

**VIBRATION ANALYSIS OF BEAMS SUBJECTED TO
MOVING LOADS**

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Programme : Aeronautics and Astronautics Engineering

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JUNE 2008

**HAREKETLİ YÜKLERE MARUZ KİRİŞLERİN
TİTREŞİM ANALİZİ**

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PREFACE

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ABBREVIATIONS

PDE	: Partial Differential Equation
ODE	: Ordinary Differential Equation
FEM	: Finite Element Method

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NOMENCLATURE

A	: Cross-sectional area, m^2
L	: Length, m
I	: Area moment of inertia, m^4
E	: Young modulus, N/m^2
G	: Shear modulus, N/m^2
κ	: Shear correction coefficient
ρ	: Density, kg/m^3
v	: Speed of the load, m/s
v_{cr}	: Critical speed of the load, m/s
a	: Acceleration, m/s^2
w	: Transverse deflection, m
w_0	: Static transverse deflection, m
u_x	: Horizontal displacement of the cross-section, m
ϕ	: Angle of rotation, rad
γ	: Angle of distortion due to shear, rad
$P(t)$	: Time dependent load, N
P_0	: Constant load, N
x	: Axial direction
z	: Vertical direction
t	: Time, s
ω_i	: Natural frequency for the i^{th} mode, 1/s
S	: Non-dimensional time
$\delta(t)$	: Dirac-Delta function
$g(t)$	: Function representing the kinematics of the force
h	: Length of the line segments, m
η	: Damping factor
c	: Damping coefficient, $kN.s/m^2$
k_L	: Linear foundation stiffness, MN/m^2
k_{NL}	: Non-linear foundation stiffness, MN/m^4

VIBRATION ANALYSIS OF BEAMS SUBJECTED TO MOVING LOADS

SUMMARY

Moving load concept has an important place in vibration studies, because it has a substantial effect on the dynamic stresses in structures that cause them to vibrate intensively. The concept has a large variety of applications in engineering. In this study, dynamic response of homogeneous, isotropic Euler-Bernoulli and Timoshenko type beams under varying magnitude of moving loads is investigated. Moving loads are taken as concentrated and they are assumed as constant, linearly increasing and linearly decreasing. Firstly, a model of moving load on a plain, simply-supported beam's vibration characteristics are investigated considering different types of motion such as constant velocity, accelerated and decelerated cases. Effects of the velocity, type of the force, damping, shear deformation and rotary inertia to the vibration characteristics are examined. The differences between the Euler-Bernoulli and Timoshenko beam theory approaches on the moving load problem are emphasized. Secondly, the dynamic response of a simply-supported Euler-Bernoulli beam, with uniform cross-section and finite length supported by a viscoelastic foundation and subjected to a concentrated constant load, are investigated by a numerical solution technique named method of lines. Viscoelastic foundation models are taken both linear and nonlinear. Effects of the nonlinearity in the foundation stiffness are investigated. Time response diagrams of the beams are graphically expressed for various speeds and damping factors.

HAREKETLİ YÜKLERE MARUZ KİRİŞLERİN TİTREŞİM ANALİZİ

ÖZET

Hareketli yük problemi yapılarda yoğun titreşime neden olan dinamik gerilmelerin oluşmasında büyük etkiye sahip olduğundan titreşim konuları içinde önemli bir yer teşkil etmektedir. Problem, mühendislikte geniş uygulama alanı bulmaktadır. Bu çalışmada çeşitli büyüklüklerde, hareketli yüklere maruz homojen, izotropik Euler-Bernoulli ve Timoshenko tipi kirişlerin bu yüklere karşı dinamik cevabı incelenmiştir. Hareketli yükler tekil olarak alınmış olup sabit, doğrusal artan ve doğrusal azalan olarak kabul edilmiştir. İlk olarak, basit mesnetli, yalın bir kirişin ona etki eden tekil yüke karşı titreşim davranışları, sabit hızlı ve ivmeli hareket için incelenmiştir. Yükün hızının, tipinin, sönümlemenin, kayma ve dönmenin titreşim davranışına etkileri araştırılmıştır. Hareketli yük probleminde Euler-Bernoulli ve Timoshenko kiriş teorisi yaklaşımlarının birbirinden farkları vurgulanmıştır. İkinci olarak, düzgün kesit alanına sahip, sonlu uzunlukta, basit mesnetli ve viskoelastik bir taban üzerine oturtulmuş Euler-Bernoulli kirişinin titreşim davranışı, çizgiler metodu olarak bilinen bir sayısal yöntem yardımıyla incelenmiştir. Viskoelastik taban modelleri lineer ve nonlineer olarak alınmıştır. Taban katılığındaki nonlineerliğin titreşim davranışına olan etkileri incelenmiştir. Kirişlerin çeşitli hızlar ve sönümleme oranlarında zamana bağlı çökme değerleri grafiksel olarak sunulmuştur.

1. INTRODUCTION

Engineering design and analysis processes require two important topics to be considered. One is the static characteristics and the other is the dynamic characteristics of engineering structures. Static consideration involves with force and moment equilibrium at zero velocity cases. Dynamic consideration studies vibration characteristics at constant velocity, accelerated or decelerated motion cases. One of the most challenging problems that arise in vibration studies is the circumstances including moving loads. “Moving loads” are basically defined as the loads in transport, manufacturing, mechanical and civil engineering that vary in both time and space. Moving load phenomena has an important place in vibration studies, because it has a substantial effect on the dynamic stresses in structures that cause them to vibrate intensively. The phenomena have a large variety of applications in engineering.

The first and the foremost application area of the moving loads is transport engineering that covers both railway and road transportation. Railway engineering mostly interests in the vibration studies on the rail-wheel interactions including contact mechanics, rail-soil interactions by modeling them as beams and foundations. Highway engineering studies, mainly focused on the bridge-vehicle interactions, railway bridges are included.

Another application field of the phenomena is manufacturing engineering. Turning can be given as an example of the moving load concept, because in turning a cutting tool is moved in the axial direction against a workpiece that is fast. During high velocity machining operations in turning and milling, excessive vibration and chatter may occur due to the rotational speed of the workpiece or axial motion of the tool. Due to this reason, the effects of the motion of the load on the vibration must be considered not to cause large deformations and fracture.

In the area of fluid-structure interaction, flow induced vibrations occur in piping systems are mathematically modeled by using moving load phenomena. The fluid

flowing inside the pipe can be considered as a kind of moving load, and exerts a lot of influences on the dynamic characteristic of a pipe. To give an example, for a water hammer, a sudden increase in pipe pressure, which brings about pressure waves, travel along the pipe at sonic speeds. Dynamic stresses grew in the pipe wall cause pipe failures in the wake of the pressure wave. [1]

Buildings that are subjected to moving pressure waves can be served as a model of moving load phenomena. Modeling of the moving pressure wave is very hard. A transient wave load is neither a quasi-static load, which is essentially indicated by the lowest frequency, nor a shock or impulse loading. It is between these two extreme circumstances. [2]

1.1 Literature Review

To many researchers the investigation of moving loads that are constant and varying time is of great importance. Many researchers conducted different studies that are varied in different parameters and conditions in this area. Studies in literature can be classified with respect to the type of the load, kinematics of the load, solution techniques used, different beam theories, different boundary conditions and damping. Classified literature review is presented below.

1.1.1 Classification with respect to the Load Types

Many loading types can be introduced in moving load phenomena on beams. A concentrated force or force system are mostly introduced in literature. Moreover, there are plenty of studies concerning the harmonic, linearly varying, impulsive, random, distributed, pressure wave loads and moving masses that include the inertial effects of the mass itself. Below, the researches including different load types are expressed.

Trethewey and Rieker [3] performed the finite element analysis of an elastic beam structure subjected to a moving distributed load. They defined the distributed load as an equivalent set of discrete point masses over the entire beam structure. Bryja and Sniady [4] defined the moving load model as the passage of a train of concentrated forces with random amplitudes. Zibdeh and Hilal [5] investigated the random vibration of a simply supported laminated composite coated beam traversed by a random moving load. Rieker and Hilal and Mohsen [6] studied the transverse vibration of beams excited by a moving harmonic load. Zibdeh and Rackwitz [7]

investigated the response of beams simply supported and with the general boundary conditions subjected to a stream of random moving loading systems of Poissonian pulse type, i.e., with mutually independent identically distributed force amplitudes arriving at the beam at independent random times. Kunow-Baumhauer [1] analyzed the response of a beam to a transient pressure wave load. Şenalp et al. [8, 9] investigated and compared the effects of linearly increasing and linearly decreasing moving loads to constant magnitude loads.

1.1.2 Classification with respect to the Kinematics of the Load

Speed of the load is one of the most significant factors affecting the dynamic response of the beam. Kinematically, the properties of moving loads can be identified as constant, accelerated and decelerated. In some cases decelerated motion Studies relating to the kinematics of the moving loads are as follows.

Zibdeh and Hilal [10] investigated the vibration analysis of beams with generally boundary conditions traversed by a moving force. The moving load is assumed to move with accelerating, decelerating and constant velocity type of motions. In the study, they showed the effects of type of the motion. Michaltsos [11] investigated the dynamic behavior of a single span beam subjected to loads moving with variable speeds. Jong-Shyong and Po-Yun [12] studied on the dynamic behaviour of a finite railway under the high-speed multiple moving forces. In the study, the influence of the moving-load speed, the total number of moving loads and the spacing between any two adjacent moving loads on the dynamic characteristics of the continuous railway and the discontinuous railway is studied. Carr and Greif [13] investigated the vertical dynamic response of railroad tracks induced by high speed trains.

1.1.3 Classification with respect to the Solution Techniques

Analytical solution techniques, in contribution of Fourier transformations, Green functions and integral equations, Laplace transformation, method of modal expansion (assumed modes method), are the most used techniques those introduced in literature. As numerical methods, finite element method is the most powerful and common method since it is possible to apply FEM in case of difficult boundary conditions. Additionally, Galerkin method, Runge Kutta methods are used to solve the equations of motion.

N. A. Krýlov [14] using the method of expansion of the eigenfunctions. S. P.

Timoshenko [15], A. N. Lowan [16] and N. G. Bondar [17] made a new approach to the problem by the contribution of Green's functions and integral equations. A large variety of studies about vibration of solids and structures under moving forces and loads can be found in the textbook of Frýba [18]. In the textbook, there are wide range of moving load problems solved by a combination of Fourier transformations and Laplace transformation. In recent years, Olsson [19] studied the dynamics of a beam subjected to a constant force moving at a constant speed and presented analytical and finite element solutions. Metrikine and Shamalta [20] performed a theoretical approach to the steady-state dynamic response of an embedded railway track to a moving train. Thambiratnam and Zhuge [21] studied the dynamics of beams on an elastic foundation and subjected to moving loads by using the finite element method. Wang [22] analyzed the multi-span Timoshenko beams subjected to a concentrated moving force by using the mode superposition method and performed a comparison between the Euler-Bernoulli and Timoshenko beams. Zheng et al. [23] analyzed the vibration of a multi span non uniform beam subjected to a moving load by using modified beam vibration functions as the assumed modes based on Hamilton's principle. The modified beam vibration functions satisfy the zero deflection conditions at all the intermediate point supports as well as the boundary conditions at the two ends of the beam. Numerical results are presented for both uniform and non uniform beams under moving loads of various velocities. Wang and Lin [24] investigated the vibration of multi-span Timoshenko frames due to moving loads by using the modal analysis.

1.1.4 Classification with respect to Different Beam Theories

Euler-Bernoulli and Timoshenko beam theories are widely used. Rayleigh beam theory is mostly used in studies that are covered by manufacturing engineering. Long and slender beams, such as, rails, large bridge spans, pavements can be modeled by Euler-Bernoulli theory. For thick and short beams like short bridge spans and workpieces, Timoshenko theory gives more reliable results than Euler-Bernoulli theory. Modeling the moving load problem on long and rotating beams, Rayleigh beam theory works best. Literature works based on Euler-Bernoulli and Timoshenko beam theories are mentioned in the sections 1.1.1, 1.1.2, 1.1.3 and 1.1.5. In this section, studies mostly concerning Rayleigh beam theory will be presented with their

specific treatments.

Lee et al. and Katz [25] investigated the vibration of a rotor shaft as a beam based on Euler-Bernoulli, Rayleigh and Timoshenko beam theories under a transverse load moving at constant velocity. Argento and Morano [26] introduced a deflection-dependent force of Katz [25] for the moving load. Zibdeh and Juma [27] treated the moving load as a random force and applied Rayleigh beam theory. Ouyang and Wang [28] presented a dynamic model for the vibration of a rotating Rayleigh beam subjected to a three-directional load acting on their surface of the beam and moving in the axial direction. They included the bending moment produced by the axial force.

1.1.5 Classification with respect to Boundary Conditions and Damping

Damping concept can be treated in different ways. For a beam, external damping is proportional to the mass of the beam. Internal damping is proportional to the bending stiffness of the beam. [10] On the other hand, linear and non-linear viscoelastic foundations (spring and dashpot systems) can be used to model elastic bearings, ballasts, soil, etc. Some of the studies including the effect of damping on the transverse vibrations of beams are indicated below.

Hong and Kim [29] presented the modal analysis of multi span Timoshenko beams connected or supported by resilient joints with damping. The results are compared with finite element method. Chen et al. [30] calculated the response of an infinite Timoshenko beam on a viscoelastic foundation to a harmonic moving load. Savin [31] calculated the dynamic amplification factor and the characteristic response spectrum for weakly damped beams. Kim [32] investigated the vibration and stability of an infinite Euler-Bernoulli beam resting on a Winkler foundation when the system is subjected to a static axial force and a moving load with either constant or harmonic amplitude variations. Kargarnovin and Younesian [33] studied the response of a Timoshenko beam with uniform cross-section and infinite length supported by a generalized Pasternak viscoelastic foundation subjected to an arbitrary distributed harmonic moving load. Zhu and Law [34] investigated the response of multi-span bridges with elastic bearings. The effect of different vertical and rotational stiffness of the springs that model the elastic bearing is analyzed. Again, Kargarnovin and Younesian [35] analyzed the dynamic response of infinite Timoshenko and Euler-

Bernoulli beams on nonlinear viscoelastic foundations to harmonic moving loads. Kocaturk and Simsek [36] studied the vibration of viscoelastic beams subjected to compressive force and a concentrated moving harmonic force. For the viscoelastic material Kelvin-Voigt model is used.

1.2 The Scope of the Study

In this study, dynamic response of homogeneous, isotropic Euler-Bernoulli and Timoshenko type beams under varying magnitude of moving loads is investigated. Moving loads are assumed as constant, linearly increasing and decreasing. Firstly, a model of moving load on a plain simply-supported beam's vibration characteristics are investigated considering different types of motion such as constant velocity, accelerated and decelerated cases. Effects of the velocity, type of the force, damping, shear deformation and rotary inertia to the vibration characteristics are examined. Next, the dynamic response of a simply-supported Euler-Bernoulli beam with uniform cross-section and finite length supported by a viscoelastic foundation subjected to a concentrated constant load are investigated by a numerical solution technique named method of lines. Effects of the non-linearity in foundation stiffness are analyzed. Time response diagrams of the beams are graphically expressed for various speeds and damping factors.

2. BEAM THEORIES

In general, a structural member that supports loads perpendicular to its longitudinal axis is referred to as a “beam”. [37] It has a length considerably longer than its cross-sectional dimensions. It carries loads that are usually perpendicular to the longitudinal axis of the beam, and thus the loads are at right angles to the length.

Early researchers found that the bending effect is the most dominant factor in a transversely vibrating beam. Jacob Bernoulli (1654-1705) introduced the direct proportionality between the curvature of an elastic beam and the bending moment at any point. Daniel Bernoulli (1700-1782) formulated the differential equation of motion of a vibrating beam. Leonhard Euler (1707-1783) used Jacob Bernoulli’s theory in his analysis of the elastic beams subjected to several loading conditions. As it is simple and it provides reasonable engineering approximations, the Euler-Bernoulli beam theory (The Classical beam theory) is commonly used. Rayleigh beam theory adds the rotational effect to the Euler-Bernoulli theory. Rotational effect is caused by the rotation of the cross-section of the beam. Shear beam theory adds the shear deformation effect to the Euler-Bernoulli theory. In 1921, Ukrainian/Russian-born scientist, S.P. Timoshenko proposed a new beam theory that includes the effects of shear and rotation to the Euler-Bernoulli beam theory. The differences between these four beam theories can be expressed below.

