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KARMAŞIK DALGA PROBLEMLERİNİN ÇÖZÜMÜNDE YENİ BİR YÖNTEM

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LOCAL INTRINSIC MODES
A NEW METHOD TO SOLVE NON-SEPARABLE WAVE PROBLEMS

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II

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LEVENT SEVGI

III

CONTENTS

ABSTRACT	IV
OZET	V
SECTION 1. Local Intrinsic Mode Theory	1
1.1. Spectral Synthesis Of 1D. Wave Functions	1
1.2. Spectral Synthesis Of 2D. Wave Functions	6
SECTION 2. Wedge With Non-penetrable Boundaries	8
2.1. LIM Formulation	8
2.2. Reduction To IM And Exact Solution	12
2.3. Numerical Test Results	15
2.3.1) Direct Numerical Integration	18
2.3.2) Steepest Descent Path (SDP) Evaluation	21
2.3.3) Asymptotic Evaluation Through Cut-Off	24
SECTION 3. Wave Propagation In A Graded Index Taper	29
3.1. Formulation, Solution and Reduction To Closed Form (exact) Solutions	29
3.2. Comparison With The Table Values Of Bessel Functions	38
SECTION 4. Longitudinally Narrowing Surface Duct	41
4.1. Weak Depth Dependence	41
4.2. Strong Depth Dependence	47
4.3. Numerical Results	48
SECTION 5. Bilinear Widening Surface Duct	54
5.1. Formulation Of The Problem	54
5.2. Numerical Results	62
5.3. Failure Of The LIM Solution Through Turning Point	66
5.4. Propagation In Antiguiding Region	72
SECTION 6. Conclusion	76
Appendix A : Steepest Descent Path for Weak Depth Dependence	78
Appendix B : Defining The Integration Contour C_β	80
References	83

IV

ABSTRACT

LOCAL INTRINSIC MODES :

A NEW METHOD TO SOLVE NON-SEPARABLE WAVE PROPAGATION

Wave guiding environments with transversely and longitudinally varying properties occur in many natural and man made settings. In such environments, it is desirable to seek propagation modes that maintain their transverse shape in a suitably defined analytical reference frame because the propagation of general wavefield can then be organized systematically around these basis elements. The definition of these basis elements are strongly tied to the separability of the wave equation. Although the mode concept is "ill-defined" for nonseparable wave problems it is still possible to construct approximate mode fields if the longitudinal variations are "sufficiently slow". Local separability can be applied for these weakly separable wave problems which gives local modes (adiabatic modes, AM) satisfying the range independent wave equation for each range point. In (x,z) coordinate space due to the longitudinal (i.e., z) variations, the transverse amplitude depends weakly on z , and its longitudinal traveling behavior is characterized by an accumulated phase in the form of a phase integral $\int \beta_m(z) dz$, where $\beta_m(z)$ is the modal eigenvalue propagation coefficient of the range independent mode corresponding to the local conditions at $z=z'$. Like exact modes in range independent environments, the AM can be normalized (i.e., they have finite energy) providing they belong to the class of "proper" modes whose transverse fields either decay at $x \rightarrow \pm\infty$ or are confined to a finite x domain by impenetrable boundaries. Within the context of coupled mode theory for range dependent propagation the smoothly adapting AM account for the lowest order (nearest neighbor) coupling between the range independent modes. Higher order coupling, or partial reflections due to the range dependent environments are neglected in the AM approximation.

The ability to normalize the AM fails when the range dependence leads to local mode cut-off and(or) trapped-to-leaky and guiding-to-antiguinding transitions. The AM failure is due to the spectral deficiency of the AM field in these transitional domains; it can be repaired by constructing a spectral β -continuum around the range dependent point spectrum $\beta_m(z)$. The extended spectrum is synthesized from that of the AM modes, and the resulting wave object, termed a local intrinsic mode (LIM) provides the required uniformization of the AM passing through the transitional domains.

ÖZET

YEREL ÖZMODLAR

KARMAŞIK DALGA PROBLEMLERİNİN ÇÖZÜMÜNDE YENİ BİR YÖNTEM

Hem enine (kesit içi) hem de boyuna (ışınım doğrultusu boyunca) değişimlerin var olduğu kılavuzlanmış dalga ışınlama problemleri doğal ve yapay birçok ortamda önemli bir problem takımını oluşturmaktadır. Böyle ortamlarda uygun koordinat sisteminde kesit içi ışınlama modlarının bulunabilmesi genel alan dağılımlarının bu temel yapı taşları etrafında yapılanmasını olası ve kolay kılmaktadır. Bu temel yapı taşlarının bulunabilmesi verilen ortamda dalga denkleminin değişkenlerine ayrıştırılabilir olması na sıkı sıkıya bağlıdır. Dalga denkleminin değişkenlerine ayrıştırılabildiği bir koordinat sisteminin bulunmadığı durumlarda MOD kavramı sağlıklı tanımlanamaz. Ancak ışınlama eksenini boyunca değişkenlerin "yeterince yavaş" olduğu varsayılabildiğinde (ayrıştırılamamanın zayıf olması durumunda) dalga denklemi yerel olarak ayrıştırılıp yaklaşık modlar inşa edilebilir. Yerel modlar (adiabatic modes: AM) denilen bu yapı taşları dalga denkleminin WKB çözümünden başka bir şey değildir [1,2]. WKB, ışınlama doğrultusunda herhangi bir noktadan geçen ve kesiti o noktadaki ortam kesitine eşit paralel levhalı dalga kılavuzu modlarının bulunması yöntemidir. İki boyutlu (x,z) koordinat sisteminde kesit (x), ışınlama doğrultusu (z) olarak alındığında AM çözümleri her z_0 noktasında değişkenlerine ayrıştırılmış dalga denklemini sağlarlar. Bu AM kendilerini, yansıma ve diğer modlara kuplaj olmaksızın, yavaş değişen ortam koşullarına uyarlayabilmektedirler. Dağılıma denkleminin $(k^2 = k_x^2 - \beta_m^2(z))$ olduğu göz önüne alındığında ışınlama doğrultusunda AM faz değişimi $\int \beta_m(z) dz$ şeklindedir. Klasik modlarda olduğu gibi AM de enerji korunumunu sağlamak amacıyla normalize edilmektedirler.

VI

Ancak ışınım boyunca "kesim" bölgesi (cut-off), "kılavuzlanma-sızıntı" bölgesi (guiding-to-leaky) ve "kılavuzlanma-ışınım" bölgesi (guiding-to antiguiding) gibi dalga ışınımı problemlerindeki kritik bölgelere yaklaşıldığında AM enerji korunumu sağlayamaz. Dalga ışınımı problemlerinde kesim kesitin daralmasından, kılavuzlanma-sızıntı ortam parametrelerinin kesitte bilineer değişmesinden, kılavuzlanma-ışınım ise boyuna değişmelerin bir z_0 noktasından sonra işaret değiştirmesinden kaynaklanmaktadır.

Yakın geçmişte, AM nin bu yetersizliğini gidermek amacıyla "özmodlar" (IM) yöntemi dielektrik kama problemine uygulanmıştır [3]. Düzlem dalgalarının süperpozisyonu ile oluşturulan bir biçim IM integrali kama içinde daralan kesit yönünde ilerleyen modların kesime girip kama dışına ışın demeti şeklinde ışımlarını doğru olarak açıklayabilmektedirler. Ayrıca geçerli oldukları bölgelerde AM çözümleri, IM integralinin yaklaşık yöntemlerle (semer noktası, durağan faz noktası yöntemleri) analitik olarak indirgenmesiyle elde edilmektedirler. Düzlem dalgalarının süperpozisyonu yerine, kritik bölgelerden yeterince uzakta, AM ile oluşturulan spektral IM integrali horn antenlerde alan dağılımı [4] ve optik dalga kılavuzlarında iletim [5] problemlerine de başarıyla uygulanmıştır.

Bu çalışmada IM yöntemi "yerel özmodlar" (LIM) şeklinde geliştirilmiş, önce kapalı tam çözümlerin bulunduğu kanonik örnek yapılarda çözümlerin doğruluğu gösterilmiş, daha sonra değişkenlerine ayrıştırılamayan kritik bölgelerin olduğu dalga problemleri çözülmüştür.

İlk bölüm yerel özmodlar (LIM) integralinin bir ve iki boyutlu yapılarda oluşturulmasına ayrılmıştır.

VII

LIM integrali, AM spektral bileşenleriyle oluşturulan integranda iki spektral keyfi fonksiyonun (biri spektral faz diğeri spektral genlik fonksiyonu) eklenmesi, spektral genlik fonksiyonunun dalga denklemini asmpototik açılımda ikinci mertebede sağlayacak şekilde seçilmesi ve faz fonksiyonunun ise integrandın semer noktasının istenen gözlem noktasına gelecek şekilde seçilmesiyle oluşturulur. Bunu yapmadaki temel amaç LIM integrali durağan faz yöntemiyle alındığında elde edilen sonucun yerel mod (AM) sonucuyla orantılı olmasını sağlamaktır. Seçilen bu keyfi genlik ve faz fonksiyonları aslında yerel modların genlik ve faz fonksiyonundan başka birşey değildirler.

İkinci ve üçüncü bölümlerde kapalı tam çözümleri bulunan iki kanonik yapıda LIM integrali ele alınmıştır. İlki tam yansıtıcı sınırlara sahip homojen dielektrikle doldurulmuş kama problemidir. İkincisi kesitte doğal sınırları bulunmayan ancak ortam kırılma indisi enine modların oluşmasına olanak sağlayacak şekilde değişen yapıdır. Bu yapının paralel tam yansıtıcı kamadan tek farkı modların tam yansıtıcı düzlemler arasında değil oluşan kostikler arasında hapsolmesidir. Her iki kanonik problemde düzlem polar koordinat sisteminde değişkenlerine ayrıştırılabilir tip-tendir. Yerel modların oluşturulabilmesi için radyal uzaklık ve bu uzaklıktaki yay parçası koordinat bileşenleri olarak alınmış ve problem değişkenlerine ayrıştırılamayan tipe dönüştürülmüştür. Bu sistemde enine rezonans prensibi kullanılarak dalga vektörünün açısal ve radyal bileşenleri bulunmuştur. Kama probleminde enine modlar sinüs fonksiyonuyla diğesinde ise Gausyen-Hermit fonksiyonuyla gösterilmektedir. Boyuna değişimler ise kaynak ve gözlem noktalarının konumlarına bağlı olarak Bessel ya da Hankel fonksiyonlarıyla verilmektedirler. Bu bölümlerde LIM integrali birinci bölümde anlatılan esaslarla oluşturulmuş, boyuna

VIII

dalga sayısı spektral domeninden uygun dönüşümlerle açısal spektrum domenine geçilip, bu domende integral yolunun uygun ötelenmesiyle Bessel ya da Hankel integral gösterilimleri elde edilip kapalı tam çözümlere ulaşılmıştır. LIM integralinin tam çözümleri verdiği analitik olarak gösterildikten sonra integralin sayısal olarak alınması için uygun bilgisayar modülleri anlatılmış ve hesaplama üç değişik yöntemle gerçekleştirilip gerek birbirleri ile gerekse tam çözümlerle karşılaştırılması yapılmıştır. İlk sayısal hesaplama LIM integralinin gerek orijinal yolu üzerinde gerekse en dik iniş yolu üzerinde alınması ve her iki sonucun da Bessel fonksiyonlarının tablo değerleriyle karşılaştırması olmuştur. Özellikle büyük argüman ve mertebeli Bessel fonksiyon değerlerini tablolarla bulmak olası olmadığından LIM integralini hesaplayan modüller (WLIM, FOR, WMODE, FOR) Bessel fonksiyonlarının hesabında rahatlıkla kullanılabilirler. Bu bölümlerde yapılan son sayısal çalışma integralin kesim bölgesinde asimtotik olarak hesaplanmasıdır.

İlk üç bölümde yöntem açıklanıp güvenilirliği kanonik yapılarda hem analitik hem de sayısal olarak gösterildikten sonra dördüncü bölümde gerek sualtı ses dalga ışıınımı gerekse troposfer tabakasında elektromagnetik dalga ışıını-mında görülen ve ortam parametrelerinin enine ve boyuna "daralan ışıınım kanalı" (narrowing duct) oluşturacak şekilde değişmesi problemi ele alınmıştır. LIM integrali analitik olarak oluşturulmuş ve kanonik test bölümlerinde uygulanan integrasyon teknikleriyle yerel özmodların gerek enine gerekse boyuna değişimleri kesim bölgesindeki davranışlarıyla beraber gösterilmiştir. Modların ayrıca üç boyutlu grafikleri de verilmiştir.

Beşinci bölümde ise ışıınım doğrultusu boyunca "genişleyen bilineer kanal" problemi incelenmiştir. "Kılavuzlanma-sızıntı" ve "Kılavuzlanma-ışınım" kritik bölgelerinin

IX

analitik ve sıyasal hesaplarının yapıldığı bu bölüm dördüncü bölüm gibi oldukça karmaşık ve zor problemler sınıfına girmektedir. LIM integrali (x,z) koordinat sisteminde modların kırılma indisinin enine bilineer boyuna lineer değişiminden ötürü belirli bir gözlem noktasına kadar tam kılavuzlandığını ve bu noktadan sonra boyuna doğrultuda kesime girip enine tabana sızdığını tam olarak verebilmektedir. Kılavuzlanma-sızıntı geçişi bilineer profille incelenmiştir. Burada profil kesitte belirli bir uzaklığa (kanal genişliği) kadar kılavuzlayıcı bu uzaklıktan sonra ise yayıcı özelliğe sahiptir. Yerel modların (AM) geçerli olduğu bölgede LIM integrali oluşturulduktan sonra geçiş bölgesinde bu çözümün birbiçim olduğu gösterilmiştir. Geçiş bölgesinin uzağında modlar kesitte üst sınır ile kostikleri arasında tuzaklanmaktadır. Tuzaklanmış modların kostik genişlikleri kanal genişliğinden oldukça küçüktürler. Sızıntıya uğramış modların ise kostik genişlikleri kanal genişliği mertebesinde-dir.

Kılavuzlanma-ışımaya geçişi yine LIM integralinin yerel modların geçerli olduğu bölgelerde oluşturulması ve birbiçim sonucun geçiş bölgesinde incelenmesiyle ele alınmıştır. Bu geçiş burada (x,y) sisteminde ele alınmıştır. Bu haliyle LIM integral çözümü kılavuzlanma-ışımaya geçişini açıklayamamaktadır. Bu geçiş bölgesinde tuzaklanmış modlar ışın demeti şeklinde ışımaya uğramaktadırlar. LIM integreli bu haliyle geçiş bölgesinin ötesine analitik olarak ötelendiğinde ve asimptotik olarak hesaplandığında ışıınının olduğu hissettirecek şekilde ışıınının bir ayağını yakalayabilmektedir. Ancak ışıını tam olarak gösterebilmek içineşfaz yüzeylerini yakalayacak uygun yerel küresel koordinat sistemi sistematik olarak bulunmalıdır.

Yöntemin kanonik ve kanonik olmayan yapılarda uygulanması, problemleri "zor problemler" sınıfına sokan içerdikleri

kritik noktalarda tek bir integral gösterilimden kılavuzlanmış, sızıntıya uğramış ve tabana ısımaya başlamış modların ayrı ayrı tam olarak açıklanabilmesinin kolaylığı sonuç bölümünde verilmiştir.



SECTION ONE

LOCAL INTRINSIC MODE THEORY [6-10]

1.1) Spectral synthesis of 1D. wave functions

The solution of one-dimensional wave equation,

$$\left\{ -\frac{d^2}{dz^2} + \gamma_m^2(z) \right\} v_m(z) = 0. \quad (1.1)$$

is traditionally given by WKB method which is not uniform when the observation point moves towards "turning points" (i.e., where $\gamma_m(z)$ vanishes). $\exp(-i\omega t)$ time dependence will be assumed and suppressed throughout. A uniform solution can be postulated as,

$$v_m(z) = \int_{\beta} a(\beta) \cdot e^{i\Omega(\beta)} \cdot e^{i\beta(z - \bar{z}(\beta))} \cdot d\beta \quad (1.2)$$

Here $\bar{z}(\beta)$ is the function which assigns a point in configuration space to a particular local wavenumber β . It is the continuation into complex β -plane of the inverse of $\gamma_m(\cdot)$. That is,

$$p = \gamma_m(q) \quad \rightarrow \quad q = \bar{z}(p) \quad (1.3)$$

Amplitude and phase functions $a(\beta)$, $\Omega(\beta)$ and the integration contour are not yet specified. It can easily be seen that eq.(1.2) expresses $v_m(z)$ as a Fourier integral with the fourier transform given by,

$$a(\beta) \cdot e^{i\Omega(\beta)} \cdot e^{-i\beta \bar{z}(\beta)}$$

Further insight is gained by formally changing the variable of integration in eq.(1.2),

$$v_m(z) = \int_{\bar{z}} a(\beta(\bar{z})) \frac{d\beta}{d\bar{z}} e^{i\Omega(\beta(\bar{z}))} e^{i\beta(\bar{z})(z-\bar{z})} d\bar{z} \quad (1.4)$$

from which it is seen that each point $\bar{z}$ along the contour of the integration contributes to the field a plane wave having wave number $\beta(\bar{z})$. The plane wave is normalized to one at $z = \bar{z}$ and is weighed by a factor of,

$$a(\beta(\bar{z})) \frac{d\beta}{d\bar{z}} e^{i\Omega(\beta(\bar{z}))}$$

The justification of eq.(1.2) then is, first of all, that any sufficiently well behaved function can be written as a Fourier integral. Secondly and more physically, where the WKB solution exists it corresponds to plane waves propagating with the local wave number $\beta = \mp \gamma_m(z)$. Thus the WKB solution must be related to the appropriate part of the spectral function $a(\beta) \cdot \exp(i\Omega(\beta))$, it must be possible to reconstruct the spectral function from the WKB solution. In fact this is what is done here in this formulation.

If the solution in eq.(1.2) is carried into eq.(1.1) then,

$$\int_{\beta} (\gamma_m^2(z) - \beta^2) a(\beta) e^{i\Omega(\beta)} e^{i\beta(z-\bar{z})} d\beta = 0 \quad (1.5)$$

is obtained. In general it is not possible to satisfy eq.(1.5) exactly. However if eq.(1.5) is evaluated asymptotically, then it is possible to choose $a(\beta)$ and $\Omega(\beta)$ such that the integrand is small near the saddle points, from which most of the contribution to the integral would normally come. Thus eq.(1.5) may be satisfied in an asymptotic sense.

The first step is to choose $\Omega(\beta)$ such that at the saddle points $\beta = \mp \beta_s$,

$$\gamma_m^2(z) - \beta_s^2 = 0 \quad ; \text{or} \quad z = \bar{z}(\beta_s) \quad (1.6)$$

The total phase of the integrand is,

$$P(z, \beta) = \Omega(\beta) + \beta(z - \bar{z}(\beta)) \quad (1.7)$$

Saddle points are found by equating the derivative of eq.(1.7) to zero.

$$\frac{d}{d\beta} P(z, \beta) = \frac{d}{d\beta} \Omega(\beta) + (z - \bar{z}(\beta)) - \beta \frac{d}{d\beta} \bar{z}(\beta) = 0 \quad (1.8)$$

So arbitrary phase function $\Omega(\beta)$ can be given as,

$$\Omega(\beta) = \int_0^\beta \hat{\beta} \frac{d}{d\hat{\beta}} \bar{z}(\hat{\beta}) d\hat{\beta} \quad (1.9)$$

which makes eq.(1.5) vanish to the leading order at the saddle point. Thus, the phase term in the asymptotic construction reads $P(z, \beta) = \int_0^\beta (z - \bar{z}(\beta)) d\beta$. In a similar way $a(\beta)$ may be chosen to enforce vanishing in the next order. To proceed eq.(1.5) is integrated by parts along the steepest descent path to obtain,

$$-\int_\beta \frac{d}{d\beta} \left\{ \frac{(\gamma_m^2(z) - \beta^2) a(\beta)}{i dP/d\beta} \right\} e^{iP(z, \beta)} d\beta \approx 0 \quad (1.10)$$

This requires,

$$\lim_{\beta \rightarrow \beta_s} \frac{d}{d\beta} \left\{ \frac{(\gamma_m^2(z) - \beta^2) a(\beta)}{i dP/d\beta} \right\} = 0 \quad (1.11)$$

In order to solve $a(\beta)$ from eq.(1.11) $q(\beta)$ and

$f(z, \beta)$ are defined as,

$$q(\beta) \triangleq \beta - \beta_s \quad ; \quad f(z, \beta) \triangleq \frac{\beta^2 - \beta_s^2}{\bar{z}(\beta) - z} \quad (1.12)$$

Using Taylor's expansion of $\bar{z}(\beta)$ around $\beta = \beta_s$ gives,

$$\bar{z}(\beta) = \bar{z}(\beta_s) + \frac{d}{d\beta_s} \bar{z}(\beta) (\beta - \beta_s) + \frac{d^2}{2d\beta_s^2} \bar{z}(\beta) (\beta - \beta_s)^2 \dots \quad (1.13)$$

$$\bar{z}(\beta) - z = q(\beta) \frac{d}{d\beta_s} \bar{z}(\beta) + \frac{1}{2} q^2(\beta) \frac{d^2}{d\beta_s^2} \bar{z}(\beta) \dots \quad (1.14)$$

where $d(\cdot)/d\beta_s$ etc. are used as a short notation for $d(\cdot)/d\beta$ at $\beta = \beta_s$. By combining eqs. (1.12) and (1.13) $f(z, \beta)$ can be rewritten as,

$$f(z, \beta) = \frac{q(\beta) + 2\beta_s}{\frac{d}{d\beta_s} \bar{z}(\beta) + \frac{1}{2} q(\beta) \frac{d^2}{d\beta_s^2} \bar{z}(\beta) + \dots} \quad (1.15)$$

At the saddle point $\beta = \beta_s$ $q(\beta) = 0$ and ,

$$f(z, \beta_s) = 2\beta_s \left(\frac{d}{d\beta_s} \bar{z}(\beta) \right)^{-1} \quad (1.16)$$

$$\frac{\partial}{\partial \beta_s} f(z, \beta) = \left\{ \frac{d\bar{z}(\beta)}{d\beta_s} \right\}^{-1} - \beta_s \left\{ \frac{d\bar{z}(\beta)}{d\beta_s} \right\}^{-2} \frac{d^2 \bar{z}(\beta)}{d\beta_s^2} \quad (1.17)$$

Using eqs. (1.16) and (1.17) in eq.(1.11) gives,

$$\left\{ \left\{ \frac{d\bar{z}(\beta)}{d\beta_s} \right\}^{-1} - \beta_s \left\{ \frac{d\bar{z}(\beta)}{d\beta_s} \right\}^{-2} \frac{d^2 \bar{z}(\beta)}{d\beta_s^2} \right\} a(\beta_s) +$$

$$2\beta_s \left\{ \frac{d\bar{z}(\beta)}{d\beta_s} \right\}^{-1} \frac{d}{d\beta_s} a(\beta) = 0 \quad (1.18)$$

A straightforward integration of eq.(1.18) results to ,

$$a(\beta_s) = \left\{ \frac{1}{\beta_s} \frac{d}{d\beta_s} \bar{z}(\beta) \right\}^{1/2} \quad (1.19)$$

Since the saddle point $\beta = \beta_s$ is arbitrary eq.(1.19) is valid for all β points, that is,

$$a(\beta) = \left\{ \frac{1}{\beta} \frac{d}{d\beta} \bar{z}(\beta) \right\}^{1/2} \quad (1.20)$$

As an example for the application of the spectral synthesis method to 1D. wave equation the Airy equation is taken,

$$\frac{d^2}{dz^2} V(z) - z V(z) = 0 \quad (1.21)$$

so that $\bar{z}(\beta) = -\beta^2$. Arbitrary phase and amplitude functions in (1.2) will then be,

$$\Omega(\beta) = \int_0^\beta \hat{\beta} (-2\hat{\beta}) d\hat{\beta} = -\frac{2}{3} \beta^3 \quad (1.22)$$

$$a(\beta) = \left\{ \frac{1}{\beta} (-2\beta) \right\}^{1/2} = (-2)^{1/2} \quad (1.23)$$

Then the 1D. wave function $V_m(z)$ will be,

$$V_m(z) = \int_{\beta} (-2)^{1/2} e^{-i2\beta^3/3} e^{i\beta(z+\beta^2)} d\beta \quad (1.24)$$

If the integration path is taken to lie along real axis,

$$V_m(z) = 2(-2)^{1/2} \int_0^\infty \cos\left(\frac{1}{3} t^3 + tz\right) dt \quad (1.25)$$

is obtained which is, apart from unimportant constant, identical with one of the standard integral representations of the Airy function $Ai(z)$. Since in this example the left hand side of the eq.(1.11) vanishes for all β values, asymptotic evaluation generates exact solution. This example is important because it is a prototype of a common situation where WKB method is messy to use, namely where $\gamma_m^2(z)$ vanishes at some point but $d\gamma_m^2(z)/dz$ does not. It is shown that the spectral synthesis solution has the correct behavior automatically at such points.

