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ISTANBUL TECHNICAL UNIVERSITY * INSTITUTE OF SCIENCE AND TECHNOLOGY

DISTURBANCE DECOUPLING AND DISTURBANCE DECOUPLED
ESTIMATION IN REGULAR AND DECOMPOSED SYSTEMS

Ph.D. THESIS

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ABSTRACT

Disturbance decoupling and disturbance decoupled estimation problems are studied for finite dimensional, linear, time-invariant systems using geometric approach.

Existence conditions and design procedures of minimal order observers which are well known for systems subject to only control inputs are generalized to unknown input systems and PID class of observers. Existence conditions of full state observers are expressed in terms of strong observability and strong detectability and it is shown that their minimal orders are given by the dimensions of the smallest undetectable subspaces of nondynamic observers. A matrix algorithm is developed for the computation of minimal order observer matrices.

Solvability of disturbance decoupling and disturbance decoupled estimation problems is investigated for a decomposed system consisting of dynamic and algebraic parts. Conditions for the existence and construction methods of dynamic subsystems which give rise to a composite system satisfying the desired design objectives such as disturbance decoupling, strong observability, strong detectability, invertibility etc. are presented. It is shown that the solvability range of the problems is enlarged when formulated in a decomposed system.

ÖZET

LİNEER SİSTEMLERDE BOZUCU BASTIRMA VE BOZUCU BASTIRILMIŞ KESTİRİM PROBLEMLERİ

Bozucu bastırma ve bozucu bastırılmış kestirim problemleri sonlu boyutlu, lineer, zamanla değişmeyen sistemler için geometrik yaklaşımla incelenmekte; cebirsel ve dinamik parçalardan oluşmuş bir sistem için problemlerin çözülebilirlik sınırlarının genişlediği gösterilmektedir.

Durum denklemleri ile temsil edilen sistemin bozucu girişleri ve kontrol girişleri olmak üzere iki tür girişi; kontrol edilen çıkışlar ve ölçülen çıkışlar olmak üzere iki tür çıkışı olduğu varsayılmaktadır. Bozucu bastırma problemi (BBP) ve bozucu bastırılmış kestirim problemi (BBKP) bu sistem modeli için geometrik yaklaşımla incelenen ilk problemler olup, sistem teorisinde geometrik yaklaşımın ortaya atılmasında ve gelişmesinde önemli rol oynamışlardır [1-3].

İlk olarak Wonham ve Morse ile Basile ve Marro tarafından çözülen BBP, kontrol edilen çıkışların girişteki bozuculardan etkilenmemesini sağlamak amacıyla, ölçülen çıkışlardan kontrol girişlerine bir geri besleme kuralının bulunması olarak tanımlanabilir. Bu problem başlangıçta tüm durum vektörünün ölçülebildiği varsayımı altında çözülmüş [1, 7], sabit ve dinamik çıkış geri beslemesi durumunda çözüm ise [19-23] de ele alınmıştır. Bu çalışmalarda kapalı çevrim sistemin kararlılığı ve özdeğerlerinin yerleştirilmesi gibi ek koşullar da incelenmiştir.

Literatürde BBP'nin duali olarak bilinen BBKP, bozucuların varlığında ölçülen çıkışların ve kontrol girişlerinin işlenerek durum vektörünün veya durum vektörünün bir fonksiyonunun kestirimi olarak tanımlanabilir. Durum vektörünün tamamen ölçülemediği uygulamalarda durum geri besleme kurallarının gerçekleştirilmesi amacıyla incelenen bu problem, tüm girişlerin bilinmesi halinde [29-31] de ele alınmış, tam dereceli ve indirgenmiş dereceli gözleyici sistemlerin varlık koşulları ve tasarım yöntemleri verilmiştir. Bilinmeyen girişlerin varlığında problem, ilk olarak geometrik yaklaşımla Basile ve Marro [3] tarafından incelenmiş, durum vektörünün kestirimi için türev alıcılara dayalı bir gözleyicinin varlık koşulları ve tasarım yöntemi verilmiştir. Integral alıcılara ve hem integral hem de türev alıcılara dayalı asimtotik gözleyicilerin tasarımı daha sonraki çalışmalarda ele alınmıştır [23, 32-38, 45, 70].

Bu çalışmada yukarıda kısaca tanıtılan BBP ve BBKP ortak bir çerçeve içinde geometrik yaklaşımla incelenmekte, minimum dereceli ÇIT (çarpma-integral-türev) sınıfı gözleyiciler için kuvvetli gözlenebilirlik ve kuvvetli sezilebilirlik kavramlarına dayalı varlık koşulları ve tasarım yöntemleri verilmektedir. Cebirsel ve dinamik parçalardan oluşmuş bir sistem için problemler yeniden tanımlanmakta, çözülebilirlik sınırlarının genişlediği gösterilmektedir. Tezin ana hatları ve başlıca katkıları aşağıda bölümler halinde özetlenmiştir.

Bölüm 2'de tezde kullanılan geometrik yaklaşım için gerekli matematik kavramlar özetlenmekte ve notasyon tanıtılmaktadır. Bu kavramlar geometrik yaklaşımla ilgili literatürde [1-6] iyi bilinmekle beraber çoğunlukla dağınık biçimde ve farklı notasyonlar kullanılarak yer almaktadır. Bu bölümde topluca ve tek bir notasyon kullanılarak verilen kavramlar tezin kendi içerisinde tam olmasını sağlamak ve okuyucuya referans kolaylığı temin etmektedir. Bu amaçla lineer cebirle ilgili olarak lineer denklem sistemlerinin çözümlerinin varlığı ve tekliği için teoremler verilmekte ve geometrik yaklaşımda kullanılan değişmez alt uzaylar tanıtılmaktadır. Yönetilebilirlik, gözlenebilirlik, kararlılaştırılabilirlik ve sezilebilirlik gibi sistem teorisiyle ilgili temel kavramlar değişmez alt uzayların özellikleri olarak tanımlanmaktadır.

Bölüm 2'de tanıtılan değişmez alt uzayların uygulamasına bir örnek olarak BBP Bölüm 3'de incelenmekte, problemin sabit ve dinamik ölçüm geri beslemesi ile çözümü için gerek ve yeter koşullar ve geri besleme kurallarının bulunması için yöntemler verilmektedir. Bu bölümde yer alan sonuçlar ve tanıt teknikleri tezin ileri ki kısımlarında kullanılmaktadır.

BBKP ile ilgili özgün sonuçlar Bölüm 4'de toplanmıştır. Literatürde bilinmeyen girişli sistemler için gözleyicilerin varlık koşulları bilindiği halde, bu koşullar genellikle çeşitli alt uzaylar cinsinden verilmekte, sistemin gözlenebilirlik, sezilebilirlik gibi yapısal özellikleriyle ilişkisi bilinmemektedir. Bu açığı kapamak amacıyla ÇIT sınıfı gözleyicilerin varlığı için literatürde verilmiş olan koşullar yeniden düzenlenerek gözlenebilirlik ve sezilebilirlik kavramları ile ifade edilmeye uygun biçime getirilmiştir (teorem 4.1-4.3). Daha sonra bu koşullar tüm durum gözleyicileri için özelleştirilerek kuvvetli gözlenebilirlik ve kuvvetli sezilebilirlik [39-41] cinsinden koşullar elde edilmiştir. Böylece teorem 4.7'de kuvvetli sezilebilirliğin ÇIT gözleyicilerin varlığı için gerek ve yeter koşul olduğu gösterilmekte aynı zamanda kuvvetli sezilebilirlik için (A,E) -değişmez alt uzaylar cinsinden bir geometrik koşul

verilmektedir. Teorem 4.9'da ise ÇIT sınıfı tüm durum gözleyicilerin varlığı için kuvvetli sezilebilirlikle beraber sağlanması gereken ek koşullar verilmektedir. Bu sonuçlar [37,38] de kuvvetli gözlenebilirlik cinsinden elde edilmiş olan sonuçları daha ileri götürmekte ve ÇIT sınıfı gözleyiciler için genelleştirmektedir. Bilinmeyen girişlerin olması halinde başlangıç durumunun bilinmesi durum vektörünün her t için bilinmesine yeterli olmamaktadır. Bu düşünceden hareketle bilinen-başlangıç durumu-gözleyicilerin tasarımı teorem 4.4.'de ele alınmış, bilinen-başlangıç durumu-tüm durum gözleyicilerin varlığı için gerek ve yeter koşullar teorem 4.8'de verilmiştir. Bu sonuç tanım 4.6'da verilen bilinen-başlangıç durumu-gözlenebilirlik kavramına dayanmaktadır. Bilinmeyen girişli sistemlerin gözlenebilirliği ve sezilebilirliği ile ilgili dinamik, geometrik ve frekans tanım bölgesi koşullar Tablo 4.2'de özetlenmektedir.

Çeşitli gözleyici tiplerini birbiriyle karşılaştırmak ve durum vektörünün izdüşümünün kestirilebileceği alt uzayların sınırlarını ortaya koymak amacıyla Kısım 4.2'de sezilemez alt uzaylar tanımlanmaktadır. Sezilemez alt uzaylar kümesinin toplama ve arakesit alma işlemleri altında kapalı olduğu gösterilmekte, bilinen ve bilinmeyen başlangıç durumu gözleyiciler için en küçük sezilemez alt uzaylar teorem 4.5 ve teorem 4.6'da verilmektedir. Bu alt sınırların gözleyici yapısından tamamen bağımsız olup zamanla değişen veya sonsuz boyutlu gözleyiciler için dahi geçerli olması dikkate değer bir sonuçtur. ÇIT sınıfı gözleyicilere ilişkin en küçük sezilemez alt uzaylar daha önce verilmiş olan varlık koşulları yardımıyla hesaplanmış ve bir kafes yapı oluşturdıkları gözlenmiştir (Şekil 4.3). Bu yapı, çeşitli gözleyici tiplerinin sezilemez alt uzaylar cinsinden üstünlüklerinin ortaya konması bakımından değer taşımaktadır. Sezilemez alt uzaylar için verilmiş olan alt sınırların ÇIT gözleyiciler kullanarak erişilebileceğinin gösterilmesi de ayrıca ilgi çekici bir sonuçtur.

Gözleyici tasarımı ile ilgili olarak literatürde sık incelenen bir diğer sorun, indirgenmiş ve minimum dereceli gözleyicilerin tasarımıdır. Bu konudaki ilk çalışmalar Luenberger [29,30] tarafından yapılmış, indirgenmiş dereceli gözleyicilerin varlık koşulları ve tasarım yöntemleri verilmiştir. Daha sonraki çalışmalarda [1,43-45] Luenberger'in öne sürdüğü derecenin daha da indirgenebileceği ortaya çıkmış ve bilinen girişli sistemler için tüm durum gözleyicilerin minimum dereceleri saptanmıştır [36]. Kısım 4.4'de indirgenmiş dereceli ÇI gözleyicilerin tasarımı bilinmeyen girişli sistemler için genelleştirildikten sonra minimum dereceli ÇIT sınıfı gözleyicilerin gerçekleştirilmesi için ÇT sınıfı bir gözleyici ve indirgenmiş dereceli bir ÇI gözleyicinin

kaskad bağlanmasından oluşan bir yapı önerilmiştir (Şekil 4.4). Burada amaç durum vektörünün mümkün olduğu kadar büyük kısmını ÇT gözleyici kullanarak kestirmek ve ÇT gözleyicinin çıkışını ÇI gözleyici için giriş olarak kullanmaktır. Bu yapı kullanılarak minimum dereceli ÇIT sınıfı tüm durum gözleyicilerin kuvvetli sezilebilirlik cinsinden varlık koşulları ve tasarım yöntemi teorem 4.10'da verilmektedir. Teorem 4.11 ise minimum derecelerin ÇT sınıfı gözleyicilerin en küçük sezilemez alt uzaylarının boyutuna eşit olduğunu göstermektedir. Bu sonuç kullanılan türev alıcı sayısı ile minimum derece arasındaki kesin ilişkiyi ortaya koymakta ve türev alıcı sayısı arttıkça derecenin azaldığını göstermektedir.

Daha önceki kısımlarda geometrik dille verilmiş olan gözleyici tasarım yöntemleri Kısım 4.5'de matrissel dile çevrilmiş ve bilinmeyen girişli, minimum dereceli ÇIT sınıfı tüm durum gözleyicilerin tasarımı için bir algoritma verilmiştir. Verilen yöntem literatürde bilinen girişli sistemler için verilmiş olan yöntemlere [1,46] benzemekte ve gözleyici matrislerinin, sistem denklemlerinden durum uzayı dönüşümüyle elde edilen yeni denklemlerde görülen alt matrisler cinsinden hesaplanmasına imkan tanımaktadır. Matrissel dile çevrilen gözleyicilerin varlık koşulları ve tasarım yöntemleri sonuç teorem 4.4'de özetlenmektedir.

Bölüm 5'de BBP ve BBKP'nin çözümü frekans tanım bölgesinde polinom ve rasyonel matrisler kullanılarak incelenmekte ve problemler arasındaki duallik ilişkisi açıklanmaktadır. Sistem teorisinde frekans tanım bölgesi yaklaşım son yıllarda gelişmiş ve [51,52,56-58] in öncülük ettiği çalışmalarla çok girişli-çok çıkışlı sistemlere uygulanabilir hale gelmiştir. Bu tezde incelenen problemler açısından bu tür bir yaklaşım problemin daha iyi anlaşılması ve kimi zaman sayısal açıdan daha uygun çözümler sunması bakımından önem taşımaktadır. Bu amaçla BBP ve BBKP'nin çözülebilirliğinin rasyonel matris denklemlerinin çözümlerinin varlığına eşdeğer olduğu teorem 5.1 ve teorem 5.4'de gösterilmiştir. Bu teoremler literatürde gözlenebilirlik veya yönetilebilirlik varsayımları altında [27,58]'de verilen benzer sonuçları bu varsayımların sağlanmadığı durumlar için genelleştirmektedir. Elde edilen sonuçların bir yan ürünü olarak ÇIT sınıfı geri besleme kurallarının bulunması için [25]'de verilen yöntemden farklı bir yöntem (teorem 5.3) ve ÇIT sınıfı BBP ile ÇIT sınıfı sıfır durum gözleyici problemleri arasında bire-bir bir dual ilişki (Tablo 5.1) verilmiştir.

Kısım 5.2'de bilinmeyen girişli sistemlerin gözlenebilirliği ve sezilebilirliği için Bölüm 4'de verilmiş

olan dinamik ve geometrik kořullardan yola çıkarak, polinom sistem matrisler cinsinden kořullar elde edilmiştir. [37]'de de yeralan bu kořullara ek olarak bilinen-başlangıç durumu-gözlenebilirlik için polinom sistem matrisi cinsinden yeni bir kořul teorem 5.8'de çıkarılmıştır. Sayısal açıdan test edilmeye çok uygun olan bu kořul sistemlerin sağ ve sol terslerinin varlığının incelenmesi için de kullanılabilir. Polinom sistem matrisler cinsinden verilen kořulların sistem sıfırları cinsinden yorumu teorem 5.10'da sunulmaktadır.

Bölüm 6'da BBP ve BBKP'nin çözümü cebirsel ve dinamik parçalardan oluşmuş ayrıştırılmış bir sistem için incelenmektedir. Gerçekte pek çok fiziksel sistemin cebirsel ve dinamik parçalardan oluşmuş olduğu düşünülebilir. Örneğin bir elektrik devresinde direnç, bağımlı ve bağımsız kaynaklar cebirsel alt sistemi; kapasite, self gibi elemanlarsa dinamik alt sistemi oluştururlar. Literatürde bu tür sistem gösterilimi durum denklemlerinin çıkarılmasında veya bir sentez probleminde ilk adım olarak kullanılmaktadır. [71]'de cebirsel ve dinamik parçalardan oluşmuş bir m-kapılının yönetilebilirliği, gözlenebilirliği ve bu özelliklerin dinamik n-kapılının seçimiyle nasıl değıştikleri incelenmiştir. Ayrıştırılmış sistemlerde BBP ve transfer matrisi değışmez bırakan dinamik alt sistemlerin bulunması sorunu daha sonraki çalışmalarda ele alınmıştır [72-74].

Bu tezde BBP ve BBKP yukarıda sözü edilen çalışmalar doğrultusunda ayrıştırılmış sistemler için yeniden tanımlanmakta ve çözülebilirlik sınırlarının genişlediği gösterilmektedir. Kısım 6.2'de tanımlanan açık çevrim (kapalı çevrim) BBP'nde amaç bileşik sistemi bozucu bastırılmış kılacak dinamik alt sistemin (dinamik alt sistem ve geri besleme kuralının) seçimidir. Tanımlanan problemlerin eşdeğer olmasını sağlayan bir yeter kořul teorem 6.1'de verildikten sonra açık çevrim BBP teorem 6.2'de çözülmüştür. Teorem 6.3'de ise sistemi hem bozucu bastırılmış hem de kararlı kılacak dinamik alt sistemlerin varlığı için gerek ve yeter kořullar verilmiştir. Kapalı çevrim BBP durum geri besleme, durum ve bozucu geri besleme ve dinamik ölçüm geri besleme kontrol kuralları için teorem 6.4-6.6'da çözülmektedir. Geri besleme problemlerinin çözümü ayrıştırılmamış sistemler için verilen çözümlerle hemen her noktada çözülebilirlik açısından karşılaştırılmış, ayrıştırılmamış sistemlerdeki durumun aksine tüm geri besleme problemlerinin hemen her noktada çözülebilir olduğu gösterilmiştir (teorem 6.7). BBP'lerinin pasif ve kayıpsız dinamik alt sistemlerle çözümü için gerek ve yeter kořullar teorem 6.8'de verilmiştir. Bu sonuçlar kesin pozitif ve simetrik matrislerin temel bir özelliğini açıklayan yardımcı teorem 6.3'e dayanmaktadır.

[71]'de ayrıştırılmış sistemlerin gözlenebilirliği için elde edilmiş olan sonuçlar kısım 6.3'de bilinmeyen girişli sistemler için genelleştirilmektedir. Teorem 6.13'de bileşik sistemi kuvvetli gözlenebilir (kuvvetli sezilebilir) kılan dinamik alt sistemlerin varlığı için gerek ve yeter koşullar ve var olduğu takdirde hesaplanmasını sağlayan bir yöntem verilmiştir. Bu sonuç bilinmeyen girişlerin olmaması halinde [71]'de elde edilen sonuçla çakışmaktadır. Ayrıştırılmış sistemlerin bilinen-başlangıç durumu-gözlenebilirliği ise teorem 6.11, 12'de incelenmiştir. Bu sonuçların bir yan ürünü olarak ve duallık ilkesinden yararlanarak elde edilen ayrıştırılmış sistemlerin yönetilebilirliği, kararlılaştırılabilirliği sağ ve sol terslerinin varlığı için gerek ve yeter koşullar da bu kısımda yer almaktadır. Elde edilen tüm sonuçlar yapıcı olup dinamik alt sistemlerin tasarımı için kavramsal algoritmalar kısım 6.4'de önerilmiştir. BBP ve BBKP'nin ayrıştırılmış sistemler için çözümü Tablo 6.1 ve Tablo 6.2'de özetlenmektedir.

Bu çalışmada ele alınan problemleri ayrıştırılmış sistemler için incelemenin getirdiği üstünlük ayrıştırılmış sistemi oluşturan parçalardan dinamik alt sistemin değişken yapısından kaynaklanmaktadır. Bu açıdan bakıldığında literatürde durum geri besleme, çıkış geri besleme gibi klasik yöntemlerle incelenmiş olan model uyumlaştırma, etkileşimsiz kontrol, yaklaşık bozucu bastırma gibi problemlerin de ayrıştırılmış sistemler için incelendiğinde çözülebilirlik sınırlarının genişleyeceği öngörülebilir. Bu çalışmada elde edilen sonuçlar ve kullanılan yöntemler yukarıda sözü edilen benzer problemlerin ayrıştırılmış sistemlerde çözümü için bir temel oluşturacak niteliktedir.

CHAPTER 1

INTRODUCTION

Physical systems usually operate under the presence of unknown, unmeasurable inputs of diverse nature and sources. Noise governed by nature, man made interference or internal system parameter variations are examples of signals which can be regarded as unknown inputs affecting the system behaviour. It is inevitable that a unified theory must be developed for the analysis, synthesis and control of systems subject to unknown inputs which may be called disturbances.

Some of the frequently used approaches to deal with disturbance inputs are to treat them as random signals with known statistics or to assume that they satisfy differential equations with known coefficients, their initial conditions being unknown. While it is true that best results can be obtained by modeling disturbances as accurately as possible suitable for the specific application, none of these models is rich enough to cover all types of disturbances that can be encountered in practice.

In this work a unified theory for disturbance decoupling and disturbance decoupled estimation problems has been presented by considering disturbances as completely arbitrary signals which belong to a certain function class. The choice of the function class is not crucial. The space of piecewise continuous functions or the space of continuously differentiable functions can be taken depending on the situation. This assumption eliminates a lot of modeling labour and enables to

concentrate on structural problems thereby shedding light on the problems arising for the more restrictive selection of disturbance models.

The system is assumed to be finite dimensional, linear, time-invariant and possesses two types of inputs: control inputs, disturbance inputs and two types of outputs: measured outputs and controlled outputs. (The outputs may have components in common.) This class of systems has been largely studied in system and control theory by the classical frequency domain techniques prior to 1960s and then by the state space approaches. More recently, the rational and polynomial matrix methods have become popular through the efforts in [51, 52, 56-58]. Our treatment of the problems is mainly in time domain using state space representations and the geometric approach developed by Wonham and Basile and Marro [1-3].

For the system and disturbance model described above the disturbance decoupling problem (DDP) and disturbance decoupled estimation problem (DDEP) have been extensively studied in literature. In fact, these two problems have motivated much of the development of the geometric approach in the original works [1-3,7,8]. The objective in DDP is to find a feedback control law from the measured output to the control input so that the controlled outputs are not affected by the disturbances acting at the system input. The early papers in this area [7,17,18] have considered disturbance decoupling by constant measurement feedback and assumed that the complete state vector can be measured. Disturbance decoupling by dynamic measurement feedback has been solved in [20-23]. Stability and pole placement requirements in the closed loop system have also been considered in these works.

DDEP is concerned with the design of systems, called observers, which process the measured outputs and control inputs of the given system in order to obtain an estimate of the state vector or a function of the state vector. Various observer types have been investigated in literature. The observer employed by Basile and Marro [3] has been based on differentiators. Existence conditions and design procedures of observers using integrators or differentiators and integrators simultaneously have been given in [23,27,29-37]. One of the main points of interest in observer design problems has been to reduce the order of the observer. Design of reduced order observers for systems subject to only control inputs has been considered in [30,36,43-46].

The outline of the thesis and its main contributions are as follows.

In Chapter 2 the necessary mathematical background is reviewed and notation is introduced. This includes some of the well known linear algebra results like existence and uniqueness of solutions of linear equations and definitions and properties of invariant subspaces used in the geometric approach. The system theoretic concepts of controllability, observability, stabilizability and detectability are introduced as properties of certain invariant subspaces. The material in this chapter is standard in geometric approach literature [1-6,13,23] and provides the self containment of the thesis.

In Chapter 3 a brief survey of DDP is presented which also serves to illustrate the application of the invariant subspaces introduced in Chapter 2. Necessary and sufficient conditions for the solvability of DDP by constant and dynamic measurement feedback and procedures for constructing the feedback control laws are given in

sections 3.2, 3.3 following the works [1,7,8,17-23]. These results and proof techniques are used in subsequent chapters.

Chapter 4 is devoted to DDEP. Existence conditions and design procedures of minimal order observers which are well known for systems subject to only control inputs are generalized to unknown input systems and PID (proportional-integral-derivative) class of observers. A subspace of the state space along which the projection of the state vector can be estimated is defined as an undetectable subspace. It is shown that the class of undetectable subspaces is closed under subspace intersection hence possesses a smallest element. The smallest undetectable subspaces for known and unknown initial state observers are established. It is shown that these lower bounds can be attained by using a PID observer. A hierarchical classification of PID class of observers is given based on their undetectable subspaces which constitute a lattice structure.

Though the relation between observability, detectability and existence of observers estimating the full state vector of a system subject to only control inputs is well known [30,31] similar conditions do not appear in literature for systems excited by unknown inputs except some preliminary results reported in [37,38]. In section 4.3 the results in these works are extended to PID class of observers using the concepts of strong observability and strong detectability defined in [39-41]. By specializing the existence conditions given in section 4.1 to full state observers it is proven that strong detectability is equivalent to the existence of a full state PID observer. Additional conditions required for the existence of full state PID class of observers are also derived.

Based on the results of previous sections existence conditions and design procedures of minimal order full state PID class of observers are presented in section 4.4. It is shown that the minimal orders are given by the dimensions of the smallest undetectable subspaces of nondynamic (PD class) observers and a trade off exists between the observer order and the number of differentiators employed. If unknown inputs are not present, the minimal order given for a full state PI observer coincides with the result obtained in [36] but lower than the orders given in [30,31,45].

Finally, in section 4.5 a completely algebraic matrix algorithm is presented for the computation of the parameters of a minimal order PID class of observer which is similar to the methods given for known input systems [46]. The algorithm enables the computation of observer matrices directly in terms of the submatrices of a system which is obtained from the original one by state space and measurement space transformations and is applicable to PI observers as well as to PID observers.

The purpose of Chapter 5 is to bridge the geometric and the more recent frequency domain approaches [37,57-61] to DDP and DDEP and establish the duality between the two problems. With this objective, the equivalence of DDP and DDEP to solvability of rational matrix equations is derived in section 5.1. Geometric conditions for the existence and uniqueness of solutions of rational matrix equations are given in terms of the realization of rational matrices in state space form. These results generalize those given in [27,58,60,61] for controllable or observable realizations to possibly uncontrollable, unobservable realizations. As side results new synthesis methods are proposed for feedforward control laws [25] and inverse systems [63].

Frequency domain conditions in terms of the polynomial system matrix for strong observability and strong detectability originally reported in [37,39] are rederived in section 5.2 starting from their dynamical definitions and the geometric conditions given in Chapter 4. A new frequency domain condition is given for known-initial state-observability which is convenient to test numerically. The relationship of unknown input observability, detectability with the system zeros [65,66] is made precise. These conditions which have intuitive interpretations for single input, single output systems provide a better understanding of the problem.

In Chapter 6 DDP and DDEP are investigated for a decomposed system consisting of fixed algebraic and variable dynamic parts. Most physical systems can be viewed as the interconnection of algebraic and dynamic subsystems. An immediate example is an electrical network in which case the algebraic subsystem consists of components such as resistors, dependent and independent sources and the dynamic subsystem consists of inductors, capacitors etc. This kind of system representation is generally used to formulate state equations or as a first step in a synthesis problem. Controllability, observability of multi port networks and how these properties change with the selection of dynamic and algebraic parts have been investigated in [71]. Transfer matrix invariance and disturbance decoupling in decomposed systems have been studied subsequently in [72,73]. The advantage of considering system decomposition is in the additional degree of freedom gained by the variable structure of the dynamic subsystem. It is shown that the solvability range of the problems is enlarged when formulated in a decomposed system.