In the Euler-Bernoulli beam theory, it is assumed that plane cross-sections perpendicular to the axis of the beam remain plane and perpendicular to the axis after deformation. This assumption implies that all transverse shear strains are zero. However, the effect of transverse shear strain on the bending solutions can’t be neglected when dealing with deep beams or sandwich beams with low shear modulus, because this effect becomes relatively significant. Also, this effect should be considered when greater accuracy of the beam deflection is required. The Timoshenko beam theory is an extension of the Euler-Bernoulli beam theory to allow for the effect of transverse shear deformation by relaxing the normality assumption. In this shear-deformation beam theory, plane sections remain plane but not

necessarily normal to the longitudinal axis after deformation, thus admitting a nonzero transverse shear strain.

Physically, taking into account the added mechanisms of deformation (shear deformation and rotational inertia effects) effectively lowers the stiffness of the beam, why the result is a larger deflection under a static load and lower predicted eigenfrequencies for a given set of boundary conditions. The latter effect is more noticeable for higher frequencies as the wavelength becomes shorter, and thus the distance between opposing shear forces decreases. If the shear modulus of the beam material approaches infinity - and thus the beam becomes rigid in shear - and if rotational inertia effects are neglected, Timoshenko beam theory converges towards Euler-Bernoulli beam theory. Table 1.1 shows the basic differences between four beam theories. [38]

Table 1.1: Beam Theories

	Bending Moment	Lateral Displacement	Rotary Inertia	Shear Deformation
Euler	√	√		
Rayleigh	√	√	√	
Shear	√	√		√
Timoshenko	√	√	√	√

In the subsequent sections, general equations of motions relating to these four beam theories are derived by using Hamilton's variational principle. Some basic assumptions can be made before deriving the equations of motion as follows.

- The material is linear-elastic.
- The Poison effect is neglected.
- The angle of rotation is small so the small angle assumption can be used.
- Axial direction is considerably larger than the other two.
- The neutral and centroidal axes coincide, due to the symmetry of the cross-sectional area.

- After deformation, planes perpendicular to the neutral axis remain perpendicular.

2.1 Equations of Motion for Euler-Bernoulli Beam

Undeformed and deformed shape of a differential beam element is shown in Figure 2.1.

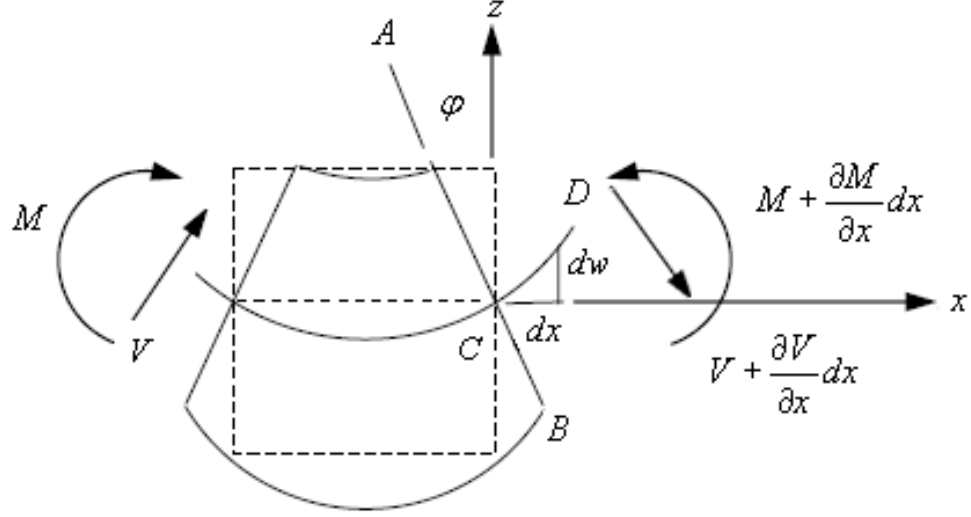


Figure 2.1: Differential Beam Element

Vertical displacement of the beam is introduced by w . Horizontal displacement of the beam is defined by u_x . The relationship between u_x and ϕ is given by the following figure and equation.

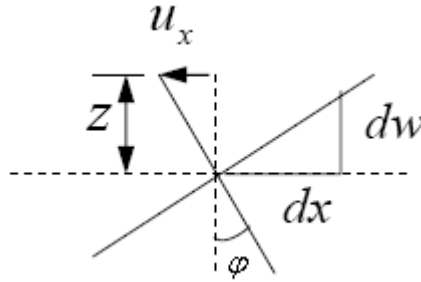


Figure 2.2: Relationship between u_x and ϕ

$$u_x = -z\phi(x) \quad (2.1)$$

In Euler-Bernoulli beam theory, plane AB is perpendicular to CD, then

$$\phi(x) = \frac{dw}{dx} \quad (2.2)$$

This yields to the following equation,

$$u_x = -z \frac{dw}{dx} \quad (2.3)$$

With the assumptions made in section 2, longitudinal (axial) strain-displacement relation is expressed as follows.

$$\varepsilon_{xx} = \frac{\partial u_x}{\partial x} \quad (2.4)$$

Substituting Equation 2.3 into Equation 2.4 leads to,

$$\varepsilon_{xx} = -z \frac{d^2 w}{dx^2} \quad (2.5)$$

The potential energy of a uniform beam due to bending is given by

$$PE_{bending} = \frac{1}{2} \int_V E \varepsilon^2 dV = \frac{1}{2} \int_0^L \left(\int_A E \varepsilon_{xx}^2 dA \right) dX \quad (2.6)$$

Substituting Equation 2.5 into Equation 2.6 we obtain,

$$PE_{bending} = \frac{1}{2} \int_0^L E \left(\frac{\partial^2 w}{\partial x^2} \right)^2 \left(\int_A z^2 dA \right) dX \quad (2.7)$$

In the equation above, $\int_A z^2 dA$ expression defines the area moment of inertia I. After

definition of the area moment of inertia, Equation 2.7 becomes,

$$PE_{bending} = \frac{1}{2} \int_0^L EI \left(\frac{\partial^2 w}{\partial x^2} \right)^2 dx \quad (2.8)$$

The kinetic energy of the beam is given by,

$$KE_{trans} = \frac{1}{2} \int_0^L \rho A \left(\frac{\partial w}{\partial t} \right)^2 dx \quad (2.9)$$

where ρ is the density of the beam and A is the cross-sectional area.

Lagrangian is defined by,

$$L = KE_{trans} - PE_{bending} \quad (2.10)$$

$$L = \frac{1}{2} \int_0^L \left[\rho A \left(\frac{\partial w}{\partial t} \right)^2 - EI \left(\frac{\partial^2 w}{\partial x^2} \right)^2 \right] dx \quad (2.11)$$

Virtual work of non-conservative forces is expressed as,

$$\delta W_{nc} = \int_0^L f(x, t) \delta w(x, t) dx \quad (2.12)$$

Hamilton's variational principle requires that, [39]

$$\int_{t_1}^{t_2} \delta L dt + \int_{t_1}^{t_2} \delta W_{nc} dt = 0 \quad (2.13)$$

Substituting Equation 2.11 and 2.12 into Equation 2.13 yields

$$\frac{1}{2} \int_{t_1}^{t_2} \delta \int_0^L \left[\rho A \left(\frac{\partial w}{\partial t} \right)^2 - EI \left(\frac{\partial^2 w}{\partial x^2} \right)^2 \right] dx dt + \int_{t_1}^{t_2} \int_0^L f(x, t) \delta w(x, t) dx dt = 0 \quad (2.14)$$

Arranging Equation 2.14, we can write

$$\int_{t_1}^{t_2} \int_0^L \left[\rho A \left(\frac{\partial w}{\partial t} \right) \delta \left(\frac{\partial w}{\partial t} \right) - EI \left(\frac{\partial^2 w}{\partial x^2} \right) \delta \left(\frac{\partial^2 w}{\partial x^2} \right) + f(x, t) \delta w \right] dx dt = 0 \quad (2.15)$$

First part of the Equation 2.15 can be determined as,

$$\int_{t_1}^{t_2} \int_0^L \rho A \left(\frac{\partial w}{\partial t} \right) \delta \left(\frac{\partial w}{\partial t} \right) dx dt = \int_0^L \left[\rho A \left(\frac{\partial w}{\partial t} \right) \delta w \right]_{t_1}^{t_2} - \int_{t_1}^{t_2} \rho A \left(\frac{\partial w}{\partial t} \right)^2 \delta w dt \quad (2.16)$$

Second part of the Equation 2.15 is given by

$$\begin{aligned}
& -EI \int_{t_1}^{t_2} \int_0^L \left(\frac{\partial^2 w}{\partial x^2} \right) \delta \left(\frac{\partial^2 w}{\partial x^2} \right) dx dt = \\
& -EI \int_{t_1}^{t_2} \left[\frac{\partial^2 w}{\partial x^2} \delta \frac{\partial w}{\partial x} \Big|_0^L - \frac{\partial^3 w}{\partial x^3} \delta w \Big|_0^L + \int_0^L \frac{\partial^4 w}{\partial x^4} \delta w dx \right] dt
\end{aligned} \tag{2.17}$$

Substituting the Equation 2.16 and 2.17 into Equation 2.15, we have the governing equation of motion of an Euler-Bernoulli beam model with general boundary conditions as follows,

$$\begin{aligned}
& - \int_{t_1}^{t_2} \int_0^L \left[\rho A \frac{\partial^2 w}{\partial t^2} + EI \frac{\partial^4 w}{\partial x^4} - f(x, t) \right] \delta w dx dt + \int_0^L \rho A \frac{\partial w}{\partial t} \delta w \Big|_{t_1}^{t_2} dx \\
& - EI \left[\int_{t_1}^{t_2} \frac{\partial^2 w}{\partial x^2} \delta \frac{\partial w}{\partial x} \Big|_0^L dt - \int_{t_1}^{t_2} \frac{\partial^3 w}{\partial x^3} \delta w \Big|_0^L dt \right] = 0
\end{aligned} \tag{2.18}$$

Equation of motion for Euler-Bernoulli beam model can be determined from the equation above as follows,

$$\rho A \frac{\partial^2 w}{\partial t^2} + EI \frac{\partial^4 w}{\partial x^4} = f(x, t) \tag{2.19}$$

General boundary conditions are,

$$\int_{t_1}^{t_2} \frac{\partial^2 w}{\partial x^2} \delta \frac{\partial w}{\partial x} \Big|_0^L dt = 0 \quad \text{and} \quad \int_{t_1}^{t_2} \frac{\partial^3 w}{\partial x^3} \delta w \Big|_0^L dt = 0 \tag{2.20}$$

$$\frac{\partial^2 w(L, t)}{\partial x^2} \delta \left(\frac{\partial w(L, t)}{\partial x} \right) = 0 \tag{2.21}$$

$$\frac{\partial^2 w(0, t)}{\partial x^2} \delta \left(\frac{\partial w(0, t)}{\partial x} \right) = 0 \tag{2.22}$$

$$\frac{\partial^3 w(L, t)}{\partial x^3} \delta (w(L, t)) = 0 \tag{2.23}$$

$$\frac{\partial^3 w(0, t)}{\partial x^3} \delta (w(0, t)) = 0 \tag{2.24}$$

In order for Equations 2.21 to 2.24 to be satisfied, four combinations of end conditions are possible.

For a free end,

$$\frac{\partial^2 w}{\partial x^2} = 0 \text{ and } \frac{\partial^3 w}{\partial x^3} = 0 \quad (2.25)$$

For a hinged end,

$$\frac{\partial^2 w}{\partial x^2} = 0 \text{ and } w = 0 \quad (2.26)$$

For a sliding end,

$$\frac{\partial w}{\partial x} = 0 \text{ and } \frac{\partial^3 w}{\partial x^3} = 0 \quad (2.27)$$

For a fixed end,

$$\frac{\partial w}{\partial x} = 0 \text{ and } w = 0 \quad (2.28)$$

In Equation 2.18 the term, $\int_0^L \rho A \frac{dw}{dt} \delta w|_{t_1}^{t_2} dx$ is expressed as,

$$\rho A \frac{\partial w(t_2, x)}{\partial t} \delta(w(t_2, x)) = 0 \quad (2.29)$$

$$\rho A \frac{\partial w(t_1, x)}{\partial t} \delta(w(t_1, x)) = 0 \quad (2.30)$$

2.2 Equations of Motion for Rayleigh Beam

Rayleigh beam theory adds the rotation effect of the cross-section of the beam by the equation below.

$$KE_{rot} = \frac{1}{2} \int_0^L \rho I \left(\frac{\partial \phi}{\partial t} \right)^2 dx \quad (2.31)$$

Considering Equation 2.2, Equation 2.31 turns into

$$KE_{rot} = \frac{1}{2} \int_0^L \rho I \left(\frac{\partial^2 w}{\partial x \partial t} \right)^2 dx \quad (2.32)$$

Lagrangian now becomes,

$$L = KE_{trans} + KE_{rot} - PE_{bending} \quad (2.33)$$

$$L = \frac{1}{2} \int_0^L \left[\rho A \left(\frac{\partial w}{\partial t} \right)^2 + \rho I \left(\frac{\partial^2 w}{\partial x \partial t} \right)^2 - EI \left(\frac{\partial^2 w}{\partial x^2} \right)^2 \right] dx \quad (2.34)$$

Applying the variational principle to KE_{rot} yields,

$$KE_{rot} = \frac{\rho I}{2} \int_{t_1}^{t_2} \int_0^L \delta \left(\frac{\partial^2 w}{\partial x \partial t} \right)^2 dx dt \quad (2.35)$$

$$KE_{rot} = \rho I \int_{t_1}^{t_2} \int_0^L \frac{\partial^2 w}{\partial x \partial t} \delta \left(\frac{\partial^2 w}{\partial x \partial t} \right) dx dt \quad (2.36)$$

$$KE_{rot} = \rho I \int_0^L \frac{\partial^2 w}{\partial x \partial t} \delta \left(\frac{\partial w}{\partial x} \right) \Big|_{t_1}^{t_2} dx - \int_{t_1}^{t_2} \int_0^L \frac{\partial^3 w}{\partial x \partial t^2} \delta \left(\frac{\partial w}{\partial x} \right) dx dt \quad (2.37)$$

$$KE_{rot} = \rho I \left[\int_0^L \frac{\partial^2 w}{\partial x \partial t} \delta \left(\frac{\partial w}{\partial x} \right) \Big|_{t_1}^{t_2} dx - \int_{t_1}^{t_2} \frac{\partial^3 w}{\partial x \partial t^2} \delta(w) \Big|_0^L dx + \int_{t_1}^{t_2} \int_0^L \frac{\partial^4 w}{\partial x^2 \partial t^2} \delta w dx dt \right] \quad (2.38)$$

Substituting Equation 2.38 into 2.18, we have

$$\begin{aligned} & - \int_{t_1}^{t_2} \int_0^L \left[\rho A \frac{\partial^2 w}{\partial t^2} + EI \frac{\partial^4 w}{\partial x^4} - \rho I \frac{\partial^4 w}{\partial x^2 \partial t^2} - f(x, t) \right] dx dt \\ & + \int_0^L \left[\rho A \frac{\partial w}{\partial t} \delta w - \rho I \frac{\partial^2 w}{\partial x \partial t} \delta \left(\frac{\partial w}{\partial x} \right) \right] \Big|_{t_1}^{t_2} dx \\ & \int_{t_1}^{t_2} \left[-EI \frac{\partial^2 w}{\partial x^2} \delta \frac{\partial w}{\partial x} + EI \frac{\partial^3 w}{\partial x^3} \delta w - \rho I \frac{\partial^3 w}{\partial x \partial t^2} \delta w \right] \Big|_0^L dt = 0 \end{aligned} \quad (2.39)$$

From Equation 2.39, the equation of motion for the Rayleigh beam can be expressed as,

$$\rho A \frac{\partial^2 w}{\partial t^2} + EI \frac{\partial^4 w}{\partial x^4} - \rho I \frac{\partial^4 w}{\partial x^2 \partial t^2} = f(x, t) \quad (2.40)$$

General boundary conditions are given by,

$$\left(EI \frac{\partial^3 w}{\partial x^3} - \rho I \frac{\partial^3 w}{\partial x \partial t^2} \right) \delta w \Big|_0^L = 0 \quad (2.41)$$

$$\frac{\partial^2 w}{\partial x^2} \delta \frac{\partial w}{\partial x} \Big|_0^L = 0 \quad (2.42)$$

2.3 Equations of Motion for Shear Beam

Shear beam theory adds the shear deformation effect to the Euler-Bernoulli theory. In shear beam model, plane AB, shown in Figure 2.1, is not perpendicular to CD anymore. Equation 2.1 holds still, but φ is now defined as,

$$\varphi(x) = \gamma + \frac{dw}{dx} \quad (2.43)$$

where γ is introduced as the angle of distortion due to shear. Longitudinal (axial) strain-displacement relation is expressed as follows,

$$\varepsilon_{xx} = -z \frac{d\varphi}{dx} \quad (2.44)$$

Total potential energy for the beam model is,

$$PE_{total} = PE_{bending} + PE_{shear} \quad (2.45)$$

$PE_{bending}$ is given by,

$$PE_{bending} = \frac{1}{2} \int_0^L EI \left(\frac{\partial \varphi}{\partial x} \right)^2 dx \quad (2.46)$$

PE_{shear} can be calculated as,

$$PE_{shear} = \frac{1}{2} \int_0^L \kappa GA \gamma^2 dx \quad (2.47)$$

where G is the shear modulus, A is the cross-sectional area, γ is the angle of distortion due to shear, κ is shear correction coefficient that is introduced to account for the difference in the constant state of shear stress in the Timoshenko beam theory and the parabolic variation of the actual shear stress through the beam depth. For a rectangular cross-section the value of κ is $5/6$.

