

Recently many studies have been carried out for the Blasius equation either semi-analytical, integral or numerical techniques, which are applied to obtain more detailed information about the physics of the flow field. Firstly in the doctoral thesis of Blasius, two dimensional incompressible flow field partial differential equations were reduced to third order nonlinear differential equation under the similarity transformation by the assumption that the same pressure distribution exists inside the boundary layer and the potential flow. Blasius obtained two series solution, one of them being around $\eta = 0$ and one for a large value of η and matched them in a suitable value [4]. Topfer solved this equation numerically in 1912 with the usage of Runge-Kutta technique [5]. In 1925, Bairstow [6] and in 1930, Goldstein [7] solved the same equation but with the aid of a slightly modified procedure. Howarth improved the solution made by Topfer and reached more accurate results in 1938 [8]. Besides numerical and analytical studies integration methods for solving Blasius equation exist [9, 10].

2.3.1. The Solution

After the similarity transformations are applied, the flat plate boundary layer equations can be reduced to the simple formed Blasius Equation, which is:

$$f'''+\frac{1}{2}ff''=0 \tag{2.17}$$

where, $f = f(\eta)$ where η is the similarity variable. The boundary conditions are:

$$f(0)=0,\ f'(0)=0,\ f'(\infty)=1 \tag{2.18}$$

The physical character of boundary layer apparently requires close (inner) and far away (outer) solution matching as was done by Blasius [4]. We applied DTM for inner and outer solutions and matched them. By applying theorems 1-5 to Eq. (2.17), the following transformed form can be obtained.

$$F(k+3)=-\frac{\sum_{l=0}^{k}(l+1)(l+2)F(l+2)F(k-l)}{2(k+1)(k+2)(k+3)} \tag{2.19}$$

Where $F(k)$ denotes to the differential transformation of $f(\eta)$.

2.3.1.1 The Inner Solution

In this solution we will obtain the solution at $\eta=0$ and the transformed B.C.'s at this point become:

$$F(0)=F(1)=0 \tag{2.20}$$

Since we don't know the value of the second derivative of f with respect to η, namely the value of the surface friction, we will call it $2c$. Here c is a constant resulting in $F(2)=c$. By using Eq. (2.19) and the transformed B.C.'s stated in Eq. (2.20), the following coefficients for $F(i)$, $i=3,4,5,6...$ are evaluated by the help of a mathematical software package, exactly in rational arithmetic.

$F(3n)=F(3n+1)=0$, $n=0,1,2,3...$

$F(5)=-\frac{2}{5!}c^2$

$F(8)=\frac{22}{8!}c^3$

$F(11)=-\frac{750}{11!}c^4$

$$F(14) = \frac{55794}{14!} c^5$$

$$F(17) = -\frac{7634274}{17!} c^6$$

$$F(20) = \frac{1731748230}{20!} c^7$$

$$F(23) = -\frac{606167920206}{23!} c^8$$

$$F(26) = \frac{310344559360578}{26!} c^9$$

Invoking Eq. (2.2), $f(\eta)$ and its derivative with respect to η can be evaluated as follows

$$f(\eta) = \sum_{k=0}^{N} F(k)\eta^k \ , \ f'(\eta) = \sum_{k=1}^{N} kF(k)\eta^{k-1} \tag{2.21}$$

where N represents a sufficiently large integer. By using Eq. (2.21), the third boundary condition given in Eq. (2.18) can be written in the following form.

$$\eta \to \infty, \ \sum_{k=1}^{N} kF(k)\eta^{k-1} = 1 \tag{2.22}$$

As one can notice that, from this form of Eq. (2.22), c can only be evaluated numerically. However, using the following transformation of variable, c can be evaluated in one step.

$$\eta = \left(\frac{x}{c}\right)^{1/3} \tag{2.23}$$

Equation (2.22) becomes:

$$x \to \infty, \ c^{2/3} \sum_{k=1}^{N} kF(k) \left(\frac{x}{c}\right)^{\frac{k-2}{3}} \frac{x^{1/3}}{c} = 1 \tag{2.24}$$

The term $kF(k)\left(\frac{x}{c}\right)^{\frac{k-2}{3}} \frac{x^{1/3}}{c}$ simplifies to a form of x only with the help of Eq. (2.24), the following relation for c can be written.

$$x \to \infty,\ c = \frac{1}{\left(\sum_{k=1}^{N} kF(k)\left(\frac{x}{c}\right)^{\frac{k-2}{3}} \frac{x^{1/3}}{c}\right)^{3/2}} \tag{2.25}$$

One can expect that for c to be a real finite number, the right side of Eq. (2.25) should converge to some value, while *x* and *N* increase. The following *c*, *x*, *N* graphs partly show this convergence. However, in the neighborhood of $x = 30$, there is some strange divergence.

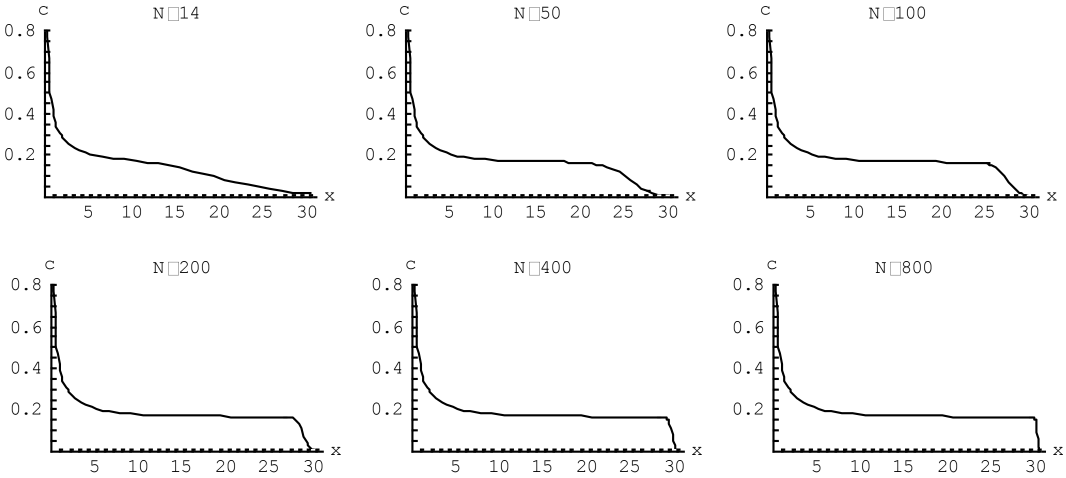

Figure 2.4. Convergence of *c*

For Eq. (2.25) to hold, *x* has to be a sufficiently large number. However we observed that, although with 800 terms, *c* doesn't converge to any value for $x > 30$. This fact can be observed from Fig. 2.4. Therefore in calculating c, we took $x = 29$ and $N = 800$. From Eq. (2.25), this resulted in $c = 0.1660637$. In comparison with the solution of Chen et. al. [11], our solution is sketched for various values of *N* in Fig 2.5.

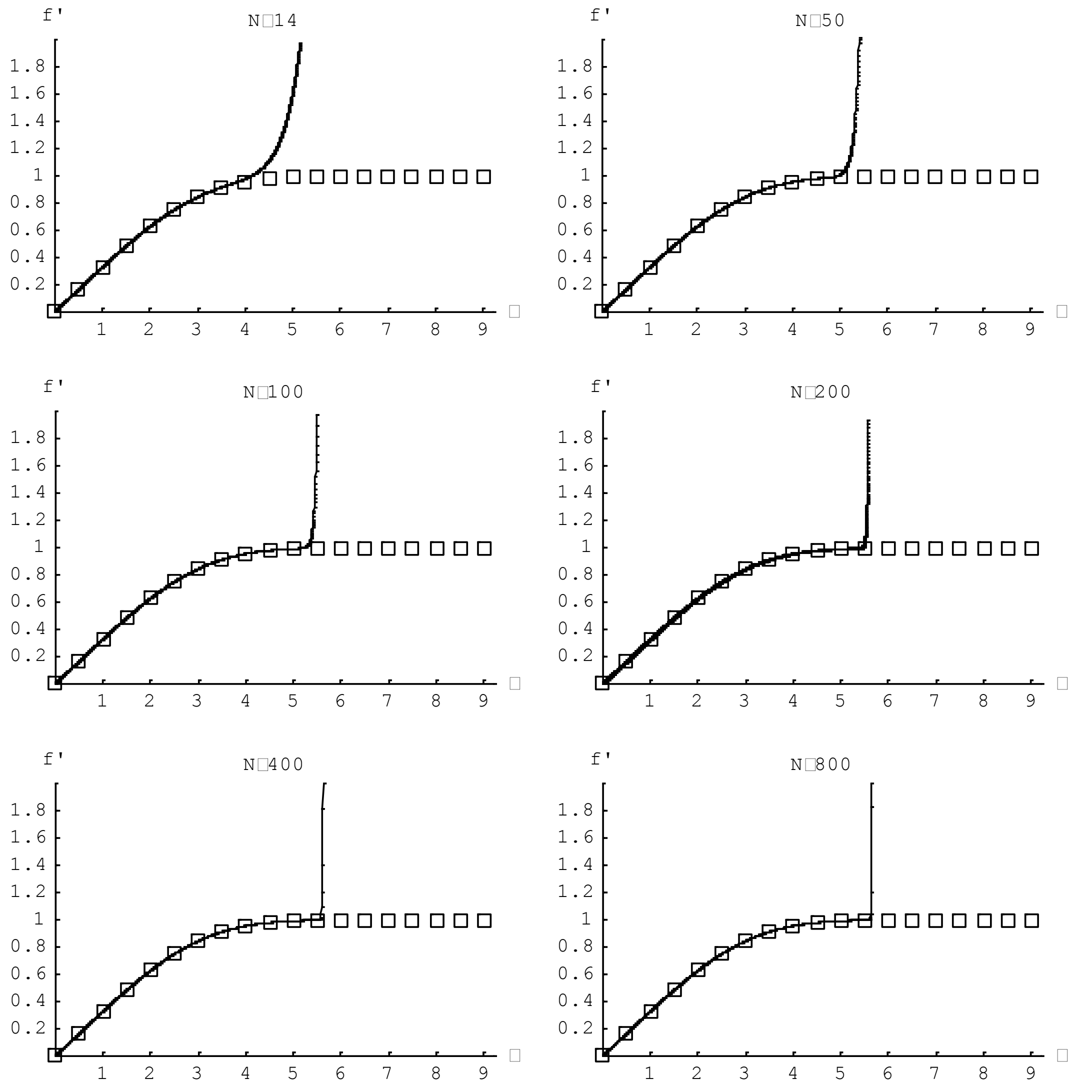

Figure 2.5. Variation of $f'(\eta)$, (W Reference [11], — Inner Solution)

The solution abruptly diverges near point $\eta = 5.7$ even for $N = 800$, which corresponds to $x \cong 30$, is a result of the divergence of c near $x = 30$. Numerical results are given below, in Table 2.2.