1.2) Spectral synthesis of 2D. wave functions

In 2D. case the solution of wave equation,

$$\left\{ \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} + k^2(x, z) \right\} W_m(x, z) = 0 \quad (1.26)$$

is postulated as,

$$W_m(x, z) = \int_{\beta} \tilde{W}_m(x, \beta) e^{i\beta(z - \bar{z}(\beta))} d\beta \quad (1.27)$$

If eq.(1.27) is substituted into eq.(1.26) ,

$$\int_{\beta} \left\{ \left(\frac{\partial^2}{\partial x^2} + k^2(x, z) - \beta^2 \right) \tilde{W}_m(x, \beta) \right\} e^{i\beta(z - \bar{z}(\beta))} d\beta = 0 \quad (1.28)$$

All done here is the transformation away the dependence of the field on z and replacement of it by dependence on the associated Fourier variable β . Assuming a slow longitudinal variation $\tilde{W}_m(x, \beta)$ can be chosen as,

$$\tilde{W}_m(x, \beta) = U_m(x, \bar{z}(\beta)) a(\beta) e^{i\Omega(\beta)} \quad (1.29)$$

where local normal mode function $U_m(x, \bar{z}(\beta))$ satisfies,

$$\left\{ \frac{\partial^2}{\partial x^2} + k^2(x, \bar{z}(\beta)) - \beta^2 \right\} U_m(x, \bar{z}(\beta)) = 0 \quad (1.30)$$

The notation of eq.(1.30) is chosen in anticipation of the fact that $a(\beta)$ and $\Omega(\beta)$ are exactly the same functions obtained in section (1.1). Although the choice of $\tilde{W}_m(x, \beta)$ causes the loss of generality it simplifies the problem and if the integral in eq.(1.27) is evaluated asymptotically x -dependence is contained only in a factor $\tilde{W}_m(x, \beta_s)$ for each saddle point.

SECTION TWO

WEDGE WITH NON-PENETRABLE BOUNDARIES

2.1) LIM formulation

In order to show that both LIM formulation and IM formulation give exact solutions when they exist, wedge with perfectly reflecting boundaries in figure(2.1) is taken into consideration since its exact solution is known. In fig.(2.1) polar coordinates are ρ and η , wedge angle is α , $H(\rho)$ is the arclength of AB at a distance ρ from the origin. Refractive index n is taken one for simplicity. Then two-dimensional scalar acoustic field function $W_q(\rho, \eta)$ satisfies the source-free wave equation,

$$\left\{ \frac{1}{\rho} \frac{\partial}{\partial \rho} \rho \frac{\partial}{\partial \rho} + \frac{1}{\rho^2} \frac{\partial^2}{\partial \eta^2} + k^2 \right\} W_q(\rho, \eta) = 0 \quad (2.1)$$

where k is the wavenumber. Acoustic pressure field function also satisfies the boundary conditions,

$$W_q(\rho, \eta) \Big|_{\eta=0, \alpha} = 0 \quad (2.2)$$

According to LIM postulation $W_q(\rho, \eta)$ function can be given as

$$W_q(\rho, \eta) = \int_{C_\beta} a(\beta) e^{i\Omega(\beta)} U_q(\bar{\rho}(\beta), \eta) e^{i\beta(\rho - \bar{\rho}(\beta))} d\beta \quad (2.3)$$

Here q is the mode index, $U_q(\bar{\rho}(\beta), \eta)$ is the local normal mode function in angular domain which satisfies,

$$i) \left\{ \frac{1}{\rho^2} \frac{\partial^2}{\partial \eta^2} + k_\eta^2 \right\} U_q(\bar{\rho}(\beta), \eta) = 0 \quad (2.4)$$

$$ii) \quad U_q(\bar{\rho}(\beta), \eta) \Big|_{\eta=0, \alpha} = 0 \quad (2.5)$$

where k_η is the angular component of the wavenumber, β is the radial component of the wavenumber, $\bar{\rho}(\beta)$ is the arbitrary phase reference point which is the inverse of $\beta = \beta(\rho)$. In fact wedge with non-penetrable boundaries is a separable problem in a plane-polar coordinate system. This means the solutions in angular domain and radial domain are independent from each other. But the idea of LIM method is to build a spectral integral with localized normal modes. In order to talk about local modes they should be longitudinally dependent. So here a new parameter $\bar{\eta} = \rho\eta$ is introduced which localizes the problem. If eq.(2.4) is thought in terms of this new local parameter it is easily seen that it is nothing but the angular part of separable wave eq.(2.1). Radial and angular components of wavenumber satisfy dispersion relation,

$$k^2 = k_\eta^2 + \beta^2 \quad (2.6)$$

Since $k = \frac{w}{v}$, where w is the radian frequency, v is the velocity of the sound in the wedge, the radial component k_η can be found by using the transverse resonance condition,

$$2 k_\eta H(\rho) - \Pi - \Pi = q' 2\Pi \quad (q' = 0, 1, 2, 3, \dots) \quad (2.7)$$

as,

$$k_\eta = \frac{q \Pi}{H(\rho)} \quad , \quad (q = 1, 2, 3, \dots) \quad (2.8)$$

This results the radial component β be,

$$\beta(\rho) = \pm \left\{ k^2 - \left(\frac{\nu}{\rho} \right)^2 \right\}^{1/2} ; \quad \nu = \frac{q \Pi}{\alpha} \quad (2.9)$$

Here ν is a modal constant and $H(\rho)$ is replaced by its value $(\rho\alpha)$. Then arbitrary phase reference point $\bar{\rho}(\beta)$ can easily be written as,

$$\bar{\rho}(\beta) = \nu (k^2 - \beta^2)^{-1/2} \quad (2.10)$$

Then the other two arbitrary amplitude and phase functions will be written via (1.19) and (1.9) ,

$$a(\beta) = \nu^{1/2} (k^2 - \beta^2)^{-3/4} \quad (2.11)$$

$$\Omega(\beta) = \frac{\nu \cdot \beta}{(k^2 - \beta^2)^{1/2}} - \nu \sin^{-1}\left(\frac{\beta}{k}\right) \quad (2.12)$$

The local normal mode function $U_q(\bar{\rho}(\beta), \eta)$ satisfying eqs.(2.4-5) can be given as,

$$U_q(\bar{\rho}(\beta), \eta) = C_0 \cdot \sin(\nu\eta) \quad (2.13)$$

where C_0 is the normalization constant evaluated from,

$$C_0 = \left\{ \int_0^{H(\rho)} \sin^2(\nu\eta) d\eta \right\}^{-1/2} \quad (2.14)$$

as,

$$C_0 = \left(\frac{2}{\alpha \cdot \bar{\rho}} \right)^{1/2} = \left(\frac{2}{\alpha \nu} \right)^{1/2} (k^2 - \beta^2)^{1/4} \quad (2.15)$$

Therefore LIM solution in eq.(2.3) can be rewritten as,

$$W_q(\rho, \eta) = \left(\frac{2}{\alpha} \right)^{1/2} \sin(\nu\eta) \int_{C_\beta} \frac{e^{iP(\beta, \rho)}}{(k^2 - \beta^2)^{1/2}} d\beta \quad (2.16)$$

where $P(\beta, \rho)$ is the total phase function given by ,

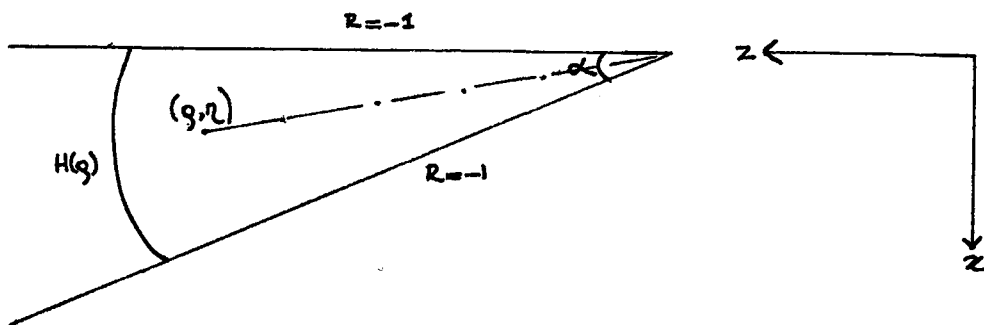


FIG.(2.1) : Wedge with perfectly reflecting boundaries in plane polar coordinates

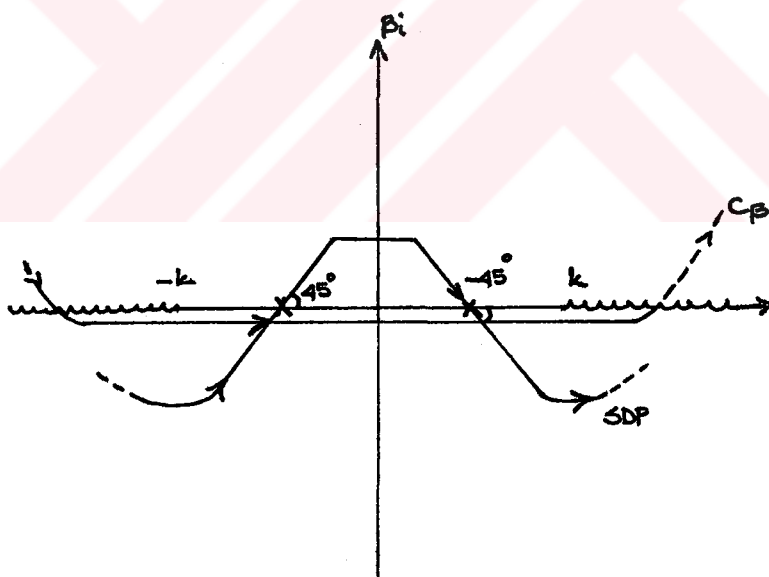


FIG.(2.2) : Branch cuts, integration contour and the steepest descent path in β -plane

$$P(\beta, \rho) = \int_0^\beta (\rho - \bar{\rho}(\beta)) d\beta = \beta \cdot \rho - \nu \cdot \sin^{-1}\left(-\frac{\beta}{k}\right) \quad (2.17)$$

The integration contour C_β and the branch cuts are shown in fig(2.2). It's well known from the exact solution of non-penetrable wedge that transverse dependence of the field is a pure sine or cosine function which is independent from radial coordinate ρ . Here in the postulation of LIM formulation first radial dependence of transverse field component is introduced but since the boundary conditions in LIM are to be satisfied at the saddle points of the integral representation this radial dependence automatically drops out as seen from eq.(2.16).

2.2) Reduction to IM and EXACT solution

Since Intrinsic Mode (IM) formulation is based on other spectral components (i.e., plane waves) in θ domain where θ is the incident angle of the plane wave to the lower boundary [6,9], the new parametrization of the structure is given in fig.(2.1). IM and also the EXACT solution in terms of Bessel and/or Hankel functions can be obtained from LIM solution in eq(2.16). In order to do that new variables are defined as,

$$\eta \triangleq \alpha - \xi \quad ; \quad \beta \triangleq k \cdot \cos\phi \quad (2.18)$$

where $\phi = \theta - \xi$. With this transformation C_β in complex β -plane maps to C_ϕ in complex ϕ -plane. Branch points at $\beta = \pm k$ become regular point on ϕ plane. Two contours are given in fig(2.3). Using $d\beta = -k \cdot \sin\phi d\phi$ and $k \cdot \sin\phi = \nu/\rho$ eq.(2.16) can reduce to,

$$W_q(\rho, \xi) = - \left(\frac{2}{\alpha}\right)^{1/2} e^{-i\nu\Pi/2} \sin(\nu\xi) \int_C e^{ik\rho\cos\phi - i\nu\phi} d\phi \quad (2.19)$$

Eq.(2.19) is the IM result obtained for non-penetrable wedge. There can be two different cases depending on the respective locations of the source and the observation points in the wedge.

- i) Downslope propagation
- ii) Upslope propagation

In the first case because of the radiation condition there are only outgoing waves (travelling waves) but in the second case where the source radial distance ρ' is greater than observation point ρ (i.e., $\rho' > \rho$), there are both positive and negative directed waves which are represented by standing waves. The exact closed-form solutions for upslope and downslope propagation (with radiation condition) are given as

$$W_q(\rho, \eta) = \begin{cases} \left(\frac{2}{\alpha}\right)^{1/2} \sin(\nu\eta) J_\nu(k\rho) & \text{upslope} \\ \left(\frac{2}{\alpha}\right)^{1/2} \sin(\nu\eta) \mathcal{H}_\nu^{(1)}(k\rho) & \text{downslope} \end{cases} \quad (2.20)$$

It is seen from fig.(2.3) that the integration contour passes both Riemann sheets in both β and ϕ -planes. Although the branch lines are crossed in β -plane there are no branch point contributions because these are not spectral branch lines. This is confirmed in ϕ -plane that $\beta = k \cos \phi$ transformation removes these branch points.

It is known that the Bessel function can be given as

$$J_\nu(k\rho) = \frac{1}{2\pi i} \int_{\omega-i\pi}^{\omega+i\pi} \exp \left[k\rho \sinh w - \nu w \right] dw \quad (2.21)$$

If the integration contour in ϕ -plane is shifted by $-\pi/2$ via $\phi \rightarrow \phi - \pi/2$ transformation and then turned $-\pi/2$ via $\phi \rightarrow -i\phi$ transformation the LIM integral reduces to

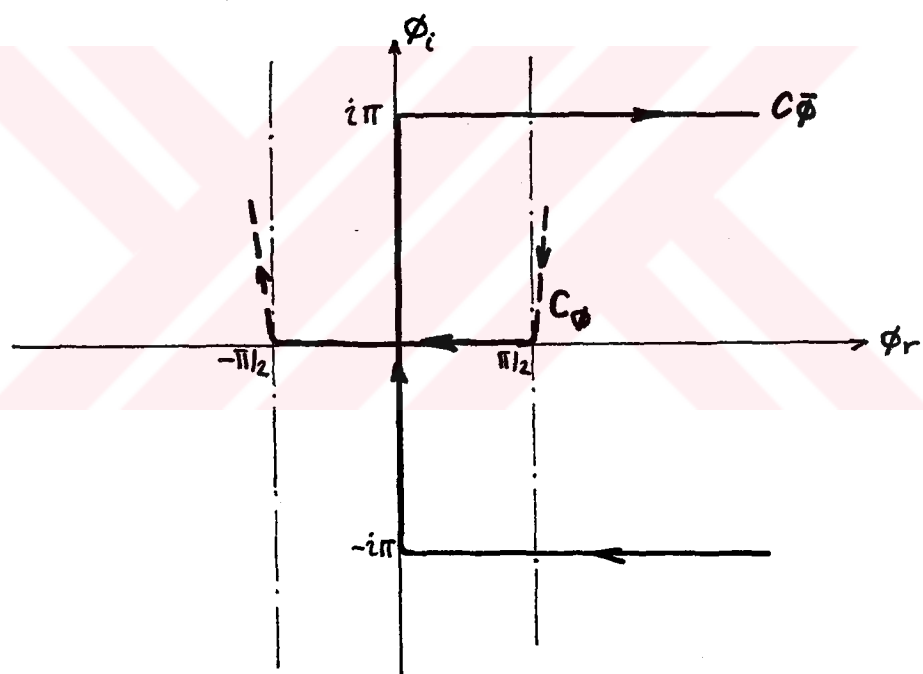
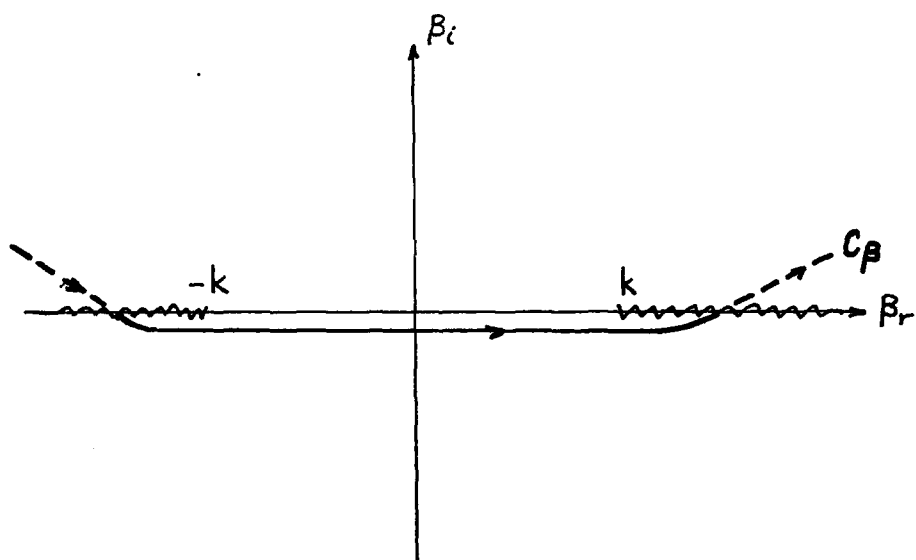


FIG.(2.3) : Conformal mapping and contour deformation

$$W_q(\rho, \eta) = i \left(\frac{2}{\alpha} \right)^{1/2} \sin(\nu \eta) \int_{\omega - i\pi}^{\omega + i\pi} \exp[k\rho \sin\phi - \nu\phi] \quad (2.22a)$$

$$W_q(\rho, \eta) = \left(\frac{2}{\alpha} \right)^{1/2} 2\pi \sin(\nu \eta) J_\nu(k\rho) \quad (2.22b)$$

As a result - besides 2π term - eq.(2.22) is the exact solution of the problem. The 2π term arises from LIM postulation since it is a Fourier-like integral representation.

2.3) Numerical Test Results[*]

It is shown in the previous section that the LIM integral in eq.(2.16) can be reduced to the exact solution given in eq.(2.20). In order to do this reduction the value at the saddle point of the arbitrary amplitude function is taken into account. Here some numerical tests are done to show the results of the LIM Method. The flow chart of the numerical computation routines is shown in fig.(2.4). Three different modules are made to do the following numerical calculations :

WLIM.FOR

This program computes the complex LIM integral given in eq.(2.3) on the integration contour shown in fig.(2.2). The main module of the program is the complex function CTABINT(X,Y,NP) which is used for numerical integration of a tabulated complex function using a Taylor's expansion of the first $(3N+1)$ points followed by a trapezoidal rule integration of the remaining $NP-(3N+1)$ points. Here X is the vector of length NP containing the values of the independent variable (stored in consecutively increasing or decreasing order) and Y is the vector of length NP containing the values of the dependent variable (ordered to correspond to the

independent variable) and NP is the number of the provided points. The program WLIM.FOR can compute both radial and angular dependencies. The number of sampling points of the function NP is determined numerically. As the number of samples increases the result of the numerical integration converges to its exact value. Here NP is taken as the number of sampling points which renders the difference between two consecutive calculation to less than a predetermined value.

WMODE.FOR

This program has a similar structure with WLIM.FOR and computes the same integral on a different integration contour that is along the Steepest Descent Path (see section 2.3.2). This SDP contour is found as it is in fig.(2.2). The necessary sampling frequency is determined by carrying the same numerical test as above. Depending on the wavenumber k (i.e., the frequency used) it is possible either to consider only a small path interval on both sides of the two saddle points or it may be necessary to extend the integration on the contour parts located on the top half plane.

WASYM.FOR

This is the third program used in this section. This program computes the asymptotic form of the same integral through cutoff where the two saddle points come closer and coincide at the origin. Since the transition through cutoff is represented with Airy functions this module contains the subroutine AIRYCA(ZZ,W,WP,N,M) which computes Airy functions and their derivatives everywhere in the complex plane. For arguments with absolute value less than 6, series representation of the Airy function is taken into account and for greater arguments asymptotic representations are utilized.

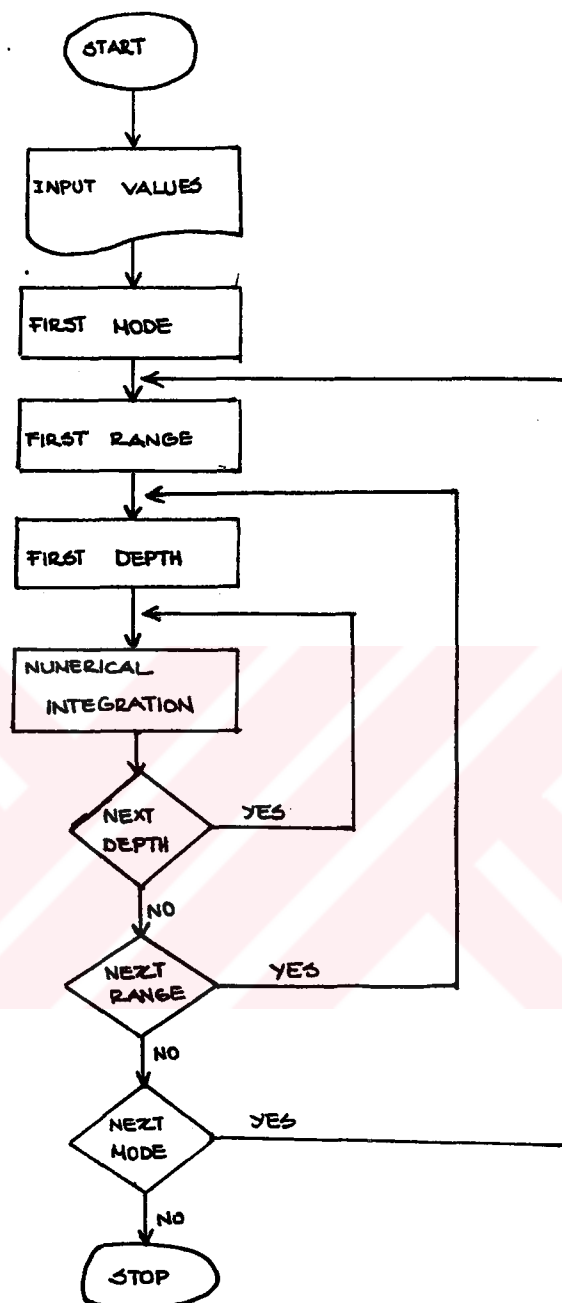


FIG.(2.4) : The flow chart of the fortran programs used in numerical evaluations

2.3.1) Direct Numerical Integration

The first numerical test is the direct numerical integration of the LIM integral given in eq.(2.16). This is a typical complex integral on a path shown in fig.(2.2). The Fortran Program used for the numerical evaluation of the LIM integral in eq.(2.16) is WLIM.FOR. This program computes angular variations by keeping radial range constant and radial variations by keeping angular depth constant. The exact solution of the problem with these boundary conditions is the combination of a SINE function and BESSEL function (for upslope propagation). Various numerical tests show that the contribution from the contour part on the real axis between $(-k, +k)$ dominates the result, therefore the contribution from the other contour parts can be neglected. The angular profile for various radial range points are given in fig.(2.5). The results show the comparison of (ρ, η) and (x, z) parametrized LIM integral computations. The solid line shows the (ρ, η) parametrized results which are the reference ones since the problem is separable in this system. One hundred equally spaced points are taken to plot these figures. The computation time is less than two minutes on IBM compatible 12 MHz AT with a 80287 math co-processor. The radial dependence of the first LIM mode for various angular depths are shown in fig.(2.6). The frequency, the speed of the sound waves and the wedge angle are taken as 25 Hz, 1500 m/sec. and 2.86° in numerical test computations. With these values the cut-off range for the first mode is nearly 600 m. from the apex. For upslope propagation the BESSEL function variation can easily be seen from the figures. It has an oscillatory character before cut-off and exponentially decaying character after cut-off.