Considering Equation 2.43, Equation 2.47 can be rewritten as,

$$PE_{shear} = \frac{1}{2} \int_0^L \kappa GA \left(\frac{\partial w}{\partial x} - \varphi \right)^2 dx \quad (2.48)$$

Then, total potential energy is given as,

$$PE_{total} = \frac{1}{2} \int_0^L \left[EI \left(\frac{\partial \varphi}{\partial x} \right)^2 + \kappa GA \left(\frac{\partial w}{\partial x} - \varphi \right)^2 \right] dx \quad (2.49)$$

Total kinetic energy for the beam model is given in Equation 2.9. Substituting Equation 2.49 and 2.9 into Equation 2.10 yields,

$$L = \frac{1}{2} \int_0^L \left[\rho A \left(\frac{\partial w}{\partial t} \right)^2 - EI \left(\frac{\partial \varphi}{\partial x} \right)^2 - \kappa GA \left(\frac{\partial w}{\partial x} - \varphi \right)^2 \right] dx \quad (2.50)$$

Application of Hamilton variational principle to Equation 2.50 yields.

$$\begin{aligned} & \frac{1}{2} \delta \int_{t_1}^{t_2} \int_0^L \left[\rho A \left(\frac{\partial w}{\partial t} \right)^2 - EI \left(\frac{\partial \varphi}{\partial x} \right)^2 - \kappa GA \left(\frac{\partial w}{\partial x} - \varphi \right)^2 \right] dx dt \\ & + \int_{t_1}^{t_2} f(x, t) \delta w dt = 0 \end{aligned} \quad (2.51)$$

Arranging the terms in Equation 2.50, we have Equation 2.55.

$$\int_{t_1}^{t_2} \int_0^L \rho A \left(\frac{\partial w}{\partial t} \right) \delta \left(\frac{\partial w}{\partial t} \right) dx dt = \int_0^L \left[\rho A \left(\frac{\partial w}{\partial t} \right) \delta w \Big|_{t_1}^{t_2} - \int_{t_1}^{t_2} \rho A \frac{\partial^2 w}{\partial t^2} \delta w dt \right] dx \quad (2.52)$$

$$-EI \int_{t_1}^{t_2} \int_0^L \left(\frac{\partial \varphi}{\partial x} \right) \delta \left(\frac{\partial \varphi}{\partial x} \right) dx dt = -EI \int_{t_1}^{t_2} \left[\frac{\partial \varphi}{\partial x} \delta \frac{\partial \varphi}{\partial x} \Big|_0^L + \int_0^L \frac{\partial^2 \varphi}{\partial x^2} \delta \varphi dx \right] dt \quad (2.53)$$

$$\begin{aligned} \int_{t_1}^{t_2} \int_0^L \kappa GA \left(\frac{\partial w}{\partial x} - \varphi \right) \delta \left(\frac{\partial w}{\partial x} - \varphi \right) dx dt &= \int_{t_1}^{t_2} \int_0^L \kappa GA \left(\frac{\partial w}{\partial x} - \varphi \right) \delta \left(\frac{\partial w}{\partial x} \right) dx dt \\ &\quad + \int_{t_1}^{t_2} \int_0^L \kappa GA \left(\frac{\partial w}{\partial x} - \varphi \right) \delta \varphi dx dt \\ &= - \int_{t_1}^{t_2} \kappa GA \left(\frac{\partial w}{\partial x} - \varphi \right) \delta w \Big|_0^L dt \\ &\quad + \int_{t_1}^{t_2} \int_0^L \kappa GA \left(\frac{\partial^2 w}{\partial x^2} - \frac{\partial \varphi}{\partial x} \right) \delta w dx dt \\ &\quad + \int_{t_1}^{t_2} \int_0^L \kappa GA \left(\frac{\partial w}{\partial x} - \varphi \right) \delta \varphi dx dt \end{aligned} \quad (2.54)$$

$$\begin{aligned} \int_{t_1}^{t_2} \int_0^L \left[-\rho A \left(\frac{\partial w}{\partial t} \right)^2 + \kappa GA \left(\frac{\partial^2 w}{\partial x^2} - \frac{\partial \varphi}{\partial x} \right) + f(x, t) \right] \delta w dx dt \\ + \int_{t_1}^{t_2} \int_0^L \left[EI \frac{\partial^2 \varphi}{\partial x^2} + \kappa GA \left(\frac{\partial w}{\partial x} - \varphi \right) \right] \delta \varphi dx dt \\ - \int_{t_1}^{t_2} \left(EI \frac{\partial \varphi}{\partial x} \delta \varphi \Big|_0^L + \kappa GA \left(\frac{\partial w}{\partial x} - \varphi \right) \delta w \Big|_0^L \right) dt = 0 \end{aligned} \quad (2.55)$$

From Equation 2.55 equations of motion for shear beam model can be derived as,

$$\rho A \left(\frac{\partial w}{\partial t} \right)^2 - \kappa GA \left(\frac{\partial^2 w}{\partial x^2} - \frac{\partial \varphi}{\partial x} \right) = f(x, t) \quad (2.56)$$

$$EI \frac{\partial^2 \varphi}{\partial x^2} + \kappa GA \left(\frac{\partial w}{\partial x} - \varphi \right) = 0 \quad (2.57)$$

Boundary conditions are derived from the equations below.

$$\frac{\partial \varphi}{\partial x} \delta \varphi \Big|_0^L = 0 \text{ and } \left(\frac{\partial w}{\partial x} - \varphi \right) \delta w \Big|_0^L = 0 \quad (2.58)$$

For a free end,

$$\frac{\partial \varphi}{\partial x} = 0 \text{ and } \left(\frac{\partial w}{\partial x} - \varphi \right) = 0 \quad (2.59)$$

For a hinged end,

$$\frac{\partial \varphi}{\partial x} = 0 \text{ and } w = 0 \quad (2.60)$$

For a sliding end,

$$\varphi = 0 \text{ and } \left(\frac{\partial w}{\partial x} - \varphi \right) = 0 \quad (2.61)$$

For a fixed end,

$$\varphi = 0 \text{ and } w = 0 \quad (2.62)$$

2.4 Equations of Motion for Timoshenko Beam

In Timoshenko beam model, plane AB, shown in Figure 2.1, is not perpendicular to CD anymore. Equation 2.1 holds still, but φ is now defined as,

$$\varphi(x) = \gamma + \frac{dw}{dx} \quad (2.63)$$

Longitudinal (axial) strain-displacement relation is expressed as follows,

$$\varepsilon_{xx} = -z \frac{d\varphi}{dx} \quad (2.64)$$

Total potential energy for the beam model is given in Equation 2.49. Total kinetic energy for the beam model is,

$$KE_{total} = KE_{trans} + KE_{rot} \quad (2.65)$$

KE_{trans} is given in Equation 2.9. KE_{rot} is expressed as,

$$KE_{rot} = \frac{1}{2} \int_0^L \rho I \left(\frac{\partial \varphi}{\partial t} \right)^2 dx \quad (2.66)$$

Considering Equation 2.9 and 2.66, total kinetic energy is given by,

$$KE_{total} = \frac{1}{2} \int_0^L \left[\rho A \left(\frac{\partial w}{\partial t} \right)^2 + \rho I \left(\frac{\partial \varphi}{\partial t} \right)^2 \right] dx \quad (2.67)$$

Lagrangian for the Timoshenko beam model,

$$L = KE_{total} - PE_{total} \quad (2.68)$$

$$L = \frac{1}{2} \int_0^L \left[\rho A \left(\frac{\partial w}{\partial t} \right)^2 + \rho I \left(\frac{\partial \varphi}{\partial t} \right)^2 - EI \left(\frac{\partial \varphi}{\partial x} \right)^2 - \kappa GA \left(\frac{\partial w}{\partial x} - \varphi \right)^2 \right] dx \quad (2.69)$$

Applying Hamilton's variational principle to Equation 2.69 yields,

$$\begin{aligned} & \frac{1}{2} \delta \int_{t_1}^{t_2} \int_0^L \left[\rho A \left(\frac{\partial w}{\partial t} \right)^2 + \rho I \left(\frac{\partial \varphi}{\partial t} \right)^2 - EI \left(\frac{\partial \varphi}{\partial x} \right)^2 - \kappa GA \left(\frac{\partial w}{\partial x} - \varphi \right)^2 \right] dx dt \\ & + \int_{t_1}^{t_2} f(x, t) \delta w dt = 0 \end{aligned} \quad (2.70)$$

Below, the variations of each term of Equation 2.70 are written, respectively.

$$\int_{t_1}^{t_2} \int_0^L \rho A \left(\frac{\partial w}{\partial t} \right) \delta \left(\frac{\partial w}{\partial t} \right) dx dt = \int_0^L \left[\rho A \left(\frac{\partial w}{\partial t} \right) \delta w \Big|_{t_1}^{t_2} - \int_{t_1}^{t_2} \rho A \frac{\partial^2 w}{\partial t^2} \delta w dt \right] dx \quad (2.71)$$

$$\int_{t_1}^{t_2} \int_0^L \frac{\partial \varphi}{\partial t} \rho I \delta \left(\frac{\partial \varphi}{\partial t} \right) dx dt = \rho I \left[\int_0^L \frac{\partial \varphi}{\partial t} \delta \varphi \Big|_{t_1}^{t_2} dx - \int_{t_1}^{t_2} \int_0^L \frac{\partial^2 \varphi}{\partial t^2} \delta \varphi dx dt \right] \quad (2.72)$$

$$-EI \int_{t_1}^{t_2} \int_0^L \left(\frac{\partial \varphi}{\partial x} \right) \delta \left(\frac{\partial \varphi}{\partial x} \right) dx dt = -EI \int_{t_1}^{t_2} \left[\frac{\partial \varphi}{\partial x} \delta \frac{\partial \varphi}{\partial x} \Big|_0^L + \int_0^L \frac{\partial^2 \varphi}{\partial x^2} \delta \varphi dx \right] dt \quad (2.73)$$

$$\begin{aligned}
\int_{t_1}^{t_2} \int_0^L \kappa G A \left(\frac{\partial w}{\partial x} - \varphi \right) \delta \left(\frac{\partial w}{\partial x} - \varphi \right) dx dt &= \int_{t_1}^{t_2} \int_0^L \kappa G A \left(\frac{\partial w}{\partial x} - \varphi \right) \delta \left(\frac{\partial w}{\partial x} \right) dx dt \\
&\quad + \int_{t_1}^{t_2} \int_0^L \kappa G A \left(\frac{\partial w}{\partial x} - \varphi \right) \delta \varphi dx dt \\
&= - \int_{t_1}^{t_2} \kappa G A \left(\frac{\partial w}{\partial x} - \varphi \right) \delta w \Big|_0^L dt \\
&\quad + \int_{t_1}^{t_2} \int_0^L \kappa G A \left(\frac{\partial^2 w}{\partial x^2} - \frac{\partial \varphi}{\partial x} \right) \delta w dx dt \\
&\quad + \int_{t_1}^{t_2} \int_0^L \kappa G A \left(\frac{\partial w}{\partial x} - \varphi \right) \delta \varphi dx dt
\end{aligned} \tag{2.74}$$

Equations 2.71, 2.72, 2.73 and 2.74 lead us to the governing equation of motion of a Timoshenko beam model with general boundary conditions.

$$\begin{aligned}
&\int_{t_1}^{t_2} \int_0^L \left[-\rho A \left(\frac{\partial w}{\partial t} \right)^2 + \kappa G A \left(\frac{\partial^2 w}{\partial x^2} - \frac{\partial \varphi}{\partial x} \right) + f(x, t) \right] \delta w dx dt \\
&+ \int_{t_1}^{t_2} \int_0^L \left[EI \frac{\partial^2 \varphi}{\partial x^2} + \kappa G A \left(\frac{\partial w}{\partial x} - \varphi \right) \right] \delta \varphi dx dt \\
&- \int_{t_1}^{t_2} \left(-\rho I \frac{\partial^2 \varphi}{\partial t^2} \delta \varphi \Big|_0^L + EI \frac{\partial \varphi}{\partial x} \delta \varphi \Big|_0^L + \kappa G A \left(\frac{\partial w}{\partial x} - \varphi \right) \delta w \Big|_0^L \right) dt = 0
\end{aligned} \tag{2.75}$$

Equations of motion for Timoshenko beam model is determined from the equation above as follows,

$$\begin{aligned}
&\int_{t_1}^{t_2} \int_0^L \left[-\rho A \left(\frac{\partial w}{\partial t} \right)^2 + \kappa G A \left(\frac{\partial^2 w}{\partial x^2} - \frac{\partial \varphi}{\partial x} \right) + f(x, t) \right] \delta w dx dt \\
&+ \int_{t_1}^{t_2} \int_0^L \left[EI \frac{\partial^2 \varphi}{\partial x^2} + \kappa G A \left(\frac{\partial w}{\partial x} - \varphi \right) \right] \delta \varphi dx dt \\
&- \int_{t_1}^{t_2} \left(-\rho I \frac{\partial^2 \varphi}{\partial t^2} \delta \varphi \Big|_0^L + EI \frac{\partial \varphi}{\partial x} \delta \varphi \Big|_0^L + \kappa G A \left(\frac{\partial w}{\partial x} - \varphi \right) \delta w \Big|_0^L \right) dt = 0
\end{aligned} \tag{2.76}$$

$$\begin{aligned}
& \int_{t_1}^{t_2} \int_0^L \left[-\rho A \left(\frac{\partial w}{\partial t} \right)^2 + \kappa GA \left(\frac{\partial^2 w}{\partial x^2} - \frac{\partial \varphi}{\partial x} \right) + f(x, t) \right] \delta w \, dx dt \\
& + \int_{t_1}^{t_2} \int_0^L \left[EI \frac{\partial^2 \varphi}{\partial x^2} + \kappa GA \left(\frac{\partial w}{\partial x} - \varphi \right) \right] \delta \varphi \, dx dt \\
& - \int_{t_1}^{t_2} \left(-\rho I \frac{\partial^2 \varphi}{\partial t^2} \delta \varphi \Big|_0^L + EI \frac{\partial \varphi}{\partial x} \delta \varphi \Big|_0^L + \kappa GA \left(\frac{\partial w}{\partial x} - \varphi \right) \delta w \Big|_0^L \right) dt = 0
\end{aligned} \tag{2.77}$$

General boundary conditions are,

$$\frac{\partial \varphi}{\partial x} \delta \varphi \Big|_0^L = 0 \text{ and } \left(\frac{\partial w}{\partial x} - \varphi \right) \delta w \Big|_0^L = 0 \tag{2.78}$$

For a free end,

$$\frac{\partial \varphi}{\partial x} = 0 \text{ and } \left(\frac{\partial w}{\partial x} - \varphi \right) = 0 \tag{2.79}$$

For a hinged end,

$$\frac{\partial \varphi}{\partial x} = 0 \text{ and } w = 0 \tag{2.80}$$

For a fixed end,

$$\varphi = 0 \text{ and } w = 0 \tag{2.82}$$

3. VIBRATION CHARACTERISTICS OF EULER-BERNOULLI AND TIMOSHENKO BEAMS SUBJECTED TO MOVING LOADS

3.1 Moving Load on a Beam without an Elastic Foundation

In Figure 3.1, a simply-supported, homogeneous, isotropic and constant cross-section beam of length, L is shown.

