NUMERICAL COMPUTATION OF 2-D LAMINAR FLOWS
IN COMPLEX GEOMETRIES

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PREFACE

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Hale AKÇAY
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NOMENCLATURE

2-D : two-dimensional
3-D : three-dimensional
AE, AW, AN, : coefficients in the discretization equations for
AS, AP momentum and pressure-correction equations
B11, B12, : coefficients of the pressure gradient terms in the discretization
B21, B22 equations for contravariant velocity fluxes
C11, C12, : coefficients of the pressure gradient terms in a unified form in
C21, C22 the discretization equation for contravariant velocity fluxes
D : diffusional conductance
F : flow rate through the cell faces
Hu : term on the right hand side of the discretization equation for
contravariant velocity fluxes
J : jacobian of the transformation
L : cavity length
mp : mass residual of a control volume in the pressure-correction
equation
P : pressure
Re : Reynolds Number
Res : residual
S : source terms
u, v : Cartesian velocity components
U, V : contravariant velocity fluxes
uL : cavity lid velocity
x, y : cartesian coordinates

Greek Symbols

α : under-relaxation factor
αP : pressure under-relaxation factor
αu, αv : velocity under-relaxation factor
β : angle between the side wall of a cavity and the horizontal line
\( \delta x, \delta y \) : distances between the grid nodes in the physical space

\( \delta \xi, \delta \eta \) : distances between the grid nodes in the computational space

\( \Delta x, \Delta y \) : distances between the cell faces in the physical space

\( \Delta \xi, \Delta \eta \) : distances between the cell faces in the computational space

\( \mu \) : dynamic viscosity of the fluid

\( \rho \) : density of the fluid

\( \Phi \) : dependent variable

**Superscripts**

* : indicates the imperfect values of the velocity and the pressure

' : indicates the correction values of the velocity and the pressure

**Subscripts**

e, w, n, s : control volume faces (east, west, north, south)

E, W, N, S, NE, : grid points

NW, SE, SW :
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KOMPLEKS GEOMETRİLERDE 2 BOYUTLU LAMİNAR AKIŞLARIN SAYISAL ÇÖZÜMLEMESİ

ÖZET


Kodun kesinliği, değişik duvar eğilimleri için üzerinde akış olan bir kuyu içindeki ve Re=10, 20 ve 40 için dairesel bir silindirin üzerindeki akıların hesaplanmasıyla doğrulanmıştır.


NUMERICAL COMPUTATION OF 2-D LAMINAR FLOWS IN COMPLEX GEOMETRIES

SUMMARY

A numerical methodology has been developed to solve steady laminar flows in two dimensional domains using curvilinear coordinates. The finite volume procedure is employed to discretize the governing equations on a collocated and structured grid arrangement. A hybrid differencing scheme is used to treat the convection-diffusion terms in the momentum equations. Both the SIMPLE and SIMPLEC algorithms are adopted for the velocity and pressure coupling. The modified version of the Rhie and Chow momentum interpolated method is selected for calculating the convective fluxes on the cell faces to get good coupling between the velocity and the pressure field.

The accuracy of the code is validated by calculating laminar flows in a lid-driven cavity with inclined walls with different angles and the steady flow past a circular cylinder for various Reynolds numbers from Re=10, 20 and 40.

This study analyzes the convergence performances of the simplified pressure-correction equation, full pressure-correction equation and the treatment of Cho and Chung on the mass flux corrections when the grid non-orthogonality becomes appreciable. The proposed methods have been tested for typical non-orthogonal two-dimensional cavity flows. The results show that the SIMPLEC algorithm is superior to the SIMPLE algorithm when simplified and full pressure-correction equation methods are used. If computational grid is not severely non-orthogonal ($\beta \geq 45^\circ$), it is more logical to use simplified version than the full one. The computer program is simpler and less memory is needed. On the other hand, full pressure-correction equation method converges fastest in a limitless range of $\alpha_p$ when the grid skewness increases. The Cho and Chung’s method serves inefficient performance if the SIMPLEC algorithm is employed. Although there is no limit to the ranges of $\alpha_p$ values, the convergence rate of the method is low.
As the performance of MSIP, LR, SIP and CGS solvers are compared, CGS converges rapidly than SIP algorithm, but there is not much difference between the performance of the MSIP and LR solvers.
1. INTRODUCTION

Numerical solution of viscous incompressible flows in complex geometries is needed for the modeling of many practical fluid flow and heat transfer problems. The meshing flexibility in the flow simulation for complex geometries has led to the development of solution algorithms based on non-orthogonal meshes. In a majority of these algorithms, the governing equations are represented in terms of generalized curvilinear co-ordinates, which are obtained by mapping the complex physical geometry into a simplified computational domain.

The difference among various numerical methods using non-orthogonal grids is in two aspects. One is the choice of the grid arrangement. The other is velocity components as the dependent variables in the momentum equations. The flow solutions have been obtained using staggered or collocated grid arrangements. The common practice is to use a staggered grid arrangement. In the staggered grid arrangement, the scalar quantities are stored at the main grid points, but the velocity components are stored at the cell faces or the corners of the control volumes. Owing to the difficulty of program making on a staggered grid arrangement, researchers adopted some algorithms to solve the pressure oscillation problem on a collocated grid system. Rhie and Chow (1983) first solved the pressure oscillation problem on a collocated grid by using the momentum interpolation technique. The comparisons of Peric et al. (1988) and Melaaen (1992) on the accuracy and convergence performance of the staggered and collocated grid methods show that there is no significant difference between the two grid systems.

The dependent variables in the momentum equations in curvilinear coordinates can be selected as Cartesian, covariant and contravariant velocity components. The expression of convective and diffusive fluxes is more complicated when the covariant velocity components. For the staggered grid system, the covariant or contravariant velocity method is the better choice for avoiding the checker-board pressure field. For the collocated grid system, the checker-board pressure field can be
avoided using the cell face momentum interpolation method regardless if the Cartesian, covariant and contravariant velocity methods are used.

The governing equations in curvilinear coordinates are obtained through the partial or complete transformations. When partial transformation is concerned, the independent coordinate variables are transformed and the dependent variables are left in the preselected orthogonal coordinate system. If both the independent and dependent variables are transformed, the approach is called complete transformation. A partial transformation leads to a strongly conservative form of the Navier-Stokes equation, in curvilinear coordinates which uses the Cartesian velocity components as the dependent variables. However, when the angles between the velocity components and the coordinate surfaces become large, since the Cartesian velocity vectors do not align with the coordinate direction, this approach may cause an increased numerical diffusion. Although a complete transformation leads to a weak conservative form of the Navier-Stokes equations, the non-Cartesian velocity components change their direction and tend to follow the grid lines. This feature makes them attractive for highly non-orthogonal geometries. Some researchers (Yang et al., 1994) proposed the strongly conservative form of the partially transformed Navier-Stokes equations. The strongly conservative form can be applied with either Cartesian, contravariant or covariant components as the dependent variables.

If the flow is incompressible the density is constant and not linked to the pressure. The non-linearities in the Navier-Stokes equations and the pressure-velocity linkage can be resolved by adopting an iterative solution procedure such as the SIMPLE algorithm of Patankar and Spalding (1972). In SIMPLE, the pressure correction is satisfactory for correcting velocities but not so good for correcting velocities but not so well for correcting pressure. SIMPLEC algorithm is identical to SIMPLE algorithm and it is more effective in faster convergence than SIMPLE algorithm.

In the present study, Cartesian velocity variables and contravariant convective fluxes are used as the dependent variables with collocated grid arrangement. The objectives of this work are to derive the discretized governing equations in general two dimensional non-orthogonal coordinate system on collocated grid, to investigate the feasibility of solving pressure equation on nine point molecule using different solvers and to apply the present model to different types of fluid flow problems and to compare the present results with the previous experimental and numerical results.
2. LITERATURE REVIEW

Study and use of the Cartesian velocity components as the dependent variables in the momentum equations have been carried out by numerous investigators. Vinkour (1974) was the first one to derive the governing equations in general coordinates using the Cartesian velocity components as the dependent variables. Rhie and Chow (1983) employed such equations to simulate the two-dimensional incompressible turbulent flows over airfoils. Braaten and Shyy (1986) investigated the consistent treatment of the continuity equation and the effects of the grid skewness on the simulations.

Some researches chose contravariant components as the dependent variables. Demirdzic et al. (1980) presented a finite volume technique which solves the semi-strong form of the Navier-Stokes equations in terms of the physical contravariant velocity components. Demirdzic et al. (1987) presented a novel and useful procedure for directly transforming the Cartesian tensor forms of the equations into general coordinates. The transformation relations were used to derive the general coordinate version of the ensemble-averaged Navier-Stokes and turbulent model equations in terms of the physical contravariant velocity components. The methodology was applied to the cross flow in a heated tube bank. Yang et al. (1988) transformed the conversation equations into those in curvilinear coordinates and then, by using special properties of the geometry, obtained a set of reasonably simple equations for the parallelepiped geometry. Yang et al. (1990) furthered their work in (Yang et al., 1988), and derived the governing equations in non-orthogonal curvilinear coordinates with contravariant velocity components as the dependent variables through a tensor transformation. The application examples were natural enclosures and horizontally closed cylinders with differentially heated ends.