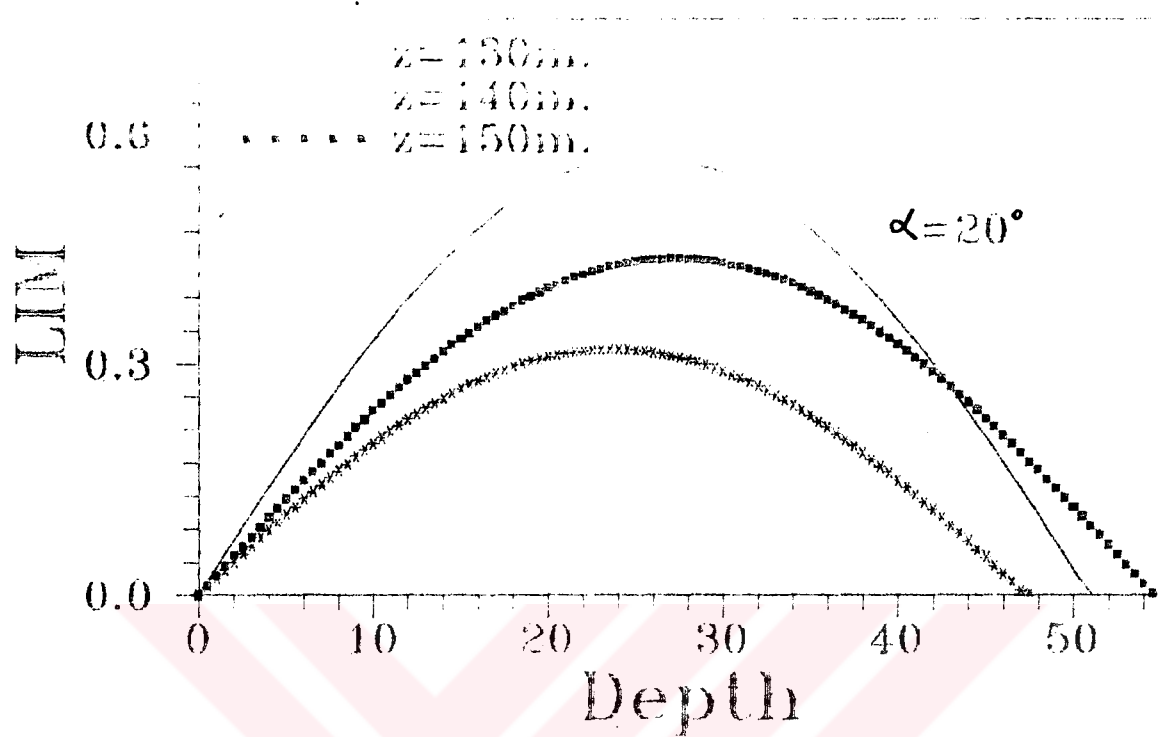


FIG.(2.5) : Actual transverse variation of the LIM_1 for different ranges

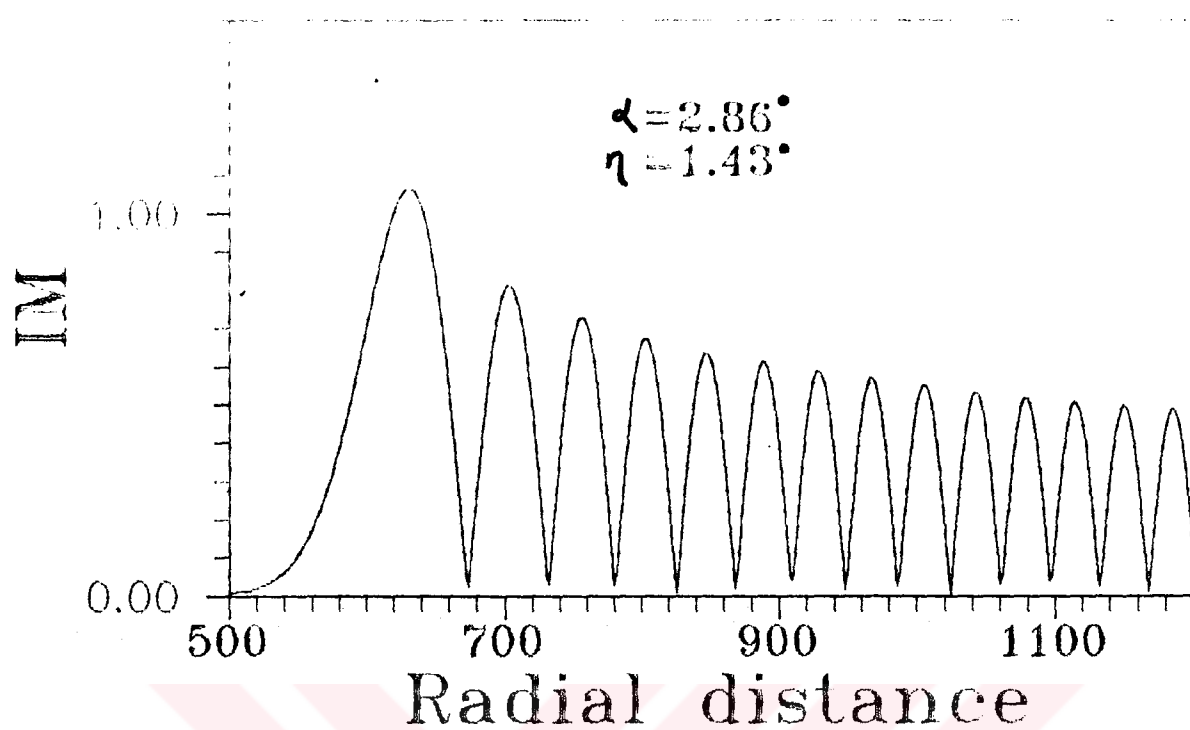


FIG.(2.6) : Radial distance variation of first LIM for certain angular depth

2.3.2) Steepest Descent Path (SDP) Evaluation

Although it is not very time consuming to compute the LIM integral directly as is done in section (2.3.1), this is not an efficient way of numerical computation in general, especially for highly oscillatory phases and for higher frequencies. For this reason the second Fortran Program WMODE.FOR is made to compute the same LIM integral on the Steepest Descent Path (SDP). Fig.(2.2) show the location of the saddle points and the SDP contours for both ranges before and after cut-off. Since ,

$$\beta_s = \pm \left\{ k^2 - \left(\frac{\nu}{\rho} \right)^2 \right\}^{1/2} \quad (2.24)$$

and,

$$P''(\rho, \beta_s) = \frac{\nu \beta_s}{(k^2 - \beta_s^2)^{3/2}} \quad (2.25)$$

$$\arg \left\{ P''(\rho, \beta_s) \right\} = \begin{cases} 0 & \text{if } \beta_s = |\beta_s| \\ \pi & \text{if } \beta_s = -|\beta_s| \end{cases} \quad (2.26)$$

SDP makes a -45° for the negative saddle point and a 45° for the positive saddle point. It is a straight line parallel to the real axis after cut-off.

The fortran program WMODE.FOR computes the same angular and radial dependencies of the LIM modes. The radial dependence of the first LIM mode is shown in fig.(2.7). It is seen from the figure that the SDP evaluation blows up before the two real saddle points coincide at the origin. This means simple 45° SDP evaluation is not valid for computations through cut-off. But for ranges before cut-off the two solutions (direct numerical integration and the SDP evaluation) agree very well. This agreement is given in fig.(2.8).

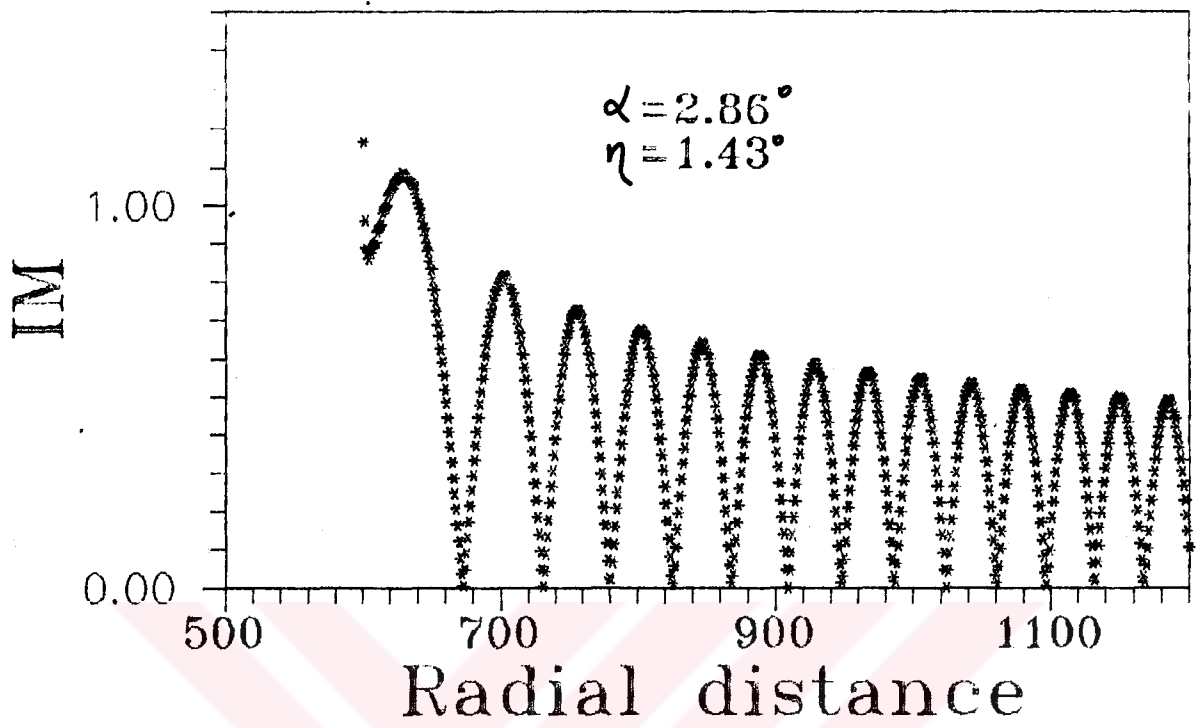


FIG.(2.7) : Radial distance variation of first LIM computed with SDP method

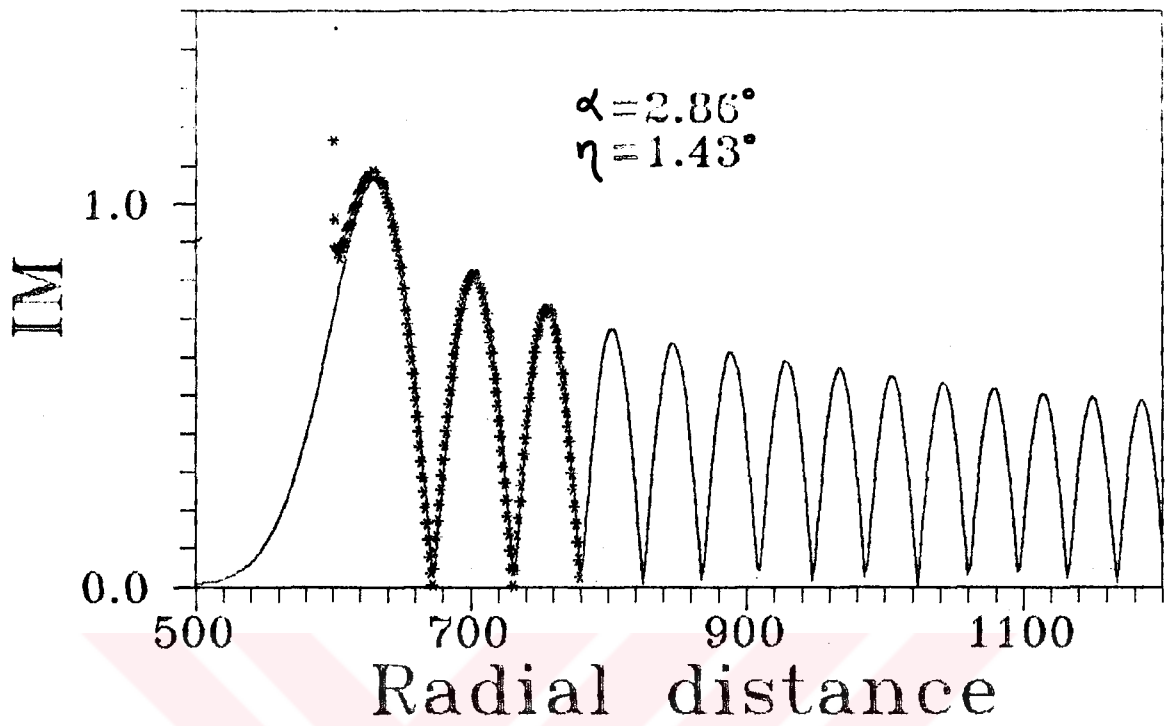


FIG.(2.8) : Comparison of the results of SDP
and direct numerical integration

2.3.3) Asymptotic Evaluation Through Cut-Off

As it is shown in section (2.3.2) the SDP evaluation blows up before cut-off. This is because the simple ($\pm 45^\circ$) steepest descent paths are valid for isolated saddle points and are not good enough when the two saddle points come closer and coincide. In order to overcome this deficiency a uniform asymptotic form is used through cut-off. Any integral in the form of,

$$I = \int_{C_\beta} f(\beta) e^{iq(\beta)} d\beta \quad (2.27)$$

having two nearby saddle points can be replaced[12] with,

$$I \approx 2\pi \left\{ \frac{2}{q^{(3)}(\beta_0)} \right\}^{1/3} f(\beta_0) \text{Ai}(x) e^{i(q(\beta_1)+q(\beta_2))/2} \quad (2.28)$$

near the caustic and,

$$I \approx 2\pi \left\{ \frac{2}{q^{(3)}(\beta_0)} \right\}^{1/3} f(\beta_0) \text{Ai}(0) e^{iq(\beta_0)} \quad (2.29)$$

at the caustic. Here β_1 and β_2 are the two saddle points, β_0 is the second order saddle point at the caustic, $q^{(3)}(\beta)$ is the third derivative of the phase term with respect to β and,

$$x \approx - \left\{ \frac{1}{2} q^{(3)}(\beta_0) \right\}^{2/3} \left\{ \frac{\beta_1 - \beta_2}{2} \right\}^2 \quad (2.30)$$

In the non-penetrable wedge case,

$$P^{(3)}(\rho, \beta) = \frac{-\nu}{(k^2 - \beta^2)^{3/2}} + \frac{-\nu\beta^2}{(k^2 - \beta^2)^{5/2}} \quad (2.31)$$

Since $\beta = \beta_0 = 0$ at the caustic,

$$P^{(3)}(\rho, \beta_0) = \frac{-\nu}{k^3} \quad (2.32)$$

and,

$$x \approx - \left\{ \frac{-\nu}{2k^3} \right\}^{2/3} \left\{ \frac{\beta_1 - \beta_2}{2} \right\}^2 \quad (2.33)$$

As a result the LIM integral in eq.(2.16) can be replaced with,

$$W_q(\rho, \eta) \approx - (2)^{5/6} (\nu)^{1/6} \left\{ \frac{1}{k\rho\alpha} \right\}^{1/2} \sin(\nu\eta) \text{Ai}(x) \\ \exp \left\{ i(P(\rho, \beta_1) + P(\rho, \beta_1)/2) \right\} \quad (2.34)$$

before the caustic and,

$$W_q(\rho, \eta) \approx - (2)^{5/6} (\nu)^{1/6} \left\{ \frac{1}{k\rho\alpha} \right\}^{1/2} \sin(\nu\eta) \text{Ai}(0) \quad (2.35)$$

at and beyond the caustic. In fact the exponent term in eq.(2.34) vanishes as in eq.(2.35) because of the symmetry between the saddle points.

Here the third Fortran program WASYM.FOR is used to evaluate eqs.(2.34) and (2.35). Fig.(2.9) show the comparison of direct numerical integration and SDP evaluation with the asymptotic form results. It is seen from these figures that this asymptotic form can be used for evaluations through cut-off. The range for the application of the SDP and Asymptotic Evaluation is quantitative and should be checked for each problem and also for each set of parameters.

Finally figs.(2.10) and (2.11) are plotted to show the 3D. plots of the first LIM mode through cut-off computed directly along the original integration contour. Figures show the same output for different angle and tilt, indicating the oscillatory and evanescent sides of the cut-off.

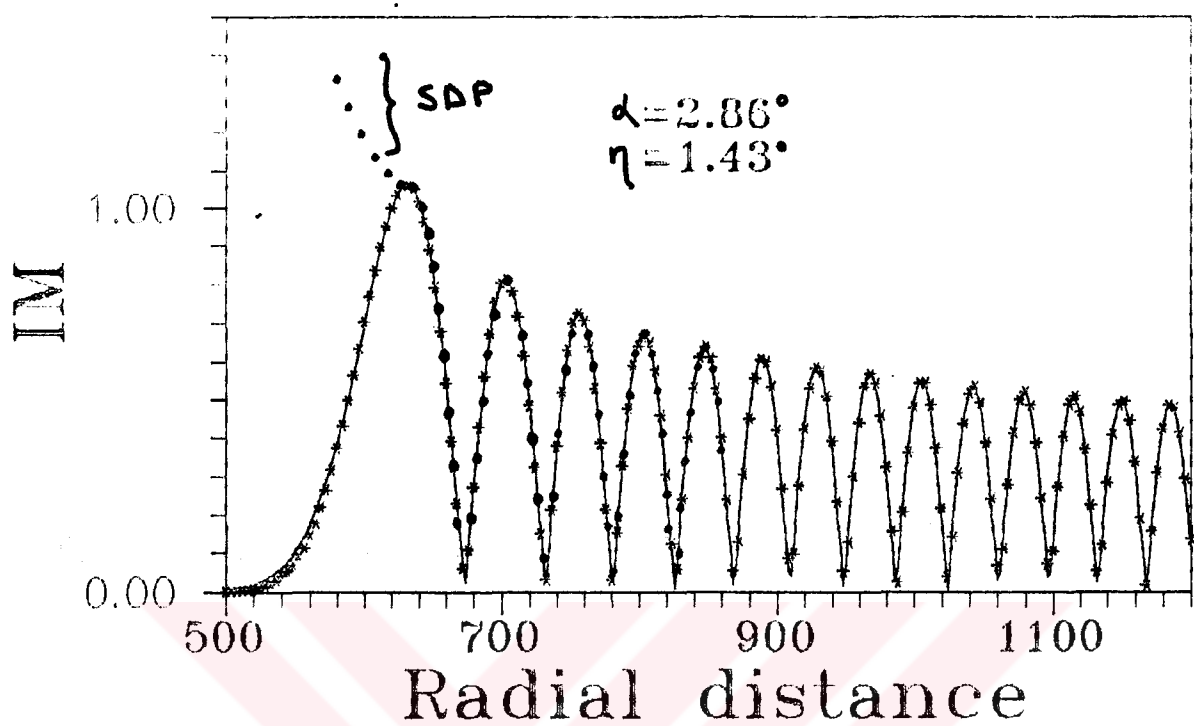


FIG.(2.9) : Comparison of the results of direct, SDP and uniform asymptotic eva.

