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ISTANBUL TECHNICAL UNIVERSITY ★ INSTITUTE OF SCIENCE AND TECHNOLOGY

FREE VIBRATION OF COMPOSITE LAMINATED CONICAL SHELL

M. Sc., THESIS

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Program: AERONAUTICS

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KOMPOZİT KONİK KABUĞUN SERBEST TİTREŞİMİ

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ABSTRACT

This study presents the dynamic equations of a stiffened composite laminated conical thin shell under the influence of initial stresses. The governing equations of truncated conical shell are based on the Donnell-Mushtari theory of thin shells including the transverse shear deformation and rotary inertia. The extension-bending coupling is considered in the derivation. The composite laminated conical shell is also reinforced at uniform intervals by elastic rings and/or stringers. The stiffening elements are relatively closely spaced, and therefore the stiffeners are "smeared out" along the conical shell. The inhomogeneity of material properties due to temperature, moisture or manufacturing processes is taken into account in the constitutive equations. A generalized variational theorem is derived so as to describe the complete set of the fundamental equations of the conical shell. Next, the uniqueness is examined in solutions of the dynamic equations of the conical shell and the boundary and initial conditions are shown to be sufficient for the uniqueness in solutions. The equations of laminated composite conical shell are solved by the use of finite difference method as an illustrative example. The accuracy of results is tested by certain earlier results and an good agreement is found.

ÖZET

KOMPOZİT KONİK KABUĞUN SERBEST TİTREŞİMLERİ

Modern uçaklar ve benzeri hava ve uzay araçlarında en yaygın kullanılan yapısal elemanlardan biri kompozit malzemeden imal edilmiş konik kabuk elemandır. Bu sebeple bu tür yapıların dinamik yükler altındaki davranışı yapının ömrü açısından önem arzetmektedir. Kompozit malzemeler için ayrı bir önem gösteren sıcaklık ve nem gibi çevresel etkiler bu tür yapıları homojenliğini bozmakta ve dinamik davranış karakterini etkilemektedir. Bu yüzden bu tür etkilerin de sözü edilen yapıların dinamik davranışı üzerine etkilerinin belirlenmesi gerekir.

Bu çalışmamızda, kompozit malzemeden yapılmış ve takviyelenmiş konik kabukların dinamik davranışı, elastik ince kabuklar için geçerli olan Donnell-Mushtari teoremlerinin basitleştirici kabulleri ile incelenmiştir. Ayrıca ince kabuklar için geçerli olan Love hipotezleri de denklemlerin elde edilmesinde dikkate alınmıştır ki bunlar :

1. Kabuğun kalınlığı diğer boyutlarına göre yeterince küçüktür
2. Birim şekil değiştirmeler ve yer değiştirmeler yeterince küçük miktardadır. Dolayısıyla birim şekil değiştirme-yerdeğiştirme denklemlerinde ikinci ve daha yüksek mertebeden terimler birinci mertebe terimlere kıyasla ihmal edilebilirler
3. Yanlamasına normal gerilme diğer normal gerilme bileşenlerine göre yeterince küçüktür ve ihmal edilebilir
4. Kabuğun referans yüzeylerinin normalleri deformasyon sonrasında da kendisine normal kalır ve bu normal üzerinde seçilen bir hattın uzunluğunda değişiklik olmaz,

Konik kabuğun temel denklemlerinin tanımlanması için genelleştirilmiş varyasyonel teorem kullanılmış, ayrıca lineerliği bozan geometrik etkiler ve ilk haldeki gerilmeler dikkate alınmıştır. Bunun yanısıra sonucun tekliği de incelenmiş ve sınır şartları ve ilk başlangıç şartlarının sonucun tekliğinde yeterli olduğu gösterilmiştir.

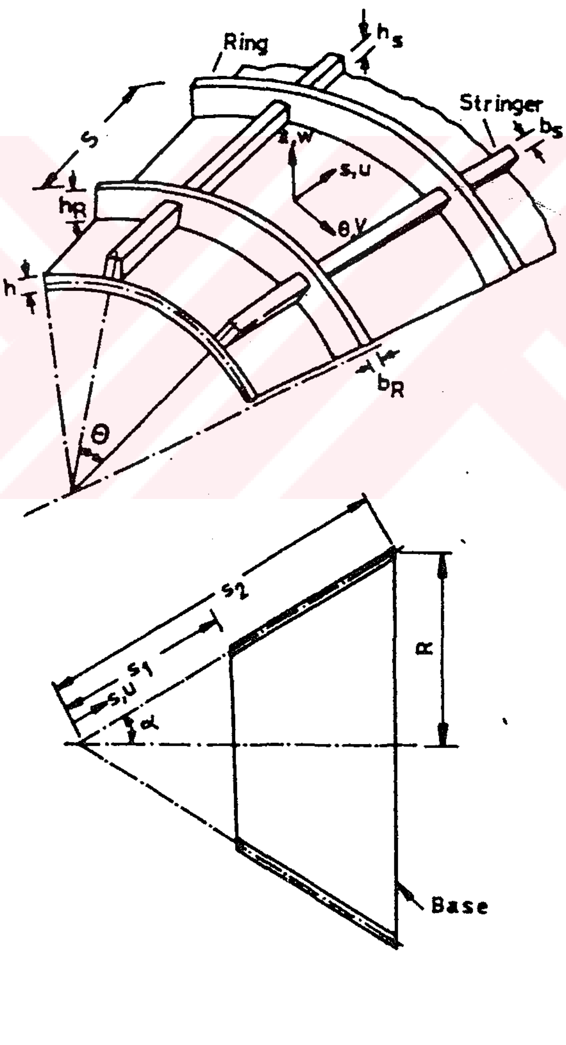
Konik kabuk yapının geometrisi Şekil 1. de gösterilmiştir :

Weierstrass teoremine göre kabuklar için yerdeğiştirme alanı şu şekildedir :

$$\begin{aligned}
U(s,\theta,z) &= u(s,\theta) + z\beta_s(s,\theta) + z^2\gamma_s(s,\theta) + \cdots \\
V(s,\theta,z) &= v(s,\theta) + z\beta_\theta(s,\theta) + z^2\gamma_\theta(s,\theta) + \cdots \\
W(s,\theta,z) &= w(s,\theta) + z\beta_z(s,\theta) + z^2\gamma_z(s,\theta) + \cdots
\end{aligned} \tag{1}$$

burada U, V ve W ; s, θ, ve z yönlerindeki yerdeğiştirme bileşenleridir. Kirchoff-Love hipotezinin basitleştirici kabulleri dikkate alınarak (1) denklemi şu şekilde yazılabilir :

$$\begin{aligned} U(s,\theta,z) &= u(s,\theta) + z\beta_s(s,\theta) \\ V(s,\theta,z) &= v(s,\theta) + z\beta_\theta(s,\theta) \\ W(s,\theta,z) &= w(s,\theta) \end{aligned} \tag{2}$$



Şekil 1. Konik yapının geometrisi ve koordinatlar

Bu yerdeğiştirme alanı gösterimi kullanılarak ince konik kabuk için kinematik bağlantılar şu şekilde yazılabilir :

$$\delta J_k = \int_T dt \iint_A \left\{ \left[\varepsilon_s - \lambda\left(\frac{\partial u}{\partial \bar{s}} + \frac{\lambda}{2}\left(\frac{\partial w}{\partial \bar{s}}\right)^2 \right)\right]\delta N_s + \left[\varepsilon_\theta - \lambda\left(\frac{u}{\bar{s}} + \frac{1}{\bar{s}}\frac{\partial v}{\partial \bar{\theta}} + \cot\alpha \frac{w}{\bar{s}} + \frac{\lambda}{2\bar{s}^2}\left(\frac{\partial w}{\partial \bar{\theta}}\right)^2 \right)\right]\delta N_\theta \right.$$
$$+ \left[\varepsilon_{s\theta} - \lambda\left(\frac{1}{\bar{s}}\frac{\partial u}{\partial \bar{\theta}} + \frac{\partial v}{\partial \bar{s}} - \frac{v}{\bar{s}} + \frac{\lambda}{\bar{s}}\frac{\partial w}{\partial \bar{s}}\frac{\partial w}{\partial \bar{\theta}} \right)\right]\delta N_{s\theta} + \left[\kappa_s - \lambda\frac{\partial \beta_s}{\partial \bar{s}} \right]\delta M_s + \left[\kappa_\theta - \lambda\left(\frac{1}{\bar{s}}\frac{\partial \beta_\theta}{\partial \bar{\theta}} + \frac{\beta_s}{\bar{s}} \right)\right]\delta M_\theta$$
$$+ \left[\kappa_{s\theta} - \lambda\left(\frac{\partial \beta_\theta}{\partial \bar{s}} - \frac{\beta_\theta}{\bar{s}} + \frac{1}{\bar{s}}\frac{\partial \beta_s}{\partial \bar{\theta}} \right)\right]\delta M_{s\theta} + \left[\varepsilon_{sz} - \lambda\left(\frac{\beta_s}{\lambda} + \frac{\partial w}{\partial \bar{s}} \right)\right]\delta Q_s$$
$$\left. + \left[\varepsilon_{\theta z} - \lambda\left(\frac{\beta_\theta}{\lambda} - \cot\alpha \frac{v}{\bar{s}} + \frac{1}{\bar{s}}\frac{\partial w}{\partial \bar{\theta}} \right)\right]\right\} dA \qquad (3)$$

Burada ε_s, ε_θ ve $\varepsilon_{s\theta}$ orta yüzeyin membran şekil değiştirmelerini ; κ_s, κ_θ ve $\kappa_{s\theta}$ eğilme sekil değişimlerini göstermektedir. $\lambda = \mathrm{Sin}\alpha \,/\, R$, konik kabuğun karakteristik bir parametresini göstermektedir. $\bar{s}$ ve $\bar{\theta}$ koordinatları şu şekilde tanımlanır : $\bar{s} = s\lambda$ ve $\bar{\theta} = \theta \mathrm{Sin}\alpha$.

Plak ve kabuk teorisinde, kuvvet ve moment bileşenlerini, gerilmeleri kabuk kalınlığı boyunca integre etmek suretiyle elde edebiliriz. Takviye elemanları da birbirine oldukça yakın yerleştirildiği için kabuk üzerinde sanki ayrı bir tabaka teşkil etmiş gibi düşünülebilir. Bu yaklaşımla bünye denklemleri varyasyonel formda şu şekilde yazılabilir

$$\delta I_c = \int_T dt \int_A \left\{ \left[N_s - \left(A_{11}\varepsilon_s + A_{12}\varepsilon_\theta + B_{11}\kappa_s + B_{12}\kappa_\theta \right)\right]\delta\varepsilon_s \right.$$
$$+ \left[N_\theta - \left(A_{12}\varepsilon_s + A_{22}\varepsilon_\theta + B_{12}\kappa_s + B_{22}\kappa_\theta \right)\right]\delta\varepsilon_\theta$$
$$+ \left[N_{s\theta} - \left(A_{33}\varepsilon_{s\theta} + B_{33}\kappa_{s\theta} \right)\right]\delta\varepsilon_{s\theta}$$
$$+ \left[M_s - \left(B_{11}\varepsilon_s + B_{12}\varepsilon_\theta + D_{11}\kappa_s + D_{12}\kappa_\theta \right)\right]\delta\kappa_s \qquad (4)$$
$$+ \left[M_\theta - \left(B_{12}\varepsilon_s + B_{22}\varepsilon_\theta + D_{12}\kappa_s + D_{22}\kappa_\theta \right)\right]\delta\kappa_\theta$$
$$+ \left[M_{s\theta} - \left(B_{33}\varepsilon_{s\theta} + D_{33}\kappa_{s\theta} \right)\right]\delta\kappa_{s\theta}$$
$$\left. + \left[Q_s - A_{55}\varepsilon_{sz} \right]\delta\varepsilon_{sz} + \left[Q_\theta - A_{44}\varepsilon_{\theta z} \right]\delta\varepsilon_{\theta z} \right\} dA = 0$$

burada

$$A_{ij} = \sum_{k=1}^{N_L} \overline{Q}_{ij}^{(k)} \left(h_k - h_{k-1}\right) + E_{ij} \qquad B_{ij} = \frac{1}{2} \sum_{k=1}^{N_L} \overline{Q}_{ij}^{(k)} \left(h_k^2 - h_{k-1}^2\right) - F_{ij}$$

$$D_{ij} = \frac{1}{3} \sum_{k=1}^{N_L} \overline{Q}_{ij}^{(k)} \left(h_k^3 - h_{k-1}^3\right) + G_{ij} \qquad i,j = 1,2,3 \tag{5}$$

$$\left(A_{44}, A_{55}\right) = \sum_{k=1}^{N_L} \int_{h_{k-1}}^{h_k} \left(\overline{Q}_{44}^{(k)}, \overline{Q}_{55}^{(k)}\right) f(z)\, dz$$

$$\begin{aligned}
\overline{Q}^{(k)}{}_{11} &= U_1 + U_2 \cos 2\theta + U_3 \cos 4\theta \\
\overline{Q}^{(k)}{}_{22} &= U_1 - U_2 \cos 2\theta + U_3 \cos 4\theta \\
\overline{Q}^{(k)}{}_{12} &= U_4 - U_3 \cos 4\theta = \overline{Q}_{21} \\
\overline{Q}^{(k)}{}_{33} &= U_5 - U_3 \cos 4\theta \\
\overline{Q}^{(k)}{}_{13} &= -\tfrac{1}{2} U_2 \sin 2\theta - U_3 \sin 4\theta = \overline{Q}_{31} \\
\overline{Q}^{(k)}{}_{23} &= -\tfrac{1}{2} U_2 \sin 2\theta + U_3 \sin 4\theta = \overline{Q}_{32}
\end{aligned} \tag{6a}$$

$$\left\{\overline{Q}^{(k)}{}_{44}, \overline{Q}^{(k)}{}_{55}\right\} = E^{(k)}{}_{\theta} \left\{ \frac{1}{2\left(1+\nu^{(k)}{}_{\theta z}\right)}, \frac{1}{2\left(1+\nu^{(k)}{}_{xz}\right)} \right\} \tag{6b}$$

burada

$$\begin{aligned}
U_1 &= \tfrac{1}{8}\left(3Q_{11} + 3Q_{22} + 2Q_{12} + 4Q_{33}\right) \\
U_2 &= \tfrac{1}{2}\left(Q_{11} - Q_{22}\right) \\
U_3 &= \tfrac{1}{8}\left(Q_{11} + Q_{22} - 2Q_{12} - 4Q_{33}\right) \\
U_4 &= \tfrac{1}{8}\left(Q_{11} + Q_{22} + 6Q_{12} - 4Q_{33}\right) \\
U_5 &= \tfrac{1}{8}\left(Q_{11} + Q_{22} - 2Q_{12} + 4Q_{33}\right)
\end{aligned} \tag{6c}$$

$$\{E_{11},E_{22},E_{12},E_{33}\} = \lambda\left\{\frac{E_S A_S}{\overline{\Theta}},\frac{E_R A_R}{\overline{S}},0,0\right\}$$

$$\{F_{11},F_{22},F_{12},F_{33}\} = 2\lambda\left\{\frac{E_S A_S e_S}{\overline{\Theta}},\frac{E_R A_R e_R}{\overline{S}},0,0\right\} \tag{7}$$

$$\{G_{11},G_{22},G_{12},G_{33}\} = \lambda\left\{\frac{E_S A_S\left(i_{\eta S}{}^2+e_S{}^2\right)}{\overline{\Theta}},\frac{E_R A_R\left(i_{\xi R}{}^2+e_R{}^2\right)}{\overline{S}},0,\left(\frac{G_S J_S}{\overline{\Theta}}+\frac{G_R J_R}{\overline{S}}\right)\right\}$$

ve

$$f(z)=\frac{5}{4}\left[1-4\left(\frac{z}{h}\right)^2\right] \tag{8}$$

Burada h, konik kabuğun kalınlığını ; E_s ve G_sJ_s elastisite modülünü ve dönme rijitliğini gösterir.

Sıcaklık ve nem etkisi gibi çevresel etkiler yüzünden elastik sabitler değişebilmektedir. Bu da malzemenin homojen özelliğinin değiştiğini gösterir. Bu yüzden malzemenin Young modülleri şu şekilde yazılabilir:

$$E_s(z)=E_{s_0}\left[1+\alpha_s\left(1+2\frac{z}{h}\right)\right] \qquad E_\theta(z)=E_{\theta_0}\left[1+\alpha_\theta\left(1+2\frac{z}{h}\right)\right] \tag{9}$$

Denge denklemi ilk ve sınır şartlar virtüel iş prensibi kullanılanarak kompozit konik kabuğun serbest titreşimi için elde edilebilir

$$\tau_s=-\frac{\partial N_s}{\partial\overline{s}}-\frac{1}{\overline{s}}\left(N_s-N_\theta\right)-\frac{1}{\overline{s}}\frac{\partial N_{s\theta}}{\partial\overline{\theta}}+\frac{m_1}{\lambda}\ddot{u}+\frac{m_2}{2\lambda}\ddot{\beta}_s-\frac{q_s}{\lambda}=0$$

$$\tau_\theta=-\frac{\partial N_{s\theta}}{\partial\overline{s}}-\frac{2}{\overline{s}}N_{s\theta}-\frac{1}{\overline{s}}\frac{\partial N_\theta}{\partial\overline{\theta}}-\frac{\cot\alpha}{\overline{s}}Q_\theta+\frac{m_1}{\lambda}\ddot{v}+\frac{m_2}{2\lambda}\ddot{\beta}_\theta-\frac{q_\theta}{\lambda}=0$$

$$\tau_z=-\frac{\partial Q_s}{\partial\overline{s}}-\frac{1}{\overline{s}}\frac{\partial Q_\theta}{\partial\overline{\theta}}-\frac{1}{\overline{s}}Q_s+\frac{\cot\alpha}{\overline{s}}N_\theta-\frac{\lambda}{\overline{s}}\frac{\partial}{\partial\overline{s}}\left(N_s^i\overline{s}\frac{\partial w}{\partial\overline{s}}\right)-\frac{\lambda}{\overline{s}^2}\frac{\partial}{\partial\overline{\theta}}\left(N_\theta^i\frac{\partial w}{\partial\overline{\theta}}\right)$$
$$-\frac{\lambda}{\overline{s}}\frac{\partial}{\partial\overline{s}}\left(N_{s\theta}^i\frac{\partial w}{\partial\overline{\theta}}\right)-\frac{\partial}{\partial\overline{\theta}}\left(N_{s\theta}^i\frac{\partial w}{\partial\overline{s}}\right)+\frac{m_1}{\lambda}\ddot{w}-\frac{q_z}{\lambda}=0 \tag{10}$$

$$\tau_{sz}=-\frac{\partial M_s}{\partial\overline{s}}-\frac{1}{\overline{s}}\left(M_s-M_\theta\right)-\frac{1}{\overline{s}}\frac{\partial M_{s\theta}}{\partial\theta}+\frac{1}{\lambda}Q_s+\frac{m_2}{2\lambda}\ddot{u}+\frac{m_{12}}{\lambda}\ddot{\beta}_s-\frac{m_s}{\lambda}=0$$

$$\tau_{\theta z}=-\frac{\partial M_{s\theta}}{\partial\overline{s}}-\frac{2}{\overline{s}}M_{s\theta}-\frac{1}{\overline{s}}\frac{\partial M_\theta}{\partial\theta}+\frac{1}{\lambda}Q_\theta+\frac{m_2}{2\lambda}\ddot{v}+\frac{m_{21}}{\lambda}\ddot{\beta}_\theta-\frac{m_\theta}{\lambda}=0$$