Instead of the contravariant components, covariant components were also chosen by some researchers as the dependent variables. For example, Galea and Markatos (1991) used covariant velocity resolute as the dependent variables. Their work was to establish a mathematical model which can describe aircraft cabin fires. Davidson and
Hedberg (1989) solved the momentum equations for covariant velocity components. They presented a mathematical derivation of the governing equations in the transformed space where a local coordinate system was set up at each grid point. Two problems of laminar flows were solved.

In the above-mentioned work, the momentum equations using the non-Cartesian velocity components as the dependent variables retain a weakly conservative form. An interesting way of obtaining a strongly conservative form of the momentum equations was given by Karki (1986), in which the discretization equations using the covariant velocity projections as the dependent variables were obtained by an algebraic manipulation of the corresponding equations for the Cartesian velocity components. Any reference to the differential form of the conservation equation for the covariant velocity projections was avoided. A variety of two-dimensional incompressible and compressible fluid flows were selected to test the proposed procedure. The procedure proposed by Karki (1986) was also adopted by Karki and Patankar (1989) to simulate two-dimensional incompressible and compressible flows, ranging from subsonic to supersonic. The strongly conservative form of the momentum equations using other non-Cartesian velocity components as the dependent variables can be obtained in the way similar to that used by Karki (1986). Darr and Vanka (1991) derived a strongly conservative formulation using contravariant velocity components as the dependent variables. The method was used to study the separated flows in driven trapezoidal cavities on staggered grids. Yang et al. (1994) proposed a general strongly conservative formulation of the Navier-Stokes equations in non-orthogonal curvilinear coordinates. In their proposed technique, the differentiation operators were directly applied to the velocity vector itself, instead of velocity components. The formulation has a clear, simple form and can be applied to the Navier-Stokes equations with either Cartesian, covariant, contravariant components or even velocity resolute as the dependent variables.

In recent years, more and more research has focused on the strongly conservative form of the Navier-Stokes equations in general non-orthogonal curvilinear coordinates. Melaaen (1992) formulated and compared two finite-volume methods for calculating flows inside complex geometries. One is based on staggered grid arrangement with covariant velocity projections as the dependent variables and the other one is based on non-staggered grid arrangement with Cartesian velocity
components as the dependent variables. Choi et al. (1993) chose Cartesian velocity components as the dependent variables. They employed both the contravariant and covariant velocity components as cell velocities to investigate the effect of different cell-face velocities on solution behavior. Sharatchandra and Rhode (1994) proposed a calculation procedure for flows in complex geometries similar that proposed by Yang et al. (1990). Covariant velocity projections were chosen as the dependent variables.

There are two kinds of grid arrangements: staggered and non-staggered (collocated). Since staggered arrangement is not adopted in the present work, attention of the review is given to the collocated arrangement.

Vanka et al. (1980) proposed a scheme in which a cell-by-cell procedure was used rather than solving the pressure-correction equation. At each pressure location, all the surrounding values of the pressure-correction equation. At each pressure location, all the surrounding values of the pressure correction were set to zero. This scheme is not widely accepted because of its intrinsic shortcoming. The first successful attempt to utilize the potential advantages of the collocated grid arrangement with curved irregular flow boundaries was made by Rhie and Chow (1983). The key idea to eliminate the pressure oscillations is to employ a special interpolation practice called momentum interpolation for evaluating the cell-face velocities. Majumdar (1988) presented more general formulations for momentum interpolation which include under-relaxation factors. He found that the converged result for any flow field considered depends on the under-relaxation factor used for the velocity. He also proposed how to implement the momentum interpolation using an iterative algorithm to achieve a unique solution that is independent of the under-relaxation factor. Miller and Schmidt (1988) rigorously developed a momentum interpolation and its implementation in the SIMPLEC algorithm. They estimated the degree of dependence of numerical solutions on the under-relaxation factor and obtained a formulation of momentum interpolation which is independent of the under-relaxation factor for a converged solution in Cartesian coordinates. Over the past 10 years, momentum interpolation has been widely used for computations of fluid flows on collocated grids. Date (1993) proposed a so-called pressure gradient interpolation. In this approach, the problem of the pressure oscillation is eliminated by interpolating the pressure gradient terms in the discretized momentum equations and the cell-face
velocities are still evaluated by linear interpolation. It seems that this method may be more attractive than the momentum interpolation because a unique solution which is independent of the under-relaxation factor for a given flow field can always be obtained.
3. SOLUTION PROCEDURE

3.1. Control Volume Approach

In finite volume method, the computational domain is divided into a set of quadrilateral volumes and the conservation laws are expressed in an integral form for each of these control volumes. Let $\delta \xi$ and $\delta \eta$ denote the distance between the grid points in $\xi$ and $\eta$ directions in the computational space (see Figure 3.1), respectively, while $\Delta \xi$ and $\Delta \eta$ denote the distance between the faces of the control volumes. $\delta x$, $\delta y$, $\Delta x$, and $\Delta y$ have the same meanings in the physical space (see Figure 3.2). As it will be seen in next section $\delta \xi$, $\delta \eta$, $\Delta \xi$, and $\Delta \eta$ will appear in the discretized transformed equations instead of $\delta x$, $\delta y$, $\Delta x$, and $\Delta y$. It would be, of course, convenient for $\delta \xi$, $\delta \eta$, $\Delta \xi$, and $\Delta \eta$ to all be unity no matter how much $\delta x$, $\delta y$, $\Delta x$, and $\Delta y$ are in the physical space.

Figure 3.1: Grid arrangement (Physical space)
In this study, the control volume boundaries are drawn first and then a grid point is placed at the geometric center of each control volume. Ghost points are used to implement the boundary conditions at the boundaries as shown in figure (3.3). This approach vanishes the need for a special discretization equation for the near-boundary control volumes. The boundary nodal values are presented as a function of the previous and the next control volume center nodal values.

3.2. Numerical Formulation

The fluid flow algorithm employed in this study is capable of solving two-dimensional laminar incompressible flows.

3.2.1. Governing Equations

The governing equations are written as Cartesian coordinate system. Because it is considerably easier to solve the transformed equations in the computational space than to solve in the physical space, governing equations must be first transformed from the physical space to the computational space.
Figure 3.3: Location of the control volume grid points and faces near boundary

The governing equations for two dimensional, steady state, incompressible flows are continuity equation and momentum equations written as

\[
\frac{\partial}{\partial x} (\rho u) + \frac{\partial}{\partial y} (\rho v) = 0
\]  \hspace{1cm} (3.1)

\[
\frac{\partial}{\partial x} (\rho uu) + \frac{\partial}{\partial y} (\rho vu) = -\frac{\partial P}{\partial x} + \frac{\partial}{\partial x} \left[ \mu \frac{\partial u}{\partial x} \right] + \frac{\partial}{\partial y} \left[ \mu \frac{\partial u}{\partial y} \right]
\]  \hspace{1cm} (3.2a)

\[
\frac{\partial}{\partial x} (\rho uv) + \frac{\partial}{\partial y} (\rho vv) = -\frac{\partial P}{\partial y} + \frac{\partial}{\partial x} \left[ \mu \frac{\partial v}{\partial x} \right] + \frac{\partial}{\partial y} \left[ \mu \frac{\partial v}{\partial y} \right]
\]  \hspace{1cm} (3.2b)

where \( \rho \) is the density, \( \mu \) is the viscosity, \( u \) and \( v \) are the velocity components in \( x \) and \( y \) directions in the physical space respectively, and \( P \) is the pressure.

With the coordinate transformation, the independent variable changes from Cartesian to curvilinear coordinates. The conversation equations can typically be written in general coordinates in the conservative form;

\[
\frac{\partial}{\partial \xi} (\rho U) + \frac{\partial}{\partial \eta} (\rho V) = 0
\]  \hspace{1cm} (3.3)
\[
\frac{\partial}{\partial \xi}(\rho U \phi) + \frac{\partial}{\partial \eta}(\rho V \phi) = \frac{\partial}{\partial \xi} \left[ \frac{\Gamma}{J} (\phi \phi_\xi - \chi \phi_\eta) \right] + \frac{\partial}{\partial \eta} \left[ \frac{\Gamma}{J} (\gamma \phi_\eta - \chi \phi_\xi) \right] + JS(\xi, \eta) \quad (3.4)
\]

where \( U \) and \( V \) are the contravariant velocity components; \( \Gamma \) is the general diffusion coefficient, and \( J \) is the Jacobian of the transformation; \( S(\xi, \eta) \) is the source term on the computational plane; with \( \chi, \phi \) and \( \gamma \) as the coefficients of the transformation.

\[
U = u y_\eta - v x_\eta, \quad V = v x_\xi - v y_\xi, \quad (3.5a)
\]

\[
J = x_\xi y_\eta - x_\eta y_\xi, \quad \chi = x_\xi x_\eta + y_\eta y_\xi, \quad (3.5b)
\]

\[
\phi = x_\eta^2 + y_\eta^2, \quad \gamma = x_\xi^2 + y_\xi^2. \quad (3.5c)
\]