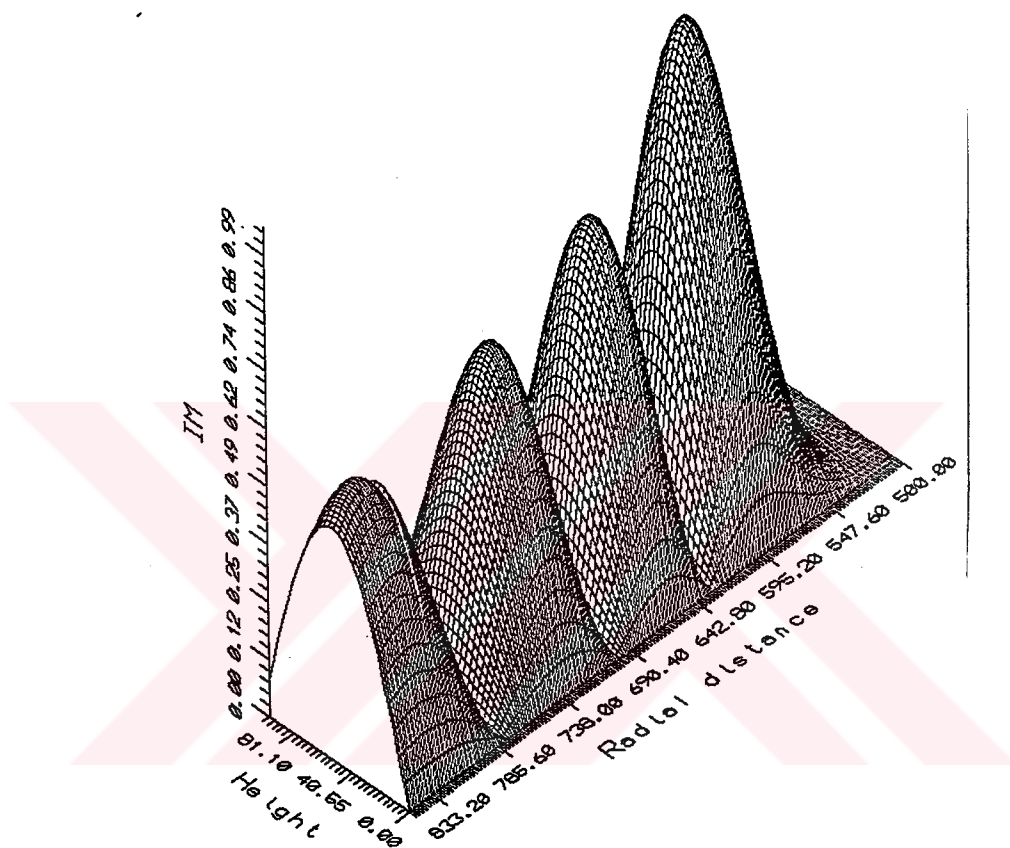


FIG.(2.10) : 3D.Plot of the first mode seen from the trapping side

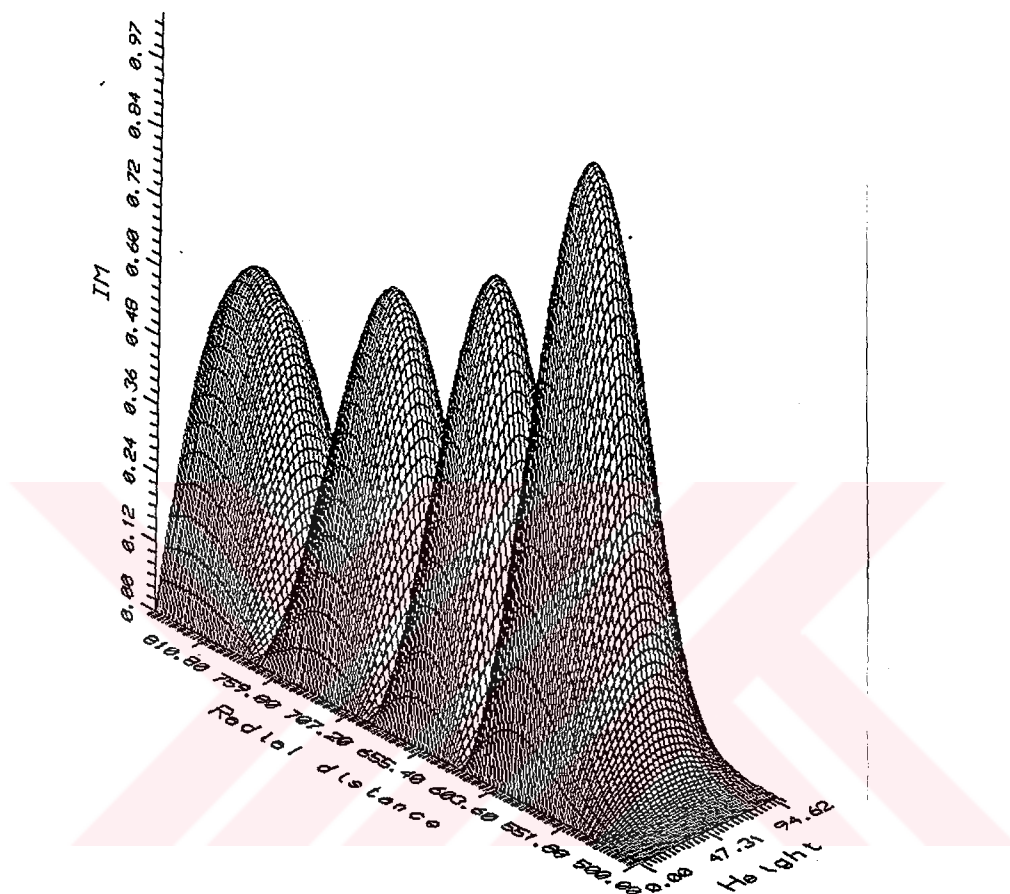


FIG.(2.11) : 3D.Plot of the first mode seen from the cut-off side

SECTION THREE

WAVE PROPAGATION IN A GRADED-INDEX TAPER[11]

3.1) Formulation, Solution And Reduction To Closed Form

Acoustic pressure field function $\psi(\rho, \eta)$ in underwater taper satisfies the wave equation in plane-polar coordinates given by,

$$\left\{ \frac{1}{\rho} \frac{\partial}{\partial \rho} \rho \frac{\partial}{\partial \rho} + \frac{1}{\rho^2} \frac{\partial^2}{\partial \eta^2} + k^2(\rho, \eta) \right\} \psi(\rho, \eta) = 0 \quad (3.1)$$

where the graded-index taper is defined with,

$$k^2(\rho, \eta) = k_0^2 \left\{ 1 - \frac{2\Delta G(\eta)}{\rho^2} \right\} \quad (3.2)$$

Here $G(\eta)$ is a 2π - periodic function of η and 2Δ is a constant with a dimension of length square. With the help of the arbitrary function $G(\eta)$ a large number of graded-index tapers can be modeled via eq.(3.2). The particular choice that is of most interest to us is,

$$G(\eta) = \frac{\sin^2(\eta)}{\cos^4(\eta)} \quad (3.3)$$

For small η angles $G(\eta) \approx \eta^2$ and eq.(3.2) becomes,

$$k^2(\rho, \eta) = k_0^2 - \frac{\alpha^2 \eta^2}{\rho^2} \quad (3.4)$$

where $\alpha = k_0 \sqrt{2\Delta}$. There are no physical boundaries in transverse domain the only boundary condition is the radiation condition when $\eta \rightarrow \pm\pi/2$ while $\rho \rightarrow \infty$. The structure and the plane -polar coordinates are shown in fig.(3.1). The solution of the problem will be found by applying LIM Theory. Since,

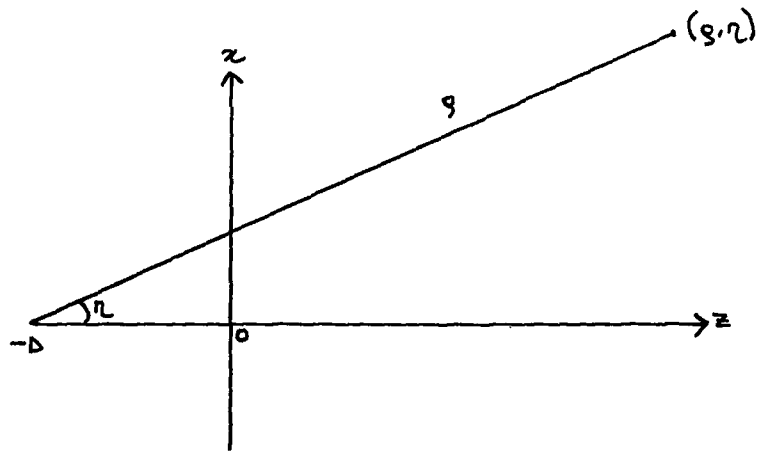


FIG.(3.1) : Graded index taper with the symmetry axis z on plane polar coordinates

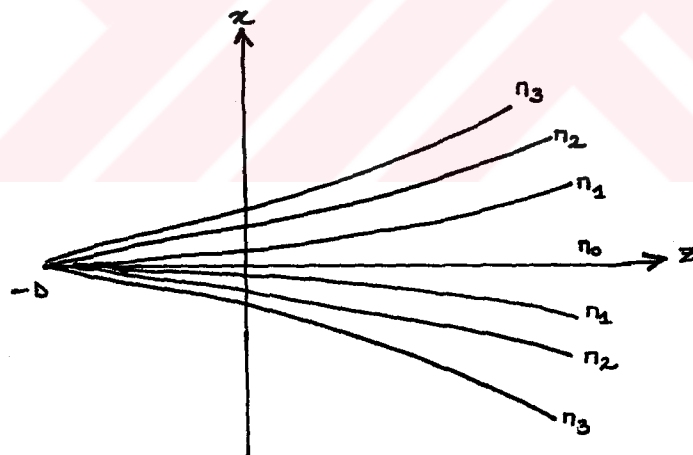


FIG.(3.2) : Constant velocity contours with the max. contour n_0 on the z -axis

$$x \triangleq \rho \sin(\eta) \quad : \quad z \triangleq \rho \cos(\eta) - D \quad (3.5)$$

the index distribution can be written in cartesian coordinates as,

$$k^2(x, z) = k_0^2 \left\{ 1 - \frac{2\Delta x^2}{(D+z)^4} \right\} \quad (3.6)$$

Constant refractive-index (sound speed) contours are in fig.(3.2). According to LIM Theory the solution of eq.(3.1) with k^2 given in eq.(3.4) can be given as,

$$\psi_m(\rho, \eta) = \int_{C_\beta} a(\beta) U_m(\bar{\rho}(\beta), \eta) e^{iP(\rho, \beta)} d\beta \quad (3.7)$$

where,

$$P(\rho, \beta) = \Omega(\beta) + \beta [\rho - \bar{\rho}(\beta)] \quad (3.8)$$

The main idea of postulation (3.7) is to construct the solution as a spectral integral with the help of the local normal mode function $U_m(\bar{\rho}(\beta), \eta)$. In this structure, modes are confined between their caustics on each side of the taper axis (i.e., $x = 0$ plane). By applying transverse resonance condition locally at a distance ρ from the origin and taking into account that each caustic causes a phase shift of $(-\pi/2)$ [12,p502],

$$2 \int_{-\xi_c}^{\xi_c} \left\{ k^2(\rho, \xi) - \beta^2 \right\}^{1/2} d\xi = (2m + 1) \pi \quad (3.9)$$

and after simplification,

$$\int_{-\xi_c}^{\xi_c} \left[\xi_c^2 - \xi^2 \right]^{1/2} d\xi = -\frac{\rho^2}{2\alpha} (2m+1) \pi \quad (3.9b)$$

can be written. Here $\xi \triangleq \rho \eta$ is introduced to localize the

transverse resonance condition and ξ_c is the local caustic distance which is equal to $\rho\eta_c$, and β is the radial component of the wavenumber,

$$\beta(\rho) = k(\rho, \eta) \big|_{\eta=\eta_c} \quad (3.10)$$

The solution of the eq.(3.9) gives the caustic angle η_c as,

$$\eta_c = \left\{ \frac{1}{\alpha} (2m + 1) \right\}^{1/2} \quad (3.11)$$

So the radial component of the wavenumber β will be,

$$\beta(\rho) = \pm k_0 \left\{ 1 - \frac{\alpha(2m+1)}{k_0^2 \rho^2} \right\}^{1/2} \quad (3.12)$$

which gives the arbitrary phase reference function $\bar{\rho}(\beta)$ as,

$$\bar{\rho}(\beta) = (\alpha(2m + 1))^{1/2} \cdot (k_0^2 - \beta^2)^{-1/2} \quad (3.13)$$

Local normal mode function $U_m(\bar{\rho}(\beta), \eta)$ satisfies,

$$\left\{ \frac{1}{\rho^2} \frac{d^2}{d\eta^2} + k^2(\bar{\rho}(\beta), \eta) - \beta^2 \right\} U_m(\bar{\rho}(\beta), \eta) = 0 \quad (3.14)$$

which is also written as,

$$\left\{ \frac{d^2}{d\eta^2} + \alpha(2m+1) - \alpha^2 \eta^2 \right\} U_m(\eta) = 0 \quad (3.15)$$

The solution of eq.(3.15) is well-known as Hermite Polynomials,

$$U_m(\eta) = C_0 e^{-\alpha\eta^2/2} \cdot H_m(\sqrt{\alpha}\eta) \quad (m = 0, 1, 2, \dots) \quad (3.16)$$

The normalization coefficient C_0 can be found from orthonormality condition. It is easy to show that Hermite Polynomials are orthogonal with the weight function $\exp(-x^2)$ in the interval $(-\infty, \infty)$,

$$\int_{-\infty}^{\infty} e^{-x^2} H_m(x) H_n(x) dx = \begin{cases} 0 & m \neq n \\ 2^m m! \sqrt{\pi} & m = n \end{cases} \quad (3.17)$$

When written for the local coordinate ξ , the normalization coefficients reads

$$C_0 = (\rho 2^m m! \sqrt{\pi})^{-1/2} \quad (3.18)$$

Two arbitrary functions $a(\beta)$ and $\Omega(\beta)$ will then be obtained as

$$\begin{aligned} a(\beta) &= \left\{ \frac{1}{\beta} \frac{\partial}{\partial \beta} \bar{\rho}(\beta) \right\}^{1/2} \\ &= [\alpha(2m+1)]^{1/2} \cdot (k_0^2 - \beta^2)^{-3/4} \end{aligned} \quad (3.19)$$

$$\begin{aligned} \Omega(\beta) &= \int_0^\beta \beta \frac{\partial}{\partial \beta} \bar{\rho}(\beta) d\beta \\ &= [\alpha(2m+1)]^{1/2} \left\{ \frac{-\beta}{(k_0^2 - \beta^2)^{1/2}} + \sin^{-1}\left(-\frac{\beta}{k_0}\right) \right\} \end{aligned} \quad (3.20)$$

Then the total phase term will be,

$$\begin{aligned} P(\rho, \beta) &= [\Omega(\beta) + \beta(\rho - \bar{\rho}(\beta))] = \int_0^\beta (\rho - \bar{\rho}(\beta)) d\beta \\ &= \beta\rho + [\alpha(2m+1)]^{1/2} \sin^{-1}\left(-\frac{\beta}{k_0}\right) \end{aligned} \quad (3.21)$$

Finally LIM solution can be written as,

$$\psi(\rho, \eta) = B_0(\eta) \int_{C_\beta} \frac{\exp\{i[\beta\rho + (\alpha(2m+1))^{1/2} \sin^{-1}(\frac{\beta}{k_0})]\}}{(k_0^2 - \beta^2)^{3/4}} d\beta \quad (3.22)$$

where,

$$B_0(\eta) = C_0 [\alpha(2m+1)]^{-1/2} e^{-\alpha\eta^2/2} H_m(\sqrt{\alpha}\eta) \quad (3.23)$$

The integration contour in eq.(3.22) is given in fig.(3.3). together with the branch cuts running from $(-\infty, k_0)$ and (k_0, ∞) . In order to get rid of these branch cuts it is convenient to map the integral onto a new complex plane λ with,

$$\beta \triangleq k_0 \cos(\lambda) \quad : \quad \nu \triangleq \left\{ \alpha(2m+1) \right\}^{1/2} \quad (3.24)$$

In this case ,

$$d\beta = -k_0 \sin(\lambda) d\lambda = -\left(\frac{\nu}{\rho}\right) d\lambda \quad (3.25)$$

Therefore in complex λ domain LIM solution will be,

$$\psi(\rho, \eta) = -\bar{B}_0(\eta) e^{-\nu\Pi/2} \int_{C_\lambda} e^{ik_0\rho\cos(\lambda)-i\nu\lambda} d\lambda \quad (3.26)$$

where,

$$\bar{B}_0(\eta) = (2^m m! \sqrt{\pi})^{-1/2} e^{-\alpha\eta^2/2} H_m(\sqrt{\alpha}\eta) \quad (3.27)$$

By mapping λ into $\bar{\lambda}$ via $\bar{\lambda} \triangleq i.\lambda$ (i.e., turning the complex plane $(\Pi/2)$ and applying Cauchy Theorem the integration contour $C_{\bar{\lambda}}$ in complex $\bar{\lambda}$ plane can be deformed to lie on real axis as shown in figs.(3.3). Then it is possible to represent the integral with the first kind of Hankel Function as,

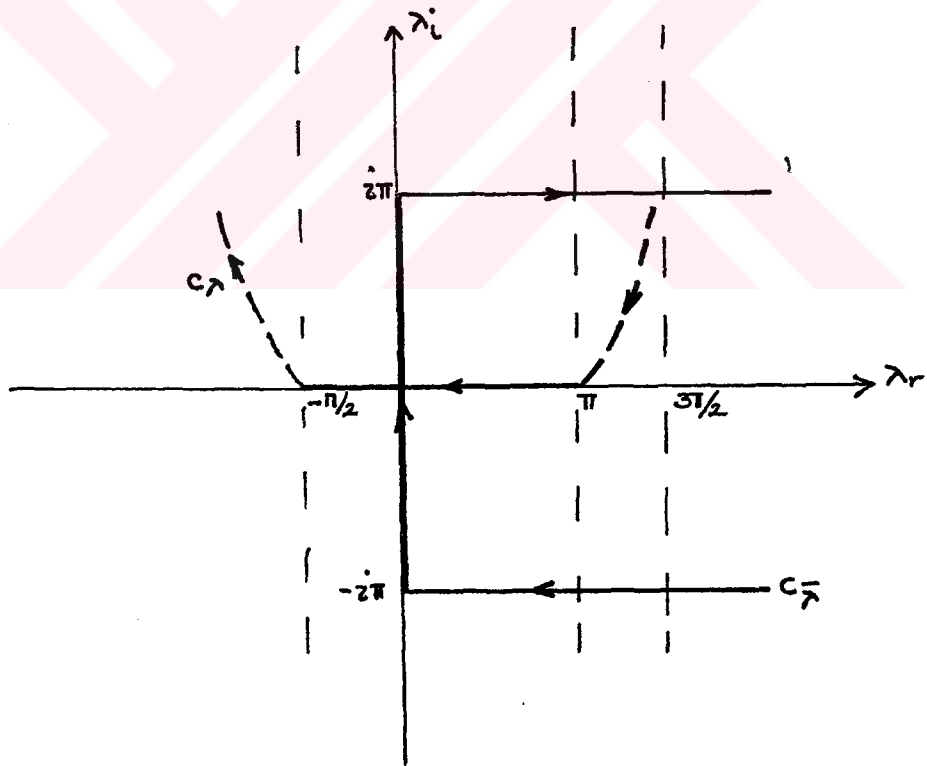
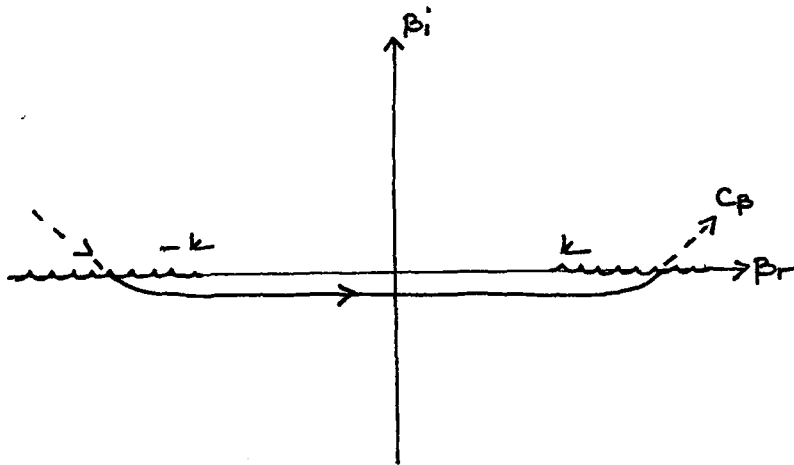


FIG.(3.3) : complex β and λ planes, integration contour in β plane and contour deformation in λ plane

$$\psi(\rho, \eta) = \bar{B}_0(\eta) \Pi x_{\nu}^{(1)}(k_0 \rho) \quad (3.28)$$

which is the exact solution of the problem. It is very interesting to note that the solution of the problem with the medium index given in the eq.(3.2) is separable.

In fact the integration contour as in fig.(3.3) traces both saddle points which correspond both forward and backward propagating modes in propagation towards cutoff. In this way propagation the modes become narrower as they move towards cutoff. But the propagation in the opposite direction has only outgoing waves which means the integration contour in fig.(3.3) passes from only one of the saddle points (depending on the reference chosen directions and the time dependence it may be the positive one or the negative one) and trace $(-\omega + i\delta) \rightarrow (-k + i\delta) \rightarrow 0 \rightarrow +i\omega$ (or $+i\omega \rightarrow 0 \rightarrow (k + i\delta) \rightarrow (\omega + i\delta)$) $\delta > 0$. In this case (3.28) is the result on the deformed contour in angular domain. For the opposite chosen sign in the LIM formulation the second kind of Hankel function is obtained as,

$$\psi(\rho, \eta) = \bar{B}_0(\eta) \Pi x_{\nu}^{(2)}(k_0 \rho) \quad (3.29)$$

which has an incoming character. Total acoustic pressure field function will be,

$$\psi(\rho, \eta) = \bar{B}_0(\eta) \Pi \left\{ x_{\nu}^{(1)}(k_0 \rho) + x_{\nu}^{(2)}(k_0 \rho) \right\} \quad (3.30)$$

The parenthesis in eq.(3.30) representing propagation in both direction (i.e., increasing ρ and decreasing ρ) causes interference which forms standing waves. Using the equality between travelling and standing waves,

$$J_{\nu}(k_0 \rho) = \frac{1}{2} \left\{ x_{\nu}^{(1)}(k_0 \rho) + x_{\nu}^{(2)}(k_0 \rho) \right\} \quad (3.31)$$

the result for narrowing surface duct can be given as,

$$\psi(\rho, \eta) = 2\pi \bar{B}_0(\eta) J_\nu(k_0 \rho) \quad (3.32)$$

The result in (3.32) can also be directly obtained from the deformed contour in complex λ -plane.

Remembering small angle approximation that is made this result explains the propagation phenomena very well in the narrowing surface duct. First of all, since caustic angle in eq.(3.11) should be small, the small angle approximation restricts the application of the results to the higher order modes to be taken into account. For the first few modes as the frequency increases ν increases to become $\nu \gg 1$. It is well known that - for $\mu \gg 1$ -,

$$J_\mu(x) \text{ is } \begin{cases} \text{oscillatory} & \text{for } x \gg \mu \\ \text{exponentially decaying} & \text{for } x \ll \mu \end{cases}$$

This leads us to conclude that the first few modes are confined to some region $\rho > \rho_c$ where ρ_c is the radial caustic given by equating the indice to argument in Bessel function as,

$$\rho_c = \frac{1}{k_0} [\alpha(2m+1)]^{1/2} \quad (3.33)$$

Thus in this model, modes are confined to the region inside a truncated wedge $(-\eta_c < \eta < \eta_c)$ and $\rho > \rho_c$. For narrowing surface duct the only necessary thing to do is to put a pressure release boundary layer on $x = 0$ plane. In that case modes are confined between surface and the caustic in regions where $\rho > \rho_c$.

3.2) Numerical Comparison With The Table Values of Bessel Function

It is shown in the previous section that the LIM integral in eq.(3.22) can be reduced to the exact, closed form solution in eq.(3.32) via suitable mapping from longitudinal wavenumber spectral domain to the angular domain. This is the key test of the reliability of the LIM integral since a Bessel function of any order can be calculated from the LIM integral as,

$$J_{\nu}(k\rho) = \frac{1}{2\pi\nu} \rho^{-1/2} \int_c^{\beta} \frac{e^{i(\beta\rho + \nu \sin^{-1}(\beta/k))}}{(k_0^2 - \beta^2)^{3/4}} d\beta \quad (3.34)$$

with a relation $\nu=[\alpha(2m+1)]^{1/2}$. Since there are only some values of the some of the high order Bessel functions in tables, the order of nine and eleven are chosen for comparison. Fig.(3.4) and Fig.(3.5) show the comparison of the computed Bessel functions of the orders nine and eleven with the table values.