(10) denklemi ile verilen denge denklemlerini yerdeğiştirmeler ve dönmeler cinsinden yazabiliriz. (3) denklemini (4) denkleminde yerine koyarsak ve daha sonra da (10) denklemine geçersek, dinamik denge denklemlerini 5 yerdeğiştirme bileşeni cinsinden aşağıdaki gibi ifade edebiliriz :

$$
\begin{aligned}
&L_{11}u+L_{12}v+L_{13}w+L_{14}\beta_s+L_{15}\beta_\theta+N_1(w)+\frac{m_1}{\lambda^2}\ddot{u}+\frac{m_2}{2\lambda^2}\ddot{\beta}_s-\frac{q_s}{\lambda^2}=0\\
&L_{21}u+L_{22}v+L_{23}w+L_{24}\beta_s+L_{25}\beta_\theta+N_2(w)+\frac{m_1}{\lambda^2}\ddot{v}+\frac{m_2}{2\lambda^2}\ddot{\beta}_\theta-\frac{q_\theta}{\lambda^2}=0\\
&L_{31}u+L_{32}v+L_{33}w+L_{34}\beta_s+L_{35}\beta_\theta+N_3(w)+\frac{m_1}{\lambda^2}\ddot{w}-\frac{q_z}{\lambda^2}=0\\
&L_{41}u+L_{42}v+L_{43}w+L_{44}\beta_s+L_{45}\beta_\theta+N_4(w)+\frac{m_2}{2\lambda^2}\ddot{u}+\frac{m_{12}}{\lambda^2}\ddot{\beta}_s-\frac{m_s}{\lambda^2}=0\\
&L_{51}u+L_{52}v+L_{53}w+L_{54}\beta_s+L_{55}\beta_\theta+N_5(w)+\frac{m_2}{2\lambda^2}\ddot{v}+\frac{m_{21}}{\lambda^2}\ddot{\beta}_\theta-\frac{m_\theta}{\lambda^2}=0
\end{aligned}
\tag{11}
$$

burada

$$
\begin{aligned}
L_{11}&=-A_{11}\frac{\partial^2}{\partial\bar{s}^2}-\frac{A_{11}}{\bar{s}}\frac{\partial}{\partial\bar{s}}+\frac{A_{22}}{\bar{s}^2}-\frac{A_{33}}{\bar{s}^2}\frac{\partial^2}{\partial\bar{\theta}^2}\\
L_{12}&=-\frac{(A_{12}+A_{33})}{\bar{s}}\frac{\partial^2}{\partial\bar{s}\partial\bar{\theta}}+\frac{(A_{22}+A_{33})}{\bar{s}^2}\frac{\partial}{\partial\bar{\theta}}\\
L_{13}&=\frac{A_{22}\cot\alpha}{\bar{s}^2}-\frac{A_{12}\cot\alpha}{\bar{s}}\frac{\partial}{\partial\bar{s}}\\
L_{14}&=-B_{11}\frac{\partial^2}{\partial\bar{s}^2}-\frac{B_{11}}{\bar{s}}\frac{\partial}{\partial\bar{s}}+\frac{B_{22}}{\bar{s}^2}-\frac{B_{33}}{\bar{s}^2}\frac{\partial^2}{\partial\bar{\theta}^2}\\
L_{15}&=\frac{(B_{22}+B_{33})}{\bar{s}^2}\frac{\partial}{\partial\bar{\theta}}-\frac{(B_{12}+B_{33})}{\bar{s}}\frac{\partial^2}{\partial\bar{s}\partial\bar{\theta}}
\end{aligned}
$$

$$L_{21} = -\frac{(A_{12}+A_{33})}{\bar{s}}\frac{\partial^2}{\partial\bar{s}\partial\bar{\theta}} - \frac{(A_{22}+A_{33})}{\bar{s}^2}\frac{\partial}{\partial\bar{\theta}}$$

$$L_{22} = -A_{33}\left[\frac{\partial^2}{\partial\bar{s}^2} + \frac{1}{\bar{s}}\frac{\partial}{\partial\bar{s}} - \frac{1}{\bar{s}^2}\right] - \frac{A_{22}}{\bar{s}^2}\frac{\partial^2}{\partial\bar{\theta}^2} + \frac{A_{44}\cot^2\alpha}{\bar{s}^2}$$

$$L_{23} = -\frac{(A_{22}+A_{44})\cot\alpha}{\bar{s}^2}\frac{\partial}{\partial\bar{\theta}}$$

$$L_{24} = -\frac{(B_{12}+B_{33})}{\bar{s}}\frac{\partial^2}{\partial\bar{s}\partial\bar{\theta}} - \frac{(B_{22}+B_{33})}{\bar{s}^2}\frac{\partial}{\partial\bar{\theta}}$$

$$L_{25} = -B_{33}\left[\frac{\partial^2}{\partial\bar{s}^2} + \frac{1}{\bar{s}}\frac{\partial}{\partial\bar{s}} - \frac{1}{\bar{s}^2}\right] - \frac{B_{22}}{\bar{s}^2}\frac{\partial^2}{\partial\bar{\theta}^2} - \frac{A_{44}\cot\alpha}{\bar{s}\lambda}$$

$$L_{31} = \frac{A_{12}\cot\alpha}{\bar{s}}\frac{\partial}{\partial\bar{s}} + \frac{A_{22}\cot\alpha}{\bar{s}^2}$$

$$L_{32} = -L_{23}$$

$$L_{33} = -A_{55}\left(\frac{\partial^2}{\partial\bar{s}^2} + \frac{1}{\bar{s}}\frac{\partial}{\partial\bar{s}}\right) - \frac{A_{44}}{\bar{s}^2}\frac{\partial^2}{\partial\bar{\theta}^2} - \frac{A_{22}\cot^2\alpha}{\bar{s}^2} - \lambda\left(\frac{\partial N_s^i}{\partial\bar{s}} + \frac{N_s^i}{\bar{s}} + \frac{1}{\bar{s}}\frac{\partial N_{s\theta}^i}{\partial\bar{\theta}}\right)\frac{\partial}{\partial\bar{s}}$$

$$-\lambda N_s^i\frac{\partial^2}{\partial\bar{s}^2} - \frac{2\lambda N_{s\theta}^i}{\bar{s}}\frac{\partial^2}{\partial\bar{s}\partial\bar{\theta}} - \lambda\left(\frac{1}{\bar{s}^2}\frac{\partial N_\theta^i}{\partial\bar{\theta}} + \frac{1}{\bar{s}}\frac{\partial N_{s\theta}^i}{\partial\bar{s}}\right)\frac{\partial}{\partial\bar{\theta}} - \frac{\lambda N_\theta^i}{\bar{s}^2}\frac{\partial^2}{\partial\bar{\theta}^2} \qquad (12)$$

$$L_{34} = \left(\frac{B_{12}\cot\alpha}{\bar{s}} - \frac{A_{55}}{\lambda}\right)\frac{\partial}{\partial\bar{s}} - \frac{A_{55}}{\lambda\bar{s}} + \frac{B_{22}\cot\alpha}{\bar{s}^2}$$

$$L_{35} = \left(\frac{B_{22}\cot\alpha}{\bar{s}^2} - \frac{A_{44}}{\lambda\bar{s}}\right)\frac{\partial}{\partial\bar{\theta}}$$

$$L_{41} = L_{14} \qquad L_{42} = L_{15}$$

$$L_{43} = \left(\frac{A_{55}}{\lambda} - \frac{B_{12}\cot\alpha}{\bar{s}}\right)\frac{\partial}{\partial\bar{s}} + \frac{B_{22}\cot\alpha}{\bar{s}^2}$$

$$L_{44} = -D_{11}\left(\frac{\partial^2}{\partial\bar{s}^2} + \frac{1}{\bar{s}}\frac{\partial}{\partial\bar{s}}\right) + \frac{D_{22}}{\bar{s}^2} - \frac{D_{33}}{\bar{s}^2}\frac{\partial^2}{\partial\bar{\theta}^2} + \frac{A_{55}}{\lambda^2}$$

$$L_{45} = \frac{(D_{22}+D_{33})}{\bar{s}^2}\frac{\partial}{\partial\bar{\theta}} - \frac{(D_{12}+D_{33})}{\bar{s}}\frac{\partial^2}{\partial\bar{s}\partial\bar{\theta}}$$

$$L_{51} = L_{24} \qquad L_{52} = L_{25} \qquad L_{53} = -L_{35}$$

$$L_{54} = -\frac{(D_{12}+D_{33})}{\bar{s}}\frac{\partial^2}{\partial\bar{s}\partial\bar{\theta}} - \frac{(D_{22}+D_{33})}{\bar{s}^2}\frac{\partial}{\partial\bar{\theta}}$$

$$L_{55} = -D_{33}\left(\frac{\partial^2}{\partial\bar{s}^2} + \frac{1}{\bar{s}}\frac{\partial}{\partial\bar{s}} - \frac{1}{\bar{s}^2}\right) - \frac{D_{22}}{\bar{s}^2}\frac{\partial^2}{\partial\bar{\theta}^2} + \frac{A_{44}}{\lambda^2}$$

Hareket denklemlerindeki lineer olmayan terimler ise

$$N_1(w) = -A_{11}\lambda \frac{\partial w}{\partial \bar{s}}\frac{\partial^2 w}{\partial \bar{s}^2} + \left(A_{12} + A_{22}\right)\frac{\lambda}{2\bar{s}^3}\left(\frac{\partial w}{\partial \bar{\theta}}\right)^2 - \left(A_{12} + A_{33}\right)\frac{\lambda}{\bar{s}^2}\frac{\partial w}{\partial \bar{\theta}}\frac{\partial^2 w}{\partial \bar{s}\partial \bar{\theta}}$$
$$-\left(A_{11} - A_{12}\right)\frac{\lambda}{2\bar{s}}\left(\frac{\partial w}{\partial \bar{s}}\right)^2 - A_{33}\frac{\lambda}{\bar{s}^2}\frac{\partial w}{\partial \bar{s}}\frac{\partial^2 w}{\partial \bar{\theta}^2}$$

$$N_2(w) = -\left(A_{12} + A_{33}\right)\frac{\lambda}{\bar{s}}\frac{\partial w}{\partial \bar{s}}\frac{\partial^2 w}{\partial \bar{s}\partial \bar{\theta}} - A_{22}\frac{\lambda}{\bar{s}^3}\frac{\partial w}{\partial \bar{\theta}}\frac{\partial^2 w}{\partial \bar{\theta}^2} - A_{33}\left(\frac{\lambda}{\bar{s}}\frac{\partial^2 w}{\partial \bar{s}^2}\frac{\partial w}{\partial \bar{\theta}} + \frac{\lambda}{\bar{s}^2}\frac{\partial w}{\partial \bar{s}}\frac{\partial w}{\partial \bar{\theta}}\right)$$
$$N_3(w) = A_{12}\frac{\lambda \cot\alpha}{2\bar{s}}\left(\frac{\partial w}{\partial \bar{s}}\right)^2 + A_{22}\frac{\lambda \cot\alpha}{2\bar{s}^3}\left(\frac{\partial w}{\partial \bar{\theta}}\right)^2$$

$$N_4(w) = -B_{11}\lambda \frac{\partial w}{\partial \bar{s}}\frac{\partial^2 w}{\partial \bar{s}^2} + \left(B_{12} + B_{22}\right)\frac{\lambda}{2\bar{s}^3}\left(\frac{\partial w}{\partial \bar{\theta}}\right)^2 - \left(B_{12} + B_{33}\right)\frac{\lambda}{\bar{s}^2}\frac{\partial w}{\partial \bar{\theta}}\frac{\partial^2 w}{\partial \bar{s}\partial \bar{\theta}}$$
$$-\left(B_{11} - B_{12}\right)\frac{\lambda}{2\bar{s}}\left(\frac{\partial w}{\partial \bar{s}}\right)^2 - B_{33}\frac{\lambda}{\bar{s}^2}\frac{\partial w}{\partial \bar{s}}\frac{\partial^2 w}{\partial \bar{\theta}^2}$$
$$N_5(w) = -\left(B_{12} + B_{33}\right)\frac{\lambda}{\bar{s}}\frac{\partial w}{\partial \bar{s}}\frac{\partial^2 w}{\partial \bar{s}\partial \bar{\theta}} - B_{22}\frac{\lambda}{\bar{s}^3}\frac{\partial w}{\partial \bar{\theta}}\frac{\partial^2 w}{\partial \bar{\theta}^2} - B_{33}\left(\frac{\lambda}{\bar{s}}\frac{\partial^2 w}{\partial \bar{s}^2}\frac{\partial w}{\partial \bar{\theta}} + \frac{\lambda}{\bar{s}^2}\frac{\partial w}{\partial \bar{s}}\frac{\partial w}{\partial \bar{\theta}}\right)$$

(13)

İlgili sınır şartları ve ilk şartların da varyasyonel formda yorumlanmasıyla bütün denklemler elde edilebilir.

Sonuçların tekliğinin belirlenmesi için elastisite teorisinde pek çok yöntemler bulunmasına karşılık burada klasik enerji yaklaşımı tercih edilmiştir.

Başlangıçta, takviyeli konik kabuğunun hareket denklemleri için muhtemel iki çözüm seti düşünülmüş ve bunların farkı alınmıştır.Çözümlerin lineerliğinden dolayı bu farkın da hareket denklemlerini sağladığı farzedilebilir. Eğer bu fark çözümün "sıfır çözüm" olduğu gösterilebilinirse, kabul edilen iki çözüm setinin birbirine denk olduğu sonucuna varılır.

(10) denklemleri ile verilen hareket denklemleri sonlu farklar yaklaşımı kullanılarak yazılan bir bilgisayar programı yardımıyla çözülmüş ve literatürdeki benzer sonuçlarla karşılaştırılmıştır. Buradan çıkan neticeler şu şekilde ifade edilebilir:

1.Kompozit konik kabuğun temel frekansı takviyesiz halde takviyeli hale nazaran daha düşük değerlerde seyretmektedir.

2. Koniğin tepe açısının artmasıyla frekans değerlerinde azalma görülmektedir.

3.Nem konsantrasyonunun artmasıyla malzeme özelliklerinde değişmeler olmakta ve malzemenin elastise modülü nem konsantrasyonu arttıkça düşmektedir. Bu da frekans değerlerinin düşük olmasına sebep olmaktadır.

CHAPTER 1

INTRODUCTION

A stiffened laminated conical shell is one of the common structural elements used in modern airplane, missile, booster and other space vehicles. The dynamic behavior of stiffened laminated conical shells under the dynamic loads and initial stresses is of considerable engineering importance for determination the failure and fatigue life of the shell. The inhomogeneity due to environmental effects such as temperature and moisture may degrade the mechanical properties of anisotropic materials and affect the dynamic behavior of the shell, and should be taken into account in the constitutive equations of the conical shell.

Studies on the vibrations of cylindrical and truncated conical shells made of isotropic and anisotropic materials were referenced by Leissa in his monograph[1] Although much literature exists on the free vibration of isotropic conical shells, e.g. references[2-5], fewer studies were found about stiffened, orthotropic or anisotropic conical shells. Singer[6] and Weingarten[7] derived the equations of motion of a Donnell-Mushtari type orthotropic shell theory. Bacon and Bert[8] showed the effect of changing the ratio of orthotropic constants upon the fundamental frequencies of shells of revolution. Cohen[9] investigated the asymmetrical free vibrations of ring-stiffened orthotropic shells of revolution. Mecitoglu and Dökmeci[10] developed a shell finite element which includes smeared stringers and rings, and examined the uniqueness on the solutions. Weingarten[11] , Goldberg et. al.[12,13], and Schneider et al.[14] studied the vibration of the conical shells subjected to initial stresses. Newton[15] obtained results for the conical shells having orthotropic material properties which vary in the meridional direction. Martin[16] studied the free vibrations of anisotropic conical shells. Heyliger and Jilani[17] considered the inhomogeneity in the free vibrations of elastic layered cylinders and spheres. Inhomogeneity of the material properties due to temperature was studied by Mecitoglu[18] for a truncated isotropic conical shell. Sivadas and Ganesan[19], and Sankaranarayanan et. al.[20] studied the laminated conical shells with variable thickness. Kayran and Vinson[21] examined the effect of transverse shear and rotary inertia on the free vibrations of truncated conical shells. Tong studied the free vibration of orthotropic[22] and laminated[23,24] conical shells using a simple and exact solution technique. Langley[25] applied a dynamic stiffness techniques to the vibration of the stiffened shell structures. Ley et. al.[26] examined the buckling of the stiffened anisotropic conical shells. Saı Ram and Sınha[27] investigated hygrothermal effects on the free vibration of laminated composite plates by finite element method. Lyrintzis and Bofilios[28] examined the effect of absorbed moisture on the dynamic response of stiffened panel structure from orthotropic material and a general external loading condition. Kollar and Patterson[29] studied a fiber-reinforced organic matrix composite cylindrical segments subjected to hygrothermal and mechanical loads. Although the uniqueness in solutions of the elastodynamic problems is very important, only a few work is found in literature examining with this subject[30].

The purpose of this paper is to derive all the dynamic equations of a composite laminated conical thin shell reinforced by stringers and rings and to examine the uniqueness in solutions of the governing shell equations. The dynamic equations of a stiffened composite laminated conical shell are derived within the frame of the Donnell-Mushtari theory of elastic thin shells. A generalized variational theorem is given so as to describe the complete set of the fundamental equations of the conical shell. The geometric nonlinearities and initial stresses are taken into account in the derivation of governing equations. The rings and/or stringers are "smeared out" along the conical shell. The inhomogeneity of material properties due to temperature, moisture or manufacturing processes is taken into account in the constitutive equations. The uniqueness is examined in solutions of the dynamic governing equations of stiffened shells, and a theorem of uniqueness is given which enumerates the initial and boundary conditions sufficient for the uniqueness. The dynamic equations of the stiffened laminated conical shell are solved by the finite difference method to obtain the vibration characteristics and an agreement is obtained with certain earlier results. Influence of the moisture on the vibration characteristics is examined.

CHAPTER 2

GOVERNING EQUATIONS

2. 1. FUNDAMENTAL EQUATIONS OF THIN ELASTIC SHELLS

2. 1. 1. BASIC ASSUMPTIONS

The basic equations which describe the behaviour of a thin elastic shell were originally derived by Love in 1888. These equations, together with the assumptions upon which they are based, form a theory of thin elastic shells which is commonly referred to as Love's first approximation. In this part, we shall derive the basic differential equations of the first approximation using, as a basis, E. Reissner's version of the Love theory.

In the three dimensional theory of elasticity, the fundamental equations occur in the three broad categories. Thus it will be recalled that in elasticity we have equations of motion which are obtained from a balance of the forces acting on some fundamental element of the medium considered that we have strain - displacement relations which are obtained from a strictly geometrical consideration of the process of the deformation, and that we have the constitutive law of elasticity (Hooke's law) which is introduced in order to provide a relationship between the stresses and the strains in the elastic medium. It will also be remembered that the solutions of the problems in the three dimensional theory of elasticity involves vast complications which have been overcome in only a few special cases. Thus, a group of simplifying assumptions that provide a reasonable description of the behaviour of the thin elastic shells was proposed by Love and has led to the development of a sub-class of the theory of elasticity known as the theory of thin elastic shells.

Love's first approximation to the theory of thin elastic shells is based upon the following postulates

1- *The shell thickness is very small in comparison with other shell dimensions such as radius of curvature, length, etc.*

2- *Strains and displacements are sufficiently small so that the quantities of second and higher order magnitude in the strain-displacement relations may be neglected in comparison with the first-order terms.*

3- *The transverse normal stress is small compared with the other normal stress components and may be neglected*

4- *Normalize to the unreformed middle surface remain straight and normal to the deformed middle surface and suffer no extension.*

The assumptions of thinness sets the stage for the entire theory. The set of Love's postulates seem to be appropriate only to thin shells and are, therefore, consequences of this postulate. As a rule of thumb, however, it is suggested that the resulting theory be applied only to shells whose thickness is everywhere less than one tenth of the radious of curvature of the reference surface.

The assumption that the deflections of the shell are small permits us to refer all derivations and calculations to the original configuration of the shell and, together with Hooke's law, assures us that the resulting theory will be a linear, elastic one.

The two remaining hypotheses of the Love theory deal with the constitutive equations of the thin elastic shells and represent the most significant features of the first approximation.