### 3.2.2. Discretization of the Transport Equations

Integrating equation (3.4) over the control volume of node \( P \) as shown in Figure (3.2) gives;

\[
\left[ \rho U \phi \Delta \eta \right]_w + \left[ \rho U \phi \Delta \xi \right]_s = \left[ \frac{\Gamma}{J} (\phi \phi_\xi - \chi \phi_\eta) \Delta \eta \right]_w + \left[ \frac{\Gamma}{J} (\gamma \phi_\eta - \chi \phi_\xi) \Delta \xi \right]_s + JS(\xi, \eta) \Delta \xi \Delta \eta \quad (3.6)
\]

Derivatives at the cell faces in the above equation can be determined using central differencing scheme, for example,

\[
\phi_\xi|_c = \frac{\phi_E - \phi_P}{\delta \xi} \quad (3.7a)
\]

\[
\phi_\eta|_c = \frac{1}{2} \left( \frac{\phi_N - \phi_S + \phi_{NE} - \phi_{SE}}{2 \delta \eta} \right) \quad (3.7b)
\]

\[
\phi_\eta|_c = \phi_N - \phi_S + \phi_{NE} - \phi_{SE} \quad (3.7b)
\]

\[
= \phi_N - \phi_S + \phi_{NE} - \phi_{SE} \quad (3.7b)
\]

Equation (3.6) becomes,

\[
F_c \phi_e - F_w \phi_w + F_n \phi_n - F_s \phi_s + (D_e + D_w + D_n + D_s) \phi_P = D_e \phi_E - D_w \phi_W + D_n \phi_N - D_s \phi_S + q_\phi \quad (3.8)
\]
where

\[ F_e = (U\Delta\eta)_e \quad F_w = (U\Delta\eta)_e \quad F_n = (V\Delta\xi)_n \quad F_s = (V\Delta\xi)_s \] (3.9)

\[ D_e = \left(\frac{\Gamma}{J} \varphi \Delta\eta/\partial\xi \right)_e \quad D_w = \left(\frac{\Gamma}{J} \varphi \Delta\eta/\partial\xi \right)_w \]
\[ D_n = \left(\frac{\Gamma}{J} \gamma \Delta\xi/\partial\eta \right)_n \quad D_s = \left(\frac{\Gamma}{J} \gamma \Delta\xi/\partial\eta \right)_s \] (3.10)

In above equations, \( F \) is the flow rate through a control volume surface; \( D \) is the orthogonal diffusional conductance. The diffusion terms which occur as a result of grid non-orthogonality, are treated as explicit terms.

The non-orthogonal terms in the momentum equation are treated as explicit terms and the source term can be expressed as

\[ q^\phi = q^\phi_s + q^\phi_{\text{curv}} \] (3.11)

\[ q^\phi_s = JS \varphi \Delta\xi \Delta\eta \] (3.12)

\( q^\phi_{\text{curv}} \) denotes the source term due to the non-orthogonality of the coordinate system.

\[ q^\phi_{\text{curv}} = (D'_e - D'_n)\phi_E + (D'_n - D'_s)\phi_W + (D'_w - D'_e)\phi_N + (D'_e - D'_w)\phi_S \\
+ (D'_n + D'_w)\phi_{NW} - (D'_s + D'_w)\phi_{SW} + (D'_e + D'_s)\phi_{SE} - (D'_n + D'_e)\phi_{NE} \] (3.13)

The ' symbol denotes the non-orthogonality and the non-orthogonal diffusion terms are as follows

\[ D'_e = \left(\frac{\rho C_{12}\Delta\eta}{4\delta\eta} \right)_e \quad D'_w = \left(\frac{\rho C_{12}\Delta\eta}{4\delta\eta} \right)_w \]
\[ D'_n = \left(\frac{\rho C_{21}\Delta\xi}{4\delta\xi} \right)_n \quad D'_s = \left(\frac{\rho C_{21}\Delta\xi}{4\delta\xi} \right)_s \] (3.14)

Discretization of the convection term in equation (3.6) is of considerable importance to the accuracy and stability for numerical computations. Central differencing scheme may fail when the mesh Reynolds number is less than two. An effective but
simple scheme to avoid the numerical instability is the upwind scheme, which is unconditionally stable. However, it may lead to large truncation error.

Hybrid scheme (36) combines the accuracy of the central differencing scheme and the stability of the upwind scheme. In this thesis, hybrid scheme (36) is employed to discretize the convection-diffusion terms. The resulting algebraic equation is

$$ A_P \phi_p = A_E \phi_E + A_W \phi_W + A_N \phi_N + A_S \phi_S + S^\phi_{\text{new}} \tag{3.15} $$

where

$$ A_E = \max(-F_c, D_c - \frac{F_c}{2}, 0) \quad A_W = \max(F_w, D_w + \frac{F_w}{2}, 0) $$

$$ A_N = \max(-F_n, D_n - \frac{F_n}{2}, 0) \quad A_S = \max(F_s, D_s + \frac{F_s}{2}, 0) \tag{3.16} $$

$$ A_P = A_E + A_W + A_N + A_S - S^\phi_p J \Delta \xi \Delta \eta $$

$S^\phi$ in equation (3.15) is linearized and is expressed as

$$ S^\phi = S^\phi_u + S^\phi_p \tag{3.17} $$

where $S^\phi_p$ must be negative.

$$ S^\phi_{\text{new}} = q^\phi + q^\phi_{\text{curv}} \tag{3.18} $$

$$ S^\phi_{\text{new}} = J S^\phi_u \Delta \xi \Delta \eta + q^\phi_{\text{curv}} \tag{3.19} $$

Usually all variables are under relaxed in iteration so as to guarantee the convergence of the solution. As a result, Equation (3.15) can be rewritten as

$$ \frac{A_P}{\alpha} \phi_p = \sum A_{nb} \phi_{nb} + S^\phi_{\text{new}} + \frac{(1-\alpha)}{\alpha} A_P \phi_p^* \tag{3.20} $$

where $\alpha$ is the relaxation factor. The superscript asterisk denotes the value of the last iteration.

Equation (3.20) is the general form for all dependent variables. For the Cartesian velocity components $u$ and $v$, equation (3.20) can be rearranged as
\[
\frac{A_P}{\alpha_u} u_P = \sum A_{nb} u_{nb} + S_{new}^u + \frac{(1 - \alpha_u)}{\alpha_u} A_P u_P^* \tag{3.21a}
\]

\[
\frac{A_P}{\alpha_u} v_P = \sum A_{nb} v_{nb} + S_{new}^v + \frac{(1 - \alpha_u)}{\alpha_u} A_P v_P^* \tag{3.21b}
\]

where \(\alpha_u\) is the relaxation factor for velocity. \(S_{new}^u\) and \(S_{new}^v\) are the corresponding source terms.

### 3.2.3. Pressure Correction Equation

In the SIMPLE-like method, pressure is calculated using the correction method. The pressure-correction equation can be derived from the continuity equation. When the momentum equations have been solved, the Cartesian, contravariant velocity components and pressure are supposed to be \(u^*, v^*, U^*, V^*\) and \(P^*\), which in general do not satisfy the continuity equation. To satisfy the continuity equation, the corresponding corrections are supposed to be \(u', v', U', V'\) and \(P'\).

\[
\begin{align*}
\nu &= u^* + u' \\
\nu &= v^* + v' \\
U &= U^* + U' \\
V &= V^* + V' \\
P &= P^* + P'
\end{align*}
\tag{3.23a}
\tag{3.23b}
\tag{3.23c}
\]

Substituting equations (3.23a) and (3.23c) into equation, you get

\[
\frac{A_P}{\alpha_u} u_P' = \sum A_{nb} u_{nb}' - (y_\eta P_{\xi}' - y_\xi P_{\eta}') \tag{3.24a}
\]

\[
\frac{A_P}{\alpha_u} v_P' = \sum A_{nb} v_{nb}' - (x_\xi P_{\eta}' - x_\eta P_{\xi}') \tag{3.24b}
\]

Wang and Kumoru stated that when the SIMPLEC method is used to approximate the pressure-velocity coupling, the convergence performance on strongly nonorthogonal grids can be greatly improved in comparison with the SIMPLE method. According to the SIMPLE-like methods, the relations between velocity corrections and pressure corrections can be written as
\[ u' = (B_{11}P'_\xi + B_{12}P'_\eta) \]  \hspace{1cm} (3.25a)

\[ \nu' = (B_{21}P'_\xi + B_{22}P'_\eta) \]  \hspace{1cm} (3.25b)

where

\[ B_{11} = -\frac{y_\eta}{A_p} \quad \quad B_{21} = -\frac{y_\xi}{A_p} \]  

\[ B_{12} = -\frac{x_\eta}{A_p} \quad \quad B_{22} = -\frac{x_\xi}{A_p} \]  

for SIMPLE

\[ B_{11} = -\frac{y_\eta}{A_p/\alpha_u - \sum A_{nb}} \quad \quad B_{21} = -\frac{y_\xi}{A_p/\alpha_u - \sum A_{nb}} \]  

\[ B_{12} = \frac{x_\eta}{A_p/\alpha_u - \sum A_{nb}} \quad \quad B_{22} = \frac{x_\xi}{A_p/\alpha_u - \sum A_{nb}} \]  

for SIMPLEXC

By substituting equations (3.25a) and (3.25b) into equation (3.24a) and (3.24b), corrections of the contravariant velocity components can be written as

\[ U'_p = (B_{11}y_\eta - B_{12}x_\eta)P'_\xi + (B_{21}y_\eta - B_{22}x_\eta)P'_\eta \]  \hspace{1cm} (3.26a)

\[ V'_p = (B_{22}x_\xi - B_{21}y_\xi)P'_\eta + (B_{12}x_\xi - B_{11}y_\xi)P'_\xi \]  \hspace{1cm} (3.26b)

where the underlined parts in equations (3.26a) and (3.26b) are usually called cross-derivatives of the pressure corrections, and for orthogonal grids they are zero.