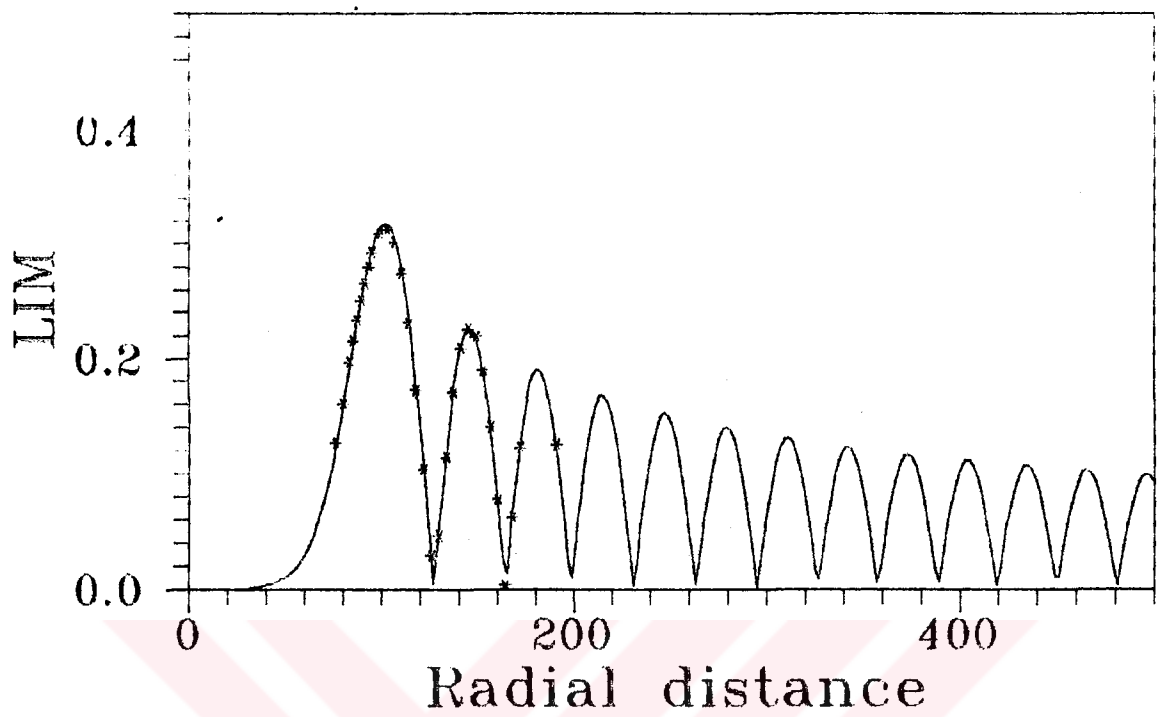


FIG.(3.4) : Comparison of LIM integral with the table values of Bessel function of order nine

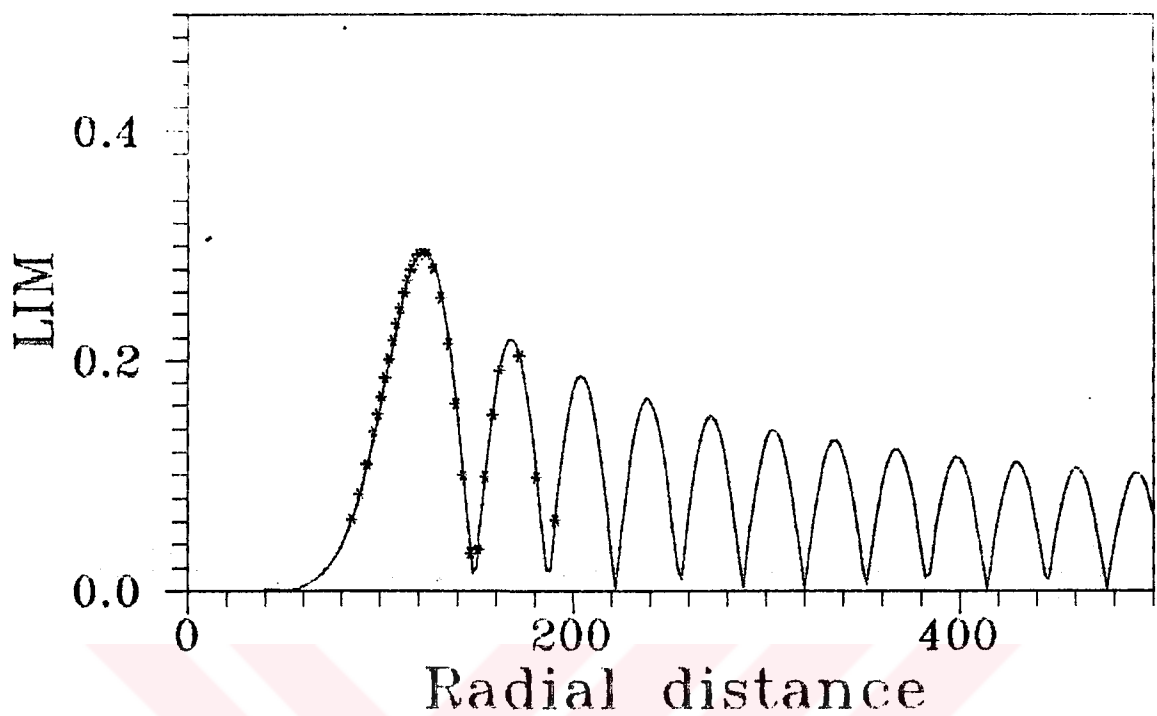


FIG.(3.5) : Comparison of LIM integral with the table values of Bessel function of order eleven

SECTION FOUR

LONGITUDINALLY NARROWING SURFACE DUCT

4.1) Weak Depth Dependence [13,15]

In the Sections II and III the LIM formulation is tested on two canonical two-dimensional guiding environments which are separable in a curvilinear (cylindrical) coordinate frame but are parameterized in terms of locally adaptive Adiabatic Modes. The first one (Section II) is a perfectly conducting homogeneous wedge waveguide. The second one (Section III) is a graded index wave guide whose refractive index varies in such a way that the solution becomes separable in cylindrical (ρ, θ) coordinates. Here the first important application of LIM Theory, which is underwater sound propagation in a longitudinally narrowing surface duct is been examined. In two dimensional cartesian coordinate system wave equation is written as,

$$\left\{ \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} + k^2(x, z) \right\} W_m(x, z) = 0 \quad (4.1a)$$

where both range and depth dependent wavenumber is given,

$$k^2(x, z) = k_0^2 \left\{ 1 - a_0 x z \right\} \quad , \quad (a_0 > 0) \quad (4.1b)$$

The structure of the problem and the depth variation of the sound speed for different ranges are shown in fig.(4.1) and fig.(4.2), respectively. Boundary conditions are,

$$i) \quad \frac{\partial}{\partial x} W_m(x, z) \Big|_{x=0} = 0 \quad (4.2a)$$

$$ii) \quad \lim_{x \rightarrow \infty} W_m(x, z) \rightarrow 0 \quad (4.2b)$$

The solution of eq.(4.1) under the boundary conditions given

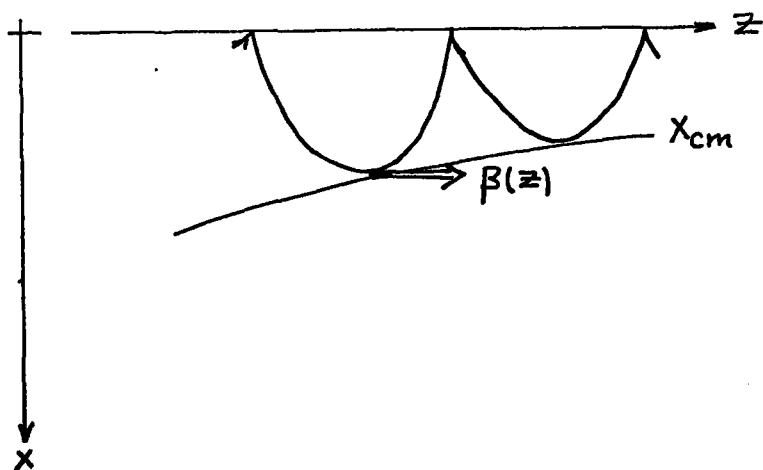


FIG.(4.1) : The surface narrowing duct

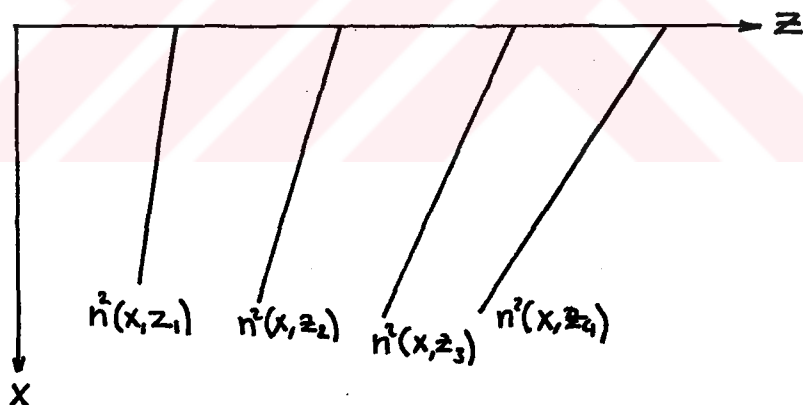


FIG.(4.2) : Variation of sound speed in different transverse cross sections

with eq.(4.2) is postulated by LIM Theory as,

$$W_m(x, z) = \int_{C_\beta} a(\beta) e^{i\Omega(\beta)} U_m(x, \bar{z}(\beta)) e^{i\beta(z - \bar{z}(\beta))} d\beta \quad (4.3)$$

Variation as in eq.(4.1b) causes trapped modes between surface and their caustics. Application of transverse resonance condition as,

$$2 \int_0^{x_{cm}} \left[k^2(x, z) - k^2(x_{cm}, z) \right]^{1/2} dx + 0 - \frac{\pi}{2} = 2m\pi \quad (4.4)$$

gives the caustic depth as,

$$x_{cm}(z) = \sigma_m (a_0 k_0^2 z)^{-1/3} \quad (4.5)$$

where,

$$\sigma_m \triangleq \left[\frac{3\pi}{2} (m + .25) \right]^{2/3} \quad (4.6)$$

Longitudinal component of the wavenumber will be,

$$\beta(z) = \pm \left\{ k_0^2 - \sigma_m (a_0 k_0^2 z)^{2/3} \right\}^{1/2} \quad (4.7)$$

The variation of β with range for the chosen parameters in Section V is given in fig.(4.3). It is clear from the figure that for each range point there corresponds two β values which are equal in magnitude but have opposite signs. As the observation point moves away the two values come closer and at a certain range point the two values coincide and become zero. This range is the cut-off range for that mode and for the given parameter values. Beyond that range β becomes complex.

Then other functions in the LIM postulation (4.3) are obtained as,

$$\bar{z}(\beta) = \frac{1}{A_0} (k_0^2 - \beta^2)^{3/2}, \quad (A_0 = a_0 k_0^2 \sigma_m^{3/2}) \quad (4.8)$$

$$a(\beta) = \left\{ \frac{-3}{A_0} (k_0^2 - \beta^2)^{1/2} \right\}^{1/2} \quad (4.9)$$

$$\begin{aligned} \Omega(\beta) = \frac{-3}{A_0} \left\{ \frac{-\beta}{4} (k_0^2 - \beta^2)^{3/2} + \frac{k_0^2 \beta}{8} (k_0^2 - \beta^2)^{1/2} \right. \\ \left. + \frac{k_0^4}{8} \sin^{-1}\left(\frac{\beta}{k_0}\right) \right\} \end{aligned} \quad (4.10)$$

The total phase term will be,

$$\begin{aligned} P(z, \beta) = \beta z + \frac{1}{A_0} \left\{ \left[\frac{2\beta^3 - 5k_0^2 \beta}{8} \right] (k_0^2 - \beta^2)^{1/2} \right. \\ \left. - \frac{3k_0^4}{8} \sin^{-1}\left(\frac{\beta}{k_0}\right) \right\} \end{aligned} \quad (4.11)$$

The local normal mode function which satisfies,

$$\left\{ \frac{d^2}{dx^2} + k^2(x, \bar{z}(\beta)) - \beta^2 \right\} U_m(x, \bar{z}(\beta)) = 0 \quad (4.12)$$

can be given as,

$$U_m(x, \beta) = C_0 \text{Ai}(\xi) \quad (4.13)$$

Here C_0 is the normalization coefficient of the Airy function and can be found as,

$$C_0 = \sigma_m^{1/4} \frac{1}{(k_0^2 - \beta^2)^{1/4} \text{Ai}(-\sigma_m)} \quad (4.14)$$

and ,

$$\xi = -\sigma_m \left\{ \frac{k_0^2 - \beta_s^2}{k_0^2 - \beta^2} + \frac{x}{\sigma_m^{3/2}} (k_0^2 - \beta^2)^{1/2} \right\} \quad (4.15)$$

As a result final LIM integral will be ,

$$W_m(x, z) = c_1 \int_{C_\beta} Ai(\xi) e^{iP(x, \beta)} d\beta \quad (4.16)$$

where,

$$c_1 = \left\{ \frac{3}{\sigma_m a_0 k_0^2} \right\}^{1/2} \frac{1}{Ai(-\sigma_m)} \quad (4.17)$$

It is interesting to look at the behavior of the Airy function with the argument as in eq.(4.15) at the saddle point. The argument of the Airy function at the saddle point will be,

$$\xi = -\sigma_m \left(1 - \frac{x}{x_{cm}} \right)$$

which gives the following behavior ;

- i) $Ai'(-\sigma_m) = 0$ at the surface (i.e., at $x = 0$)
- ii) $Ai(\xi)$ is oscillatory for $x < x_{cm}$
- iii) $Ai(\xi)$ is exponentially decaying for $x > x_{cm}$

The integration contour in complex β plane and the location of the saddle points are shown in fig(4.4). It's seen from the figure that real saddle points approach to each other at the origin as range increases and then become pure imaginary which shows us that there is a cut-off range where $\beta = 0$. The LIM integral has highly oscillatory phase term so it is very efficient to apply Steepest Descent Path Technic. Deformed contours which agree with the local SDP paths in trapped region and beyond cut-off are shown in fig.(4.4). Numerical

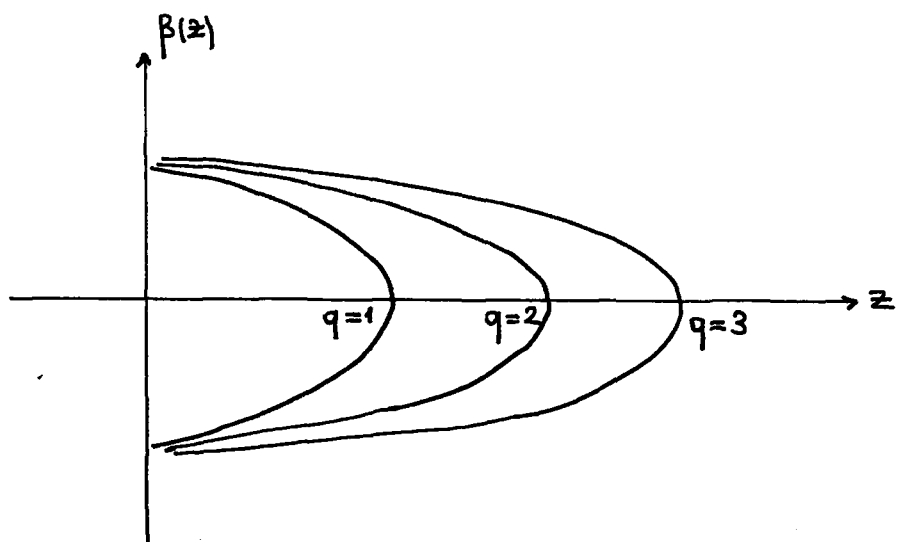


FIG.(4.3) : Variation of the longitudinal wavenumber with range

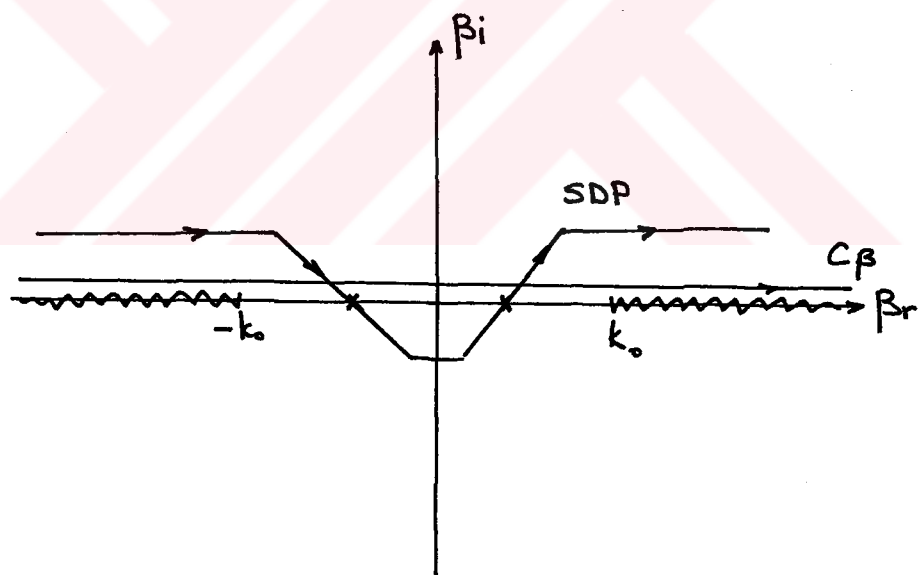


FIG.(4.4) : Complex β plane, the integration contour and steepest descent path

integration results show us that,

i) Evaluation on the SDP for guided modes shown in fig.(4.4) gives contributions from both incident and cut-off reflected waves.

ii) Evaluation on the SDP for cutoff modes shown in fig.(4.4) gives a strong exponentially decay which means there is no contribution from beyond cut-off region.

4.2) Strong Depth Dependence

Another depth profile can be obtained by introducing a strong depth dependence as,

$$k^2(x,z) = k_0^2 \left\{ 1 - a_0 x^2 z \right\} \quad (4.18)$$

In this case parameters and functions appearing in LIM integral will be,

$$x_{cm}(z) = \frac{(4n+1)^{1/2}}{(B_0 z)^{1/4}}, \quad B_0 \triangleq a_0 k_0^2 \quad (4.19)$$

$$\beta(z) = \pm \left\{ k_0^2 - (B_0 z)^{1/2} (4n+1) \right\}^{1/2} \quad (4.20)$$

$$\bar{z}(\beta) = \frac{(k_0^2 - \beta^2)^2}{B_0 (4n+1)^2} \quad (4.21)$$

$$a(\beta) = \left\{ \frac{4(k_0^2 - \beta^2)}{B_0 (4n+1)^2} \right\}^{1/2} \quad (4.22)$$

$$\Omega(\beta) = + \frac{4D_0 k_0^2}{3} \beta^3 - \frac{4D_0}{5} \beta^5, \quad D_0 \triangleq \frac{1}{B_0 (4n+1)^2} \quad (4.23)$$

Local normal mode equation will be,

$$\left\{ \frac{d^2}{dx^2} + (k_0^2 - \beta^2) - \frac{(k_0^2 - \beta^2)^2}{(4n+1)^2} x^2 \right\} U_n(x, \bar{z}(\beta)) = 0 \quad (4.24)$$

which has the solution in terms of Hermite Polynomials as,

$$U_n(x, \beta) = C_0 e^{-\alpha x^2/2} H_{2n}(\sqrt{\alpha} x) \quad (4.25)$$

where,

$$\alpha \triangleq \frac{(k_0^2 - \beta^2)}{(4n+1)} \quad (4.26)$$

$$C_0 = \left(\frac{1}{\sqrt{\alpha}} 2^n n! \sqrt{\pi} \right)^{-1/2} \quad (4.27)$$

The integration contour is shown in fig.(4.4). Since the rigid boundary condition is taken into account only the even indice Hermite polynomials appear in the formulation. Odd indice Hermite polynomials are zero at the boundary. The location of the saddle points and the integration contour are very similar in sections (4.1) and (4.2). The only difference is the adiabatic mode inserted in the LIM integral to represent the transverse behavior. In section (4.1) it is Airy function for weak range dependence and in section (4.2) it is Hermite polynomials for strong depth dependence. In both cases longitudinal variations are in the form of standing waves that occur from the interference of outgoing and caustic reflected waves. The integration beyond caustics shows that there is no appreciable contribution from this region.

4.3) Numerical Results

The two cases in section (4) are taken into account separately and final LIM integrals corresponding to both weak and strong depth dependence are evaluated numerically.

The flow chart of the programs used in numerical evaluation is given in fig.(2.5). The fortran programs used to compute the LIM integrals in this section are given below :

NLIM.FOR

This program computes the LIM integral given in eq.(4.16) on the integration contour shown in fig.(4.4). The module for numerical integration of a complex function CTABINT and the subroutine AIRYCA which are explained in section (2.3) are used here, too. As in the perfectly conducting wedge case, here the main contribution of the integral comes from the contour part on real β -axis in the interval $(-k_0, +k_0)$ where the two real saddle points lie for trapped modes. The contribution from the rest of the contour parts are negligible.

NMODE.FOR

This program is used for Steepest Descent Path computation. Both CTABINT and AIRYCA modules are used here, too. The local SDP used in the computations is shown in fig.(4.4). It makes 45° at the first saddle point which lies on negative real axis and touches the imaginary axis and then makes -45° at the second saddle point which lies on the positive real axis.

NASYM.FOR

This program is very similar to WASYM.FOR. It computes the asymptotic form of the LIM integral through cutoff, too. Here the module AIRYCA is used for both evaluation of the depth profile and the transition through cutoff.

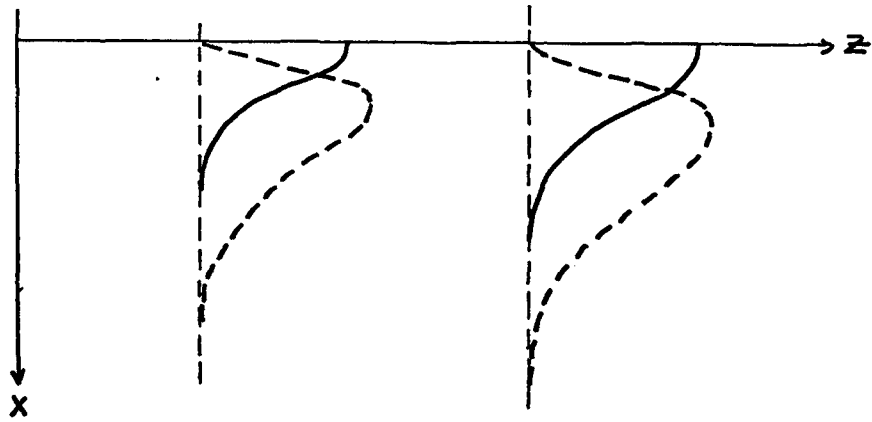


FIG.(4.5) : Transverse profiles for hard and soft boundary conditions

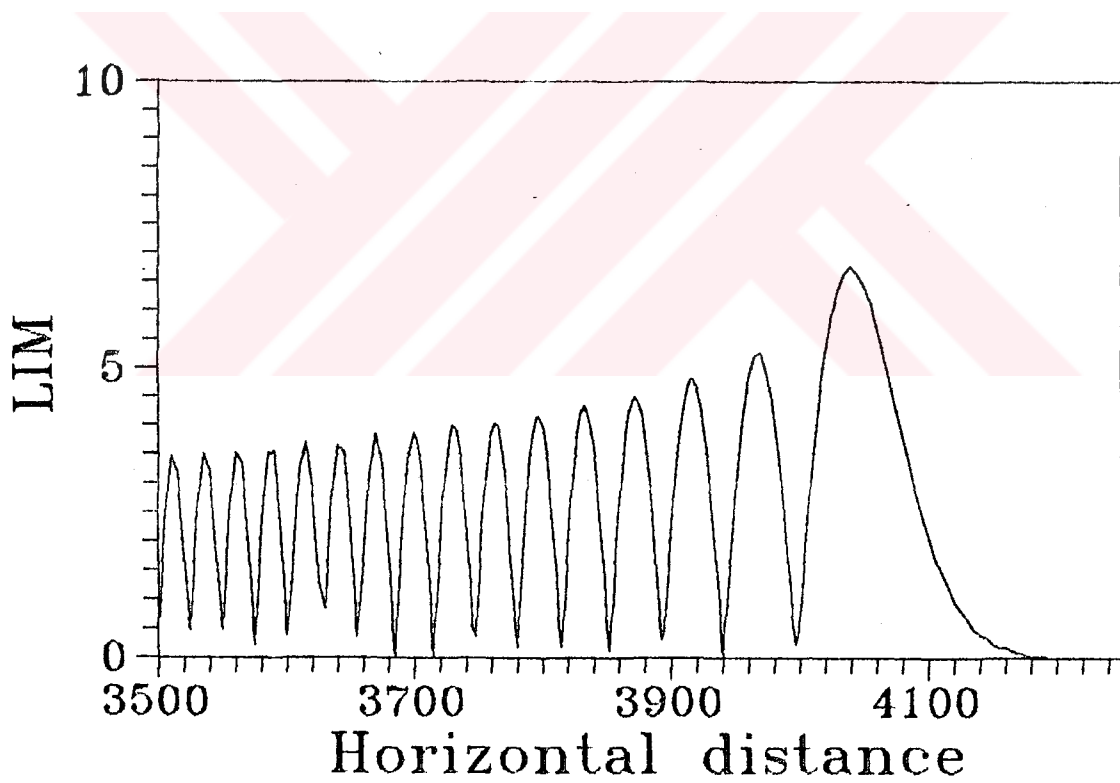


FIG.(4.6) : Longitudinal variation of first LIM near the surface

HERMIT.FOR

This program is written for the strong depth dependence case where the adiabatic depth profile is given by HERMIT polynomials. It computes the LIM integral on the SDP contour where the numerical computation is carried out by CTABINT module.