Table 2.2. Numerical Results with comparison to Chen et. al. [11] (N=600)

η	f	f ref.[11]	f'	f' ref.[11]	f''	f'' ref.[11]
0	0	0	0	0	0.3324149	0.3320571
0.5	0.0415374	0.0414928	0.1660637	0.1658851	0.3312661	0.3309107
1.0	0.1657495	0.1655716	0.3301327	0.3297798	0.3233453	0.3230069
1.5	0.3705335	0.3701382	0.4873017	0.4867890	0.3028761	0.3025803
2.0	0.6507098	0.6500239	0.6304093	0.6297654	0.2669765	0.2667514
2.5	0.9973429	0.9963104	0.7519941	0.7512593	0.2175484	0.2174115
3.0	1.3982207	1.3968070	0.8468255	0.8460440	0.1614126	0.1613603
3.5	1.8395053	1.8376970	0.9138312	0.9130400	0.1077642	0.1077726
4.0	2.3079460	2.3057450	0.9562963	0.9555179	0.0641969	0.0642341
4.5	2.7927180	2.7901320	0.9802722	0.9795140	0.0339409	0.0339809
5.0	3.2862317	3.2832720	0.9922822	0.9915417	0.0158770	0.0159068
5.5	3.7838969	3.7805700	0.9976076	0.9968786	0.0065906	0.0065786

The results are in 3 digits precision and not valid for $\eta > 5.5$ because of the divergence.

2.3.1.2. The Outer Solution

In this solution we will take $x_0 = m$, where m is a sufficiently large number satisfying the third B.C. stated in Eq. (2.18). We expect to overcome the divergence problem for high values of η. Eq. (2.19) is still valid however the transformed B.C.'s are changed to:

$$F(1)=1,\ \sum_{k=0}^{N}(-1)^k F(k)m^k = 0,\ \sum_{k=1}^{N}(-1)^{k-1} kF(k)m^{k-1} \tag{2.26}$$

Since we don't know the value of f and its second derivative with respect to η at $x = m$, we will call them a and $2b$ respectively. The following series solution up to nine terms can be obtained by using Eq. (2.19) and the first B.C. stated in Eq. (2.26).

$$F(0)=a,\ F(1)=1,\ F(2)=b$$

$$F(3)=-\frac{ab}{6}$$

$$F(4)=\frac{1}{48}b\left(a^2-2\right)$$

$$F(5)=-\frac{1}{480}b\left(a^3-6a+8b\right)$$

$$F(6)=\frac{1}{5760}b\left(a^4-12a^2+40ab+12\right)$$

$$F(7)=-\frac{1}{80640}b\left(a^5-20a^3+128a^2b+60a-176b\right)$$

$$F(8)=\frac{1}{1290240}b\left(a^6-30a^4+336a^3b+180a^2+704b^2-1344ab-120\right)$$

$$F(9)=-\frac{1}{23224320}b\left(a^7-42a^5+792a^4b+420a^3-6144a^2b+7488ab^2-840a+4128b\right)$$

With the second and the third B.C.'s stated in Eq. (2.26), a and b are evaluated numerically. In the following figures, variation of $f'(\eta)$ with respect to η for several m and N values are given.

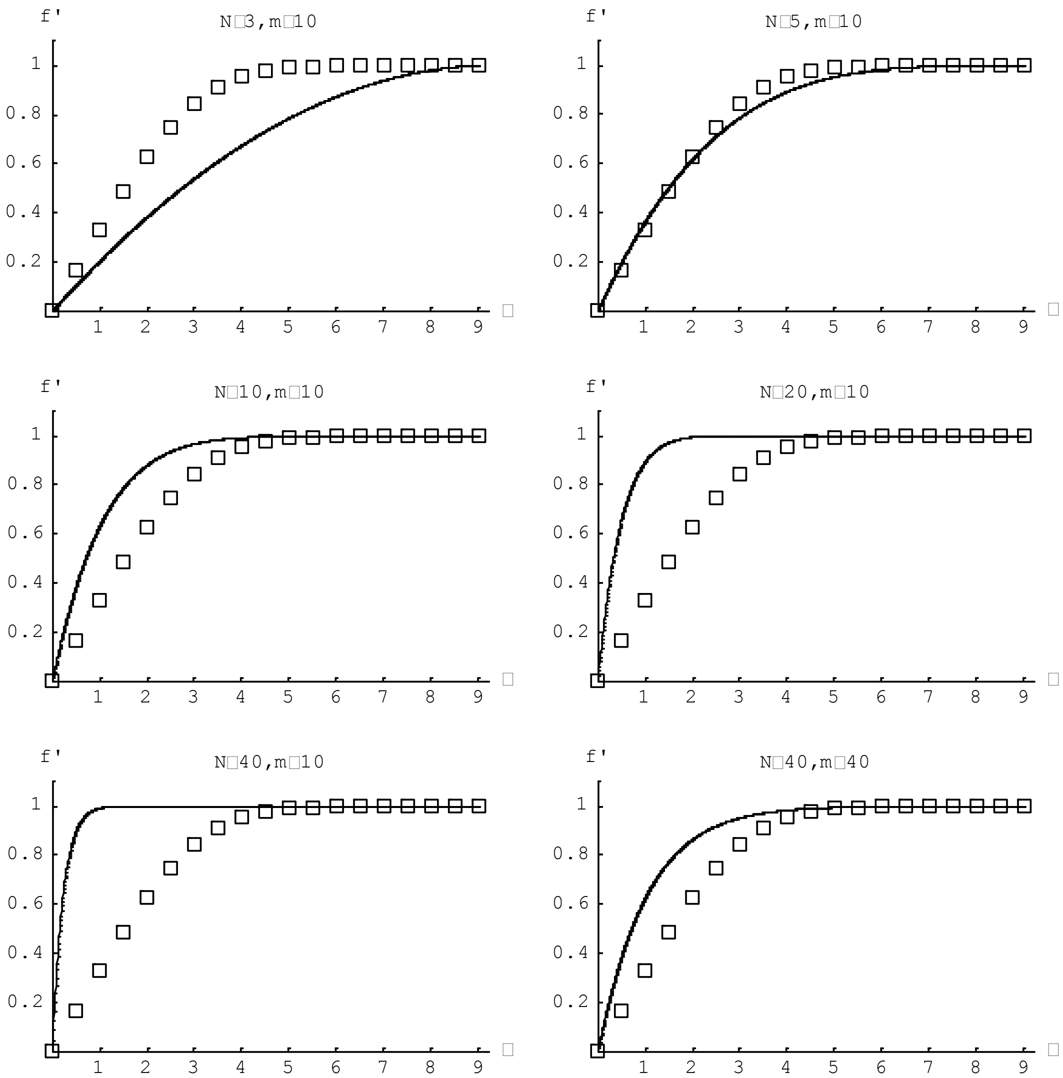

Figure 2.6. Variation of $f'(\eta)$ (W Reference [11], — Outer Solution)

From Fig. 2.6 one can deduce that there is no divergence near point $\eta = 5.5$ and no matching problem for large η, however; there is a matching problem of the solution, for small values of η, with the data in hand. The numerical data shouldn't be given for this solution because it is far from being satisfactory.

2.3.1.3. Matching of the Inner and the Outer Solutions

This method is simply, the combination of the previous two solutions in a mid point η_0. While choosing this point one thing must be carefully treated that, for $\eta < \eta_0$ the inner solution doesn't diverge and for $\eta > \eta_0$ the outer solution has no matching problem. This point is determined as $4 < \eta_0 < 5$. In this range, the change of this point doesn't affect the result peculiarly for large *N*. If we call $f(\eta)$ found in the inner solution as $f_1(\eta)$ and the one found in the outer solution as $f_2(\eta)$, the unknown constants c, a and b can be derived from the following equations numerically.

$$f_1(\eta_0) = f_2(\eta_0) \tag{2.27a}$$

$$f_1'(\eta_0) = f_2'(\eta_0) \tag{2.27b}$$

$$f_1''(\eta_0) = f_2''(\eta_0) \tag{2.27c}$$

2.3.1.3. Results

In the solution of equation system (2.27a)-(2.27c), the value of m, which appears in the outer solution, is taken as 10. For various values of N, the results for $f'(\eta)$ are presented as follows.

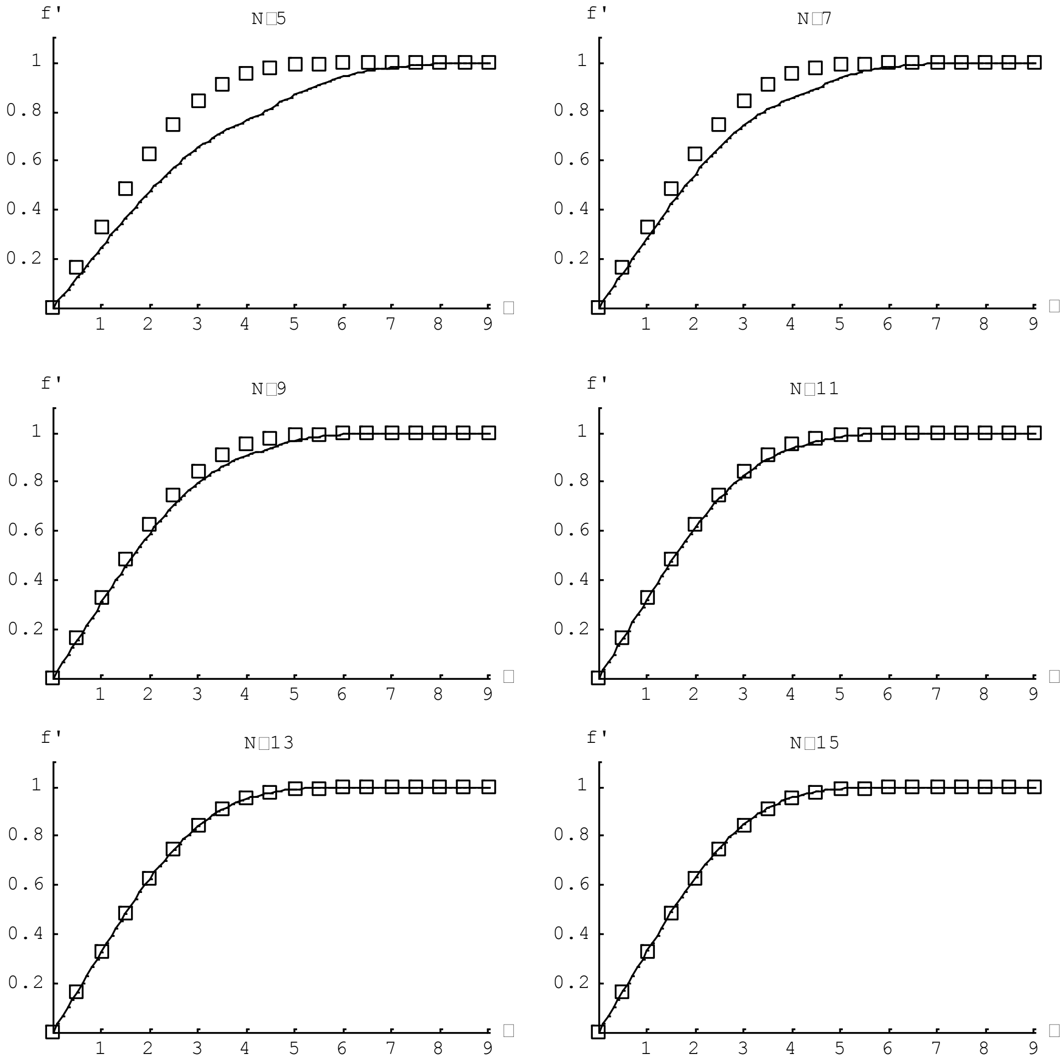

Figure 2.7. Variation of $f'(\eta)$ (W Reference [11], — Matched Inner-Outer Solution)

As it can be seen from Fig. 2.7, even by taking 13 terms, the result obtained fits the solution of Chen et. al. [11]. The values for f, f' and f'' are given below, in Table 2.3, up to $\eta = 9$.