Then the contravariant velocity components satisfying the continuity equation can be expressed as

\[ U = U^* + C_{11}P'_\xi + C_{12}P'_\eta \]  \hspace{1cm} (3.27a)

\[ V = V^* + C_{21}P'_\eta + C_{22}P'_\xi \]  \hspace{1cm} (3.27b)

where
\[C_{11} = B_{11} y_n - B_{12} x_n \quad C_{12} = B_{21} y_n - B_{22} x_n\]
\[C_{21} = B_{22} x_\xi - B_{21} y_\xi \quad C_{22} = B_{12} x_\xi - B_{11} y_\xi\]  

(3.28)

When equation (3.3) is integrated over the control volume of node P, the discretization form of the continuity equation can be written as

\[\eta \eta \eta \eta = -\xi \xi \xi \xi\]

(3.29)

On substituting the above flux corrections into the continuity equation (3.29), the general P' equation can be obtained as follows:

\[\left[ (\rho C_{11})_e + (\rho C_{11})_w + (\rho C_{22})_n + (\rho C_{22})_s \right] P'_{P} = \left( \rho C_{11} \right)_e P'_E + \left( \rho C_{11} \right)_w P'_W + \left( \rho C_{22} \right)_n P'_N + \left( \rho C_{22} \right)_s P'_S\]

\[+ \left( \rho U^* \right)_e - \left( \rho U^* \right)_w + \left( \rho V^* \right)_n - \left( \rho V^* \right)_s\]

\[+ \left( \rho C_{12} \right)_e \left( \frac{\partial P'}{\partial \eta} \right)_e - \left( \rho C_{12} \right)_w \left( \frac{\partial P'}{\partial \eta} \right)_w\]

\[+ \left( \rho C_{21} \right)_n \left( \frac{\partial P'}{\partial \xi} \right)_n - \left( \rho C_{21} \right)_s \left( \frac{\partial P'}{\partial \xi} \right)_s\]

(3.30)

Substituting equations (3.27a) and (3.27b) into equation (3.29), and neglecting the cross derivatives of pressure corrections, the pressure correction equation can be formulated as

\[A_P P'_P = A_E P'_E + A_W P'_W + A_N P'_N + A_S P'_S + m_P\]

(3.31)

where

\[A_E = (\rho C_{11})_e\]
\[A_W = (\rho C_{11})_w\]
\[A_N = (\rho C_{22})_n\]
\[A_S = (\rho C_{22})_s\]
\[A_P = A_e + A_w + A_n + A_S\]  

(3.32)
\[ m_p = \left( \rho U^* \right)_e - \left( \rho U^* \right)_w + \left( \rho V^* \right)_n - \left( \rho V^* \right)_s \]  \hspace{1cm} (3.33)

When the pressure-correction equation has been solved, equations (3.27a) and (3.27b) are used to calculate the corrections of the contravariant velocity components. Then the following relations are used to calculate the corrections of the Cartesian velocity components.

\[ u' = (U'\xi + V'\eta)/J \]  \hspace{1cm} (3.34a)

\[ v' = (U'\eta + V'\xi)/J \]  \hspace{1cm} (3.34b)

3.2.4. Momentum Interpolation Technique

On a collocated grid system exists a problem associated with the storing both the pressure and velocities at the same grid point. Therefore, non-physical oscillation of pressure or so-called checkerboard pressure field will be encountered unless some special treatments are taken. Rhie and Chow first developed a scheme based on momentum interpolation to overcome this problem. In this scheme, momentum equations are solved at main grid points for Cartesian velocity components and the cell-face velocities are obtained by the interpolation of the momentum equations on the neighboring nodes.

\[ U_p = (Hu)_p + \left( \frac{\varphi}{A_p} P_\varsigma \right)_p \]  \hspace{1cm} (3.35)

\[ U_E = (Hu)_E + \left( \frac{\varphi}{A_p} P_\varsigma \right)_E \]  \hspace{1cm} (3.36)

\[ (Hu)_e = f(Hu)_E + (1-f)(Hu)_p \]  \hspace{1cm} (3.37)

\[ U_e = f(Hu)_E + (1-f)(Hu)_p + (B_{11}P_\varsigma)_e \]  \hspace{1cm} (3.38)

\[ U_e = fU_E + (1-f)U_p + \left[ (\frac{\varphi}{A_p} P_\varsigma)_e - f(\frac{\varphi}{A_p} P_\varsigma)_E - (1-f)(\frac{\varphi}{A_p} P_\varsigma)_p \right] \]  \hspace{1cm} (3.39)

In order to get a convergent solution independent of the relaxation factor used for velocity, the following formula is adopted:
\[ U_c = fU_E + (1 - f)U_P + \left[ (\frac{\Phi}{A_p}P_\zeta)_c - f(\frac{\Phi}{A_p}P_\zeta)_E - (1 - f)(\frac{\Phi}{A_p}P_\zeta)_P \right] \] (3.40)

Similarly, the contravariant velocity \( V^*_c \) at cell face \( n \) can be calculated by

\[ V_c = fV_N + (1 - f)V_P + \left[ (\frac{\gamma}{A_p}P\eta)_n - f(\frac{\gamma}{A_p}P\eta)_N - (1 - f)(\frac{\gamma}{A_p}P\eta)_P \right] \] (3.41)

### 3.2.5. The Solution Algorithm

1. Initialize all the variables.
2. Calculate the coefficients and the source terms for the momentum equations.
3. Solve the momentum discretization equations to obtain the \( u^* \) and \( v^* \).
4. Calculate the contravariant velocity components at cell faces using the momentum interpolation method.
5. Solve the pressure correction equation.
6. Use the pressure corrections to correct the pressure field via

\[ P = P^* + \alpha P' \] (3.42)

where \( \alpha \) is the pressure relaxation factor.
7. Correct the contravariant velocity components.
8. Correct the Cartesian velocity components.
9. Repeat the steps 2-8 until the convergence is satisfied.

The solution algorithm consists of a successive solution of the discretized momentum equations, followed by correction of the mass fluxes by enforcing the continuity constraint on the obtained velocity field. This makes one outer iteration; the coefficients in the equations are then updated, and the whole process is repeated until convergence. Due to the coupling of variables and nonlinearity of the equations, it is not necessary to solve exactly the discretized equations for a given set of coefficients; these are only approximate and need to be updated. It is sufficient to perform one or two inner iterations by an iterative solver. However, the continuity constraint has to be forced to a certain level of convergence for every outer iteration.
The convergence criterion was that the sum of absolute values of residuals in all equations, normalized with $\rho U_L^2 L$ for momentum and $\rho U_L L$ for the continuity equation, falls below a critical value. The residuals, which provide a measure of the degree to which each equation is satisfied throughout the flow field, are computed for each discretization equation by summing the imbalance in the equation for all cells. The residual, Res, for momentum equations can be obtained by,

$$\text{Res} = \sum \left| A_E \phi_E + A_W \phi_W + A_N \phi_N + A_S \phi_S + S_\phi - A_P \phi_P \right|$$

(3.43)

The residual for pressure as the imbalance in the continuity equation is

$$\text{Res} = \sum \left| F_e - F_w + F_n - F_s \right|$$

(3.44)
4. COMPUTATIONAL RESULTS

The illustrative test problems have been chosen to show the accuracy, convergence rate and the capability of the algorithm to handle complex flow situations. The algorithm is first validated for the flows in lid-driven square and skewed cavities. Secondly, the algorithm has been applied for flow over a circular cylinder. This simulation illustrates the implementation of complex outlet boundary conditions for an external flow problem.

4.1. Lid-driven Cavity Flows

The laminar incompressible flow in a cavity with moving lid has served as a model problem for testing and evaluating numerical methods. In this test case as shown in Figure (4.1), laminar flows in which the fluid motion is induced by the movement of top wall are studied. The side walls of the cavity are taken to be inclined, forming an angle \( \beta \) with the horizontal plane. This problem is chosen so as to include the most important features of the complex flows encountered in engineering practice. It has also been of much interest from the view points of transition to turbulence and bifurcation phenomena. Moreover, as far as grid non-orthogonality is concerned, this test case is appropriate to cover the effect of the grid non-orthogonality on the convergence of the solutions on a quadrilateral solution domain when parallel walls are inclined.

Cavity length \( H = L = 1 \text{ m} \), density \( \rho = 1 \text{ kg/m}^3 \) and lid velocity \( U_L = 1 \text{ m/sn} \) are used in the numerical calculations. The no-slip condition at the stationary walls yields that \( u = v = 0 \). At the moving wall \( u = 1 \text{ m/sn} \) and \( v = 0 \) are described.