The definition and basic properties of decomposed systems are introduced in section 6.1. Disturbance decoupling in decomposed systems is discussed in section 6.2. The open loop and closed loop disturbance decoupling problems are defined and solved which improves and extends the results in [72-74]. The open loop disturbance decoupling is solved under the additional condition of stability. The closed loop disturbance decoupling is investigated for constant state feedback with and without disturbance feedforward and dynamic measurement feedback. A set of necessary and sufficient conditions are also obtained for the solvability of disturbance decoupling problems by passive and lossless dynamic subsystems. The results are compared with regular state space systems in terms of their generic solvability. Unlike regular systems generic solvability conditions are obtained for all feedback problems.

Solvability of DDEP in decomposed systems is investigated in section 6.3. The results in this section generalize those of [71] to unknown input systems by giving necessary and sufficient conditions for the existence of dynamic subsystems which give rise to a strongly observable, strongly detectable or known-initial state-observable composite system. A number of side results are also obtained for the controllability, stabilizability and invertibility of decomposed systems. Besides their basic significance in system and control theory the concepts of controllability, observability and input observability also play key roles in the circuit theoretic studies of passivity and losslessness [76,77].

The presentation of the thesis is original and its contributions consist of the results of Chapters 4-6 mentioned above. The main results are displayed in

tables and illustrated by numerical examples. The thesis is concluded by summarizing its contributions and suggesting related areas for further research.



CHAPTER 2

MATHEMATICAL PRELIMINARIES AND SYSTEM THEORETIC INTERPRETATIONS

Some definition and theorems associated with the geometric approach to linear system theory are collected in this chapter. The aim is to provide ease of reference and to introduce relevant notation. Detailed proofs and extensions can be found in the references cited.

Throughout the thesis uppercase is used for linear mappings and their matrix representations and lowercase for vectors. Unless otherwise stated the letters F, K, N , and Q to Z are preassigned to indicate subspaces and vector spaces. The dimension of a subspace S is denoted by $d(S)$. $S^\perp$ stands for the orthogonal complement of S . The image (kernel) of a map A is written as $\text{im } A$ ($\ker A$). If $A:S \rightarrow V$ is a linear mapping $A':V' \rightarrow S'$ represents the dual of A . The symbol $\sigma(A)$ denotes the spectrum (the set of eigenvalues) of A .

Certain invariant spaces are fundamental for a geometric approach to linear system theory and these are usually associated with the dynamical behaviour of the system described by

$$\dot{x} = Ax + Bu, \quad y = Cx \quad (2.1)$$

with $x(t) \in X = \mathbb{R}^n$ the state, $u(t) \in U = \mathbb{R}^m$ the input and $y(t) \in Y = \mathbb{R}^p$ the output and $A:X \rightarrow X$, $B:U \rightarrow X$ and $C:Y \rightarrow X$ are maps of appropriate dimensions. Although the model (2.1) corresponds to a continuous time system, all results in this chapter apply equally well to the

corresponding discrete time system model, as they are essentially algebraic properties of the triple (C, A, B) .

2.1 SOME RESULTS FROM LINEAR ALGEBRA

The first lemma is a standard result on solvability of linear matrix equations.

Lemma 2.1 : Let $M: X \rightarrow Y$, $N: X \rightarrow Z$ be arbitrary maps.

1) There exists $X: Y \rightarrow Z$ such that $XM=N$ if and only if $\ker M \subset \ker N$ 2) Assuming that it exists X is unique if and only if M is surjective.

Proof: 1) «Only if» Take $x \in \ker M$ then $Nx=XMx=0$ hence $x \in \ker N$ proving that $\ker M \subset \ker N$

«If» Let $\{x_1 \dots x_j, x_{j+1} \dots x_k\}$ be a basis for X such that $\{x_1 \dots x_j\}$ is a basis for $\ker M$. As the vectors $Mx_{j+1} \dots Mx_k$ are linearly independent and span $\text{im} M$, X_0 can be defined on $\text{im} M$ as $X_0(Mx_i)=Nx_i$, $i=j+1 \dots k$. If $\text{im} M=Y$ we can take $X=X_0$ and the definition of X is complete. Otherwise let X be an extension of X_0 from $\text{im} M$ to Y then X satisfies $XM=N$.

2) The arbitrariness of extension when M is not onto shows that X is nonunique in this case. Conversely if there exists $X_1 \neq X_2$ such that $X_1 M=N$ and $X_2 M=N$ then $(X_1-X_2)M=0$ ie, the rows of M are linearly dependent hence $\text{rank } M < d(Y)$ proving that M is not surjective.

The dual of lemma 2.1 is as follows

Lemma 2.2: Let $A: V \rightarrow S$, $B: W \rightarrow S$ be arbitrary maps.

1) There exists $X: W \rightarrow V$ such that $AX=B$ if and only if $\text{im} B \subset \text{im} A$, 2) Assuming that it exists X is unique if and only if A is injective.

Proof: Consider the dual maps $A':S' \rightarrow V'$, $B':S' \rightarrow W'$ there exists $X:W \rightarrow V$ such that $AX=B$ if and only if there exists $X':V' \rightarrow W'$ such that $X'A'=B'$. Moreover the map X satisfying $AX=B$ is unique if and only if the map X' satisfying $X'A'=B'$ is unique. The conclusion now follows from the previous lemma since $\ker A' \subset \ker B'$ if and only if $\operatorname{im} B \subset \operatorname{im} A$ and A' is surjective if and only if A is injective.

The following definitions will be needed in the sequel.

Definition 2.1: Let $\underline{S}$ be a nonempty set of subspaces, the largest or supremal element S^* of $\underline{S}$ is defined to be that member of $\underline{S}$ (when it exists) which contains every member of $\underline{S}$ i.e., 1°) $S^* \in \underline{S}$ and 2°) if $S \in \underline{S}$ then $S \subset S^*$

Definition 2.2: Let $\underline{S}$ be a nonempty set of subspaces, the smallest or infimal element S_* of $\underline{S}$ is defined to be that member of $\underline{S}$ (when it exists) which is contained in every member of $\underline{S}$ i.e., 1°) $S_* \in \underline{S}$ and 2°) if $S \in \underline{S}$ then $S_* \subset S$

It is clear from the definitions that the supremal and infimal element of a set of subspaces is unique if it exists. The next lemma gives conditions under which supremal or infimal element of a set of subspaces exists.

Lemma 2.3: Let $\underline{S}$ be a nonempty set of subspaces closed under subspace addition (intersection). Then $\underline{S}$ possesses a supremal (infimal) element.

Proof: The lemma can be proved by constructing a nondecreasing sequence of subspaces $S_1 \subset S_1 + S_2 \subset S_1 + S_2 + S_3 \dots$ when $\underline{S}$ is closed under subspace addition and a nonincreasing sequence $S_1 \supset S_1 \cap S_2 \supset S_1 \cap S_2 \cap S_3 \dots$ when $\underline{S}$ is closed under subspace intersection where $S_i \in \underline{S}$ for

$i=1,2,\dots$ and each inclusion is strict. As the spaces are finite dimensional the chains can not be continued beyond, say, k terms. Then we can set $S^*=S_1+S_2+\dots+S_k$ and $S_*=S_1\cap S_2\cap\dots\cap S_k$. Clearly $S^*(S_*)\in\underline{S}$ and contains (is contained in) every $S\in\underline{S}$.

The concept of factor space is introduced in the next definition [1].

Definition 2.3: Let $S\subset X$. The factor space X/S is defined as $X/S=\{\bar{x}:\bar{x}=x+S, x\in X\}$

For $x\in X$ the element $\bar{x}=x+S\in X/S$ is called the coset of x . It is well known that X/S is indeed a vector space with addition and multiplication by scalars defined as $\bar{x}_1+\bar{x}_2=\overline{x_1+x_2}$ and $\alpha\bar{x}=\overline{\alpha x}$ where $x_1, x_2, x\in X, \alpha\in R$.

The map $P:X\rightarrow X/S$ defined by $Px=x+S=\bar{x}$ is called the canonical projection of X on X/S . P is clearly surjective and $\ker P=S$.

2.2 A-INVARIANT SUBSPACES

Definition 2.4: A subspace $S\subset X$ is said to be A -invariant if and only if $AS\subset S$.

The class of A -invariant subspaces is denoted by $\underline{I}(A)$. Clearly the 0 subspace, the state space X itself and subspaces spanned by the eigenvectors of A are A -invariant subspaces. The family of A -invariant subspaces $\underline{I}(A)$ is closed under subspace addition and subspace intersection. Therefore supremal and infimal elements of a set of A -invariant subspaces exist according to lemma 2.3. For example the controllable subspace:

$$\langle A/\text{im } B \rangle = \bigcap_{n=0}^{n-1} \text{im } A^n B = \inf\{S: S \in \underline{I}(A), \text{im } B \subset S\} \quad (2.2)$$

is the smallest A invariant subspace containing $\text{im } B$ and the unobservable subspace:

$$\langle \ker C/A \rangle = \bigcap_{n=0}^{n-1} \ker C A^n = \sup\{S: S \in \underline{I}(A), S \subset \ker C\} \quad (2.3)$$

is the largest A -invariant contained in $\ker C[1]$. When $\langle A/\text{im } B \rangle = X$ the pair (A, B) is called controllable; when $\langle \ker C/A \rangle = 0$ the pair (C, A) is called observable.

The next lemma shows how a linear mapping A is decomposed by an A -invariant subspace.

Lemma 2.4: Assume that $AV \subset V \subset X$ and W is an arbitrary space such that $d(W) = d(X) - d(V)$. Let $V: V \rightarrow X$ be such that $\ker V = 0$ and $\text{im } V = V$ and $P: X \rightarrow W$ be such that $\ker P = V$ and $\text{im } P = W$ then there exists unique maps $A_0: V \rightarrow V$ and $\bar{A}: W \rightarrow W$ such that the diagram (2.4) commutes.

$$\begin{array}{ccc}
 V & \xrightarrow{A_0} & V \\
 V \downarrow & & \downarrow V \\
 X & \xrightarrow{\quad} & X \\
 P \downarrow & & \downarrow P \\
 W & \xrightarrow{\bar{A}} & W
 \end{array} \quad (2.4)$$

Proof: The lemma is an immediate consequence of lemmas 2.1, 2.2. and definition 2.1.

In order to give matrix representations of the maps which appear in (2.4) let $V:V \rightarrow X$ be the canonical injection defined by $Vx=x$, $x \in V$ and $P:X \rightarrow W=X/V$ be the canonical projection. In this case the map A_0 in (2.4) is the restriction of A to V and is denoted by $A:V$ and the map $\bar{A}$ is called the map induced by A in X/V and is denoted by $A:X/V$. Choose a basis $\{x_1 \dots x_k x_{k+1} \dots x_n\}$ for X such that $\{x_1 \dots x_k\}$ is a basis for V then in this basis the matrix representation of $A:X \rightarrow X$ is of the form

$$A = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} \quad (2.5)$$

where A_{11} and A_{22} are the matrix representations of $A:V$ and $A:X/V$ respectively. The block triangular structure of A in (2.5) implies, via the characteristic polynomial

$$\sigma(A) = \sigma(A:V) \dot{\cup} \sigma(A:X/V) \quad (2.6)$$

where $\dot{\cup}$ denotes union with any common elements repeated.

To investigate stability properties of invariant subspaces the complex plane is divided into two self conjugate parts C_s and C_u such that $C = C_s \cup C_u$, $C_s \cap C_u = \emptyset$ which will indicate stability and instability regions respectively. For continuous time systems the usual choice for C_s is $C_s = \{s \in \mathbb{C} : \operatorname{Re}(s) < 0\}$ and for discrete time systems $C_s = \{z \in \mathbb{C} : \operatorname{Re}(z) < 1\}$. But any other choice will not change the theory. Thus, for example one can define $C_s = \{s \in \mathbb{C} : \operatorname{Re}(s) < -\alpha \text{ for some } \alpha > 0\}$ if one wants to obtain a prescribed degree of stability.

Definition 2.5: Let C_0 denote a self conjugate subset of the complex plane. An A -invariant subspace V is said to be internally (externally) stable with respect to C_0 if and only if $\sigma(A:V) \subset C_0$ ($\sigma(A:X/V) \subset C_0$).

It is clear from definition 2.5 that $V \in \underline{I}(A)$ is internally (externally) stable with respect to C_0 if and only if $\sigma(A_{11}) \subset C_0$ ($\sigma(A_{22}) \subset C_0$) in the representation (2.5).

The duality between internally and externally stable invariant subspaces is expressed in the next lemma.

Lemma 2.5: Let V be an internally (externally) stable A -invariant with respect to C_0 then $V^\perp$ is an externally (internally) stable A' -invariant with respect to C_0 .

Proof: Assume that V is an internally stable A -invariant with respect to C_0 . Define the injection $V:V \rightarrow X$ such that $\text{im} V = V$. Then by lemma 2.4 and definition 2.5 there exists A_0 such that $AV = VA_0$ and $\sigma(A_0) \subset C_0$. Taking transpose of the last equation it follows from lemma 2.1 that $\ker V' = V^\perp$ is A' -invariant and externally stable with respect to C_0 since $\sigma(A':X'/V^\perp) = \sigma(A_0') = \sigma(A_0) \subset C_0$. The other half of the lemma can be proven similarly and is therefore omitted.

Geometric conditions for an A -invariant subspace to be externally (internally) stable are given below.

Lemma 2.6: Let $A:X \rightarrow X$ has minimal polynomial $a(s) = a_0(s)a_b(s)$ where the zeros of $a_0(s)$, $a_b(s)$ lie in C_0 , C_b respectively. An A -invariant subspace $V \subset X$ is externally stable with respect to C_0 if and only if $\ker a_b(A) \subset V$.

Proof: Let $P: X \rightarrow X/V$ be the canonical projection and $\bar{A}: X/V \rightarrow X/V$ be the map induced by A in X/V . By definition $\bar{A}$ satisfies $\bar{A}P = PA$.

«If» Let $\bar{x} = Px \in X/V$ be arbitrary $a_g(\bar{A})\bar{x} = a_g(\bar{A})Px = Pa_g(A)x$ but since $a_g(A)x \in \ker a_b(A) \subset V = \ker P$ we have $a_g(\bar{A})\bar{x} = 0$. Since $\bar{x}$ was arbitrary it follows that $a_g(\bar{A}) = 0$. Hence the minimal polynomial of $\bar{A}$ divides $a_g(s)$ so that $\sigma(\bar{A}) \subset C_g$ which proves that V is externally stable with respect to C_g .

«Only if» Let $x \in \ker a_b(A)$. Since the polynomials $a_g(s)$, $a_b(s)$ are coprime there exists polynomials $p_1(s)$, $p_2(s)$ such that $1 = p_1(s)a_g(s) + p_2(s)a_b(s)$ hence $x = p_1(A)a_g(A)x + p_2(A)a_b(A)x = p_1(A)a_g(A)x$. So $\bar{x} = Px = Pp_1(A)a_g(A)x = p_1(\bar{A})a_g(\bar{A})\bar{x}$ But $\sigma(\bar{A}) \subset C_g$ implies that $a_g(\bar{A}) = 0$ so that $Px = 0$ i.e., $\ker a_b(A) \subset \ker P = V$

The dual of lemma 2.6 for internally stable A -invariants is as follows:

Lemma 2.7: Let $A: X \rightarrow X$ and $\alpha(s) = a_g(s)a_b(s)$ be defined as in lemma 2.6. An A -invariant subspace $V \subset X$ is internally stable with respect to C_g if and only if $V \subset \ker a_g(A)$.

Proof: By lemma 2.5 V is an internally stable A -invariant if and only if $V^\perp$ is an externally stable A' -invariant. Thus by lemma 2.6 $\ker a_b(A') \subset V^\perp$. Taking orthogonal complements and using the fact that $\text{ima}_b(A) = \ker a_g(A)$ we have $V \subset \ker a_g(A)$ which proves the lemma.

It follows from lemmas 2.6 and 2.7 that $\ker a_g(A)$ ($\ker a_b(A)$) is the largest (smallest) A -invariant subspace which is internally (externally) stable with respect to C_g . These results are summarized by writing

$$X_g(A) = \ker \alpha_g(A) = \sup\{V: V \in \underline{I}(A), \sigma(A:V) \subset C_g\} \quad (2.7)$$

$$X_b(A) = \ker \alpha_b(A) = \inf\{V: V \in \underline{I}(A), \sigma(A:X/V) \subset C_g\} \quad (2.8)$$

As an application of the foregoing ideas the following algebraic definitions can be given for the stabilizability and detectability of (2.1)

Definition 2.5: Let C_g denote the open left half complex plane. (2.1) is stabilizable if and only if $\langle A/\text{im}B \rangle$ is externally stable with respect to C_g .

Definition 2.6: Let C_g denote the open left half complex plane. (2.1) is detectable if and only if $\langle \ker C/A \rangle$ is internally stable with respect to C_g .

It can be easily shown from the above definitions and the spectral assignability property of controllable (observable) pairs that stabilizability (detectability) of (2.1) is equivalent to the existence of $F: X \rightarrow U$ ($G: Y \rightarrow X$) such that $\sigma(A+BF) \subset C_g$ ($\sigma(A+GC) \subset C_g$). Finally, geometric characterizations of stabilizability and detectability are given, based on lemmas 2.6 and lemma 2.7. The conditions are as given in [1] but the proof is simpler.

Lemma 2.8: Let C_g denote the open left half complex plane. 1) (2.1) is detectable if and only if $\langle \ker C/A \rangle \subset X_g(A)$ 2) (2.1) is stabilizable if and only if $X_b(A) \subset \langle A/\text{im}B \rangle$

Proof: The conclusion follows from definitions 2.5, 2.6 and lemmas 2.6 and 2.7.

Loosely speaking, the lemma states that the system is detectable if and only if the unobservable modes are stable and stabilizable if and only if the unstable modes are controllable.

2.3 (A,B)-INVARIANT SUBSPACES

The concept of (A,B)-invariance is a natural generalization of A-invariance, originally introduced by Wonham and, Basile and Marro [1,2]. The formal definition is as follows:

Definition 2.7: A subspace $V \subset X$ is (A,B)-invariant if and only if $AV \subset V + \text{im} B$.

The set of all (A,B)-invariant subspaces in a given subspace $K \subset X$ is denoted by $\underline{V}(A,B;K)$. It follows from Definition 2.4 that any A-invariant subspace is automatically (A,B)-invariant. The essential fact about an (A,B)-invariant subspace is that it can be made (A+BF)-invariant by a suitable choice of the map F.

Lemma 2.9: Let $V \subset X$. There exists an $F: X \rightarrow U$ such that $(A+BF)V \subset V$ if and only if $V \in \underline{V}(A,B;X)$

Proof: «Only if» Let $\{v_1, \dots, v_k\}$ be a basis for V . By assumption there exists $w_i \in V$ $i=1,2,\dots,k$ such that $(A+BF)v_i = w_i$, $i=1,2,\dots,k$ or $Av_i = w_i - BFv_i \in V + \text{im} B$. Hence $V \in \underline{V}(A,B;X)$ by definition.

«If» Let V be (A,B)-invariant. By definition there exists $w_i \in V$ and $u_i \in U$ such that $Av_i = w_i + Bu_i$, for $i=1,2,\dots,k$. Define $F_0: V \rightarrow U$ by $F_0 v_i = -u_i$, $i=1,2,\dots,k$ and let F be any extension of F_0 from V to X then $(A+BF)v_i = w_i$, hence F will have the required property.

The mapping F constructed in the sufficiency part of Lemma 2.9 represents a state feedback of the form $u = Fx$ for the system (2.1). If V is an (A,B)-invariant subspace the class of $F: X \rightarrow U$ such that $(A+BF)V \subset V$ is denoted by $\underline{F}(V)$. Lemma 2.9 is called the feedback characterization of (A,B)-invariant subspaces the

dynamic characterization related to trajectories of (2.1) is given in the next lemma.

Lemma 2.10: Let $x(t, x_0, u)$ denote the unique solution of (2.1) corresponding to the initial state x_0 and input u . A subspace V is (A, B) -invariant if and only if for all $x_0 \in V$ there exists a continuous input $u: [0, \infty) \rightarrow U$ such that $x(t, x_0, u) \in V$ for all $t \geq 0$.

Proof: «Only if» Assume that V is (A, B) -invariant. By lemma 2.9 there exists $F: X \rightarrow U$ such that $(A + BF)V \subset V$. Let $u = Fx$ then for any $x_0 \in V$ $x(t, x_0, u) = e^{(A+BF)t}x_0 \in V$ $t \geq 0$.

«If» Assume that for any $x_0 \in V$ there exists a continuous input u such that $x(t, x_0, u) \in V$ $t \geq 0$. Let P be a mapping such that $\ker P = V$ then $Px(t) = 0$ $t \geq 0$. Differentiating with respect to time we have $\dot{P}x(t) = 0$ $t \geq 0$ thus $\dot{x}(t) \in V$ $t \geq 0$. For $t=0$ we can write $\dot{x}(0) = Ax_0 + Bu(0) \in V$ hence $Ax_0 \in V + \text{im} B$ which proves that $V \in \underline{V}(A, B; X)$.

Given the system (2.1) consider the closed loop system formed by means of state feedback F and the connection of a gain matrix G at the system input as in Fig.(2.1).

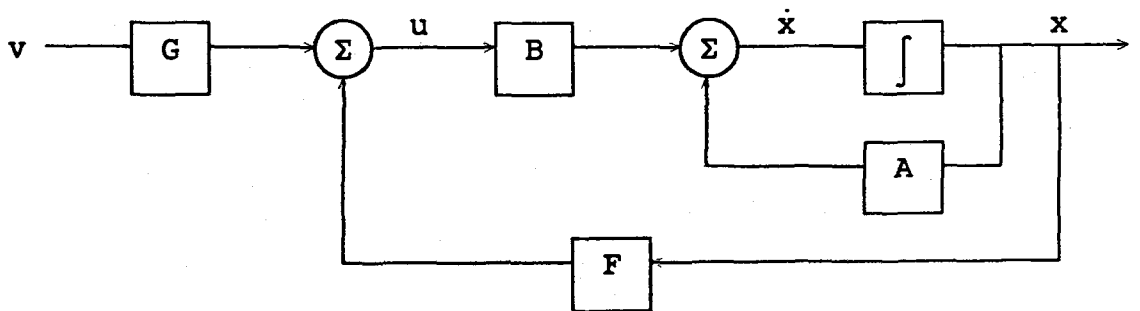


Fig. 2.1: Closed loop system

The controllability subspaces are defined as follows

Definition 2.8: A subspace $R \subset X$ is an (A,B) -controllability subspace if there exists maps $F:X \rightarrow U$ and $G:U \rightarrow U$ such that $R = \langle A+BF/imBG \rangle$ ie., R is the controllable subspace of the pair $(A+BF, BG)$ which is obtained as described in Fig.2.1.

The notation $\underline{R}(A,B;K)$ is used to denote the class of (A,B) -controllability subspaces contained in a given subspace K . Several facts follow easily from definition 2.8. First, if R is an (A,B) -controllability subspace then R is (A,B) -invariant moreover every state $x_1 \in R$ can be reached from the initial state $x_0 \in R$ along a controlled state trajectory that is entirely contained in R . This is the dynamical characterization of a controllability subspace. See Fig.2.2.

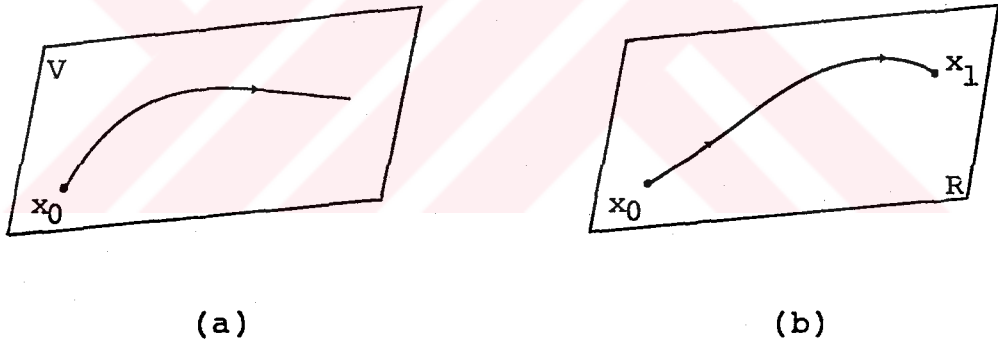


Fig. 2.2: (a) An (A,B) -invariant subspace
(b) An (A,B) -controllability subspace.

By exploiting the equivalence between controllability and spectral assignability the class of controllability subspaces can be characterized by the following basic property.

Theorem 2.1 [1] Let $R \subset X$ be a subspace with $d(R)=k \geq 1$. For every symmetric set Ω of k complex numbers there exists a map $F:X \rightarrow U$ such that $(A+BF)R \subset R$ and $\sigma[A+BF:R] = \Omega$ if and only if $R \in \underline{R}(A,B;X)$.

It is this design property of controllability subspaces that makes the concept central in applications. As a special case of theorem 2.1, if the pair (A,B) is controllable $\langle A/\text{im}B \rangle = X$ is a controllability subspace so that $\sigma[A+BF:X] = \sigma(A+BF)$ can be arbitrarily assigned by suitable choice of F . The well known pole placement property of controllable pairs is recovered.

The classes of (A,B) -invariant and (A,B) -controllability subspaces are closed under subspace addition. Their supremal elements can be computed by the recursive algorithms:

1. Supremal (A,B) -invariant subspace contained in K :
 $\underline{V^*}(A,B;K)$

Define the sequence of subspaces V^k $k=0,1,2,\dots$ according to

$$V^0 = X, \quad V^{k+1} = K \cap A^{-1}(\text{im}B + V^k) \quad (2.9)$$

The sequence V^k is nonincreasing that is, $V^{k+1} \subset V^k$. For some $k \leq d(K)+1$ $V^{k+1} = V^k = \underline{V^*}(A,B;K)$ can be used as a stopping rule.

2. Supremal (A,B) -controllability subspace contained in K :
 $\underline{R^*}(A,B;K)$

Let $V^* = \underline{V^*}(A,B;K)$. Define the sequence R^k $k=0,1,2,\dots$ according to

$$R^0 = 0, \quad R^{k+1} = V^* \cap (AR^k + \text{im}B) \quad (2.10)$$

Induction shows that the sequence R^k is nondecreasing and for some $k \geq d(V^*)$ $R^{k+1} = R^k = \underline{R^*}(A,B;K)$

It should be noted that (2.9) reduces to (2.3), the unobservable subspace algorithm, if $B=0$, $K=\ker C$ and (2.10) reduces to (2.2), the controllable subspace, if $K=X$.

Internally and externally stabilizable (A,B) -invariant subspaces which are generalizations of internally and externally stable A -invariants are introduced in the next definition.