It is necessary to note that the fourth hypothesis that concerns the preservation of the normal element represents an extension to the case of the thin elastic shell of the familiar Bernouli-Euler hypothesis of beam theory which states that " plane sections remain plane." The assumption of the preservation of the normal implies that all the strain components in the direction of the normal to the reference surfaces ($\varepsilon_n, \gamma_{1n}, \gamma_{2n}$) vanish and the shearing stress components (τ_{1n}, τ_{2n}) also vanish.

Our restriction to the consideration of thin shell makes reasonable the third assumption of the Love theory that is expressed as $\sigma_n = 0$.

It is expected that this assumption will be generally valid except in the vicinity of the highly concentrated loads[31].

2. 1. 2 DISPLACEMENT FIELD

We can define, in the shell space, the following displacement vector:

$$\mathbf{U}(\alpha_1,\alpha_2,\zeta) = U(\alpha_1,\alpha_2,\zeta)\mathbf{t}_1 + V(\alpha_1,\alpha_2,\zeta)\mathbf{t}_2 + W(\alpha_1,\alpha_2,\zeta)\mathbf{n} \tag{2.1}$$

where $\mathbf{t}_1, \mathbf{t}_2, \mathbf{n}$ are unit vectors along α_1, α_2 and the normal to the reference surface, respectively, and U, V, W are the components of the displacement vector in the corresponding orthogonal coordinate directions. We can expand this function in Maclaurin series form with neighbour of $\zeta = 0$ and obtain following functions :

$$\begin{aligned} U(\alpha_1,\alpha_2,\zeta) &= u(\alpha_1,\alpha_2) + \zeta.u'(\alpha_1,\alpha_2) + \zeta^2.u''(\alpha_1,\alpha_2) + \cdots \\ V(\alpha_1,\alpha_2,\zeta) &= v(\alpha_1,\alpha_2) + \zeta.v'(\alpha_1,\alpha_2) + \zeta^2.v''(\alpha_1,\alpha_2) + \cdots \\ W(\alpha_1,\alpha_2,\zeta) &= w(\alpha_1,\alpha_2) + \zeta.w'(\alpha_1,\alpha_2) + \zeta^2.w''(\alpha_1,\alpha_2) + \cdots \end{aligned} \tag{2.2}$$

where u, v and w displacements components of the middle surface and a prime denotes a derivative with respect to ζ. The assumption on the preservation of the normal element implies that the displacements are linearly distributed throughout the thickness of the shell. Thus, we may assume that the displacement components are represented by the following relationships:

$$\begin{aligned} U(\alpha_1,\alpha_2,\zeta) &= u(\alpha_1,\alpha_2) + \zeta.u'(\alpha_1,\alpha_2) \\ V(\alpha_1,\alpha_2,\zeta) &= v(\alpha_1,\alpha_2) + \zeta.v'(\alpha_1,\alpha_2) \\ W(\alpha_1,\alpha_2,\zeta) &= w(\alpha_1,\alpha_2) \end{aligned} \tag{2.3}$$

where a prime denotes a derivative with respect to ζ . The quantities u, v and w represent the components of the displacement vector of a point on the reference surface, and the quantities u' and v' represent the rotations of tangents to the reference surface oriented along the parametric lines α_1, α_2 respectively. The rotations u' and v' which will henceforth be denoted by β_1 and β_2 .

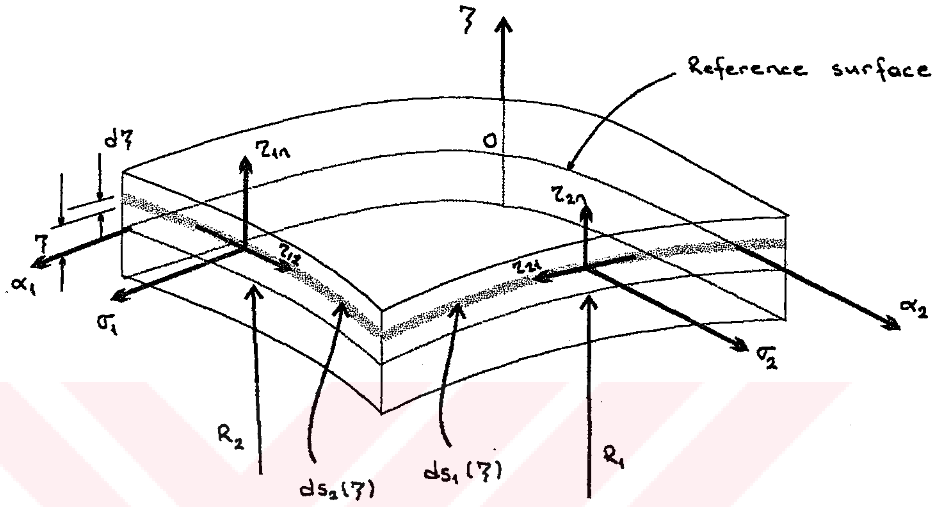


Figure 2.1 Differential element of a shell

2. 1. 3 STRAIN DISPLACEMENT RELATIONS

In the text on the theory of elasticity, in an orthogonal curvilinear coordinate system. the normal and shearing strain components are related to the components of the displacement vector by

$$\varepsilon_i = \frac{\partial}{\partial \alpha_i}\left(\frac{u_i}{\sqrt{g_i}}\right) + \frac{1}{2g_i}\sum_{k=1}^{3}\frac{\partial g_i}{\partial \alpha_k}\frac{u_k}{\sqrt{g_k}} \qquad i = 1,2,3 \qquad (2.4)$$

$$\varepsilon_{ij} = \frac{1}{\sqrt{g_i g_j}}\left[g_i \frac{\partial}{\partial \alpha_j}\left(\frac{u_i}{\sqrt{g_i}}\right) + g_j \frac{\partial}{\partial \alpha_i}\left(\frac{u_j}{\sqrt{g_j}}\right)\right] \qquad i = 1,2,3 \quad i \neq j \qquad (2.5)$$

where, for the application to shells

$$\begin{array}{lll} \alpha_1 = \alpha_1 & \alpha_2 = \alpha_2 & \alpha_3 = \zeta \\ u_1 = U & u_2 = V & u_3 = W \\ g_1 = A_1^2(1+\zeta/R_1)^2 & g_2 = A_2^2(1+\zeta/R_2)^2 & g_3 = 1 \end{array} \tag{2.6}$$

where A_1 and A_2 basic form parameters or lame parameters, R_1 and R_2 radius of curvature, g_1, g_2 and g_3 components of basic form of the surface.

If we substitute Eqs. (2.6) into Eqs (2.5), we obtain the following strain displacement equations in the shell space:

$$\begin{aligned} \varepsilon_1 &= \frac{1}{A_1}\frac{\partial}{\partial\alpha_1}\left(u+\zeta\beta_1\right)+\frac{v+\zeta\beta_2}{A_1A_2}\frac{\partial A_1}{\partial\alpha_2}+\frac{w}{R_1}+\frac{1}{2A_1^2}\left(\frac{\partial w}{\partial\alpha_1}\right)^2 \\ \varepsilon_2 &= \frac{1}{A_2}\frac{\partial}{\partial\alpha_2}\left(v+\zeta\beta_2\right)+\frac{u+\zeta\beta_1}{A_1A_2}\frac{\partial A_2}{\partial\alpha_1}+\frac{w}{R_2}+\frac{1}{2A_2^2}\left(\frac{\partial w}{\partial\alpha_2}\right)^2 \\ \varepsilon_{12} &= \frac{A_2}{A_1}\frac{\partial}{\partial\alpha_1}\left(\frac{v+\zeta\beta_2}{A_2}\right)+\frac{A_1}{A_2}\frac{\partial}{\partial\alpha_2}\left(\frac{u+\zeta\beta_1}{A_1}\right)+\frac{1}{A_1A_2}\left(\frac{\partial w}{\partial\alpha_1}\right)\left(\frac{\partial w}{\partial\alpha_2}\right) \\ \varepsilon_{1n} &= \frac{1}{A_1}\frac{\partial w}{\partial\alpha_1}-\frac{u}{R_1}+\beta_1 \\ \varepsilon_{2n} &= \frac{1}{A_2}\frac{\partial w}{\partial\alpha_2}-\frac{v}{R_2}+\beta_2 \end{aligned} \tag{2.7}$$

The rotations β_1 and β_2 can be determined from the hypothesis that $\varepsilon_{1n} = \varepsilon_{2n} = 0$

$$\begin{aligned} \beta_1 &= \frac{u}{R_1}-\frac{1}{A_1}\frac{\partial w}{\partial\alpha_1} \\ \beta_2 &= \frac{v}{R_2}-\frac{1}{A_2}\frac{\partial w}{\partial\alpha_2} \end{aligned} \tag{2.8}$$

The strains can also be expressed in the form

$$\begin{aligned} \varepsilon_1 &= \varepsilon_1^0+\zeta\kappa_1 \\ \varepsilon_2 &= \varepsilon_2^0+\zeta\kappa_2 \\ \varepsilon_{12} &= \varepsilon_{12}^0+\zeta\kappa_{12} \end{aligned} \tag{2.9}$$

where

$$\varepsilon_1^0 = \frac{1}{A_1}\frac{\partial u}{\partial \alpha_1} + \frac{v}{A_1 A_2}\frac{\partial A_1}{\partial \alpha_2} + \frac{w}{R_1} + \frac{1}{2A_1^2}\left(\frac{\partial w}{\partial \alpha_1}\right)^2$$
$$\varepsilon_2^0 = \frac{1}{A_2}\frac{\partial v}{\partial \alpha_2} + \frac{u}{A_1 A_2}\frac{\partial A_2}{\partial \alpha_1} + \frac{w}{R_2} + \frac{1}{2A_2^2}\left(\frac{\partial w}{\partial \alpha_2}\right)^2 \tag{2.10}$$
$$\varepsilon_{12}^0 = \frac{A_2}{A_1}\frac{\partial}{\partial \alpha_1}\left(\frac{v}{A_2}\right) + \frac{A_1}{A_2}\frac{\partial}{\partial \alpha_2}\left(\frac{u}{A_1}\right) + \frac{1}{A_1 A_2}\left(\frac{\partial w}{\partial \alpha_1}\right)\left(\frac{\partial w}{\partial \alpha_2}\right)$$

$$\kappa_1 = \frac{1}{A_1}\frac{\partial \beta_1}{\partial \alpha_1} + \frac{\beta_2}{A_1 A_2}\frac{\partial A_1}{\partial \alpha_2}$$
$$\kappa_2 = \frac{1}{A_2}\frac{\partial \beta_2}{\partial \alpha_2} + \frac{\beta_1}{A_1 A_2}\frac{\partial A_2}{\partial \alpha_1} \tag{2.11}$$
$$\kappa_{12} = \frac{A_2}{A_1}\frac{\partial}{\partial \alpha_1}\left(\frac{\beta_2}{A_2}\right) + \frac{A_1}{A_2}\frac{\partial}{\partial \alpha_2}\left(\frac{\beta_1}{A_1}\right)$$

The quantities $\varepsilon_1^0, \varepsilon_2^0$ and ε_{12}^0 represent the normal and shearing strains of the reference surface. The quantities κ_1 and κ_2 represent a linearly distributed bending component of a strain and, in analogy to the theory of beams, it is readily seen that κ_1 and κ_2 represent the changes in the curvature of the reference surface during deformation. The quantity κ_{12} represents the torsion of the reference surface during deformation

2. 1. 4 STRESS- STRAIN RELATIONS

For isotropic linear elastic medium, the stress-strain relations:

$$\varepsilon_1 = \frac{\sigma_1}{E} - \frac{\nu}{E}\sigma_2 - \frac{\nu}{E}\sigma_n \qquad \gamma_{12} = \frac{\tau_{12}}{G}$$
$$\varepsilon_2 = \frac{\sigma_2}{E} - \frac{\nu}{E}\sigma_1 - \frac{\nu}{E}\sigma_n \qquad \gamma_{1n} = \frac{\tau_{1n}}{G} \tag{2.12}$$
$$\varepsilon_n = \frac{\sigma_n}{E} - \frac{\nu}{E}\sigma_1 - \frac{\nu}{E}\sigma_2 \qquad \gamma_{2n} = \frac{\tau_{2n}}{G}$$

where $\sigma_1, \sigma_2, \sigma_n$ are the normal stresses along three mutually perpendicular directions, $\varepsilon_1, \varepsilon_2, \varepsilon_n$ are the corresponding normal strains, and $\tau_{12}, \tau_{1n}, \tau_{2n}$ and $\gamma_{12}, \gamma_{1n}, \gamma_{2n}$ are , respectively, the shearing stresses and strains. Finally E,G and ν are the elastic constants (Young's modulu, shear modulu and Poisson's ratio) along the three coordinate directions.

As a consequence of the third and fourth postulates of the Love theory, the system of stress-strain relations is reduced to the following two dimensional constitutive law thin elastic shells

$$\begin{aligned} \varepsilon_1 &= \frac{\sigma_1}{E} - \frac{\nu}{E}\sigma_2 \\ \varepsilon_2 &= \frac{\sigma_2}{E} - \frac{\nu}{E}\sigma_1 \\ \gamma_{12} &= \frac{\tau_{12}}{G} \end{aligned} \tag{2.13a}$$

or

$$\begin{aligned} \sigma_1 &= \frac{E}{1-\nu^2}(\varepsilon_1 + \nu\varepsilon_2) \\ \sigma_2 &= \frac{E}{1-\nu^2}(\varepsilon_2 + \nu\varepsilon_1) \\ \tau_{12} &= \frac{E}{2(1+\nu)}\gamma_{12} \end{aligned} \tag{2.13b}$$

Since the strains, and thereby the stresses, have been shown to be linearly distributed across the thickness of a thin elastic shell, it is convenient to integrate the stress distributions thorough the thickness of the shell and to replace the usual consideration of stresses by a consideration of statically equivalent stress resultants and stresses resultants and stress couples. By performing such integrations, the variations with respect to ζ are completely eliminated to give a two-dimensional theory.

$$\begin{Bmatrix} N_1 \\ N_{12} \\ Q_1 \end{Bmatrix} = \int_\zeta \begin{Bmatrix} \sigma_1 \\ \tau_{12} \\ \tau_{1n} \end{Bmatrix} (1+\zeta/R_2)d\zeta$$
$$\begin{Bmatrix} N_2 \\ N_{21} \\ Q_2 \end{Bmatrix} = \int_\zeta \begin{Bmatrix} \sigma_2 \\ \tau_{21} \\ \tau_{2n} \end{Bmatrix} (1+\zeta/R_1)d\zeta \tag{2.14a}$$

$$\begin{Bmatrix} M_1 \\ M_{12} \end{Bmatrix} = \int_\zeta \begin{Bmatrix} \sigma_1 \\ \tau_{12} \end{Bmatrix} (1+\zeta/R_2)\zeta d\zeta$$
$$\begin{Bmatrix} M_2 \\ M_{21} \end{Bmatrix} = \int_\zeta \begin{Bmatrix} \sigma_2 \\ \tau_{21} \end{Bmatrix} (1+\zeta/R_1)\zeta d\zeta \tag{2.14b}$$

Where N_1, N_2, N_{12} ve N_{21} normal force resultants, Q_1 and Q_2 shear forces resultants, M_1 , M_2 , M_{12} and M_{21} moment resultants.

The reference surface is taken as the middle surface (that is, $-h/2 \leq \zeta \leq h/2$), the forces and moment equations may be written such as

$$\begin{aligned}
N_1 &= C(\varepsilon_1^0 + \nu\varepsilon_2^0) \\
N_2 &= C(\varepsilon_2^0 + \nu\varepsilon_1^0) \\
N_{12} &= N_{21} = C(1-\nu)\gamma_{12}^0 / 2 \\
M_1 &= D(\kappa_1 + \nu\kappa_2) \\
M_2 &= D(\kappa_2 + \nu\kappa_1) \\
M_{12} &= M_{21} = D(1-\nu)\kappa_{12} / 2
\end{aligned} \tag{2.15}$$

where

$$\begin{aligned}
C &= Eh/(1-\nu^2) \\
D &= Eh^3/12(1-\nu^2)
\end{aligned} \tag{2.16}$$

are, respectively, the extensional rigidity and the bending rigidity of a thin shell composed of a single isotropic elastic layer.

2. 2. FUNDAMENTAL EQUATIONS OF LAMINATED COMPOSITE CONICAL SHELL

Consider a conical shell as shown in Figure 2. 2, where R indicates the radius of the cone at the large end, α denotes the semivertex angle of the cone and L is the cone length along its generator. h_s and h_r are the height of the stringers and rings, respectively. b_s and b_r are the width of the stringers and rings, respectively. Θ is the angular spacing of the stringers and S is the spacing of the rings. We use the s-θ coordinate system; s is measured along the cone's generator starting at the cone vertex and θ is the circumferential coordinate.

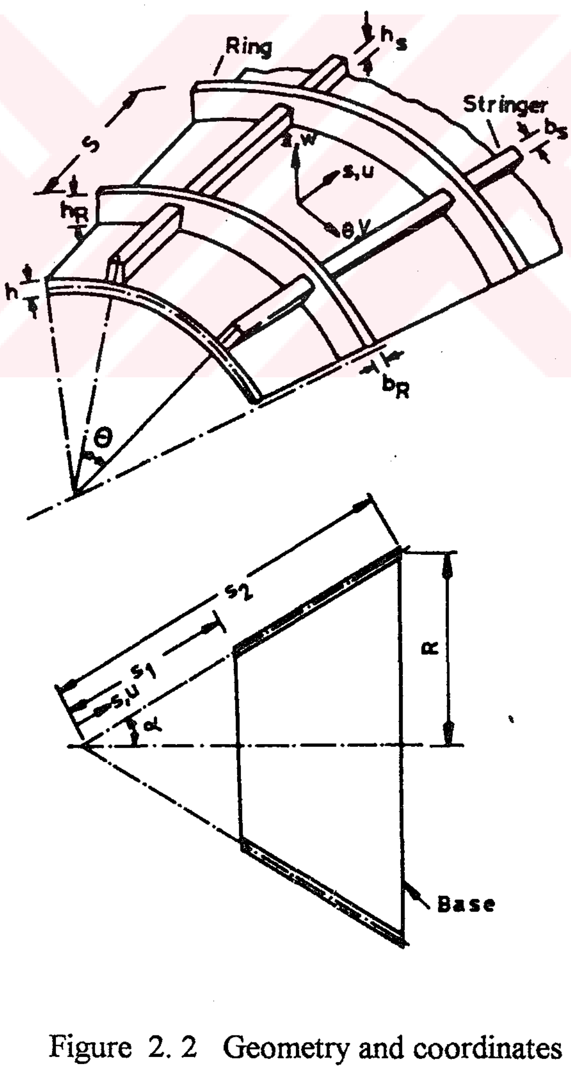


Figure 2. 2 Geometry and coordinates

2. 2. 1. DISPLACEMENT FIELD

Eqs. (2. 17) may be obtained by substituting α_1=s, α_2=θ, ζ=z, u=β_s, v=β_θ into Eqs. (2. 3) because of the using s-θ coordinate system. (Figure 2. 2)

$$
\begin{aligned}
U(s,\theta,z) &= u(s,\theta) + z\beta_S(s,\theta) \\
V(s,\theta,z) &= v(s,\theta) + z\beta_\theta(s,\theta) \\
W(s,\theta,z) &= w(s,\theta)
\end{aligned}
\qquad (2.\ 17)
$$

where u, v, and w are the components of displacement at the middle surface in the s, θ, and normal directions, respectively, and β_S and β_θ are the rotations of the normal to the middle surface during deformation about the s and θ axes, respectively.