Three cases with different inclinations were studied: \( \beta = 90^\circ, 45^\circ, 30^\circ \). In this study, the Reynolds numbers based on the velocity of the moving lid and the side length of the cavity are 100 and 1000 respectively. In general, the convergence criterion was that the sum of absolute values of residuals in all equations falls below \( 10^{-6} \).
Figure 4.1: Geometrical description and boundary conditions for lid-driven flow

4.1.1. Test Case 1 ($\beta=90^\circ$)

The first test case of the lid driven cavity flow is the square cavity. Figure (4.2) shows the streamlines for (a) Re=100 and (b) Re=1000. As the Re number increases the strength of the corner vortices increase.

The computed profiles for $u$ and $v$ velocities along the mid planes of the cavity are shown in Figure (4.3) and Figure (4.4) for Re=100 and 1000, respectively. The results are in good agreement with the benchmark results of Ghia et al. (1982). Ghia et al. (1982) adopted a fine grid of 129 x 129 in their computation. In this study, for Re=100, results obtained with 80 x 80 grid matches with that of benchmark results. For Re=1000, a finer mesh, 160x 160 grid arrangement provides more agreeable results compared with that of Ghia et al. (1982).

Figure 4.2: Predicted streamlines in the square cavity
Figure 4.3: Variation of the centerline velocity profiles in case $\beta=90$, Re=100 (a) U component; (b) V component
Figure 4.4: Variation of the centerline velocity profiles in case $\beta=90$, $Re=1000$ as a function of grid fineness. (a) $U$ component; (b) $V$ component
4.1.2. Test Case 2 ($\beta=45^\circ$)

The inclined cavity ($\beta\neq 90^\circ$) is a more complex configuration. Figure (4.5) presents the computed streamlines for (a) Re=100, (b) Re=1000. For Re=100 the main vortex fills almost the whole cavity. For Re=1000, the primary vortex fills only about a third of the cavity. The largest vortex is the second one.

The results of $u$ and $v$ velocities at $\beta=45^\circ$ along the mid plane of the cavity for Re=100 are shown in Figure (4.6). The $u$ and $v$ velocity profiles are in good agreement with the results of Demirzdic et al. (1992) on the 80 x 80 and 160 x 160 grid arrangements.

For Re=1000, it is seen that with increasing grid points, the computed profiles of both $u$ and $v$ become closer to the benchmark data. Figure (4.7) shows that 160 x 160 grid gives solutions nearly as accurate as the benchmark solutions of the Demirzdic et al. (1992) which were obtained on 320 x 320 grid.

![Figure 4.5: Predicted streamlines in inclined cavity ($\beta=45^\circ$): (a) Re=100; (b) Re=1000](image)
Figure 4.6: Variation of the centerline velocity profiles in case $\beta=45$, Re=100: (a) U component; (b) V component
Figure 4.7: Variation of the centerline velocity profiles in case $\beta=45$, $Re=1000$: (a) U component; (b) V component
4.1.3. Test Case 3 ($\beta=30^\circ$)

Figure (4.8) presents the computed streamlines for (a) $Re=100$, (b) $Re=1000$. As inclination angle increases, the number of corner vortices in the flow domain increases.

Figures (4.9a) and (4.9b) present the u-velocity profiles along a vertical line and the v-velocity profiles along a horizontal line passing through the geometric center of the cavity for $Re=100$ respectively. Strong grid dependence is observed up to 80 x 80 CVs in the case of $Re=100$ and up to 160 x 160 CVs for $Re=1000$. These profiles are in good agreement with that of Demirzdic (1992). Figure (4.10) also shows the computed velocity profiles for $Re=1000$.

Figure 4.8: Predicted streamlines in inclined cavity ($\beta=30^\circ$): (a) $Re=100$; (b) $Re=1000$
**Figure 4.9:** Variation of the centerline velocity profiles in case $\beta=30$, $Re=100$ as a function of grid fineness. (a) U component; (b) V component
Figure 4.10: Variation of the centerline velocity profiles in case $\beta=30$, Re=1000 as a function of grid fineness. (a) U component; (b) V component
4.2. Flow Past a Circular Cylinder

A numerical investigation of the steady separated flow past a circular cylinder is carried out for the purpose of validation. This flow has been used as a benchmark to examine the accuracy of the numerical methods and codes for a long time. It is well-known that the flow displays immensely different patterns as the Reynolds number changes. The flow is steady with two symmetric vortices on each side of the wake centerline up to \( \text{Re} \approx 47 \), where \( \text{Re} \) is the Reynolds number based on the diameter of the cylinder. For larger Reynolds numbers the flow is still laminar, but it becomes unsteady and asymmetric (Williamson, 1989). In this study, two-dimensional laminar flow past a circular cylinder is simulated at \( \text{Re}=10, 20 \) and \( 40 \) where the flow satisfies steady and axis symmetric conditions.

The cylinder is modeled as a semi-circle and a semi-circle flow domain is created around the solid body. The free-stream boundary is located 40 diameters from the center of the cylinder. The cylinder diameter is 0.1 meter.

The inlet velocity field is specified as uniform flow where \( U=1 \text{ m/sn}, \ V=0 \) at the inflow portion of the outer boundary. At the outflow portion of the outer boundary, both \( U \) and \( V \) velocities are extrapolated from the interior with zero gradient boundary condition. The no-slip condition is specified on the body surface (\( U=V=0 \)). At the lower boundary symmetry condition is prescribed where \( V = \delta U/\delta y = 0 \). At the body surface and at the inflow boundary, the boundary side coefficient in the pressure correction equation is set to zero. At the outflow, the pressure correction is taken to be zero. The flow domain is seen in Figure 4.10.

![Figure 4.11: The flow domain](image)
The grid used in simulations is of structured C-type. Dependence of the grid resolution is demonstrated for the Re=20 case. Three grid sizes are investigated, 41×55, 69×93 and 91×121. The smallest grid cell in the fine grid is \( \Delta \xi \times \Delta \eta = 0.00174 \times 0.00144 \) and in the coarse grid \( 0.00392 \times 0.00503 \). The wake length, \( L_s \), is 1.72, 1.80 and 1.81 for the coarse, medium and fine grid respectively. \( L_s \) is the separation bubble length, measured as the distance from the rear of the cylinder to the point where the streamwise velocity is zero and normalized with the cylinder diameter. The variations are small, and in order to limit the computational cost the medium grid is chosen for the simulations. Figure 4.11 shows the entire and close-up views of a typical grid around the half cylinder with 69×93 grid points.

Figure 4.12: Computational grid (a) Entire grid; (b) Grid detail near the cylinder
Figure 4.12 illustrates the streamlines when flow is steady. In all cases a vortices develops behind the cylinder for the symmetric case is perfectly aligned. The length of the wake grows approximately linearly with Re as stated in Dennis & Chang (1970).

![Streamlines for different Re values](image)

**Figure 4.13:** Calculated streamlines for steady flow past a circular cylinder

Some quantitative parameters for the recirculation region, such as the length of wake, $L_s$, from the rear point of the cylinder to the end of the wake, separation angle $\theta_s$ and drag coefficient $C_d$; are calculated for the comparison with the experimental and computational works from other researchers.

$$C_d = \frac{D}{\rho U_\infty^2 r}$$  \hspace{1cm} (4.1)

where $D$ is the total drag on the cylinder, $\rho$ is the density. The total drag can be obtained by integrating the total stress component in the direction $\zeta$ of around the surface of the cylinder. If $p$ is the pressure, $u$ and $v$ are the velocities then the total drag on the cylinder surface formula in the transformed grid is expressed as
Table 4.1 compares the above mentioned quantitative parameters we obtained with previously published results. All these flow parameters agree well with the results of previous listed studies for all three Reynolds numbers studied.

Table 4.1: Comparison of length of the recirculation region ($L_s$), separation angle ($\theta_s$) and drag coefficient ($C_d$) for Re = 10, 20 and 40

<table>
<thead>
<tr>
<th>Re</th>
<th>References</th>
<th>$L_s$</th>
<th>$\theta_s$</th>
<th>$C_d$</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>Takami and Keller (1969)</td>
<td>0.249</td>
<td>29.3</td>
<td>2.80</td>
</tr>
<tr>
<td></td>
<td>Dennis and Chang (1970)</td>
<td>0.252</td>
<td>29.6</td>
<td>2.85</td>
</tr>
<tr>
<td></td>
<td>Fornberg (1980)</td>
<td>--</td>
<td>--</td>
<td>--</td>
</tr>
<tr>
<td></td>
<td>Present</td>
<td>0.246</td>
<td>28.3</td>
<td>2.73</td>
</tr>
<tr>
<td>20</td>
<td>Takami and Keller (1969)</td>
<td>0.935</td>
<td>43.7</td>
<td>2.01</td>
</tr>
<tr>
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<td>43.7</td>
<td>2.05</td>
</tr>
<tr>
<td></td>
<td>Fornberg (1980)</td>
<td>0.91</td>
<td>--</td>
<td>2.00</td>
</tr>
<tr>
<td></td>
<td>Present</td>
<td>0.91</td>
<td>43.4</td>
<td>1.98</td>
</tr>
<tr>
<td>40</td>
<td>Takami and Keller (1969)</td>
<td>2.32</td>
<td>53.6</td>
<td>1.54</td>
</tr>
<tr>
<td></td>
<td>Dennis and Chang (1970)</td>
<td>2.35</td>
<td>53.8</td>
<td>1.52</td>
</tr>
<tr>
<td></td>
<td>Fornberg (1980)</td>
<td>2.24</td>
<td>--</td>
<td>1.49</td>
</tr>
<tr>
<td></td>
<td>Present</td>
<td>2.22</td>
<td>53.4</td>
<td>1.49</td>
</tr>
</tbody>
</table>