Definition 2.9: An (A,B) -invariant subspace V is said to be internally (externally) stabilizable with respect to C_g if there exists $F \in \underline{F}(V)$ such that $\sigma(A+BF:V) \subset C_g$ ($\sigma(A+BF:X/V) \subset C_g$)

The family of internally stabilizable (A,B) -invariant subspaces with respect to C_g which are contained in K will be denoted by $\underline{V}_g(A,B;K)$. This set also possesses the supremal element denoted by $V_g^*(A,B;K)$. Several methods are given in literature to compute $V_g^*(A,B;K)$. The following closed form expression is suitable for our purposes.

3. Supremal internally stabilizable (A,B) -invariant subspace in $K: V_g^*(A,B;K)$

Let $V^* = \sup \underline{V}(A,B;K)$, $R^* = \sup \underline{R}(A,B;K)$. Choose $F \in \underline{F}(V^*)$ and write $m(s)$ for the minimal polynomial of $A+BF$. Factor $m(s) = m_g(s)m_b(s)$ where the zeros of m_g (m_b) lie in C_g (C_b). Then

$$V_g^*(A,B;K) = R^* + V^* \cap \ker m_g(A+BF) \quad (2.11)$$

If V is an (A,B) -invariant subspace theorem 2.1 gives only partial answer to the question of assignability of the eigenvalues of $A+BF$ when F is restricted to $\underline{F}(V)$. The complete freedom in assigning

the eigenvalues of $A+BF$ by $F \in \underline{F}(V)$ is described in the next theorem.

Theorem 2.2 [13] Let $V \in \underline{V}(A,B;X)$, $R^* = \sup R(A,B;V)$ and $S = V + \langle A/\text{im} B \rangle$. For $F \in \underline{F}(V)$ $\sigma(A+BF) = \sigma_f \dot{\cup} \sigma_0$ where $\sigma_f = \sigma(A+BF:R^*) \dot{\cup} \sigma(A+BF:S/V)$ can be arbitrarily assigned by suitable choice of $F \in \underline{F}(V)$ and $\sigma_0 = \sigma(A+BF:V/R^*) \dot{\cup} \sigma(A+BF:X/S)$ is fixed for all $F \in \underline{F}(V)$.

The situation with the spectrum of $A+BF$ is summarized in the spectral lattice diagram of Figure 2.3. From theorem 2.2 it is easy to conclude for instance that an (A,B) -invariant subspace V is internally (externally) stabilizable with respect to C_g if and only if for $F \in \underline{F}(V)$ $\sigma(A+BF:V/R^*) \subset C_g$ ($\sigma(A+BF:X/S) = \sigma(A:X/S) \subset C_g$)

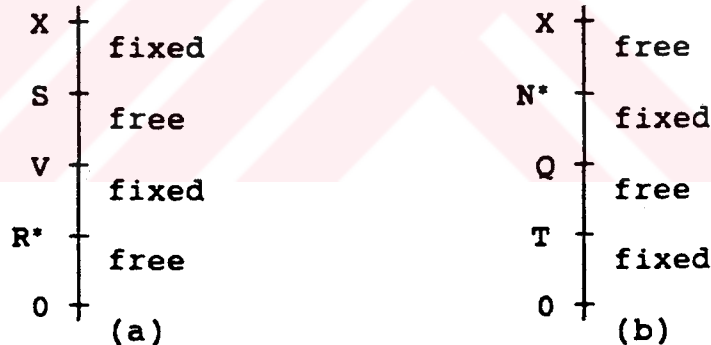


Fig. 2.3: (a) Lattice Diagram for $\sigma(A+BF)$ $F \in \underline{F}(V)$
 (b) Lattice Diagram for $\sigma(A+GC)$ $G \in \underline{G}(Q)$

2.4 (C,A) -INVARIANT SUBSPACES

(C,A) -invariant subspaces are developed as the dual of (A,B) -invariant subspaces and usually employed in measurement feedback and observer design problems. Thus all results in this section except lemma 2.11 which

expresses the duality between (A,B) -invariant and (C,A) -invariant subspaces are completely dual of those given in the previous section.

Definition 2.10: A subspace $Q \subset X$ is (C,A) -invariant if and only if $A(Q \cap \ker C) \subset Q$

The set of (C,A) -invariant subspaces containing a given subspace K is denoted by the symbol $\underline{Q}(C,A;K)$

Lemma 2.11: Let $Q \in \underline{Q}(C,A;K)$ then $(Q^\perp \in \underline{V}(A',C';K^\perp))$ and conversely let $V \in \underline{V}(A,B;K)$ then $(V^\perp \in \underline{Q}(B',A';K^\perp))$

Proof: Let $T, N \subset X$ and $AT \subset N$ then $(AT)^\perp \supset N^\perp$. Thus $(n, At) = (A'n, t) = 0$ for every $n \in N$ and $t \in T$ so that $A'N^\perp \subset T^\perp$. Repeating the same argument for subspaces $N^\perp, T^\perp$ and map A' satisfying $A'N^\perp \subset T^\perp$ it can be concluded that $AT \subset N$ if and only if $A'N^\perp \subset T^\perp$. Applying this result to the situation at hand $A(\ker C \cap Q) \subset Q$ if and only if $A'Q^\perp \subset (\ker C \cap Q)^\perp = Q^\perp + \text{im } C'$. The converse now follows by a simple change of symbols.

The dual of Lemma 2.9 for (C,A) -invariant subspaces is as follows

Lemma 2.12: Let $Q \subset X$. There exists a map $G: Y \rightarrow X$ such that $(A+GC)Q \subset Q$ if and only if $Q \in \underline{Q}(C,A;0)$

Proof: «Only if» Suppose $(A+GC)Q \subset Q$. Take any $x \in \ker C \cap Q$. By hypotheses there exists $q \in Q$ such that $(A+GC)x = Ax = q \in Q$ thus $A(\ker C \cap Q) \subset Q$.

«If» Assume that $Q \in \underline{Q}(C,A;0)$. Let $\{q_1 \dots q_j \dots q_k\}$ be a basis for Q such that $\{q_1 \dots q_j\}$ is a basis for $Q \cap \ker C$. Then $Aq_i = s_i$ for $i=1 \dots j$ and $Aq_i = r_i$ for $i=j+1 \dots k$ where $s_i \in Q$ by assumption. Noting that the vectors $Cq_{j+1} \dots Cq_k$ are linearly independent a map $G: Y \rightarrow X$ can be defined

such that $GCq_i = -r_i$ for $i=j+1 \dots k$. Then $(A+GC)q_i = Aq_i = s_i \in Q$ for $i=1 \dots j$ and $(A+GC)q_i = 0 \in Q$ for $i=j+1 \dots k$ as desired.

If Q is a (C,A) -invariant subspace the set of maps $G:Y \rightarrow X$ such that $(A+GC)Q \subset Q$ is denoted by $\underline{G}(Q)$. The map G is called an output injection and lemma 2.12 is the output injection characterization of (C,A) -invariant subspaces.

The dynamic characterization of (C,A) -invariance will be omitted since it will not be needed in the thesis and the dual of controllability subspaces will be given in the next definition.

Definition 2.11: Consider the system $\Sigma_{G,H}: \dot{x} = (A+GC)x + Bu, y = HCx$ obtained from (2.1) by means of the output injection $G:Y \rightarrow X$ and the endomorphism $H:Y \rightarrow Y$. A subspace $N \subset X$ is a (C,A) -complementary observability subspace if there exists maps G and H such that $N = \langle \ker HC / A+GC \rangle$ i.e., N is the unobservable subspace of $\Sigma_{G,H}$.

The set of (C,A) complementary observability subspaces containing a given subspace K is denoted by $\underline{N}(C,A;K)$. It follows from definition 2.11 that complementary observability subspaces are $A+GC$ -invariant hence by lemma 2.12 they are automatically (C,A) -invariant. Moreover they have spectral assignability properties which follow easily from the eigenvalue placement property of observable pairs.

Theorem 2.3: [13,23]: Let N be a subspace with $d(N)=k \geq 1$. For every symmetric set Ω of $n-k$ complex numbers there exists a map $G:Y \rightarrow X$ such that $(A+GC)N \subset N$ and $\sigma(A+GC: X/N) = \Omega$ if and only if $N \in \underline{N}(C,A;0)$.

The sets $\underline{Q}(C,A;K)$ and $\underline{N}(C,A;K)$ are closed under subspace intersection their infimal elements can be computed by dualizing the recursions (2.9) and (2.10).

4. Infimal (C,A) -invariant subspace containing $K:Q_*(C,A;K)$

Define the sequence of subspaces Q_k $k=0,1,2 \dots$ as

$$Q_0=0, \quad Q_{k+1}=K+A(Q_k \cap \ker C) \quad (2.12)$$

The sequence Q_k $k=0,1,2 \dots$ is nondecreasing that is $Q_k \subset Q_{k+1}$ and reaches its limit $\inf Q(C,A;K)$ for $k \geq n-d(K)+1$.

5. Infimal (C,A) -complementary observability subspace containing $K:N_*(C,A;K)$

Let $Q_* = \inf Q(C,A;K)$. Define the sequence N_k $k=0,1,2 \dots$ as

$$N_0=X, \quad N_{k+1}=Q_*(A^{-1}N_k) \cap \ker C \quad (2.13)$$

The sequence N_k $k=0,1,2 \dots$ is nonincreasing and reaches its limit $N_*(C,A;K)$ in at most $n-d(Q_*)$ steps.

The internally and externally stabilizable (C,A) -invariant subspaces are defined as follows.

Definition 2.12: A (C,A) -invariant subspace Q is said to be externally (internally) stabilizable with respect to C_0 if there exists $G \in \underline{G}(Q)$ such that $\sigma(A+GC:X/Q) \subset C_0$. ($\sigma(A+GC:Q) \subset C_0$).

The class of externally stabilizable (C,A) -invariant subspaces with respect to C_0 and containing K is denoted by $\underline{Q}_0(C,A;K)$.

It follows from lemma 2.5, the feedback characterization of (A,B) -invariant and output injection characterization of (C,A) -invariant subspaces that,

internally stabilizable (A,B) -invariants and externally stabilizable (C,A) -invariants are duals of each other. More precisely $V \in \underline{V}_g(A,B;K)$ if and only if $V^\perp \in \underline{Q}_g(B',A';K^\perp)$. Thus algorithm 3 of the previous section can be dualized to obtain a closed form expression for $\inf \underline{Q}_g(C,A;K)$.

6. Infimal externally stabilizable (C,A) -invariant subspace containing $K:Q_{g*}(C,A;K)$

Let $Q_* = \inf \underline{Q}(C,A;K)$, $N_* = \inf \underline{N}(C,A;K)$. Choose $G \in \underline{G}(Q_*)$ and write $m(s)$ for the minimal polynomial of $A+GC$. Factor $m(s)$ as $m(s) = m_g(s) \cdot m_b(s)$ where the zeros of $m_g(m_b)$ lie in $C_g(C_b)$. Then

$$Q_{g*}(C,A;K) = N_* \cap (Q_* + \ker m_b(A+GC)) \quad (2.14)$$

The final result of this section is the dual of theorem 2.2 and describes the spectrum of $A+GC$ when $G \in \underline{G}(Q)$.

Theorem 2.4 [13] Let $Q \in \underline{Q}(C,A;0)$, $N_* = \inf \underline{N}(C,A;Q)$ and $T = Q \cap \langle \ker C/A \rangle$. For $G \in \underline{G}(Q)$ $\sigma(A+GC) = \sigma_g \dot{\cup} \sigma_b$ where $\sigma_g = \sigma(A+GC: X/N_*) \dot{\cup} \sigma(A+GC: Q/T)$ can be arbitrarily assigned by $G \in \underline{G}(Q)$ and $\sigma_b(\sigma(A+GC: T) \dot{\cup} \sigma(A+GC: N_*/Q))$ is fixed for all $G \in \underline{G}(Q)$.

The content of theorem 2.4 is summarized in Fig.2.3b

CHAPTER 3

DISTURBANCE DECOUPLING PROBLEM

In this chapter disturbance decoupling problem (DDP) is briefly reviewed for finite dimensional linear, time-invariant systems. The results presented and the techniques used will find application in subsequent chapters.

Basically DDP amounts to finding a feedback control law so that the controlled outputs of the system are not affected by the disturbances acting at the input. (Precise problem statement will be given in section 3.1) This problem has been widely investigated in the geometric approach literature and various solvability conditions and synthesis procedures have been obtained. In [1,17] disturbance decoupling by constant measurement feedback is considered and it is assumed that the whole state vector is accessible for measurement; only recently the problem has been solved by dynamic measurement feedback. [21-23]. Apart from its practical motivation the reason of the wide interest in DDP can be explained by the fact that they serve as model problems for introducing (A,B) -invariant and (C,A) -invariant subspaces and illustrating their application. DDP has also been treated in [26-28] with this objective where the concept of almost (A,B) -invariant subspaces is applied to the solution of approximate DDP.

The chapter is organized as follows. In section 3.1 basic definitions and the problem statement are given. Solvability of DDP by constant measurement feedback is considered in section 3.2. Finally the solution of the problem by dynamic measurement feedback is presented in section 3.3.

3.1 PROBLEM STATEMENT

Consider the linear, time invariant system described by

$$\begin{aligned}\dot{x} &= Ax + Bu + Eq \\ y &= Cx \\ z &= Dx\end{aligned}\tag{3.1}$$

with $x(t) \in X = R^n$ the state, $u(t) \in U = R^m$ the control input, $q(t) \in F = R^r$ the disturbance, $y(t) \in Y = R^p$ the measured output and $z(t) \in Z = R^l$ the controlled output. It is assumed that the function $q: [0, \infty) \rightarrow F$ is piecewise continuous.

Definition 3.1: The system (3.1) is said to be disturbance decoupled (relative to the pair q, z) if and only if the forced response.

$$z(t) = \int_0^t De^{A(t-s)}Eq(s)ds = 0 \text{ for all } q \text{ and } t \geq 0 \tag{3.2}$$

According to definition 3.1 the controlled outputs are affected only by the initial conditions and the control inputs when (3.1) is disturbance decoupled. Geometric and transfer matrix conditions for (3.1) to be disturbance decoupled are given in lemma 3.1.

Lemma 3.1: The following statements are equivalent

1. (3.1) is disturbance decoupled.
2. The transfer matrix $D(sI - A)^{-1}E = 0$ for all $s \in \mathbb{C}$
3. There exists $V \in \underline{I}(A)$ such that $\text{im} E \subset V \subset \ker D$

Proof «(1) $\Rightarrow$ (2)» Follows by taking Laplace transform of (3.2).

«(2) $\Rightarrow$ (3)» Expanding $(sI-A)^{-1}$ into powers of s^{-1} according to $(sI-A)^{-1} = s^{-1}I + s^{-2}A + s^{-3}A^2 + \dots$ and equating the coefficient of s^{-k} $k=1,2,\dots$ to zero yields $DA^{k-1}E=0$ $k=1,2,\dots$ which proves that $\langle A/\text{im}E \rangle \subset \ker D$. The conclusion follows by noting that $\text{im}E \subset \langle A/\text{im}E \rangle \in \underline{I}(A)$

«(3) $\Rightarrow$ (1)» Assume that $AV \subset V$ and $\text{im}E \subset V \subset \ker D$. Let $P: X \rightarrow X/V$ be the canonical projection. By lemma 2.4 there exists $\bar{A}: X/V \rightarrow X/V$ such that $\bar{A}P = PA$ and $\bar{D}: X/V \rightarrow Z$ such that $\bar{D}P = D$. Hence $\int_0^t De^{A(t-s)}Eq(s)ds = \int_0^t \bar{D}\bar{e}^{\bar{A}(t-s)}PEq(s)ds = 0$ for all q and $t \geq 0$ since $\text{im}E \subset \ker P = V$

The Disturbance Decoupling Problem which is depicted in Fig.3.1 applies when the system (3.1) is not disturbance decoupled and is the problem of finding a feedback control law from the measurements y to the control inputs u such that the closed loop system consisting of the plant and the feedback compensator is disturbance decoupled.

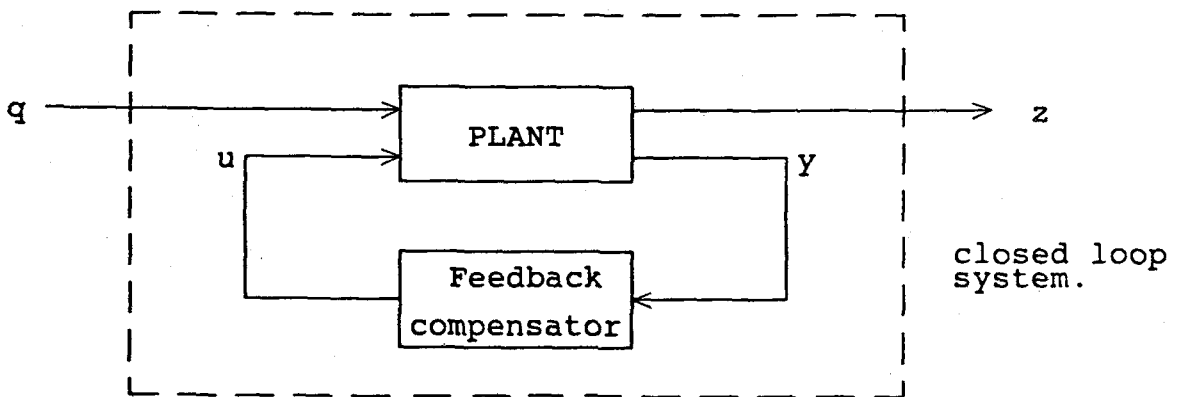


Fig.3.1. Disturbance Decoupling Problem (DDP)

3.2 DISTURBANCE DECOUPLING BY CONSTANT MEASUREMENT FEEDBACK

Consider the system (3.1) and assume that all state variables are available for measurement ie., $C=I$. The closed loop system resulting from the feedback control law $u=Fx$ is described by

$$\begin{aligned}\dot{x} &= (A+BF)x + Eq \\ z &= Dx\end{aligned}\tag{3.3}$$

The problem of finding a feedback of the form $u=Fx$ such that (3.3) is disturbance decoupled is called disturbance decoupling by state feedback and solvability conditions have been originally obtained in [1,2,7]. Their main result is presented in theorem 3.1.

Theorem 3.1: Disturbance decoupling by state feedback is solvable if and only if

$$\text{im}E \subset V^*(A, B; \ker D)\tag{3.4}$$

Proof: «Only if» Assume that (3.3) is disturbance decoupled. By lemma 3.1 there exists a subspace V such that $(A+BF)V \subset V$ and $\text{im}E \subset V \subset \ker D$. It follows from lemma 2.9 that $V \in \underline{V}(A, B; \ker D)$ and the conclusion follows by noting that $\text{im}E \subset V \subset V^*(A, B; \ker D)$.

«If» Write $V^*=V^*(A, B; \ker D)$ and by lemma 2.9 choose $F: X \rightarrow U$ such that $(A+BF)V^* \subset V^*$. Since $\text{im}E \subset V^* \subset \ker D$, $u=Fx$ is the desired control law.

It should be noted that the sufficiency proof of theorem 3.1 is constructive and offers a method for synthesizing the control law $u=Fx$ when (3.4) is satisfied. It is possible to weaken condition (3.4) by assuming that the disturbance is also available as a

feedforward term in the control law. The closed loop system resulting from the feedback control law $u = Fx + Rq$ is given by

$$\begin{aligned}\dot{x} &= (A + BF)x + (E + BR)q \\ z &= Dx\end{aligned}\tag{3.5}$$

The next lemma is needed to give the solution of disturbance decoupling by state feedback and disturbance feedforward.

Lemma 3.2: Given arbitrary maps $E: F \rightarrow X$, $B: U \rightarrow X$ and a subspace $V \subset X$ there exists a map $R: F \rightarrow U$ such that $\text{im}(E + BR) \subset V$ if and only if

$$\text{im} E \subset V + \text{im} B\tag{3.6}$$

Proof: «Only if» Assume that $\text{im}(E + BR) \subset V$. For any $q \in F$ there exists $v \in V$ such that $(E + BR)q = v$ or $Eq = v - BRq \in V + \text{im} B$ which proves (3.6)

«If» Let $\{q_1, \dots, q_r\}$ be a basis for F . By (3.6) there exists $v_i \in V$ $i = 1 \dots r$ such that $Eq_i = v_i + Bu_i$, $i = 1, \dots, r$. Define $R: F \rightarrow U$ by $Rq_i = -u_i$, $i = 1 \dots r$ then R satisfies $\text{im}(E + BR) \subset V$

Theorem 3.2 [18]: Disturbance decoupling problem is solvable by a control law $u = Fx + Rq$ (state feedback and disturbance feedforward) if and only if

$$\text{im} E \subset V^*(A, B; \ker D) + \text{im} B\tag{3.7}$$

Proof: The following statements are equivalent

1. There exist $F: X \rightarrow U$ and $R: F \rightarrow U$ such that (3.5) is disturbance decoupled.
2. There exist $F: X \rightarrow U$, $R: F \rightarrow U$ and $V \in \underline{I}(A + BF)$ such that $\text{im}(E + BR) \subset V \subset \ker D$.

3. There exists $V \in \underline{V}(A, B; \ker D)$ such that $\text{im } E \subset V + \text{im } B$.
 where the equivalence of 1. to 2. follows from lemma 3.1 and that of 2. to 3. from lemma 2.9 and lemma 3.2. (3.7) is established by noting that $V^*(A, B; \ker D) = \sup \underline{V}(A, B; \ker D)$.

In theorems 3.1 and 3.2 it was assumed that all state variables are available for measurement, which is quite restrictive in applications. Hence the control laws should be of measurement feedback type. The closed loop system obtained by applying the control law $u = Ky$ to (3.1) is given by

$$\begin{aligned} \dot{x} &= (A + BKC)x + E\dot{y} \\ Z &= Dx \end{aligned} \quad (3.8)$$

Solvability of DDP by constant measurement feedback is considered in theorems 3.3 and 3.4.

Theorem 3.3 [19] Disturbance decoupling by constant measurement feedback is solvable if and only if

$$\underline{V}(A, B; \ker D) \cap \underline{Q}(C, A; \text{im } E) \neq \emptyset \quad (3.9)$$

Proof: «Only if» Assume that the closed loop system (3.8) obtained by the control law $u = Ky$ is disturbance decoupled. By lemma 3.1 there exists $V \in \underline{I}(A + BKC)$ such that $\text{im } E \subset V \subset \ker D$. Taking $F = KC$ and $G = BK$ it is seen that $V \in \underline{I}(A + BF) \cap \underline{I}(A + GC)$. Thus by lemma 2.9 and lemma 2.12 $V \in \underline{V}(A, B; \ker D) \cap \underline{Q}(C, A; \text{im } E)$

«If» Let $V \in \underline{V}(A, B; \ker D) \cap \underline{Q}(C, A; \text{im } E)$ it will be shown that there exists $K: Y \rightarrow U$ such that $(A + BKC)V \subset V$. To prove this statement let $\{x_1 \dots x_j, x_{j+1} \dots x_k\}$ be a basis for V such that $\{x_1 \dots x_j\}$ is a basis for $V \cap \ker C$. Since $V \in \underline{V}(A, B; \ker D)$ there exists $v_i \in V$ and $u_i \in U$ such that $Ax_i = v_i + Bu_i$, $i = 1 \dots k$. Noting that the vectors $Cx_{j+1}, \dots, Cx_k$

are linearly independent $K:Y \rightarrow U$ can be defined so that $K(Cx_i) = -u_i$, $i=j+1 \dots k$. Then we have $(A+BKC)x_i = v_i \in V$ for $i=j+1, \dots, k$ but also $(A+BKC)x_i = Ax_i \in V$ for $i=1, \dots, j$ because $Cx_i = 0$. Hence $u = Ky$ is the desired control law.

Once a subspace $V \in \underline{V}(A, B; \ker D) \cap \underline{Q}(C, A; \text{im } E)$ is given the construction of a feedback map K achieving disturbance decoupling is described in theorem 3.3. However condition (3.8) can not be verified easily unless some additional assumptions are satisfied. One such assumption is presented in the next theorem.

Theorem 3.4 [19]: Assume that $A(\ker C \cap \ker D) \subset \ker C \cap \ker D$ then disturbance decoupling by constant measurement feedback is solvable if and only if

$$\text{im } E \subset V^*(A, B; \ker D) \quad (3.10)$$

Proof: «Only if» The necessity of (3.10) has already been established in theorem 3.1.

«If» Write $V^* = V^*(A, B; \ker D)$. By assumption $\ker C \cap \ker D \in \underline{V}(A, B; \ker D)$ hence $\ker C \cap \ker D \subset \ker C \cap V^*$. Conversely since $V^* \subset \ker D$, $\ker C \cap V^* \subset \ker C \cap \ker D$. Thus it is proven that $\ker C \cap V^* = \ker C \cap \ker D$. Then we can write $A(\ker C \cap V^*) = A(\ker C \cap \ker D)$ $\ker C \cap \ker D = \ker C \cap V^* \subset V^*$ which shows that $V^* \in \underline{Q}(C, A; \text{im } E)$ and the conclusion follows from theorem 3.3.

Unlike theorem 3.3, theorem 3.4 gives an easy to check sufficient condition for the solution of DDP by constant measurement feedback but the general case remains unsolved.

3.3 DISTURBANCE DECOUPLING BY DYNAMIC MEASUREMENT FEEDBACK

The feedback structures considered in section 3.2 were all linear, time-invariant and memoryless. In this section, dynamic processing of the measurements will be allowed before feedback. Thus it will be assumed that the control input is synthesized by means of the dynamic compensator

$$\begin{aligned}\dot{w} &= Nw + My \\ u &= Lw + Ky\end{aligned}\tag{3.11}$$

where $w(t) \in W$ is the state of the compensator; the order of the compensator is $d(W)$.

Combining equations (3.1) and (3.11) gives rise to the closed loop system (3.12)

$$\begin{aligned}\dot{x}_c &= A_c x_c + E_c q \\ z &= D_c x_c\end{aligned}\tag{3.12}$$

$$\text{where } x_c(t) = \begin{bmatrix} x(t) \\ w(t) \end{bmatrix} \in X \oplus W, \quad A_c = \begin{bmatrix} A+BKC & BL \\ MC & N \end{bmatrix}, \quad E_c = \begin{bmatrix} E \\ 0 \end{bmatrix}, \quad D_c = [D \ 0]$$

At this point it is appropriate to show how the closed loop system (3.12) can be realized by applying constant measurement feedback to an extension of (3.1). For this purpose the concept of dynamic extension is introduced in the next definition.

Definition 3.2: Let Σ denote the system given by (3.1). The system Σ_e defined by

$$\begin{bmatrix} \dot{x} \\ \dot{w} \end{bmatrix} = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ w \end{bmatrix} + \begin{bmatrix} B & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} + \begin{bmatrix} E \\ 0 \end{bmatrix} q \quad (3.13)$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} C & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} x \\ w \end{bmatrix}, \quad z = [D \ 0] \begin{bmatrix} x \\ w \end{bmatrix}$$

and considered as a system with state $x_e(t) = \begin{bmatrix} x(t) \\ w(t) \end{bmatrix} \in X \oplus W$,

input $u_e(t) = \begin{bmatrix} u(t) \\ v(t) \end{bmatrix} \in U \oplus W$ and measurements $y_e(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}$

$\in Y \oplus W$ is called an extension of Σ .