Table 2.3. Numerical results for the matched inner-outer solution

η	f	f ref.[11]	f'	f'ref.[11]	f''	f''ref.[11]
0	0	0	0	0	0.33205734	0.3320571
0.5	0.04149282	0.0414928	0.16588525	0.1658851	0.33091095	0.3309107
1.0	0.16557173	0.1655716	0.32978002	0.3297798	0.32300712	0.3230069
1.5	0.37013853	0.3701382	0.48678929	0.4867890	0.30258050	0.3025803
2.0	0.65002437	0.6500239	0.62976574	0.6297654	0.26675155	0.2667514
2.5	0.99631111	0.9963104	0.75125970	0.7512593	0.21741158	0.2174115
3.0	1.39680823	1.3968070	0.84604444	0.8460440	0.16136032	0.1613603
3.5	1.83769859	1.8376970	0.91304038	0.9130400	0.10777264	0.1077726
4.0	2.30574642	2.3057450	0.95551823	0.9555179	0.06423412	0.0642341
4.5	2.79013435	2.7901320	0.97951429	0.9795140	0.03398089	0.0339809
5.0	3.28327366	3.2832720	0.99154190	0.9915417	0.01590680	0.0159068
5.5	3.78057189	3.7805700	0.99687882	0.9968786	0.00657859	0.0065786
6.0	4.27962092	4.2796190	0.99897287	0.9989727	0.00240204	0.0024020
6.5	4.77932233	4.7793210	0.99969907	0.9996988	0.00077411	0.0007741
7.0	5.27923881	5.2792370	0.99992160	0.9999214	0.00022017	0.0002202
7.5	5.77921803	5.7792170	0.99998186	0.9999816	0.00005526	0.0000553
8.0	6.27921343	6.2792120	0.99999627	0.9999959	0.00001224	0.0000122
8.5	6.77921253	6.7792110	0.99999932	0.9999989	0.00000239	0.0000024
9.0	7.27921237	7.2792110	0.99999989	0.9999995	0.00000041	0.0000004

The results, except the last 1-2 digits, are the same as found by Chen et. al. [11]. In their calculations, they changed this boundary value problem into two initial value problems by the usage of direct transformation method [10]. They evaluated from the first initial value problem $A = 2.08541$, where the constant c that appears in our calculations is related to this value as $c = (A)^{-3/2}/2$. We used their method with 25 terms and the same domain size, $H = 0.5$, and evaluated this value as $A = 2.0854092$ resulting in $c = 0.16602867$ which is exactly the value that we obtained for $f''(0)/2$, as given in Table 2.3. We checked our solution with their method and revealed out that, taking 90 terms for f_1 and f_2, ten-digit accuracy is achieved.

2.4. The Flow Over a Wedge

In this section, we studied Falkner and Skan's similarity type solution for the two-dimensional incompressible laminar flow over a wedge. Similarity transformations are of great importance in reducing a system of partial differential equations in a single ordinary differential equation. Two dimensional conservation equations can be reduced to the following well-known equation with the related similarity transformations.

$$f''' + \alpha f f'' + \beta(1 - f'^2) = 0 \tag{2.28}$$

with the B.C.'s

$$f(0)=0,\ f'(0)=0,\ f'(\infty)=1 \tag{2.29}$$

This equation was first given by Falkner and Skan [12]. Then Hartree [13] made further detailed studies on the solution. The constants α and β determine the potential velocity distribution and for the particular case $\alpha=1$, Falkner-Skan equation represents the flow over a wedge with a wedge angle of $\pi\beta$. Since the potential velocity is

$$U(x)=Cx^{\beta/(2-\beta)} \tag{2.30}$$

, the only physically meaningful solution, where the velocity at infinity is finite, is obtained for $\beta=0$, namely the flat plate case, where $U=U_\infty$. Blasius [4] firstly discussed this case in his doctoral thesis. The case of $\beta=1$ gives the stagnation point flow, which was firstly solved by Heimenz [14] and then improved by Howarth [15]. This type of flow has been of great interest by several authors since impinging jets are still used to achieve high levels of heat transfer in industrial processes.

Since Navier-Stokes equations are obtained with the assumption of continuum media, rarefied gases can't be investigated with this equation for a value of Knudsen number (Kn) higher than 0.1 [16]. For the value of $0.1<Kn<0.01$ the no slip B.C. can't be used and should be replaced with the following expression [17].

$$u_t = A_p \frac{\partial u_t}{\partial n} \tag{2.31}$$

Where u_t is the tangent velocity, n the normal direction to the wall and A_p is a value related with the mean free path. For $Kn<0.01$ the viscous flow is recovered and the no slip condition is valid.

It is possible to divide studies relevant to the similarity type boundary layer flow into two main groups under the mathematical sense. In the first group authors tried to proof existence and uniqueness of the solution by using purely analytical techniques such as integral methods [18,19]. However in the second group studies, authors are focused on numerical treatment of the equation by using numerical, finite difference and analytical methods. In the numerical methods authors frequently prefer the Runge-Kutta technique.

However the missing B.C., namely the skin friction at the surface is determined with shooting method, which is an iterative cost inefficient one. For the no slip condition the method of invariant imbedding can be used to evaluate the missing B.C. [10]. This method is a non-iterative one however it can't be used for slip boundary condition. Finite difference methods can be divided into two groups being implicit and explicit. In the implicit finite difference method, the equation is solved in the given domain at one step and the solution process is non-iterative as was on the method of invariant imbedding. The explicit finite difference method uses the shooting technique to determine the missing B.C. Presently, the mostly used analytical method is the perturbation method. It usually requires making assumptions on the asymptotic behavior of the differential equation. This method gives fast analytical results however, as a result of the assumptions, precision is usually lost and unsuitable choices of the small parameter, which perturbs the equation, results in badly effects [20].

2.4.1 The Solution

For the flow over a wedge, Falkner-Skan Equation is:

$$f''' + ff'' + \beta(1 - f'^2) = 0 \tag{2.32}$$

where, $f = f(\eta)$ and η is the similarity variable. The values of $\beta = 0$ and $\beta = 1$ represent flat plate boundary layer and impinging jet flows on a flat surface respectively. The Boundary conditions are given below as:

$$f(0) = 0,\quad f'(0) = \gamma f''(0),\quad f'(\infty) = 1 \tag{2.33}$$

where, γ is the relative importance of slip to viscous effects. For $\gamma = 0$ the no slip boundary condition is recovered and for $\gamma = \infty$ the flow is entirely potential. In this section a wide range of γ will be discussed. By applying theorems 1-5 to Eq. (2.32), the following recurrence relation can be obtained.

$$\begin{aligned} F(k+3) = -\frac{1}{(k+1)(k+2)(k+3)}\Bigg(&\sum_{l=0}^{k}(l+1)(l+2)F(l+2)F(k-l) \\ &+\beta\delta(k) - \beta\sum_{l=0}^{k}(l+1)(k-l+1)F(l+1)F(k-l+1)\Bigg) \end{aligned} \tag{2.34}$$

Since the problem is defined in an infinite domain, as was done by several authors [4,21], the solution requires the matching of an inner and an outer solution. As was previously done in section 2.3, the inner solution will be obtained near point $\eta = 0$ and the outer near $\eta = m$, where m is a sufficiently large number satisfying the third of Eq. (2.33), namely the far field boundary condition.

2.4.1.1. The Inner Solution

This solution will be obtained near point $\eta = 0$ and will be called as $f_1(\eta)$. The following transformed B.C.'s can be obtained from Eqs. (2.33) and (2.1) as follows:

$$F_1(0) = 0\,,\; F_1(1) = 2\gamma F_1(2),\; F_1(2) = c \tag{2.35}$$

where $F_1(k)$ denotes to the differential transformation of the inner solution. Since we don't know the value of the second derivative of f_1 with respect to η, namely the value of the surface friction, we will call it $2c$. Here c is a constant resulting in $F_1(2) = c$. By using Eq. (2.34) and the transformed B.C.'s stated in Eq. (2.35), the following coefficients for $F(i)$, up to $i = 5$, are evaluated by using Mathematica 4.0, exactly in rational arithmetic.

$$F_1(3) = \frac{1}{3!}\beta\left(4c^2\gamma^2 - 1\right)$$

$$F_1(4) = \frac{2^2}{4!}c^2\gamma(2\beta - 1)$$

$$F_1(5) = \frac{2^2}{5!}c\left(c(2\beta - 1) - \beta\gamma(\beta - 1) + 4c^2\beta\gamma^3(\beta - 1)\right)$$

By using Eq. (2.2), the inner solution can be obtained as follows:

$$f_1(\eta) = \sum_{k=0}^{N_1} F_1(k)\eta^k + O\left(\eta^{N_1+1}\right) \tag{2.36}$$

where N_1 represents the number of terms calculated for $F_1(k)$ and $O\left(\eta^{N_1+1}\right)$ means the order η^{N_1+1}. For $\eta < 1$ this term is negligible because η^{N_1+1} is a small number, however for large $\eta's$, although its coefficient is small, neglecting of this term results in divergence in the solution as was observed in the solution of the Blasius Equation in section 2.3.

2.4.1.2. The Outer Solution

The solution obtained here will be called as $f_2(\eta)$. In this solution we will take $\eta_0 = m$, where m is a sufficiently large number satisfying the third B.C. stated in Eq. (2.33). Since we don't know the value of f_2 and its second derivative with respect to η at $\eta = m$, we will call them a and $2b$ respectively. The recurrence relation given in Eq. (2.34) is still valid, however the transformed B.C.'s are changed to:

$$F_2(0) = a\,, F_2(1) = 1\,, F_2(2) = b \tag{2.37}$$

The following series solution up to 5 terms can be obtained by using Eqs. (2.34) and (2.37).

$$F_2(3) = -\frac{2}{3!}ab$$

$$F_2(4) = \frac{2}{4!}b\left(a^2 + 2\beta - 1\right)$$

$$F_2(5) = \frac{2}{5!}b\left(3a - a^3 - 2b - 4a\beta + 4b\beta\right)$$

By using Eq. (2.2), the outer solution can be obtained as follows.

$$f_2(\eta) = \sum_{k=0}^{N_2} F_2(k)(\eta - m)^k + O\left((\eta - m)^{N_2+1}\right) \tag{2.38}$$

Where N_2 represents the number of terms calculated for $F_2(k)$. As was explained in the inner solution, the term $O\left((\eta - m)^{N_2+1}\right)$ represents here the order of $(\eta - m)^{N_2+1}$. For $\eta \to m$ this term is negligible because $(\eta - m)^{N_2+1}$ is a small number, however for small $\eta' s$, neglecting of this term results in matching problems in the solution as was seen in the solution of the Blasius Equation in section 2.3. Therefore we will match these inner and outer solutions at a point, namely η_0, where $f_1(\eta)$ is valid on the region $0 < \eta < \eta_0$ and $f_2(\eta)$ on the region $\eta_0 < \eta < m$. There exist numerous studies obtaining asymptotic solutions for the outer solution with several assumptions that won't be discussed here [21]. However one thing must be carefully treated that, to obtain the outer solution given in Eq. (2.38), no assumptions are made. Therefore it is the exact analytical series solution for the case $N_2 \to \infty$ and $m \to \infty$.