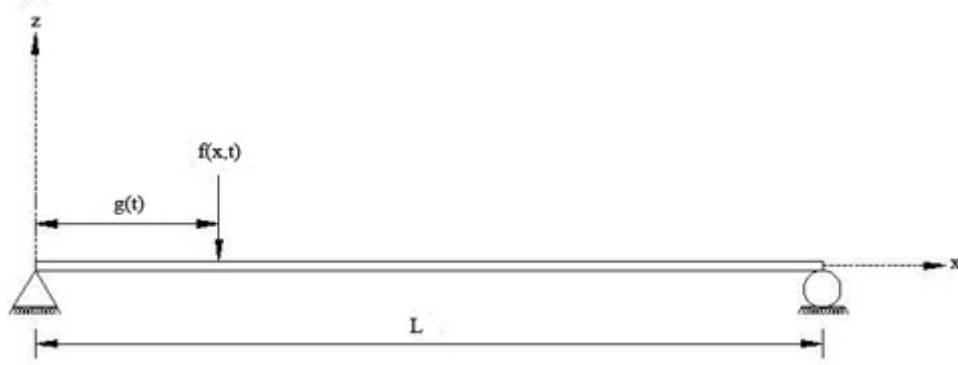


Figure 3.1: Simply-supported Beam Subjected to a Concentrated Moving Force

The force $f(x,t)$ is expressed as,

$$f(x,t) = \delta(x - g(t))P(t) \quad (3.1)$$

where $\delta(x-g(t))$ is Dirac-Delta function, $P(t)$ refers to the force for the considered case. $g(t)$ represents the kinematics of the moving force as follows,

$$g(t) = x_0 + vt + \frac{1}{2}at^2 \quad (3.2)$$

where x_0 is the starting point of the force ($x_0=0$), v is the initial speed and a is the constant acceleration. Dirac-Delta function $\delta(x)$ is thought of as a unit concentrated force acting at point $x=0$. Dirac-Delta function is defined as,

$$\int_a^b \delta(x - \xi)f(x)dx = f(\xi), \text{ for } a < \xi < b \quad (3.3)$$

Three force cases are considered as follows,

$$\text{Constant Force: } P(t) = P_0$$

Linearly Increasing Force: $P(t) = P_0 \left(\frac{t}{\bar{t}} \right)$

Linearly Decreasing Force: $P(t) = P_0 \left(1 - \frac{t}{\bar{t}} \right)$

where $\bar{t}$ is traverse time of force over beam.

3.1.1 Euler-Bernoulli Beam

The problem is governed by the following differential equation below: [18]

$$EI \frac{\partial^4 w(x,t)}{\partial x^4} + \rho A \frac{\partial^2 w(x,t)}{\partial t^2} + \eta \frac{\partial w(x,t)}{\partial t} = f(x,t) \quad (3.4)$$

where EI, ρ , A, η and $w(x,t)$ are the flexural rigidity, the density, the cross-sectional area, the damping coefficient and the transverse deflection of the beam at point x and time t, respectively. Simply-supported beam boundary and initial conditions are,

$$w(0,t) = w(L,t) = \frac{\partial^2 w(0,t)}{\partial x^2} = \frac{\partial^2 w(L,t)}{\partial x^2} = 0 \quad (3.5)$$

$$w(x,0) = \frac{\partial w(x,0)}{\partial t} = 0 \quad (3.6)$$

To solve Equation 3.4 method of modal expansion is used. By using the method of modal expansion, transverse deflection $w(x,t)$ can be assumed as,

$$w(x,t) = \sum_{i=1}^{\infty} X_i(x) T_i(t) \quad (3.7)$$

where X_i is the i^{th} normal mode vibration of a uniform beam, expressed as,

$$X_i(x) = \sin\left(\frac{\lambda_i}{L} x\right) + A_i \cos\left(\frac{\lambda_i}{L} x\right) + B_i \sinh\left(\frac{\lambda_i}{L} x\right) + C_i \cosh\left(\frac{\lambda_i}{L} x\right) \quad (3.8)$$

For a simply-supported beam Equation 3.8 reduces to the following form,

$$X_i(x) = \sin\left(\frac{\lambda_i}{L} x\right) \quad (3.9)$$

where $\lambda_i = i\pi$ (eigenvalues).

Substituting Equation 3.9 into Equation 3.7, and Equation 3.7 into Equation 3.4, then multiplying by $X_k(x)$ and integrating over the length of the beam, we have,

$$\begin{aligned} \sum_{i=1}^{\infty} T_i(t) \int_0^L EI \left(\frac{\partial^4 X_i}{\partial x^4} \right) X_k dx + \sum_{i=1}^{\infty} \ddot{T}_i(t) \int_0^L \mu X_i X_k dx \\ + 2\omega_b \sum_{i=1}^{\infty} \dot{T}_i(t) \int_0^L \mu X_i X_k dx = \int_0^L X_k f(x, t) dx \end{aligned} \quad (3.10)$$

Orthogonality conditions requires that,

$$\int_0^L X_i X_k dx = 0, \quad i \neq k \quad (3.11)$$

Rearranging the terms in Equation 3.10, we have

$$M_i = \int_0^L \mu X_i^2(x) dx \quad ; \quad H_i = \int_0^L EI \left(\frac{\partial^4 X_i}{\partial x^4} \right) X_i dx \quad (3.12)$$

Using these relations, we obtain the differential equation of the i^{th} mode of the generalized deflection as,

$$\ddot{T}_i(t) + 2\omega_i \xi_i \dot{T}_i(t) + \omega_i^2 T_i(t) = Q_i(t) \quad (3.13)$$

where,

$$\omega_i = \sqrt{\frac{H_i}{M_i}} = \left(\frac{\lambda_i}{L} \right)^2 \sqrt{\frac{EI}{\mu}} \quad ; \quad \xi_i = \frac{\omega_b}{\omega_i} \quad (3.14)$$

$$Q_i(t) = \frac{1}{M_i} \int_0^L X_i(x) f(x, t) dx \quad (3.15)$$

$Q_i(t)$ is defined as for the three different force cases,

$$P(t) = P_0 \rightarrow Q_i = \frac{P_0}{M_i} X_i[g(t)] \quad (3.16)$$

$$P(t) = P_0 \left(\frac{t}{\tau} \right) \rightarrow Q_i = \frac{P_0}{M_i} X_i[g(t)] \left(\frac{t}{\tau} \right) \quad (3.17)$$

$$P(t) = P_0 \left(1 - \frac{t}{\tau} \right) \rightarrow Q_i = \frac{P_0}{M_i} X_i[g(t)] \left(1 - \frac{t}{\tau} \right) \quad (3.18)$$

Solution of the Equation 3.13 is,

$$T_i(t) = \frac{P_0 e^{-\xi_i \omega_i t}}{M_i \omega_i \sqrt{1 - \xi_i^2}} \int_0^t e^{-\xi_i \omega_i \tau} \sin(\omega_i \sqrt{1 - \xi_i^2} \tau) Q_i(\tau) d\tau \quad (3.19)$$

Equation 3.19 can be solved by using a fourth-order Runge-Kutta method. Solution of Equation 3.19 provides the time part of Equation 3.7 for the Euler-Bernoulli beam. By multiplying the time part of Equation 3.7 with the mode shape function $X_i(x)$ the deflection curve of the beam is found. Time response diagrams of the beams are graphically expressed in the results chapter.

3.1.2 Timoshenko Beam

With the addition of shear deformation and rotary inertia effects, governing equation of the transverse vibration of the beam becomes, [40]

$$\begin{aligned} EI \frac{\partial^4 w}{\partial x^4} - \left(f(x,t) - \rho A \frac{\partial^2 w}{\partial t^2} - \eta \frac{\partial w}{\partial t} \right) - \rho A r^2 \frac{\partial^4 w}{\partial x^2 \partial t^2} \\ + \frac{EI}{\kappa AG} \frac{\partial^2}{\partial x^2} \left(f(x,t) - \rho A \frac{\partial^2 w}{\partial t^2} - \eta \frac{\partial w}{\partial t} \right) \\ - \frac{\rho A r^2}{\kappa AG} \frac{\partial^2}{\partial t^2} \left(f(x,t) - \rho A \frac{\partial^2 w}{\partial t^2} - \eta \frac{\partial w}{\partial t} \right) = 0 \end{aligned} \quad (3.20)$$

where κ , G , and r are the shear correction coefficient, the shear modulus, the radius of gyration, respectively. Boundary conditions are the same as in Equation 3.5, and initial conditions are as follows,

$$w(x,0) = \frac{\partial w(x,0)}{\partial t} = \frac{\partial^2 w(x,0)}{\partial t^2} = \frac{\partial^3 w(x,0)}{\partial t^3} = 0 \quad (3.21)$$

Considering Timoshenko beam model, the method of modal expansion will be used as in the Euler-Bernoulli beam model. Substituting Equation 3.7 into Equation 3.20

by employing the orthogonality conditions, the differential equation for $T_i(t)$ is found as follows,

$$\begin{aligned} \frac{d^4 T_i(t)}{dx^4} + \sigma \frac{d^3 T_i(t)}{dx^3} + \beta \frac{d^2 T_i(t)}{dx^2} + \psi \frac{dT_i(t)}{dx} + \mu T_i(t) &= \frac{2\kappa G}{r^2 \rho^2 AL} \int_0^L X_i(x) f(x, t) dx \\ - \frac{2EI}{r^2 \rho^2 A^2 L} \int_0^L X_i(x) \frac{\partial^2 f(x, t)}{\partial x^2} dx &+ \frac{2}{\rho^2 A^2 L} \int_0^L \rho A X_i(x) \frac{\partial^2 f(x, t)}{\partial t^2} dx \end{aligned} \quad (3.22)$$

where,

$$\sigma = \frac{\eta}{\rho A} \quad (3.23)$$

$$\beta = \frac{\kappa G}{r^2 \rho L^2} \left[L^2 + r^2 \lambda^2 \left(1 + \frac{E}{\kappa G} \right) \right] \quad (3.24)$$

$$\psi = \frac{\kappa G \eta}{r^2 \rho^2 A} \left[\frac{\lambda EI}{\kappa AG} + 1 \right] \quad (3.25)$$

$$\mu = \frac{EI \kappa G}{r^2 \rho^2 A} \left(\frac{\lambda}{L} \right)^4 \quad (3.26)$$

Substituting Equation 3.1 and Equation 3.9 into the right side of the Equation 3.22 yields to,

$$\begin{aligned} \frac{d^4 T_i(t)}{dx^4} + \sigma \frac{d^3 T_i(t)}{dx^3} + \beta \frac{d^2 T_i(t)}{dx^2} + \psi \frac{dT_i(t)}{dx} + \mu T_i(t) \\ = P(t) \sin \left(\frac{\lambda v}{L} t \right) \left[\frac{2\kappa G}{r^2 \rho^2 AL} + \frac{2EI}{r^2 \rho^2 A^2 L} \left(\frac{\lambda}{L} \right)^2 - \frac{2}{\rho AL} \left(\frac{\lambda v}{L} \right)^2 \right] H(L - vt) \end{aligned} \quad (3.27)$$

for $0 \leq t \leq \frac{L}{v}$

$H(x)$ is the Heaviside unit step function and it can be expressed as,

$$H(x) = \int_{-\infty}^x \delta(t) dt \quad (3.28)$$

Equation 3.27 can be solved by Mathematica. Solution of Equation 3.27 provides the time part of Equation 3.7 for the Timoshenko beam. By multiplying the time part of

Equation 3.7 with the mode shape function $X_i(x)$ the deflection curve of the beam is found. Time response diagrams of the beams are graphically expressed in the results chapter.

3.2 Moving Load on a Beam with an Elastic Foundation

Viscoelastic foundation models consist of dashpots and springs. In early investigations, the railway track is usually assumed to be linear in order to simplify the track model, even though the rail pad and ballast are essentially non-linear. [41] Here two elastic foundation models, linear and nonlinear, are used. The moving force $f(x, t)$ is expressed as,

$$f(x, t) = \delta(x - g(t))P_0 \quad (3.29)$$

$g(t)$ represents the kinematics of the moving force (constant velocity) as follow,

$$g(t) = vt \quad (3.30)$$

3.2.1 Linear Foundation Model

In Figure 3.2, a simply-supported, homogeneous, isotropic and constant cross-section beam of length L with a linear viscoelastic foundation, is shown. The Euler-Bernoulli beam is assumed to be initially at rest and undeformed.

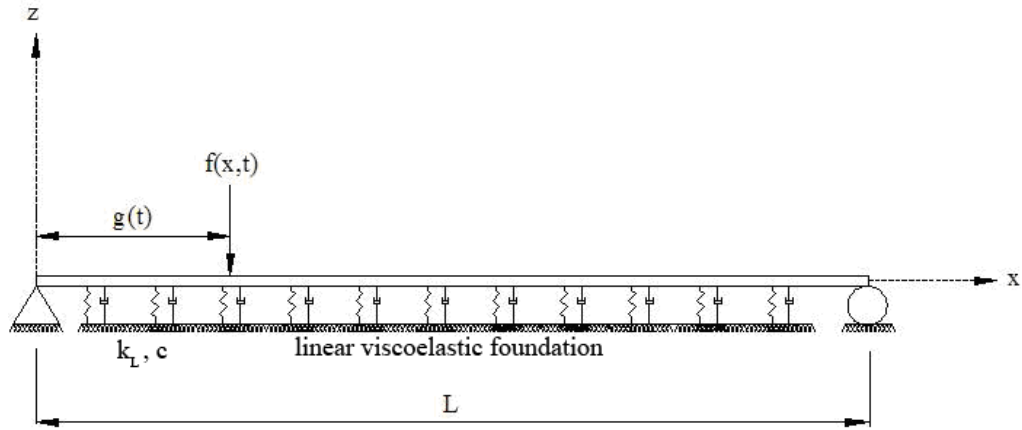


Figure 3.2: Simply-supported Beam on a Linear Viscoelastic Foundation

The problem is governed by the following differential equation below: [35]

$$EI \frac{\partial^4 w(x, t)}{\partial x^4} + \rho A \frac{\partial^2 w(x, t)}{\partial t^2} + k_L w(x, t) + c \frac{\partial w(x, t)}{\partial t} = f(x, t) \quad (3.31)$$

where k_L is the linear foundation stiffness and c is the damping coefficient of the foundation. Simply-supported beam boundary and initial conditions are the same as expressed in Equation 3.5 and 3.6. To solve Equation 3.31, numerical method of lines is used. Method of Lines is used to obtain the transverse deflection responses. In this method, one replaces the spatial differentiation by appropriate finite differences on a discrete set of nodes for numerical solution of time-dependent partial differential equations [42]. In this procedure, one can obtain a set of ordinary differential equations from Equation 3.31. Spatial derivatives are replaced by appropriate finite differences on a discrete set of nodes as follows,

$$w_k''(t) = -\left(\frac{EI}{\rho A}\right) \frac{w_{k+2}(t) - 4w_{k+1}(t) + 6w_k(t) - 4w_{k-1}(t) + w_{k-2}(t)}{h^4} - \left(\frac{k_L}{\rho A}\right) w_k(t) - \left(\frac{c}{\rho A}\right) w_k'(t) + \left(\frac{P_0 \delta(kh - g(t))}{\rho A}\right) \quad (3.32)$$

where $k=3,4,5,\dots,n-1$; h is the length of a line segment ($h=L/n$), n is the number of line segments shown in Figure 3.3.