(3.13) will be written compactly as $\dot{x}_e = A_e x_e + B_e u_e + E_e q$, $y_e = C_e x_e$. Physically the extension of Σ is obtained by adding $d(W)$ auxilliary integrators to Σ and taking the integrator outputs as new measurements. The control law $u_e = K_e y_e$ applied to Σ_e with K_e partitioned as $K_e = \begin{bmatrix} K & L \\ M & N \end{bmatrix}$ yields the closed loop system (3.12).

Two mappings between X and $X^e = X \oplus W$ which play a fundamental role in the sequel are the projection $P: X^e \rightarrow X$ defined by

$$P \begin{bmatrix} x \\ w \end{bmatrix} = x \quad (3.14)$$

and the embedding $S: X \rightarrow X^e$ defined by

$$Sx = \begin{bmatrix} x \\ 0 \end{bmatrix} \quad (3.15)$$

For a subspace $V^e \subset X^e$ one has

$$V_p = PV^e = \{x \in X: \exists w \in W \text{ s.t. } \begin{bmatrix} x \\ w \end{bmatrix} \in V^e\} \quad (3.16)$$

$$V_i = S^{-1}V^e = \{x \in X: \begin{bmatrix} x \\ 0 \end{bmatrix} \in V^e\} \quad (3.17)$$

Geometrically, V_p may be viewed as the projection of V^e on the space X and V_i as the intersection of V^e with space X . The next lemma will be used to establish the necessity part of the main theorem of this section.

Lemma 3.3: [21] Let $V \subset X \oplus W$ be an A_c -invariant subspace then i) $V_i = S^{-1}V$ is (C, A) -invariant, ii) $V_p = PV$ is (A, B) -invariant and iii) $V_i \subset V_p$

The constructive sufficiency part of theorem 3.5 is based on the following lemma which describes how A_c -invariant subspaces can be produced by dynamic extension.

Lemma 3.4 [21, 23]: Let V_1 be (C, A) -invariant, V_2 be (A, B) -invariant with $V_1 \subset V_2$. Then there exists an extension space W of dimension: $d(W) = d(V_2) - d(V_1)$ and an A_c -invariant subspace $V \subset X \oplus W$ such that $V_i = S^{-1}V$ and $V_2 = PV$.

Proof: Let W be a linear space of dimension: $d(V_2) - d(V_1)$ and let $R: V_2 \rightarrow W$ be a surjection such that $\ker R = V_1$.

Introduce $V = \left\{ \begin{bmatrix} x \\ Rx \end{bmatrix} : x \in V_2 \right\} \subset X \oplus W$. It is clear that

$V_i = S^{-1}V$, $V_2 = PV$. Moreover $A_e V \subset V + \text{im } B_e$ and

$V \cap \ker C_e = V_1 \cap \ker C \oplus 0$ so that $A_e(V \cap \ker C_e) \subset V$. Thus

$V \in \underline{V}(A_e, B_e; X^e) \cap \underline{Q}(C_e, A_e; 0)$ and it is proven as in theorem 3.3 that there exists $K_e: Y \oplus W \rightarrow U \oplus W$ such that $(A_e + B_e K_e C_e)V \subset V$. Setting $A_c = A_e + B_e K_e C_e$ the theorem is established.

Theorem 3.5 for the solvability of disturbance decoupling by dynamic measurement feedback follows as a consequence of the foregoing lemmas.

Theorem 3.5 [21,23]: The problem of disturbance decoupling by dynamic measurement feedback is solvable if and only if

$$Q_*(C, A; \text{im} E) \subset V^*(A, B; \ker D) \quad (3.18)$$

Proof: «Only if» Assume that (3.12) is disturbance decoupled. By lemma 3.1 there exists $V \subset X \oplus W$ such that $A_c V \subset V$ and $\text{im} E_c \subset V \subset \ker D_c$. Hence $S^{-1}(\text{im} E_c) = \text{im} E \subset S^{-1}V$ and $PV \subset P(\ker D_c) = \ker D$ which, in view of lemma 3.3, implies that $Q_*(C, A; \text{im} E) \subset S^{-1}V \subset PV \subset V^*(A, B; \ker D)$

«If» Let $V_1 = Q_*(C, A; \text{im} E)$ and $V_2 = V^*(A, B; \ker D)$ and assume that $V_1 \subset V_2$. Let W be a space such that $d(W) = d(V_2) - d(V_1)$ and construct $V \subset X \oplus W$ and $K_e = \begin{bmatrix} K & L \\ M & N \end{bmatrix}: Y \oplus W \rightarrow U \oplus W$ as

in lemma 3.4. The compensator matrices designed in this way make (3.12) disturbance decoupled since $\text{im} E_c \subset V \subset \ker D_c$ and $V \in \underline{I}(A_c)$

Remark 3.1: The order of the feedback compensator that can be designed using the procedure described in theorem 3.5 is $d(W) = d(V^*(A, B; \ker D)) - d(Q_*(C, A; \text{im} E))$. This gives an upper bound for the compensator order. The minimal extension order is given by

$$n_e^* = \min\{d(V_2) - d(V_1) : V_1 \in \underline{Q}(C, A; \text{im} E), V_2 \in \underline{V}(A, B; \ker D), V_1 \subset V_2\}$$

A lower bound for n_e^* is obviously zero in which case V_1 coincides with V_2 so that disturbance decoupling by constant measurement feedback is solvable. Thus constant measurement feedback can be viewed as a compensator of order zero. The problem of finding the minimal extension order has not been solved in literature and is open for future research.

Remark 3.2: Suppose that all state variables are available for measurement. In this case one wishes to know if dynamic state feedback brings any improvement on the solvability of DDP. This can be investigated by setting $C=I$ in (3.18) giving $Q.(C,A;imE)=imE \subset V^*(A,B;kerD)$ which is identical to (3.4). Thus it is seen that dynamic state feedback is equivalent to constant state feedback as far as the solvability of DDP is concerned.

Remark 3.3: As a final comment it can be proven that a system which is not disturbance decoupled can not be made so by any form of measurement feedback if the measurements are related to controlled outputs by $y=Kz$. This statement can be justified by letting $kerD \subset kerC$. In this case (3.18) reduces to $Q.(C,A;imE)=\langle A/imE \rangle \subset V^*(A,B;kerD) \subset kerD$ which means that the original system is already disturbance decoupled.

CHAPTER 4

DISTURBANCE DECOUPLED ESTIMATION PROBLEM

The problem of estimating the states of linear, time invariant systems is considered in the presence of unknown disturbances. Though this problem is one of the first problems in the literature, attacked by the geometric approach [3] and studied extensively after the original works in [29-33], the theory is not fully matured as in the case of known input systems and; there still remains some open problems. In particular, (i) Existence conditions for full state observers are not often given in terms of the observability, detectability of the observed system comparable to conditions given for known input systems. (ii) The undetectable subspaces of different types of observers and the relationships among them are not known. Neither the lower bounds for the undetectable subspaces are established. (iii) Conditions for the existence of minimal order unknown input observers, the minimal orders and their design procedures are unknown.

In this chapter answers will be provided for the questions raised above together with an up to date complete account of deterministic observer theory. The system is assumed to be described by

$$\dot{x}=Ax+Bu+Eq \quad , \quad y=Cx \quad , \quad z=Dx \quad (4.1)$$

where $x(t) \in X=R^n$ is the state, $u(t) \in U=R^m$ the control input, $q(t) \in F=R^r$ the disturbance input, $y(t) \in Y=R^p$ and $z(t) \in Z=R^l$ are the measured and the to be estimated outputs respectively.

The plant equations are assumed to be known, the disturbance q and/or the initial state $x(0)$ are unknown. The objective is to determine the to be estimated outputs z from the measurements $y(t)$, $t \in [0, \infty)$ and the inputs $u(t)$, $t \in [0, \infty)$. The device used to achieve this task is called an observer. See Fig.4.1.

Definition 4.1: A system driven by the measurements y and inputs u of (4.1) and giving the output $\hat{z}$ is said to be an observer for (4.1) if the estimation error defined by $e = \hat{z} - z$ satisfies.

$$\lim_{t \rightarrow \infty} e(t) = 0 \text{ for all } x(0) \in X, w(0) \in W, q \text{ and } u \quad (4.2)$$

where $w(0)$ represents the initial state of the observer.

When (4.2) is satisfied $\hat{z}$ is called the estimate of z .

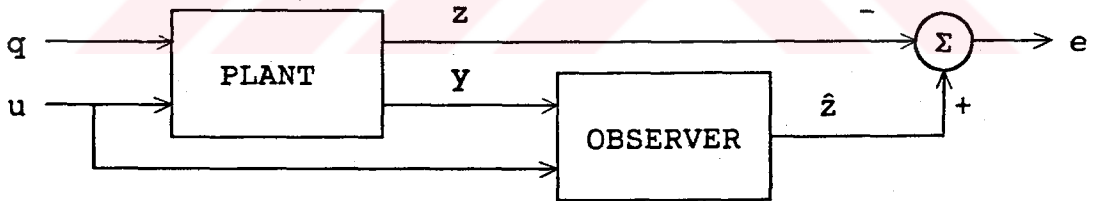


Fig. 4.1 Disturbance Decoupled Estimation Problem (DDEP)

We will restrict our attention to finite dimensional, linear, time-invariant PID (proportional-integral-derivative) class of observers which can be described by

$$\dot{w} = Nw + My + Ru \quad (4.3)$$

$$\hat{z} = Lw + \sum_{i=0}^{k-1} \left(K_i \frac{d^i}{dt^i} y + S_i \frac{d^i}{dt^i} u \right)$$

(4.3) includes the following six types of observers as special cases:

- 1) Cheap observer: $\hat{z}=0$
- 2) P-observer: $\hat{z}=K_0 y + S_0 u$
- 3) PD-observer : $\hat{z} = \sum_{i=0}^k (K_i \frac{d^i}{dt^i} y + S_i \frac{d^i}{dt^i} u)$ where k is unrestricted.
- 4) I-observer : $\dot{\hat{w}} = N\hat{w} + M y + R u$, $\hat{z} = L\hat{w}$
- 5) PI-observer : $k=1$ in (4.3)
- 6) PID-observer : k is unrestricted in (4.3).

The number of differentiators used in a PD (PID)-observer is indicated by a subscript k as PD_k (PID_k). The disturbance q and the input u are assumed to be differentiable sufficiently many times in order that expressions involving time derivatives of u and q are valid. The order of a dynamic (I, PI, PID) observer is by definition the dimension of the state space W of (4.3).

Throughout this chapter C_o represents the open left half complex plane and C_b the right half plane including the imaginary axis. Also, to shorten the notation the arguments of the subspaces will be omitted and the symbols V^* , Q_* , R^* and N_* will be used to denote the subspaces $V^*(A, E; \ker C)$, $Q_*(C, A; \text{im } E)$, $R^*(A, E; \ker C)$ and $N_*(C, A; \text{im } E)$ respectively. Recall from Chapter 2 that the sequence of subspaces V^k $k=0, 1, 2, \dots$ defined by $V^0 = X$, $V^{k+1} = \ker C \cap A^{-1}(V^k + \text{im } E)$ converges to V^* . This sequence plays a central role in the results related to PID class of observers and will be referred to repeatedly in this chapter. Finally, V_b^* denotes the largest internally stabilizable (A, E) -invariant subspace with respect to C_b which is contained in $\ker C$.

The chapter is organized as follows. In section 4.1 existence conditions of PID class of observers are given in a common framework in terms of well defined subspaces of the state space X . In section 4.2 the concept of undetectable subspaces is introduced. The precise relationships between the undetectable subspaces of different types of observers and their lower bounds are established. In section 4.3 existence conditions of full state observers are expressed in terms of strong observability, strong detectability which are generalizations of familiar observability, detectability to unknown input systems. Based on the results of previous sections, existence conditions and design procedures of minimal order full state PID class of observers are presented in section 4.4. Finally, a matrix algorithm is given for the computation of minimal order observer matrices in section 4.5. Relation with previous work and original contributions are mentioned in each section.

4.1 EXISTENCE CONDITIONS FOR PID CLASS OF OBSERVERS

The purpose of this section is to give existence conditions of PID class of observers for the system described by (4.1). Though similar conditions have been obtained in literature [23,35,36], the conditions given here are expressed in a form which makes it possible to give existence conditions of full state observers in terms of observability, detectability of (4.1). The sufficiency parts of theorems 4.1. -theorem 4.4 provide construction procedures of the observer parameters. These are included for the sake of completeness and since they will be needed in subsequent sections.

For any $f: [0, \infty) \rightarrow F$ write $f_k = [f', \frac{d}{dt} f', \dots, \frac{d^{k-1}}{dt^{k-1}} f']'$: $[0, \infty) \rightarrow F^k$ and define matrices E_k, B_k, C_k $k \geq 1$ by

$$E_k = \begin{bmatrix} 0 & 0 & \dots & 0 \\ CE & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ CA^{k-2}E & CA^{k-3}E & \dots & CE \end{bmatrix}, B_k = \begin{bmatrix} 0 & 0 & \dots & 0 \\ CB & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ CA^{k-2}B & CA^{k-3}B & \dots & CB \end{bmatrix},$$

$$C_k = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{k-1} \end{bmatrix} \quad (4.4)$$

With these definitions the combined plant-observer equations (4.1)-(4.3) can be written in the compact form

$$\begin{aligned} \dot{x}_c &= A_c x_c + B_c q_c \\ e &= D_c x_c + K E_k q_{k-1} + (S + K B_k) u_{k-1} + S_{k-1} \frac{d^{k-1}}{dt^{k-1}} u \end{aligned} \quad (4.5)$$

where $x_c(t) = \begin{bmatrix} x(t) \\ w(t) \end{bmatrix} \in X \oplus W = X^e$, $q_c(t) = \begin{bmatrix} u(t) \\ q(t) \end{bmatrix} \in U \oplus F = U^e$, are

the state and the input of (4.5) and $A_c = \begin{bmatrix} A & 0 \\ MC & N \end{bmatrix}$, $B_c = \begin{bmatrix} B & E \\ R & 0 \end{bmatrix}$,

$D_c = [KC_k - D \quad L]$, $K = [K_0, K_1, \dots, K_{k-1}]$, $S = [S_0, S_1, \dots, S_{k-2}]$ are maps of appropriate dimensions.

In lemma 4.1 necessary and sufficient conditions for (4.3) to serve as an observer for (4.1) are given in terms of the parameters of (4.5).

Lemma 4.1: Consider the plant observer system described by (4.5). $e(t) \rightarrow 0$ as $t \rightarrow \infty$ for all $x(0)$, $w(0)$, q and u if and only if

$$\begin{array}{ll} (a) \quad KE_k = 0 & (b) \quad S + KB_k = 0 \\ (c) \quad S_{k-1} = 0 & (d) \quad \langle A_c / \text{im} B_c \rangle + X_b(A_c) \subset \ker D_c \end{array} \quad (4.6)$$

Proof: «If» Let $V = \langle A_c / \text{im} B_c \rangle + X_b(A_c)$ and $P: X^* \rightarrow X^*/V$ the canonical projection. Since the subspace V is A_c -invariant by lemma 2.4 and 2.6 there exists $\bar{A}_c: X^*/V \rightarrow X^*/V$ such that $\bar{A}_c P = P A_c$ and $\sigma(\bar{A}_c) \subset C_g$ and it follows from lemma 2.1 that there exists $\bar{D}: X^*/V \rightarrow Z$ such that $\bar{D}P = D_c$. Then the equation governing the error e can be written in terms of $\bar{x} = Px_c$ as $\dot{\bar{x}} = \bar{A}_c \bar{x}$, $e = \bar{D} \bar{x}$. Thus $e(t) \rightarrow 0$ as $t \rightarrow \infty$ for all $x_c(0)$, q_c since $\sigma(\bar{A}_c) \subset C_g$.

«Only if» Conditions (4.6a, b, c) and $\langle A_c / \text{im} B_c \rangle \subset \ker D_c$ follows from the requirement that the transfer function from q_c to e must be identically zero. To prove that $X_b(A_c) \subset \ker D_c$ let $S: X_b(A_c) \rightarrow X^*$ be the canonical injection. By lemma 2.4 and lemma 2.7 there exists $A_1: X_b(A_c) \rightarrow X_b(A_c)$ such that $A_c S = S A_1$, and $\sigma(A_1) \subset C_b$. For $x_c(0) = Sx_0$ and $q_c = 0$ the solution of (4.5) is $e(t) = D_c S e^{A_1 t} x_0$. Now, if $\text{im} S = X_b(A_c) \not\subset \ker D_c$ there exists $x_c(0) = Sx_0$ such that $e(t) \not\rightarrow 0$ as $t \rightarrow \infty$ contradicting the assumption. This proves the lemma.

To derive existence conditions for specific types of observers some preliminary results are needed. Lemma 4.2 gives an algebraic characterization of V^* ; the same characterization is also stated in [40] for discrete time systems.

Lemma 4.2: Define the sequence of subspaces V^k $k=1,2,\dots$ by $V^1=\ker C$, $V^{k+1}=\ker C \cap A^{-1}(V^k + \operatorname{im} E)$ and let E_k , C_k $k \geq 1$ be defined as in (4.4) then $V^k = C_k^{-1}(\operatorname{im} E_k)$ $k \geq 1$ and $V^* = C_k^{-1}(\operatorname{im} E_k)$ for $k > d(\ker C)$

Proof: The proof is by induction. The claim clearly holds for $k=1$ since $V^1 = C^{-1}(0) = \ker C$. Assume that $V^{k-1} = C_{k-1}^{-1}(\operatorname{im} E_{k-1}) = \{x : C_{k-1}x = E_{k-1}r \text{ for some } r\}$. Since $V^k = \{x : \exists q \in F \text{ such that } Ax + Eq \in V^{k-1}, Cx = 0\}$ by using the induction assumption we have $V^k = \{x : \exists q \in F \text{ such that } C_{k-1}(Ax + Eq) = E_{k-1}r \text{ for some } r, Cx = 0\}$. Writing the last expression explicitly $V^k = \{x : C_k x = E_k r \text{ for some } r\} = C_k^{-1}(\operatorname{im} E_k)$ which completes the proof of the first part. $V^* = C_k^{-1}(\operatorname{im} E_k)$ for $k > d(\ker C)$ follows since the sequence V^k $k \geq 1$ converges to V^* in at most $d(\ker C) + 1$ steps. (See algorithm 2.9).

The next lemma is an extension of the dual of lemma 3.2.

Lemma 4.3: Let $C: X \rightarrow Y$, $D: X \rightarrow Z$ be arbitrary maps and $T \subset X$, $S \subset Y$ be given subspaces. There exists $K: Y \rightarrow Z$ such that $T \subset \ker(KC - D)$ and $S \subset \ker K$ if and only if $C^{-1}(S) \cap T \subset \ker D$.

Proof «Only if» Let $x \in C^{-1}(S) \cap T$ then $Cx \in S \subset \ker K$ hence $(KC - D)x = -Dx = 0$ which proves that $x \in \ker D$. Thus $C^{-1}(S) \cap T \subset \ker D$.

«If» Let $K_0: Y \rightarrow Z$ be such that $\ker K_0 = S$ and let $C_0 = K_0 C$. Then $\ker C_0 = C^{-1}(\ker K_0) = C^{-1}S$. Choose a basis $\{x_1 \dots x_i, x_{i+1} \dots x_t\}$ for T such that $\{x_1 \dots x_i\}$ is a basis for $T \cap C^{-1}S$ and define $K_1: Z \rightarrow Z$ so that $K_1(C_0 x_j) = Dx_j$ $j = i+1 \dots t$. Finally, let $K = K_1 K_0$ then K satisfies $S \subset \ker K$ and $T \subset \ker(KC - D)$.

Existence conditions of nondynamic (PD class) observers are given in the theorem below.

Theorem 4.1: Let $T_c = X_b(A) + \langle A/\text{im}E + \text{im}B \rangle$. There exists a PD_{k-1} ($k \geq 1$) observer if and only if

$$V^k \cap T_c \subset \ker D \quad (k \geq 1) \quad (4.7)$$

Proof: The following statements are equivalent.

- 1) There exists a PD_{k-1} observer.
- 2) There exists $K: Y^k \rightarrow Z$ such that $T_c \subset \ker(KC_k - D)$, $\text{im}E_k \subset \ker K$
- 3) $C_k^{-1}(\text{im}E_k) \cap T_c \subset \ker D$

The equivalence of statements 1) and 2) follows from lemma 4.1 since $\langle A_c/\text{im}B_c \rangle + X_b(A_c) = T_c$ and $D_c = KC_k - D$ for a PD_{k-1} observer. Notice that the remaining conditions (4.6b, c) can always be satisfied by choosing $S = -KB_k$ and $S_{k-1} = 0$. The equivalence of statements 2) and 3) follows from lemma 4.3 and the conclusion follows from statement 3) by applying lemma 4.2.

Solvability conditions for cheap, P and PD observers are immediately obtained as a special case of theorem 4.1.

Corollary 4.1:

- 1) There exists a cheap observer if and only if $T_c \subset \ker D$
- 2) There exists a P-observer if and only if $\ker C \cap T_c \subset \ker D$
- 3) There exists a PD-observer if and only if $V^* \cap T_c \subset \ker D$

Proof: 1) Follows from (4.7) by letting $C=0$. 2) is identical to (4.7) for $k=1$ and 3) follows since the sequence V^k $k \geq 1$ reaches its limit V^* in at most $d(\ker C) + 1$ steps.

It is clear that the number of differentiators employed does not exceed $d(\ker C)$ if there exists a PD observer at all. However it is preferred to test (4.7) for each k in order to minimize the number of differentiators used. The condition in statement 3) of corollary 4.1 is weaker than the one given in [3] for the existence of a PD-observer since a dead-beat PD-observer is designed in [3] instead of an asymptotic one. In view of the construction given in lemma 4.3 the proof of theorem 4.1 is completely constructive.

In order to give existence conditions of dynamic observers the projection $P: X^e \rightarrow X$ and the embedding $S: X \rightarrow X^e$ are defined as in (3.14), (3.15). The following lemma is standard and also appears in [36].

Lemma 4.4: Let $V \subset X^e$ be an A_c -invariant subspace and let $S: X \rightarrow X^e$ be the embedding defined as in (3.15) then $V_1 = S^{-1}V$ is (C, A) -invariant.

Proof: Take $x \in V_1 \cap \ker C$; then $\begin{bmatrix} x \\ 0 \end{bmatrix} \in V$ and $\begin{bmatrix} A & 0 \\ MC & N \end{bmatrix} \begin{bmatrix} x \\ 0 \end{bmatrix} = \begin{bmatrix} Ax \\ 0 \end{bmatrix} \in V$ since V is A_c -invariant. So $Ax \in V_1$ which proves

that $A(V_1 \cap \ker C) \subset V_1$. Hence V_1 is (C, A) -invariant by definition 2.10.

A fundamental relationship between the spaces V^* , Q_* and R^* originally established in [10, lemma 1.1] is rederived in the next lemma.

Lemma 4.5: The following relations hold between the subspaces V^* , Q_* and R^* and N_* .

$$(a) R^* = V^* \cap Q_* \quad \text{and} \quad (b) N_* = V^* + Q_* \quad (4.8)$$

Proof: Consider the subspace sequences $Q_0 = 0$, $Q_{k+1} = \text{im} E + A(Q_k \cap \ker C)$; $R^0 = 0$, $R^{k+1} = V^* \cap (AR^k + \text{im} E)$ and $V^0 = X$, $V^{k+1} = \ker C \cap A^{-1}(V^k + \text{im} E)$. The sequences converge respectively to Q_* , R^* and V^* . To prove (4.8a) it is enough to show that $R^k = V^* \cap Q_k$ $k=0, 1, 2, \dots$. The proof is by induction. $R^1 = V^* \cap Q_1$ holds for $i=0, 1$. Assume that $R^k = V^* \cap Q_k$ we have $R^{k+1} = V^* \cap (AR^k + \text{im} E) = V^* \cap (A(V^* \cap Q_k) + \text{im} E) \subset V^* \cap (A(\ker C \cap Q_k) + \text{im} E) = V^* \cap Q_{k+1}$. For the reverse inclusion let $x \in V^* \cap Q_{k+1}$, then $x = At + b$ for some $t \in Q_k \cap \ker C$, $b \in \text{im} E$. But since $x \in V^*$ $At + b \in V^*$, $At \in V^* + \text{im} E$ or $t \in A^{-1}(V^* + \text{im} E)$. Hence $t \in Q_k \cap \ker C \cap A^{-1}(V^* + \text{im} E) = Q_k \cap V^* = R^k$ and $x \in (AR^k + \text{im} E) \cap V^* = R^{k+1}$ which proves that $V^* \cap Q_{k+1} \subset R^{k+1}$. Thus it is proven that $R^k = V^* \cap Q_k$ for all $k \geq 0$, hence the conclusion.

To prove (4.8b) note that $N_* = (R^*(A', C'; \ker E'))^\perp$ the conclusion then follows from (4.8a) and lemma 2.11.

Necessary and sufficient conditions for the existence of I-observers are presented in theorem 4.2 which is a modified version of [23, proposition 4].

Theorem 4.2: There exists an I-observer if and only if

$$Q_* + V_b^* \subset \ker D \quad (4.9)$$

Proof: «Only if» Since the matrices K_i , S_i $i \geq 0$ in (4.3) are constrained to zero for an I-observer it follows from lemma 4.1 that $\langle A_c / \text{im} B_c \rangle + X_b(A_c) \subset \ker[-D \quad L]$. Defining $S: X \rightarrow X^*$ as in (3.15) and operating with S^{-1} on both sides it follows from lemma 4.4 that $Q_* \subset \ker D$. For the necessity of $V_b^* \subset \ker D$ see the proof of theorem 4.5.