2.4.1.3. Matching of the Inner and the Outer Solutions and Results

After the calculation of $f_1(\eta)$ and $f_2(\eta)$ in terms of a, b, c, γ, β and m, these two solutions can be matched to give numerical values of a, b and c as follows:

$$f_1(\eta_0) = f_2(\eta_0) \tag{2.39a}$$

$$f_1'(\eta_0) = f_2'(\eta_0) \tag{2.39b}$$

$$f_1''(\eta_0) = f_2''(\eta_0) \tag{2.39c}$$

Equation System (2.39a)-(2.39c) is solved numerically for various values of γ and β. This technique is superior to the shooting method because it is non iterative and cost efficient. Once f_1 and f_2 are calculated, just by changing γ and β, C(2) continuous analytical solutions can be obtained very precisely. And after having an analytical form, pressure, streamlines or velocity components can be easily evaluated.

From the previous studies it is observed that, for the third B.C. stated in Eq. (8) to satisfy, it is not necessary to take very large m. Even for $\beta = 0$, which is the value that the boundary layer has the maximum thickness, at point $\eta \cong 3.5$ the value of $f' \cong 0.99$ is reached. For Heimenz flow ($\beta = 1$) this value is reached at $\eta \cong 2.4$. In the solution of Eqs. (2.39a)-(2.39c), η_0 is taken at a range of $2 < \eta_0 < 3$, m is taken 7.5 for all β's and to assure 6 digits accuracy, $N_1 = N_2 = 90$ is taken. Since $F_1(k)$ and $F_2(k)$ for $k > 5$ are large expressions, we are not able to give them here. However they can easily be obtained from Eqs. (2.34), (2.35) and (2.37) as discussed in the preceding paragraphs. Since it is the most important parameter, variation of $f''(0)$ with respect to β and γ is given below in Table 2.4.

Table 2.4. Variation of $f''(0) = 2c$ with respect to β and γ

	β (Wedge Angle/ π)							
γ (Slip Factor)	0	0.25	0.5	0.75	1	1.25	1.5	2
0.0	0.469600	0.731941	0.927680	1.090442	1.232588	1.360309	1.477224	1.687218
0.4	0.448073	0.618707	0.731768	0.817334	0.886346	0.944189	0.993950	1.076379
0.8	0.405074	0.515241	0.583034	0.631547	0.668969	0.699200	0.724407	0.764604
1.2	0.360702	0.435194	0.479151	0.509635	0.532585	0.550767	0.565684	0.589021
1.6	0.321479	0.374450	0.404932	0.425672	0.441059	0.453108	0.462899	0.478047
2.0	0.288333	0.327661	0.349929	0.364892	0.375886	0.384430	0.391330	0.401928
2.4	0.260586	0.290830	0.307767	0.319048	0.327282	0.333646	0.338763	0.346583
2.8	0.237286	0.261218	0.274514	0.283313	0.289703	0.294623	0.298567	0.304570
3.2	0.217569	0.236954	0.247659	0.254709	0.259810	0.263725	0.266856	0.271608
3.6	0.200732	0.216740	0.225540	0.231313	0.235477	0.238666	0.241210	0.245064
4.0	0.186222	0.199658	0.207017	0.211829	0.215292	0.217939	0.220047	0.223236
4.4	0.173609	0.185042	0.191285	0.195358	0.198282	0.200514	0.202290	0.204971
4.8	0.162556	0.172399	0.177762	0.181253	0.183756	0.185663	0.187178	0.189464
5.2	0.152798	0.161361	0.166016	0.169041	0.171206	0.172855	0.174164	0.176135
5.6	0.144125	0.151641	0.155720	0.158366	0.160258	0.161697	0.162838	0.164556
6.0	0.136369	0.143018	0.146621	0.148956	0.150623	0.151889	0.152894	0.154404
6.4	0.129395	0.135318	0.138524	0.140598	0.142079	0.143202	0.144093	0.145431
6.8	0.123091	0.128401	0.131272	0.133127	0.134450	0.135454	0.136249	0.137442
7.2	0.117367	0.122154	0.124739	0.126409	0.127598	0.128500	0.129214	0.130285
7.6	0.112147	0.116484	0.118824	0.120334	0.121409	0.122224	0.122869	0.123836
8.0	0.107367	0.111315	0.113444	0.114816	0.115793	0.116533	0.117118	0.117995
8.4	0.102976	0.106584	0.108529	0.109781	0.110672	0.111347	0.111880	0.112680
8.8	0.098927	0.102238	0.104021	0.105169	0.105985	0.106603	0.107091	0.107823
9.2	0.095182	0.098231	0.099872	0.100928	0.101679	0.102246	0.102695	0.103367
9.6	0.091709	0.094526	0.096041	0.097016	0.097708	0.098231	0.098645	0.099264
10.0	0.088480	0.091090	0.092493	0.093395	0.094036	0.094520	0.094902	0.095475

One can see from Table 2.4 that, as gamma gets larger, shear stress on the wall gets smaller and the flow becomes to be potential. Another thing that should be carefully treated is that, as gamma increases, deviation with β decreases. For example, the relative difference between $c(0,0)$ and $c(2,0)$ is %259.28 however, this difference is only %7.91 between $c(0,10)$ and $c(2,10)$.

The graphical representation of the variation of the shear stress on the wall with respect wedge angle and slip factor is given below in Fig. 2.8.

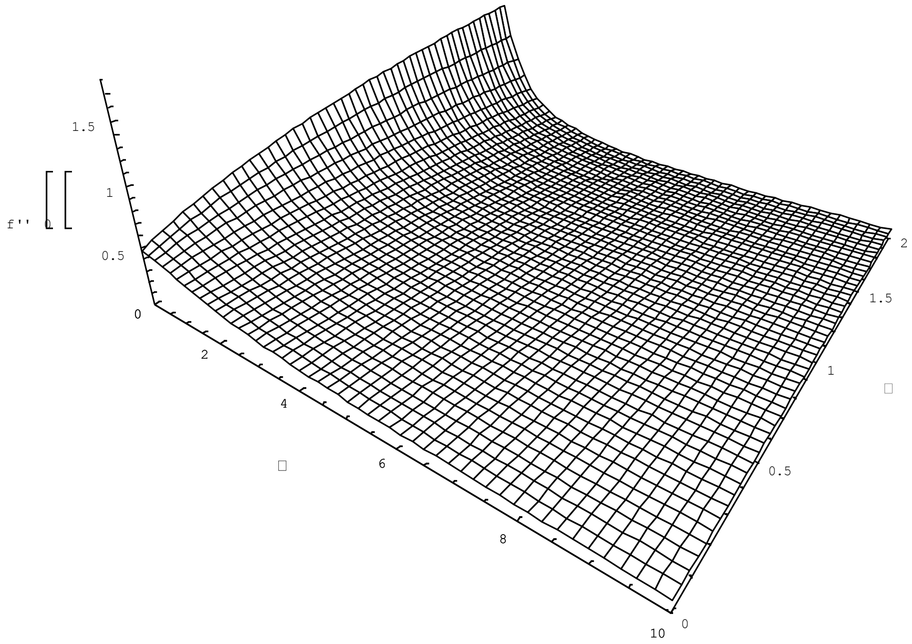

Figure 2.8. Variation of $f''(0)$ with respect to β and γ

For several values of γ and β, $f'(\eta)$ is given below.

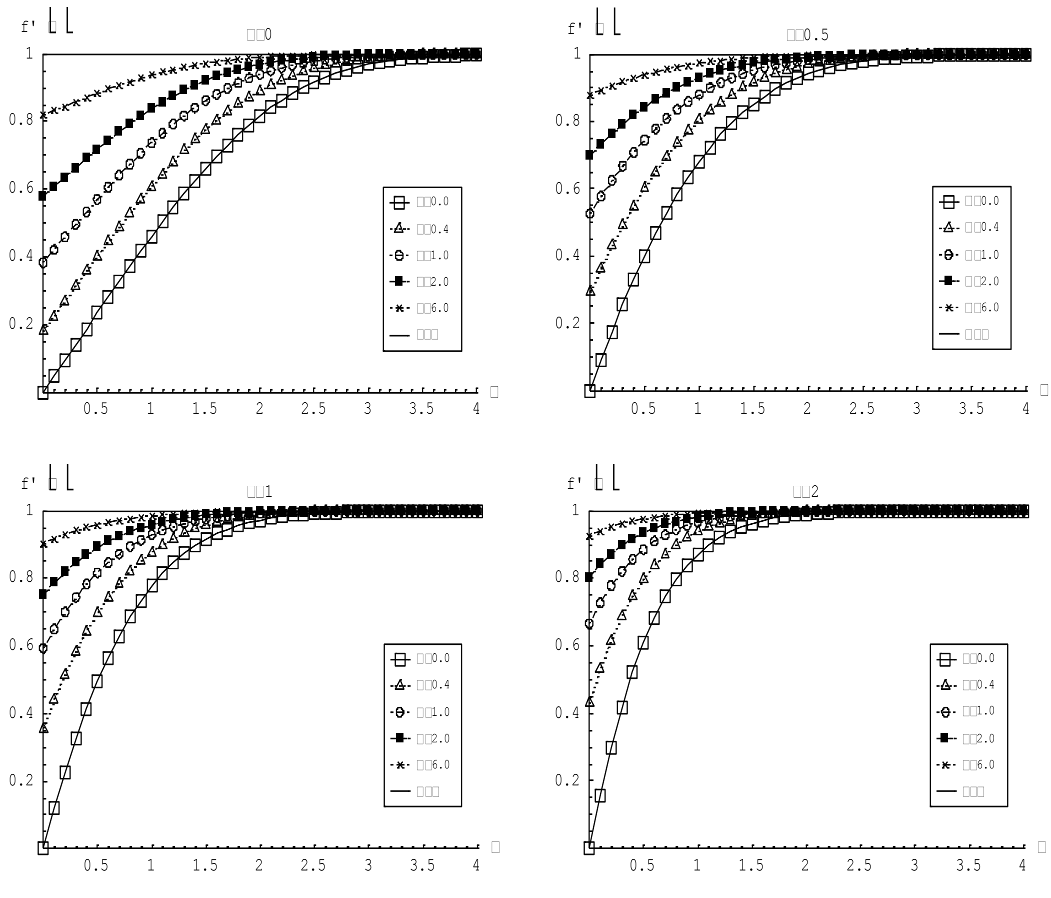

Figure 2.9. Variation of $f'(\eta)$ with respect to η, for constant wedge angles

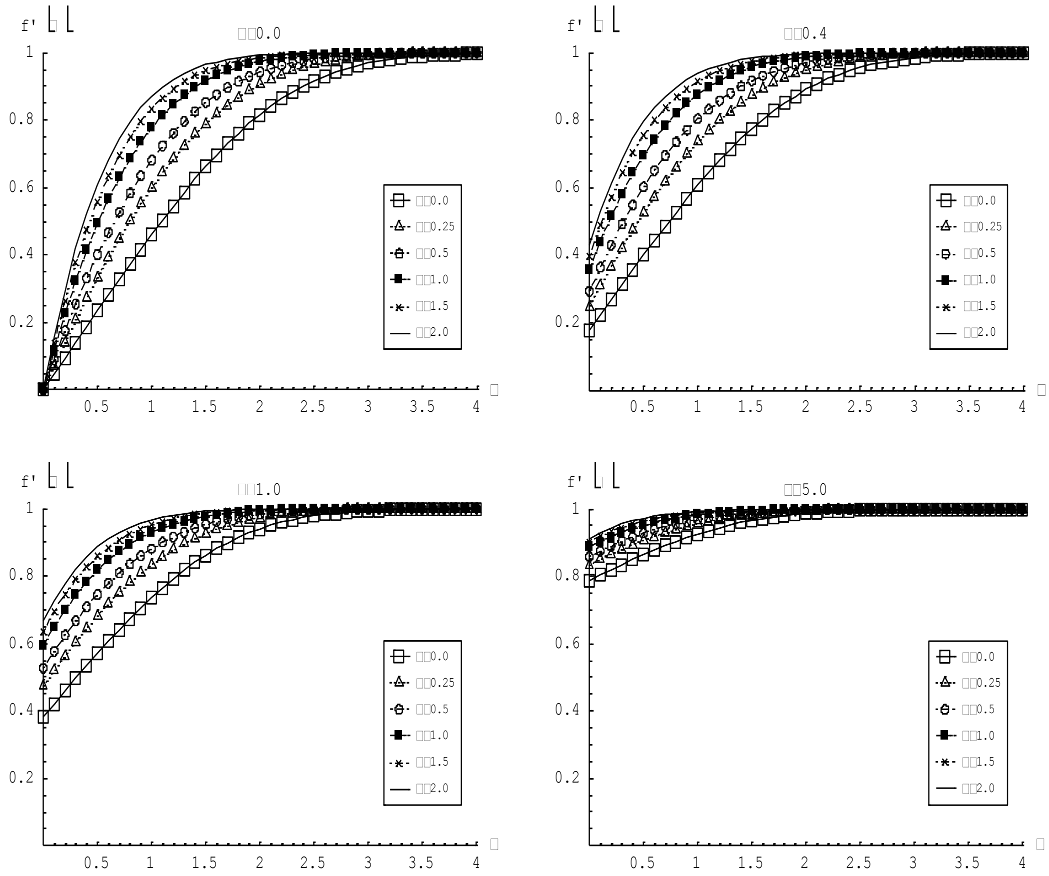

Figure 2.10. Variation of $f\,'(\eta)$ with respect to η, for constant slip factors