2. 2. 2 KINEMATICAL RELATIONS

Kinematical relations of a thin truncated conical shell can be approximated by the use of preceding representation of displacement field and by taking into consideration Eqs. (2. 10) and (2. 11). The geometrical nonlinearities are taken into account in the formulation. Now, we define a functional J_k for the kinematical relations whose first variation is given by

$$
\begin{aligned}
\delta J_k = \int_T dt \int_A \int \Bigg\{ & \left[\varepsilon_s - \lambda\left(\frac{\partial u}{\partial \bar{s}} + \frac{\lambda}{2}\left(\frac{\partial w}{\partial \bar{s}}\right)^2 \right) \right] \delta N_s + \left[\varepsilon_\theta - \lambda\left(\frac{u}{\bar{s}} + \frac{1}{\bar{s}}\frac{\partial v}{\partial \bar{\theta}} + \cot\alpha \frac{w}{\bar{s}} + \frac{\lambda}{2\bar{s}^2}\left(\frac{\partial w}{\partial \bar{\theta}}\right)^2 \right) \right] \delta N_\theta \\
& + \left[\varepsilon_{s\theta} - \lambda\left(\frac{1}{\bar{s}}\frac{\partial u}{\partial \bar{\theta}} + \frac{\partial v}{\partial \bar{s}} - \frac{v}{\bar{s}} + \frac{\lambda}{\bar{s}}\frac{\partial w}{\partial \bar{s}}\frac{\partial w}{\partial \bar{\theta}} \right) \right] \delta N_{s\theta} + \left[\kappa_s - \lambda \frac{\partial \beta_s}{\partial \bar{s}} \right] \delta M_s + \left[\kappa_\theta - \lambda\left(\frac{1}{\bar{s}}\frac{\partial \beta_\theta}{\partial \bar{\theta}} + \frac{\beta_s}{\bar{s}} \right) \right] \delta M_\theta \\
& + \left[\kappa_{s\theta} - \lambda\left(\frac{\partial \beta_\theta}{\partial \bar{s}} - \frac{\beta_\theta}{\bar{s}} + \frac{1}{\bar{s}}\frac{\partial \beta_s}{\partial \bar{\theta}} \right) \right] \delta M_{s\theta} + \left[\varepsilon_{sz} - \lambda\left(\frac{\beta_s}{\lambda} + \frac{\partial w}{\partial \bar{s}} \right) \right] \delta Q_s \\
& + \left[\varepsilon_{\theta z} - \lambda\left(\frac{\beta_\theta}{\lambda} - \cot\alpha \frac{v}{\bar{s}} + \frac{1}{\bar{s}}\frac{\partial w}{\partial \bar{\theta}} \right) \right] \delta Q\theta \Bigg\} dA
\end{aligned}
\qquad (2.\ 18)
$$

where, ε_s, ε_θ, and $\varepsilon_{s\theta}$ are the membrane strains of the middle surface and κ_s, κ_θ, and $\kappa_{s\theta}$ the bending strains. These membrane and bending strains are equivalent to the strains which stated before by Eqs. (2.10) and (2.11). $\lambda = \sin\alpha/R$ is a characteristic parameter of the conical shell. The coordinates $\overline{s}$ and $\overline{\theta}$ are defined as $\overline{s} = s\lambda$ and $\overline{\theta} = \theta\sin\alpha$, respectively. The strain-displacement equations of the Donnell-Mushtari shell theory are obtained vanishing the first variation of functional.

2. 2. 3 CONSTITUTIVE EQUATIONS

In the plate and shell theory, as we stated by Eqs. (2. 14), it is convenient to introduce the force and moment resultants by integrating the stresses over the shell thickness. The constitutive equations of an anisotropic material relate the force and moment resultants to the membrane and bending strains. In the Eqs. (2. 15) constitutive equations are written for isotropic material. Here, the bending-stretching coupling and anisotropy are considered in the constitutive equations. The stiffening elements are relatively closely spaced, and therefore the stiffeners are "smeared out" along the conical shell. The material properties of the laminated conical shell are changed to take into account the "smeared out" stiffeners. Hence, the constitutive equations of the stiffened laminated conical shell with stretching-bending coupling are given in variational form as

$$
\begin{aligned}
\delta I_c = \int_T dt \int_A \Big\{ & \left[N_s - \left(A_{11}\varepsilon_s + A_{12}\varepsilon_\theta + B_{11}\kappa_s + B_{12}\kappa_\theta\right)\right] \delta\varepsilon_s \\
& + \left[N_\theta - \left(A_{12}\varepsilon_s + A_{22}\varepsilon_\theta + B_{12}\kappa_s + B_{22}\kappa_\theta\right)\right] \delta\varepsilon_\theta \\
& + \left[N_{s\theta} - \left(A_{33}\varepsilon_{s\theta} + B_{33}\kappa_{s\theta}\right)\right] \delta\varepsilon_{s\theta} \\
& + \left[M_s - \left(B_{11}\varepsilon_s + B_{12}\varepsilon_\theta + D_{11}\kappa_s + D_{12}\kappa_\theta\right)\right] \delta\kappa_s \\
& + \left[M_\theta - \left(B_{12}\varepsilon_s + B_{22}\varepsilon_\theta + D_{12}\kappa_s + D_{22}\kappa_\theta\right)\right] \delta\kappa_\theta \\
& + \left[M_{s\theta} - \left(B_{33}\varepsilon_{s\theta} + D_{33}\kappa_{s\theta}\right)\right] \delta\kappa_{s\theta} \\
& + \left[Q_s - A_{55}\varepsilon_{sz}\right]\delta\varepsilon_{sz} + \left[Q_\theta - A_{44}\varepsilon_{\theta z}\right]\delta\varepsilon_{\theta z} \Big\} dA = 0
\end{aligned}
\tag{2. 19}
$$

where

$$A_{ij}=\sum_{k=1}^{N_L}\overline{Q}_{ij}^{(k)}\left(h_k-h_{k-1}\right)+E_{ij} \qquad B_{ij}=\frac{1}{2}\sum_{k=1}^{N_L}\overline{Q}_{ij}^{(k)}\left(h_k^2-h_{k-1}^2\right)-F_{ij}$$

$$D_{ij}=\frac{1}{3}\sum_{k=1}^{N_L}\overline{Q}_{ij}^{(k)}\left(h_k^3-h_{k-1}^3\right)+G_{ij} \qquad i,j=1,2,3 \tag{2. 20a}$$

$$\left(A_{44},A_{55}\right)=\sum_{k=1}^{N_L}\int_{h_{k-1}}^{h_k}\left(\overline{Q}_{44}^{(k)},\overline{Q}_{55}^{(k)}\right)f(z)dz$$

$$\begin{aligned}
\overline{Q}^{(k)}{}_{11} &= U_1 + U_2\cos 2\theta + U_3\cos 4\theta \\
\overline{Q}^{(k)}{}_{22} &= U_1 - U_2\cos 2\theta + U_3\cos 4\theta \\
\overline{Q}^{(k)}{}_{12} &= U_4 - U_3\cos 4\theta = \overline{Q}_{21} \\
\overline{Q}^{(k)}{}_{33} &= U_5 - U_3\cos 4\theta \\
\overline{Q}^{(k)}{}_{13} &= -\tfrac{1}{2}U_2\sin 2\theta - U_3\sin 4\theta = \overline{Q}_{31} \\
\overline{Q}^{(k)}{}_{23} &= -\tfrac{1}{2}U_2\sin 2\theta + U_3\sin 4\theta = \overline{Q}_{32}
\end{aligned} \tag{2.20b}$$

$$\left\{\overline{Q}^{(k)}{}_{44},\overline{Q}^{(k)}{}_{55}\right\}=E^{(k)}{}_{\theta}\left\{\frac{1}{2\left(1+\nu^{(k)}{}_{\theta z}\right)},\frac{1}{2\left(1+\nu^{(k)}{}_{xz}\right)}\right\} \tag{2. 20c}$$

where

$$\begin{aligned}
U_1 &= \tfrac{1}{8}(3Q_{11}+3Q_{22}+2Q_{12}+4Q_{33}) \\
U_2 &= \tfrac{1}{2}(Q_{11}-Q_{22}) \\
U_3 &= \tfrac{1}{8}(Q_{11}+Q_{22}-2Q_{12}-4Q_{33}) \\
U_4 &= \tfrac{1}{8}(Q_{11}+Q_{22}+6Q_{12}-4Q_{33}) \\
U_5 &= \tfrac{1}{8}(Q_{11}+Q_{22}-2Q_{12}+4Q_{33})
\end{aligned} \tag{2. 20d}$$

$$\{E_{11},E_{22},E_{12},E_{33}\} = \lambda\left\{\frac{E_S A_S}{\overline{\Theta}},\frac{E_R A_R}{\overline{S}},0,0\right\}$$

$$\{F_{11},F_{22},F_{12},F_{33}\} = 2\lambda\left\{\frac{E_S A_S e_S}{\overline{\Theta}},\frac{E_R A_R e_R}{\overline{S}},0,0\right\}$$

$$\{G_{11},G_{22},G_{12},G_{33}\} = \lambda\left\{\frac{E_S A_S\left(i_{\eta S}^{\,2}+e_S^{\,2}\right)}{\overline{\Theta}},\frac{E_R A_R\left(i_{\xi R}^{\,2}+e_R^{\,2}\right)}{\overline{S}},0,\left(\frac{G_S J_S}{\overline{\Theta}}+\frac{G_R J_R}{\overline{S}}\right)\right\} \tag{2. 21}$$

and

$$f(z) = \frac{5}{4}\left[1-4\left(\frac{z}{h}\right)^2\right] \tag{2. 22}$$

The force and moment statements above are written by taking into account s-θ coordinate system. Here, h is the total wall thickness of the conical shell. E_S, and $G_S J_S$ are the modulus of elasticity, and torsional stiffness of a stringer, respectively. A_S is the cross-sectional area of stringer and i_S is the radius of gyration of the stringer cross-sectional area about a centroidal axis parallel to the θ axis, respectively. e_S is the distance to the centroid of the stringer cross-section from the shell middle surface (eccentricity). The subscript R indicates the properties of a ring. In Eqs. (2. 21), $\overline{\Theta} = \Theta \sin\alpha$ and $\overline{S} = S\lambda$.

2. 2. 3. 1 MOISTURE EFFECT IN CONSTITUTIVE EQUATIONS

The matrix materials, such as epoxy resins, that are commonly used for present-day composites absorb moisture from the atmosphere by what is essentially a diffusion process. Under ambient temperature conditions, the rate of diffusion is quite slow: it takes times of the order of months for a laminate kept in a humid atmosphere to achieve an equilibrium moisture distribution. The amount of moisture in a laminate is generally expressed in terms of the percentage increase in the laminate eight; for graphite/epoxy laminates exposed for long times in humid atmospheres, weight increases of the order of 1% are encountered[32].

The presence of moisture in a laminate can significantly affect certain of its structural properties, considerable attention has been devoted to establishing theoretical procedures for predicting the moisture content. Consider an initially dry laminate immersed in a constant humid atmosphere. It is assumed that the moisture enters the laminate only through its flat faces. If z denotes the coordinate in the thickness direction, here measured from one face of the laminate, and t the time; it can be seen that, for sufficiently long times, the moisture concentration approaches a uniform value across the depth of the laminate.(Figure 2. 3)

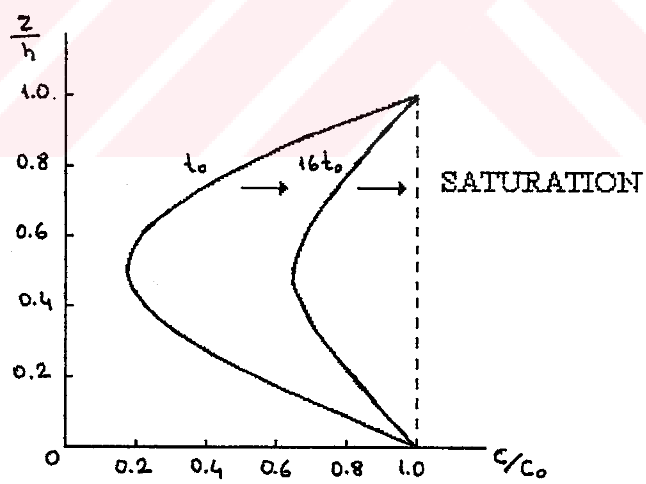


Figure 2. 3 Moisture / time profiles in a laminate

There are two principal effects to changes in the hygrothermal environment on mechanical behavior of polymer composites :

1. Matrix-dominated properties such as stiffness and strength under transverse, off-axis or shear loading are altered. Increased temperature causes a gradual softening of the polymer matrix material up to a point. If the temperature is increased beyond the so-called “glass transition” region (indicating a transition from glassy behavior to rubbery behavior), however, the polymer becomes too soft for use as a structural material.(Figure 2. 4) Plasticization of the polymer by absorbed moisture causes a reduction in the glass transition temperature, T_g and a corresponding degradation of composite properties.

2. Hygrothermal expansions or contractions change the stress and strain distributions in the composite. Increased temperature and/or moisture content causes swelling of the polymer matrix, whereas reduced temperature and/or moisture content causes contractions. Since the fibers are not affected as much by hygrothermal conditions, the selling or contractions of the matrix is resisted by the fibers and residual stresses develop in the composite. A similar effect at the laminate level is due to differential expansions or contractions of constituent laminae.[33]

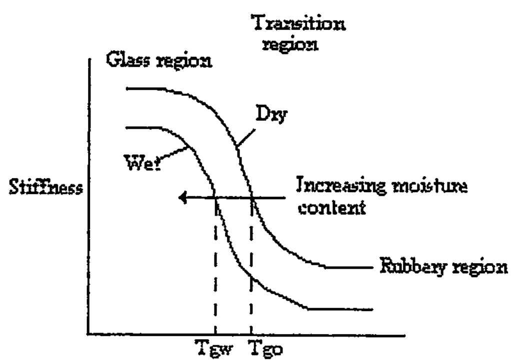


Figure 2. 4 Variation of the stiffness with the temperature for a typical polymer showing the glass transition temperature T_g and the effect of absorbed moisture on T_g. Note: T_{go}=“dry” T_g and T_{gw}=“wet” T_g.

2. 2. 3. 2 EFFECT OF MOISTURE AND TEMPERATURE ON STRUCTURAL PERFORMANCE

It transpires that the main effect of moisture on structural performance comes not from the moisture alone, but rather when it is taken in combination with elevated temperatures such as occur in high-speed flight. Composite components on a supersonic fighter aircraft might be expected to experience temperatures in the approximate range -50 to 125 ^{0}C depending on speed and altitude; it is the high end of this range that is seen as being of most importance.

When a fiber / epoxy composite is in a "wet" condition having a moisture content around, say, 1.2%, then the glass transition temperature of the epoxy matrix may be reduced by around 40-50 ^{0}C . Under these circumstances, the maximum temperature seen in service may be approaching the glass transition temperature of the composite matrix. This is where the main potential for reduction in structural performance arises. However, by no means are all of the structural properties of a composite reduced significantly from this cause. As would be expected from the above argument, it is only those properties strongly dependent on the matrix performance that are affected.

The tensile strength and extensional stiffness of fiber-dominated laminates (i.e., laminates with a significant proportion of fibers in the principal stress directions) are little affected by temperature. The situation is different for matrix-dominated laminates, where substantial reductions in tensile strength and stiffness can occur.[32]

As it mentioned above; elastic constants or material density may have a spatial change due to environmental causes such as temperature, moisture effect or manufacturing processes. Therefore, the material of the conical shell is inhomogeneous in nature and it should be considered in the constitutive relations. For purposes of demonstration, a simple linear variation in the Young modulus through laminates of the conical shell is considered[17], described by the equation

$$E_s(z) = E_{s_0}\left[1+\alpha_s\left(1+2\frac{z}{h}\right)\right] \qquad E_\theta(z) = E_{\theta_0}\left[1+\alpha_\theta\left(1+2\frac{z}{h}\right)\right] \tag{2. 23}$$

where α_s and α_θ are the parameters. E_{s_0} and E_{θ_0} are the Young modules of the inner surface of truncated conical shell.

2. 2. 4 EQUILIBRIUM EQUATIONS

Now, the equilibrium equations, boundary and initial conditions for the vibration of stiffened composite conical shell with initial stresses are derived from the virtual work principle. The rotary inertia of conical shell and stiffeners is taken into account. Thus the virtual work principle is applied to the stiffened composite conical shell under the influence of the initial stresses and the following equation is obtained.

$$\begin{aligned}
\delta J_e = \int_T dt \int_A \int \Big[& (N_s + N_s^i)\delta\varepsilon_s + (N_\theta + N_\theta^i)\,\delta\varepsilon_\theta + (N_{s\theta} + N_{s\theta}^i)\,\delta\varepsilon_{s\theta} + (M_s + M_s^i)\,\delta\kappa_s \\
& + (M_\theta + M_\theta^i)\,\delta\kappa_\theta + (M_{s\theta} + M_{s\theta}^i)\,\delta\kappa_{s\theta} + (Q_s + Q_s^i)\,\delta\varepsilon_{sz} + (Q_\theta + Q_\theta^i)\,\delta\varepsilon_{\theta z}\Big]\, dA \\
- \int_A \int \Big[& (q_s + q_s^i)\,\delta u + (q_\theta + q_\theta^i)\,\delta v + (q_z + q_z^i)\,\delta w + (m_s + m_s^i)\,\delta\beta_s + (m_\theta + m_\theta^i)\delta\beta_\theta\Big]\, dA \\
- \int_A \int \Big[& m_1(\dot{u}\delta\dot{u} + \dot{v}\delta\dot{v} + \dot{w}\,\delta\dot{w}) + \frac{m_2}{2}(\dot{u}\delta\dot{\beta}_s + \dot{\beta}_s\delta\dot{u} + \dot{v}\delta\dot{\beta}_\theta + \dot{\beta}_\theta\delta\dot{v}) \\
& + m_{12}\dot{\beta}_s\delta\dot{\beta}_s + m_{21}\dot{\beta}_\theta\delta\dot{\beta}_\theta\Big]\, dA = 0
\end{aligned} \tag{2. 24}$$

where the superscript i indicates the initial stresses and the dot above the symbols denotes the derivation with respect to time. $q_s^i, q_\theta^i, q_z^i, m_s^i$, and m_θ^i are the static loads which cause the initial stresses, and q_s, q_θ, q_z, m_s, and m_θ are the dynamic loads. In Eq. (2. 24) the parameters related with the mass of the conical shell are

$$m_1 = \rho_c h + \frac{\lambda \rho_S A_S}{\overline{\Theta}} + \frac{\lambda \rho_R A_R}{\overline{S}} \qquad m_2 = \frac{2\lambda \rho_S A_S e_S}{\overline{\Theta}} + \frac{2\lambda \rho_R A_R e_R}{\overline{S}}$$

$$m_{12} = \frac{\rho_c h^3}{12} + \frac{\lambda \rho_S A_S}{\overline{\Theta}}\left(e_S^{\,2} + i_S^{\,2}\right) + \frac{\lambda \rho_R A_R}{\overline{S}}\left(e_R^{\,2} + i_{pR}^{\,2}\right) \tag{2. 25}$$

$$m_{21} = \frac{\rho_c h^3}{12} + \frac{\lambda \rho_S A_S}{\overline{\Theta}}\left(e_S^{\,2} + i_{pS}^{\,2}\right) + \frac{\lambda \rho_R A_R}{\overline{S}}\left(e_R^{\,2} + i_R^{\,2}\right)$$

where ρ_c, ρ_s, and ρ_r are the material densities of the shell, stringer and ring, respectively. i_{pS} and i_{pR} are the polar radius of gyration of the stringer and the ring about the centroidal axes, respectively.