The dimensionless pressure coefficient distribution over the cylinder surface is selected in order to verify the accuracy of the current code and the dimensionless pressure coefficient is defined as

$$C_P(\theta) = \frac{P(\theta) - P_\infty}{\frac{1}{2} \rho U^2}$$  \hspace{1cm} (4.3)$$

where $P(\theta)$ is the static pressure on the cylinder surface and $P_\infty$ is the far stream static pressure. The computed pressure coefficient distributions over the cylinder surface for Re=10 and 40 are shown in Figure 4.13. The results are in well agreement with the numerical results of the Dennis & Chang (1970).
Figure 4.14: Pressure coefficient distribution over a cylinder
5. ANALYSIS OF CONVERGENCE PERFORMANCE

The discretized form of the pressure correction equation in curvilinear coordinates on the collocated grid arrangement results an asymmetric coefficients matrix. The coefficient matrix has nonzero coefficients on nine diagonals for 2-D flows. However, since solving this matrix is very complex and expensive, many researchers studied on the treatment of non-orthogonal terms in the pressure correction equation and reduction the computational cost.

When the grid is highly skewed, previous researchers proposed that the under relaxation factors have extremely narrow range for convergence of the solution. Although there is a few studies on the convergence rate and range with SIMPLE algorithm, the effect of the SIMPLEC algorithm on the treatment of non-orthogonal terms in the \( p' \) equation with existing methods is not totally investigated.

An investigation on the convergence behavior of the simplified pressure correction equation method, full correction equation method and the Cho and Chung’s method are applied to four different geometries with different skew angles.

Since most of the computing time in fluid flow predictions is spent on solving the pressure-correction equation, in the second part of this section various solvers are compared to have an efficient solver for it.

5.1. Treatment of Non-orthogonal Terms in the Pressure Correction Equation

There are a couple of treatment methods widely used for non-orthogonal terms in the \( p' \) equation. The First is the simplified \( p' \) equation method in which the non-orthogonal terms in the pressure correction equation are neglected. Since solving the full pressure correction equation most authors often used the simplified method. When the grid is highly skewed, neglecting the non-orthogonal often leads to numerical divergence unless the relaxation factors are carefully selected. Therefore solving the full pressure correction equation may be required if the grid non-orthogonality becomes appreciable. However both methods have defects. Cho and
Chung (1994) introduce a new technique in order to solve the pressure correction equation in wide ranges of relaxation factors.

5.1.1. Simplified Pressure-Correction Equation Method

When non-orthogonal grids are employed, the pressure correction equation is even more time consuming. Therefore most authors have introduced simplifications to this equation to make it easier to solve. The underlined terms in the equation (1) are neglected to solve the pressure correction equation on a 5 point computational molecule in two dimensional case. The resulting simplified pressure-correction equation has a diagonally dominant and symmetric matrix.

5.1.2. Full Pressure-Correction Equation Method

Unless the non-orthogonal terms of the pressure correction equation are neglected, the discretized equation is solved on 9 point computational molecule in a two dimensional case. First, Peric (1990) studied the feasibility of solving the full pressure correction equation.

In this study, full p' equation method is adopted as a case to investigate the contribution of the method to the convergence range by various solvers. The discretized p' equation is rewritten as follows for the case of unneglected pressure terms.

\[
A_P p'_P = A_E p'_E + A_W p'_W + A_N p'_N + A_S p'_S +
A_{NE} p'_{NE} + A_{NW} p'_{NW} + A_{SE} p'_{SE} + A_{SW} p'_{SW} + m_P
\]

(5.1)

where

\[
A_E = \left( \frac{\rho C_{11} \Delta \eta}{\delta \xi} \right)_e, \quad A_W = \left( \frac{\rho C_{11} \Delta \eta}{\delta \xi} \right)_w
\]

\[
A_N = \left( \frac{\rho C_{22} \Delta \xi}{\delta \eta} \right)_n, \quad A_S = \left( \frac{\rho C_{22} \Delta \xi}{\delta \eta} \right)_s
\]

\[
A_{NE} = \left( \frac{\rho C_{12} \Delta \eta}{4 \delta \eta} \right)_e + \left( \frac{\rho C_{21} \Delta \xi}{4 \delta \xi} \right)_n, \quad A_{NW} = \left( \frac{\rho C_{12} \Delta \eta}{4 \delta \eta} \right)_w + \left( \frac{\rho C_{21} \Delta \xi}{4 \delta \xi} \right)_n
\]

\[
A_{SE} = \left( \frac{\rho C_{21} \Delta \eta}{4 \delta \eta} \right)_e + \left( \frac{\rho C_{21} \Delta \xi}{4 \delta \xi} \right)_s, \quad A_{SW} = \left( \frac{\rho C_{21} \Delta \eta}{4 \delta \eta} \right)_w + \left( \frac{\rho C_{21} \Delta \xi}{4 \delta \xi} \right)_s
\]
\[ A_p = A_E + A_W + A_N + A_S \]

**5.1.3. Treatment of Cho and Chung**

Cho and Chung introduce a new technique to separate the non-orthogonal terms into explicit and implicit ones and thus this new method guarantees that the implicit terms are larger than the explicit terms in the discretized form of pressure correction equation. Rearranging the non-orthogonal terms in the \( p' \) equation, the underlined terms are adopted to have a form

\[
\begin{align*}
(pU)_e \cdot \left( \frac{\partial p'}{\partial \eta} \right)_e - (pU)_w \cdot \left( \frac{\partial p'}{\partial \eta} \right)_w \\
+ (pV)_n \cdot \left( \frac{\partial p'}{\partial \xi} \right)_n - (pV)_s \cdot \left( \frac{\partial p'}{\partial \xi} \right)_s
\end{align*}
\]

\( (a) \)

\[
-\alpha \left( A'_E P'_E + A'_W P'_W + A'_N P'_N + A'_S P'_S - A'_P P'_P \right)
\]

\( (b) \)

\[
+\alpha \left( A'_E P'_E + A'_W P'_W + A'_N P'_N + A'_S P'_S - A'_P P'_P \right)
\]

\( (c) \)

\[
(5.2)
\]

\( (d) \)

In the above equation, the term (5.2d) is treated as an implicit part and terms (a), (b), (c) are treated as explicit terms. Detailed expressions for the coefficients of this method can be found in Cho et al (1994).

**5.1.4. Study on the Convergence Properties**

This study analyzes the performance of the three method mentioned above when the grid skewness becomes appreciable. Four test cases of the lid cavity flow with different inclination angle are studied to investigate the feasibility of solving pressure correction equation using one of methods listed in the section 4.3 when the grid skewness increases. Moreover, the effect of the grid skewness on the number of iterations for both SIMPLE and SIMPLEC methods are investigated.

The results are compared on 20 x 20 CV grid at Re=100. The convergence criterion is that the sum of absolute values of normalized residuals falls below \( 10^{-5} \).

Calculation is performed for various under relaxation parameters for velocities and pressure, \( \alpha_u \) and \( \alpha_p \).
The convergence rate depends very strongly on the under relaxation factors $\alpha_u$ and $\alpha_p$ and grid skewness. Firstly, SIMPLE algorithm is employed to study the convergence rate of three mentioned model as the grid skewness becomes appreciable.

Figure (5.1) shows the number of outer iterations required for convergence with the method of Cho & Chung (1994) with the simplified and full pressure correction methods for $\beta=90^\circ$. In the case of orthogonal grid the convergence properties of simplified and full $p'$ equations are similar. The comparison between the simplified and the treatment of Cho & Chung also shows the similar results. The optimal convergence rate is obtained for $\alpha_u = 0.9$ in the range of $0.1 < \alpha_p < 0.3$.

![Figure 5.1: Comparison of convergence properties for cavity $\beta=90^\circ$ (SIMPLE) (a) Simplified Pressure-correction method (b) Cho & Chung’s method](image)

Figure 5.2 shows the results for $\beta=60^\circ$ with the simplified treatment in comparison with the full treatment and the Cho & Chung’s treatment. As the $\alpha_u$ increases, the convergence range for $\alpha_p$ decreases for all of the methods. However, Cho & Chung method converges for a wider range of $\alpha_p$. Both simplified and full pressure-correction methods converge at the same iteration number. The treatment of Cho & Chung needs more iteration for convergence for smaller $\alpha_p$.

As seen Figure (5.3), all models serve similar results for $\beta=45^\circ$ as in that for $\beta=60^\circ$. It can be seen from figures that as the grid skewness increases, the convergence range of $\alpha_p$ with the simplified method becomes very narrow. As the grid non-ortogonality becomes more significant the difference between the convergence rates with three
different methods becomes more noticeable. When $\alpha_u$ increases, converge rate remains invariant for the simplified method, while for full $p'$ equation and the Cho & Chung’s treatment convergence slightly decreases. The results for $\alpha_u = 0.8$ allows reasonable number of iterations in a wider range of $\alpha_p$. Although, $\alpha_u = 0.9$ has the fastest convergence rate in the case of full pressure-correction method, the usable range of $\alpha_p$ is between 0.15 and 0.3 for $\beta=60^\circ$ and between 0.18 to 0.28 for $\beta=45^\circ$.