For the first case (i.e., for weak depth dependence) the computational parameters are chosen as $a_0 = .0001$ $f = 100$ Hz. , $V_0 = 1500$ m/sec.. For these chosen values it is seen from the fig.(4.2) that cut-off range is nearly at 4.1 km.for the first mode. Both depth and range profiles are obtained and given in figs.(4.5-8). For all three type of calculations depth profile is drawn with nearly 100-150 depth points for the first few modes. This number should be increased as the mode number increases in order to display the profile accurately. For the range profile only the first few peaks are included to show the cutoff transition clearly. Nearly 2 minutes are needed for calculations with IBM PC (AT) having 12 MHz. speed. But direct numerical integration needs more than 12 hours with the same PC to get the same profiles.

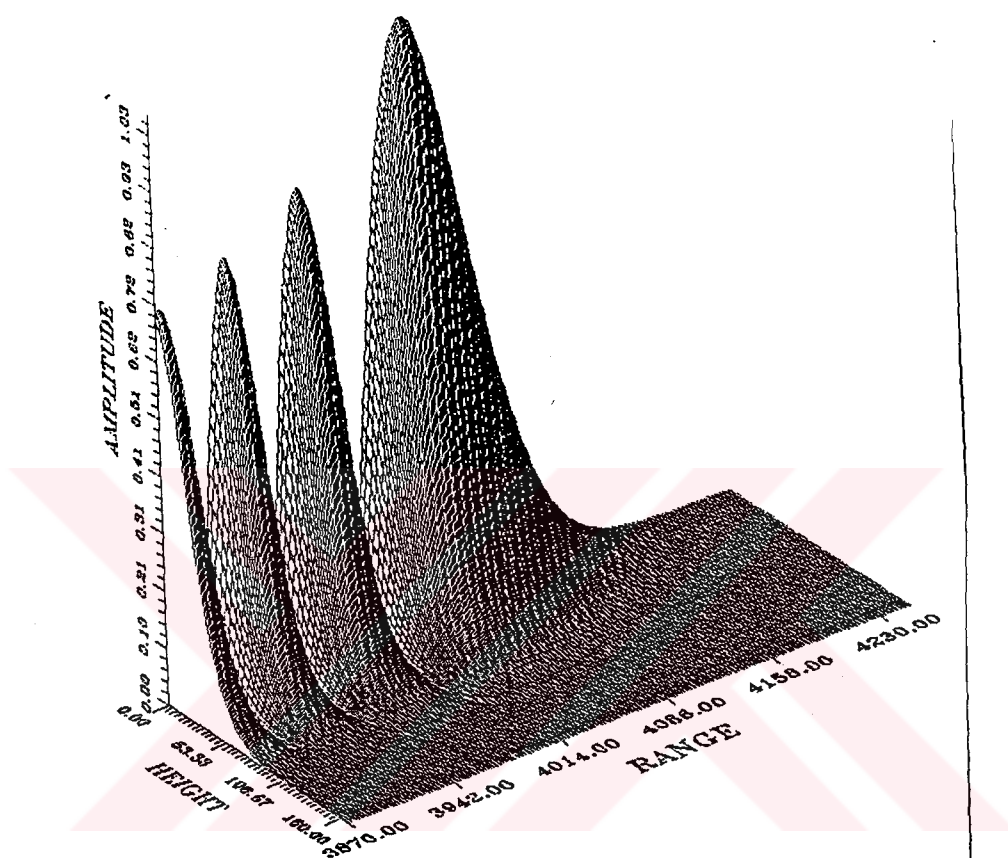


FIG.(4.7) : 3D.Plot of the first mode

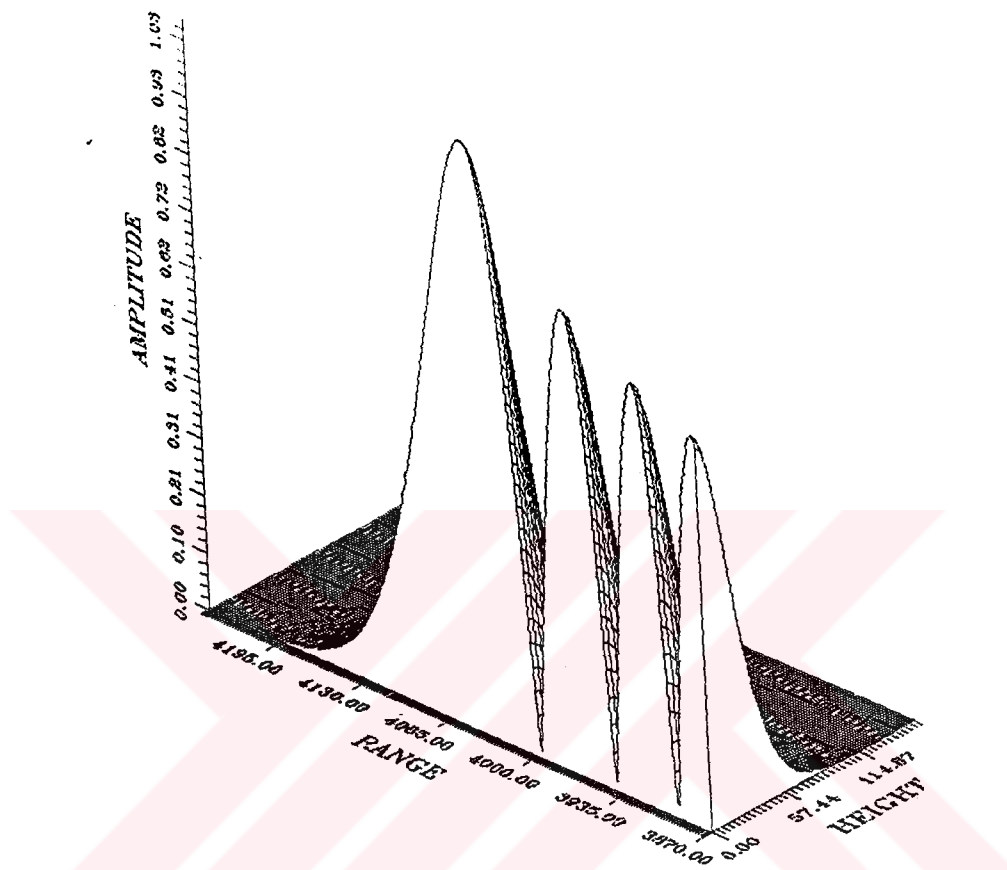


FIG.(4.8) : 3D.Plot of the first mode with different angle and tilt

SECTION FIVE

BILINEAR WIDENING DUCT

5.1) Formulation Of The Problem

It is shown in the previous section that the LIM integral gives a smooth solution near and beyond the caustic which is a critical range for narrowing duct problems. Another critical range occurs in widening bilinear ducts which is called " leaky range ". Fig.(5.1) and fig.(5.2) shows the structure and the mode types of the problem. The LIM integral $W_m(x,z)$ satisfies the wave equation in 2D. cartesian coordinate system given by eq.(4.1a) where the range and depth dependent wavenumber is given with,

$$k^2(x,z) = \begin{cases} k_0^2 \left[1 + a_0 x z \right] & \dots\dots x \leq x_d \\ b_0(z) \times & \dots\dots x \geq x_d \end{cases} \quad (5.1)$$

Here,

$$b_0(z) \triangleq \frac{k_0^2}{x_d} \left[1 + a_0 x_d z \right] \quad (5.2)$$

and x_d is the duct depth shown in fig.(5.1). The first portion of the above profile causes mode trapping and the second portion causes mode leakage. The negative range region (i.e., $z < 0$) is the " guiding " region and the positive range region (i.e., $z > 0$) is the " anti-guiding " region. So ($z = 0$) corresponds to " turning point ". In guiding region modes which satisfies $x_{cm} \ll x_d$ are trapped between the upper boundary and their caustics but the modes which gives $x_{cm} \approx x_d$ are the leaky modes. Since the caustic depths (i.e., x_{cm}) of the trapped modes broadens with the increasing range (i.e., towards to the turning point), as the observation point moves towards to the turning point the number of trapped modes decrease and more and more high

order modes become leaky. By using the transverse resonance condition the caustic depth is found as,

$$x_{cm}(z) = -\sigma_m (a_0 k_0^2 z)^{-1/3} \quad (5.3)$$

where σ_m is to be determined via (5.13) and (5.14). Since the caustic depth is real for trapped modes and becomes complex for leaky modes the second Riemann Sheet in the three layered complex Z-domain is chosen so that for guiding region for trapped modes $-Z \Rightarrow Z \cdot \exp(3\pi)$ is taken to give the caustic depth as,

$$x_{cm}(Z) = \sigma_m (a_0 k_0^2 z)^{-1/3} \quad (5.4)$$

The standard 2D. LIM integral given in eq.(4.3) is then valid also in this case, with the local normal mode equation ,

$$\left\{ \frac{d^2}{dx^2} + k_i^2(x, \bar{z}(\beta)) - \beta^2 \right\} U_{m_i}(x, \bar{z}(\beta)) = 0 \quad (5.5)$$

where the index "i" represents the two different depth portions (i.e., $i=1$ is the portion between the upper boundary and the duct depth x_d , $i=2$ is the remaining portion $x > x_d$). Writing the equation as,

$$\left\{ \frac{d^2}{dx^2} + A_i x + B_i \right\} U_{m_i}(x, \bar{z}(\beta)) = 0 \quad (5.6)$$

and introducing a new variable ξ as,

$$\xi_i \triangleq -A_i^{-2/3}(A_i x + B_i) \quad (5.7)$$

one obtains,

$$\left\{ \frac{d^2}{d\xi_i^2} - \xi_i \right\} U_{m_i}(\xi_i) = 0 \quad (5.8)$$

which has a solution in form of a combination of the Airy functions. In eq.(5.7) the parameters are ,

$$A_1 \triangleq a_0 k_0^2 \bar{z} \quad A_2 \triangleq b_0(\bar{z}) \quad (5.9)$$

$$B_1 \triangleq k_0^2 - \beta_s^2 \quad B_2 \triangleq -\beta_s^2 \quad (5.10)$$

Boundary conditions for bilinear profile are,

$$i) \quad U_{m_1}(\xi_1) \Big|_{x=0} = 0 \quad (5.11a)$$

$$ii) \quad U_{m_1}(\xi_1) \Big|_{x=x_d} = U_{m_2}(\xi_2) \Big|_{x=x_d} \quad (5.11b)$$

$$iii) \quad \frac{d}{d\xi} U_{m_1}(\xi_1) \Big|_{x=x_d} = \frac{d}{d\xi} U_{m_2}(\xi_2) \Big|_{x=x_d} \quad (5.11c)$$

The solution of eq.(5.8) satisfying the boundary conditions in eqs.(5.11a-c) can be given as,

$$U_{m_1}(\xi_1) = Ai(\xi_1) + R(x_d, \bar{z}, \sigma_m) Bi(\xi_1) \quad (5.12a)$$

$$U_{m_2}(\xi_2) = T(x_d, \bar{z}, \sigma_m) W_1(\xi_2) \quad (5.12b)$$

Here $R(x_d, \bar{z}, \sigma_m)$ and $T(x_d, \bar{z}, \sigma_m)$ are the reflection and the transmission coefficients at the duct boundary and σ_m is the root of the equation for the first boundary condition, that is ,

$$Ai(-\sigma_m) + R(x_d, \bar{z}, \sigma_m) Bi(-\sigma_m) = 0 \quad (5.13)$$

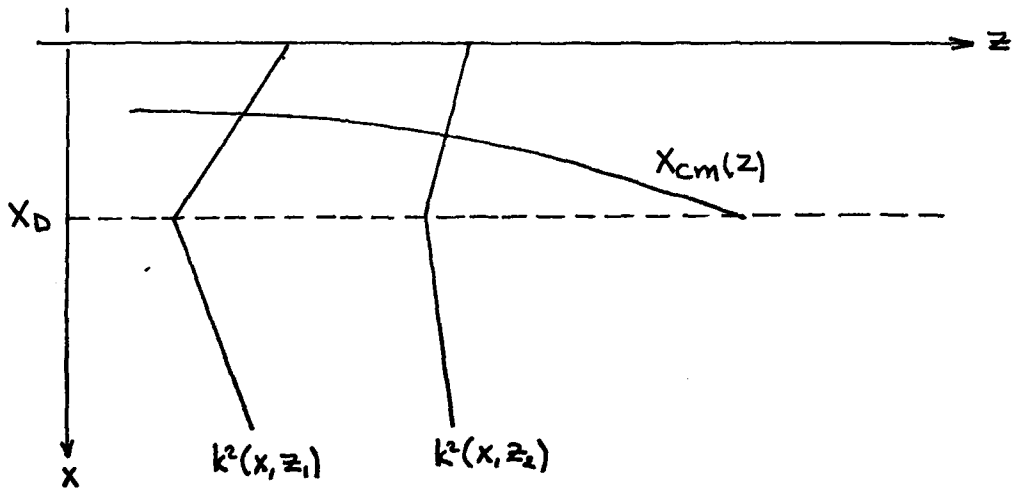


FIG.(5.1) : The structure of the problem

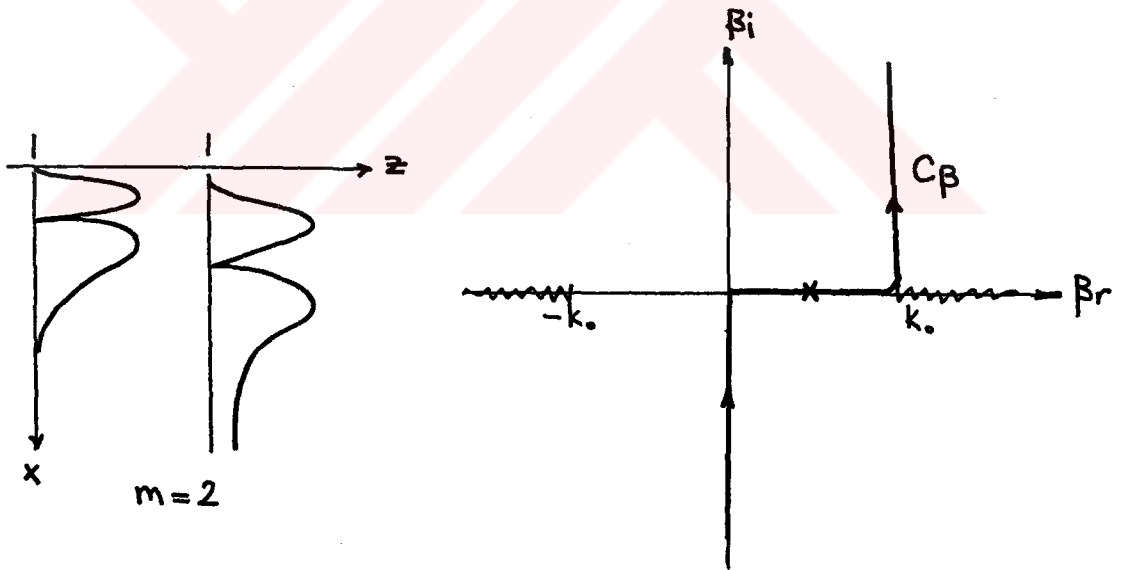


FIG.(5.2) : The shapes of trapped and leaky modes and the integration contour in β plane

The reflection and the transmission coefficients are obtained from the second and the third boundary conditions as,

$$R(x_d, \bar{z}, \sigma_m) = \frac{x \text{Ai}'(\xi_{1d}) W_1(\xi_{2d}) - \gamma \text{Ai}(\xi_{1d}) W_1'(\xi_{2d})}{-x \text{Bi}'(\xi_{1d}) W_1(\xi_{2d}) + \gamma \text{Bi}(\xi_{1d}) W_1'(\xi_{2d})} \quad (5.14)$$

$$T(x_d, \bar{z}, \sigma_m) = \frac{\text{Ai}(\xi_{1d}) + R(x_d, \bar{z}, \sigma_m) \text{Bi}(\xi_{1d})}{W_1(\xi_{2d})} \quad (5.15)$$

$$x \triangleq \left. \frac{d\xi_1}{dx} \right|_{x=x_d} = - (a_0 k_0^2 \bar{z})^{1/3} \quad (5.16)$$

$$\gamma \triangleq \left. \frac{d\xi_2}{dx} \right|_{x=x_d} = - \left[\frac{k_0}{x_d} + (a_0 k_0^2 \bar{z}) \right]^{1/3} \quad (5.17)$$

$$\xi_{1d} \triangleq x x_d - \sigma_m \quad (5.18)$$

$$\xi_{2d} \triangleq \gamma x_d - \frac{k_0^2 - \sigma_m x^2}{\gamma^2} \quad (5.19)$$

Table (5.1) shows some of the roots of eq.(5.13). Here the parameters a_0 , k_0 , $\bar{z}$ are chosen as 10^{-4} , $2\pi/15$, 1500, respectively. The first five modes are then trapped and sixth and seventh modes are leaky. It is interesting to see that the roots of trapped modes are rather well approximated by (4.6) and they are almost equal to the roots of $\text{Ai}(-\sigma_m) = 0$ which are real. The real parts of the roots of leaky modes are slightly different than the roots of Airy function and the imaginary parts are not negligible anymore. Table (5.2) shows the variation of the roots of eq.(5.13) for σ_m for $m=89$ for different observation points with the parameters $a_0=8.9 \cdot 10^{-8}$, $k_0=2\pi$, $x_d=100$. It is seen that in well trapped region the imaginary parts of the roots are almost zero.

TABLE (5.1) : The roots of $Ai(-\sigma_m) + R(x_d, \sigma_m, \bar{z}) Bi(-\sigma_m) = 0$ representing the trapped and leaky modes for bilinear profile

m	σ_m	Roots of (4.6)
1	(2.338107 , $-2.9527 \cdot 10^{-8}$)	2.3381
2	(4.087952 , $-4.6243 \cdot 10^{-9}$)	4.0879
3	(5.520526 , $-1.7407 \cdot 10^{-9}$)	5.5205
4	(6.811103 , $9.7294 \cdot 10^{-8}$)	6.7867
5	(7.954243 , $-8.3421 \cdot 10^{-6}$)	7.9441
6	(9.021342 , $8.6007 \cdot 10^{-2}$)	9.0226
7	(10.029440 , $-6.2199 \cdot 10^{-2}$)	10.0402

TABLE (5.2) : Roots of eigenvalue equation of mode 89 for various observation points.

MODE NUMBER : 89

RANGE	ROOT OF $Ai(-\sigma_m) + R.Bi(-\sigma_m) = 0$
-75 km.	(55.934297 , $2.899731 \cdot 10^{-8}$)
-65 km.	(55.934297 , $-2.989755 \cdot 10^{-8}$)
-60 km.	(55.934297 , $-3.954721 \cdot 10^{-8}$)
-55 km.	(55.934052 , $-4.181754 \cdot 10^{-3}$)
-53 km.	(55.928683 , $-5.364346 \cdot 10^{-1}$)
-52 km.	(55.928683 , $-5.364346 \cdot 10^{-1}$)
-51 km.	(55.924320 , $-1.541757 \cdot 10^{-1}$)
-50 km.	(55.924248 , $-3.672841 \cdot 10^{-1}$)
-49.5 km.	(55.930012 , $-5.481932 \cdot 10^{-1}$)
-49 km.	(55.951612 , $-7.983145 \cdot 10^{-1}$)

TABLE (5.3) : Module of the reflection coefficient of three different modes for various range points.

RANGE	R_{50}	R_{55}	R_{61}
-1500m.	$2.026 \cdot 10^{-32}$	$6.376 \cdot 10^{-5}$	$6.316 \cdot 10^{-1}$
-1400m.	$3.586 \cdot 10^{-29}$	$3.169 \cdot 10^{-3}$	$7.504 \cdot 10^{-1}$
-1300m.	$6.807 \cdot 10^{-26}$	$1.922 \cdot 10^{-1}$	$8.150 \cdot 10^{-1}$
-1200m.	$1.367 \cdot 10^{-22}$	$7.033 \cdot 10^{-1}$	1.819
-1100m.	$2.858 \cdot 10^{-19}$	$7.925 \cdot 10^{-1}$	$4.978 \cdot 10^{-1}$
-1000m.	$6.025 \cdot 10^{-16}$	1.143	2.476

Table (5.3) shows the absolute value of the reflection coefficient for the modes $m=50$, 55 and 61 for different observation range points. In this case the parameters are chosen $a_0=10^{-5}$, $k_0=2\pi/15$, $x_d=300$. This renders mode-50 well-trapped, mode-55 almost trapped and mode-61 leaky at the starting observation point (i.e., range = -1500 m.) As the observation point moves towards the turning point it is known from (5.4) that the caustic depth of each mode broadens and approaches to the duct depth where the trapped mode becomes leaky. It is interesting to note that the reflection coefficient of well-trapped modes are almost zero, but unnegligible for leaky modes. So adiabatic depth profile can be chosen only as the first kind of the Airy function for well-trapped modes but the whole solution as in eq.(5.12) must be taken into account for leaky modes or for the ranges where the trapped modes become leaky.