Integrating Eq. (2. 24) by parts, one arrives the following variational equation

$$\begin{aligned}
\delta J_e = \int_T dt \int_A \int \Bigg\{ & \left[-\frac{\partial}{\partial \bar{s}}\left(N_s + N_s^i\right) - \frac{1}{\bar{s}}\left[\left(N_s + N_s^i\right) - \left(N_\theta + N_\theta^i\right)\right] - \frac{1}{\bar{s}}\frac{\partial}{\partial \bar{\theta}}\left(N_{s\theta} + N_{s\theta}^i\right) + \frac{m_1}{\lambda}\ddot{u} + \frac{m_2}{2\lambda}\ddot{\beta}_s - \frac{q_s + q_s^i}{\lambda} \right]\delta u \\
& + \left[-\frac{\partial}{\partial \bar{s}}\left(N_{s\theta} + N_{s\theta}^i\right) - \frac{2}{\bar{s}}\left(N_{s\theta} + N_{s\theta_i}\right) - \frac{1}{\bar{s}}\frac{\partial}{\partial \bar{\theta}}\left(N_\theta + N_\theta^i\right) - \frac{\cot\alpha}{\bar{s}}\left(Q_\theta + Q_\theta^i\right) + \frac{m_1}{\lambda}\ddot{v} + \frac{m_2}{2\lambda}\ddot{\beta}_\theta - \frac{q_\theta + q_\theta^i}{\lambda} \right]\delta v \\
& + \Bigg[-\frac{\partial}{\partial \bar{s}}\left(Q_s + Q_s^i\right) - \frac{1}{\bar{s}}\frac{\partial}{\partial \bar{\theta}}\left(Q_\theta + Q_\theta^i\right) - \frac{1}{\bar{s}}\left(Q_s + Q_s^i\right) + \frac{\cot\alpha}{\bar{s}}\left(N_\theta + N_\theta^i\right) \\
& \qquad - \frac{\lambda}{\bar{s}}\frac{\partial}{\partial \bar{s}}\left[\left(N_s + N_s^i\right)\bar{s}\frac{\partial w}{\partial \bar{s}}\right] - \frac{\lambda}{\bar{s}^2}\frac{\partial}{\partial \bar{\theta}}\left[\left(N_\theta + N_\theta^i\right)\frac{\partial w}{\partial \bar{\theta}}\right] \\
& \qquad - \frac{\lambda}{\bar{s}}\frac{\partial}{\partial \bar{s}}\left[\left(N_{s\theta} + N_{s\theta}^i\right)\frac{\partial w}{\partial \bar{\theta}}\right] - \frac{\partial}{\partial \bar{\theta}}\left[\left(N_{s\theta} + N_{s\theta}^i\right)\frac{\partial w}{\partial \bar{s}}\right] + \frac{m_1}{\lambda}\ddot{w} - \frac{q_z + q_z^i}{\lambda} \Bigg]\delta w \\
& + \Bigg[-\frac{\partial}{\partial \bar{s}}\left(M_s + M_s^i\right) - \frac{1}{\bar{s}}\left[\left(M_s + M_s^i\right) - \left(M_\theta + M_\theta^i\right)\right] - \frac{1}{\bar{s}}\frac{\partial}{\partial \bar{\theta}}\left(M_{s\theta} + M_{s\theta}^i\right) \\
& \qquad + \frac{1}{\lambda}\left(Q_s + Q_s^i\right) + \frac{m_2}{2\lambda}\ddot{u} + \frac{m_{12}}{\lambda}\ddot{\beta}_s - \frac{m_s + m_s^i}{\lambda} \Bigg]\delta\beta_s \\
& + \Bigg[-\frac{\partial}{\partial \bar{s}}\left(M_{s\theta} + M_{s\theta}^i\right) - \frac{2}{\bar{s}}\left(M_{s\theta} + M_{s\theta}^i\right) - \frac{1}{\bar{s}}\frac{\partial}{\partial \bar{\theta}}\left(M_\theta + M_\theta^i\right) \\
& \qquad + \frac{1}{\lambda}\left(Q_\theta + Q_\theta^i\right) + \frac{m_2}{2\lambda}\ddot{v} + \frac{m_{21}}{\lambda}\ddot{\beta}_\theta - \frac{m_\theta + m_\theta^i}{\lambda} \Bigg]\delta\beta_\theta \Bigg\} dA = 0
\end{aligned}$$

(2. 26)

The variational statement (2. 26) may be divided into the two parts as follow

$$\delta J_e = \delta J_e^i + \delta J_e^d \tag{2. 27}$$

where J_e^i defines a functional of the initial stresses as a function of static loads, and J_e^d defines a functional related with the vibration behavior as a function of dynamic loads. We may write the parts of Eq. (2. 27) in this form

$$\begin{aligned}
\delta J_e^i &= \int_T dt \iint_A \left(\tau_s^i \delta u + \tau_\theta^i \delta v + \tau_z^i \delta w + \tau_{sz}^i \delta\beta_s + \tau_{\theta z}^i \delta\beta_\theta\right) dA \\
\delta J_e^d &= \int_T dt \iint_A \left(\tau_s \delta u + \tau_\theta \delta v + \tau_z \delta w + \tau_{sz} \delta\beta_s + \tau_\theta \delta\beta_\theta\right) dA
\end{aligned} \tag{2. 28}$$

where

$$\begin{aligned}
\tau_s^i &= -\frac{\partial N_s^i}{\partial \bar{s}} - \frac{1}{\bar{s}}\left(N_s^i - N_\theta^i\right) - \frac{1}{\bar{s}}\frac{\partial N_{s\theta}^i}{\partial \bar{\theta}} - \frac{q_s^i}{\lambda} = 0 \\
\tau_\theta^i &= -\frac{\partial N_{s\theta}^i}{\partial \bar{s}} - \frac{2}{\bar{s}} N_{s\theta}^i - \frac{1}{\bar{s}}\frac{\partial N_\theta^i}{\partial \bar{\theta}} - \frac{\cot\alpha}{\bar{s}} Q_\theta^i - \frac{q_\theta^i}{\lambda} = 0 \\
\tau_z^i &= -\frac{\partial Q_s^i}{\partial \bar{s}} - \frac{1}{\bar{s}}\frac{\partial Q_\theta^i}{\partial \bar{\theta}} - \frac{1}{\bar{s}} Q_s^i + \frac{\cot\alpha}{\bar{s}} N_\theta^i - \frac{q_z^i}{\lambda} = 0 \\
\tau_{sz}^i &= -\frac{\partial M_s^i}{\partial \bar{s}} - \frac{1}{\bar{s}}\left(M_s^i - M_\theta^i\right) - \frac{1}{\bar{s}}\frac{\partial M_{s\theta}^i}{\partial \theta} + \frac{1}{\lambda} Q_s^i - \frac{m_s^i}{\lambda} = 0 \\
\tau_{\theta z}^i &= -\frac{\partial M_{s\theta}^i}{\partial \bar{s}} - \frac{2}{\bar{s}} M_{s\theta}^i - \frac{1}{\bar{s}}\frac{\partial M_\theta^i}{\partial \theta} + \frac{1}{\lambda} Q_\theta^i - \frac{m_\theta^i}{\lambda} = 0
\end{aligned} \tag{2. 29}$$

This set is usually solved by stress functions for the unknown initial stresses. Once these are known, following vibration equations are solved:

$$\tau_s = -\frac{\partial N_s}{\partial \bar{s}} - \frac{1}{\bar{s}}\left(N_s - N_\theta\right) - \frac{1}{\bar{s}}\frac{\partial N_{s\theta}}{\partial \bar{\theta}} + \frac{m_1}{\lambda}\ddot{u} + \frac{m_2}{2\lambda}\ddot{\beta}_s - \frac{q_s}{\lambda} = 0$$
$$\tau_\theta = -\frac{\partial N_{s\theta}}{\partial \bar{s}} - \frac{2}{\bar{s}}N_{s\theta} - \frac{1}{\bar{s}}\frac{\partial N_\theta}{\partial \bar{\theta}} - \frac{\cot\alpha}{\bar{s}}Q_\theta + \frac{m_1}{\lambda}\ddot{v} + \frac{m_2}{2\lambda}\ddot{\beta}_\theta - \frac{q_\theta}{\lambda} = 0$$
$$\tau_z = -\frac{\partial Q_s}{\partial \bar{s}} - \frac{1}{\bar{s}}\frac{\partial Q_\theta}{\partial \bar{\theta}} - \frac{1}{\bar{s}}Q_s + \frac{\cot\alpha}{\bar{s}}N_\theta - \frac{\lambda}{\bar{s}}\frac{\partial}{\partial \bar{s}}\left(N_s^i \bar{s}\frac{\partial w}{\partial \bar{s}}\right) - \frac{\lambda}{\bar{s}^2}\frac{\partial}{\partial \bar{\theta}}\left(N_\theta^i \frac{\partial w}{\partial \bar{\theta}}\right)$$
$$- \frac{\lambda}{\bar{s}}\frac{\partial}{\partial \bar{s}}\left(N_{s\theta}^i \frac{\partial w}{\partial \bar{\theta}}\right) - \frac{\partial}{\partial \bar{\theta}}\left(N_{s\theta}^i \frac{\partial w}{\partial \bar{s}}\right) + \frac{m_1}{\lambda}\ddot{w} - \frac{q_z}{\lambda} = 0$$
$$\tau_{sz} = -\frac{\partial M_s}{\partial \bar{s}} - \frac{1}{\bar{s}}\left(M_s - M_\theta\right) - \frac{1}{\bar{s}}\frac{\partial M_{s\theta}}{\partial \theta} + \frac{1}{\lambda}Q_s + \frac{m_2}{2\lambda}\ddot{u} + \frac{m_{12}}{\lambda}\ddot{\beta}_s - \frac{m_s}{\lambda} = 0$$
$$\tau_{\theta z} = -\frac{\partial M_{s\theta}}{\partial \bar{s}} - \frac{2}{\bar{s}}M_{s\theta} - \frac{1}{\bar{s}}\frac{\partial M_\theta}{\partial \theta} + \frac{1}{\lambda}Q_\theta + \frac{m_2}{2\lambda}\ddot{v} + \frac{m_{21}}{\lambda}\ddot{\beta}_\theta - \frac{m_\theta}{\lambda} = 0$$

(2. 30)

Note that we have neglected N_s, N_θ, and $N_{s\theta}$ in the third to sixth terms of τ_z since they are small when compared to N_s^i, N_θ^i and $N_{s\theta}^i$. This is a good assumption for most engineering problems of this type.

Finally, we may write the equilibrium equations given by Eqs. (2. 30) in terms of the displacements and rotations. On substituting Eqs. (2. 18) into Eqs. (2. 19), and further into Eqs. (2. 30), we may express the equations of dynamic equilibrium in terms of the five displacements, namely

$$L_{11}u + L_{12}v + L_{13}w + L_{14}\beta_s + L_{15}\beta_\theta + N_1(w) + \frac{m_1}{\lambda^2}\ddot{u} + \frac{m_2}{2\lambda^2}\ddot{\beta}_s - \frac{q_s}{\lambda^2} = 0$$
$$L_{21}u + L_{22}v + L_{23}w + L_{24}\beta_s + L_{25}\beta_\theta + N_2(w) + \frac{m_1}{\lambda^2}\ddot{v} + \frac{m_2}{2\lambda^2}\ddot{\beta}_\theta - \frac{q_\theta}{\lambda^2} = 0$$
$$L_{31}u + L_{32}v + L_{33}w + L_{34}\beta_s + L_{35}\beta_\theta + N_3(w) + \frac{m_1}{\lambda^2}\ddot{w} - \frac{q_z}{\lambda^2} = 0$$
$$L_{41}u + L_{42}v + L_{43}w + L_{44}\beta_s + L_{45}\beta_\theta + N_4(w) + \frac{m_2}{2\lambda^2}\ddot{u} + \frac{m_{12}}{\lambda^2}\ddot{\beta}_s - \frac{m_s}{\lambda^2} = 0$$
$$L_{51}u + L_{52}v + L_{53}w + L_{54}\beta_s + L_{55}\beta_\theta + N_5(w) + \frac{m_2}{2\lambda^2}\ddot{v} + \frac{m_{21}}{\lambda^2}\ddot{\beta}_\theta - \frac{m_\theta}{\lambda^2} = 0$$

(2. 31)

where

$$L_{11} = -A_{11}\frac{\partial^2}{\partial \bar{s}^2} - \frac{A_{11}}{\bar{s}}\frac{\partial}{\partial \bar{s}} + \frac{A_{22}}{\bar{s}^2} - \frac{A_{33}}{\bar{s}^2}\frac{\partial^2}{\partial \bar{\theta}^2}$$

$$L_{12} = -\frac{\left(A_{12}+A_{33}\right)}{\bar{s}}\frac{\partial^2}{\partial \bar{s}\partial \bar{\theta}} + \frac{\left(A_{22}+A_{33}\right)}{\bar{s}^2}\frac{\partial}{\partial \bar{\theta}}$$

$$L_{13} = \frac{A_{22}\cot\alpha}{\bar{s}^2} - \frac{A_{12}\cot\alpha}{\bar{s}}\frac{\partial}{\partial \bar{s}} \tag{2. 32a}$$

$$L_{14} = -B_{11}\frac{\partial^2}{\partial \bar{s}^2} - \frac{B_{11}}{\bar{s}}\frac{\partial}{\partial \bar{s}} + \frac{B_{22}}{\bar{s}^2} - \frac{B_{33}}{\bar{s}^2}\frac{\partial^2}{\partial \bar{\theta}^2}$$

$$L_{15} = \frac{\left(B_{22}+B_{33}\right)}{\bar{s}^2}\frac{\partial}{\partial \bar{\theta}} - \frac{\left(B_{12}+B_{33}\right)}{\bar{s}}\frac{\partial^2}{\partial \bar{s}\partial \bar{\theta}}$$

$$L_{21} = -\frac{\left(A_{12}+A_{33}\right)}{\bar{s}}\frac{\partial^2}{\partial \bar{s}\partial \bar{\theta}} - \frac{\left(A_{22}+A_{33}\right)}{\bar{s}^2}\frac{\partial}{\partial \bar{\theta}}$$

$$L_{22} = -A_{33}\left[\frac{\partial^2}{\partial \bar{s}^2} + \frac{1}{\bar{s}}\frac{\partial}{\partial \bar{s}} - \frac{1}{\bar{s}^2}\right] - \frac{A_{22}}{\bar{s}^2}\frac{\partial^2}{\partial \bar{\theta}^2} + \frac{A_{44}\cot^2\alpha}{\bar{s}^2}$$

$$L_{23} = -\frac{\left(A_{22}+A_{44}\right)\cot\alpha}{\bar{s}^2}\frac{\partial}{\partial \bar{\theta}} \tag{2. 32b}$$

$$L_{24} = -\frac{\left(B_{12}+B_{33}\right)}{\bar{s}}\frac{\partial^2}{\partial \bar{s}\partial \bar{\theta}} - \frac{\left(B_{22}+B_{33}\right)}{\bar{s}^2}\frac{\partial}{\partial \bar{\theta}}$$

$$L_{25} = -B_{33}\left[\frac{\partial^2}{\partial \bar{s}^2} + \frac{1}{\bar{s}}\frac{\partial}{\partial \bar{s}} - \frac{1}{\bar{s}^2}\right] - \frac{B_{22}}{\bar{s}^2}\frac{\partial^2}{\partial \bar{\theta}^2} - \frac{A_{44}\cot\alpha}{\bar{s}\lambda}$$

$$L_{31} = \frac{A_{12}\cot\alpha}{\bar{s}}\frac{\partial}{\partial \bar{s}} + \frac{A_{22}\cot\alpha}{\bar{s}^2}$$

$$L_{32} = -L_{23}$$

$$\begin{aligned} L_{33} = {} & -A_{55}\left(\frac{\partial^2}{\partial \bar{s}^2} + \frac{1}{\bar{s}}\frac{\partial}{\partial \bar{s}}\right) - \frac{A_{44}}{\bar{s}^2}\frac{\partial^2}{\partial \bar{\theta}^2} - \frac{A_{22}\cot^2\alpha}{\bar{s}^2} - \lambda\left(\frac{\partial N_s^i}{\partial \bar{s}} + \frac{N_s^i}{\bar{s}} + \frac{1}{\bar{s}}\frac{\partial N_{s\theta}^i}{\partial \bar{\theta}}\right)\frac{\partial}{\partial \bar{s}} \\ & - \lambda N_s^i\frac{\partial^2}{\partial \bar{s}^2} - \frac{2\lambda N_{s\theta}^i}{\bar{s}}\frac{\partial^2}{\partial \bar{s}\partial \bar{\theta}} - \lambda\left(\frac{1}{\bar{s}^2}\frac{\partial N_\theta^i}{\partial \bar{\theta}} + \frac{1}{\bar{s}}\frac{\partial N_{s\theta}^i}{\partial \bar{s}}\right)\frac{\partial}{\partial \bar{\theta}} - \frac{\lambda N_\theta^i}{\bar{s}^2}\frac{\partial^2}{\partial \bar{\theta}^2} \end{aligned}$$

$$L_{34} = \left(\frac{B_{12}\cot\alpha}{\bar{s}} - \frac{A_{55}}{\lambda}\right)\frac{\partial}{\partial \bar{s}} - \frac{A_{55}}{\lambda\bar{s}} + \frac{B_{22}\cot\alpha}{\bar{s}^2}$$

$$L_{35} = \left(\frac{B_{22}\cot\alpha}{\bar{s}^2} - \frac{A_{44}}{\lambda\bar{s}}\right)\frac{\partial}{\partial \bar{\theta}}$$

(2. 32c)

$$L_{41} = L_{14} \qquad L_{42} = L_{15}$$

$$L_{43} = \left(\frac{A_{55}}{\lambda} - \frac{B_{12}\cot\alpha}{\bar{s}}\right)\frac{\partial}{\partial\bar{s}} + \frac{B_{22}\cot\alpha}{\bar{s}^2}$$

$$L_{44} = -D_{11}\left(\frac{\partial^2}{\partial\bar{s}^2} + \frac{1}{\bar{s}}\frac{\partial}{\partial\bar{s}}\right) + \frac{D_{22}}{\bar{s}^2} - \frac{D_{33}}{\bar{s}^2}\frac{\partial^2}{\partial\bar{\theta}^2} + \frac{A_{55}}{\lambda^2} \tag{2. 32d}$$

$$L_{45} = \frac{(D_{22} + D_{33})}{\bar{s}^2}\frac{\partial}{\partial\bar{\theta}} - \frac{(D_{12} + D_{33})}{\bar{s}}\frac{\partial^2}{\partial\bar{s}\partial\bar{\theta}}$$

$$L_{51} = L_{24} \qquad L_{52} = L_{25} \qquad L_{53} = -L_{35}$$

$$L_{54} = -\frac{(D_{12} + D_{33})}{\bar{s}}\frac{\partial^2}{\partial\bar{s}\partial\bar{\theta}} - \frac{(D_{22} + D_{33})}{\bar{s}^2}\frac{\partial}{\partial\bar{\theta}} \tag{2. 32e}$$

$$L_{55} = -D_{33}\left(\frac{\partial^2}{\partial\bar{s}^2} + \frac{1}{\bar{s}}\frac{\partial}{\partial\bar{s}} - \frac{1}{\bar{s}^2}\right) - \frac{D_{22}}{\bar{s}^2}\frac{\partial^2}{\partial\bar{\theta}^2} + \frac{A_{44}}{\lambda^2}$$

and the nonlinear terms in the equations of motion given by Eqs. (2. 31) are

$$N_1(w) = -A_{11}\lambda\frac{\partial w}{\partial\bar{s}}\frac{\partial^2 w}{\partial\bar{s}^2} + (A_{12} + A_{22})\frac{\lambda}{2\bar{s}^3}\left(\frac{\partial w}{\partial\bar{\theta}}\right)^2 - (A_{12} + A_{33})\frac{\lambda}{\bar{s}^2}\frac{\partial w}{\partial\bar{\theta}}\frac{\partial^2 w}{\partial\bar{s}\partial\bar{\theta}}$$
$$-(A_{11} - A_{12})\frac{\lambda}{2\bar{s}}\left(\frac{\partial w}{\partial\bar{s}}\right)^2 - A_{33}\frac{\lambda}{\bar{s}^2}\frac{\partial w}{\partial\bar{s}}\frac{\partial^2 w}{\partial\bar{\theta}^2}$$

$$N_2(w) = -(A_{12} + A_{33})\frac{\lambda}{\bar{s}}\frac{\partial w}{\partial\bar{s}}\frac{\partial^2 w}{\partial\bar{s}\partial\bar{\theta}} - A_{22}\frac{\lambda}{\bar{s}^3}\frac{\partial w}{\partial\bar{\theta}}\frac{\partial^2 w}{\partial\bar{\theta}^2} - A_{33}\left(\frac{\lambda}{\bar{s}}\frac{\partial^2 w}{\partial\bar{s}^2}\frac{\partial w}{\partial\bar{\theta}} + \frac{\lambda}{\bar{s}^2}\frac{\partial w}{\partial\bar{s}}\frac{\partial w}{\partial\bar{\theta}}\right)$$

$$N_3(w) = A_{12}\frac{\lambda\cot\alpha}{2\bar{s}}\left(\frac{\partial w}{\partial\bar{s}}\right)^2 + A_{22}\frac{\lambda\cot\alpha}{2\bar{s}^3}\left(\frac{\partial w}{\partial\bar{\theta}}\right)^2$$

$$N_4(w) = -B_{11}\lambda\frac{\partial w}{\partial\bar{s}}\frac{\partial^2 w}{\partial\bar{s}^2} + (B_{12} + B_{22})\frac{\lambda}{2\bar{s}^3}\left(\frac{\partial w}{\partial\bar{\theta}}\right)^2 - (B_{12} + B_{33})\frac{\lambda}{\bar{s}^2}\frac{\partial w}{\partial\bar{\theta}}\frac{\partial^2 w}{\partial\bar{s}\partial\bar{\theta}}$$
$$-(B_{11} - B_{12})\frac{\lambda}{2\bar{s}}\left(\frac{\partial w}{\partial\bar{s}}\right)^2 - B_{33}\frac{\lambda}{\bar{s}^2}\frac{\partial w}{\partial\bar{s}}\frac{\partial^2 w}{\partial\bar{\theta}^2}$$

$$N_5(w) = -(B_{12} + B_{33})\frac{\lambda}{\bar{s}}\frac{\partial w}{\partial\bar{s}}\frac{\partial^2 w}{\partial\bar{s}\partial\bar{\theta}} - B_{22}\frac{\lambda}{\bar{s}^3}\frac{\partial w}{\partial\bar{\theta}}\frac{\partial^2 w}{\partial\bar{\theta}^2} - B_{33}\left(\frac{\lambda}{\bar{s}}\frac{\partial^2 w}{\partial\bar{s}^2}\frac{\partial w}{\partial\bar{\theta}} + \frac{\lambda}{\bar{s}^2}\frac{\partial w}{\partial\bar{s}}\frac{\partial w}{\partial\bar{\theta}}\right)$$

(2. 33)

The nonlinear equations can be extraordinarily difficult to solve. The equations raise mathematical problems in their own right. Fortunately, the nonlinear terms in Eqs. (2. 31) may be neglected in the most engineering problems.