**Figure 5.2:** Comparison of convergence properties for cavity $\beta=60^\circ$ (SIMPLE) (a) Simplified Pressure-correction method (b) Full Pressure-correction method (c) Cho and Chung’s method
Figure 5.3: Comparison of convergence properties for cavity $\beta=45^\circ$. (SIMPLE) (a) Simplified Pressure-correction method (b) Full Pressure-correction method (c) Cho & Chung’s method

In the case of $\beta=30^\circ$ the treatment of Cho & Chung converges at a wider range of $\alpha_p$ than the full treatment, but it requires more iterations to reach converged solution. Results show that the applicability of a solution procedure that employs a simplified pressure-correction equation is severely restricted by the grid non-ortogonality. However, full $p'$ equation and the Cho & Chung method seem to have the same convergence range of $\alpha_p$, being nearly independent of the inclined angle. The obtained convergence characteristic of the Cho and Chung method is unlike from the results in their study. This is thought as a result of the non-orthogonal pressure terms put into the momentum interpolation equation for this treatment only.
Secondly, using SIMPLEC algorithm a parametric study is conducted to introduce the effect of under relaxation factors on the convergence rate for various pressure-correction equation solution methods as the grid skewness increases. There is a few investigation regarding the effect of nonorthogonality using the SIMPLEC algorithm and no work is available in the literature to make the comparison between the solution of pressure-correction equation with three methods mentioned in Section 5.1. Computations are employed to analyze the convergence properties of these methods for a fixed value of $\alpha_u$ in the range 0.2 to 1 of $\alpha_p$ values.
Figures (5.5) to (5.8) demonstrates the ranges of $\alpha_p$ values for $\beta = 90^\circ$, $60^\circ$, $45^\circ$, $30^\circ$. For each $\beta$ value $\alpha_u$ varies from 0.6 to 0.9. There is almost no difference between the convergence rate of simplified pressure-correction equation method and that of Cho and Chung’s method when the cavity is square ($\beta=90^\circ$). Faster convergence is achieved when $\alpha_u = 0.8$ for both methods.

When SIMPLEC algorithm is employed the range of $\alpha_p$ enlarges, in fact the convergence range is not effected by the grid orthogonality and $\alpha_u$ values if the full pressure-correction and Cho and Chung’s methods are employed. This finding is valid for the simplified version in the case of $\beta > 45^\circ$. In the case of $\beta < 45^\circ$, the convergence range widens quite reasonably as compared to the SIMPLE algorithm. Moreover, in all cases SIMPLE algorithm stated that the range of usable $\alpha_p$ factors becomes narrower as $\alpha_u$ increases. The SIMPLE algorithm to the contrary the larger $\alpha_u$ value is, the wider the convergence interval is achieved when the SIMPLEC algorithm is used. It is obvious when the convergence properties for simplified pressure-correction equation at $45^\circ$ and $30^\circ$ are investigated for both algorithms.
As the grid skewness increases, Cho and Chung’s method converges appreciably slower. Although converge rates of the simplified and full pressure-correction methods are close, full pressure-correction converges faster for every \( \alpha_u \) and \( \alpha_p \) values. The fastest converge rate is achieved especially when \( \alpha_p = 1 \) and \( \alpha_u = 0.8 \) for every methods. It can be also seen that the full and simplified pressure-correction equation leads to faster convergence for values of \( \alpha_u = 0.7 \) and 0.8. However, the fastest convergence is achieved with the Cho & Chung’s method for \( \alpha_u = 0.6 \) and 0.7 values for the case \( \beta = 60^\circ, 45^\circ \) and \( 30^\circ \).
To sum up, the SIMPLEC algorithm is superior to the SIMPLE algorithm when simplified and full pressure-correction equation methods are used. If computational grid is not severely non-orthogonal ($\beta > 45^\circ$), it is more logical to use simplified version than the full one. The computer program is simpler and less memory is needed. On the other hand, full pressure-correction equation method converges fastest in a limitless range of $\alpha_p$ when the grid skewness increases. The Cho and Chung’s method serves inefficient performance if the SIMPLEC algorithm is employed. Although there is no limit to the ranges of $\alpha_p$ values, the convergence rate

**Figure 5.7:** Comparison of convergence properties for cavity $\beta = 45^\circ$ (SIMPLEC)  
(a) Simplified Pressure-correction method (b) Full Pressure-correction method (c) Cho & Chung’s method
of the method is low. The simplified pressure-correction equation method has reasonable convergence rate and the method allows wide range of under-relaxation factors even for seriously non-orthogonal grids when the SIMPLEC algorithm is employed. The methods are compared with the convergence properties. To examine the computing times of different solvers a systematical study is done in the next section.

Figure 5.8: Comparison of convergence properties for cavity $\beta=30^\circ$ (SIMPLEC) 
(a) Simplified Pressure-correction method (b) Full Pressure-correction method (c) Cho & Chung’s method
5.2. Analysis of Iterative Solvers

The result of the discretized process, according to each one of the techniques, discussed earlier, is a system of linear algebraic equations, having such a structure that its coefficients originate from tri-, penta- or nine-diagonal matrixes that depend on the particular approach used. More difficult problem with the full pressure equation is the solver. In a 2-D case the coefficient matrix has nine nonzero diagonals and is not symmetric. The solution procedures are several, among them: the strongly implicit procedure (SIP) method (Stone, 1968), the modified strongly implicit (MSI) method (), the RL (right-to-left) and LR (left-to-right) solvers () and the conjugate gradient method (CGS).

5.2.1. Review of Iterative Solvers

The conjugate gradient method generates a sequence of conjugate (orthogonal) vectors. These vectors are the residuals of the iterations. They are also the gradients of a quadratic functional, the minimization of which is equivalent to solving the linear system. CG is an extremely effective method when the coefficient matrix is symmetric positive definite, since storage for only a limited number of vectors is required.

In the CG method two coupled two-term recurrences are used; one that updates residuals using a search direction vector and done updating the search direction with a newly computed residual. This makes the Conjugate Gradient Method quite attractive computationally. The preconditioned Conjugate gradient method for the solution of $A\Phi=b$ with preconditioner can be summarized as:

Initialize by setting: $k = 0$, $p^0 = 0$ and

$$\rho^0 = S - A\phi^0$$  \hspace{1cm} (5.1)

defining $M = C^{-1}$ where $C$ is the preconditioning matrix

$$Mz^k = \rho^{k-1}$$  \hspace{1cm} (5.2)

In the above formula $\rho$ is the residual, $p$ is the search direction, $z$ is an auxiliary vector.
The following steps are calculated as the counter \( k \) advances, \( \alpha \) and \( \beta \) are parameters used in constructing the new solution, residual, and search direction.

\[
s^k = \rho^{k-1} z^k \quad (5.3)
\]

\[
\beta^k = s^k / s^{k-1} \quad (5.4)
\]

\[
p^k = z^k + \beta^k p^{k-1} \quad (5.5)
\]

\[
\alpha^k = s_k / (p^k \cdot A p^k) \quad (5.6)
\]

\[
\phi^k = \phi^{k-1} + \alpha^k p^k \quad (5.7)
\]

\[
\rho^k = \rho^{k-1} - \alpha^k A p^k \quad (5.8)
\]

The strongly implicit procedure (SIP) for solving a non-symmetric sparse linear system of equations proposed by Stone (1968) is designed for five-diagonal matrices. The algorithm is an example of the incomplete \([L][U]\) factorization, which requires less effort than the complete factorization. The objective is to replace the coefficients matrix \([A]\) by a modified form \([A+P]\). Thus, the modified matrix can be decomposed into upper and lower triangular matrices indicate \([U]\) and \([L]\), respectively.

For the linear system of \([A]\Phi=S\), an iterative procedure is defined as

\[
[A + P] \Phi^{n+1} = S + [P] \Phi^n \quad (5.9)
\]

We can define a increment vector

\[
\delta^{n+1} = \phi^{n+1} - \phi^n \quad (5.10)
\]

and a residual vector

\[
R^n = [A] \Phi^n - S \quad (5.11)
\]

so that replacing \([A+P]\) by the \([L][U]\) product in the equation (5.9) gives

\[
[L][U] \delta^{n+1} = -R^n \quad (5.12)
\]
defining an intermediate vector $W^{n+1}$ by

$$W^{n+1} = [U] \delta^{n+1}$$  \hfill (5.13)

the solution procedure can be again written as a two step process:

Step 1: $[L] W^{n+1} = -R^n$ \hfill (5.14)

Step 2: $[U] \delta^{n+1} = W^{n+1}$ \hfill (5.15)

The process represented by equations (5.14) and (5.15) consist of forward substitution to determine $W^{n+1}$ and a backward substitution to obtain $\delta^{n+1}$. The coefficients remain unchanged for the iterative process. The right-hand side of the Step 1 is then updated and the solution procedure is repeated.

Another method for the iterative solution of nine-diagonal matrices is developed by Peric (1987). His algorithm derives from SIP.