2.4.2. Domain Splitting Technique

This technique simply splits the domain into smaller ones and at each domain applies the DTM. At each domain the B.C.'s and their transformed forms are given below.

$$\eta=\eta_i,\ \begin{cases} f_i=a_i,\ F_i(0)=a_i \\ f_i\,'=b_i,\ F_i(1)=b_i \\ f_i\,''=2c_i,\ F_i(2)=c_i \end{cases} \tag{2.40}$$

Where $F_i(k)$ denotes to the differential transformation of $f(\eta)$ at $\eta=\eta_i$. With the help of Eq. (2.2), a_{i+1}, b_{i+1} and c_{i+1} can be driven as follows:

$$a_{i+1}=\sum_{k=0}^{N}F_i(k)(\eta_{i+1}-\eta_i)^k \tag{2.41a}$$

$$b_{i+1} = \sum_{k=1}^{N} kF_i(k)(\eta_{i+1} - \eta_i)^{k-1} \tag{2.41b}$$

$$c_{i+1} = \frac{1}{2}\sum_{k=2}^{N} k(k-1)F_i(k)(\eta_{i+1} - \eta_i)^{k-2} \tag{2.41c}$$

Where N is the number of terms calculated and $H = \eta_{i+1} - \eta_i$ is the domain size of each cell. In our calculations we observed that taking $N = 7$ and $H = 0.25$ gives 6-digit accuracy. However one can take more number of terms and a smaller domain size to have better results. From Eqs. (2.34) and (2.40) the following values for $F(k)$ up to $k = 7$ can be evaluated.

$$F_i(3) = \frac{1}{3!}[-2\,a_i\,c_i + \beta\,(b_i^2 - 1)]$$

$$F_i(4) = \frac{1}{4!}[a_i^2\,c_i - 2\,b_i\,c_i + \beta\,(a_i - a_i\,b_i^2 + 4\,b_i\,c_i)]$$

$$F_i(5) = \frac{1}{5!}[-2\,a_i^3\,c_i + 6\,a_i\,b_i\,c_i - 4\,c_i^2 + \beta\,(-a_i^2 + 2\,b_i + a_i^2\,b_i^2 - 2\,b_i^3 - 8\,a_i\,b_i\,c_i + 8\,c_i^2)$$
$$+\beta^2\,(-2\,b_i + 2\,b_i^3)]$$

$$F_i(6) = \frac{1}{6!}[a_i^4\,c_i - 12\,a_i^2\,b_i\,c_i + 6\,b_i^2\,c_i + 20\,a_i\,c_i^2 + \beta\,(a_i^3 - 5\,a_i\,b_i - a_i^3\,b_i^2 + 5\,a_i\,b_i^3 + 8\,c_i$$
$$+12\,a_i^2\,b_i\,c_i - 24\,b_i^2\,c_i - 32\,a_i\,c_i^2) + \beta^2\,(4\,a_i\,b_i - 4\,a_i\,b_i^3 - 12\,c_i + 20\,b_i^2\,c_i)]$$

$$F_i(7) = \frac{1}{7!}[a_i^5\,c_i + 20\,a_i^3\,b_i\,c_i - 30\,a_i\,b_i^2\,c_i - 64\,a_i^2\,c_i^2 + 44\,b_i\,c_i^2 + \beta\,(-a_i^4 + 9\,a_i^2\,b_i - 8\,b_i^2$$
$$+a_i^4\,b_i^2 - 9\,a_i^2\,b_i^3 + 8\,b_i^4 - 38\,a_i\,c_i - 16\,a_i^3\,b_i\,c_i + 98\,a_i\,b_i^2\,c_i + 88\,a_i^2\,c_i^2 - 128\,b_i\,c_i^2)$$
$$+\beta^2\,(-4 - 6\,a_i^2\,b_i + 20\,b_i^2 + 6\,a_i^2\,b_i^3 - 16\,b_i^4 + 52\,a_i\,c_i - 76\,a_i\,b_i^2\,c_i + 80\,b_i\,c_i^2)$$
$$+\beta^3\,(6 - 16\,b_i^2 + 10\,b_i^4)]$$

Once the missing boundary condition of the first cell $F_1(2) = c_1$ is determined, a_2, b_2 and c_2 are obtained straightforward. In general, with the values of a_i, b_i and c_i, a_{i+1}, b_{i+1} and c_{i+1} can be easily evaluated with Eqs. (2.41a)-(2.41c). To determine $F_1(2)$, we offer shooting for $F_1(2) = c_1$, calculate each $F_i(k)$ for $k = 1, 2, 3, ..., 7$ and at $\eta = m$, m being a large number, then check if $f\,'(m)$ is equal to 1 or not. If it is equal, it is clear that we have the solution in hand and if it is not, we have to repeat the procedure. Although this is a trial-error type of evaluation, it gives fast and precise solutions. The change of precision with respect to the domain size H is given below in Table 2.5 for $\beta = 1$ and $\gamma = 0$.

Table 2.5. Variation of $f''(0)$ with H for $\beta = 1$ and $\gamma = 0$

H	1	0.5	0.25	0.125	0.1
$f''(0)$	1.23120360	1.23254644	1.23258626	1.23258762	1.23258764

We won't give additional data found with this technique because it is exactly the same as found with the matching technique.

2.5. The Flow Over a Rotating Infinite Disc

Rotary system type flow has received much attention since the beginning of the 19th Century. This attention is being carried both by practical and theoretical interests. Theoretical results are mostly obtained under the similarity approach, which requires to be checked against the experimentally obtained ones.

The problem of flow field between rotating discs is of great interest in many practical and engineering aspects. Mainly, requirement of high temperatures in the turbine stage of a gas turbine engine to achieve high thermal efficiencies, cooling of the air is essential to ensure for extending the life of turbine discs and blades. It is vital to know how flow and thermal fields are at every stage for a safe and effective work life, in the operation of the rotary type machine systems.

For an accurate determination of temperature distribution, firstly the flow field must be solved as precisely as possible. Since the governing equations, namely the momentum equations, are highly nonlinear and coupled, it is hard to obtain exact analytical solutions for the full problem. Although the numerical solutions for the unsimplified cases are widespread and well-known, in order to gain physical insight, we need to have analytical solutions which are available only for simplified cases.

If we have a look at the history of the studies on rotary systems, in 1921 Von Karman considered the case of infinite discs by simplifying the governing equations with the use of axisymmetry [22]. After that, Cochran [23] derived the numerical solutions of the governing equations found by von Karman. Soon after that, Bödewadt used von Karman's analysis to consider the problem of a stationary disc with the outer flow in solid body rotation [24]. The similarity transformation that was obtained by Von Karman was used by Batchelor to examine the condition where two discs rotate with different angular velocities [25]. Batchelor assumed that the fluid rotates with constant angular velocity between the two boundary layers on the wall.

The rotating cores predicted by Batchelor were not observed in experiments performed by Stewartson [26]. He stated that a boundary layer would only exist on the rotating disc and the core wouldn't rotate when a disc is stationary or when they turn in opposite directions. Both conditions were examined for different values of Re and aspect ratios and for the case of laminar flow, Bertela and Gori have made numerical solutions [27].

Though numerous studies have been recently devoted to the investigation of the instabilities, the first attempt for the prediction of the turbulent flow was made by Morse [28]. He used an elliptic solver and a low turbulence Reynolds number k-ε model. The case of co-rotating discs with radial through-flow and enclosed discs were studied by Iacovides et al. [29]. After that, they tested four turbulence models for numerical computations of the flow in enclosed rotor-stator systems [30]. Elena and Schiestel explored the turbulent flows in rotor-stator systems with or without source flow [31]. They suggested that the second order models provide a crucial level of closure. Their offer was in agreement with the numerical calculations made by Randriamampianina [32]. Cheah et al. obtained detailed measurements of mean velocity and the turbulent flow field [35].

2.5.1. Theoretical Model

Concerning the axially symmetrical flows, where there is no deviation with respect to the angular coordinate θ, the conservation equations can be written as follows:

$$u\frac{\partial u}{\partial r}+w\frac{\partial u}{\partial z}-\frac{v^2}{r}=-\frac{1}{\rho}\frac{\partial p}{\partial r}+\nu\left[\frac{\partial^2 u}{\partial r^2}+\frac{\partial}{\partial r}\left(\frac{u}{r}\right)+\frac{\partial^2 u}{\partial z^2}\right] \tag{2.42a}$$

$$u\frac{\partial v}{\partial r}+w\frac{\partial v}{\partial z}+\frac{uv}{r}=\nu\left[\frac{\partial^2 v}{\partial r^2}+\frac{\partial}{\partial r}\left(\frac{v}{r}\right)+\frac{\partial^2 v}{\partial z^2}\right] \tag{2.42b}$$

$$u\frac{\partial w}{\partial r}+w\frac{\partial w}{\partial z}=-\frac{\partial p}{\partial z}+\nu\left[\frac{\partial^2 w}{\partial r^2}+\frac{1}{r}\frac{\partial w}{\partial r}+\frac{\partial^2 w}{\partial z^2}\right] \tag{2.42c}$$

$$\frac{\partial(ru)}{\partial r}+\frac{\partial(rw)}{\partial z}=0 \tag{2.42d}$$

where u is the radial, v is the circumferential and w is the axial component of the velocity, ν is the kinematic viscosity and p is the pressure. Since we are concerned with the flow over a single free disc, the slip boundary conditions can be stated as follows:

$$u = \frac{2-\sigma}{\sigma}\lambda\frac{\partial u}{\partial z},\ v = r\Omega + \frac{2-\sigma}{\sigma}\lambda\frac{\partial v}{\partial z}, w = 0 \text{ at } z = 0 \tag{2.43a}$$

$$u \to 0,\ v \to 0 \text{ as } z \to \infty \tag{2.43b}$$

Where σ is the tangential momentum accommodation coefficient and λ is the mean free path. This kind of slip B.C. is valid for $0.1 < Kn < 0.01$. In 1921, Von Karman suggested the following similarity solution [22]:

$$u = \Omega r F(\zeta),\ v = \Omega r G(\zeta),\ w = \sqrt{\Omega \nu} H(\zeta),\ p = -\rho\Omega\nu P(\zeta) \tag{2.44}$$