2. 2. 5 BOUNDARY AND INITIAL CONDITIONS

The related boundary conditions at both ends of the conical shell may be expressed in variational form as follow

$$\delta J_b = \int_T dt \left\{ \oint_{C_f} \left[\left(N_s - N_s^*\right)\delta u + \left(N_{s\theta} - N_{s\theta}^*\right)\delta v + \left(Q_s - Q_s^*\right)\delta w + \left(M_s - M_s^*\right)\delta\beta_s + \left(M_{s\theta} - M_{s\theta}^*\right)\delta\beta_\theta \right] \frac{\bar{s}d\bar{\theta}}{\lambda} \right.$$
$$\left. + \oint_{C_d} \left[\left(u - u^*\right)\delta N_s + \left(v - v^*\right)\delta N_{s\theta} + \left(w - w^*\right)\delta Q_s + \left(\beta_s - \beta_s^*\right)\delta M_s + \left(\beta_\theta - \beta_\theta^*\right)\delta M_{s\theta} \right] \frac{\bar{s}d\bar{\theta}}{\lambda} \right\}$$

(2. 34)

where C_f is the boundary part that the forces are prescribed and C_d is the boundary part that the displacements are prescribed. The asterisk denotes the prescribed values of displacements and forces.

The initial conditions are given in the variational form as follow

$$\begin{aligned}
\delta J_i = \int_A \Big\{ & \left[u(\bar{s},\bar{\theta},t_0) - u^*(\bar{s},\bar{\theta})\right] \delta\dot{u}(\bar{s},\bar{\theta}) + \left[v(\bar{s},\bar{\theta},t_0) - v^*(\bar{s},\bar{\theta})\right] \delta\dot{v}(\bar{s},\bar{\theta}) \\
& + \left[w(\bar{s},\bar{\theta},t_0) - w^*(\bar{s},\bar{\theta})\right] \delta\dot{w}(\bar{s},\bar{\theta}) + \left[\beta_s(\bar{s},\bar{\theta},t_0) - \beta_s^*(\bar{s},\bar{\theta})\right] \delta\dot{\beta}_s(\bar{s},\bar{\theta}) \\
& + \left[\beta_\theta(\bar{s},\bar{\theta},t_0) - \beta_\theta^*(\bar{s},\bar{\theta})\right] \delta\dot{\beta}_\theta(\bar{s},\bar{\theta}) + \left[\dot{u}(\bar{s},\bar{\theta},t_0) - \dot{u}^*(\bar{s},\bar{\theta})\right] \delta u(\bar{s},\bar{\theta}) \\
& + \left[\dot{v}(\bar{s},\bar{\theta},t_0) - \dot{v}^*(\bar{s},\bar{\theta})\right] \delta v(\bar{s},\bar{\theta}) + \left[\dot{w}(\bar{s},\bar{\theta},t_0) - \dot{w}^*(\bar{s},\bar{\theta})\right] \delta w(\bar{s},\bar{\theta}) \\
& + \left[\dot{\beta}_s(\bar{s},\bar{\theta},t_0) - \dot{\beta}_s^*(\bar{s},\bar{\theta})\right] \delta\beta_s(\bar{s},\bar{\theta}) + \left[\dot{\beta}_\theta(\bar{s},\bar{\theta},t_0) - \dot{\beta}_\theta^*(\bar{s},\bar{\theta})\right] \delta\beta_\theta(\bar{s},\bar{\theta}) \Big\}\, dA
\end{aligned}$$

(2. 35)

2. 2. 6 GENERALIZED VARIATIONAL THEOREM

The fundamental equations of the conical shell may be stated with a generalized variational statement[34]. Now, let us define a functional J whose first variation is given by

$$\delta J = \delta J_e^d + \delta J_b + \delta J_k + \delta J_c + \delta J_i + \delta J_e^i + \delta J_b^i + \delta J_k^i + \delta J_c^i \tag{2. 36}$$

with

$$\Lambda_c = \left(u, v, w, \beta_s, \beta_\theta \in C_{12} \varepsilon_s, \varepsilon_\theta, \varepsilon_{s\theta}, \varepsilon_{sz}, \varepsilon_{\theta z}; \kappa_s, \kappa_\theta, \kappa_{s\theta} \in C_{00} \right.$$
$$\left. N_s, N_\theta, N_{s\theta}; M_s, M_\theta, M_{s\theta}, Q_s, Q_\theta \in C_{10}\right) \tag{2. 37}$$

where J_b^i, J_k^i, and J_c^i are the corresponding functional related with boundary conditions, kinematical relations and constitutive relations for initial stresses, respectively. Then, of all the admissible states Λ_c, only those which admit the functional J have zero first variation, if and only if, they satisfy the equilibrium equations, the displacement-strain equations, the constitutive equations, the boundary conditions and the initial conditions as appropriate Euler equations. The variational theorem $\delta J(\Lambda_c) = 0$ evidently generates, by the use of the fundamental lemma of the calculus of variations, the complete set of the basic equations of stiffened composite shell with initial stresses.

2. 3 UNIQUENESS OF SOLUTION

In the previous section, a set of two-dimensional, differential, approximate equations of Donnell's type is derived for the dynamic response of a laminated conical shell with stringers and rings. The two-dimensional governing equations of stiffened laminated conical shell are constructed by the use of the virtual work principle within the limits of the well-known Kirchhoff-Love hypotheses of thin shells. Now, the boundary and initial conditions are obtained which are sufficient to ensure the uniqueness in solutions of the dynamical governing equations. Of the several arguments to be used to establish the uniqueness of solutions in elasticity[35], the classical energy argument which relies on the positive definiteness of kinetic and potential energies is used in this study. Kirchhoff[36] used the energy argument at establishing uniqueness in elastostatics, so did Neumann[37] in elastodynamics and Weiner[38] in thermoelasticity. A uniqueness theorem of Neumann's type is proved for solutions of the initial mixed-boundary value problems defined by the two-dimensional governing equations of stiffened cylindrical shell[39]. Mecitoglu and Dökmeci[10] examined the uniqueness in solutions of the dynamic governing equations of stiffened cylindrical shells.

To begin with, consider two possible sets of solutions to the governing equations of stiffened conical shell, namely,

$$\Lambda_\beta = \big(u, v, w, \beta_s, \beta_\theta \in C_{12} \varepsilon_s, \varepsilon_\theta, \varepsilon_{s\theta}, \varepsilon_{sz}, \varepsilon_{\theta z}; \kappa_s, \kappa_\theta, \kappa_{s\theta} \in C_{00}$$
$$N_s, N_\theta, N_{s\theta}; M_s, M_\theta, M_{s\theta}, Q_s, Q_\theta \in C_{10}\big)_\beta \quad \beta = 1,2 \tag{2. 38}$$

Let the difference set of two solutions be denoted by $\Lambda = \Lambda_2 - \Lambda_1$. The difference set of solutions apparently satisfies all the governing equations of stiffened conical shell due to their linearity. It will be shown that the homogeneous linear governing equations possess only the zero solution, that is, the two set of solutions (2. 38) are equivalent under the pertinent boundary and initial conditions. In so doing, we introduce a relation of the form

$$\Gamma = \int_T \left(\Gamma_s + \Gamma_\theta + \Gamma_z + \Gamma_{sz} + \Gamma_{\theta z}\right) dt = 0 \tag{2. 39}$$

with

$$\Gamma_s = \int_A \int \left(\tau_s + \tau_s^i\right) \dot{u} dA \qquad \Gamma_\theta = \int_A \int \left(\tau_\theta + \tau_\theta^i\right) \dot{v} dA \qquad \Gamma_z = \int_A \int \left(\tau_z + \tau_z^i\right) \dot{w} dA$$

$$\Gamma_{sz} = \int_A \int \left(\tau_{sz} + \tau_{sz}^i\right) \dot{\beta}_s dA \qquad \Gamma_{\theta z} = \int_A \int \left(\tau_{\theta z} + \tau_{\theta z}^i\right) \dot{\beta}_\theta dA$$

(2. 40)

where τ_s, τ_θ, τ_z, τ_{sz} , $\tau_{\theta z}$ are defined by Eqs. (2. 30) and $\tau_s^i, \tau_\theta^i, \tau_z^i, \tau_{sz}^i, \tau_{s\theta}^i$ are defined by Eqs. (2. 29).

Now, let us calculate the rates of the strain and kinetic energies of the stiffened conical shell in terms of the displacement components. The rate of the strain energy is expressed with respect to the difference set of the solutions in the form

$$\dot{U} = \int_A \int \Big[\left(N_s + N_s^i\right) \dot{\varepsilon}_s + \left(N_\theta + N_\theta^i\right) \dot{\varepsilon}_\theta + \left(N_{s\theta} + N_{s\theta}^i\right) \dot{\varepsilon}_{s\theta} + \left(M_s + M_s^i\right) \dot{\kappa}_s$$
$$+ \left(M_\theta + M_\theta^i\right) \dot{\kappa}_\theta + \left(M_{s\theta} + M_{s\theta}^i\right) \dot{\kappa}_{s\theta} + \left(Q_s + Q_s^i\right) \dot{\varepsilon}_{sz} + \left(Q_\theta + Q_\theta^i\right) \dot{\varepsilon}_{s\theta}\Big] dA$$

(2. 41)

in terms of the difference set of the solutions. Using the strain displacement Eqs. (2. 18) in Eq. (2. 41) and integrating the equation by parts, one arrives at the rate of the form

$$
\begin{aligned}
\delta\dot{U} = \lambda\int_A\int \Bigg\{ & \left[-\frac{\partial}{\partial\bar{s}}\left(N_s+N_s^i\right)-\frac{1}{\bar{s}}\left[\left(N_s+N_s^i\right)-\left(N_\theta+N_\theta^i\right)\right]-\frac{1}{\bar{s}}\frac{\partial}{\partial\bar{\theta}}\left(N_{s\theta}+N_{s\theta}^i\right)\right]\dot{u} \\
& +\left[-\frac{\partial}{\partial\bar{s}}\left(N_{s\theta}+N_{s\theta}^i\right)-\frac{2}{\bar{s}}\left(N_{s\theta}+N_{s\theta}^i\right)-\frac{1}{\bar{s}}\frac{\partial}{\partial\bar{\theta}}\left(N_\theta+N_\theta^i\right)-\frac{\cot\alpha}{\bar{s}}\left(Q_\theta+Q_\theta^i\right)\right]\dot{v} \\
& +\Bigg[-\frac{\partial}{\partial\bar{s}}\left(Q_s+Q_s^i\right)-\frac{1}{\bar{s}}\frac{\partial}{\partial\bar{\theta}}\left(Q_\theta+Q_\theta^i\right)-\frac{1}{\bar{s}}\left(Q_s+Q_s^i\right)+\frac{\cot\alpha}{\bar{s}}\left(N_\theta+N_\theta^i\right) \\
& \qquad -\frac{\lambda}{\bar{s}}\frac{\partial}{\partial\bar{s}}\left[\left(N_s+N_s^i\right)\bar{s}\frac{\partial w}{\partial\bar{s}}\right]-\frac{\lambda}{\bar{s}^2}\frac{\partial}{\partial\bar{\theta}}\left[\left(N_\theta+N_\theta^i\right)\frac{\partial w}{\partial\bar{\theta}}\right] \\
& \qquad -\frac{\lambda}{\bar{s}}\frac{\partial}{\partial\bar{s}}\left[\left(N_{s\theta}+N_{s\theta}^i\right)\frac{\partial w}{\partial\bar{\theta}}\right]-\frac{\partial}{\partial\bar{\theta}}\left[\left(N_{s\theta}+N_{s\theta}^i\right)\frac{\partial w}{\partial\bar{s}}\right]\Bigg]\dot{w} \\
& +\left[-\frac{\partial}{\partial\bar{s}}\left(M_s+M_s^i\right)-\frac{1}{\bar{s}}\left[\left(M_s+M_s^i\right)-\left(M_\theta+M_\theta^i\right)\right]-\frac{1}{\bar{s}}\frac{\partial}{\partial\bar{\theta}}\left(M_{s\theta}+M_{s\theta}^i\right)+\frac{1}{\lambda}\left(Q_s+Q_s^i\right)\right]\dot{\beta}_s \\
& +\left[-\frac{\partial}{\partial\bar{s}}\left(M_{s\theta}+M_{s\theta}^i\right)-\frac{2}{\bar{s}}\left(M_{s\theta}+M_{s\theta}^i\right)-\frac{1}{\bar{s}}\frac{\partial}{\partial\bar{\theta}}\left(M_\theta+M_\theta^i\right)+\frac{1}{\lambda}\left(Q_\theta+Q_\theta^i\right)\right]\dot{\beta}_\theta\Bigg\}dA \\
& +\oint_C \psi_\theta\bar{s}\,d\bar{\theta}+\int_L \psi_s\,d\bar{s}
\end{aligned}
$$

(2. 42)

In this equation, the quantities of the form

$$
\begin{aligned}
\psi_s = \frac{1}{\lambda}\Bigg\{ & \left(N_{s\theta}+N_{s\theta}^i\right)\dot{u}+\left(N_\theta+N_\theta^i\right)\dot{v}+\left[\left(Q_\theta+Q_\theta^i\right)+\frac{\lambda}{\bar{s}}\left(N_\theta+N_\theta^i\right)\frac{\partial w}{\partial\bar{\theta}}+\lambda\left(N_{s\theta}+N_{s\theta}^i\right)\frac{\partial w}{\partial\bar{s}}\right]\dot{w} \\
& +\left(M_{s\theta}+M_{s\theta}^i\right)\dot{\beta}_s+\left(M_\theta+M_\theta^i\right)\dot{\beta}_\theta\Bigg\}_{\bar{\theta}=0}^{2\pi\sin\alpha}
\end{aligned}
$$

(2. 43)

and

$$
\begin{aligned}
\psi_\theta = \frac{1}{\lambda}\Bigg\{ & \left(N_s+N_s^i\right)\dot{u}+\left(N_{s\theta}+N_{s\theta}^i\right)\dot{v}+\left[\left(Q_s+Q_s^i\right)+\frac{\lambda}{\bar{s}}\left(N_{s\theta}+N_{s\theta}^i\right)\frac{\partial w}{\partial\bar{\theta}}\right]\dot{w} \\
& +\left(M_s+M_s^i\right)\dot{\beta}_s+\left(M_{s\theta}+M_{s\theta}^i\right)\Bigg\}\Bigg|_{\bar{s}=\bar{s}_1}^{1}
\end{aligned}
$$

(2. 44)

are introduced.

Likewise, the rate of the total kinetic energy may be written as follows

$$\dot{K} = \int_A \int \left[m_1(\dot{u}\ddot{u} + \dot{v}\ddot{v} + \dot{w}\ddot{w}) + \frac{m_2}{2}\left(\dot{u}\ddot{\beta}_s + \dot{\beta}_s\ddot{u} + \dot{v}\ddot{\beta}_\theta + \dot{\beta}_\theta\ddot{v}\right) + m_{12}\dot{\beta}_s\ddot{\beta}_s + m_{21}\dot{\beta}_\theta\ddot{\beta}_\theta \right] dA$$

(2. 45)

in terms of the difference set of the solutions. Integrating this equation by parts, one arrives at the rate of the form

$$\dot{K} = \int_A \int \left[\left(m_1\ddot{u} + \tfrac{1}{2}m_2\ddot{\beta}_s\right)\dot{u} + \left(m_1\ddot{v} + \tfrac{1}{2}m_2\ddot{\beta}_\theta\right)\dot{v} + m_1\ddot{w}\right.$$
$$\left. + \left(\tfrac{1}{2}m_2\ddot{u} + m_{12}\ddot{\beta}_s\right)\dot{\beta}_s + \left(\tfrac{1}{2}m_2\ddot{v} + m_{21}\ddot{\beta}_\theta\right)\dot{\beta}_\theta\right] dA \qquad (2.\ 46)$$

In view of the energy rates (2. 42) and (2. 46), the relation (2. 39) is expressed as

$$\Gamma = \int_T (\dot{U} + \dot{K})dt - \int_T dt\left(\oint_C \psi_\theta \bar{s}d\bar{\theta} + \int_L \psi_s d\bar{s}\right) = 0 \qquad (2.\ 47)$$

An integration of this equation with respect to time yields

$$K(t_1) + U(t_1) = K(t_0) + U(t_0) + \int_T \psi dt \qquad (2.\ 48)$$

with

$$\psi = \oint_C \psi_\theta \bar{s}d\bar{\theta} + \int_L \psi_s d\bar{s} \qquad (2.\ 49)$$

The kinetic and strain energy densities are positive-definite, by definition, and initially zero; so that the total kinetic energy and strain energy, K and U, calculated by integration from the difference set of solutions for the stiffened conical elastic shell have the same properties. Thus, it follows from Eq. (2. 47) that

$$K(t_1) = U(t_1) = K(t_0) = U(t_0) = 0 \qquad (2.\ 50)$$

This implies a trivial solution for the difference set of solutions, Λ_c , since the remaining term Ψ of Eq. (2. 47) vanishes in view of Eq. (2. 35). Hence, the uniqueness is ensured in solutions of the governing equations of stiffened conical elastic shell. The following theorem is then concluded.

Theorem - Given the regular region of a stiffened cylindrical elastic shell in the Euclidean three-dimensional space, then there exist at most one set of single-valued solutions, Λ_C, namely,

$$\Lambda_c = \big(u, v, w, \beta_s, \beta_\theta \in C_{12} \varepsilon_s, \varepsilon_\theta, \varepsilon_{s\theta}, \varepsilon_{sz}, \varepsilon_{\theta z}; \kappa_s, \kappa_\theta, \kappa_{s\theta} \in C_{00}$$
$$N_s, N_\theta, N_{s\theta}; M_s, M_\theta, M_{s\theta}, Q_s, Q_\theta \in C_{10}\big)$$

which satisfies all the governing equations of the stiffened shell, provided that kinetic and strain energies are positive-definite, and boundary (2. 34) and initial conditions (2. 35) are prescribed. C_{mn} refers to the function with derivatives of order up to and including (m) and (n) with respect to space coordinates $(\overline{s}, \overline{\theta})$ and time t.