Schneider and Zedan (1981) proposed an alternative procedure for establishing the $[L] [U]$ matrices. The new method is called modified strongly implicit method (MSIP) by the authors. The algorithm is developed to handle the nine diagonal coefficient matrix representation of algebraic equations and treats the five diagonals matrices as a special case. Their modified strongly implicit method does not reduce to the SIP, if applied to a five-diagonal matrix.

The basic two-step algorithm remains the same as given in equations (5.14) and (5.15) for the LR solver of Peric (1987) and MSIP solver of Schneider and Zedan (1981). The improvement results from extending the approach of Stone to a nine-point formula and approximating the extra diagonals in the product matrix of the $[L]•[U]$.

Stone and Peric approximates two extra diagonals in the following form:
\[ \phi_{NW} = \alpha(\phi_W + \phi_N - \phi_P) \]  
(5.16)

\[ \phi_{SE} = \alpha(\phi_E + \phi_S - \phi_P) \]  
(5.17)

where \( \alpha \) is a parameter in the range 0-1.

When the two triangular matrices are chosen such that they have nonzero coefficients on the same diagonals as matrix \([A]\), the product matrix has nonzero coefficients on four extra diagonals. The following expressions are used for the extra diagonals corresponding to corner points, by Schneider and Zedan (1981):

\[ \phi_{NN} = \alpha(2\phi_N - \phi_P) \]  
(5.18)

\[ \phi_{SS} = \alpha(2\phi_S - \phi_P) \]  
(5.19)

For the remaining two points Schneider and Zedan (1981) adopted the below approximation formulas:

\[ \phi_{NNW} = \alpha(2\phi_N + \phi_W - 2\phi_P) \]  
(5.20)

\[ \phi_{SSE} = \alpha(2\phi_S + \phi_E - 2\phi_P) \]  
(5.21)

Schneider and Zedan (1981) declared that the approximations in equation (5.20) and (5.21) resulted in the reduction of sensitivity to \( \alpha \).

5.2.2. Performance of Various Solvers

Laminar flows are commonly used to test the performance of numerical algorithms. Since the effects of the pressure-velocity coupling which usually controls the convergence of the algorithm are evident for such flows. For the solution approach adopted in the fluid flow code used here, the convergence rate is a function of the underrelaxation factors \( \alpha \) employed. In the light of the results in the section 5.1, the SIMPLEC algorithm is employed in the solution of laminar flow in the lid-driven cavity flow with 30\(^{\circ}\) inclined walls in which the convergence rates deteriorate as compared to less inclined grids. The flow domain is same in the Figure (4.1). The flow domain is divided into 20x20 uniform grids. For all the results presented here,
these values were used for all the iterative solvers- SIP, CG, MSIP and LR algorithms. SIP and CG algorithms are adopted for solution of the simplified pressure-correction equation which results a symmetric five diagonal coefficient matrix. On the other hand, the unsymmetrical nine diagonal coefficient matrix of the discretized full pressure-correction equation is solved by MSIP and LR algorithms.

The numbers of outer iterations required to reach converged solution presented in Figure 5.9 for various solvers for the same convergence criterion ($1 \times 10^{-4}$) using the 2x2x3 inner iteration combination. As clearly seen in the figures the convergence rate of the LR and MSIP is higher than that of CGS and SIP. It is the result of the employed full pressure algorithm with LR and MSIP solvers. To make a clear comparison, CGS and SIP solvers should be analyzed together and LR and MSIP together.

Figure 5.9: Comparison of various solvers as a function of underrelaxation parameters
The convergence rate of the CGS is slightly higher than SIP algorithm. However, the convergence range of the SIP becomes superior to CGS as the $\alpha_u$ decreases. If it is thought that the solvers show an increase in the convergence rate when $\alpha_u=0.7$, CGS may be advantageous. As seen in the Figure 5.9 in the case of $\alpha_u=0.7$ and 0.6, CGS does not converge for the $\alpha_p=0.8$, but anyhow the performance of the SIP is inefficient above this range. The disadvantage of SIP solver is the needed to change the parameter $\alpha$ from case to case, since the dependence of the convergence rate on it is problem dependent. The parameter $\alpha$ is adjusted to maintain the highest convergence rate. CGS does not have any parameter that needs to be adjusted.

The MSIP and LR are less sensitive the underrelaxation parameters. They show similar convergence characteristics. The parameter $\alpha$ is less sensitive with MSIP and it does not need to be changed during iterations. The minimum numbers of outer iterations are maintained when $\alpha$ is equal to 1. The LR solver generally needs $\alpha$ adjusted between iterations.

To see the performances of the adapted solvers as the grid size increases, iteration are performed on five different grids- 20x20, 40x40, 60x60, 80x80, 100x100. The momentum equation system is relaxed at $\alpha_u=0.8$ which maintains the fastest convergence for used grid sizes in the procedure used in the code. The pressure underrelaxation factors are selected to ensure the minimum iteration numbers for each grid sizes. The numbers of iterations using the SIP, CG, MSIP and LR algorithms are presented in Table 5.1 on different meshes.

<table>
<thead>
<tr>
<th>Grid sizes</th>
<th>CGS</th>
<th>SIP</th>
<th>LR</th>
<th>MSIP</th>
</tr>
</thead>
<tbody>
<tr>
<td>20x20</td>
<td>156</td>
<td>174</td>
<td>119</td>
<td>122</td>
</tr>
<tr>
<td>40x40</td>
<td>372</td>
<td>421</td>
<td>268</td>
<td>276</td>
</tr>
<tr>
<td>60x60</td>
<td>662</td>
<td>722</td>
<td>409</td>
<td>405</td>
</tr>
<tr>
<td>80x80</td>
<td>1004</td>
<td>1041</td>
<td>580</td>
<td>578</td>
</tr>
<tr>
<td>100x100</td>
<td>1389</td>
<td>1609</td>
<td>768</td>
<td>767</td>
</tr>
</tbody>
</table>

Table 5.1 represents that the sensitivity to the grid refinement of the LR and MSIP is lower than the others. The SIP algorithm shows the highest sensitivity and fails to be an efficient algorithm on the finer grid solutions.
6. CONCLUSIONS AND DISCUSSION

6.1. Conclusions

In this thesis, a numerical method has been developed to calculate the fluid flows in complex geometries. In this method, the Cartesian velocity fluxes were used as the dependent variables in the momentum equations using body fitted coordinates. The discretization equations for contravariant velocity fluxes ensure a strongly conservative form which avoids complete transformation and is derived from the Cartesian velocity components. Non-staggered grid arrangement was adopted, where the velocity components and the pressure are stored at the same grid point. A modified version of the momentum interpolation method proposed by Rhie and Chow (1983) was developed to prevent the splitting of the pressure field. No under-relaxation factor appears in the present formulation. Throughout the study, a hybrid differencing scheme was used to treat the convection-diffusion terms in the momentum equations.

In the test problem 1, the comparisons show that excellent agreements were obtained between the results and the benchmark solutions for severely inclined lid-driven cavity flows. The validation of the algorithm was also maintained by comparing the existing numerical results of flow past a circular cylinder with the results obtained by this study. The emphasis was placed on the study of the range variation of the pressure under-relaxation factor, \( \alpha_p \), for lid-driven cavity flows where the non-orthogonal terms of \( p' \) equation were omitted.

The effect of the grid skewness on the number of iterations for both SIMPLE and SIMPLEC methods are investigated. The computations are performed to investigate the feasibility of solving pressure correction equation using one of methods listed in the section 4.3 when the grid skewness increases.

The results showed that the SIMPLEC algorithm is superior to the SIMPLE algorithm when simplified and full pressure-correction equation methods are used. If computational grid is not severely non-orthogonal (\( \beta >45^\circ \)), it is more logical to use
simplified version than the full one. The computer program is simpler and less memory is needed. On the other hand, full pressure-correction equation method converges fastest in a limitless range of $\alpha_p$ when the grid skewness increases. The Cho and Chung’s method serves inefficient performance if the SIMPLEC algorithm is employed. Although there is no limit to the ranges of $\alpha_p$ values, the convergence rate of the method is low. The simplified pressure-correction equation method has reasonable convergence rate and the method allows wide range of under-relaxation factors even for seriously non-orthogonal grids when the SIMPLEC algorithm is employed.

6.2. Recommendations

The present study was limited to two-dimensional, laminar, incompressible fluid flows. The following three suggestions are provided for the refinement and extension of the solution method presented in this thesis.

1. The present study can be easily extended to three-dimensional case.
2. As most of the fluid flows in real life are turbulent, problems studied here should be extended to the turbulent flow regime.
3. Compressible flow problems should be studied using the proposed method. To evaluate the density at the cell faces, density can be linearized for subsonic flows and upwinded for supersonic flows.
REFERENCES


RESUME

Hale Akçay was born in Adapazari, Sakarya in 1980. She graduated Sakarya Anatolian High School in 1998. She entered Istanbul Technical University Mechanical Engineering Department in 1998, and she received a Bachelor of Science degree in Mechanical Engineering from the Istanbul Technical University in 2003. She has been a student in the ITU Mechanical Faculty Thermofluids division. She has been holding a position as a teaching and research assistant in the same faculty since May 2004.