«If» Let $X = X_1 \oplus X_2 \oplus X_3 \oplus X_4$ where $X_1 = R^* = Q_* \cap V^*$, $X_1 \oplus X_2 = V^*$, $X_1 \oplus X_3 = Q_*$. In a basis adopted to this

decomposition the maps A, E, C have the following matrix representations:

$$A = \begin{bmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ 0 & A_{22} & A_{23} & A_{24} \\ A_{31} & A_{32} & A_{33} & A_{34} \\ 0 & 0 & A_{43} & A_{44} \end{bmatrix} \quad E = \begin{bmatrix} E_1 \\ 0 \\ E_3 \\ 0 \end{bmatrix} \quad (4.10)$$

$$C = \begin{bmatrix} 0 & 0 & C_3 & C_4 \end{bmatrix}$$

The zero blocks in E , and C are due to the fact that $\text{im} E \subset Q_*$ and $V^* \subset \ker C$ respectively and the presence of zeros in A_{21} , A_{41} and A_{42} positions are explained by R^* and V^* being (A, E) -invariant subspaces. Let $V: X \rightarrow W$ be a surjection whose matrix representation in the basis chosen for X is $V = \begin{bmatrix} 0 & P & 0 & 0 \\ 0 & 0 & 0 & I \end{bmatrix}$ where P is a full row rank

matrix such that $\ker P = X_b(A_{22})$. It follows from lemma 2.7 and the structure of V that $\ker V = Q_* + V_b^*$. Let $N_* = Q_* + V^*$ and choose $G \in \underline{G}(Q_*)$ such that $\sigma(A + GC: X/N_*) = \Omega \subset C_g$. Such a choice is possible in view of theorem 2.3. The matrix representation of $A + GC$ in the same basis becomes.

$$A + GC = \begin{bmatrix} A_{11} & A_{12} & A_{13} + G_1 C_3 & A_{14} + G_1 C_4 \\ 0 & A_{22} & 0 & A_{24} + G_2 C_4 \\ A_{31} & A_{32} & A_{33} + G_3 C_3 & A_{34} + G_3 C_4 \\ 0 & 0 & 0 & A_{44} + G_4 C_4 \end{bmatrix} \quad \text{where } \sigma(A_{44} + G_4 C_4) = \Omega$$

The unique matrix N satisfying $NV = V(A + GC)$ is found to be

$$N = \begin{bmatrix} \bar{A}_{22} & P(A_{24}+G_2C_4) \\ 0 & A_{44}+G_4C_4 \end{bmatrix} \text{ where } \bar{A}_{22} \text{ is such that } PA_{22}=\bar{A}_{22}P$$

and $\sigma(\bar{A}_{22}) \subset C_g$ by lemma 2.6. Thus $\sigma(N)=\sigma(\bar{A}_{22}) \cup \sigma(A_{44}+G_4C_4) \subset C_g$. This completes the construction of the matrix N in (4.3). The remaining parameters of the I-observer are chosen as follows: Let $M=-VG$, $R=VB$ and take L such that $LV=D$. Then it is easily verified that $e_1=w-Vx$ is governed by $\dot{e}_1=Ne_1$ and $e=\hat{z}-z=Le_1 \rightarrow 0$ as $t \rightarrow \infty$ for all $x(0)$, $w(0)$, q and u since $\sigma(N) \subset C_g$.

In the sufficiency proof of theorem 4.2 the major effort is devoted to establish that $Q_*+V_b^* \in \underline{Q}_g(C,A;imE)$. In fact, it can be proved that $Q_*+V_b^* = \inf \underline{Q}_g(C,A;imE)$. The construction of the observer matrices based on an externally stabilizable (C,A) -invariant subspace is well known in the literature [23, 36].

Remark 4.1: The exponents of convergence in (4.2) is of practical importance in observer design problems. It is usually desired that the response of the estimator system be rapid compared with the time constants of the plant if it will be used to implement a feedback control law. It follows from the construction given in theorem 4.2 that the set of error poles always include the stable eigenvalues of A_{22} in the representation (4.10). In the notation of theorem 2.4 this is the fixed part of the spectrum of $A+GC:X/Q$ for $G \in \underline{G}(Q)$ and $Q=Q_*+V_b^*$. If it is required to assign all error poles arbitrarily condition (4.9) should be replaced with

$$Q_*+V^* \subset \ker D \quad (4.11)$$

The necessity of (4.11) is proven in [23] sufficiency follows from theorem 2.3 and the construction given in theorem 4.2.

The structure of a PID_k observer is shown in Fig.4.2. It consists of a PD_k -observer and an I-observer. z_1 and z_2 denote respectively the outputs which can be estimated by the PD_k and the I-observers. $\hat{z}$ is obtained as a linear combination of $\hat{z}_1$ and $\hat{z}_2$. The precise solvability condition is given in theorem 4.3.

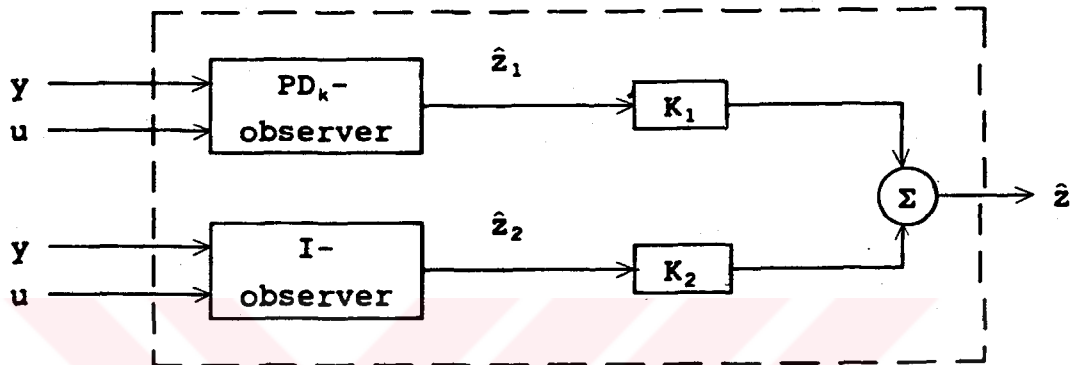


Fig. 4.2. Structure of a PID_k -observer

Theorem 4.3: There exists a PID_{k-1} observer if and only if

$$V^* \cap Q_* + V_b^* \subset \ker D \quad k \geq 1 \quad (4.12)$$

Proof: «Only if» Let $S: X \rightarrow X^*$ be defined as in (3.15) and let $V = \langle A_c / \text{im} B_c \rangle + X_b(A_c)$. Then by lemma 4.4 $V_1 = S^{-1}V$ is (C, A) -invariant and contains $\text{im} E$. Thus it follows from lemma 4.1 that there exists $K: Y^k \rightarrow Z$ such that $Q_* \subset V_1$, $S^{-1}\ker D_c = \ker(KC_k - D)$ and $\text{im} E_k \subset \ker K$. From this condition $Q_* \cap V^* \subset \ker D$ follows by lemma 4.2 and lemma 4.3. For $V_b^* \subset \ker D$ see the proof of theorem 4.5.

«If» Let $D_1: X \rightarrow \text{im} D_1$ and $D_2: X \rightarrow \text{im} D_2$ be such that $\ker D_1 = V^*$ and $\ker D_2 = Q_* + V_b^*$. By theorems 4.1 and 4.2 there exists a PD_{k-1} observer estimating $z_1 = D_1 x$ and an

I-observer estimating $z_2 = D_2 x$. Thus $v = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} D_1 \\ D_2 \end{bmatrix} x$

can be estimated. Let $\hat{v}$ be an estimate for v . On the other hand, since $\ker \begin{bmatrix} D_1 \\ D_2 \end{bmatrix} = V^k \cap (Q_* + V_b^*) = V^k \cap Q_* + V_b^* \subset \ker D$

there exists $K = [K_1 \ K_2] : \text{im} D_1 \oplus \text{im} D_2 \rightarrow Z$ such that $K_1 D_1 + K_2 D_2 = D$. Then $\hat{z} = K \hat{v}$ is an estimate for $z = Dx$ since $e = \hat{z} - z = K(\hat{v} - v) \rightarrow 0$ as $t \rightarrow \infty$.

As a corollary to theorem 4.3 we have

Corollary 4.2: 1) There exists a PI-observer if and only if $\ker C \cap Q_* + V_b^* \subset \ker D$

2) There exists a PID-observer if and only if $V_b^* \subset \ker D$

Proof: 1) Follows from (4.12) for $k=1$.

2) For $k > d(\ker C)$ $V^k \cap Q_* = V^* \cap Q_* = R^* \subset V_b^*$ and (4.12) reduces to $V_b^* \subset \ker D$

Since the sequence of subspaces V^k $k \geq 1$ is monotonically decreasing condition (4.12) of theorem 4.3 gets weaker as the number of differentiators increased reaching its lower limit for the PID-observer which employs at most $d(\ker C)$ differentiators. This fact is elaborated in section 4.2 when different observer types are compared in terms of their undetectable subspaces. A disadvantage of the construction given in theorem 4.3 is that it may contain redundancy ie., the state of the I-observer and the output of the PD-observer may identify the same linear combinations of the state components. An observer structure which does not suffer from this drawback is suggested in section 4.4.

In theorems 4.1-4.3 it was assumed that the initial state of (4.1) is unknown. By relaxing this assumption, namely, assuming that the initial state is known it is possible to enlarge the solvability ranges. The problem is still nontrivial in this case because of the unknown inputs in (4.1). The definition of a known-initial state observer is given below.

Definition 4.2: A system driven by the measurements y and inputs u of (4.1) and giving the output $\hat{z}$ is said to be a known-initial state-observer for (4.1) if for every initial state $x(0)$ of (4.1) there exists an initial state $w(0)$ of the system which depends on $x(0)$, u and y such that the estimation error $e = \hat{z} - z$ satisfies

$$\lim_{t \rightarrow \infty} e(t) = 0 \quad \text{for all } q \text{ and } u \quad (4.13)$$

The difference of the above definition from definition 4.1 is that here, it is not required that (4.13) holds for arbitrary $x(0)$ and $w(0)$ but since knowledge of $x(0)$ is available we are allowed to choose $w(0)$ accordingly. This extra degree of freedom enables to consider even unstable dynamical systems as observer candidates causing the solvability range to extend for this class of observers. For nondynamic observers a similar improvement in solvability is possible if $x(0)$ is known and fixed and is discussed in Chapter 5 in connection with the solvability of rational function equations. Existence conditions of known-initial state-dynamical observers are presented in theorem 4.4.

Theorem 4.4: 1) There exists a known-initial state I-observer if and only if

$$Q_* \subset \ker D \quad (4.14)$$

- 2) There exists a known-initial state PID_{k-1} -observer if and only if

$$V^k \cap Q_* \subset \ker D \quad k \geq 1 \quad (4.15)$$

Proof: «Only if» 1) and 2) Note that for a known-initial state-observer conditions (4.6a-c) of lemma 4.1 should be satisfied and (4.6d) is replaced by $\langle A_c / \text{im} B_c \rangle \subset \ker D_c$. These follow from the requirement that the estimation error must be decoupled from the inputs u, q . Then, since $\langle A_c / \text{im} B_c \rangle$ is an A_c -invariant subspace reasoning as in theorem 4.2 and theorem 4.3 the necessity of (4.14) and (4.15) is established.

«If» 1) and 2) Follow the construction procedure given in theorem 4.2. Take $Q = Q_*$, $d(W) = n - d(Q)$ and $V: X \rightarrow W$ such that $\ker V = Q$. Choosing the parameters N, M, R, L as in theorem 4.2 it is easily verified that the error $e_1 = w - Vx$ still satisfies $\dot{e}_1 = Ne_1$ and $e = \hat{z} - z = Le_1$. By starting the observer system with the initial state $w(0) = Vx(0)$ the dead-beat behaviour $e(t) = 0 \quad t \geq 0$ is obtained. This proves the sufficiency of (4.14) for known-initial state I-observers. For $PID_{k-1} \quad k \geq 1$ observers a known-initial state I-observer is used to estimate the output $z_2 = D_2 x$ where $\ker D_2 = Q_*$. The proof then follows as in theorem 4.3.

The existence conditions of known-initial state PI and PID observers are recovered from theorem 4.4 as a corollary.

Corollary 4.3: 1) There exists a known-initial state PI-observer if and only if

$$\ker C \cap Q_* \subset \ker D \quad (4.16)$$

2) There exists a known-initial state PID-observer if and only if

$$R^* \subset \ker D \quad (4.17)$$

Proof: 1) (4.16) is equivalent to (4.15) for $k=1$

2) For $k > d(\ker C)$ $V^k = V^*$ and (4.15) is equivalent to (4.17) in view of lemma 4.5. See also theorem 4.6 for an alternate proof.

The solvability conditions for PID class of observers are summarized in Table 4.1.

Table 4.1. Existence conditions and smallest undetectable subspaces of PID class of observers

	OBSERVER TYPE	NECESSARY AND SUFFICIENT CONDITIONS FOR THE EXISTENCE	
NONDYNAMIC	Cheap	$T_c = X_b(A) + \langle A/\text{im}E + \text{im}B \rangle \subset \ker D$	
	P	$T_p = \ker C \cap (X_b(A) + \langle A/\text{im}E + \text{im}B \rangle) \subset \ker D$	
	PD	$T_{PD} = V^* \cap (X_b(A) + \langle A/\text{im}E + \text{im}B \rangle) \subset \ker D$	
		$x(0)$ KNOWN	$x(0)$ UNKNOWN
DYNAMIC	I	$T_{KI} = Q_* \subset \ker D$	$T_I = Q_* + V_b^* \subset \ker D$
	PI	$T_{KPI} = \ker C \cap Q_* \subset \ker D$	$T_{PI} = \ker C \cap Q_* + V_b^* \subset \ker D$
	PID	$T_{KPID} = R^* \subset \ker D$	$T_{PID} = V_b^* \subset \ker D$

4.2 UNDETECTABLE SUBSPACES

In this section the six types of observers introduced previously will be compared in terms of their undetectable subspaces whose definition is given below. The precise relationships among the undetectable subspaces of PID class of observers and the smallest undetectable subspaces for known and unknown initial state observers will be established.

Definition 4.3: A subspace $T \subset X$ is an undetectable subspace (a known-initial state-undetectable subspace) if there exists an observer (a known-initial state-observer) such that the projection of the state vector of (4.1) along T can be estimated.

The set of undetectable subspaces (known-initial state-undetectable subspaces) will be denoted by $\underline{T}$ ($\underline{T}_k$). It is clear that $\underline{T} \subset \underline{T}_k$. Two simple properties of undetectable subspaces immediately follow from the definition.

Property 4.1: Any subspace S which contains an undetectable subspace T is itself an undetectable subspace.

Proof: Let S be a projection along S and T be a projection along T . Since $\ker T \subset \ker S$ there exists a map H such that $HT=S$. Thus if $\hat{z}$ is an estimate for Tx then $H\hat{z}$ is an estimate for Sx since $H\hat{z}-Sx=H(\hat{z}-Tx) \rightarrow 0$ as $t \rightarrow \infty$. This proves that S is an undetectable subspace.

Property 4.2: If T_1, T_2 are undetectable subspaces so is $T_1 \cap T_2$.

Proof: Let T_1 , T_2 and T_3 be projections along T_1 , T_2 and $T_1 \cap T_2$ respectively. Since $\ker \begin{bmatrix} T_1 \\ T_2 \end{bmatrix} = \ker T_3$ there exists a

map H such that $H \begin{bmatrix} T_1 \\ T_2 \end{bmatrix} = T_3$. Thus if $\hat{z}_1$, $\hat{z}_2$ are estimates

of $T_1 x$ and $T_2 x$ then $E \begin{bmatrix} \hat{z}_1 \\ \hat{z}_2 \end{bmatrix}$ is an estimate for $T_3 x$ which

shows that $T_3 = \ker T_3$ is an undetectable subspace.

Because of properties 4.1 and 4.2 the set of undetectable subspaces is closed under subspace addition and subspace intersection therefore supremal and infimal elements of the sets $\underline{T}$ and $\underline{T}_k$ exists by lemma 2.3. The state space X is clearly the supremal element of both sets; the infimal elements which are denoted by T_* and T_{k*} are given in the theorems below.

Theorem 4.5: The smallest undetectable subspace is $T_* = V_b^*$

Proof: It must be proven that 1°) $V_b^* \in \underline{T}$ and 2°) $T \in \underline{T}$ implies $V_b^* \subset T$. By corollary 4.2 the projection of the state vector along V_b^* can be estimated by a PID-observer this establishes that V_b^* is an undetectable subspace. 2°) will be proved by contradiction. Assume that T is a subspace such that $V_b^* \not\subset T$. Let D be a map such that $\ker D = T$. Choose $F \in \underline{F}(V^*)$ such that $\sigma(A + EF: V_b^*) \subset C_b$. Write $A_1 = A + EF: V_b^*$ and let $S: V_b^* \rightarrow X$ be the canonical injection. For $x(0) = Sx_0$, $q = Fx$ and $u = 0$ the solution of (4.1) is $z(t) = De^{(A+EF)t} Sx_0 = DSe^{A_1 t} x_0$ and $y(t) = 0$ $t \geq 0$. Since $DS \neq 0$ and $\sigma(A_1) \subset C_b$ there exists $x(0) \in V_b^*$ such that $z(t)$ does not converge to zero as $t \rightarrow \infty$. On the other hand for $x(0) = 0$, $u = 0$, $q = 0$ we

have $z(t)=0$ $t \geq 0$ and $y(t)=0$ $t \geq 0$ since $y(t)=0$ $t \geq 0$ and $u(t)=0$ $t \geq 0$ in both cases it is clear that (4.2) can not be satisfied, for we must have $\hat{z}(t) \neq 0$ in the first case and $\hat{z}(t) \rightarrow 0$ in the second. This contradicts the existence of an observer for $z=Dx$ (ie., $T \notin \underline{T}$) proving the theorem.

It follows from theorem 4.5 that $V_b^* \subset \ker D$ is a necessary condition for the existence of an observer estimating $z=Dx$. This fact has been used in the proof of theorem 4.2, 4.3.

Theorem 4.6: The smallest known-initial state-undetectable subspace is $T_{K^*} = R^*$

Proof: The proof is similar to that of theorem 4.5. By corollary 4.3 there exists a known-initial state PID observer such that the projection of the state vector along R^* can be estimated. This proves that $R^* \in \underline{T}_K$. Next, let $z=Dx$ where $R^* \not\subset \ker D$. It can be shown as in the previous theorem that there exists $F_1 \in \underline{F}(V^*)$ and $x_0 \in R^*$ such that the solution of (4.1) for $x(0)=x_0$, $q=F_1x$, $u=0$ satisfies $z(t) \neq 0$ as $t \rightarrow \infty$ while $y(t)=0$ for $t \geq 0$. Since $\sigma(A+EF:R^*)$ can be arbitrarily assigned by $F \in \underline{F}(V^*)$ there exists $F_2 \in \underline{F}(V^*)$ such that $\sigma(A+EF_2:R^*) \subset C$, and the solution of (4.1) for $x(0)=x_0$, $q=F_2x$, $u=0$ satisfies $z(t) \rightarrow 0$ as $t \rightarrow \infty$ and $y(t)=0$ for $t \geq 0$. Since $y(t)=0$ $t \geq 0$, $u(t)=0$ $t \geq 0$ and $x(0)=x_0$ in both cases a known-initial state observer which estimates $z=Dx$ does not exist for we must have $\hat{z} \neq 0$ when $q=F_1x$ and $\hat{z} \rightarrow 0$ when $q=F_2x$. This establishes that $\ker D \notin \underline{T}_K$ proving the theorem.

It is worth noting that theorem 4.5 and theorem 4.6 are extremely general and do not depend on the particular observer structure. The observers may be even nonlinear, time-varying or infinite dimensional. Nothing

is assumed for the observer system except that it has a unique solution for a given initial state and input.

By restricting the observers to the PID class the smallest undetectable subspaces for the six types of observers introduced can be computed from the necessary and sufficient conditions given in section 4.1. These are presented along with each solvability condition in Table 4.1.

The foregoing discussion shows that the smallest undetectable subspaces are uniquely defined however any complement of this subspace can be taken as a «detectable subspace» since the projection of the state vector on any complement of the undetectable subspace can be estimated. In a space equipped with inner product it is natural to take the orthogonal complement of the undetectable subspace and consider only orthogonal projections in order to define the detectable subspace uniquely. This point of view can be encountered in some references e.g [32, 33] but will not be pursued any further in this work. Some remarks concerning undetectable subspaces of PID-class of observers are in order here.

Remark 4.2: The smallest undetectable subspaces for known and unknown initial state PID class of observers form a lattice structure as shown in Figure 4.3 where a rising branch from T_α to T_β means $T_\alpha \subset T_\beta$. The lattice diagram provides a hierarchical classification of PID class of observers in terms of their undetectable subspaces. It is seen that the PID observer has the smallest «smallest undetectable» subspace among all observers belonging to the PID observer class. Moreover it follows from theorems 4.5 and 4.6 that even outside this class no observer can have a smaller undetectable subspace, the PID-observer is the best in this sense and

that nothing would be gained by introducing new types of observers such as nonlinear, distributed etc.

Remark 4.3: It is interesting to note some extremal cases. The undetectable subspaces when unknown inputs are not present can be obtained from the ones given in Table 4.1 by letting $E=0$. In this case the subspaces for I, PI and PID observers coincide and are given by $T_k=0$ when $x(0)$ is known, and by $T=\langle \ker C/A \rangle \cap X_b(A)$ when $x(0)$ is unknown. These results agree with the ones given in literature for $q=0$ case. Another extremal case can be created by taking $C=0$, $B=0$ which simulates the situation where there are no measurements and no control inputs. In this case $T=T_c=\langle A/\text{im}E \rangle + X_b(A)$ for all six types of observers.

Remark 4.4: While it is clear that the undetectable subspaces of PD_k and PID_k observers get smaller as k is increased, because of the obvious disadvantage of using differentiators, it is important to know whether it is worth to construct PID observers and how much benefit is gained by using a PID-observer instead of a PI-observer. This question can be precisely answered by comparing the smallest undetectable subspaces of PI and PID observers. From Table 4.1 it follows that $T_{PI}=T_{PID}+T_{KPI}$ which means, the smallest undetectable subspace of a PI-observer is larger than of the PID-observer by exactly $T_{KPI}=\ker C \cap Q$, and that they are equal if and only if $T_{KPI} \subset T_{PID}$. This is the case if $\ker C \cap \text{im}E=0$ for instance. Under the latter condition a PI-observer is as good as a PID-observer as far as their undetectable subspaces are concerned. However in section 4.4 it will be shown that the order of a full state PID_k observer decreases by increasing k . Hence there may still be a reason to prefer a PID-observer to a PI-observer although their undetectable subspaces are identical.

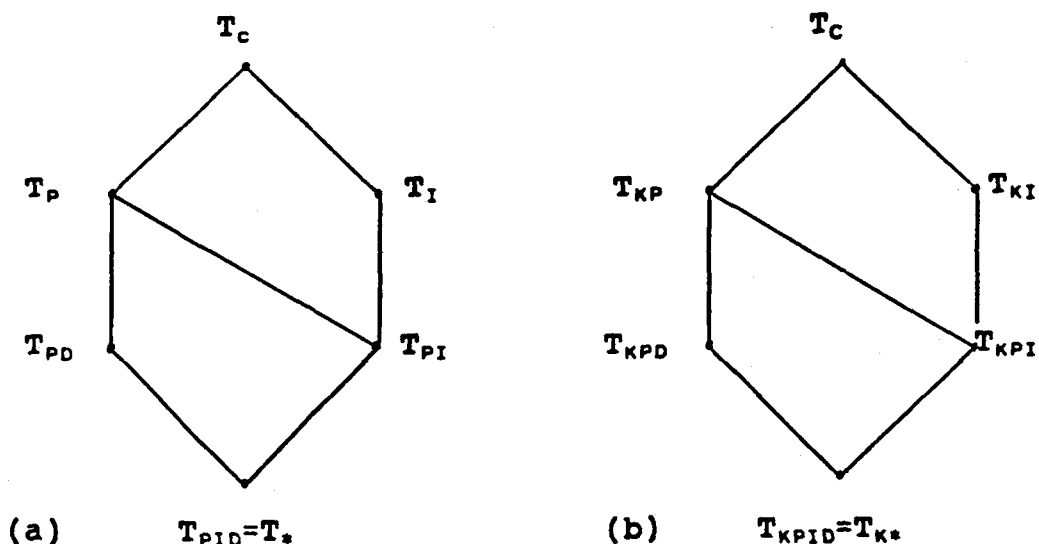


Fig. 4.3. Lattice diagram of smallest undetectable subspaces of PID class of observers (a) $x(0)$ unknown (b) $x(0)$ known.

4.3 OBSERVABILITY, DETECTABILITY OF UNKNOWN INPUT SYSTEMS

Though the relation between observability and the existence of observers estimating the state vector of a linear, time-invariant system subject to only control inputs is well known [29-31] similar conditions do not appear in literature for systems excited by unknown disturbances except some preliminary results reported in [37,38]. In this section the work in [38] will be extended to PID class of observers using the concepts of strong observability, strong detectability whose definitions are given below. The notation $x(t, x_0, q)$, $y(t, x_0, q)$ etc. denotes the state and output trajectories of (4.1) corresponding to the initial state x_0 , disturbance input q and $u=0$.

Definition 4.4: The system (4.1) is strongly observable if $x_0=0$ whenever $y(t, x_0, q)=0$ for $t \geq 0$.

Definition 4.5: The system (4.1) is strongly detectable if $x(t, x_0, q) \rightarrow 0$ as $t \rightarrow \infty$ whenever $y(t, x_0, q) = 0$ for $t \geq 0$.

The dynamical definitions of strong observability and strong detectability stated above have been originally given for discrete time systems in [39-41]. Frequency domain conditions have been derived later in [37]. It can be easily verified that these definitions reduce to the familiar definitions of observability and detectability if unknown inputs are taken to be zero.

Equivalent conditions for strong observability which give more insight to the dynamics of a strongly observable system are given in the next lemma.

Lemma 4.6: The following statements are equivalent.

- (i) The system (4.1) is strongly observable.
- (ii) For every x_0 and every q $y(t, x_0, q) = 0$ for $t \geq 0$ implies $x(t, x_0, q) = 0$ for $t \geq 0$
- (iii) Let $d = E q$ the mapping $(x_0, d) \rightarrow y(., x_0, q)$ is injective

Proof: «(i) $\Rightarrow$ (ii)» Assume that $y(t, x_0, q) = 0$ $t \geq 0$. Take $t_1 > 0$ and let $x_1 = x(t_1, x_0, q)$. Define $q_1(t) = q(t + t_1)$ $t \geq 0$ then $y(t, x_1, q_1) = C e^{At} x_1 + \int_0^t C e^{A(t-T)} E q_1(T) dT$. Substituting x_1 and q_1 and making the change of variables $s = T + t_1$ it follows that $y(t, x_1, q_1) = y(t + t_1, x_0, q) = 0$ $t \geq 0$. Thus by strong observability $x_1 = 0$. Since t_1 was arbitrary $x(t, x_0, q) = 0$ for all $t \geq 0$.

«(ii) $\Rightarrow$ (i)» Follows from definition 4.4 since $x(t, x_0, q) = 0$ $t \geq 0$ implies $x_0 = 0$

«(ii) $\Leftrightarrow$ (iii)» Since the mapping $(x_0, d) \rightarrow y(., x_0, q)$ is linear it is injective if and only if $y(t, x_0, q) = 0$ $t \geq 0$ implies $x_0 = 0$, $d = 0$ which is equivalent to (ii).