The general LIM integral form is,

$$W_m(x, z) = \int_{C_\beta} c(\beta) a(\beta) e^{i\Omega(\beta)} U_m(x, \bar{z}(\beta)) e^{i\beta(z - \bar{z}(\beta))} d\beta \quad (5.20)$$

where the arbitrary phase function is given as,

$$\Omega(\beta) = \int_0^\beta \beta \frac{d}{d\beta} \bar{z}(\beta) d\beta \quad (5.21)$$

$$\bar{z}(\beta) = \frac{1}{A_0} (k_0^2 - \beta^2)^{3/2}, \quad (A_0 = a_0 k_0^2 \sigma_m^{3/2}) \quad (5.22)$$

The normalization coefficient $c(\beta)$ of the local normal mode function is found as,

$$c(\beta) = \sigma_m^{1/4} \frac{1}{(k_0^2 - \beta^2)^{1/4} \text{Ai}'(-\sigma_m)} \quad (5.23)$$

while the arbitrary amplitude function $a(\beta)$ is given as,

$$a(\beta) = \left\{ \frac{-3}{A_0} (k_0^2 - \beta^2)^{1/2} \right\}^{1/2} \quad (5.24)$$

The saddle points are found from,

$$\beta(z) = \pm \left\{ k_0^2 - \sigma_m (a_0 k_0^2 z)^{2/3} \right\}^{1/2} \quad (5.25)$$

Since there are only outgoing waves in the direction of the broadening duct, the integration contour C_β is chosen to traverse only the saddle point located on the positive real axis between the origin and the branch point as $i\omega \rightarrow 0 \rightarrow k \rightarrow k - i\omega$. The saddle points are real for trapped modes since the root σ_m in the eq.(5.25) are real, but they are located in the first quadrant for leaky modes since the root σ_m are complex. As the observation point moves towards the origin the duct broadens and higher order trapped modes become

leaky. The mode profiles in trapped and leaky region are shown in fig.(5.3). It is shown analytically and numerically that the amplitudes of leaky modes strongly decay with the range and hence the leaky modes don't contribute to the total field in long distance propagation. Fig.(5.4a,b) shows the range dependence of the mode 89 for three different depth points. In the first part of the figure the depth point is taken as the duct depth which is 100 m. for this particular case. It is seen that mode 89 is well trapped for the ranges smaller than -55 km. but leaks to the bottom after that point. The two other parts of the figure show how energy of the mode leaks as a beam shifts with the range.

5.2) Numerical Results

The programs made for the computations in this section are given below.

ROOT.FOR

This program computes in general the complex root of a given function. The main routine is the MULLER Algorithm and the subroutine MYFN. The subroutine MYFN computes the function given in eq.(5.13) and the subroutine MULLER tries to find the root of this function (i.e., σ_m) by consecutive iterations with a given tolerance. Since the function here is the combination of Airy functions there are infinite number of roots which are complex in general. These roots depend on the duct depth x_d , the observation range point and the mode number m . The program may compute the roots of any number of modes in any number of observation range points.

BILIN.FOR

This program computes the LIM integral given in eq.(5.20). It

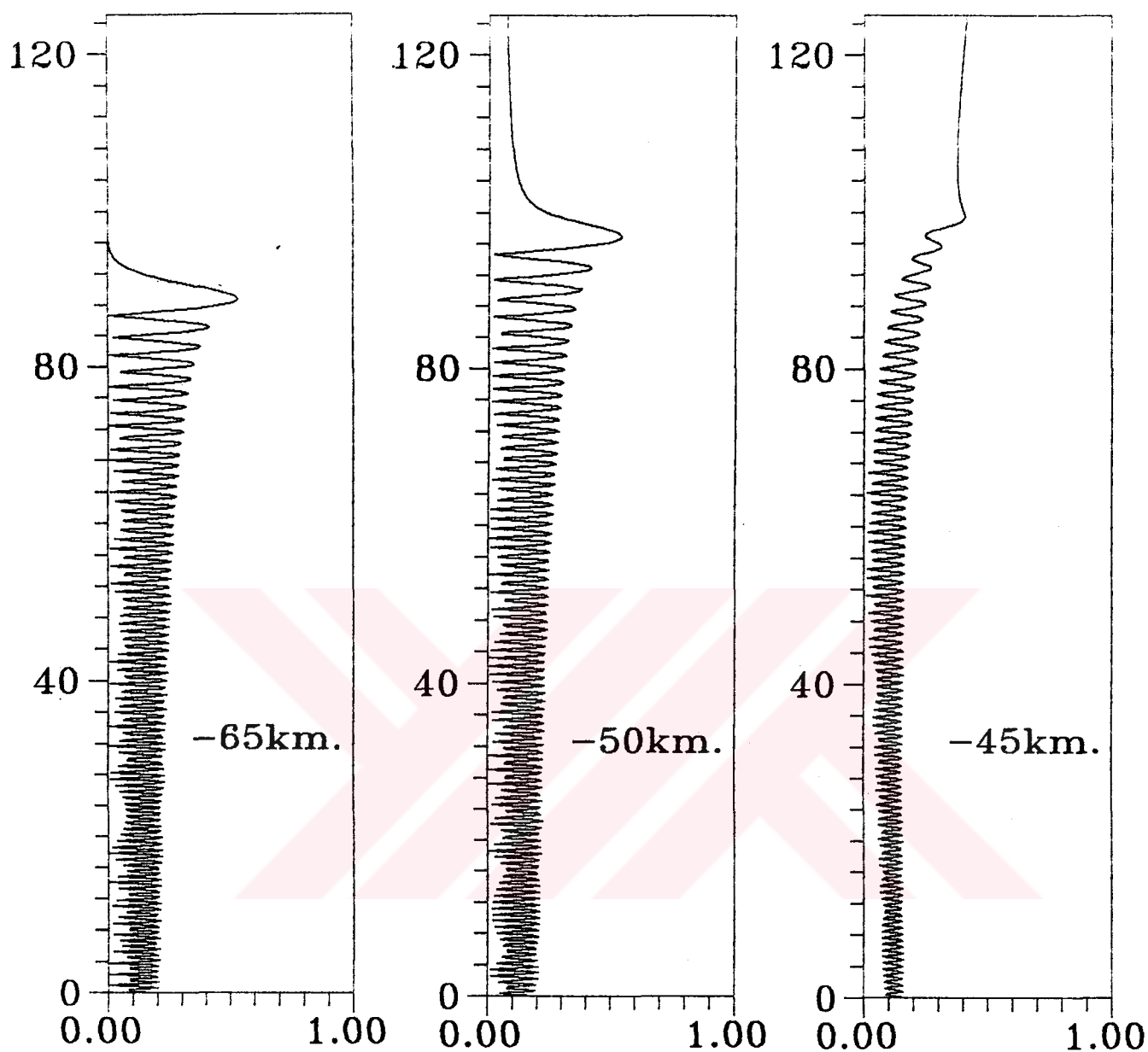


FIG.(5.3) : Transverse variation of the mode 89 for three different ranges

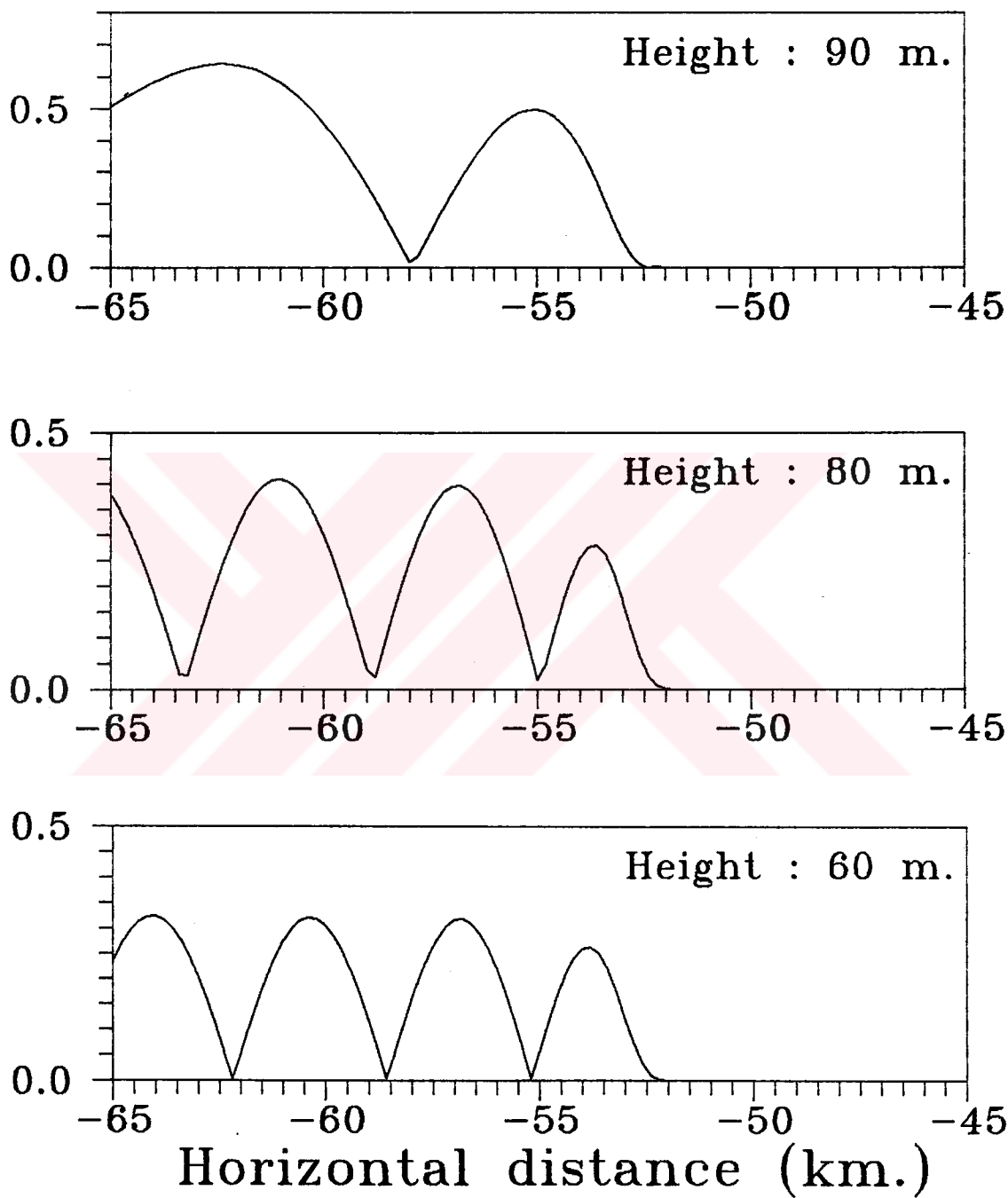


FIG.(5.4a) : Longitudinal variation of mode 89 for three different horizontal segments (troposphere)

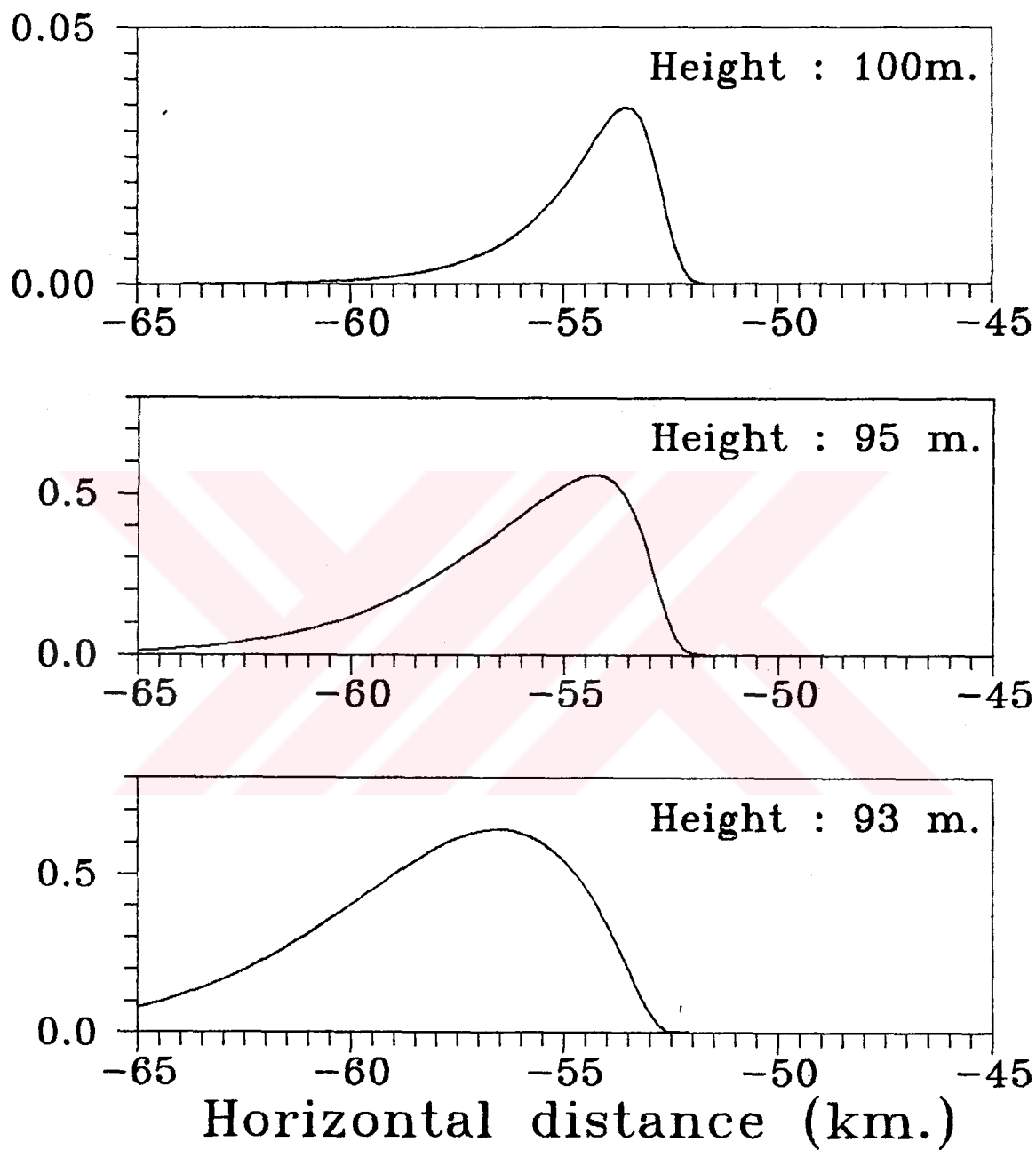


FIG.(5.4b) : Longitudinal variation of mode 89 for three different horizontal segments (troposphere)

uses the subroutines AIRYCA and MYFUN for the construction of depth profile. It uses SDP evaluation in guiding range and saddle point evaluation for the leaky range. The complex root σ_m is asked as an input for computation of any depth profile in a given observation range point. For the range profile it calls and runs the program ROOT.FOR first. Then it keeps the roots of eq.(5.13) for each range points in an input data file "root.dat". Taking these roots as an input it computes the range profile.

The first computation is made to compare SDP evaluation with only the saddle point contribution. With the parameters taken as $a_0 = 8.9 \cdot 10^{-8}$, $f = 1500$ Hz., $X_d = 100$ m., $V_0 = 1500$ m/sec. the difference between the two results are almost negligible. The second computation is to find the depth profile of different modes for different observation points. Figure (5.3) show the depth profile of mode 89 for three different observation points. It is clearly seen that the trapped modes broaden and become leaky as the observation point moves towards the turning point (i.e., towards $Z = 0$). Fig.(5.4a,b) shows the range dependence of mode 89. It is seen that when the caustic depth is almost equal to the duct depth the trapped mode leaks to the bottom as a beam. Since it is clearly seen from the figure that the maximum of the beam shifts as the observation point moves towards the turning point this beam also propagates towards the turning point. Finally 3D. plots of the fifth and twenty-fifth modes are plotted in figs.(5.5-8).

5.3) Failure Of The LIM Solution Through Turning Point

It is shown in previous sections that LIM parameterization uniformize Adiabatic Mode spectrum through critical points such as cutoff in a narrowing duct and leaky range in a broadening bilinear duct propagation. There is another

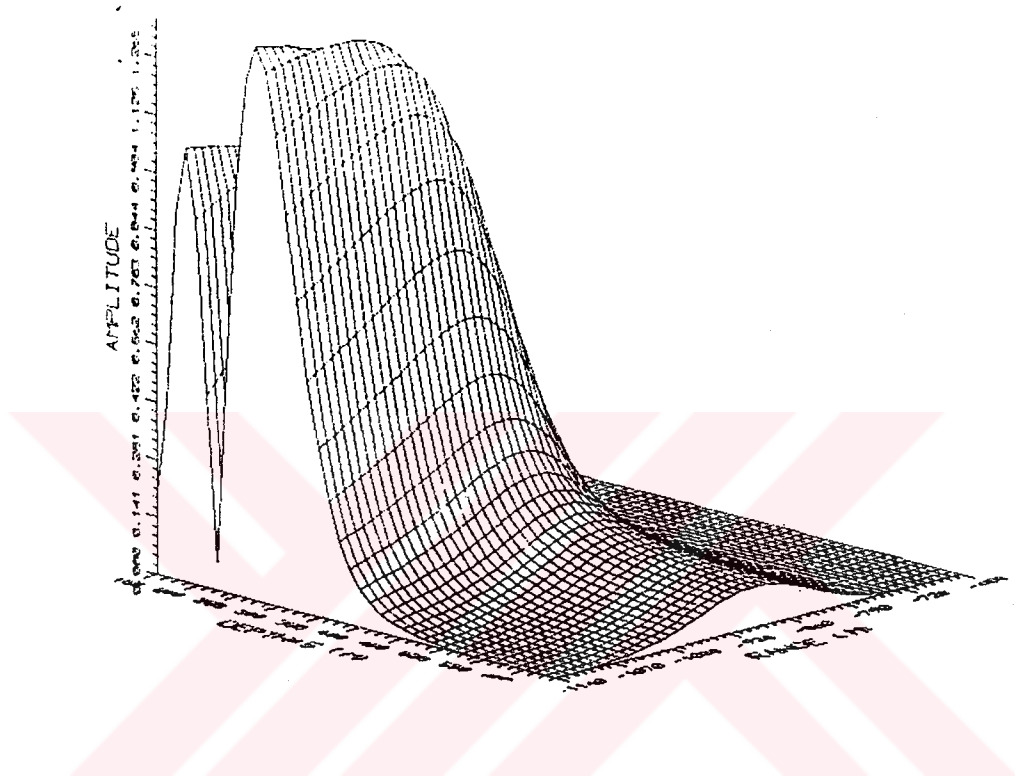


FIG.(5.5) : 3D.Plot of the fifth mode showing trapped-to-leaky transition. Only the last two cycles before caustic is shown