CHAPTER 3

METHOD OF SOLUTION

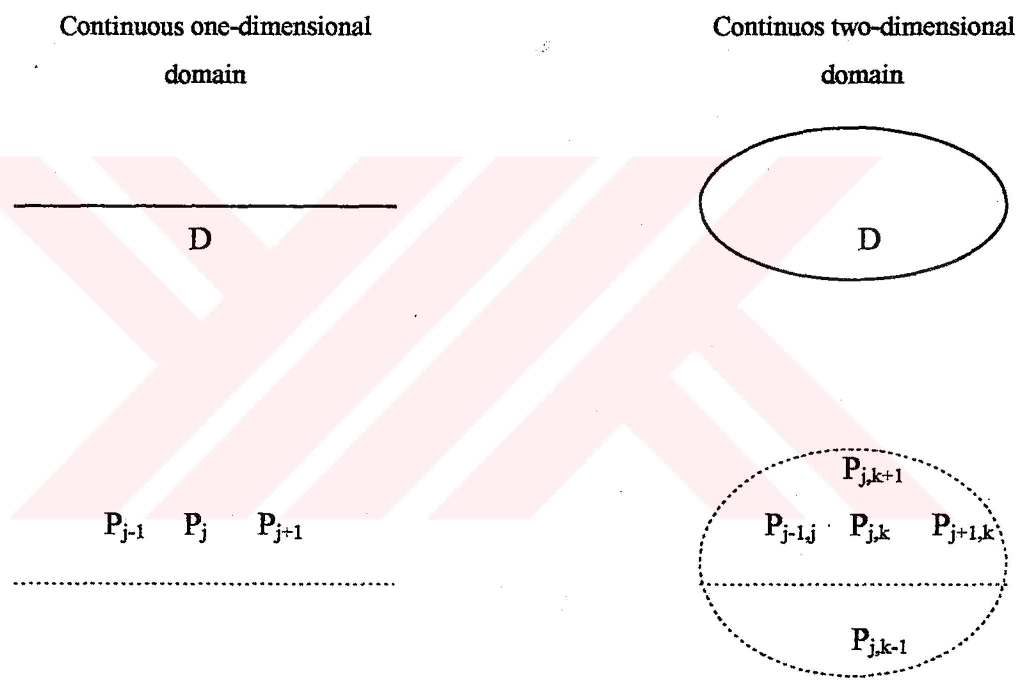
There are many methods in order to solve the equations of motion both analytically and numerically. Among the analytical methods are Galerkin's method, collocation method and so on. Development of computers has enabled us to use numerical methods efficiently in the dynamic analysis of structures which are impossible to treat analytically. Most widely used of numerical methods are finite element method and finite difference method. In this chapter it will be given brief information about the finite difference method

3. 1. FINITE DIFFERENCE METHOD

The finite difference method is a purely mathematical technique. The equations of motion have to be known for the structure that is to be investigated. By contrast, the finite element approach will not require knowledge of the differential equations of motion once the element stiffness and mass matrices are defined.[40]

In finite difference method, the basic approximation involves the replacement of a continuous domain D by a pattern of discrete points within D as shown in Figure (3. 1). Instead of obtaining a continuous solution for ψ defined through D we find approximations to ψ only at these isolated points. When necessary, intermediate values, derivatives, or integrals may be obtained from our discrete solution by interpolation techniques.

As we stated above shortly, the reduction of the governing equation and boundary conditions for the continuous domain to the governing equations for the discrete replacement may be accomplished *physically* or *mathematically*. In the *physical* approach the discrete model is invested with lumped physical characteristics of the continuous system. The governing equations are then obtained from the physical laws applied directly to the lumped-parameter model. In the *mathematical* approach the continuous formulation is reduced to a discrete formulation by simple replacing derivatives with finite-difference approximations.



(a) Discrete replacement of one-dimensional form

(b) Discrete replacement of two-dimensional form

Figure 3. 1 Approximation of a continuous domain by an array of discrete points.

In preparation for replacing differential equations with finite-difference relations, we can write several basic finite difference expressions in one and two dimensions. In one dimension we consider only the case of equally spaced points, and in two dimension we consider only the square network, or lattice. In both cases the spacing dimension will be called *h*.

In Fig. 3. 1(a) let the points . . . . , P_{j-1}, P_j, P_{j+1}, . . . be separated by a distance *h* and let the value of $\psi(x)$ at the P_j be denoted by ψ_j. Then, for instance, the first derivative at P_j may be approximated by the following finite-difference expression,

$$[d\psi / dx]_j = [\psi_{j+1} - \psi_{j-1}] / 2h \qquad (3.\ 1)$$

The second derivative is obtained from

$$[d^2\psi/dx^2]_j = [\{d\psi / dx\}_{j+1} - \{d\psi/dx\}_{j-1}] / 2h \qquad (3.\ 2)$$

or, after substitution of Eq. (3. 1),

$$[d^2\psi/dx^2]_j = [\psi_{j+1} - 2\psi_j + \psi_{j-1}] / h^2 \qquad (3.\ 3)$$

We now turn to the problem of replacing the differential equation and boundary conditions of a given equation with algebraic equations for a discrete approximation. At each internal point the finite difference approximation to the governing differential equation provides an algebraic equation connecting the values of ψ at the several neighboring points. Exceptional situations can arise only near the boundaries. It is then necessary to introduce finite different approximations to the given boundary conditions and thereby to eliminate to need for a value that lies outside D.[41]

The dynamic equations of a stiffened composite laminated conical thin shell which are given Eqs. (2. 31) are solved by finite difference method by using a computer program written in FORTRAN language. To solve the equations of motion which characterized the linear free vibrations of stiffened composite laminated conical shell without the influence of initial stresses, we assume displacement functions of the form

$$u(\bar{s},\bar{\theta},t)=\sum_{n=1}^{\infty}\hat{u}(\bar{s})\cos\frac{n\bar{\theta}}{\sin\alpha}\cos\omega t \qquad v(\bar{s},\bar{\theta},t)=\sum_{n=1}^{\infty}\hat{v}(\bar{s})\sin\frac{n\bar{\theta}}{\sin\alpha}\cos\omega t$$

$$w(\bar{s},\bar{\theta},t)=\sum_{n=1}^{\infty}\hat{w}(\bar{s})\cos\frac{n\bar{\theta}}{\sin\alpha}\cos\omega t$$

$$\beta_s(\bar{s},\bar{\theta},t)=\sum_{n=1}^{\infty}\hat{\beta}_s(\bar{s})\cos\frac{n\bar{\theta}}{\sin\alpha}\cos\omega t \qquad \beta_\theta(\bar{s},\bar{\theta},t)=\sum_{n=1}^{\infty}\hat{\beta}_\theta(\bar{s})\sin\frac{n\bar{\theta}}{\sin\alpha}\cos\omega t$$

(3. 4)

with the harmonic motion assumption. Since the shell is axisymmetric (e.g., no cutouts) and has axisymmetric boundary conditions, then the vibration modes uncouple with respect to θ and the summations on the n is not considered in Eqs. (3. 4). Substituting these equations into (2. 31) and canceling the initial stresses, nonlinear terms and dynamic external loads we obtain five ordinary differential equations.

$$-A_{11}\hat{u}''-\frac{A_{11}}{\bar{s}}\hat{u}'+\frac{A_{22}+A_{33}\psi^2}{\bar{s}^2}\hat{u}-\frac{A_{12}+A_{33}}{\bar{s}}\psi\hat{v}'+\frac{A_{22}+A_{33}}{\bar{s}^2}\psi\hat{v}-\frac{A_{12}\cot\alpha}{\bar{s}}\hat{w}'+\frac{A_{22}\cot\alpha}{\bar{s}^2}\hat{w}$$
$$-B_{11}\hat{\beta}_s''-\frac{B_{11}}{\bar{s}}\hat{\beta}_s'+\frac{B_{22}+B_{33}\psi^2}{\bar{s}^2}\hat{\beta}_s-\frac{B_{12}+B_{33}}{\bar{s}}\psi\hat{\beta}_\theta'+\frac{B_{22}+B_{33}}{\bar{s}^2}\psi\hat{\beta}_\theta=\frac{\omega^2}{\lambda^2}\left(m_1\hat{u}+\tfrac{1}{2}m_2\hat{\beta}_s\right)$$

$$\frac{A_{12}+A_{33}}{\bar{s}}\psi\hat{u}'+\frac{A_{22}+A_{33}}{\bar{s}^2}\psi\hat{u}-A_{33}\hat{v}''-A_{33}\frac{1}{\bar{s}}\hat{v}'+\frac{A_{22}\psi^2+A_{33}+A_{44}\cot^2\alpha}{\bar{s}^2}\hat{v}$$
$$+\frac{(A_{22}+A_{44})\cot\alpha}{\bar{s}^2}\psi\hat{w}+\frac{B_{12}+B_{33}}{\bar{s}}\psi\hat{\beta}_s'+\frac{B_{22}+B_{33}}{\bar{s}^2}\psi\hat{\beta}_s$$
$$-B_{33}\hat{\beta}_\theta''-B_{33}\frac{1}{\bar{s}}\hat{\beta}_\theta'+\left(\frac{B_{22}\psi^2+B_{33}}{\bar{s}^2}-\frac{A_{44}\cot\alpha}{\bar{s}\lambda}\right)\hat{\beta}_\theta=\frac{\omega^2}{\lambda^2}\left(m_1\hat{v}+\tfrac{1}{2}m_2\hat{\beta}_\theta\right)$$

$$\frac{A_{12}\cot\alpha}{\bar{s}}\hat{u}'+\frac{A_{22}\cot\alpha}{\bar{s}^2}\hat{u}+\frac{(A_{22}+A_{44})\cot\alpha}{\bar{s}^2}\psi\hat{v}-A_{55}\hat{w}''-A_{55}\frac{1}{\bar{s}}\hat{w}'+\frac{A_{44}\psi^2-A_{22}\cot^2\alpha}{\bar{s}^2}\hat{w}$$
$$+\left(\frac{B_{12}\cot\alpha}{\bar{s}}-\frac{A_{55}}{\lambda}\right)\hat{\beta}_s'+\left(\frac{B_{22}\cot\alpha}{\bar{s}^2}-\frac{A_{55}}{\lambda\bar{s}}\right)\hat{\beta}_s+\left(\frac{B_{22}\cot\alpha}{\bar{s}^2}-\frac{A_{44}}{\lambda\bar{s}}\right)\psi\hat{\beta}_\theta=\frac{\omega^2}{\lambda^2}m_1\hat{w}$$

$$-B_{11}\hat{u}''-\frac{B_{11}}{\bar{s}}\hat{u}'+\frac{B_{22}+B_{33}\psi^2}{\bar{s}^2}\hat{u}-\frac{B_{12}+B_{33}}{\bar{s}}\psi\hat{v}'+\frac{B_{22}+B_{33}}{\bar{s}^2}\psi\hat{v}$$
$$+\left(\frac{A_{55}}{\lambda}-\frac{B_{12}\cot\alpha}{\bar{s}}\right)\hat{w}'+\frac{B_{22}\cot\alpha}{\bar{s}^2}\hat{w}-D_{11}\left(\hat{\beta}_s''+\frac{1}{\bar{s}}\hat{\beta}_s'\right)+\left(\frac{D_{22}+D_{33}\psi^2}{\bar{s}^2}+\frac{A_{55}}{\lambda^2}\right)\hat{\beta}_s$$
$$-\frac{D_{12}+D_{33}}{\bar{s}}\psi\hat{\beta}_\theta'+\frac{D_{22}+D_{33}}{\bar{s}^2}\psi\hat{\beta}_\theta=\frac{\omega^2}{\lambda^2}\left(m_{12}\hat{\beta}_s+\tfrac{1}{2}m_2\hat{u}\right)$$

$$\frac{B_{12}+B_{33}}{\bar{s}}\psi\hat{u}'+\frac{B_{22}+B_{33}}{\bar{s}^2}\psi\hat{u}-B_{33}\left(\hat{v}''+\frac{1}{\bar{s}}\hat{v}'\right)+\left(\frac{B_{22}\psi^2+B_{33}}{\bar{s}^2}-\frac{A_{44}\cot\alpha}{\bar{s}\lambda}\right)\hat{v}$$
$$+\left(\frac{B_{22}\cot\alpha}{\bar{s}^2}-\frac{A_{44}}{\lambda\bar{s}}\right)\psi\hat{w}+\frac{D_{12}+D_{33}}{\bar{s}}\psi\hat{\beta}_s'+\frac{D_{22}+D_{33}}{\bar{s}^2}\psi\hat{\beta}_s$$
$$-D_{33}\left(\hat{\beta}_\theta''+\frac{1}{\bar{s}}\hat{\beta}_\theta'\right)+\left(\frac{D_{22}\psi^2+D_{33}}{\bar{s}^2}+\frac{A_{44}}{\lambda^2}\right)\hat{\beta}_\theta=\frac{\omega^2}{\lambda^2}\left(m_{21}\hat{\beta}_\theta+\tfrac{1}{2}m_2\hat{v}\right) \tag{3. 5}$$

where the prime above the symbols denotes its ordinary derivative with respect to $\bar{s}$.

The finite difference method is applied to solve resulting ordinary differential equations together with the boundary conditions at the small end

$$u=v=w=\beta_s=M_{s\theta}=0 \tag{3. 6a}$$

and at the large end

$$N_s=v=w=M_s=M_{s\theta}=0 \tag{3. 6b}$$

The central difference formulas are used to obtain the derivatives with respect to $\bar{s}$.

CHAPTER 4

NUMERICAL RESULTS

The analysis described in the previous sections is applicable to the free vibration of laminated conical shells. The axisymmetric vibration of antisymmetric cross-plied laminated cones is considered in the illustrative example and the material parameters of each layer are taken to be

$$E_s = 15E_\theta \qquad \nu_{s\theta} = 0.25 \qquad \nu_{xz} = \nu_{\theta z} = 0.3 \tag{4.1}$$

The geometrical property of the truncated conical shell is taken to be L/R = 0.5. Following stiffening parameters are used for the stiffened shell case; $E_S = E_R = 0.52E_s$, $\rho_S = \rho_R = 1.73\rho_c$, $h_S = h_R = 5h$, $b_S = b_R = h$, $N_S = N_R = 6$. Here, h_S, b_S and N_S are the depth, width and the number of stringers, respectively. The subscript R denotes the properties of a ring.

The frequencies used in comparison are calculated in the case of n=1 and N=30. n and N denote the number of circumferential waves and number of grid points respectively.

Before presenting the results, let us introduce a frequency parameter defined as

$$\Omega = \omega R \sqrt{\frac{\rho_c h}{A_{11}}} \tag{4.2}$$

where A_{11} is determined for the unstiffened conical shell. The frequency rates used in Fig. (5. 1) -(5. 3) are calculated as Ω / Ω_0 . Ω_0 is the first value of the frequencies. Lamina material properties at the elevated moisture concentrations [27] used in the present analysis are presented in Table 4. 1 .

TABLE 4. 1

Elastic moduli of graphite/epoxy lamina at different moisture concentrations ;

$G_{13} = G_{12}$, $G_{23} = 0.5G_{12}$, $\nu = 0.3$, $\beta_1 = 0$ and $\beta_2 = 0.44$

Elastic Moduli (Gpa)	Moisture concentration						
	0.00	0.25	0.50	0.75	1.00	1.25	1.50
E_1	130	130	130	130	130	130	130
E_2	9.5	9.25	9.0	8.75	8.5	8.5	8.5
G_{12}	6.0	6.0	6.0	6.0	6.0	6.0	6.0

The results are given in Figures (5. 1) - (5. 9). The present results are compared with those obtained from power series approach given by Tong [24], for the unstiffened laminated conical shell in figures (5. 1) - (5. 3) for different semivertex angles and relatively agreement is found.

The different frequency parameter values are given in figure (5. 4) for different semivertex angles. It can be realized that the higher semivertex angle the lower frequency parameter.

The fundamental frequency of the laminated truncated conical shell decreases with the stringers and rings. This case is shown in figures (5. 5) - (5. 7) for different semivertex angles. The reason for the reducing effect of stiffeners on the natural frequencies of the conical shell can be explained as the relative contribution of the stiffeners to the mass of the laminated conical shell is much more than the contribution of the stiffeners to the stiffness of the conical shell.

The effect of uniform moisture concentration on frequency parameter is given with figure (5. 8). It can be seen that easily in figure (5. 8), the frequency values reduce with the increase in moisture concentration for the laminated conical shell. The result shows a good agreement with the study which on the free vibration of laminated composite plates [27].

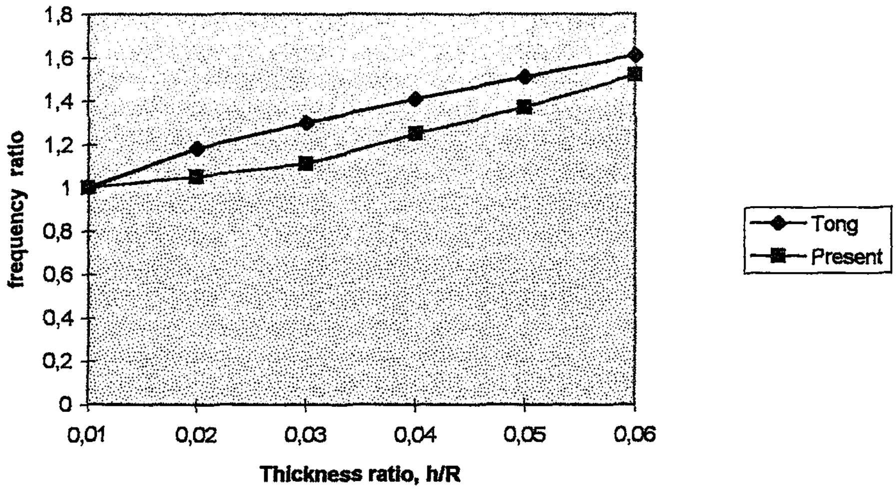


Figure 5. 1 Change of frequency ratio with thickness ratio for $\alpha = 30^0$ and L / R=0.5

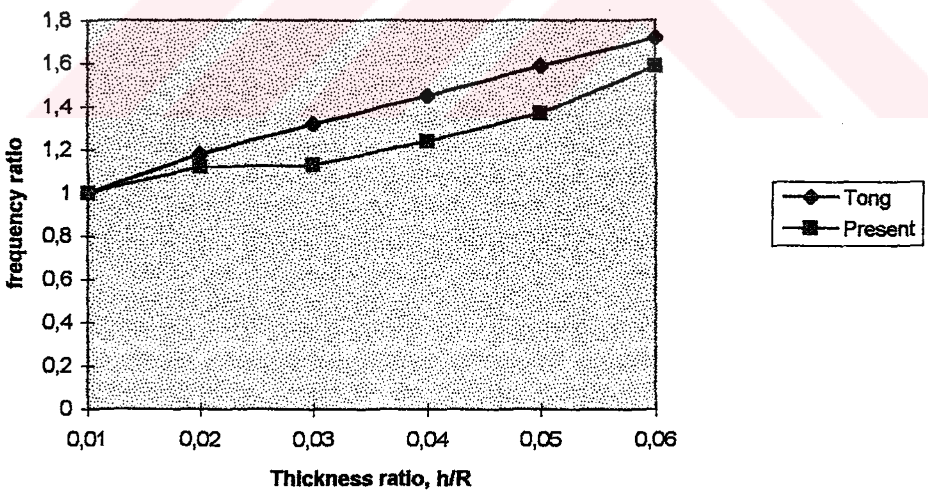


Figure 5. 2 Change of frequency ratio with thickness ratio for $\alpha = 45^0$ and L / R=0.5