The fact that the mapping $(x_0, d) \rightarrow y(., x_0, q)$ is injective, immediately implies that x_0 and d can be uniquely determined from the measurements $y(t, x_0, q)$ $t \geq 0$ when the system is strongly observable. Thus strong observability is closely related to the construction of unknown input observers. It is well known that strong observability is equivalent to the geometric condition $V^* = 0$ [41]. A geometric condition for strong detectability is given in the next theorem.

Theorem 4.7: Let $z=x$ in (4.1). The following statements are equivalent.

- (i) There exists a PID-observer for (4.1).
- (ii) (4.1) is strongly detectable.
- (iii) $V_b^* = 0$

Proof: «(ii)' $\Rightarrow$ (i)» Follows from definition 4.1 by taking $z=x$. If (4.1) is not strongly detectable (4.2) can not be satisfied for all $x(0)$ and q as the argument at the end of the proof of theorem 4.5 shows.

«(iii)' $\Rightarrow$ (ii)» Assuming that $V_b^* \neq 0$ there exists x_0 and q such that $y(t, x_0, q) = 0$ $t \geq 0$ while $x(t, x_0, q) \not\rightarrow 0$ as $t \rightarrow \infty$ contradicting strong detectability. See the proof of theorem 4.5.

«(iii) $\Rightarrow$ (i)» Follows from corollary 4.2 by taking $D=I$

It follows from their dynamical definitions and the geometric conditions that the strong versions of observability and detectability reduce to the ordinary observability and detectability when unknown inputs are not present. Namely, for $E=0$ $V^*=0$ if and only if $\langle \ker C/A \rangle = 0$ and $V_b^*=0$ if and only if $\langle \ker C/A \rangle \cap X_b(A) = 0$.

Another observability condition which do not have a counterpart for known input systems and which is related

to the construction of known-initial state-observers is introduced in the following definition.

Definition 4.6: The system (4.1) is known-initial state-observable if «for every q , $y(t,0,q)=0 \ t \geq 0$ implies $x(t,0,q)=0 \ t \geq 0$ »

A system which is not subject to unknown inputs is trivially known-initial state-observable since the condition of definition 4.6 is always satisfied when $q=0$. Known-initial state-observability is closely related to left invertibility of (4.1) which is defined as [52,53].

Definition 4.7: The system (4.1) is left invertible if « $q_1 \neq q_2$ implies $y(\cdot,0,q_1) \neq y(\cdot,0,q_2)$ » or equivalently for linear systems « $y(t,0,q)=0 \ t \geq 0$ implies $q=0$ »

Lemma 4.7: (4.1) is left invertible if and only if it is known-initial state-observable and $\ker E=0$.

Proof: «Only if» Take q such that $Eq=0$ then $y(t,0,q)=0$ for $t \geq 0$ and by left invertibility $q=0$ thus $\ker E=0$. Let q be such that $y(t,0,q)=0 \ t \geq 0$ then by left invertibility $q=0$ which implies that $x(t,0,q)=0 \ t \geq 0$ thus (4.1) is known-initial state-observable by definition 4.6.

«If» Assume that $y(t,0,q)=0$ for $t \geq 0$ by assumption $x(t,0,q)=0$ for $t \geq 0$ which implies that $Eq=0$ and $q=0$ since $\ker E=0$.

The next theorem which is the analog of theorem 4.7 establishes the equivalence of known-initial state-observability and the existence of known initial state observers.

Theorem 4.8: Let $z=x$ in (4.1). The following statements are equivalent.

- (i) There exists a known-initial state PID observer for (4.1)
- (ii) (4.1) is known-initial state-observable.
- (iii) $R^*=0$

Proof: «(i) $\Leftrightarrow$ (iii)» Follows from corollary 4.3 by taking $D=I$

«(ii) $\Rightarrow$ (iii)» Take $x_0 \in R^*$. It follows from the dynamical characterization of R^* that there exists $t_0 > 0$ and q such that $x(t_0, 0, q) = x_0$ and $x(t, 0, q) \in R^* \subset \ker C$ for $t \geq 0$. Therefore $y(t, 0, q) = 0$ $t \geq 0$ which implies that $x(t, 0, q) = 0$ $t \geq 0$. Thus $x_0 = 0$ since x_0 was arbitrary it follows that $R^* = 0$.

«(iii) $\Rightarrow$ (ii)» Assume that $y(t, 0, q) = 0$ $t \geq 0$. Since R^* is the reachable subspace in $\ker C$ $x(t, 0, q) \in R^*$ for $t \geq 0$ and the conclusion follows since $R^* = 0$.

The following implications hold between the three types of observability defined above: Strong observability $\Rightarrow$ strong detectability $\Rightarrow$ known-initial state-observability. These can be verified from the dynamical definitions or from the geometric conditions derived, since $R^* \subset V_b^* \subset V^*$. The foregoing results are summarized in Table 4.2 where the frequency domain conditions in terms of the system matrix $P(s) = \begin{bmatrix} sI - A & E \\ C & 0 \end{bmatrix}$

are also included for comparison purposes. The frequency domain conditions for strong observability and strong detectability have been given in [37]; the condition for known-initial state-observability will be established in theorem 5.8.

We proceed to give existence conditions for full state PID_k $k \geq 0$ observers in terms of strong observability and strong detectability. Though these follow immediately from the general conditions given in section 4.1 they can not be easily deduced from the ones usually given in literature. It should be noted that a full state I-observer does not exist unless $E=0$.

Theorem 4.9: Let $z=x$ in (4.1).

- 1) There exists a PID_{k-1} $k \geq 1$ observer if and only if
 - (a) $V^k \cap Q_* = 0$ and (4.18)
 - (b) (4.1) is strongly detectable
- 2) There exists a PID_{k-1} $k \geq 1$ observer with freely assignable error poles if and only if
 - (a) $V^k \cap Q_* = 0$ and (4.19)
 - (b) (4.1) is strongly observable

Proof: (4.18) follows from theorem 4.3 and theorem 4.7 by taking $D=I$. (4.19) follows similarly by invoking remark 4.1 and the geometric condition for strong observability.

Despite the fact that observability or detectability is sufficient for the existence of full state PID_{k-1} $k \geq 1$ observer when $q=0$ it is clear from theorem 4.9 that this is no longer the case when $q \neq 0$. An alternative type of detectability which is equivalent to strong detectability and the condition $\ker C \cap \text{im } E = 0$ has been suggested in [37] to preserve the equivalence between detectability and the existence of full state PI-observers.

Table 4.2 Observability, detectability of unknown input systems.

	Dynamical definition	Geometric condition	Frequency domain condition	Iff condition for the existence of full state
Strong observability	$y(t, x_0, q) = 0 \quad t \geq 0 \implies x(t, x_0, q) = 0 \quad t \geq 0$	$V^* = 0$	$\text{rank } P(s) = n + \text{rank } E$ $V \in C_b$	PID observer with freely assignable error poles
Strong detectability	$y(t, x_0, q) = 0 \quad t \geq 0 \implies x(t, x_0, q) \rightarrow 0$ as $t \rightarrow \infty$	$V_b^* = 0$	$\text{rank } P(s) = n + \text{rank } E$ $V \in C_b$	PID observer
Known-initial state-observability	$y(t, 0, q) = 0 \quad t \geq 0 \implies x(t, 0, q) = 0 \quad t \geq 0$	$R^* = 0$	$\exists s \in C \text{ s.t. rank } P(s) = n + \text{rank } E$	Known-initial state PID observer

4.4 MINIMAL ORDER OBSERVERS

The order of the PI-observer designed by the procedure described in theorem 4.3 is $n-d(Q)$ where Q is a subspace satisfying $Q \in \underline{Q}_g(C, A; \text{im} E)$ and $\ker C \cap Q \subset \ker D$. Thus an upper bound for the observer order is $n - \text{rank} E$. Still, this number may be unnecessarily large for if the to be estimated output is the state itself, the projection of the state vector along T_p (the smallest undetectable subspace for P-observers) could be immediately determined by nondynamic processing of the measurements alone. Hence an observer of order $d(T_p)$ would suffice to recover the remaining components of the state. This idea has already been applied to the design of PI-observers of order $d(\ker C)$ in [30]. Later, this order has been further reduced in [1] and [36, 43-45]. In this section the results in these works are generalized to include the presence of unknown inputs and order reduction by using PID observers is investigated. The minimal orders for full state PID_k , $k \geq 0$ observers are established. It is shown that they are given by the dimensions of smallest undetectable subspaces of PD_k , $k \geq 0$ observers.

The common practice followed in literature for dealing with minimal order observer design problems is to convert the problem to an equivalent dynamic cover problem [44-45]. The question then reduces to finding a cover with minimal dimension. Though this approach proved to be useful when the subspace to be covered is one dimensional (ie., when the to be estimated output is a single linear functional of the state) the general problem remains largely unsolved. In this section a different approach is followed for finding the minimal orders of full state observers which does not require the use of cover techniques. However, for observers

estimating an output which is different from the state the minimal orders given are only upper bounds.

Some preliminary results which by themselves are significant are given first.

Lemma 4.8: Assume that $\ker C \cap \text{im} E = 0$. Let S be a subspace such that $\ker C \subset S$ and $S \oplus \text{im} E = X$ and let P be a projection on S along $\text{im} E$. Then

- (a) $V^* = \langle \ker C / PA \rangle$ and (4.20)
 (b) $V_b^* = \langle \ker C / PA \rangle \cap X_b(PA)$

Proof: Let $V \in \underline{V}(A, E; \ker C)$ then by definition $AV \subset V + \text{im} E$, $V \subset \ker C$. Thus $PAV \subset PV = V \subset \ker C$ which means that $V \in \underline{I}(PA; \ker C)$. Hence $\underline{V}(A, E; \ker C) \subset \underline{I}(PA; \ker C)$. For the reverse inclusion, let $V \in \underline{I}(PA; \ker C)$ then $PAV \subset V$, $V \subset \ker C$. Operating with P^{-1} (the functional inverse of P) $P^{-1}(PAV) = \text{im} E + AV \subset P^{-1}V = V + \text{im} E$ or $AV \subset V + \text{im} E$. Thus $\underline{I}(PA; \ker C) \subset \underline{V}(A, E; \ker C)$ and it follows that $\underline{V}(A, E; \ker C) = \underline{I}(PA; \ker C)$ from which (4.20a) can be established by taking the supremal elements of both sets.

To prove (4.20b), note that $R^* = 0$ since $\text{im} E \cap V^* = \text{im} E \cap \ker C = 0$. On the other hand $\text{im}(I-P)A \subset \text{im} E$ hence there exists $F: X \rightarrow F$ such that $EF = (I-P)A$. For this choice of F we have $(A - EF)V^* = PAV^* \subset V^*$ which shows that $F \in \underline{F}(V^*)$. The conclusion now follows from (2.11).

Lemma 4.8 offers a method for the easy computation of supremal (A, E) -invariant and supremal internally stabilizable (A, E) -invariant subspaces in $\ker C$ when $\ker C \cap \text{im} E = 0$. In this sense the result is complementary to the work in [12] since the formula given in [12] can not be applied when $\text{rank} E < \text{rank} C$ whereas the assumption of lemma 4.8 generically holds in this case.

Lemma 4.8 allows to express the existence conditions given in theorem 4.9 in terms of the detectability and observability of the pair (C, PA) .

Lemma 4.9: Let $z=x$ in (4.1) and assume that $\ker C \cap \text{im } E = 0$. Define the projection P as in lemma 4.8. Then

- 1) There exists a PI-observer if and only if the pair (C, PA) is detectable.
- 2) There exists a PI-observer with freely assignable error poles if and only if the pair (C, PA) is observable.

Proof: Follows from theorem 4.9 ($k=1$), lemma 4.8 and the definitions of observability detectability.

It follows from the discussion at the beginning of the section that the larger the dimension of $Q \in \underline{Q}_g(C, A; \text{im } E)$ the smaller the order of the PI-observer. Moreover for full state PI-observers we must have $\ker C \cap Q = 0$. Thus one is tempted to take Q such that $Q \oplus \ker C = X$. The eigenvalues of $A + GC: X/Q$ for $G \in \underline{G}(Q)$ and Q is a complement of $\ker C$ is described in [21, lemma 5.2]. A matrix proof of this statement is given in the next lemma.

Lemma 4.10: Assume that $Q \oplus \ker C = X$ and let V be a projection on $\ker C$ along Q . Then

$$\sigma(A + GC: X/Q) = \sigma(VA: \ker C) \text{ for all } G \in \underline{G}(Q) \quad (4.21)$$

Proof: Choose a basis for X as a union of bases for Q and $\ker C$ respectively. With respect to this basis the matrix representations of the maps A and C are $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$

$C=[C_1 \ 0]$. For $G=\begin{bmatrix} G_1 \\ G_2 \end{bmatrix}$ chosen such that $(A+GC)Q \subset Q$ the

matrix representation of $A+GC$ becomes $A+GC=\begin{bmatrix} A_{11}+G_1C_1 & A_{12} \\ 0 & A_{22} \end{bmatrix}$

Thus it follows that the matrix representation of the

maps $VA:\ker C$ and $A+GC:X/Q$ are both given by A_{22} . This proves (4.21).

The following lemma which is an extension of [1, lemma 3.5] brings us very close to the main result.

Lemma 4.11: 1) Assume that $\ker C \cap \text{im} E = 0$ and $V_b^* = 0$. There exists a subspace Q such that $Q \oplus \ker C = X$, $\text{im} E \subset Q$ and $\sigma(A+GC:X/Q) \subset C_g$ for all $G \in \underline{G}(Q)$

2) Assume that $\ker C \cap \text{im} E = 0$ and $V^* = 0$. For any given symmetric set Ω of $d(\ker C)$ complex numbers there exists a subspace Q such that $Q \oplus \ker C = X$, $\text{im} E \subset Q$ and $\sigma(A+GC:X/Q) = \Omega$ for all $G \in \underline{G}(Q)$.

Proof: 1) Let S be a subspace such that $\ker C \oplus \text{im} E \oplus S = X$. Let P be a projection on $\ker C \oplus S$ along $\text{im} E$ and V be a projection on $\ker C$ along $\text{im} E \oplus S$. Assume that for some $x \in \ker C$ $VPx = sx$ and $(I-V)Px = 0$ then also $Px = sx$ and by the detectability of the pair (C, PA) (cf. lemma 4.8) it follows that $s \in C_g$ or $x = 0$. Define maps $A_0:\ker C \rightarrow \ker C$ and $C_0:\ker C \rightarrow \ker V$ such that $A_0x = VPx$, $x \in \ker C$ and $C_0x = (I-V)Px$, $x \in \ker C$. It is proven that the pair (C_0, A_0) is detectable. So there exists $G_0:\ker V \rightarrow \ker C$ such that $\sigma(A_0 + G_0C_0) \subset C_g$. Let $G:X \rightarrow X$ be an extension of G_0 such that $Gx = G_0x$, $x \in \ker V$ and $Gx = 0$, $x \notin \ker V$. Take $W = VP + G(I-V)P$. It can be verified that $W^2 = W$ hence W is a projection on $\text{im} W = \ker C$ along $Q = \ker W \supset \text{im} E$ and the conclusion follows by applying lemma 4.10 since $\sigma(WA:\ker C) = \sigma(A_0 + G_0C_0) \subset C_g$.

2) is proven similarly by replacing «detectable» with «observable» and choosing G_0 such that $\sigma(A_0 + G_0C_0) = \Omega$.

The conditions for the existence of a PI-observer of order $d(\ker C)$ are presented in lemma 4.12 which can be viewed as a generalization of the well known construction given by Luenberger [29,30] to unknown input systems. An observer of order $d(\ker C)$ is called a «reduced order» observer. The term «minimal order observer» is often misused in literature to refer Luenberger observers of order $d(\ker C)$. It will be shown in the sequel that the order of a minimal order PI-observer is in general lower than this number.

Lemma 4.12: Let $z=x$ in (4.1). 1) There exists a PI-observer of order $d(\ker C)$ if and only if $\ker C \cap \text{im } E = 0$ and (4.1) is strongly detectable.
2) There exists a PI-observer with freely assignable error poles of order $d(\ker C)$ if and only if $\ker C \cap \text{im } E = 0$ and (4.1) is strongly observable.

Proof: The necessity of conditions has already been proven in theorem 4.9 for $k=1$. Sufficiency is established by constructing the subspace Q satisfying the conditions of lemma 4.11 and designing PI-observers of order $n-d(Q)=d(\ker C)$ following the proof of sufficiency of theorem 4.3.

The construction of minimal order PID_k $k \geq 0$ observers is considered in theorem 4.10. It will be proven later on that the orders given are in fact minimum. The structure leading to the synthesis of minimal order PID_k observers is depicted in Fig.4.4. The basic idea is to estimate as large part of the state vector as possible by a PD_k observer for which no dynamics is needed and use the output of the PD_k observer as an input to a reduced order PI-observer to recover the whole state vector.

Theorem 4.10: Let $z=x$ in (4.1). There exists a PID_{k-1} $k \geq 1$ observer of order $d(V^k \cap T_c)$ if and only if $\text{im} E \cap V^k = 0$ and (4.1) is strongly detectable.

Proof: «Only if» Follows from theorem 4.9.

«If» Consider $\Sigma_0: \dot{x} = Ax + Bu + Eq, y_0 = C_0 x, z = x$ where C_0 is such that $\ker C_0 = V^k \cap T_c$. Let V_0 be the largest internally stabilizable (A, E) -invariant subspace with respect to C_0 and which is contained in $\ker C_0$. Since $\ker C_0 \subset \ker C$ $V_0 = V_0^* = 0$ hence Σ_0 is strongly detectable. Moreover $\ker C_0 \cap \text{im} E = V^k \cap \text{im} E = 0$. Thus by lemma 4.12 there exist a PI observer of order $d(V^k \cap T_c)$ for Σ_0 . Suppose that this observer is described by $\dot{w}_0 = Nw_0 + My_0 + Su, \hat{z}_0 = Lw_0 + Ky_0$ where $w_0 \in W_0$ and $d(W_0) = d(\ker C_0)$. On the other hand, by theorem 4.1 there exists a PD_{k-1} observer estimating y_0 i.e., there exists $K_0, K_1 \dots K_{k-1}$ and $S_0, S_1 \dots S_{k-1}$ such that

$$\hat{y}_0 = \sum_{i=0}^{k-1} K_i y^{(i)} + S_i u^{(i)} \text{ and } (\hat{y}_0 - y_0) \rightarrow 0 \text{ as } t \rightarrow \infty. \text{ We claim}$$

that the system described by $\Sigma: \dot{w} = Nw + M\hat{y}_0 + Su, \hat{z} = Lw + K\hat{y}_0$ is a PID_{k-1} observer of order $d(V^k \cap T_c)$ for (4.1). To prove this, define $r_1 = w_0 - w$ and $r = \hat{z}_0 - \hat{z}$. It can be verified that r_1 and r satisfy $\dot{r}_1 = Nr_1 + M(y_0 - \hat{y}_0)$ and $r = Lr_1 + K(y_0 - \hat{y}_0)$. Since $\sigma(N) \subset C_0$ and $(y_0 - \hat{y}_0) \rightarrow 0$ it follows that $r \rightarrow 0$ for $t \rightarrow \infty$ as well. Thus we have $e = \hat{z} - z = (\hat{z}_0 - z) - r \rightarrow 0$ as $t \rightarrow \infty$ since each term on the right converges to zero separately. This establishes that Σ is indeed an observer for (4.1).

Note that the derivative terms in the state equation of the observer can be eliminated and the equations can be put into the standard form (4.3) by a change of variables as in [50] if desired.

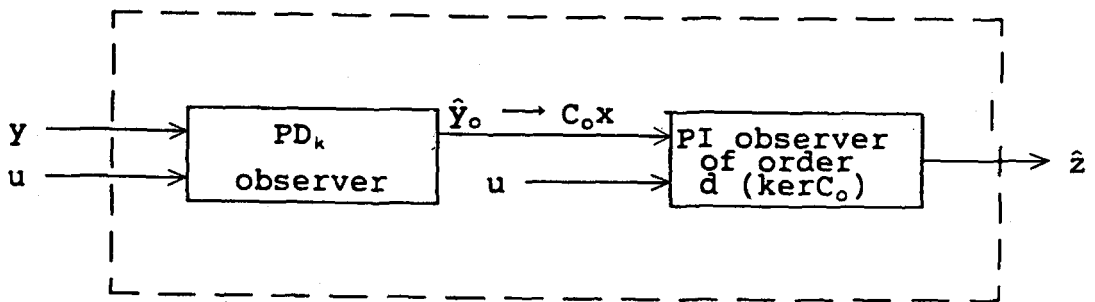


Fig.4.4. Realization of a minimal order PID_k -observer.

As a corollary to theorem 4.10 we can state

Corollary 4.4: There exists a PID observer of order $d(V^k \cap T_c)$ if and only if (4.1) is strongly detectable.

Proof: Since strong detectability implies that $\text{im} E \cap V^* = 0$ the conclusion follows from theorem 4.10 by taking $k > d(\ker C)$.

Remark 4.5: Since the sequence of subspaces $V^k \cap T_c$, $k=1,2,\dots$ is monotonically decreasing the order of a PID_{k-1} observer decreases as the number of differentiators increased. Hence there is a trade off between the observer order and the number of differentiators employed.

Theorem 4.10 also shows that in what sense a reduced order Luenberger observer is minimal. As $V^1 \cap T_c \subset \ker C$ the order of a PI observer is in general lower than the order of a reduced order observer. They are equal to each other if the pair $(A, (B, E))$ is controllable or $\sigma(A) \subset C_o$ for example.

The next lemma will be used in the proof of theorem 4.11.

Lemma 4.13: Consider the combined plant-observer equations (4.5). Define the projection $P: X \oplus W \rightarrow X$ and the injection $S: X \rightarrow X \oplus W$ as in (3.14), (3.15). Assume that the pair (L, N) is observable. Let $V = \langle A_c / \text{im} B_c \rangle + X_b(A_c)$, $V_i = S^{-1}V$ and $V_p = PV$ then

$$(a) V_i \subset V_p \quad \text{and} \quad (b) V_p = T_c = \langle A / \text{im} E + \text{im} B \rangle + X_b(A) \quad (4.22)$$

Proof: $V_i \subset V_p$ and $\langle A / \text{im} E + \text{im} B \rangle = P \langle A_c / \text{im} B_c \rangle$ follow immediately from the definitions of P and S . To prove $X_b(A) = PX_b(A_c)$, assume that the minimal polynomial of A is $\alpha(s) = \alpha_g(s)\alpha_b(s)$ where the zeros of α_b and α_g lie in C_b and C_g respectively. Because of the block triangular structure of A_c the minimal polynomial $m(s)$ of A_c is the product of the minimal polynomials $n(s)$ of N and $\alpha(s)$ of A . Moreover the zeros of $n(s)$ are in C_g for an observable observer hence $X_b(A_c) = \ker \alpha_b(A_c)$. Using the triangular form of A_c we can write $\alpha_b(A_c) = \begin{bmatrix} \alpha_b(A) & 0 \\ Z & \alpha_b(N) \end{bmatrix}$ where Z is a matrix which depends on A, N, M and C . Since $\sigma(N) \subset C_g$ $\det(\alpha_b(N)) \neq 0$. Now, take $x_c \in X_b(A_c)$ then $x = Px_c \in \ker \alpha_b(A)$; next take $x \in \ker \alpha_b(A)$ then $\begin{bmatrix} x \\ w \end{bmatrix} \in X_b(A_c)$ for $w = -\alpha_b(N)^{-1}Zx$ thus $x \in PX_b(A_c)$. This establishes that $PX_b(A_c) = X_b(A)$. The conclusion follows by noting that $PV = PX_b(A_c) + P \langle A_c / \text{im} E_c \rangle$

The minimal orders of full state PID_k $k \geq 0$ observers are established in the next theorem which is an extension of [36, theorem 4] to the case where unknown inputs are present and observers with differentiators are allowed. It is shown that the orders given in theorem 4.10 are indeed minimal.

Theorem 4.11: Assume that there exists a full state PID_{k-1} observer then its minimal order is given by $d(T_{PD_{k-1}})$ where $T_{PD_{k-1}} = V^k \cap T_c$ is the smallest undetectable subspace for PD_{k-1} observers.

Proof: It has already been proven in theorem 4.10 that under conditions (4.18) a PID_{k-1} observer of order $d(V^k \cap T_c)$ exists. To show that this number is a lower bound assume without loss of generality that the observer is observable. (otherwise a lower order observer can be designed by deleting the unobservable part [36].) Define $P: X \oplus W \rightarrow X$, $S: X \rightarrow X \oplus W$ as in (3.14,15) and $P_1: X \oplus W \rightarrow W$ by $P_1 \begin{bmatrix} x \\ w \end{bmatrix} = w$. Let $V = \langle A_c / \text{im} B_c \rangle + X_b(A_c)$, $V_1 = S^{-1}V$ and $V_p = PV$. Suppose that W is the state space of a full state PID_{k-1} observer then

$$d(W) \geq d(P_1 V) = d(V) - d(\ker P_1 \cap V) = d(V) - d(V_1) \geq d(V_p) - d(V_1) \quad (4.23)$$

Since $V_1 \subset V_p$ by lemma 4.13, it follows that $V_1 + V^k \subset V_p + V^k$ and $d(V_p) - d(V_1) \geq d(V_p \cap V^k) - d(V_1 \cap V^k)$. On the other hand it can be shown as in the necessity proof of theorem 4.3 that $V_1 \cap V^k = 0$ for full state PID_{k-1} observers. Thus by (4.23) $d(W) \geq d(V_p \cap V^k) = d(T_c \cap V^k)$ where the equality $V_p = T_c$ is justified by lemma 4.13. This proves the desired result. The fact that $T_c \cap V^k$ is the smallest undetectable subspace for PD_{k-1} observers follows from Definition 4.3 and theorem 4.1.

If unknown inputs are not present ($E=0$), the minimal order given for a full state PID_{k-1} observer in theorem 4.11 coincides with the result obtained in [36] for $k=1$ but is lower than the orders given in [30,45].

The ideas presented in this section are also applicable to the design of reduced order known-initial

state PID_{k-1} observers with minor modifications. In particular it can be shown that under the condition $V^k \cap Q_* = 0$ there exists a full state, known-initial state- PID_{k-1} observer of order $d(V^k)$. The proof of this statement is similar to theorem 4.10 and is therefore omitted.

4.5 A MATRIX ALGORITHM FOR THE DESIGN OF MINIMAL ORDER OBSERVERS

Based on the synthesis procedures of the previous sections a computational matrix algorithm can be easily developed for the construction of minimal order PID_{k-1} observer matrices. The procedure to be described in this section resembles in many respects to the algebraic approaches given for known input systems in [1,46]. In literature, there also appeared a number of algebraic approaches for the calculation of observer parameters for unknown input systems. Some of the recent works in this area are [47-49] which give synthesis methods for full order PI-observers using generalized matrix inverses. The main features of our approach that distinguishes it from existing related work can be stated as follows.