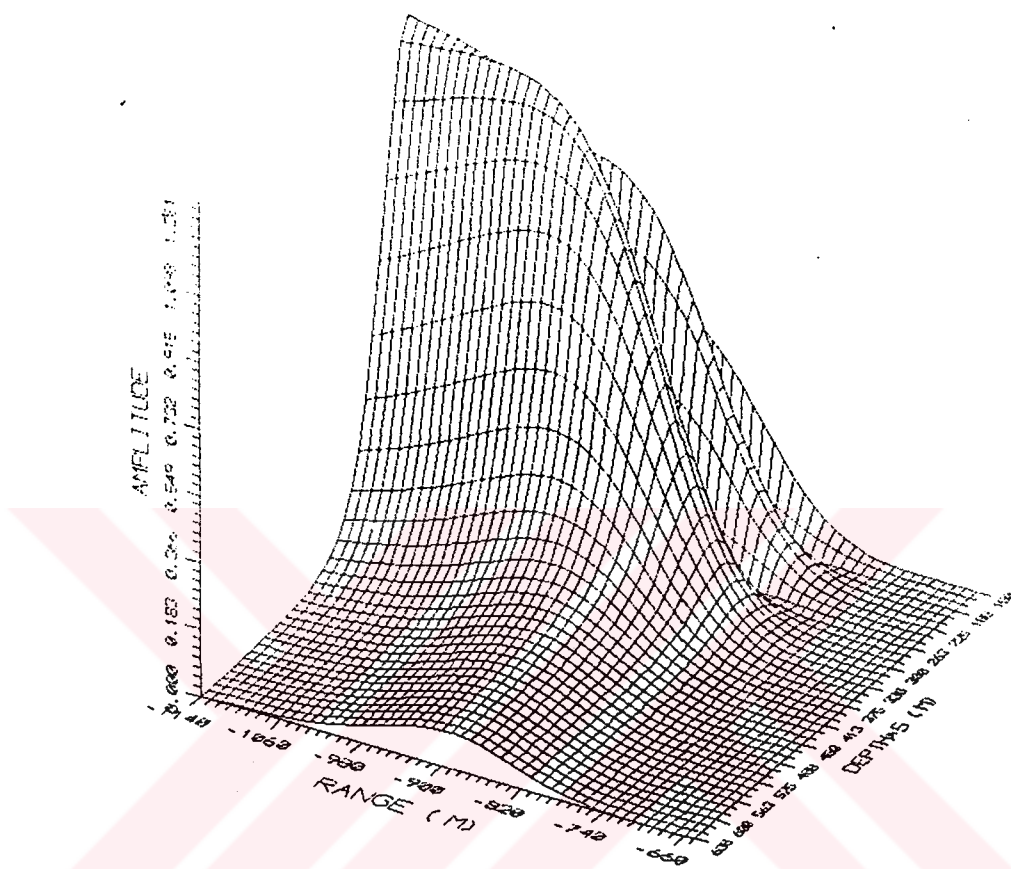


FIG.(5.6) : 3D.Plot of the fifth mode with different angle and tilt

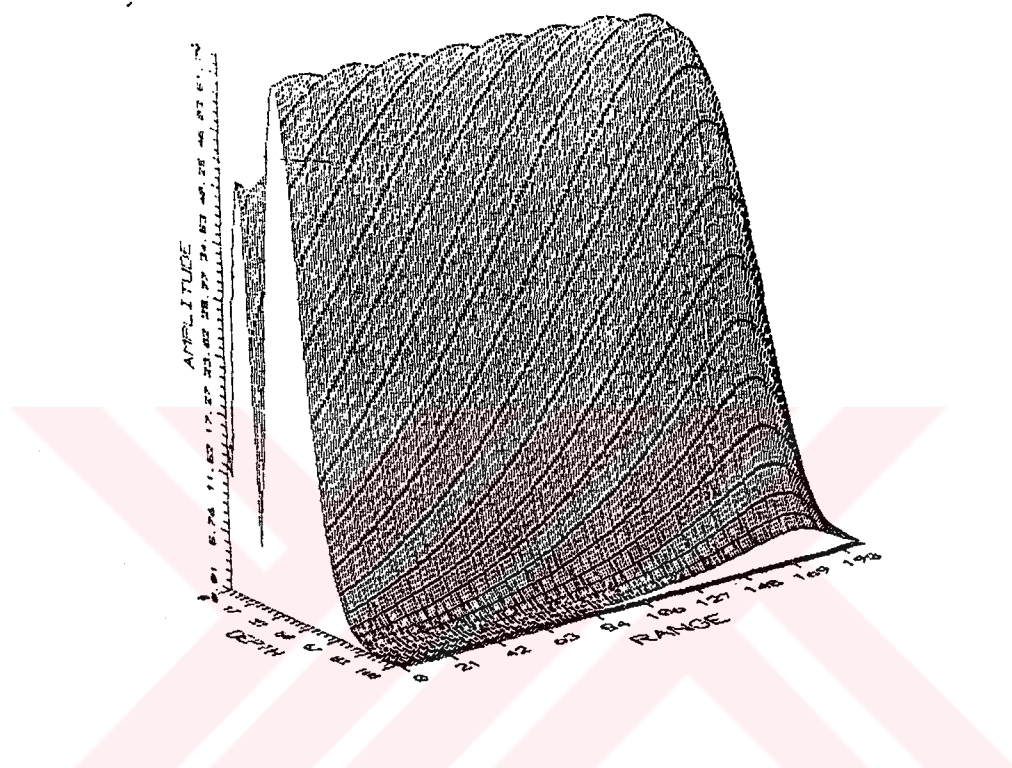
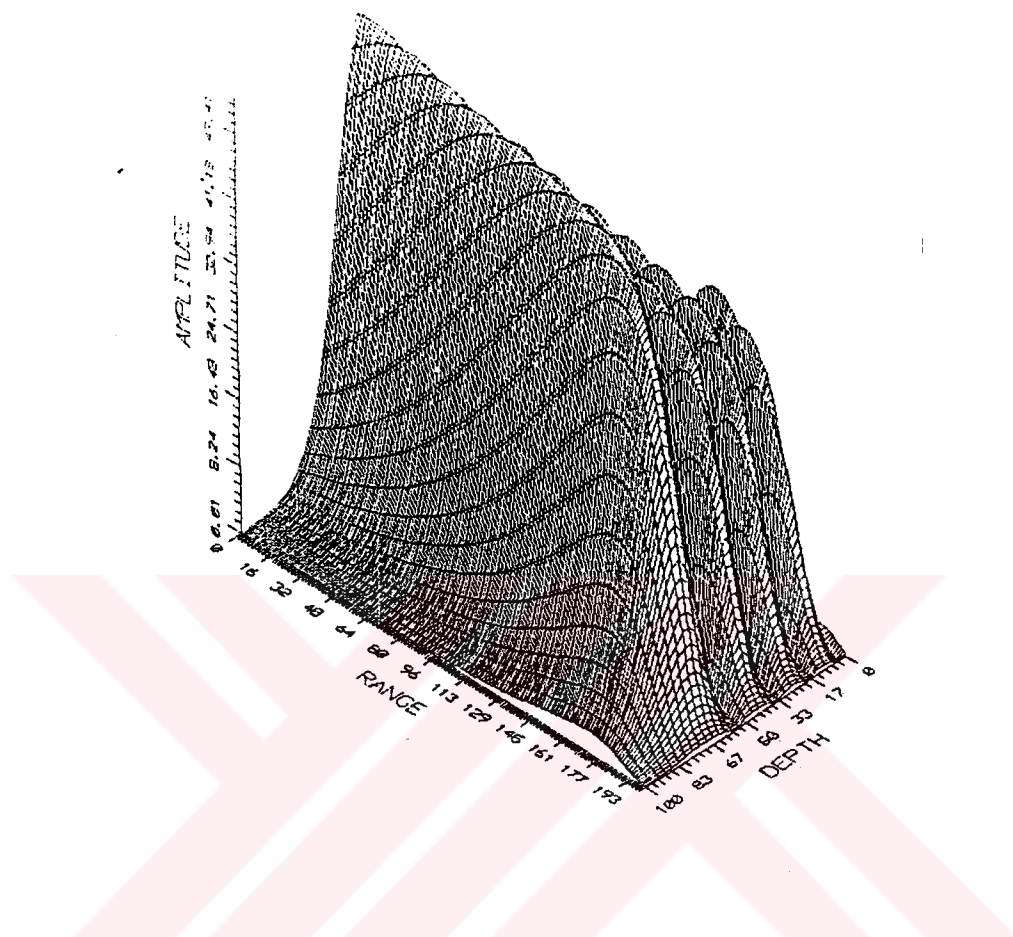


FIG.(5.7) : 3D.Plot of the twenty fifth mode



important critical point in ducted propagation which is guiding-to-antiguinding transition called the " turning point". It occurs in wave propagation problem with the wavenumber profile given,

$$k^2(x,z) = k_0^2 (1 + a_0 x z) \quad (5.26)$$

with the same boundary conditions as in section (4.1). Here negative Z - values cause mode trapping between the upper boundary and their caustics. Since there is no lower boundary infinite number of modes are trapped and propagate in Z - direction. The main difference of this problem and bilinear broadening duct problem is that the cause of leaky modes are different. In section (5.1) the modes leak to the bottom as a beam because of the change in profile at the duct depth X_d . But here the modes leak to the bottom as a beam during and after the transition through turning point which occurs by the change of the sign of the range coordinate at $z=0$. The caustic depth of each mode is found same as in (5.3).

$$X_{cm}(z) = - \sigma_m (a_0 k_0^2 z)^{-1/3} \quad (5.27)$$

Since the profiles are the same with the problem in section (4.1) it may directly be shown that the modal dependence is,

$$U_m(x, \bar{z}(\beta)) = C_0(\beta) Ai(1 - \frac{x}{X_{cm}(z)}) \quad (5.28)$$

The same LIM parameterization is tested for this problem too and found that it fails to describe the guiding-to-antiguinding transition through $z = 0$. The main reason of this failure can be directly seen from the eq.(5.27). As the observation point moves towards the turning point the caustic depth of each mode goes to infinity and modes never touch the turning point. Since the depth

dependence of the LIM parameterization are derived from the local Adiabatic Modes which are known to adopt themselves to the changing smooth environments, it is to be expected that Adiabatic Modes are not sufficient as an input near the guiding-to-antiguinding transition where the variations are not smooth anymore. However this guiding-to-antiguinding transition can be handled by Parabolic Equation Method which is pure numerical. By putting the adiabatic profile as an input to the PE algorithm it was shown that[11] the modes detach and leak as beams into deep water after passing through the transition region.

5.4) Propagation In Antiguinding Region

The propagation problem with the profile given in eq.(5.26) represent "narrowing duct" if $z < 0$ (i.e., in guiding region) and the propagation direction is taken towards the $z \rightarrow -\infty$, which is explained in details in section (4.1). But the same problem represents guiding-to-antiguinding "widening duct" if the propagation direction is taken towards the $z \rightarrow +\infty$. In this case $z < 0$ and $z > 0$ are the guiding and the antiguinding regions, respectively with the "turning point" at $z=0$. LIM integral with the adiabatic input in guiding region for this guiding-to-antiguinding problem can be given as in eqs.(4.15-17). The LIM integral accounts for the propagation in guiding region but fails to account for the transition through the "turning point" which is explained in section (5.3). Here, it is shown that with the analytic continuation applied to the eqs.(4.15-17) the LIM integral also accounts for the leakage in antiguinding region as a mode detachment from the surface. From the guiding region (i.e., $-z = z \exp(i3\pi)$) to the antiguinding region (i.e., $z = z \exp(i2\pi)$) the caustic depth of each mode in eq.(4.5) becomes complex as

$$\bar{x}_{cm}(z) = -\sigma_m(a_0 k_0^2 z)^{-1/3} = x_{cm} \cdot e^{-i5\pi/3} \quad (5.29)$$

while the argument of the Airy function in eq.(4.15) becomes,

$$\xi = -\sigma_m \left\{ 1 - \frac{x}{x_{cm}} e^{-i\pi/3} \right\} \quad (5.30)$$

Using the relation between σ and α of $W_1(\alpha)$ and $Ai(\sigma)$,

$$-\sigma_m = \alpha_m e^{i2\pi/3} \quad (5.31)$$

the argument in eq.(5.30) can be rewritten as,

$$\xi = e^{i2\pi/3} \left\{ \alpha_m - \frac{x}{(a_0 k_0^2 z)^{-1/3}} \right\} \quad (5.32)$$

From the definition of the Airy functions,

$$Ai(e^{i2\pi/3} \xi) = -\frac{1}{2} e^{-2\pi/3} W_1(\xi) \quad (5.33)$$

By using this relation the depth function given as in eq.(4.13) can be reorganized as,

$$U_m(x, z) = -\frac{1}{2} e^{-2\pi/3} W_1 \left\{ \alpha_m - \frac{x}{(a_0 k_0^2 z)^{-1/3}} \right\} \quad (5.34)$$

which explains the leakage of the modes in antiguiding region. The locations of the saddle points are,

$$\beta_s = \left\{ k_0^2 - \sigma_m (a_0 k_0^2 z)^{2/3} e^{i4\pi/3} \right\}^{1/2} \quad (5.35)$$

which are given in fig.(5.2) with the suitable integration contour. The saddle point evaluation of the LIM integral reduces the solution in antiguiding region to,

$$W_m(x, z) = - .5 \sigma_m^{1/4} \frac{e^{i7\pi/12}}{Ai(-\sigma_m)} \beta_s (k_0^2 - \beta_s^2)^{-1/2} e^{i\Omega(\beta_s)} \cdot W_1 \left\{ \alpha_m - \frac{x}{(a_0 k_0^2 z)^{-1/3}} \right\} \quad (5.36)$$

with

$$\Omega(\beta_s) = \frac{-3}{(a_0 k_{0m}^2)^{3/2}} \int_0^{\beta_s} \beta^2 (k_0^2 - \beta^2) d\beta \quad (5.37)$$

This asymptotically reduced mode function is calculated numerically in antiguiding region for various range points and the results are plotted in fig.(5.9). In this figure the discrete values are the LIM integral results which show a good agreement with the parabolic equation method in [15] as the observation point moves away from the transition region. Fig.(5.10) shows the constant amplitude curves taken from the LIM integral computation results. These constant amplitude curves give the creeping rays which are tangent to the wavevector in this antiguiding region.

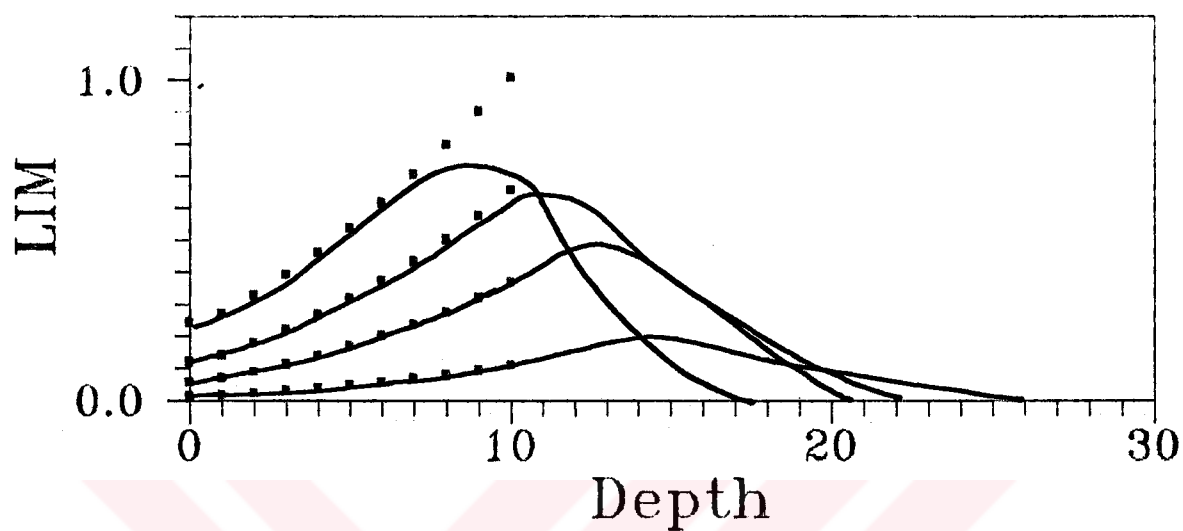


FIG.(5.9) : Transverse variatin in antiguiding region

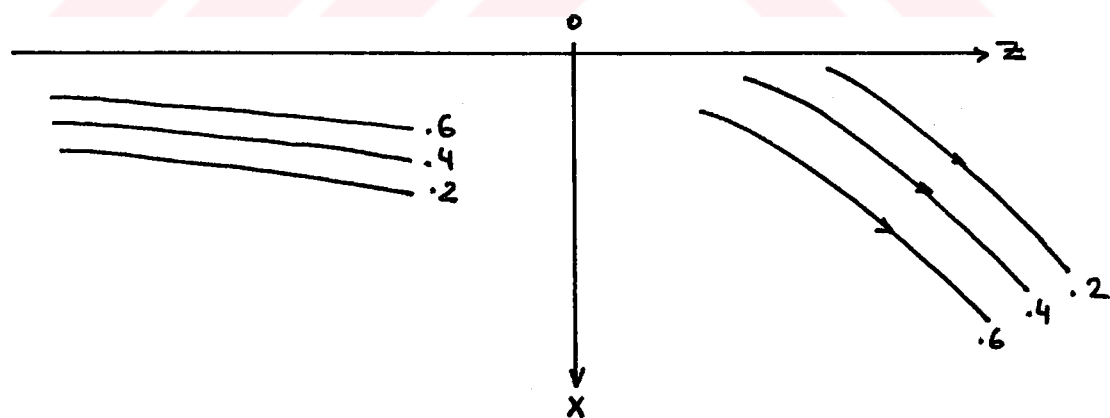


FIG.(5.10) : Constant amplitude curves

SECTION SIX

CONCLUSION

In this study, first the local intrinsic mode (LIM) formalism is tested for propagation in longitudinally varying environments against canonical configurations with simple exact-closed form- solutions in an "appropriate" separable coordinate system. From these canonical tests, for trapping between fixed boundaries as well as trapping by refraction, it has been shown how to parametrize LIM and how to reduce LIM integral analytically to the closed form solutions by suitable conformal mapping from longitudinal wavenumber spectral domain to the angular spectral domain. This suggests that the spectral content of the LIM is adequate for treating regular and cutoff transition phenomena even under noncanonical conditions because the basic LIM formulation does not rely on the "suitably chosen"[16] coordinate separability. It has also been shown that the LIM reduce to the simple adiabatic modes (AM) in weakly varying "regular" guiding environments and that the LIM uniformize the AM in propagation -to -evanescent cutoff transition and propagation -to -leaky transition where the AM fails. Then, numerical calculations have been carried out to test the accuracy of the LIM integral and the various asymptotic approximations such as SDP technic and uniform asymptotic form of the LIM integral in terms of standard functions. Because the integrand of the LIM integral is highly oscillatory, numerical integration along the original integration path is intensive due to the fine sampling required for stability; however it is found that the effective integral interval is restricted to the "visible spectrum" (propagating mode) range $\beta \leq k_0$. Integration, in the complex β plane, along the local steepest descent paths (SDP) through the saddle points ($\pm\beta_{sm}$) reduce the oscillations substantially, causes the integrand to decay rapidly away from the saddle pts. and makes computations more

from the saddle points, and makes computations more efficient. Except the transition regions, SDP numerical results agree very well with the direct integration results as well as the table values of the Bessel functions in the closed form exact solutions. Since the angular functions in canonical test sections are β -independent, the LIM integrals here actually compute Bessel or Hankel functions. So the main reliability and dependability of the LIM integration routine is carried out against these functions with their table values. The simple ($\pm 45^\circ$) SDP is not valid for the observation points near the transition ranges. Uniform asymptotic solutions which explicit and involve standard functions overcome the failure of the simple SDP evaluation through the transition domains.

In the last two sections (i.e., Sec.IV and Sec.V) two different noncanonical wave problems involving critical transition ranges are formulated with LIM integral. Both analytical and numerical results show that LIM representation accounts for the propagation- to-evanescent cutoff and propagation-to-leaky transitions. But the LIM integral with discrete adiabatic modes transverse profile fail to explain the guiding-to-antiguinding transition where the caustic depth of each mode goes to infinity making the transverse domain unbounded. Nevertheless, analytical continuation of the LIM spectra from the strongly trapping side to antiguiding side (both being away from the transition range) explains the initial growth of the beam-type leaky mode in antiguiding region. This study shows how to overcome critical ranges in wave propagation problems in complex environments by using LIM theory. Also a powerful and computationally efficient algorithm is developed for the computation of any kind of Bessel function with large argument and order. Finally it is concluded that asymptotic evaluation provides a simple, same efficient and accurate scheme in parameter ranges considered

APPENDIX A : Steepest Descent Path for weak depth dependence

If the Taylor's expansion is applied to the phase term of the LIM integral around the saddle point β_s ,

$$iP(z, \beta) \approx iP(z, \beta_s) + \frac{i}{2} P''(z, \beta_s) (\beta - \beta_s)^2 \quad (A.1)$$

is obtained. Here prime denotes the derivative with respect to β . Eq.(A.1) can also be written as,

$$iP(z, \beta) \approx iP(z, \beta_s) + \left| \frac{1}{2} P''(z, \beta_s) (\beta - \beta_s)^2 \right| e^{i2\psi} \quad (A.2)$$

where

$$\psi = \frac{\pi}{4} + \arg\left\{ (\beta - \beta_s) \right\} + \frac{1}{2} \arg\left\{ P''(z, \beta_s) \right\} \quad (A.3)$$

Since $\arg\left\{ P''(z, \beta_s) \right\}$ is constant, ψ changes only with $(\beta - \beta_s)$. The Steepest Descent Path (SDP) is a certain integration contour where the phase term behaves like,

$$\sim e^{-|P(z, \beta)|}$$

Thus, in a certain neighborhood of the saddle point $\beta = \beta_s$ in order to obtain this behavior ψ should be equal to $(\mp \pi/2)$ (i.e., $\cos(2\psi) = -1$). The second derivative of the phase term at the saddle point is found as,

$$P''(z, \beta_s) = \frac{3 \beta_s (k_0^2 - \beta_s^2)^{1/2}}{2 a_0 k_0^2 \sigma_m^{3/2}} \quad (A.4)$$

In the guiding region saddle points lie in symmetric pairs on the real β -axis between $\mp k_0$. So at the positive saddle point the argument of $P''(z, \beta_s)$ will be zero but at the negative saddle point it will be π . This gives,

$$\arg (\beta - \beta_s) = \begin{cases} \pi/4 & \text{at } \beta = +|\beta_s| \\ -\pi/4 & \text{at } \beta = -|\beta_s| \end{cases} \quad (\text{A.5})$$

As a result SDP makes a 45° angle with positive real β -axis and a -45° angle with the negative real β -axis as shown in fig.(4.4). The situation is different beyond cut-off since saddle points lie on the imaginary β -axis on both sides of the real β -axis. In this case SDP should pass through the saddle point which is on the positive imaginary axis as in fig.(4.4).

Appendix B : Defining the integration contour C_β

The integration contour C_β in eq.(5.20) is an infinite contour where the integrand converges as the β goes to infinity in complex β -plane. In order to satisfy this requirement the behavior of the integrand must be checked as β goes to infinity. Since as β goes to infinity the integral term in (5.21) of the total phase function dominates the integrand, it is sufficient to investigate this term which is equal to,

$$\Omega(\beta) = \frac{-3}{a_0 k_0^2 \sigma_m^{3/2}} \int_0^\beta \beta^2 (k_0^2 - \beta^2)^{1/2} d\beta \quad (B.1)$$

Writing $\Omega(\beta) = \text{Re}\{\Omega(\beta)\} + i.\text{Im}\{\Omega(\beta)\}$ will give,

$$i.\Omega(\beta) = i.\text{Re}\{\Omega(\beta)\} - \text{Im}\{\Omega(\beta)\} \quad (B.2)$$

which gives the convergence condition as,

$$\text{Im}\{\Omega(\beta)\} \geq 0 \quad (B.3)$$

which can also be written as,

$$\text{Im} \left\{ \frac{-3}{a_0 k_0^2 \sigma_m^{3/2}} \int_0^\beta \beta^2 (k_0^2 - \beta^2)^{1/2} d\beta \right\} \geq 0 \quad (B.4)$$

The roots of some of the modes for different observation points are given in table (5.1-2). In table (5.1) the roots of the first seven modes are given. It is seen that the roots are in the form of $\sigma_m = \sigma_{mr} - i \sigma_{mi}$ which has a positive real part and a negative imaginary part. Since,

$$\frac{-3}{a_0 k_0^2 \sigma_m^{3/2}} = -a - i b \quad (a \gg b) \quad (B.5)$$

where a and b are real numbers, eq.(B.4) can be rewritten as,

$$\text{Im} \left\{ (-a - i b) (\text{Re}\{f.. \partial \beta\} + i \text{Im}\{f.. \partial \beta\}) \right\} \geq 0 \quad (B.6)$$

which is equal to,

$$-a \cdot \text{Im}\{f.. \partial \beta\} - b \cdot \text{Re}\{f.. \partial \beta\} \geq 0 \quad (B.7)$$

For trapped modes since the root σ_m is real b is zero. This means only the first term of eq.(B.7) gives the convergence condition, but for leaky modes the whole condition must be satisfied. The integrand of the phase integral is,

$$\beta^2 (k_0^2 - \beta^2)^{1/2} = (\text{Re}\{\beta^2\} + i \text{Im}\{\beta^2\})(\text{Re}\gamma + i \text{Im}\gamma) \quad (B.8)$$

where

$$\text{Re}\gamma \triangleq \text{Re} (k_0^2 - \beta^2)^{1/2}, \quad \text{Im}\gamma \triangleq \text{Im} (k_0^2 - \beta^2)^{1/2} \quad (B.9)$$

Eq.(B.7) and (B.8) give the total convergence condition as,

$$\text{Re}\{\beta^2\} \cdot \text{Im}\gamma + \text{Im}\{\beta^2\} \cdot \text{Re}\gamma < 0 \quad (B.10)$$

$$\text{Re}\{\beta^2\} \cdot \text{Re}\gamma - \text{Im}\{\beta^2\} \cdot \text{Im}\gamma < 0 \quad (B.11)$$

Since the propagation towards the turning point is considered here it is known from the physics of the structure that the caustic depth of each mode broadens in this direction. Also there is only one-way propagation and there is no reflected wave. The locations of the saddle points for both trapped and leaky modes are shown in fig.(5.3). It is tested that the integration contour in this figure satisfies the necessary conditions given by eqs.(B.10-11). Naming the integration

segment on the second quadrant number one, the segment on the positive real axis number two, the segment on the fourth quadrant number three helps to check the convergence conditions via the tables given below.

$$\text{Re}\sqrt{} = \text{Re} \{ k_0^2 - \beta_r^2 + \beta_i^2 - 2 \beta_r \beta_i \}^{1/2}$$

$$\text{Im}\sqrt{} = \text{Im} \{ k_0^2 - \beta_r^2 + \beta_i^2 - 2 \beta_r \beta_i \}^{1/2}$$

$$\text{Re}\{\beta^2\} = \beta_r^2 - \beta_i^2 \quad \text{Im}\{\beta^2\} = 2 \beta_r \beta_i$$

SEGMENT	$\text{Re}\{\beta^2\} \cdot \text{Im}\sqrt{}$	+	$\text{Im}\{\beta^2\} \cdot \text{Re}\sqrt{}$	NEGATIVE	
one	-----	+	--	++	(-)
three	-----	++	-	++	(-)

For leaky modes the same check is repeated for the real part of the integrand as in table below.

SEGMENT	$\text{Re}\{\beta^2\} \cdot \text{Re}\sqrt{}$	-	$\text{Im}\{\beta^2\} \cdot \text{Im}\sqrt{}$	NEGATIVE	
one	-----	++	--	+	(-)
three	-----	++	-	++	(-)

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- [*] The routines in this study can be obtained from the author (i.e., Levent Sevgi Technical University of Istanbul, Dept. of EE MASLAK/ ISTANBUL TURKEY)

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