Where, $\zeta = z\sqrt{\Omega/\nu}$ is the new defined independent variable. With this new variable and the flow field variables defined, the governing equations given in Eqs. (2.42a) - (2.42d) simplify to the following ordinary, coupled and nonlinear set of differential equations:

$$F'' = F^2 - G^2 + F'H \tag{2.45a}$$

$$G'' = 2FG + G'H \tag{2.45b}$$

$$P' = HH' - H'' \tag{2.45c}$$

$$H' = -2F \tag{2.45d}$$

And the B.C.'s can be obtained from Eqs. (2.43a) and (2.43b) as follows:

$$G(0) = 1 + \omega G(0)',\ F(0) = \omega F'(0),\ H(0) = 0 \tag{2.46a}$$

$$F(\infty) = 0,\ G(\infty) = 0 \tag{2.46b}$$

Where, $\omega = \left[\left(2-\sigma\right)\lambda\Omega^{1/2}\right]/\sigma\nu^{1/2}$. To change this infinite domain problem to a finite one, Benton [34] defined the following coordinate transformation.

$$\xi = e^{-c\zeta} \tag{2.47}$$

and the following dependent variables

$$F(\zeta) = c^2 f(\xi),\ G(\zeta) = c^2 g(\xi),\ H(\zeta) = -c\left[1 - h(\xi)\right] \tag{2.48}$$

Then Eqs. (2.45a) - (2.45c) may be written as

$$\xi^2 f'' = f^2 - g^2 - \xi f' h \tag{2.49a}$$

$$\xi^2 g'' = 2fg - \xi g' h \tag{2.49b}$$

$$\xi h' = 2f \tag{2.49c}$$

The B.C.'s given in Eqs. (2.46a) and (2.46b) become as follows:

$$f(1) = \omega f'(1),\ g(1) = \frac{1}{c^2} - \omega c g'(1),\ h(1) = c \tag{2.50a}$$

$$f(0) = 0,\ g(0) = 0,\ h(0) = 0 \tag{2.50b}$$

2.5.2. The Solution

In the solution of the differential equation system (2.49a) - (2.49c) with the B.C.'s given in (2.50a) and (2.50b), we will apply DTM at the point $\xi = 0$. By using the theorems 1-5, Eqs. (2.49a) - (2.49c) transforms to:

$$\overline{F}(k) = \frac{1}{k(k-1)} \sum_{l=0}^{k} \overline{F}(l)\overline{F}(k-l) - \overline{G}(l)\overline{G}(k-l) - l\overline{F}(l)\overline{H}[k-l] \tag{2.51a}$$

$$\overline{G}(k) = \frac{1}{k(k-1)} \sum_{l=0}^{k} 2\overline{F}(l)\overline{G}(k-l) - l\overline{G}(l)\overline{H}(k-l) \tag{2.51b}$$

$$\overline{H}(k) = \frac{2}{k}\overline{F}(k) \tag{2.51c}$$

Where $\overline{F}(k)$, $\overline{G}(k)$ and $\overline{H}(k)$ denote to the differential transform of $f(\xi)$, $g(\xi)$ and $h(\xi)$ respectively. To be able to evaluate the dependent variables, we need to know the missing B.C.'s $f'(0)$ and $g'(0)$. In the study made by Owen et. al. [35], for the no-slip case, shooting method is used to determine these unknown B.C.'s. Since this is an iterative and cost inefficient technique, we will obtain the $\overline{F}(k)$, $\overline{H}(k)$ and $\overline{G}(k)$ for $k = 2,3,....,N$ in terms of $f'(0)$ and $g'(0)$, which will be called as f_1 and g_1 respectively, then by using the B.C.'s given in Eq. (2.50a) for $\xi = 1$, f_1, g_1 and c

will be evaluated numerically. With the new defined ones, the B.C.'s given in Eq. (2.50b) for $\xi = 0$ are transformed to

$$\overline{F}(0) = 0\,,\ \overline{G}(0) = 0\,,\ \overline{H}(0) = 0\,,\ \overline{F}(1) = f_1\,,\ \overline{G}(1) = g_1 \qquad (2.52)$$

and the B.C.'s given in Eq. (2.50a) for $\xi = 1$ become:

$$\sum_{k=0}^{N}\overline{F}(k) = \omega\sum_{k=1}^{N}k\overline{F}(k)\,,\ \sum_{k=0}^{N}\overline{G}(k) = \frac{1}{c^2} - \omega c\sum_{k=1}^{N}k\overline{G}(k)\,,\ \sum_{k=0}^{N}\overline{H}(k) = c \qquad (2.53)$$

By using Eqs. (2.51a) - (2.51c) and (2.52), $\overline{F}(k)$, $\overline{H}(k)$ and $\overline{G}(k)$ up to $k = 4$ are evaluated as follows:

$$\overline{F}(2) = -\left(f_1^2 + g_1^2\right)/2$$

$$\overline{F}(3) = f_1\left(f_1^2 + g_1^2\right)/4$$

$$\overline{F}(4) = -\left(17f_1^4 + 18f_1^2g_1^2 + g_1^4\right)/144$$

$$\overline{G}(2) = 0$$

$$\overline{G}(3) = -g_1\left(f_1^2 + g_1^2\right)/12$$

$$\overline{G}(4) = f_1g_1\left(f_1^2 + g_1^2\right)/18$$

$$\overline{H}(1) = 2f_1$$

$$\overline{H}(2) = -\left(f_1^2 + g_1^2\right)/2$$

$$\overline{H}(3) = f_1\left(f_1^2 + g_1^2\right)/6$$

$$\overline{H}(4) = -\left(17f_1^4 + 18f_1^2g_1^2 + g_1^4\right)/288$$

Since the terms get larger as k increases, we are giving the terms up to $k = 4$, however one can easily obtain higher order terms. After the calculation of these terms and using Eq. (2.2), $f(\xi)$, $g(\xi)$ and $h(\xi)$ are evaluated as follows

$$f(\xi) = \sum_{k=0}^{N}\overline{F}(k)\xi^k\,,\ g(\xi) = \sum_{k=0}^{N}\overline{G}(k)\xi^k\,,\ h(\xi) = \sum_{k=0}^{N}\overline{H}(k)\xi^k \qquad (2.54)$$

Where, N is the number of terms calculated. After evaluating $f(\xi)$, $g(\xi)$ and $h(\xi)$, the original dependant variables $F(\zeta)$, $G(\zeta)$ and $H(\zeta)$ can be easily obtained by using Eqs. (2.47) and (2.48). Also $P(\zeta)$ can be obtained by integrating Eq. (2.45c) as follows:

$$P(\zeta) - P_0 = \frac{H^2}{2} - H' \quad (2.55)$$

2.5.3. Results

In the calculations made, to assure a 6-digits accuracy, it was enough to take $N = 35$. For $\omega = 0$, convergence of c versus N is given in Table 2.6.

Table 2.6. Convergence of c

N	6	10	15	20	25	30	35
c	0.899460	0.889049	0.883855	0.884523	0.884471	0.884474	0.884474

The numerical data for the flow field variables with comparison to Owen et. al. [35] for $\omega = 0$, is given in Table 2.7. and Table 2.8.

Table 2.7. Numerical results of F, G and H for $\omega = 0$ with comparison to ref. [35]

ζ	F	F ref.[35]	G	G ref.[35]	H	H ref.[35]
0.0	0.000000	0.0000	1.000000	1.0000	0.000000	0.0000
0.5	0.153623	0.1536	0.707580	0.7076	-0.091880	-0.0919
1.0	0.180156	0.1802	0.476627	0.4766	-0.265473	-0.2655
1.5	0.155862	0.1559	0.313213	0.3132	-0.435675	-0.4357
2.0	0.118851	0.1189	0.203349	0.2033	-0.573200	-0.5732
2.5	0.084755	0.0848	0.131272	0.1313	-0.674465	-0.6745
3.0	0.058120	0.0581	0.084523	0.0845	-0.745244	-0.7452
4.0	0.025668	0.0257	0.034945	0.0349	-0.825059	-0.8251
5.0	0.010893	0.0109	0.014433	0.0144	-0.859605	-0.8596
6.0	0.004549	0.0045	0.005960	0.0060	-0.874147	-0.8741
7.0	0.001887	0.0019	0.002461	0.0025	-0.880200	-0.8802
8.0	0.000781	0.0008	0.001026	0.0010	-0.882707	-0.8827
9.0	0.000323	0.0003	0.000420	0.0004	-0.883744	-0.8837
10.0	0.000133	0.0001	0.000173	0.0002	-0.884173	-0.8842

Table 2.8 Numerical results of F', G' and $P-P_0$ for $\omega=0$ with comparison to [35]

ζ	F'	F' ref.[35]	G'	G' ref.[35]	$P-P_0$	$P-P_0$ ref.[35]
0.0	0.510233	0.5102	-0.615922	-0.6159	0.000000	0.0000
0.5	0.146744	0.1467	-0.532142	-0.5321	0.311467	0.3115
1.0	-0.015704	-0.0157	-0.391136	-0.3711	0.395555	0.3955
1.5	-0.069322	-0.0693	-0.267712	-0.2677	0.406629	0.4066
2.0	-0.073889	-0.0739	-0.177130	-0.1771	0.401981	0.4020
2.5	-0.061186	-0.0612	-0.115333	-0.1153	0.396963	0.3970
3.0	-0.045462	-0.0455	-0.074544	-0.0745	0.393936	0.3939
4.0	-0.021640	-0.0216	-0.030892	-0.0309	0.391697	0.3917
5.0	-0.009450	-0.0094	-0.012764	-0.0128	0.391247	0.3112
6.0	-0.003992	-0.0040	-0.005271	-0.0053	0.391165	0.3912
7.0	-0.001664	-0.0017	-0.002177	-0.0022	0.391150	0.3912
8.0	-0.000690	-0.0007	-0.000900	-0.0009	0.391148	0.3911
9.0	-0.000285	-0.0003	-0.000371	-0.0004	0.391147	0.3911
10.0	-0.000118	-0.0001	-0.000153	-0.0002	0.391147	0.3911

The results are completely in agreement with the ones obtained by [35]. The primary flow field parameters $F'(0)$, $G'(0)$ and $H(\infty)=-c$ for several values of the slip factor ω are given below in Table 2.9 and reported in Figs. 2.11-2.13. Numerical values of f_1 and g_1 are also reported in Table 2.9.

Table 2.9. Variation of primary flow field parameters with respect to ω

ω	0	0.5	1.5	4.0	10.0	100.0
$F'(0)$	0.510233	0.223848	0.084615	0.025259	0.006813	0.000180
$G'(0)$	-0.615922	-0.502810	-0.323134	-0.170059	-0.081030	-0.009556
$H(\infty)$	-0.884474	-0.842393	-0.745844	-0.615427	-0.487585	-0.242885
f_1	1.182245	0.942791	0.843786	0.791410	0.765040	0.740937
g_1	1.536777	1.191324	1.018746	0.915420	0.858998	0.804151

As one can see from Eq. (2.55), since $H'(\infty)=0$ and $H(\infty)=-c$, $P(\infty)-P_0=c^2/2$ can be easily obtained from Table 2.9.