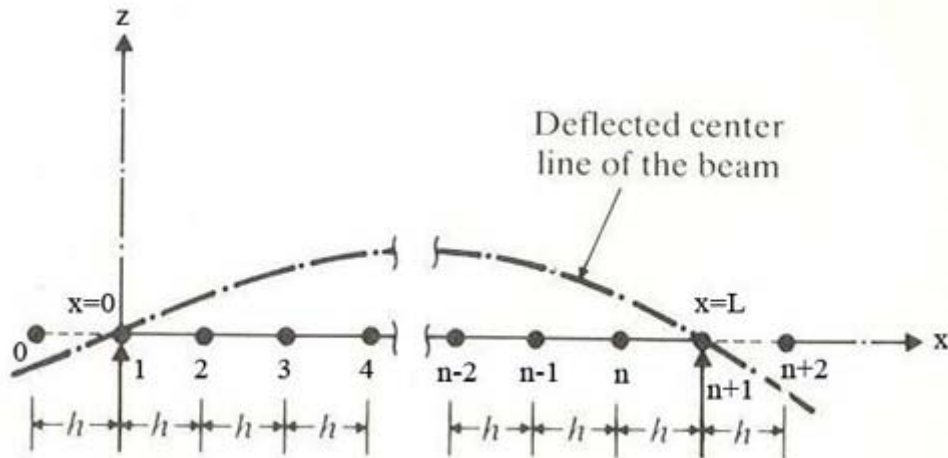


Figure 3.3: Line Segments of Beam with Simply-Supported Ends

Boundary conditions in Equation 3.5 in terms of finite differences can be indicated as,

$$w_1(t) = 0 \quad (3.33)$$

$$w_{n+1}(t) = 0 \quad (3.34)$$

$$w_1''(t) = \frac{w_1(t) - 2w_2(t) + w_3(t)}{h^2} = 0 \quad (\text{Forward Finite Difference}) \quad (3.35)$$

$$w_{n+1}''(t) = \frac{w_{n+1}(t) - 2w_n(t) + w_{n-1}(t)}{h^2} = 0 \quad (\text{Backward Finite Difference}) \quad (3.36)$$

Initial conditions are discretized as,

$$w_p(0) = 0 \quad (3.37)$$

$$\dot{w}_p(0) = 0 \quad (3.38)$$

where $p=1,2,3,\dots,n+1$.

The ordinary differential equation system with ‘the boundary’ and initial conditions indicated in Equation 3.32-3.38 are solved by a 4th order Runge-Kutta method. MATHEMATICA software is used for solution procedure. Time response histories of the beams are presented in the results and discussion section.

3.2.2 Nonlinear Foundation Model

In Figure 3.3, viscoelastic foundation is modeled by the combination of linear and cubic nonlinear springs. The Euler-Bernoulli beam is assumed to be initially at rest and undeformed.

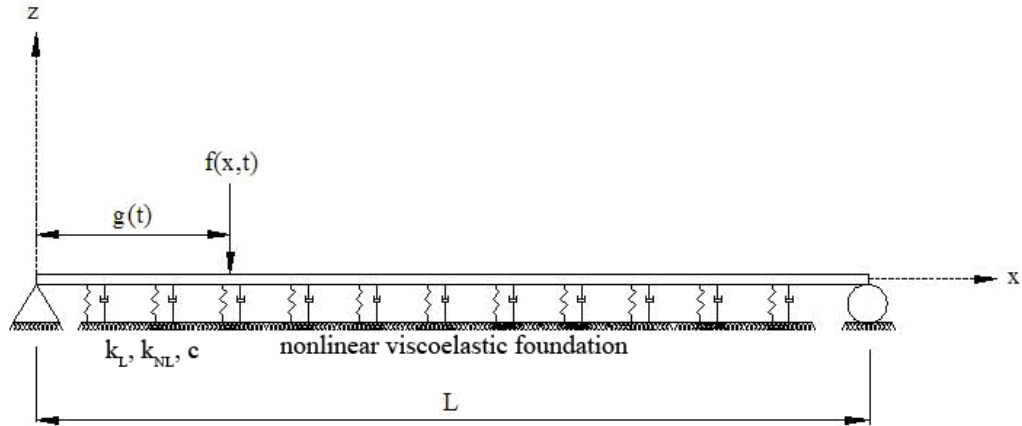


Figure 3.4: Simply-supported Beam on a Nonlinear Viscoelastic Foundation

The problem is governed by the following differential equation below: [35]

$$\begin{aligned}
& EI \frac{\partial^4 w(x,t)}{\partial x^4} + \rho A \frac{\partial^2 w(x,t)}{\partial t^2} + k_L w(x,t) \\
& + k_{NL} w^3(x,t) + c \frac{\partial w(x,t)}{\partial t} = f(x,t)
\end{aligned} \tag{3.39}$$

where k_{NL} is the nonlinear foundation stiffness.

Simply-supported beam boundary and initial conditions are the same as expressed in Equation 3.5 and 3.6. The same approach, method of lines, as in the Equation 3.31, is used for the solution of Equation 3.39. Spatial derivatives are replaced by appropriate finite differences on a discrete set of nodes as follows,

$$\begin{aligned}
w_k''(t) = & - \left(\frac{EI}{\rho A} \right) \frac{w_{k+2}(t) - 4w_{k+1}(t) + 6w_k(t) - 4w_{k-1}(t) + w_{k-2}(t)}{h^4} \\
& - \left(\frac{k_L}{\rho A} \right) w_k(t) - \left(\frac{k_{NL}}{\rho A} \right) w_k^3(t) - \left(\frac{c}{\rho A} \right) w_k'(t) + \left(\frac{P_0 \delta(kh - g(t))}{\rho A} \right)
\end{aligned} \tag{3.40}$$

Since boundary and initial conditions are the same as in linear foundation case, discretizations of them are the same expressed in Equations 3.33-3.38. The ordinary differential equation system with ‘the boundary’ and initial conditions indicated in Equation 3.40 and Equations 3.33-3.38 are solved by the same approach as explained in the previous section. Time response histories of the beams are presented in the results and discussion section.

4. SOLUTION TECHNIQUES

Two different solution techniques are used to solve the equations of motion presented in Equation 3.4, 3.20 and 3.29. For the model without foundation indicated in section 3.1, method of modal expansion technique is used. For the model with viscoelastic foundation considered in section 3.2, method of lines technique is performed. Introduction of these two methods are given in the sections below.

4.1 Method of Modal Expansion (Assumed Modes Method)

Method of modal expansion method is also known as separation of variables in mathematics. Method is applicable for linear problems, because its basis requires the usage of superposition principle. We can then hope to find so many solutions of the homogeneous differential equation that any other solution can be represented as a linear combination of these. Application of separation of variables is indicated briefly below.

Assume that we have a separable, linear partial differential equation of second order. Firstly, by applying method of separation of variables we will have two ordinary differential equations. The equation below shows the assumed solution of the partial differential equation as product of two different variables.

$$u(x, t) = F(x)G(t) \quad (4.1)$$

Secondly, the solutions of two ordinary equations that satisfy the boundary conditions must be determined. And lastly, using Fourier series, the solutions must be composed to get a solution of the called partial differential equation that also satisfies the initial conditions.

4.2 Numerical Method of Lines

Method of lines is a technique for numerical solution of time-dependent partial differential equations in which one replaces the spatial differentiation by appropriate finite differences on a discrete set of nodes. [42] Applying numerical method of lines, partial differential equation is transformed into a set of ordinary differential

equations. Spatial derivatives are replaced by appropriate finite differences on a discrete set of nodes. Finite difference approach can be classified in three groups.

4.2.1 Forward Finite Difference Method

Using Taylor series expansion, values of f_{k+1} , f_{k+2} , f_{k+3} , f_{k+4} and f_{k+5} can be calculated as,

$$f_{k+1} = f_k + hf'_k + \frac{h^2}{2!} f''_k + \frac{h^3}{3!} f'''_k + \frac{h^4}{4!} f^{iv}_k \quad (4.2)$$

$$f_{k+2} = f_k + (2h)f'_k + \frac{(2h)^2}{2!} f''_k + \frac{(2h)^3}{3!} f'''_k + \frac{(2h)^4}{4!} f^{iv}_k \quad (4.3)$$

$$f_{k+3} = f_k + (3h)f'_k + \frac{(3h)^2}{2!} f''_k + \frac{(3h)^3}{3!} f'''_k + \frac{(3h)^4}{4!} f^{iv}_k \quad (4.4)$$

$$f_{k+4} = f_k + (4h)f'_k + \frac{(4h)^2}{2!} f''_k + \frac{(4h)^3}{3!} f'''_k + \frac{(4h)^4}{4!} f^{iv}_k \quad (4.5)$$

$$f_{k+5} = f_k + (5h)f'_k + \frac{(5h)^2}{2!} f''_k + \frac{(5h)^3}{3!} f'''_k + \frac{(5h)^4}{4!} f^{iv}_k \quad (4.6)$$

Here, h represents the interval between the nodes.

4.2.1.1 Forward Finite Difference of Order O(h)

Arranging the Equation 4.2, we have

$$f'_k = \frac{f_{k+1} - f_k}{h} - \frac{h}{2!} f''_k \quad (4.7)$$

$$f'_k = \frac{f_{k+1} - f_k}{h} - O(h) \quad (4.8)$$

Multiplying Equation 4.2 by -2 and adding it with Equation 4.3 we obtain the second derivative of order O(h) as,

$$f''_k = \frac{f_k - 2f_{k+1} + f_{k+2}}{h^2} + O(h) \quad (4.9)$$

To find f_k''' Equation 4.2, 4.3 and 4.4 is multiplied by 3, -3, 1, respectively and added each other.

$$f_k''' = \frac{-f_k + 3f_{k+1} - 3f_{k+2} + f_{k+3}}{h^3} + O(h) \quad (4.10)$$

f_k^{iv} can be found multiplying Equation 4.2 Equation 4.3, Equation 4.4 and Equation 4.5 by -4, 6, -4, 1, respectively and adding each other.

$$f_k^{iv} = \frac{f_k - 4f_{k+1} + 6f_{k+2} - 4f_{k+3} + f_{k+4}}{h^4} \quad (4.11)$$

Table 4.1 summarizes the forward finite difference of order $O(h)$ of the derivatives one to four below.

Table 4.1: Forward Finite Difference of Order $O(h)$

	f_k	f_{k+1}	f_{k+2}	f_{k+3}	f_{k+4}
hf_k'	-1	1			
$h^2 f_k''$	1	-2	1		
$h^3 f_k'''$	-1	3	-3	1	
$h^4 f_k^{iv}$	1	-4	6	-4	1

4.2.1.2 Forward Finite Difference of Order $O(h)^2$

Multiplying Equation 4.2 by 4 and subtracting Equation 4.3 from it, we have f_k' as,

$$f_k' = \frac{-3f_k + 4f_{k+1} - f_{k+2}}{2\Delta} + O(\Delta)^2 \quad (4.12)$$

Multiplying Equation 4.2, 4.3 and 4.4 by -5, 4, -1, respectively and adding them, we have f_k'' as,

$$f_k'' = \frac{2f_k - 5f_{k+1} + 4f_{k+2} - f_{k+3}}{h^2} + O(h)^2 \quad (4.13)$$

Multiplying Equation 4.2, 4.3, 4.4 and 4.5 by 18, -24, 14, -3, respectively and adding them, we have f_k''' as,

$$f_k''' = \frac{-5f_k + 18f_{k+1} - 24f_{k+2} + 14f_{k+3} - 3f_{k+4}}{2h^3} + O(h)^2 \quad (4.14)$$

Multiplying Equation 4.2, 4.3, 4.4, 4.5 and 4.6 by -14, 26, -24, 11, -2, respectively and adding them, we have f_k^{iv} as,

$$f_k^{iv} = \frac{3f_k - 14f_{k+1} + 26f_{k+2} - 24f_{k+3} + 11f_{k+4} - 2f_{k+5}}{h^4} + O(h)^2 \quad (4.15)$$

Table 4.2 summarizes the forward finite difference of order $O(h)^2$ of the derivatives one to four below.

Table 4.2: Forward Finite Difference of Order $O(h)^2$

	f_k	f_{k+1}	f_{k+2}	f_{k+3}	f_{k+4}	f_{k+5}
$2hf_k'$	-3	4	-1			
h^2f_k''	2	-5	4	-1		
$2h^3f_k'''$	-5	18	-24	14	-3	
$h^4f_k^{iv}$	3	-14	26	-24	11	-2

4.2.2 Backward Finite Difference Method

Using Taylor series expansion, values of f_{k-1} , f_{k-2} , f_{k-3} , f_{k-4} and f_{k-5} can be calculated as,

$$f_{k-1} = f_k - hf_k' + \frac{h^2}{2!}f_k'' - \frac{h^3}{3!}f_k''' + \frac{h^4}{4!}f_k^{iv} \quad (4.16)$$

$$f_{k-2} = f_k - (2h)f'_k + \frac{(2h)^2}{2!}f''_k - \frac{(2h)^3}{3!}f'''_k + \frac{(2h)^4}{4!}f^{iv}_k \quad (4.17)$$

$$f_{k-3} = f_k - (3h)f'_k + \frac{(3h)^2}{2!}f''_k - \frac{(3h)^3}{3!}f'''_k + \frac{(3h)^4}{4!}f^{iv}_k \quad (4.18)$$

$$f_{k-4} = f_k - (4h)f'_k + \frac{(4h)^2}{2!}f''_k - \frac{(4h)^3}{3!}f'''_k + \frac{(4h)^4}{4!}f^{iv}_k \quad (4.19)$$

$$f_{k-5} = f_k - (5h)f'_k + \frac{(5h)^2}{2!}f''_k - \frac{(5h)^3}{3!}f'''_k + \frac{(5h)^4}{4!}f^{iv}_k \quad (4.20)$$

4.2.2.1 Backward Finite Difference of Order O(h)

Arranging the Equation 4.16, we have

$$f'_k = \frac{f_k - f_{k-1}}{h} + O(h) \quad (4.21)$$

Multiplying Equation 4.16 by -2 and adding it with Equation 4.17 we obtain the second derivative of order O(h) as,

$$f''_k = \frac{f_k - 2f_{k-1} + f_{k-2}}{h^2} + O(h) \quad (4.22)$$

To find f'''_k Equations 4.16, 4.17 and 4.18 is multiplied by -3, 3, -1, respectively and added each other.

$$f'''_k = \frac{-f_{k-3} + 3f_{k-2} - 3f_{k-1} + f_k}{h^3} + O(h) \quad (4.23)$$

f^{iv}_k can be found multiplying Equations 4.16, 4.17, 4.18 and 4.19 by -4, 6, -4, 1, respectively and adding each other.

$$f^{iv}_k = \frac{f_{k-4} - 4f_{k-3} + 6f_{k-2} - 4f_{k-1} + f_k}{h^4} + O(h) \quad (4.24)$$

Table 4.3 indicates the forward finite difference of order O(h) of the derivatives one to four below.