1. The existence conditions of minimal order observers can be given in terms of the structural properties of the observed system (like strong observability, strong detectability etc.) and they can easily be tested by elementary operations on matrices derived from the system parameters during the algorithm.
2. The observer matrices can be calculated directly in terms of the submatrices of a system which is obtained from (4.1) by state space and measurement space transformations. The method does not require the use of generalized matrix inverses.

3. PID_k $k \geq 1$ observers are treated in the same way as PI observers.

Since the proposed method is a naive translation of the constructive steps of theorems 4.1, 4.3 and 4.10 no claim is made regarding numerical behaviour of the algorithm. It can be further improved with respect to numerical conditioning and organization of construction steps in order to develop an efficient software package.

The input of the algorithm are the matrices A, B, E, C , of (4.1) and D_1, D_2 satisfying $\ker D_1 = V^k$, $\ker D_2 = T_c$. The output are the matrices of the minimal order, full state PID_{k-1} observer described by

$$\dot{\hat{w}} = N\hat{w} + M\hat{y}_0 + Su \quad (4.24)$$

$$\hat{z} = L\hat{w} + K\hat{y}_0 \quad (4.25)$$

$$\text{where } \hat{y}_0 = \sum_{i=0}^{k-1} K_i y^{(i)} + S_i u^{(i)} \quad (4.26)$$

The equations (4.24)-(4.26) can be converted to the standard form (4.3) by the change of variables [50]

$$s = w - \sum_{i=0}^{k-2} \sum_{j=i+1}^{k-1} N^{j-i-1} (MK_j y^{(i)} + MS_j u^{(i)}) \quad (4.27)$$

The given data must satisfy

$$\text{im} E \cap \ker D_1 = 0 \quad \text{and} \quad (4.28)$$

$$V_b^*(A, E; \ker C) = 0 \quad (4.29)$$

These conditions are tested in steps 1 and 5 of the algorithm. The construction of the PI-observer part (4.24), (4.25) is completed in steps 1-8 and the matrices of the PD-observer (4.26) are calculated in steps 9-13.

Step 1: Check if $\text{rank} D_1 E = \text{rank} E$ and stop otherwise.

Step 2: By row operations represented by $H_0 = [H_1 \ H_2]$

obtain a full row rank matrix D_0 such that $D_0 = H_0 \begin{bmatrix} D_1 \\ D_2 \end{bmatrix} = H_1 D_1 + H_2 D_2$. Store H_1 for future use.

Step 3: Express matrices A, B, E, D_0 and $D = I$ with respect to new bases suitable to the direct sum decomposition $X = \text{im} E \oplus S \oplus \ker D_0$. For this purpose, let $T = [T_1 \ T_2 \ T_3]$ be a matrix such that the columns of T_1, T_2, T_3 are bases for $\text{im} E, S$ and $\ker D_0$ respectively. Transform A, B, E, D_0, D according to:

$$A^* = T^{-1} A T = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \quad E^* = T^{-1} E = \begin{bmatrix} E_{11} \\ 0 \\ 0 \end{bmatrix} \quad B^* = T^{-1} B = \begin{bmatrix} B_{11} \\ B_{21} \\ B_{31} \end{bmatrix}$$

$$D_0^* = D_0 T = [D_{11} \ D_{12} \ 0] \quad D^* = D T = T$$

where the matrices are partitioned according to new bases.

Step 4: Write $H = [D_{11} \ D_{12}]$ and calculate H^{-1} . (Since D_0 is full row rank the matrix H is nonsingular.)

Note 4.1: In the basis selected in step 3 the projections V, P and the maps A_0, C_0 defined in lemma 4.11 have the matrix representations

$$V = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & I \end{bmatrix}, \quad P = \begin{bmatrix} 0 & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}, \quad A_0 = A_{33}, \quad C_0 = \begin{bmatrix} 0 \\ A_{23} \end{bmatrix}$$

Step 5: Check whether the pair (A_{23}, A_{33}) is detectable. It follows from the argument in the proof of lemma 4.11

that this is a necessary condition for the existence of full state PID_{k-1} observers.

Step 6: By an eigenvalue placement algorithm construct a matrix G_0 such that $\sigma(A_{33}+G_0A_{23}) \subset C_g$ and let

$$N=A_{33}+G_0A_{23} \quad (4.30)$$

Note 4.2: After selection of G_0 the projection W and the subspace $Q=\ker W$ in lemma 4.11 are determined as

$$W = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & G_0 & I \end{bmatrix} \quad Q = \ker W = \{x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} : G_0x_2 + x_3 = 0\}$$

The remaining parameters of the observer are constructed following the proof of sufficiency of theorem 4.2 and theorem 4.3. For this, one needs a full row rank matrix R such that $\ker R = Q$. Such a matrix is easily given as $R = [0 \ G_0 \ I]$ by taking the last row of W . A matrix G satisfying $(A^*+GD_0^*)Q \subset Q$ is computed by solving G from $R(A^*+GD_0^*)Q = 0$. This gives

$$G = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ G_{31} & G_{32} \end{bmatrix} H^{-1} \quad \text{where} \quad G_{31} = -G_0A_{21} - A_{31} \quad \text{and}$$

$$G_{32} = G_0A_{23}G_0 + A_{33}G_0 - G_0A_{22} - A_{32}.$$

Step 7: Matrices $S=RB^*$ and $M=-RG$ are given by

$$S = G_0B_{21} + B_{31} \quad (4.31)$$

$$M = [M_1 \ M_2] H^{-1} \quad \text{where} \quad (4.32)$$

$$M_1 = G_0A_{21} + A_{31} \quad (4.33)$$

$$M_2 = A_{32} + G_0A_{22} - A_{33}G_0 - G_0A_{23}G_0 \quad (4.34)$$

Step 8: By direct calculation, it can be verified that matrices L and K satisfying $LR+KD_0^*=D^*=T$ are

$$L=T_3 \quad (4.35)$$

$$K=[T_1 \ T_2-T_3G_0] \ H^{-1} \quad (4.36)$$

Step 9: Construct matrices C_k, E_k, B_k defined by (4.4).

Note 4.3: Since $V^k=C_k^{-1}(iE_k)=\ker D_1$, $\ker[C_k \ E_k] \subset \ker[D_1 \ 0]$ and there exists a matrix $\bar{K}$ such that $\bar{K}[C_k \ E_k]=[D_1 \ 0]$ by lemma 2.1. It follows that $\hat{z}_1=\bar{K}y_k-\bar{K}B_k u_{k-1}$ is a dead-beat PD_{k-1} observer for $z_1=D_1x$.

Step 10: Solve $\bar{K}$ from $\bar{K}[C_k \ E_k]=[D_1 \ 0]$ and let $\bar{S}=-\bar{K}B_k$.

Step 11: Partition $\bar{K}, \bar{S}$ as $\bar{K}=[\bar{K}_0 \ \bar{K}_1 \dots \bar{K}_{k-1}]$, $\bar{S}=[\bar{S}_0 \ \bar{S}_1 \dots \bar{S}_{k-2}]$ where each $\bar{K}_i, (\bar{S}_i)$ has $p(m)$ columns.

Step 12: Let

$$K_i=H_1\bar{K}_i \quad i=0,1..k-1 \quad (4.37)$$

$$S_i=H_1\bar{S}_i \quad i=0,1..k-2; \ S_{k-1}=0 \quad (4.38)$$

where H_1 has been determined in step 2.

The foregoing algorithm can be summarized by the following corollary which gives algebraic conditions for the existence and construction of minimal order, full state, PID_{k-1} $k \geq 1$ observers.

Corollary 4.4: There exists a full state PID_{k-1} $k \geq 1$ observer for (4.1) if and only if $\text{rank} D_1 E = \text{rank} E$ and the pair (A_{23}, A_{33}) is detectable. Under these conditions the parameters of the minimal order observer (4.24)-(4.26) are given by (4.30)-(4.38).

The following example illustrates the design procedure described in this section.

Example 4.1: Assume that the system (4.1) is given by

$$A = \begin{bmatrix} -1 & 1 & 0 & 2 \\ 0 & 0 & 1 & 0 \\ 2 & -1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \quad E = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \quad C = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$D=I$

1. Minimal order PI-observer: Since $V^1 = \ker C$ and $T_c = R^4$ the matrix D_1 satisfying $\ker D_1 = V^1$ is taken as $D_1 = C$. This gives $D_0 = C$ and $H_1 = I$. Denoting the i^{th} unit vector by e_i , the transformation T which brings the matrices A, E, B and D_0 into the form described in step 3 can be expressed as $T = [T_1 \ T_2 \ T_3]$ where $T_1 = e_3$, $T_2 = e_4$ and $T_3 = [e_1 \ e_2]$. Thus we have

$$A^* = \begin{bmatrix} 0 & 1 & 2 & -1 \\ 1 & 0 & 0 & 1 \\ 0 & 2 & -1 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \quad E^* = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad B^* = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \quad D_0^* = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$D^* = T$

from which the submatrices A_{23}, A_{33} are $A_{23} = \begin{bmatrix} 0 & 1 \end{bmatrix}$,

$$A_{33} = \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}.$$

Clearly $\text{rank} D_1 E = \text{rank} E = 1$ and the pair (A_{23}, A_{33}) is detectable hence there exists a PI-observer of order $d(V^1 \cap T_c) = 2$.

Choosing $G_0 = \begin{bmatrix} 0 \\ -2 \end{bmatrix}$ the eigenvalues of $N = A_{33} + G_0 A_{23}$ are placed at $s_1 = -1$ and $s_2 = -2$. Note that the eigenvalue at $s_1 = -1$ is fixed for any choice of G_0 .

The remaining parameters of the observer are found from (4.31)-(4.38). The equations of the PI-observer are given by

$$\dot{w} = \begin{bmatrix} -1 & 1 \\ 0 & -2 \end{bmatrix} w + \begin{bmatrix} 0 & 4 \\ -1 & -4 \end{bmatrix} y + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$\hat{z} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} w + \begin{bmatrix} 0 & 0 \\ 0 & 2 \\ 1 & 0 \end{bmatrix} y$$

2. Minimal order PID-observer: As $V^2=V^*=\text{span}(e_1)$ a PID observer of order $d(V^* \cap T_c)=1$ can be designed using at most one differentiator.

Following the steps of the algorithm, the equations of the PID observer after the change of variables (4.27) are given by

$$\dot{s} = -s + \begin{bmatrix} -1 & 1 \end{bmatrix} y$$

$$\hat{z} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} s + \begin{bmatrix} 0 & 1 \\ -1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} y + \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \dot{y}$$

CHAPTER 5

FREQUENCY DOMAIN SOLUTION OF DISTURBANCE DECOUPLING AND DISTURBANCE DECOUPLED ESTIMATION PROBLEMS

In this chapter solvability of disturbance decoupling and disturbance decoupled estimation problems (DDP and DDEP) are discussed in frequency domain using rational and polynomial matrices. The interest in frequency domain techniques in multivariable control problems started with the works in [51-53,56-58] as an alternative to state space approaches which have proved succesful in the solution of a variety of synthesis problems. For the problems concerned in this thesis the frequency domain approach provides a deeper insight and allows to establish a duality relationship between the classes of disturbance decoupling and disturbance decoupled estimation problems which is also apparent from the duality of (A,B) and (C,A) -invariant subspaces. On the other hand the frequency domain conditions are sometimes more convenient from the numerical point of view for testing the existence and construction of a solution. It is evident from the recent works [28,69,70] that a combined use of geometric and frequency domain methods is the most effective way of solving problems arising in linear system theory. In view of the current trends in control theory it is believed that a study of DDP and DDEP would be incomplete if it did not include the polynomial and rational matrix approaches developed in [56-62]. With this objective, the equivalence of DDP and DDEP to solvability of rational matrix equations is established in section 5.1 and in section 5.2 frequency

domain conditions for the observability and detectability of unknown input systems are presented.

The notation for rational matrices and vectors to be used in this chapter is as follows. $R^n[s]$ and $R^n(s)$ denote, respectively, the set of polynomials and the set of rational functions with coefficients in R^n in the indeterminate s . The set of proper and strictly proper rational functions with coefficients in R^n are denoted by $R_0^n(s)$ and $R_+^n(s)$ respectively. Analogous notation is used for the set of $m \times n$ matrices with entries in $R[s]$ or $R(s)$. It is well known that [53] $R^n(s)$ is an n -dimensional vector space over the field $R(s)$ while $R^n[s]$, $R_0^n(s)$ and $R_+^n(s)$ are modules over the rings $R[s]$, $R_0(s)$ and $R_+(s)$ respectively.

5.1. SOLVABILITY OF RATIONAL MATRIX EQUATIONS

In this section necessary and sufficient conditions for the existence and uniqueness of solutions of rational matrix equations and their relationship with the solvability of DDP and DDEP are investigated. The basic equation considered in this section is

$$X(s)M(s)=N(s) \quad (5.1)$$

and its dual form

$$M'(s)X'(s)=N'(s) \quad (5.2)$$

where $M(s) \in R_+^{p \times r}(s)$, $N(s) \in R_+^{1 \times r}(s)$ are given and $X(s)$ the unknown. It will be shown that the solvability of (5.1) and (5.2) over different classes is equivalent to solvability of DDEP and DDP for appropriately defined systems.

In literature, solvability of (5.1) or (5.2) has been extensively studied in connection with the exact model matching problem [52,54,55]. In [52,58] conditions for the existence of solutions are given in terms of certain degree and cancellation conditions on the matrices $M(s)$, $N(s)$ and in [56] the solutions are given in terms of the kernel of a polynomial matrix. The approach followed in this section is similar to those of [27,55,60,61] and is based on a relatization of the rational matrices $M(s)$, $N(s)$ in state space form. The results given for the solvability of (5.1), (5.2) are rederived using known-initial state-observer theory and conditions given for controllable or observable realizations of $M(s)$, $N(s)$ in [27,58] are generalized to possibly uncontrollable, unobservable realizations. As side results alternative synthesis methods are presented for feedforward control laws [25] and inverse systems [63] and a one to one correspondence is established for a class of DDEPs and their dual DDPs.

Consider the rational matrix equation (5.1) and realize $\begin{bmatrix} M(s) \\ N(s) \end{bmatrix}$ in state space form as

$$\begin{bmatrix} M(s) \\ N(s) \end{bmatrix} = \begin{bmatrix} C \\ D \end{bmatrix} (sI-A)^{-1}E \quad (5.3)$$

Thus $M(s)=C(sI-A)^{-1}E$, $N(s)=D(sI-A)^{-1}E$ and (5.1) is equivalent to

$$X(s)C(sI-A)^{-1}E=D(sI-A)^{-1}E \quad (5.4)$$

In order to carry the results obtained in Chapter 4 define the dynamical system Σ by

$$\Sigma: \quad \begin{aligned} \dot{x} &= Ax + Eq \\ y &= Cx, \quad z = Dx \end{aligned} \quad (5.5)$$

where $x(t) \in \mathbb{R}^n$, $y(t) \in \mathbb{R}^p$, $z(t) \in \mathbb{R}^l$, $q(t) \in \mathbb{R}^r$ denote the state, the measurement, the to be estimated outputs and the disturbance respectively.

Definition 5.1: A system with initial state $w(0)$, input y and giving the output $\hat{z}$ is said to be a zero-state-observer (ZSO) for Σ if for $x(0)=0$, $w(0)=0$ the error $e=\hat{z}-z$ satisfies $e(t)=0$ for all q and $t \geq 0$.

The PID class of ZSO's are defined as in (4.3). In section 4.1 it has been pointed out that knowledge of $x(0)$ does not enlarge the solvability range of the observer problem for nondynamic observers and this has been explained by the lack of a state description for a nondynamic observer. By taking the initial state of the plant to be zero this difference between solvability conditions of dynamic and nondynamic observers disappears and both of them can be expressed by the requirement of zero transfer matrix from the disturbance q to the error e . The transfer matrix and geometric conditions for the existence of PID class of ZSO's are presented in theorem 5.1 which readily follows from the results in section 4.1.

Theorem 5.1: Consider the rational matrix equation (5.1) and any realization of $M(s)$, $N(s)$ as in (5.3). Then the following statements hold true.

1. (5.1) is solvable over $\mathbb{R}^{l \times p} \iff$ There exists a P-ZSO for $\Sigma \iff \ker C \cap \langle A / \text{im } E \rangle \subset \ker D$
2. (5.1) is solvable over $\mathbb{R}^{l \times p}[s] \iff$ There exists a PD-ZSO for $\Sigma \iff \langle A / \text{im } E \rangle \cap V^*(A, E; \ker C) \subset \ker D$
3. (5.1) is solvable over $\mathbb{R}_+^{l \times p}(s) \iff$ There exists an I-ZSO for $\Sigma \iff Q_*(C, A; \text{im } E) \subset \ker D$

4. (5.1) is solvable over $R_0^{1 \times p}(s) \iff$ There exists a PI-ZSO for $\Sigma \iff \ker C \cap Q_*(C, A; \text{im } E) \subset \ker D$
5. (5.1) is solvable over $R^{1 \times p}(s) \iff$ There exists a PID-ZSO for $\Sigma \iff R^*(A, E; \ker C) \subset \ker D$

Proof: Only the proof of statement 5 will be given since the others can be proven similarly. Assume that there exists a PID-ZSO for Σ . Denoting the transfer function from y to $\hat{z}$ of the observer by $T(s)$, the transfer function from q to $e = \hat{z} - z$ can be written as $T(s)M(s) - N(s) = 0$. Thus $X(s) = T(s) \in R^{1 \times p}(s)$ is a solution of (5.1). Conversely, let $X(s) \in R^{1 \times p}(s)$ be a solution of (5.1). Determine $L, N, M, K_0, K_1, \dots$ by realizing $X(s)$ in the form $X(s) = L(sI - N)^{-1}M + K_0 + K_1s + \dots$. Then the system described by $\dot{w} = Nw + My$, $\hat{z} = Lw + K_0y + K_1\dot{y} + \dots$ is a PID-ZSO for Σ since the transfer function from q to $e = \hat{z} - z$ is identically zero. This completes the proof of the first equivalence in 5. The geometric condition $R^*(A, E; \ker C) \subset \ker D$ for the existence of PID-ZSO's follows from corollary 4.3.

The geometric conditions in statements 3 and 4 of theorem 5.1 are identical to those given in [57,58] which are proved using (ξ, w) representations of (A, B) -invariant subspaces. The proof given in theorem 5.1 using ZSO theory is more elementary. The condition in statement 2 reduces to the one given in [27] under the assumption of controllability of the pair (A, E) .

Theorem 5.1 not only presents existence conditions for the solution of (5.1) but also gives a method for constructing a solution through designing a ZSO. Since there are standard techniques [51-53] to realize transfer matrices $M(s)$, $N(s)$ in the form (5.3) the method suggested is straightforward and may find application in the exact model matching problem or in algebraic techniques for control system design. The

following example illustrates the solution of (5.1) via theorem 5.1.

Example 5.1: Consider the equation (5.1) with $M(s) = \frac{1}{s+1}$, $N(s) = \frac{1}{(s+2)(s+3)}$ which is solvable over $R_0(s)$ by inspection. Realizing $M(s)$, $N(s)$ as in (5.3) we have

$$A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix} \quad E = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad C = [0 \quad -1 \quad 2] \quad D = [1 \quad 0 \quad 0]$$

It is easily verified that the condition in statement 4 of theorem 5.1 is satisfied. Following the constructive proof of theorem 4.4 a PI-ZSO is designed as

$$\begin{aligned} \dot{w} &= Nw + My \\ z &= Lw + K_0 y \end{aligned} \tag{5.6}$$

where $N = \begin{bmatrix} -1 & 2 \\ 0 & -1 \end{bmatrix}$, $M = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $L = [1, 2]$ and $K_0 = 1$. The transfer function $L(sI - N)^{-1}M + K_0 = \frac{s^2 + 5s + 6}{(s+1)^2}$ of (5.6) is the solution of (5.1).

Solvability conditions for equation (5.2) or an arbitrary rational matrix equation which can be converted into the form (5.1) immediately follow from theorem 5.1. Before presenting these conditions it is appropriate to introduce the dual Σ' of the system (5.5) and a class of DDP's defined for Σ' whose solutions lead to the solvability of a rational matrix equation of the form (5.2).

Definition 5.2: Consider the dynamical system $\Sigma: \dot{x} = Ax + Bu + Eq$ and assume that it is represented by the triple

of matrices $\left(\begin{bmatrix} C \\ D \end{bmatrix}, A, [B, E] \right)$. The dual of Σ , denoted by Σ' , is obtained by transposing all three matrices in this representation and interchanging $[C', D']$ and $[B, E]'$. Thus Σ' is described by the equations $\Sigma': \dot{s} = A's + C'v + D'd$, $w = B's$, $r = E's$.

It should be noted that except suitable naming of input and output variables in accordance with the partitioning of input and output matrices this is the familiar definition of duality encountered in linear systems [14].

The «PID class of DDP's» or its equivalent the «DDP by feed forward control» [25] is defined for the system Σ' as follows.

Definition 5.3: Consider the system

$$\begin{aligned} \Sigma': \quad \dot{s} &= A's + C'v + D'd \\ r &= E's \end{aligned} \quad (5.7)$$

where $s(t) \in \mathbb{R}^n$ is the state, $v(t) \in \mathbb{R}^p$ the control input, $d(t) \in \mathbb{R}^1$ is the disturbance input and $r(t) \in \mathbb{R}^r$ is the controlled output. It is assumed that the disturbance d is continuously differentiable sufficiently many times. Define the PID class of control laws by (1)-(5)

1. P -control law: $v = F_0 d$
2. PD -control law: $v = F_0 d + F_1 \dot{d} + F_2 \ddot{d} + \dots$
3. I -control law: $v = F s$
4. PI -control law: $v = F s + F_0 d$
5. PID-control law: $v = F s + F_0 d + F_1 \dot{d} + F_2 \ddot{d} + \dots$

Let $H_{c1}(s)$ denote the transfer matrix from d to r of the closed loop system obtained by applying one of the

control laws in (1)-(5). The P (PD) (I) (PI) (PID)-DDP is said to be solvable if there exists a P (PD) (I) (PI) (PID)-control law such that $H_{c_1}(s)=0$ for all $s \in \mathbb{C}$.

The reason for defining the PID class of DDP's for the system Σ' rather than the original system Σ will become clear when the duality of PID-ZSO problem and PID-DDP is established. The solvability conditions of I and PI-DDP's have already been presented in section 3.2 under the names of DDP with state feedback and DDP with state feedback and disturbance feedforward. Solvability of PID-DDP is considered in [25,27]. In theorem 5.2 and theorem 5.3 a different synthesis method will be given for PD and PID-control laws based on the solvability of rational matrix equations and the design of ZSOs.

Theorem 5.2: Consider the system described by (5.7)

1. P-DDP is solvable if and only if $\text{im}D' \subset \langle \ker E'/A' \rangle + \text{im}C'$
2. PD-DDP is solvable if and only if $\text{im}D' \subset \langle \ker E'/A' \rangle + Q_*(E', A'; \text{im}C')$

Proof: 1. From lemma 3.1 it follows that P-DDP is solvable if and only if there exists F_0 such that $\text{im}(D' + C'F_0) \subset \langle \ker E'/A' \rangle$. Thus the conclusion follows from lemma 3.2.

2. PD-DDP is solvable if and only if there exist matrices $F_0, F_1, F_2, \dots$ such that $E'(sI - A')^{-1}(C'F_0 + C'F_1s + C'F_2s^2 + \dots) + E'(sI - A')^{-1}D' = 0$ for all $s \in \mathbb{C}$. Letting $M(s) = C(sI - A')^{-1}E$, $N(s) = D(sI - A')^{-1}E$ it follows that PD-DDP is solvable if and only if there exists $F(s) \in \mathbb{R}^{1 \times p}[s]$ such that $F(s)M(s) = N(s)$. But by theorem 5.1 the latter is equivalent to $\langle A/\text{im}E \rangle \cap V^*(A, E; \ker C) \subset \ker D$. Taking orthogonal complements of subspaces and making use of lemma 2.11 the result follows.

By theorem 5.2 the construction of a PD control law reduces to the design of a PD-ZSO for which a matrix algorithm is presented in section 4.5. In view of the recursion (2.9) and lemma 4.2 the number of differentiation steps required for the solution of PD-DDP is at most $n - \text{rank} C'$ where $\text{rank} C'$ represents the number of independent control inputs.

The next lemma will be used in the proof of theorem 5.3.

Lemma 5.1: $Q_*(E', A'; \text{im} C') = Q_*(E', A' + C'F; \text{im} C')$ for any $F: R^n \rightarrow R^p$.

Proof: Let $Q \in Q(E', A'; \text{im} C')$ and take any $F: R^n \rightarrow R^p$. Then $(A' + C'F)(Q \cap \ker E') \subset A'(Q \cap \ker E') + C'F(Q \cap \ker E') \subset Q$ which proves that $Q \in Q(E', A' + C'F; \text{im} C')$. Conversely, let $Q \in Q(E', A' + C'F; \text{im} C')$ repeating the same argument it is proven that $Q \in Q(E', A'; \text{im} C')$. This establishes that $Q(E', A'; \text{im} C') = Q(E', A' + C'F; \text{im} C')$ for any $F: R^n \rightarrow R^p$ hence the conclusion.

Theorem 5.3: PID-DDP is solvable for the system described by (5.7) if and only if

$$\text{im} D' \subset V^*(A', C'; \ker E') + Q_*(E', A'; \text{im} C') \quad (5.8)$$

Proof: The following statements are equivalent.

1. There exists a PID control law such that (5.7) is disturbance decoupled.
2. There exists $F: R^n \rightarrow R^p$ such that $\text{im} D' \subset \langle \ker E' / A' + C'F \rangle + Q_*(E', A' + C'F; \text{im} C')$
3. There exists $F: R^n \rightarrow R^p$ such that $\text{im} D' \subset \langle \ker E' / A' + C'F \rangle + Q_*(E', A'; \text{im} C')$
4. $\text{im} D' \subset V^*(A', C'; \ker E') + Q_*(E', A'; \text{im} C')$

where the equivalence of 1 to 2 follows from theorem 5.2 and 2 to 3 from lemma 5.1.

The condition given in theorem 5.3 is identical to the one given in [25] for DDP by feedforward control. However the construction of the PID-control law using theorems 5.2 and 5.3 is entirely different and is believed to be easier. It is useful to outline the steps leading to the construction of the PID-control law which achieves disturbance decoupling in (5.7).

Construction of PID control law:

Step 1: Compute $V^*(A', C'; \ker E')$, $Q_*(E', A'; \text{im } C')$ and verify (5.8).

Step 2: Choose $F: R^n \rightarrow R^p$ such that $(A' + C'F)V^*(A', C'; \ker E') \subset V^*(A', C'; \ker E')$ and write $A'_F = A' + C'F$.

Step 3: Design a PD-ZSO for the system $\dot{x} = A_F x + E_F u$, $y = Cx$, $z = Dx$. This can be done either by solving $X(s) \in R^{1 \times p}[s]$ from the rational matrix equation $X(s)C(sI - A_F)^{-1}E = D(sI - A_F)^{-1}E$ or following the procedure described in Note 4.3 of the algorithm presented in section 4.5. Assume that the PD-ZSO is described by $\hat{z} = -F_0' y - F_1' \dot{y} - F_2' \ddot{y} - \dots$

Step 4: The PID control law is given by $v = Fs + F_0 d + F_1 \dot{d} + F_2 \ddot{d} + \dots$

The next example illustrates the solution of PID-DDP.