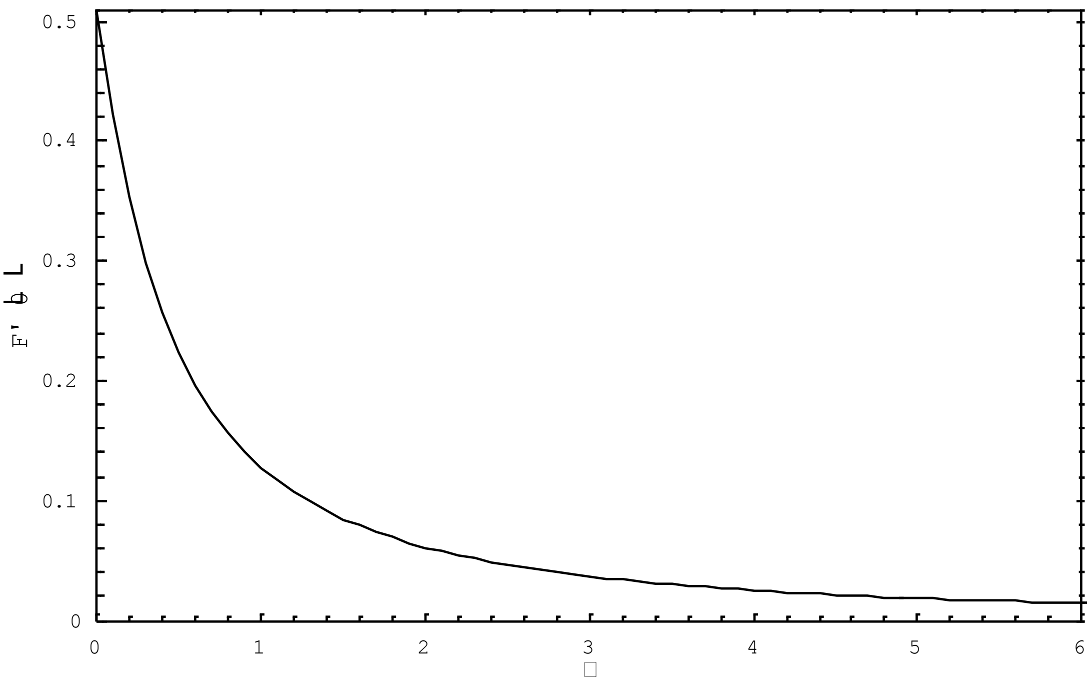

Figure 2.11. Variation of $F'(0)$ with ω

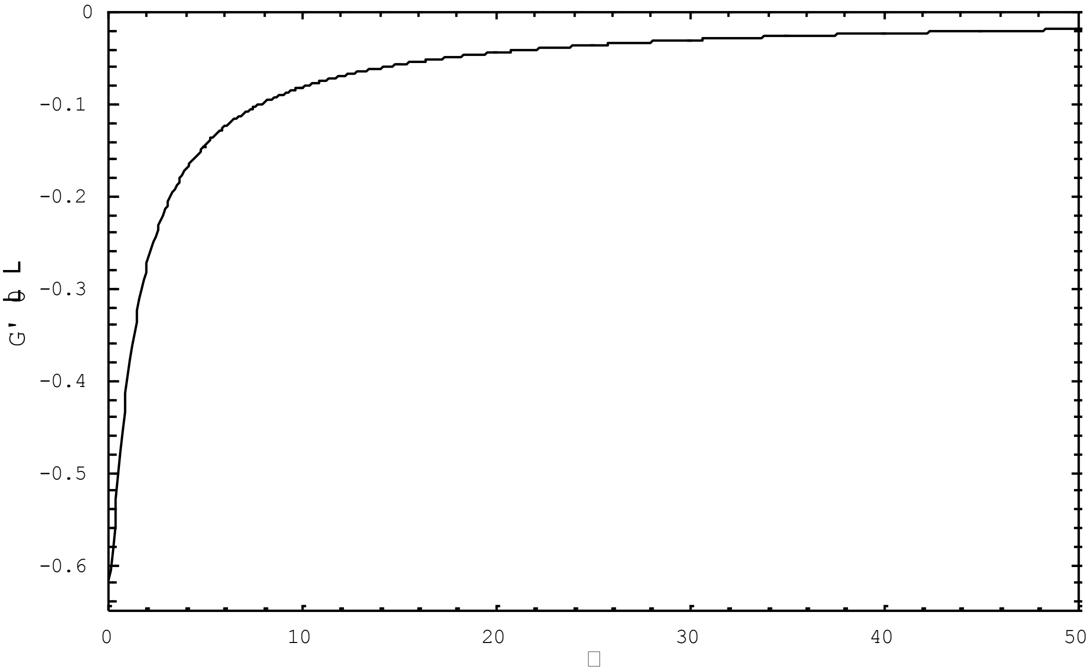

Figure 2.12. Variation of $G'(0)$ with ω

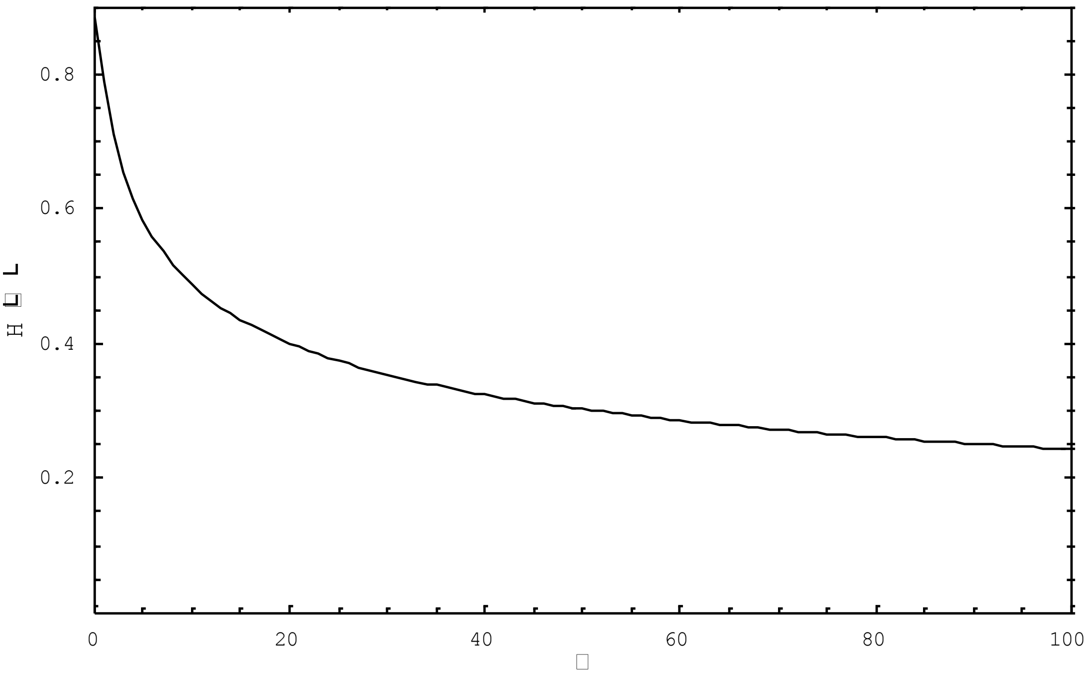

Figure 2.13. Variation of $H'(\infty)$ with ω

As the slip factor ω increases, the magnitude of surface skin friction and the inflow rate at infinity decrease. This is quite natural since the rotating disc acts like a centrifugal fan and owing to the centrifugal forces, throws the fluid that sticks on it. A fluid stream compensates this thrown fluid, which is in the axial direction. When ω increases, less amount of fluid can stick and the rotating disc loses its efficiency to transfer its circumferential momentum to the fluid particles. The fluid loses circumferential velocity therefore the centrifugal forces that throw the fluid outwards decrease. Since the disc throws less fluid away, less amount of fluid stream in the axial direction exists. Variations of the velocities on the rotating disc, where $\zeta = 0$, are given below in Figs. 2.14 and 2.15.

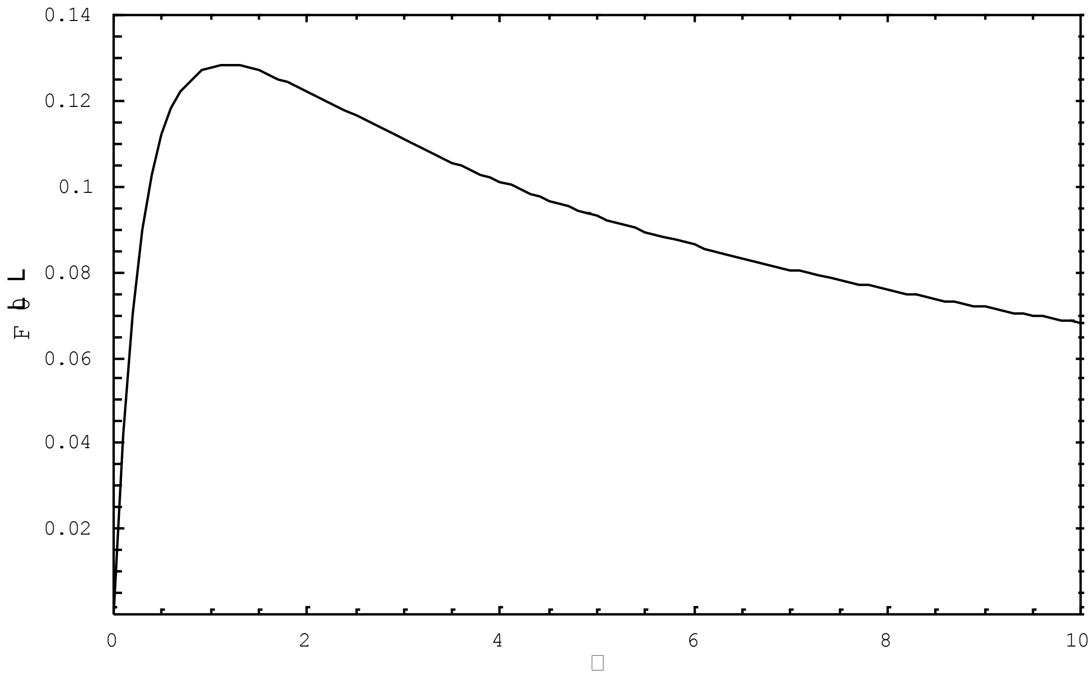

Figure 2.14. Variation of the radial velocity with respect to ω at $\zeta = 0$

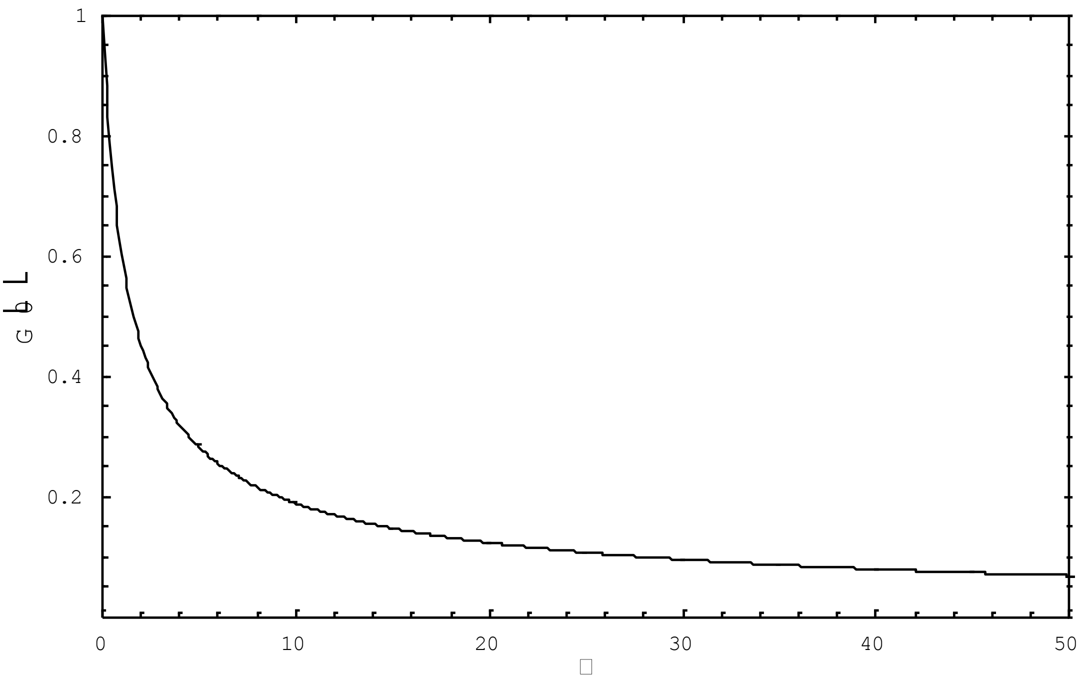

Figure 2.15. Variation of the circumferential velocity with respect to ω at $\zeta = 0$

The highest value of the radial velocity on the surface is 0.128440 and this value is reached at $\omega = 1.1586$. As one can see from Fig. 2.15, the maximum circumferential velocity on the surface is at $\omega = 0$, where the no-slip condition is present. This is a result of the negative velocity gradient of the circumferential component of the velocity in z direction. Variations of F, G, H and P for several values of ω are given below in Figs. 2.16-2.19.