Table 4.3: Backward Finite Difference of Order $O(h)$

	f_{k-4}	f_{k-3}	f_{k-2}	f_{k-1}	f_k
hf'_k				-1	1
$h^2 f''_k$			1	-2	1
$h^3 f'''_k$		-1	3	-3	1
$h^4 f^{iv}_k$	1	-4	6	-4	1

4.2.2.2 Backward Finite Difference of Order $O(h)^2$

Multiplying Equation 4.16 by -4 and adding with Equation 4.17, we have f'_k as,

$$f'_k = \frac{3f_k - 4f_{k-1} + f_{k-2}}{2h} + O(h)^2 \quad (4.25)$$

Multiplying Equation 4.16, 4.17 and 4.18 by -5, 4, -1, respectively and adding them, we have f''_k as,

$$f''_k = \frac{2f_k - 5f_{k-1} + 4f_{k-2} - f_{k-3}}{h^2} + O(h)^2 \quad (4.26)$$

Multiplying Equation 4.16, 4.17, 4.18 and 4.19 by -18, 24, -14, 3, respectively and adding them, we have f'''_k as,

$$f'''_k = \frac{5f_k - 18f_{k-1} + 24f_{k-2} - 14f_{k-3} + 3f_{k-4}}{2h^3} + O(h)^2 \quad (4.27)$$

Multiplying Equation 4.16, 4.17, 4.18, 4.19 and 4.20 by 14, 26, -24, 11, -2, respectively and adding them, we have f^{iv}_k as,

$$f^{iv}_k = \frac{3f_k - 14f_{k-1} + 26f_{k-2} - 24f_{k-3} + 11f_{k-4} - 2f_{k-5}}{h^4} + O(h)^2 \quad (4.28)$$

Table 4.4 summarizes the forward finite difference of order $O(h)^2$ of the derivatives one to four below.

Table 4.4: Backward Finite Difference of Order $O(h)^2$

	f_{k-5}	f_{k-4}	f_{k-3}	f_{k-2}	f_{k-1}	f_k
$2hf'_k$				1	-4	3
$h^2 f''_k$			-1	4	-5	2
$2h^3 f'''_k$		3	-14	24	-18	5
$h^4 f^{iv}_k$	-2	11	-24	26	-14	3

4.2.3 Central Finite Difference Method

4.2.3.1 Central Finite Difference Method of Order $O(h)^2$

Subtracting Equation 4.16 from Equation 4.2,

$$\begin{aligned}
 f_{k+1} - f_{k-1} &= f_k + hf'_k + \frac{h^2}{2!} f''_k + \frac{h^3}{3!} f'''_k + \frac{h^4}{4!} f^{iv}_k \\
 &\quad - f_k + hf'_k - \frac{h^2}{2!} f''_k + \frac{h^3}{3!} f'''_k - \frac{h^4}{4!} f^{iv}_k
 \end{aligned} \tag{4.29}$$

$$f_{k+1} - f_{k-1} = 2hf'_k + \frac{h^3}{3} f'''_k \tag{4.30}$$

$$f'_k = \frac{f_{k+1} - f_{k-1}}{2h} - \frac{h^2 f'''_k}{6} \tag{4.31}$$

$$f'_k = \frac{f_{k+1} - f_{k-1}}{2h} - O(h)^2 \tag{4.32}$$

Adding Equations 4.2 and 4.16,

$$\begin{aligned}
 f_{k+1} + f_{k-1} &= f_k + hf'_k + \frac{h^2}{2!} f''_k + \frac{h^3}{3!} f'''_k + \frac{h^4}{4!} f^{iv}_k \\
 &\quad + f_k - hf'_k + \frac{h^2}{2!} f''_k - \frac{h^3}{3!} f'''_k + \frac{h^4}{4!} f^{iv}_k
 \end{aligned} \tag{4.33}$$

$$f_{k+1} + f_{k-1} = 2f_k + h^2 f''_k + \frac{h^4}{12} f^{iv}_k \tag{4.34}$$

$$f_k'' = \frac{f_{k+1} - 2f_k + f_{k-1}}{h^2} - \frac{h^2}{12} f_k^{iv} \quad (4.35)$$

$$f_k'' = \frac{f_{k+1} - 2f_k + f_{k-1}}{h^2} + O(h)^2 \quad (4.36)$$

To find the term f_k''' Equations 4.17, 4.16, 4.2 and 4.3 are multiplied by -1, 2, -2, 1, respectively and added each other. f_k^{iv} can be found by multiplying Equations 4.17, 4.16, 4.2 and 4.3 by 1, -4, -4, 6 and adding them.

$$f_k''' = \frac{-f_{k-2} + 2f_{k-1} - 2f_{k+1} + f_{k+2}}{2h^3} + O(h)^2 \quad (4.37)$$

$$f_k^{iv} = \frac{f_{k-2} - 4f_{k-1} + 6f_k - 4f_{k+1} + f_{k+2}}{h^4} + O(h)^2 \quad (4.38)$$

Table 4.5: Central Finite Difference of Order $O(h)^2$

	f_{k-2}	f_{k-1}	f_k	f_{k+1}	f_{k+2}
$2hf_k'$		-1	0	1	
$h^2 f_k''$		1	-2	1	
$2h^3 f_k'''$	-1	2	0	-2	1
$h^4 f_k^{iv}$	1	-4	6	-4	6

4.2.3.2 Central Finite Difference Method of Order $O(h)^4$

Multiplying Equations 4.17, 4.16, 4.2 and 4.3 by 1, -8, 8, -1, respectively and adding them, f_k' can be found as

$$f_k' = \frac{f_{k-2} - 8f_{k-1} + 8f_{k+1} - f_{k+2}}{12h} + O(h)^4 \quad (4.39)$$

Multiplying Equations 4.17, 4.16, 4.2 and 4.3 by -1, 16, 16, -1, respectively and adding them, f_k'' can be found as

$$f_k'' = \frac{-f_{k-2} + 16f_{k-1} - 30f_k + 16f_{k+1} - f_{k+2}}{12h^2} + O(h)^4 \quad (4.40)$$

Multiplying Equations 4.18, 4.17, 4.16, 4.2, 4.3 and 4.4 by 1, -8, 13, -13, 8, -1 respectively and adding them, f_k''' can be found as

$$f_k''' = \frac{f_{k-3} - 8f_{k-2} + 13f_{k-1} - 13f_{k+1} + 8f_{k+2} - f_{k+3}}{8h^3} + O(h)^4 \quad (4.41)$$

Multiplying Equations 4.18, 4.17, 4.16, 4.2, 4.3 and 4.4 by -1, 12, -39, -39, 12, -1 respectively and adding them, f_k^{iv} can be found as

$$f_k^{iv} = \frac{-f_{k-3} + 12f_{k-2} - 39f_{k-1} + 56f_k - 39f_{k+1} + 12f_{k+2} - f_{k+3}}{6h^4} + O(h)^4 \quad (4.42)$$

Table 4.6: Central Finite Difference of Order $O(h)^4$

	f_{k-3}	f_{k-2}	f_{k-1}	f_k	f_{k+1}	f_{k+2}	f_{k+3}
$12hf_k'$		1	-8	0	8	-1	
$12h^2f_k''$		-1	16	-30	16	-1	
$8h^3f_k'''$	1	-8	13	0	-13	8	-1
$6h^4f_k^{iv}$	-1	12	-39	56	-39	12	-1

5. RESULTS AND DISCUSSION

Results are given for an example beam [43] with properties below:

Table 5.1: Beam Properties

Young Modulus, E	$20.6 \times 10^{10} \text{ N/m}^2$
Shear Modulus, G	77 Gpa
Density, ρ	7850 kg/m^3
Cross-Sectional Area, A	$6.37 \times 10^{-3} \text{ m}^2$
Area Moment of Inertia, I	$2.037 \times 10^{-5} \text{ m}^4$
Length, L	500 m
Shear Correction Coefficient, κ	0.40
Foundation Stiffness (Linear), k_L	$3.73 \times 10^7 \text{ N/m}^2$
Foundation Stiffness (Non-linear), k_{NL}	$3.73 \times 10^{14} \text{ N/m}^4$

5.1 Results for Moving Load on a Beam without an Elastic Foundation

Euler-Bernoulli and Timoshenko beam models are both simply supported, due to the symmetry, maximum deflections will occur at the mid-span of the beams, that is $L/2$.

To make a good comparison, dynamic deflections are non-dimensionalized by static

deflections which is $w_0 = \frac{P_0 L^3}{48EI}$ and time variable is non-dimensionalized $\left(S = \frac{t}{\bar{t}} \right)$

by the total time of the load, $\bar{t}$, in order to travel from the left-end to right-end of the

beam. Non-dimensional speed parameter is $\alpha = \frac{v}{v_{cr}}$, where v_{cr} is the critical speed,

$v_{cr} = \frac{\omega_i L}{\pi}$. [18] For the accelerated motion, the force is assumed to be at rest at $x=0$

and reaches the speed v at $x=L$. For the constant velocity motion, the force is travelling at constant speed v along the beam. For the decelerated motion, the force is assumed to be at a speed v at $x=0$ and stops at $x=L$. Results are graphically displayed for all force and motion cases with several speed parameters and damping coefficients. Vertical axis represents the ratio of maximum dynamic deflection to static deflection, horizontal axis represents the non-dimensional time S .

5.1.1 Constant Velocity Motion with All Force Cases

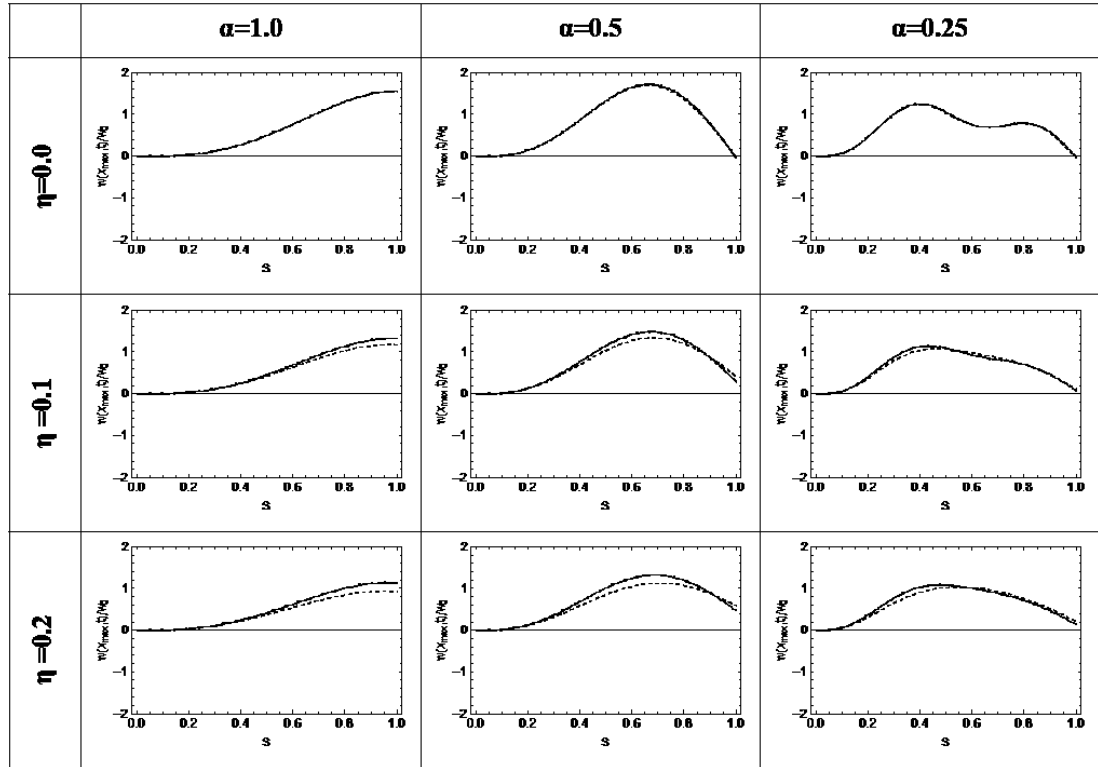


Figure 5.1: Dynamic response of the beam under moving constant load with constant velocity. Straight Line (Timoshenko), Dashed Line (Euler-Bernoulli)

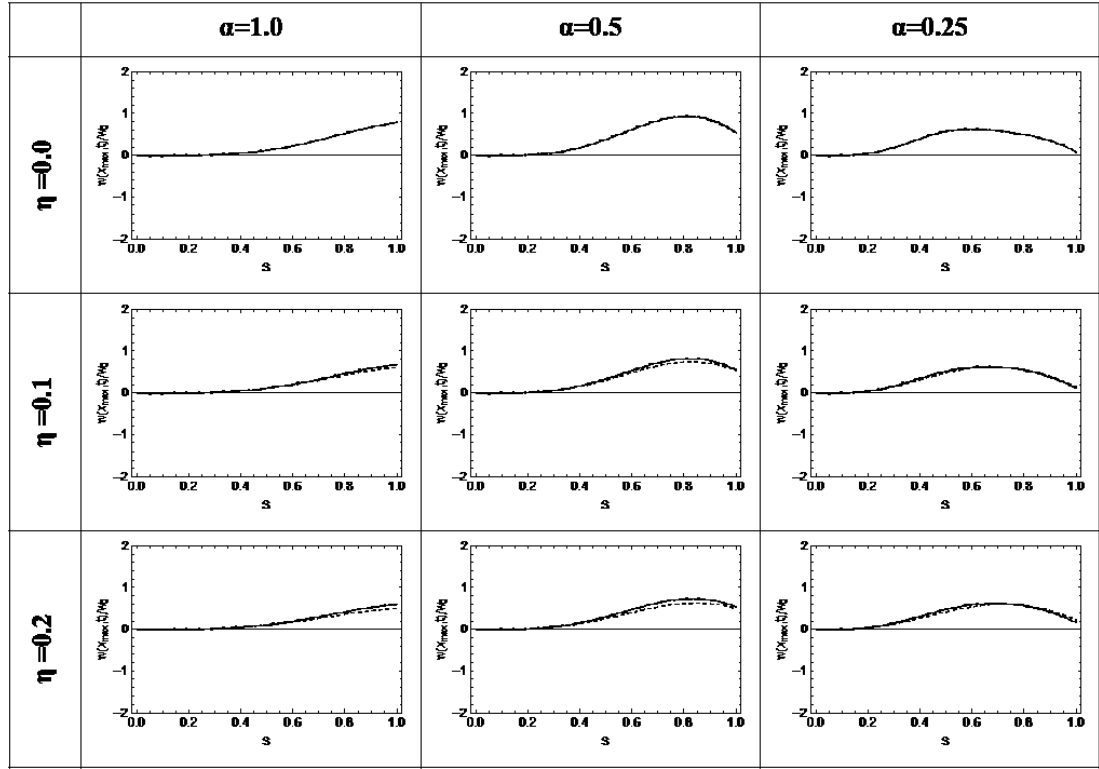


Figure 5.2: Dynamic response of the beam under moving linearly increasing load with constant velocity. Straight Line (Timoshenko), Dashed Line (Euler-Bernoulli)