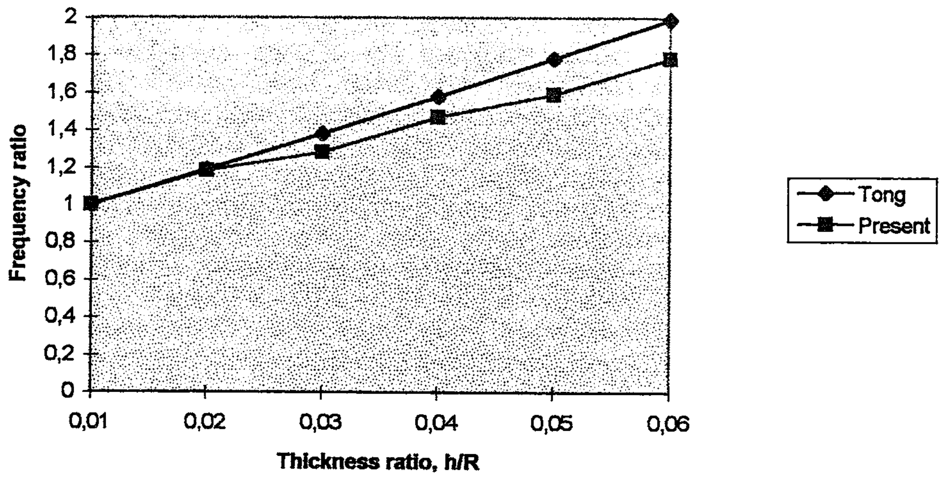


Figure 5. 3 Change of frequency ratio with thickness ratio for $\alpha = 60^0$ and L / R=0.5

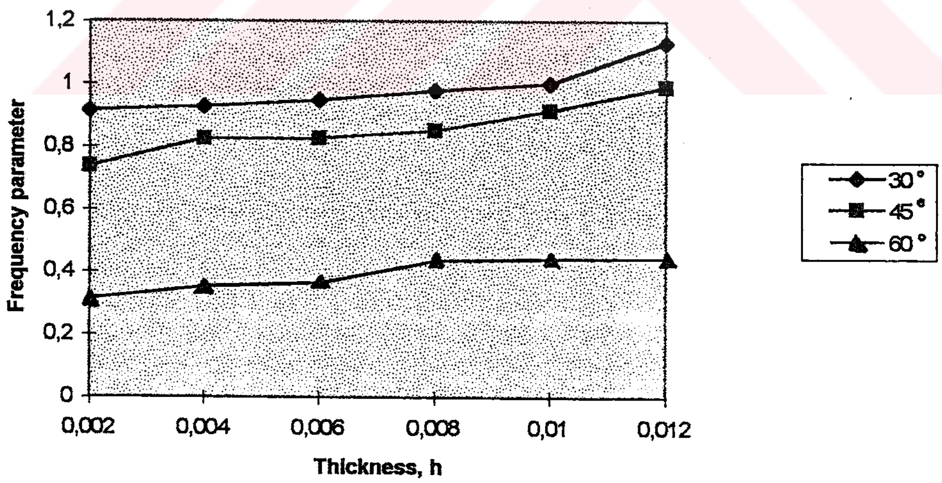


Figure 5. 4 Change of frequency parameter with thickness for different semivertex angles

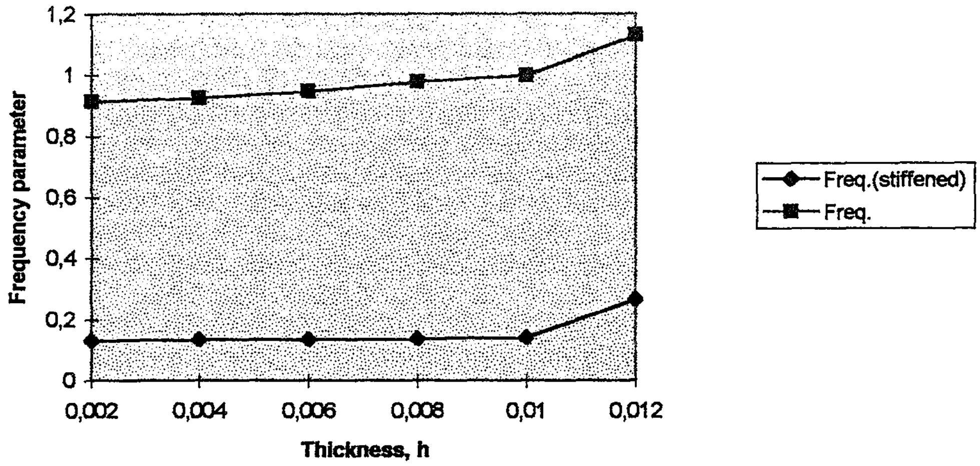


Figure 5. 5 Change of frequency parameter for stiffened and unstiffened case ($\alpha=30^0$ L / R=0.5)

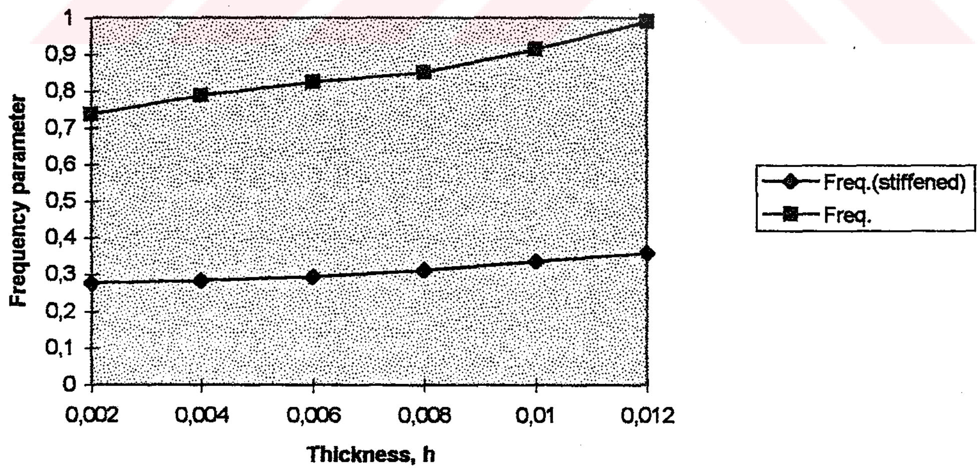


Figure 5. 6 Change of frequency parameter for stiffened and unstiffened case ($\alpha=45^0$ L / R=0.5)

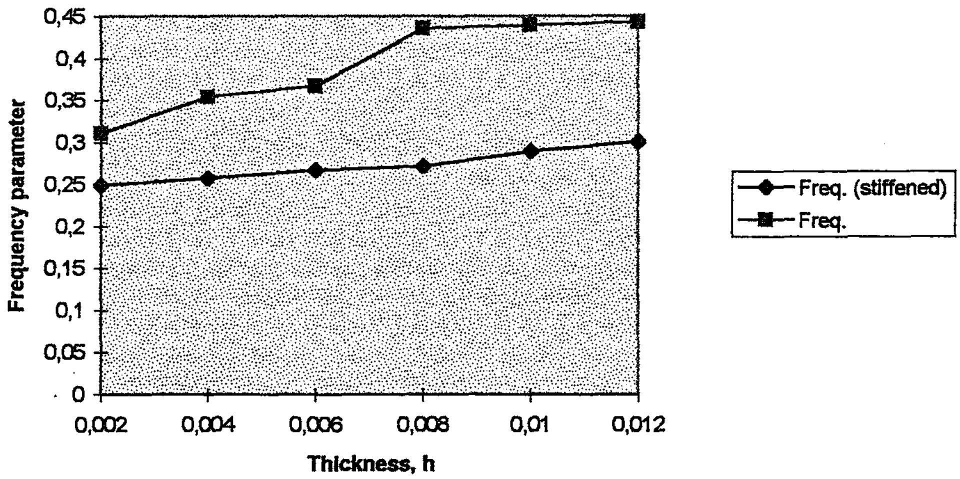


Figure 5. 7 Change of frequency parameter for stiffened and unstiffened case (α=60^0 L / R=0.5)

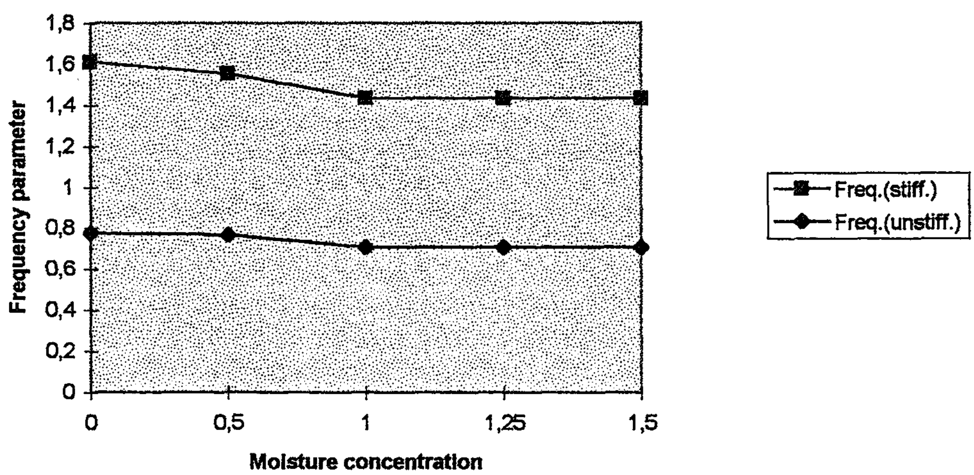


Figure 5. 8 Change of frequency parameter with moisture concentration (α = 45^0, h= 0.002)

CHAPTER 5

CONCLUSIONS

In this study, all the dynamic equations of a composite laminated conical thin shell reinforced by stringers and rings are derived in the variational form and the uniqueness is examined in solutions of the governing shell equations. The dynamic equations of a stiffened composite laminated conical shell are based on the Donnell-Mushtari theory of elastic thin shells. A generalized variational theorem is given so as to describe the complete set of the fundamental equations of the conical shell. The geometric nonlinearities and initial stresses are taken into account in the derivation of governing equations. The rings and/or stringers are "smeared out" along the conical shell. The inhomogeneity of material properties is considered in the constitutive equations. The uniqueness is examined in solutions of the dynamic governing equations of stiffened shells, and a theorem of uniqueness is given which enumerates the initial and boundary conditions sufficient for the uniqueness. The dynamic equations are solved by the finite difference method to obtain the free vibration characteristics of a composite laminated shell and a agreement is obtained with certain earlier results. Influence of the moisture on the vibration characteristics is examined.

In the uniqueness theorem, the boundary and initial conditions which render Ψ to zero are shown to be sufficient for the uniqueness in solutions of the governing equations of stiffened conical elastic shell. The conditions (2. 34) and (2. 35), and also, to specify one member of each products in Ψ of (2. 49) ensure a unique solution for the governing equations. Besides, the sufficient conditions can be expressed in terms of the stress resultants as well as the displacement components, and they can be obtained by logarithmic convexity arguments with no restrictions on the positive-definiteness of energies[42].

In the illustrative example the free vibration of the stiffened laminated composite conical shell is studied numerically by the finite difference method. The isotropic stiffeners reduce the fundamental frequencies of the laminated conical shell with all the semivertex angles. Because the mass contribution of the stiffeners is greater than the stiffness contribution of them. Also the effect of moisture on the composite laminated shell is investigated and found that the natural frequency reduces with the increase in uniform moisture concentration.

Finally, it is concluded that the present method can be employed to establish the dynamic governing equations of a general stiffened laminated shell and to prove the uniqueness theorem for the solution of the equations. A numerical and parametric study can be performed by accounting the complicating effects such as initial stresses, inhomogeneity and nonlinearity..

REFERENCES

1. **LEISSA, A.W.**, "Vibration of Shells," NASA SP-288, 1973.

2. **SAUNDERS,. H WISNIEWSKI, E.J. AND PASLEY, P.R.** "Vibration of Conical Shells," *Journal of Acoustical Society of America*, Vol. 32, 1960, pp.765-772.

3. **DREHER, J.F., and LEISSA, A.W.**, "Axisymmetric Vibration of Thin Conical Shells," *Developments in Theoretical and applied Mechanics*, Vol. 4, 1970, pp.163-181.

4. **IRIE, T., YAMADA, G. and KANEKO, Y.**, "Free Vibration of a Conical Shell with Variable Thickness," *Journal of Sound and Vibration*, Vol. 82, 1983, pp. 83-94.

5. **IRIE, T., YAMADA, G. and KANEKO, Y.**, "Natural Frequencies of Truncated Conical Shells," *Journal of Sound and Vibration*, Vol. 92, 1984, pp. 447-453.

6. **SINGER, J.**, "A Donnell Type Theory for Bending and Buckling of Orthotropic Conical Shells," *Technion-Israel Institute of Technology*, TAE Report 19, 1962.

7. **WEINGARTEN, V.I.**, "Free Vibrations of Ring-Stiffened Conical Shells," *AIAA Journal*, Vol. 3, No. 8, 1965, pp. 1475-1481.

8. **BACON, M.D. and BERT, C.W.**, "Unsymmetric Free Vibrations of Orthotropic Sandwich Shells of Revolution," *AIAA Journal*, Vol. 5, No. 3, 1967, pp. 413-417.

9. **COHEN, G.A.**, "Computer Analysis of Asymmetric Free Vibrations of Ring Stiffened Orthotropic Shells of Revolution," *AIAA Journal*, Vol. 3, No. 12, 1965, pp. 2305-2312.

10. **MECİTOĞLU, Z. and DÖKMECİ, M.C.**, "Vibration Analysis of Stiffened Cylindrical Thin Shells", *Proc. 17th Congress of the ICAS*, Stockholm, 1990, pp. 986-993.

11. **WEINGARTEN, V.I.**, "The Effect of Internal and External Pressure on the Free Vibrations of Conical Shells," *International Journal of Mechanical Sciences*, Vol. 8, No. 2, 1966, pp. 47-56.

12. **GOLBERG, J. E.., BOGDANOF J.L. and ALSPAUGH, D.W.** "On the Calculation of the Modes and Frequencies of Vibration of Pressurized Conical Shells," *Proc. AIAA 5th Annual Structures and Materials Conference*, 1964, pp. 243-249.

13. **GOLBERG, J. E.., BOGDANOF J.L. and ALSPAUGH, D.W.** , "Modes and Frequencies of Pressurized Conical Shells," *Journal of Aircraft*, Vol. 1, No. 6, 1964, pp. 372-374.

14. **SCHEINEDER, M.H.., SNELL, R.F.., TRACY, J.J., and POWERS, D.R.**, "Buckling and Vibration of Externally Pressurized Conical Shells with Continuous and Discontinuous Rings," *AIAA Journal*, Vol. 29, No. 9, 1991, pp. 1515-1522.

15.**NEWTON, R.A.**, "Free Vibrations of Rocket Nozzles," *AIAA Journal*, Vol. 4 No. 7, 1966, pp. 1303-1305.

16. **MARTIN, R.E.**, "Free Vibrations of Anisotropic Conical Shells," *AIAA Journal*, Vol. 7, No. 5, 1969, pp. 960-962.

17. **HEYLIGER, P.R. and JILANI, A.**, "The Free Vibrations of Inhomogeneous Elastic Cylinders and Spheres," *International Journal of Solids and Structures*, Vol. 29, No. 22, 1992, pp. 2689-2708.

18. **MECİTOĞLU, Z.** "Free Vibrations of a Conical Shell with Temperature Dependent Material Properties," *The First International Symposium on Thermal Stresses and Related Topics*, Shizuoka University, Hamamatsu, Japan, June 1995.

19. **SIVADAS, K.R. and GANESAN, N.**, "Vibration Analysis of Thick Composite Clamped Conical Shells of Varying Thickness," *Journal of Sound and Vibration* Vol. 152, 1992, pp. 27-37.

20. **SANKARANARAYANAN, N., CHANDRASEKARAN, K. and RAMAIYAN, C.**, "Free Vibrations of Laminated Conical Shells of Variable Thickness," *Journal of Sound and Vibration*, Vol. 123, 1988, pp. 357-371.

21. **KAYRAN, A. and VINSON, J.R.**, "Free Vibration Analysis of Laminated Composite Truncated Circular Conical Shells," *AIAA Journal*, Vol. 28, No. 7, 1990, pp. 1259-1269.

22. **TONG, L.**, "Free Vibration of Orthotropic Conical Shells", *International Journal of Engineering Sciences*, Vol. 31, No. 5, 1993, pp. 719-733.

23. **TONG, L.**, "Free Vibration of Laminated Conical Shells," *International Journal of Mechanical Sciences*, Vol. 35, No. 1, 1993, pp. 47-61.

24. TONG, L., "Free Vibration of Laminated Conical Shells Including Transverse Shear Deformation," *International Journal of Solids and Structures*, Vol. 31, No. 4, 1994, pp. 443-456.

25. LANGLEY, R.S., "A Dynamic Stiffness Technique for the Vibration Analysis of Stiffened Shell Structures", *Journal of Sound and Vibration*, Vol. 156, No. 3, 1992, pp. 521-540.

26. LEY, R.P., JOHNSON, E.R., and GÜRDAL, Z., "Buckling of Imperfect, Anisotropic, Ring- Stiffened Cylinders Under Combined Loads," *AIAA Journal*, Vol. 32, No. 6, 1994, pp. 1302-1309.

27. SAI RAM, K.S., and SINHA, P.K., *"Hygrothermal effects on the free vibration of laminated composite conical plates"* Journal of Sound and Vibration (1992), vol 158(1), pp. 133-148

28. LYRINTZIS, C,S., and BOFILIOS, A.D., *"Moisture effect on the response of orthotropic stiffened panel structures "* AIAA Journal, Vol 28, No 12, pp. 2117-2124

29. KOLLAR, P.L., and PATTERSON, J.M., *"Composite cylindrical segments subjected to hygrothermal and mechanical loads"* Int. J. Solids Structures Vol. 30, No 18, pp. 3525-3545

30. KNOPS, R.J. and PAYNE, L.E., *Uniqueness Theorems in Linear Elasticity*, Springer Verlag, New York 1972.

31. KRAUS, H., *"Thin Elastic Shells"*, McGraw Hill, New York, 1967

32. HOSKIN, B.C., BAKER, A.A., "*Composite Materials for Aircraft Structures,* AIAA Education Series, 1986"

33. GIBSON, R.F., *"Principles of Composite Material Mechanics" , McGraw-Hill,Inc., 1994 pp.132*

34. DÖKMECİ, M.C., "Theory of Vibrations of Coated Thermopiezoelectric Laminae," *Journal of Mathematical Physics*, Vol. 19, 1978, pp. 109-126.

35. DÖKMECİ, M.C., and SARIGÜL, N., "Certain Integral and Differential Types of Variational Principles of Mathematical Physics," *Abstracts of International Congress of Mathematics*, pp. 269, Berkeley, California, August 1986; *AIAA Journal*, Vol. 22, No. 8, August 1984, pp. 1173-1175, and IEEE TRANS. UFFC, Vol. 35, No. 6, Nov. 1988, pp. 775-787.

36.. KIRCHHOFF, G., "Über das Gleichgewicht und die Bewegung einer elastischen Scheibe," *J. Reine Angew. Math.*, Vol. 40, 1850, pp. 51-58.

37. NEUMANN, F., *Vorlesungen über die Theorie der Elastizitat der festen Körper und des Lichtathers*, Teubner, Leipzig, 1885.

38. WEINER, J.H., "A Uniqueness Theorem for the Coupled Thermoelastic Problem," *Quarterly of Applied Mathematics*, Vol. 15, 1957, pp. 102-105.

39. DÖKMECİ M.C "A Generalized Variational Theorem in Elastodynamics with Application Shell Theory", *Meccanica*, Vol.8, 1970, pp. 252-260.

40. SOEDEL, W., " *Vibrations of Shells and Plates*" Marcel Dekker Inc., New York, 1993

41. CRANDALL, S.H., *"Engineering Analysis"* McGraw-Hill Company Inc, 1956 pp. 243-247

42. DÖKMECİ M.C. "Vibration and Stability Analysis of Laminated Composites", *AEK Tek.* 2940-3-77, 1980.

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