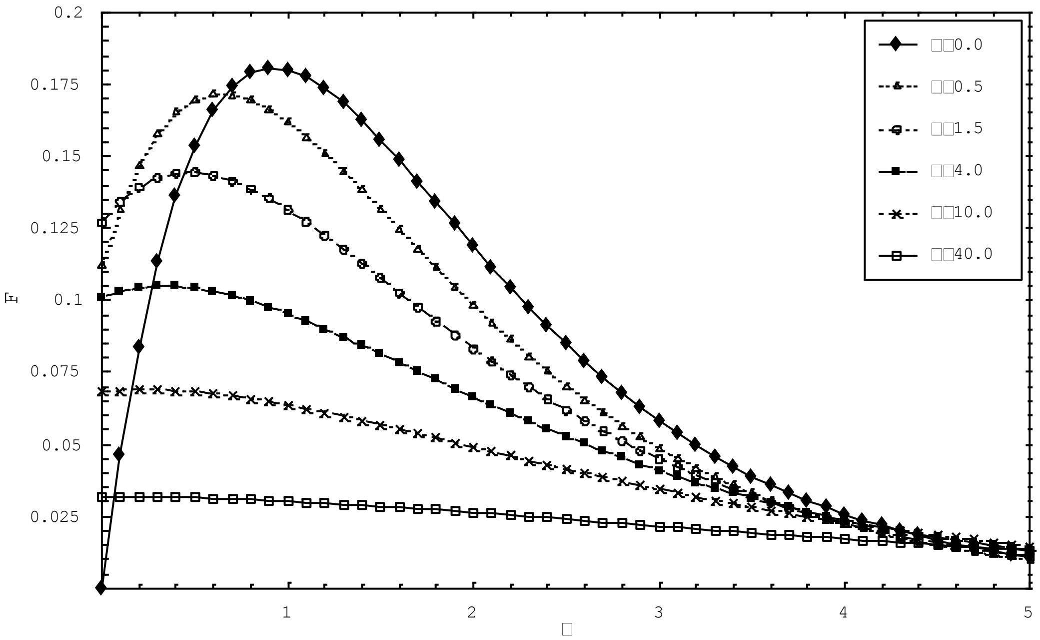

Figure 2.16. Radial component of the velocity for several values of ω

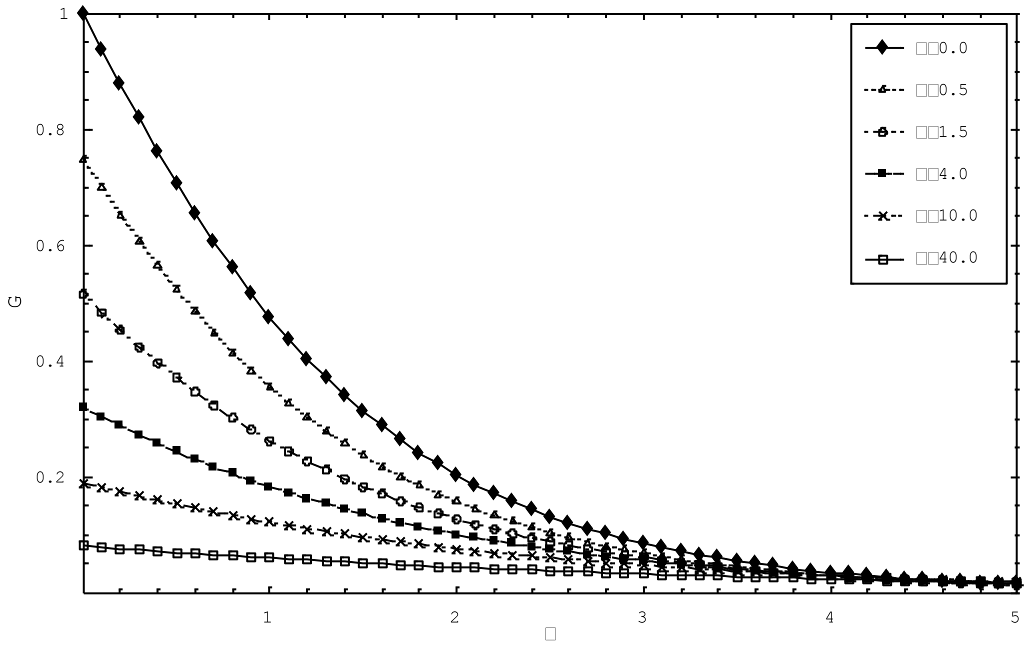

Figure 2.17. Circumferential component of the velocity for several values of ω

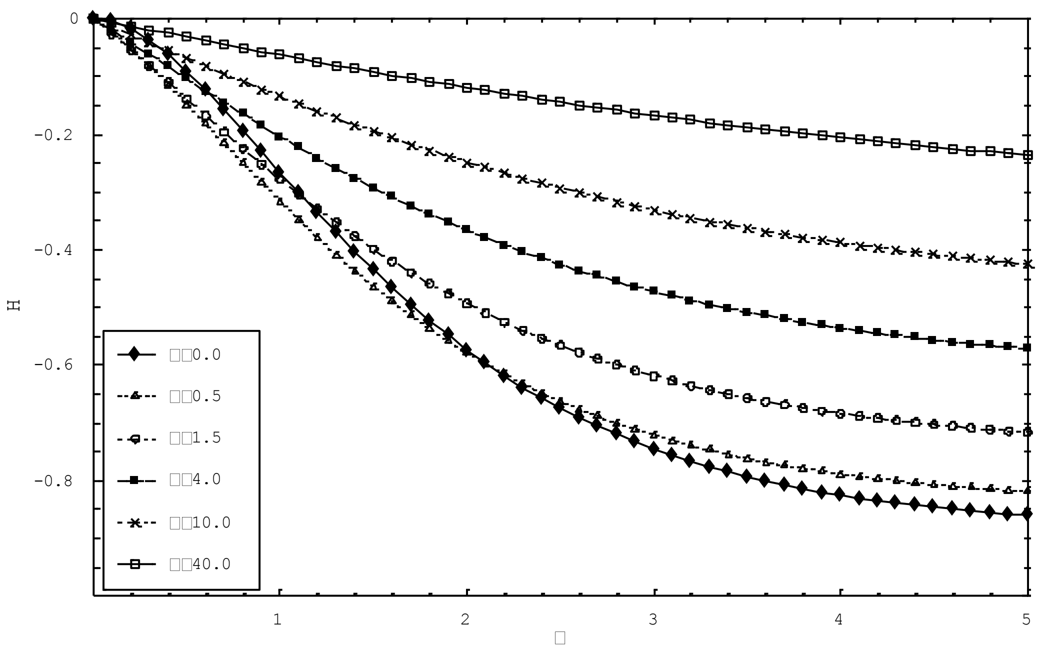

Figure 2.18. Axial component of the velocity for several values of ω

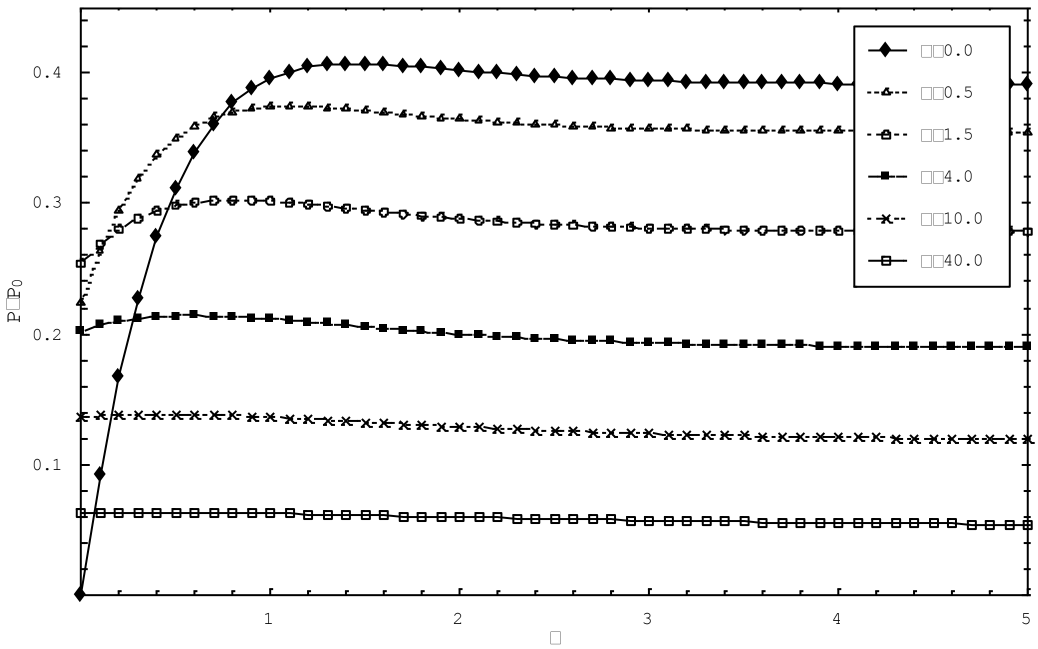

Figure 2.19. Variation of Pressure for several values of ω

From the figures above, the decreasing effect of the slip factor ω on velocity and pressure field can be easily seen. In the limiting case of $\omega \to \infty$, when the flow is entirely potential, the rotating disc has no ability on rotating the fluid particles therefore all the velocities are zero and pressure is constant and equal to P_0.

3. RESULTS AND DISCUSSION

In this study we applied one dimensional Differential Transform Method to several engineering problems. In literature, this tool is used mostly in eigenvalue or initial value problems especially being defined in finite domains. It is observed that, this semi analytical-numerical technique can be applied to BVP's and eigenvalue problems by the help of a mathematical software package very easily. We used Mathematica 4.0, which allows symbolic calculations, in all our derivations. We obtained very precise results which are in good agreement with literature.

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