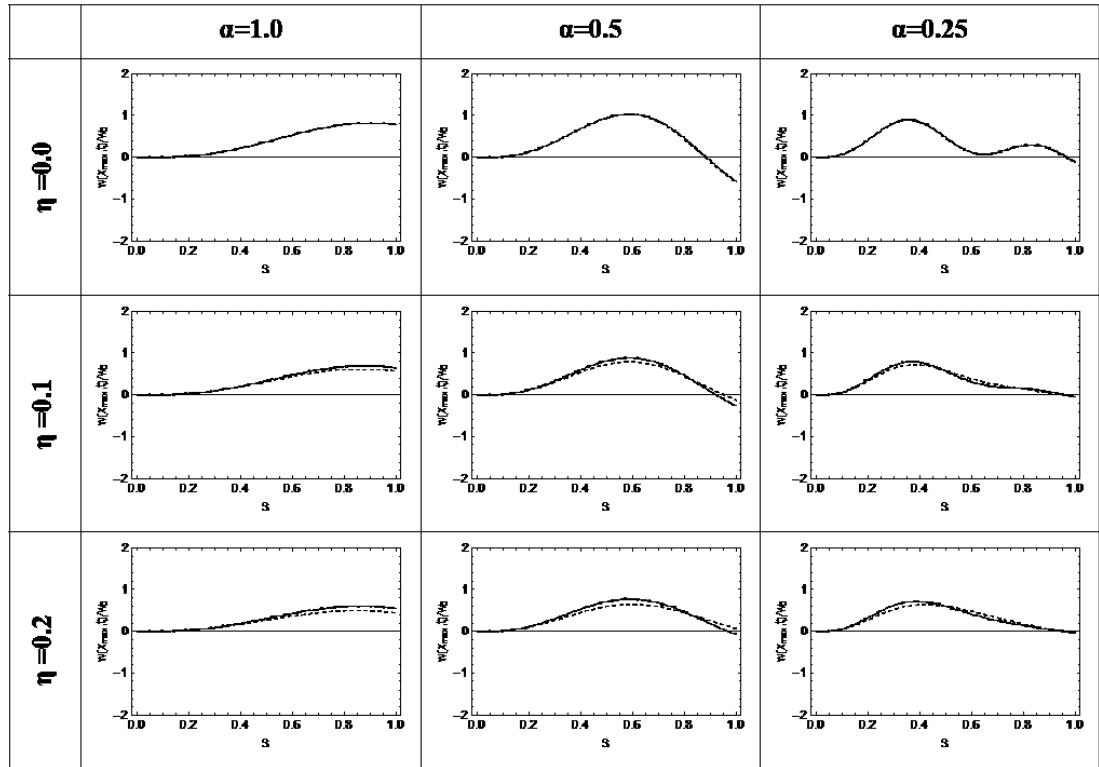


Figure 5.3: Dynamic response of the beam under moving linearly decreasing load with constant velocity. Straight Line (Timoshenko), Dashed Line (Euler-Bernoulli)

5.1.2 Accelerated Motion Case with All Force Cases

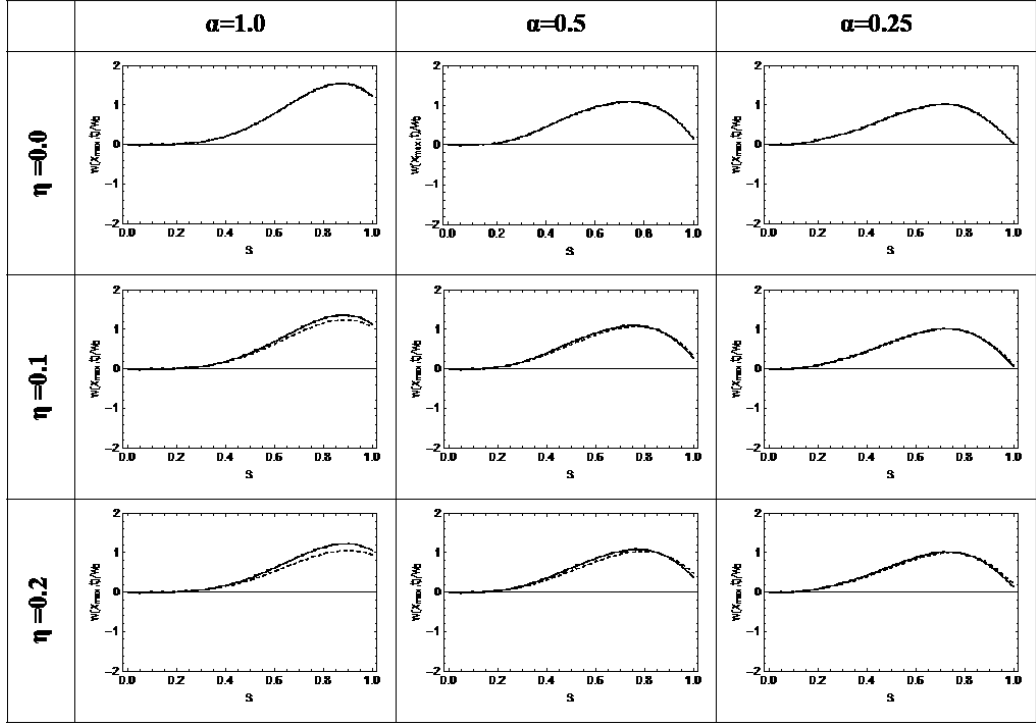


Figure 5.4: Dynamic response of the beam under moving constant load with acceleration. Straight Line (Timoshenko), Dashed Line (Euler-Bernoulli)

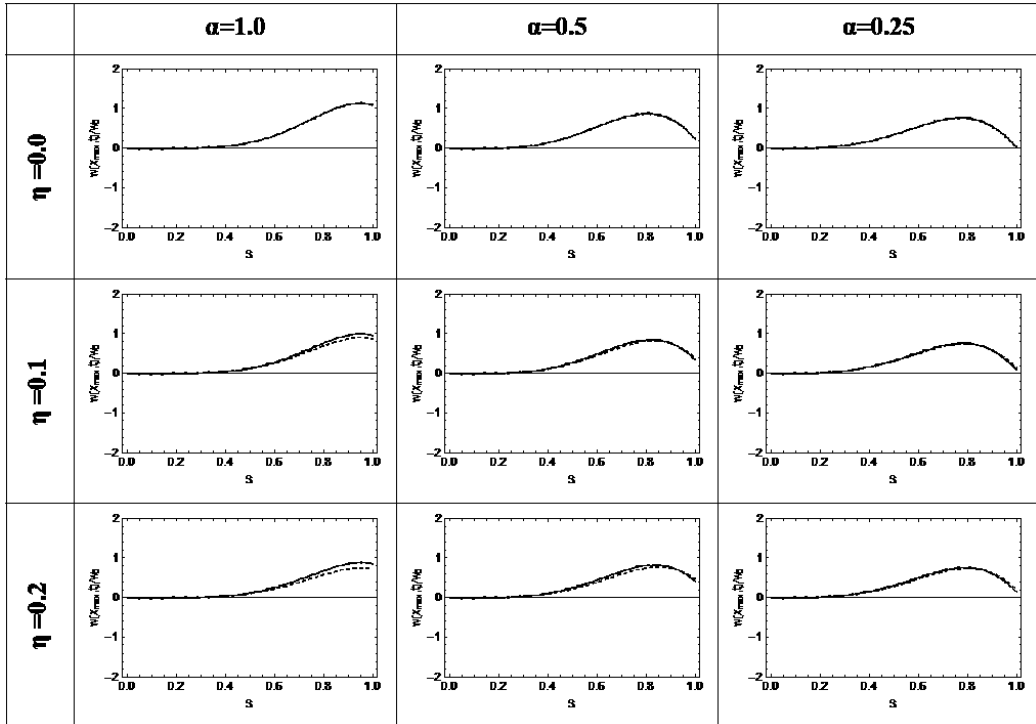


Figure 5.5: Dynamic response of the beam under moving linearly increasing load with acceleration. Straight Line (Timoshenko), Dashed Line (Euler-Bernoulli)

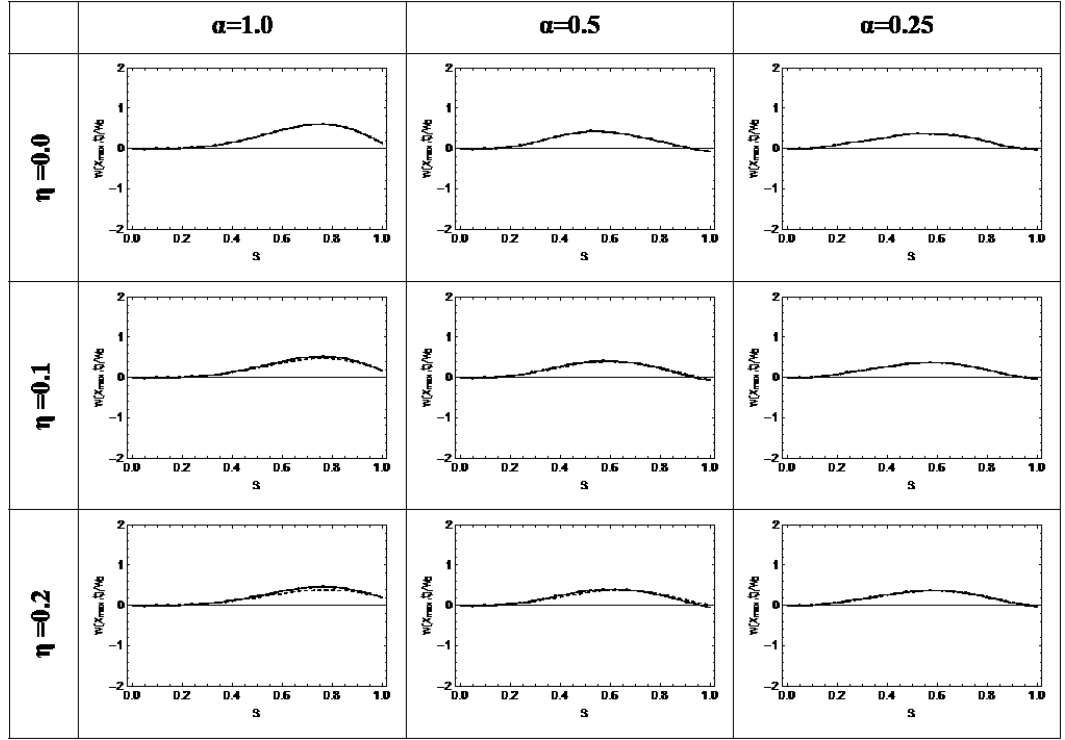


Figure 5.6: Dynamic response of the beam under moving linearly decreasing load with acceleration. Straight Line (Timoshenko), Dashed Line (Euler-Bernoulli)

5.1.3 Decelerated Motion Case with All Force Cases

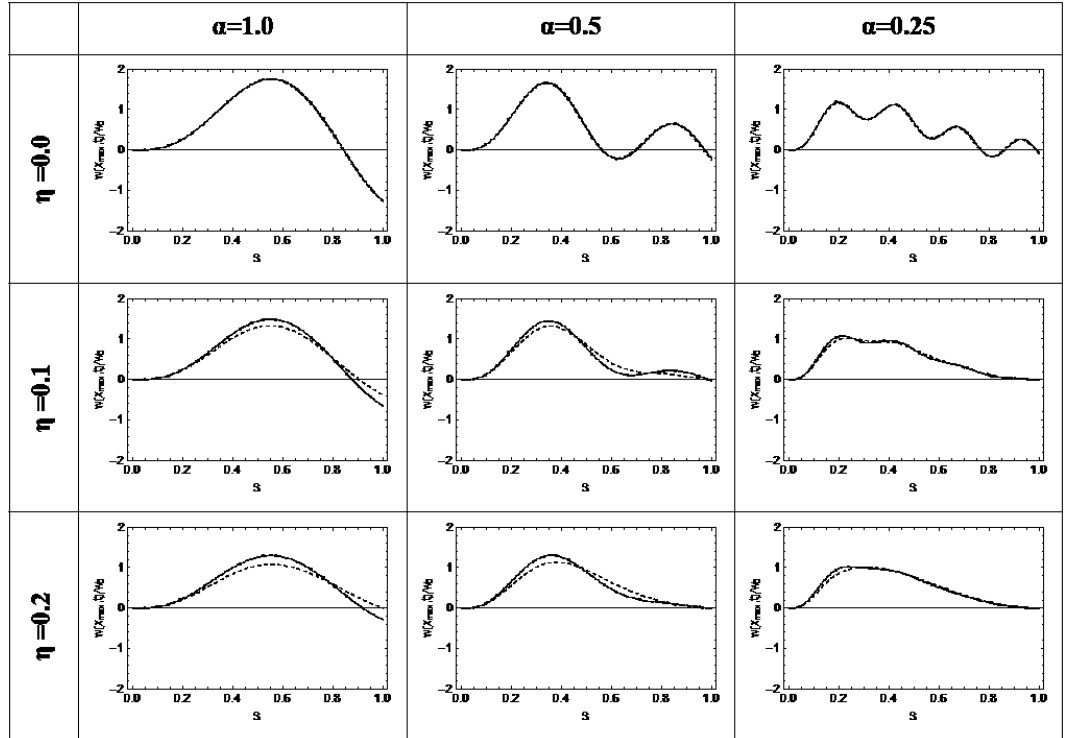


Figure 5.7: Dynamic response of the beam under moving constant load with deceleration. Straight Line (Timoshenko), Dashed Line (Euler-Bernoulli)

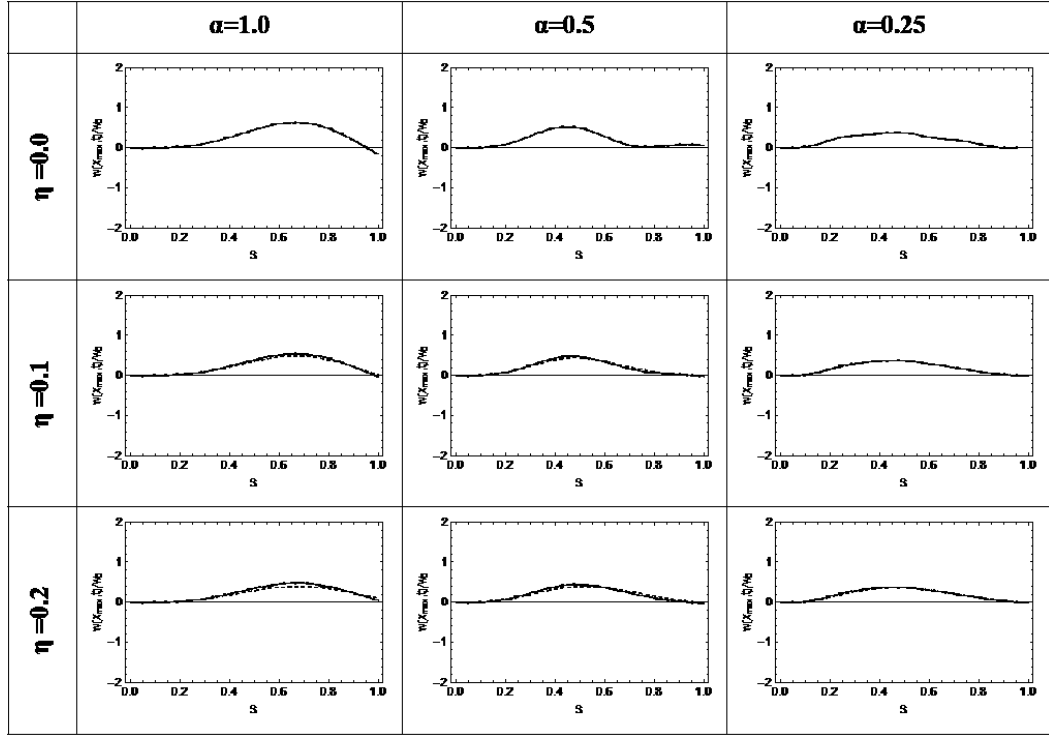


Figure 5.8: Dynamic response of the beam under moving linearly increasing load with deceleration. Straight Line (Timoshenko), Dashed Line (Euler-Bernoulli)

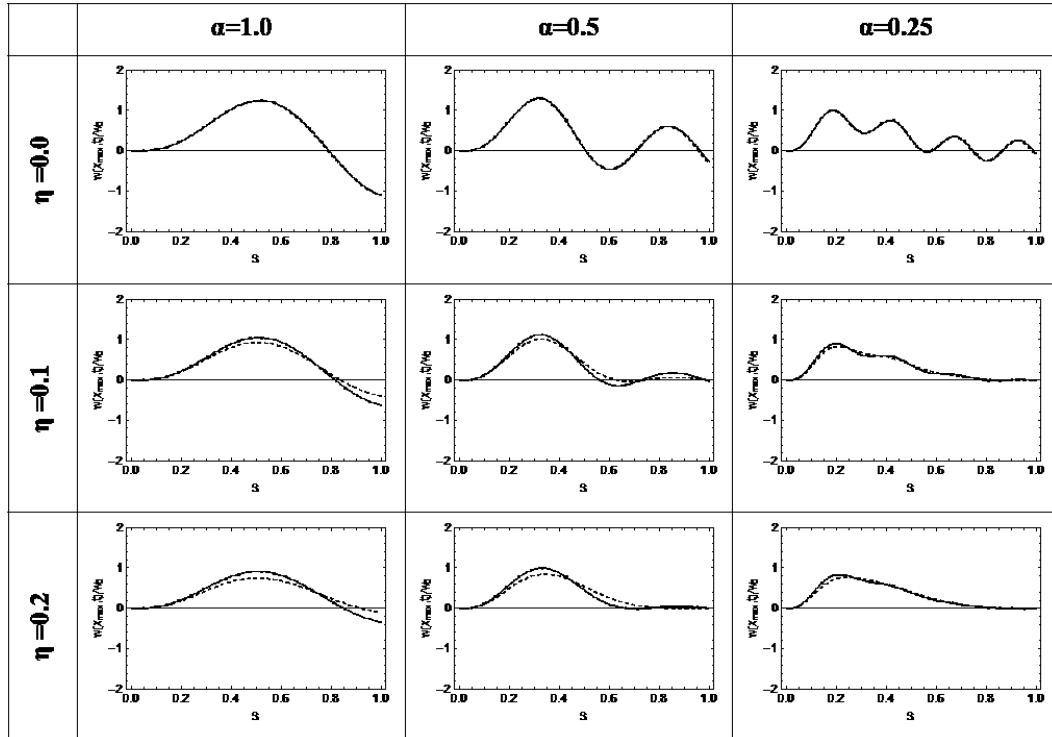


Figure 5.9: Dynamic response of the beam under moving linearly decreasing load with deceleration. Straight Line (Timoshenko), Dashed Line (Euler-Bernoulli)

5.2 Conclusions

From the Figures 4.1 to 4.9, the conclusions below can be deduced. [32, 33]

- For all types of motion and all force cases that include damping, the ratios of the maximum dynamic to maximum static deflections found by Timoshenko beam theory are higher than the ratios found by Euler-Bernoulli beam theory. This means that the moving load problem can be more critical when the shear deformation and rotary inertia effects are taken into account.
- It is verified that damping is inversely proportional to the dynamic response. On the other hand, when damping ratio is increased, the differences in the response of two models rise. It shows that Timoshenko beam theory based mathematical models are stiffer than Euler-Bernoulli based ones in terms of transverse vibration.
- The most critical situation for the moving load problem is decelerated motion with a constant magnitude load case shown in Figure 4.7. For $\alpha=1.0$ and $\eta=0.0$, the ratio of maximum dynamic deflection to maximum static deflection is approximately 1.8.
- It is observed that, increasing the speed of the load (increasing α) delays the time at which maximum response occurs.
- Linearly increasing and linearly decreasing loads both have an attenuating effect on the response of the beam compared to the constant load.

5.3 Results for Moving Load on a Beam with an Elastic Foundation

In order to use the technique, method of lines, the length of the beam is divided into 100 line segments. ($n=100$) The concentrated force, P_0 is assumed as 98000 N. [43] Since the beam is simply-supported at both ends, maximum deflections for a single concentrated load will occur at the mid-span of the beam.

The moving force velocities of 50, 250 and 500 m/s are considered. At each velocity, undamped and damped dynamic responses for linear and nonlinear viscoelastic foundation models are investigated as in Figures 5.10-5.12. For the linear foundation model, damping ratios are determined by considering the critical damping coefficient c_{cr} . The damping ratio ξ is expressed as,

$$\xi = \frac{c}{c_{cr}} \quad (5.1)$$

where

$$c_{cr} = 2\sqrt{k_L \rho A} \quad (5.2)$$

Deflection response histories of beams with two foundation models at the force speeds of 50, 250 and 500 m/s are plotted below. Black line represents the linear foundation model and the red line represents nonlinear foundation model.

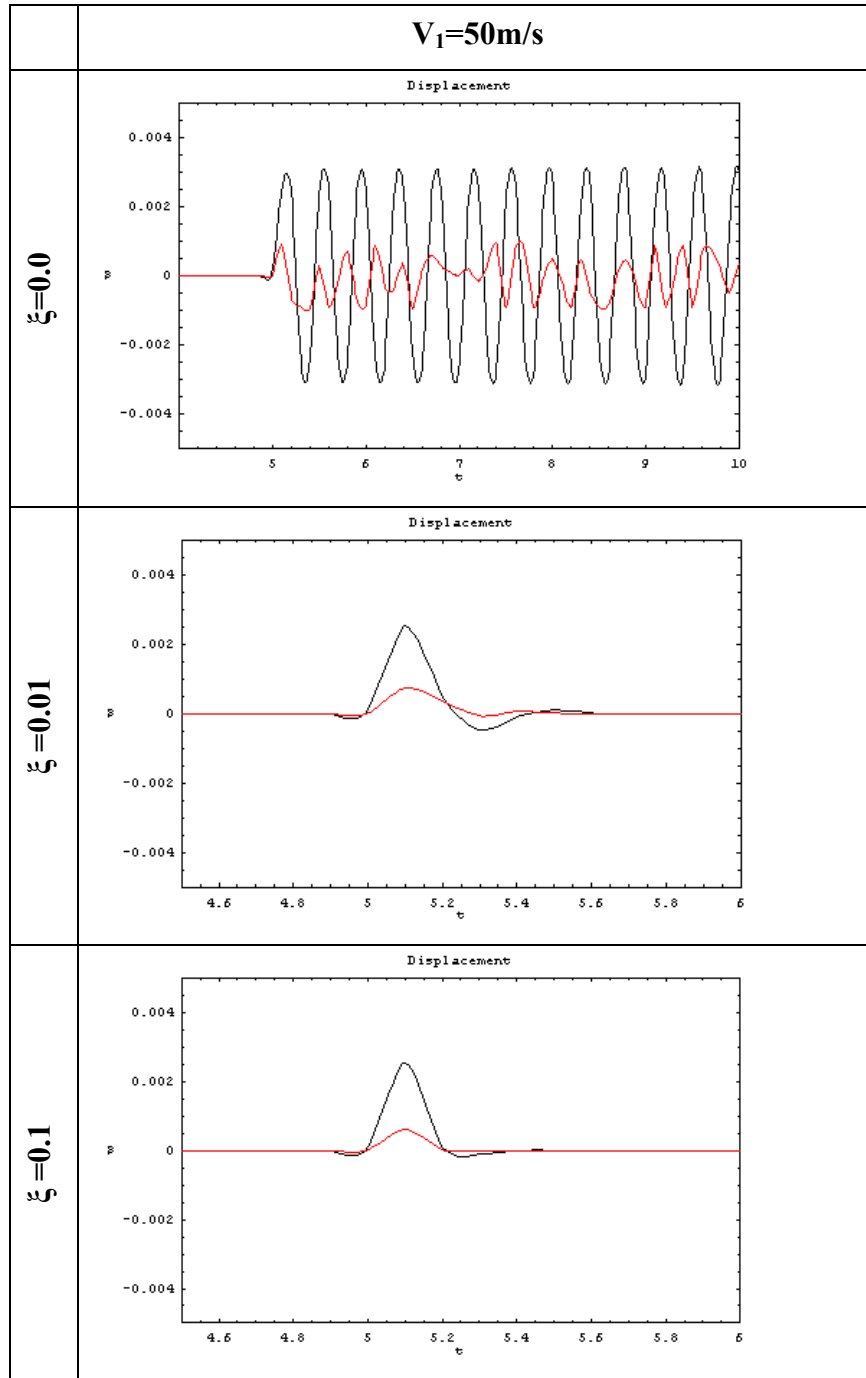


Figure 5.10: Deflection response histories of beams with two foundation models at the force speed of 50 m/s

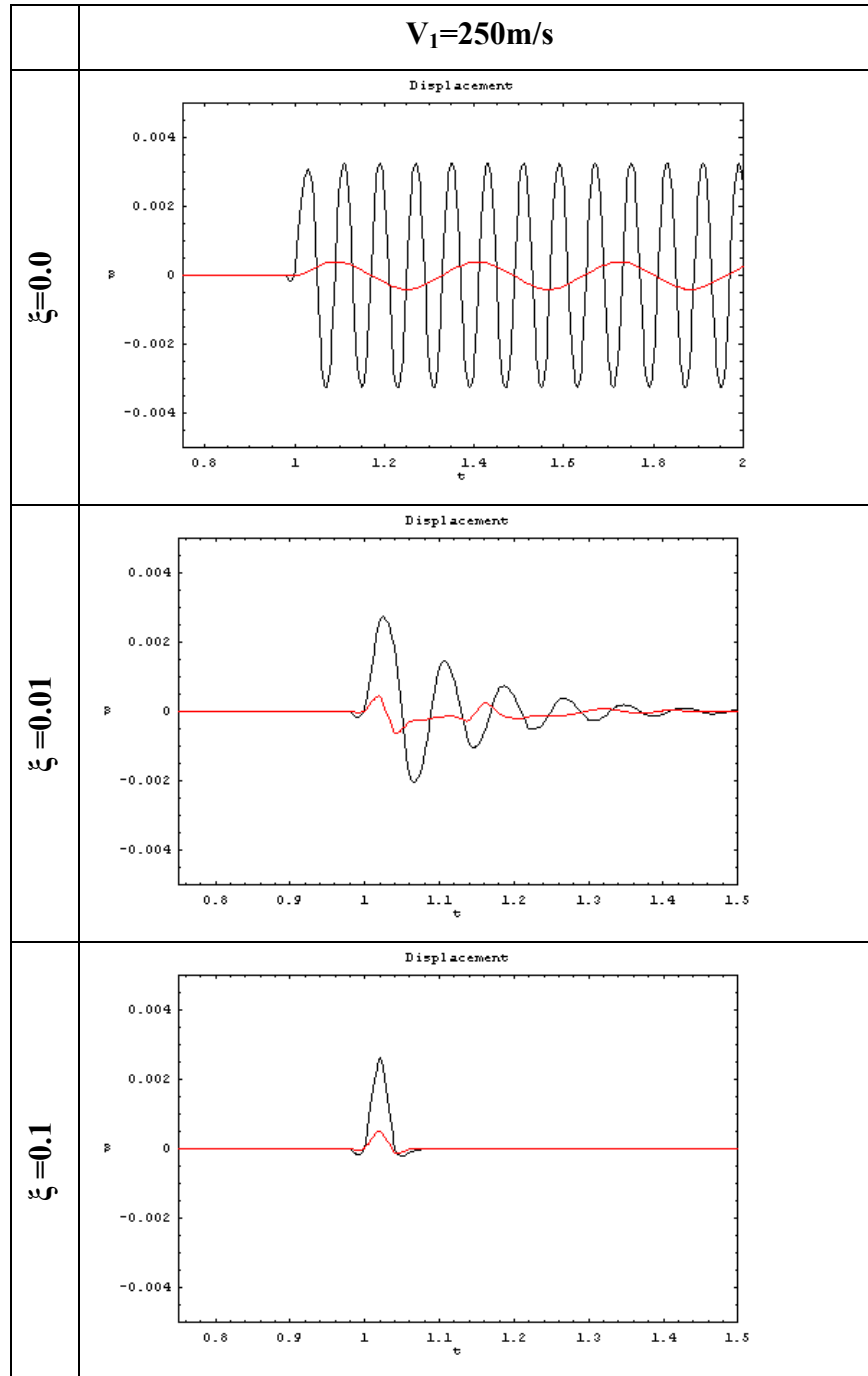


Figure 5.11: Deflection response histories of beams with two foundation models at the force speed of 250 m/s

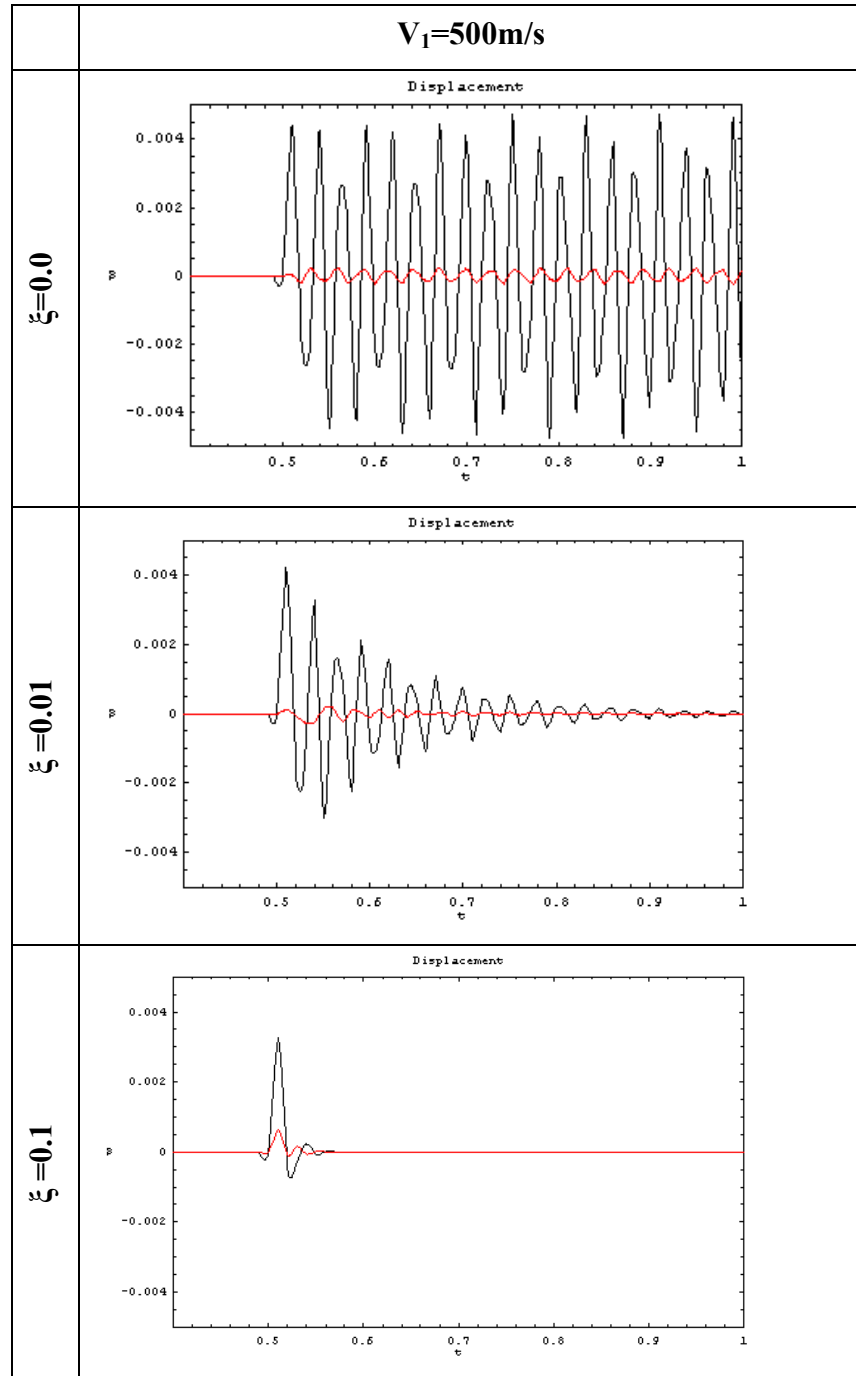


Figure 5.12: Deflection response histories of beams with two foundation models at the force speed of 500 m/s.

5.4 Conclusions

Reference [43] investigated the dynamic behavior of a finite railway under the high-speed multiple moving forces by using finite element method. The transverse

deflection for undamped ($\xi=0.0$) cases in Figures 5.10-5.12 are in close agreement with the transverse deflection found in [43].

It can be seen that the transverse deflection response amplitudes for the nonlinear foundation models are significantly smaller than those for the linear foundation models.

For the linear undamped cases, there is a periodicity in the dynamic response for all moving speeds, but, for the nonlinear undamped cases, the periodicity vanishes for moving speed of 50 m/s as shown in Figures 5.10-5.12.

As the damping ratio (ξ) increases, the dynamic deflections decrease and die out very quickly for all moving speeds in both models in Figures 5.10-5.12.

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RESUME

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