Example 5.2: Assume that the system (5.7) is given by

$$\dot{s} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} s + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} v + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} d$$

$$r = [0 \ 0 \ 1] s$$

The subspaces $V^*(A', C'; \ker E')$ and $Q_*(E', A'; \operatorname{im} C')$ are computed as $V^*(A', C'; \ker E') = \operatorname{im} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $Q_*(E', A'; \operatorname{im} C') = \operatorname{im} \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$. Hence (5.8) is satisfied. $F \in \underline{F}(V^*(A', C'; \ker E'))$ is

chosen as $F = [0 \ -1 \ 0]$. The matrix $X(s) \in \mathbb{R}^{1 \times p}[s]$ satisfying $X(s)C(sI - A_f)^{-1}E = D(sI - A_f)^{-1}E$ is found to be $X(s) = -F_0' - F_1's$ where $F_0 = [-1 \ 0 \ 0]$ and $F_1 = [0 \ 0 \ -1]$. Thus the PID control law is given by $v = Fs + F_0d + F_1\dot{d}$ with F, F_0 and F_1 as above.

In practice it is quite unlikely that the derivatives of d be available to use as feedforward terms in the feedback control law. Even if the disturbance is measurable it is generally not recommended to use differentiators in order to obtain the derivatives of the disturbance. The real significance of the conditions derived for PID-DDP is their applicability in approximate DDP. In fact in [27] it is shown that the solvability of almost DDP by state feedback is equivalent to the solvability of PID-DDP and condition (5.8) is derived using this equivalence.

The frequency domain solvability conditions for the PID class of DDPs immediately follow from theorems 5.1-5.3 and theorems 3.1-3.2. The proof of theorem 5.4 provides more insight by characterizing the solution of (5.2) in terms of the solution of PID-DDP.

Theorem 5.4: Consider the rational matrix equation (5.2) and any realization of $M(s)$, $N(s)$ as in (5.3). Then the following statements hold true for the system (5.7)

1. P -DDP is solvable if and only if there exists $X'(s) \in \mathbb{R}^{p \times 1}$ such that (5.2) is satisfied.
2. PD -DDP is solvable if and only if there exists $X'(s) \in \mathbb{R}^{p \times 1}[s]$ such that (5.2) is satisfied.
3. I -DDP is solvable if and only if there exists $X'(s) \in \mathbb{R}_{+}^{p \times 1}(s)$ such that (5.2) is satisfied.
4. PI -DDP is solvable if and only if there exists $X'(s) \in \mathbb{R}_0^{p \times 1}(s)$ such that (5.2) is satisfied.
5. PID-DDP is solvable if and only if there exists $X'(s) \in \mathbb{R}^{p \times 1}(s)$ such that (5.2) is satisfied.

Proof: Only the proof of statement 5 will be given since the others can be considered as special cases of 5.

«Only if» Assume that PID-DDP is solvable by the control law $v = Fs + F_0 d + F_1 \dot{d} + \dots$. Using the identity $(sI - A' - C'F)^{-1} = (sI - A')^{-1}[I + C'F(sI - A' - C'F)^{-1}]$ [60,61] the closed loop transfer function matrix from d to r can be written as $N'(s) + M'(s)F(s) \equiv 0$ where $F(s) = (sI - A' - C'F)^{-1}(D' + C'F_0 + sC'F_1 + \dots) + F_0 + sF_1 + \dots$ is the transfer function from d to v in the closed loop system. Thus $X'(s) = -F(s)$ is the solution of (5.2).

«If» Follows from theorem 5.1 and theorem 5.3.

The foregoing theorems establish the equivalence between the PID class of ZSO design problems defined for the system (5.5) and the PID class of DDPs defined for the dual system (5.7). Moreover the solution of either class of problems can be used to characterize the solution of the rational matrix equation (5.1) or (5.2). These results are summarized in Table 5.1.

Table 5.1. Frequency domain and geometric conditions for the solvability of DDP and existence of zero state observers.

System for which DDEP is defined $\dot{x}=Ax+Eq$ $y=Cx, \quad z=Dx$	System for which DDP is defined $\dot{s}=A's+C'v+D'd$ $r=E's$	$X(s)C(sI-A)^{-1}E=$ $D(sI-A)^{-1}E$ is solvable over	Geometric conditions
$\exists \text{ P -ZSO}$	P -DDP is solvable	$R^{1 \times p}$	$\ker C \cap \langle A/imE \rangle \subset \ker D$
$\exists \text{ PD -ZSO}$	PD -DDP is solvable	$R^{1 \times p}[s]$	$V^*(A, E; \ker C) \cap \langle A/imE \rangle \subset \ker D$
$\exists \text{ I -ZSO}$	I -DDP is solvable	$R_+^{1 \times p}(s)$	$Q_*(C, A; imE) \subset \ker D$
$\exists \text{ PI -ZSO}$	PI -DDP is solvable	$R_0^{1 \times p}(s)$	$\ker C \cap Q_*(C, A; imE) \subset \ker D$
$\exists \text{ PID-ZSO}$	PID-DDP is solvable	$R^{1 \times p}(s)$	$R^*(A, E; \ker C) \subset \ker D$

Since $R^n(s)$ is a vector space, standard existence and uniqueness tests for solutions to linear matrix equations over a vector space are applicable to (5.1). However it is also possible to give explicit geometric conditions for the uniqueness of solutions in terms of the realization (5.3) and using the concepts of left and right system invertibility. Left invertibility of the system (5.5) has been defined in section 4.3 and its relationship with known-initial state-observability has been made clear in lemma 4.7. The dual notion of right invertibility is defined below.

Definition 5.4: The system Σ' described by (5.7) is right invertible if for every function $\hat{f}: [0, \infty) \rightarrow R^r$ there exists an input v such that with the initial state $s(0)=0$ and $d=0$ the corresponding output trajectory $r(t)=\hat{f}(t)$ for $t \geq 0$.

It is well known that left invertibility of (5.5) is equivalent to the right invertibility of the dual system (5.7) [63]. Geometric and transfer matrix conditions for left and right system invertibility are exhibited in the next theorem.

Theorem 5.5 [69]: The following statements are equivalent.

- (i) The system Σ described by (5.5) is left invertible.
- (ii) The system Σ' described by (5.7) is right invertible.
- (iii) $R^*(A, E; \ker C) = 0$ and $\ker E = 0$
- (iv) There exists $X(s) \in R^{r \times p}(s)$ such that $X(s)C(sI-A)^{-1}E = I$,

Remark 5.1: A system with transfer function $X(s)$ ($X'(s)$) satisfying $X(s)C(sI-A)^{-1}E = I$, is called a left (right) inverse of $\Sigma(\Sigma')$. Let $z=x$ in (5.5), if the transfer

function of the PID-ZSO for Σ is denoted by $T(s)$ the transfer function of the left inverse system is given by $X(s) = E^L (sI - A)T(s)$ where E^L is any matrix satisfying $E^L E = I_r$. Thus the methods developed for the design of PID-ZSO's can be applied to the construction of left and right inverse systems. In particular, it is possible to design a reduced order inverse system (a system of order $d(\ker C)$) using the procedure described in section 4.4.

As an immediate consequence of theorem 5.5 the following result can be stated for the uniqueness of solutions to (5.1).

Theorem 5.6 Consider the rational matrix equation (5.1) and any realization of $M(s)$, $N(s)$ as in (5.3). The solution of (5.1) is unique if and only if $R^*(A', C'; \ker E') = 0$ and $\ker C' = 0$.

Proof: The general solution of (5.1) can be written as $X(s) = X_p(s) + X_h(s)$ with $X_p(s)$ being a particular solution of (5.1) and $X_h(s)$, the general solution of the homogenous equation $X(s)M(s) = 0$. It follows that the solution of (5.1) is unique if and only if $X(s)M(s) = 0$ implies $X(s) = 0$. By lemma 2.1 the latter is equivalent to the existence of $T(s) \in \mathbb{R}^{p \times r}(s)$ such that $T(s)M'(s) = I_p$. Thus the conclusion follows from theorem 5.5 with E', A', C' playing the roles of C, A, E respectively.

Theorems 5.1 and 5.6 completely solve the problem of existence and uniqueness of solutions to (5.1) just like lemma 2.1 does for equations over vector spaces.

5.2 FREQUENCY DOMAIN CONDITIONS FOR UNKNOWN INPUT OBSERVABILITY AND DETECTABILITY

In the previous section existence conditions of PID class of zero state observers have been given in terms of the solvability of the rational matrix equation (5.1). However stability properties of the observers have not been taken into account in these results. Though it is not difficult to give similar conditions in this case by requiring the solution to be a stable rational matrix it is more convenient to give conditions in terms of the polynomial system matrix since they have nice interpretations in terms of the (finite) zeros of (5.5) and there are fast, numerically reliable algorithms in literature for the computation of zeros of multivariable systems [11, 67].

In literature frequency domain conditions for strong observability and strong detectability have been given which can be viewed as generalizations of the well known PBH tests [37, 53 pp.135]. In theorem 5.7 these results are rederived starting from the dynamical definitions and geometric conditions given in section 4.3 and in theorem 5.8 a new frequency domain condition for known-initial state-observability is presented which can also be used to characterize left, right system invertibility and uniqueness of solutions to equations (5.1), (5.2).

Consider the system described by (5.5). By taking Laplace transforms the state equation and the measurement equation can be rewritten in the form

$$\begin{bmatrix} sI-A & E \\ C & 0 \end{bmatrix} \begin{bmatrix} x_L(s) \\ q_L(s) \end{bmatrix} = \begin{bmatrix} x(0) \\ y_L(s) \end{bmatrix} \quad (5.9)$$

where $x_L(s)$, $q_L(s)$, $y_L(s)$ denote the Laplace transforms of $x(t)$, $q(t)$ and $y(t)$ respectively. The matrix $P(s) = \begin{bmatrix} sI-A & E \\ C & 0 \end{bmatrix}$ which appears in (5.9) is called the system matrix [51] and will play a key role in the sequel.

A characterization of (A,E) -invariant subspaces in terms of closed loop eigenvalues and eigenvectors is given in the next lemma.

Lemma 5.2 [64]. Let V be an (A,E) -invariant subspace such that $d(V)=v$. Then there exists a set of complex numbers $z_1, z_2, \dots, z_s$; integers $r_1, r_2, \dots, r_s$ and vectors $x_1^1, x_1^2, \dots, x_1^{r_1} \in \mathbb{R}^n$, $q_1^1, q_1^2, \dots, q_1^{r_1} \in \mathbb{R}^r$, $1 \leq i \leq s$, such that

$$Ax_i^k = z_i x_i^k + x_i^{k-1} - E q_i^k, \quad x_i^0 = 0, \quad i=1, 2, \dots, s; \quad k=1, 2, \dots, r_i$$

and $V = \text{span} (x_i^k, i=1, 2, \dots, s; k=1, 2, \dots, r_i)$

Proof: Choose $F: \mathbb{R}^n \rightarrow \mathbb{R}^r$ such that $(A+EF)V \subset V$. Let $V: V \rightarrow \mathbb{R}^n$ be such that $\ker V = 0$ and $\text{im} V = V$. By lemma 2.2 there exists $A_0: V \rightarrow V$ such that

$$(A+EF)V = VA_0 \quad (5.10)$$

Assume that the Jordan canonical form of A_0 is given by

$$A_c = \text{blockdiag} (J_1, J_2, \dots, J_s) \text{ where } J_i = \begin{bmatrix} z_i & 1 & 0 & \dots & 0 \\ 0 & z_i & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & z_i \end{bmatrix}_{r_i \times r_i}$$

Hence there exists a nonsingular matrix T such that $A_0 = TA_c T^{-1}$. Defining $V_c = VT$ and $U_c = -FVT$ (5.10) can be written as $AV_c = V_c A_c + EU_c$. Denoting the columns of V_c by x_i^k $i=1, 2, \dots, s$; $k=1, 2, \dots, r_i$, defining q_i^k as $q_i^k = -Fx_i^k$ and writing the last equation in columns the result is established.

It should be noted that the complex numbers $z_1, z_2, \dots, z_s$ introduced in lemma 5.2 are the eigenvalues of the map $A+EF:V$ for $F \in \underline{F}(V)$. The complete freedom in assigning these numbers is described in theorem 2.2. This observation will be useful when proving theorem 5.7, 5.8.

Theorem 5.7: 1) The system described by (5.5) is strongly observable if and only if $\text{rank}P(s) = n + \text{rank}E$ for all $s \in C$.

2) The system described by (5.5) is strongly detectable if and only if $\text{rank}P(s) = n + \text{rank}E$ for all $s \in C_b$.

Proof: Parts 1) and 2) will be proved together.

«Only if» Assume that for $s_0 \in C$ ($s_0 \in C_b$) $\text{rank}P(s_0) < n + \text{rank}E$. Then there exists $x_0 \in \mathbb{R}^n$, $g \in \mathbb{R}^r$ with $x_0 \neq 0$ such that $P(s_0) \begin{bmatrix} x_0 \\ g \end{bmatrix} = 0$. Let $q(t) = -ge^{s_0 t}$ and $x(0) = x_0$ then the solution of (5.5) is given by $x(t) = x_0 e^{s_0 t}$ and $y(t) = Cx(t) = 0$ $t \geq 0$. Since $y(t) = 0$ $t \geq 0$ but $x_0 \neq 0$ ($x(t)$ does not converge to zero as $t \rightarrow \infty$) (5.5) is not strongly observable (strongly detectable) proving the necessity of part 1 (part 2).

«If» Assume that (5.5) is not strongly observable (detectable) we have $V^*(A, E; \ker C) \neq 0$ ($V_b^*(A, E; \ker C) \neq 0$). By lemma 5.2 there exists $s_0 \in C$ ($s_0 \in C_b$), $0 \neq x_0 \in \mathbb{R}^n$ and $q \in \mathbb{R}^r$ such that $P(s_0) \begin{bmatrix} x_0 \\ q \end{bmatrix} = 0$ which proves that $\text{rank}P(s_0) < n + \text{rank}E$.

It follows from theorem 5.7 that $\text{rank}E \leq \text{rank}C$ is a necessary condition for strong observability and strong

detectability for otherwise $\text{rank}P(s) < n + \text{rank}E$ for all $s \in \mathbb{C}$. When $\text{rank}P(s) < n + \text{rank}E$ for all $s \in \mathbb{C}$ the triple (C, A, B) is called degenerate [62]. The next theorem shows that this situation happens whenever the system (5.5) is not known-initial state-observable.

Theorem 5.8: The following statements are equivalent.

- (i) The system described by (5.5) is known-initial state-observable.
- (ii) $R^*(A, E; \ker C) = 0$.
- (iii) There exists $s \in \mathbb{C}$ such that $\text{rank}P(s) = n + \text{rank}E$.

Proof: Since the equivalence of (i) to (ii) is established in theorem 4.8 we only need to prove the equivalence of (ii) to (iii).

«(iii) $\Rightarrow$ (ii)» Assume that $R^*(A, E; \ker C) \neq 0$. Take any $z \in \mathbb{C}$. By theorem 2.1 there exists $F: \mathbb{R}^n \rightarrow \mathbb{R}^r$ such that $(A + EF)R^*(A, E; \ker C) \subset R^*(A, E; \ker C)$ and $z \in \sigma(A + EF: R^*(A, E; \ker C))$. Thus reasoning as in lemma 5.2 it follows that there exists $0 \neq x_0 \in \mathbb{R}^n$, $q \in \mathbb{R}^r$ such that $P(z) \begin{bmatrix} x_0 \\ q \end{bmatrix} = 0$. Since z was arbitrary it is proven that $\text{rank}P(s) < n + \text{rank}E$ for all $s \in \mathbb{C}$.

«(ii) $\Rightarrow$ (iii)» Assume that $\text{rank}P(s) < n + \text{rank}E$ for all $s \in \mathbb{C}$. Pick complex numbers $s_1 \neq s_2 \neq \dots \neq s_k$ and compute $x_i \in \mathbb{R}^n$ $i=1, 2, \dots, k$ such that $x_i \neq 0$ for all i and for some $q_i \in \mathbb{R}^r$

$$P(s_i) \begin{bmatrix} x_i \\ q_i \end{bmatrix} = 0 \quad i=1, 2, \dots, k \quad (5.11)$$

The number k is the smallest integer for which the vectors $x_1, x_2, \dots, x_k$ are linearly dependent. Making use of the identity $(s_i I - A) = (s_i I - A) + (s_i - s_k)I$ [65] the k equations (5.11) can be written in the form

$$\begin{aligned} (s_k I - A)x_1 + (s_1 - s_k)x_1 + Eq_1 &= 0 & i=1, 2, \dots, k-1 \\ (s_k I - A)x_k + Eq_k &= 0 \end{aligned} \quad (5.12)$$

where $x_i \in V^*(A, E; \ker C)$ for $i=1, 2, \dots, k$. Since $x_1, x_2, \dots, x_k$ are linearly dependent there exists $a_i \in R$ $i=1, 2, \dots, k$ not all zero such that $\sum_{i=1}^k a_i x_i = 0$. Thus from (5.12) it follows that $\sum_{i=1}^{k-1} a_i (s_i - s_k)x_i + E \sum_{i=1}^k a_i q_i = 0$. It is clear from the linear independence of $x_1, x_2, \dots, x_{k-1}$ that $0 \neq x = \sum_{i=1}^{k-1} a_i (s_i - s_k)x_i$, $x \in \text{im} \text{ENV}^*(A, E; \ker C)$ hence $R^*(A, E; \ker C) \neq 0$ by algorithm (2.10)

From a numerical point of view the frequency domain condition of theorem 5.8 is most suitable to decide whether a system is known-initial state-observable, since it can be tested by computing the rank of $P(s)$ for at most $n+1$ distinct complex numbers.

In order to express the results of the foregoing theorems in terms of the zeros of (5.5) a few definitions are needed.

Definition 5.5: Let $M(s) \in R^{m \times n}(s)$. The number of linearly independent rows (columns) of $M(s)$ considered as vectors over $R^n(s)$ ($R^m(s)$) is called the normal rank of $M(s)$ and is denoted by $\text{nr} \text{rank} M(s)$.

The zeros of (5.5) are defined by means of the Smith form of the polynomial matrix $P(s)$ which is introduced next.

Theorem 5.9 [15]: Let $A(s)$ be a polynomial matrix of normal rank k , then $A(s)$ may be transformed by a series of elementary row and column operations into a matrix $S(s)$ called the Smith form as $A(s)$ having the form $S(s) = \begin{bmatrix} S^*(s) & 0 \\ 0 & 0 \end{bmatrix}$ where $S^*(s) = \text{diag}[s_1(s), s_2(s) \dots s_k(s)]$ and $s_i(s)$ divides $s_{i+1}(s)$ without remainder for $i=1, 2, \dots, k-1$.

Though it is generally accepted that the zeros of a single input-single output system are the roots of the numerator polynomial of its scalar transfer function, there are a number of equivalent or slightly different definitions of zeros in the multivariable case. A concise summary of these definitions and the relationships among them can be found in the works [65,66]. The following definition due to Rosenbrock is suitable for our purposes.

Definition 5.6: The zeros (also called the invariant zeros) of the system $\dot{x} = Ax + Eq$, $y = Cx$ are the roots, counting multiplicities, of the nonzero polynomials on the diagonal of the Smith form of the system matrix $P(s)$.

When the pair (A, E) is controllable and the pair (C, A) is observable the zeros defined via definition 5.6 coincide with the roots of the numerator polynomials in the Smith-McMillan form of the transfer matrix $C(sI - A)^{-1}E$ [51] thus reducing to the familiar definition of zero in the single input-single output case.

Based on these definitions the conditions for unknown input observability and detectability of (5.5) can be stated as follows.

Theorem 5.10: Let Z denote the set of zeros of the system $\Sigma: \dot{x} = Ax + Eq, y = Cx$ with q representing the disturbance input and y the measured output.

- 1) Σ is known-initial state-observable if and only if $\text{nrnk}P(s) = n + \text{rank}E$.
- 2) Σ is strongly observable if and only if $\text{nrnk}P(s) = n + \text{rank}E$ and $Z = \emptyset$.
- 3) Σ is strongly detectable if and only if $\text{nrnk}P(s) = n + \text{rank}E$ and $Z \subset C_0$.

Proof: The proof follows from theorem 5.7-5.9 by noting that $\text{rank}P(s) < n + \text{rank}E$ if and only if $\text{nrnk}P(s) < n + \text{rank}E$ or $s \in Z$.

It is interesting to note the implications of theorem 5.10 for single disturbance-single measurement systems. Assuming that the system Σ is minimal strong observability corresponds to the transfer function $C(sI - A)^{-1}E$ being all pole type and strong detectability corresponds to the minimum phase condition (zeros lie in open left half plane).

The proofs of theorems 5.7, 5.8 and 5.10 are not constructive like the geometric conditions given in Chapter 4 or the transfer matrix conditions given in section 5.1. For constructive solvability conditions in terms of polynomial system matrices see [37] for instance.

CHAPTER 6

SOLVABILITY OF DISTURBANCE DECOUPLING AND DISTURBANCE DECOUPLED ESTIMATION PROBLEMS IN DECOMPOSED SYSTEMS

In this chapter solvability of disturbance decoupling and disturbance decoupled estimation problems will be investigated for a decomposed system consisting of separated algebraic and dynamic parts. It will be shown that the solvability range of the problems is extended by considering system decomposition. This kind of system representation has been originally suggested in [71] to investigate controllability, observability of linear m -ports. Transfer matrix invariance and disturbance decoupling in decomposed systems have been subsequently studied in [72,73]. In the present work the results in these studies are improved and extended in a number of ways. The treatment of the problems closely follows [75].

The definition and general properties of decomposed systems are presented in section 6.1. A complete set of results for disturbance decoupling in decomposed systems are given in section 6.2. The open loop and closed loop disturbance decoupling problems are defined. The open loop disturbance decoupling which is called filtering problem is solved under the additional condition of stability while the closed loop disturbance decoupling problem is investigated for the feedback structures introduced in Chapter 3. Solvability of both problems with the restriction to passive and lossless dynamic subsystems is also considered. The results are compared with regular (undecomposed) state space systems in terms of their generic solvability. Section 6.3 is devoted to

disturbance decoupled estimation in decomposed systems. The problem is solved in three different settings assuming that i) the initial state is unknown, the inputs are known ii) the initial state is known, the inputs are unknown and finally iii) both the inputs and the initial state may be unknown. The chapter is concluded in section 6.4 with the conceptual algorithms outlining some of the synthesis procedures developed. Summarizing tables and numerical examples which exhibit the advantage of considering system decomposition are included in each section.

6.1 SYSTEMS CONSISTING OF DYNAMIC AND ALGEBRAIC PARTS

We consider linear, time invariant systems composed of dynamic and algebraic parts as shown in Fig.6.1. The algebraic subsystem N_A is defined by

$$\begin{aligned} w &= -F_1 \bar{x} - F_2 u - E_1 q \\ y &= C \bar{x} \quad , \quad z = D \bar{x} \end{aligned} \quad (6.1)$$

and is assumed to be fixed by construction. The dynamic subsystem N_D is described by

$$\dot{\bar{v}} = \bar{w} \quad , \quad x = H \bar{v} \quad (6.2)$$

and can be made to vary by suitable choice of nonsingular H matrices. When the subsystems N_D and N_A are interconnected subject to the constraints

$$\bar{w} = -w \quad \text{and} \quad \bar{x} = x \quad (6.3)$$

they give rise to the state space description

$$\begin{aligned} \dot{x} &= H F_1 x + H F_2 u + H E_1 q \\ y &= C x \quad , \quad z = D x \end{aligned} \quad (6.4)$$

or to the description

$$\dot{v} = F_1 H v + F_2 u + E_1 q \quad (6.5)$$

$$y = C H v, \quad z = D H v$$

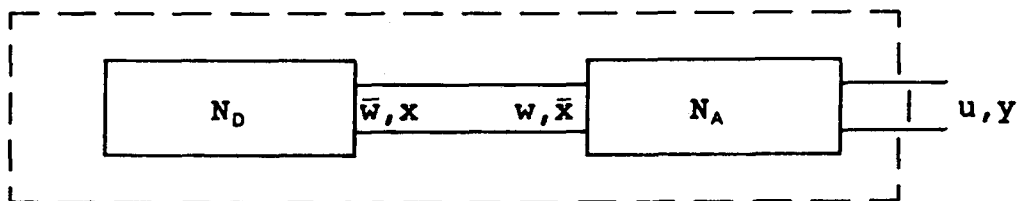


Fig.6.1. Decomposed system.

Since the matrix H defining the dynamic subsystem N_D is nonsingular, the state equations (6.4) and (6.5) are equivalent under the transformation of variables $x = H v$.

This kind of system representation arises naturally in electrical networks where for example N_A consists of all algebraic components such as resistors, dependent and independent sources and N_D consists of inductors, capacitors etc. In this case the variables $(\bar{x}, w)$ denote hybrid pairs of port voltages and port currents and the elements of v are flux or charge variables depending on $\bar{w}$. The constraints (6.3) are simply Kirchoff's current and voltage laws resulting from the interconnection of N_A with N_D .

Remark 6.1: An n -port network described by the state representation (6.2) is called a generalized capacitive/inductive n -port in literature. The matrix H is positive definite and symmetric if (6.2) is to represent a passive, lossless n -port [76,77].

A motivating example which illustrates the decomposition of an arbitrary electrical network into its dynamic and algebraic parts is given below.

Example 6.1: Consider the circuit in Fig.6.2a. The circuit can be redrawn as in Fig.6.2b by extracting the capacitor, inductor and connecting them externally to the algebraic three port. The hybrid description of N_A is

$$w = - \begin{bmatrix} -G & -1 \\ 1+rG & 0 \end{bmatrix} \bar{x} - \begin{bmatrix} G \\ -rG \end{bmatrix} u, \quad y = i = [-G \quad 0] \bar{x} + Gu$$

and N_D is described by

$$\dot{v} = \bar{w}, \quad x = \begin{bmatrix} 1/C & 0 \\ 0 & 1/L \end{bmatrix} v$$

$$\text{where } w = \begin{bmatrix} i_1 \\ v_2 \end{bmatrix}, \quad \bar{x} = \begin{bmatrix} \bar{v}_1 \\ \bar{i}_2 \end{bmatrix}, \quad x = \begin{bmatrix} v_1 \\ i_2 \end{bmatrix}, \quad \bar{w} = \begin{bmatrix} \bar{i}_1 \\ \bar{v}_2 \end{bmatrix} \text{ and } v = \begin{bmatrix} q \\ \phi \end{bmatrix}$$

A variation in the parameters of N_A or a variation of the form $u + \delta u$ in u can be modeled as a disturbance q .

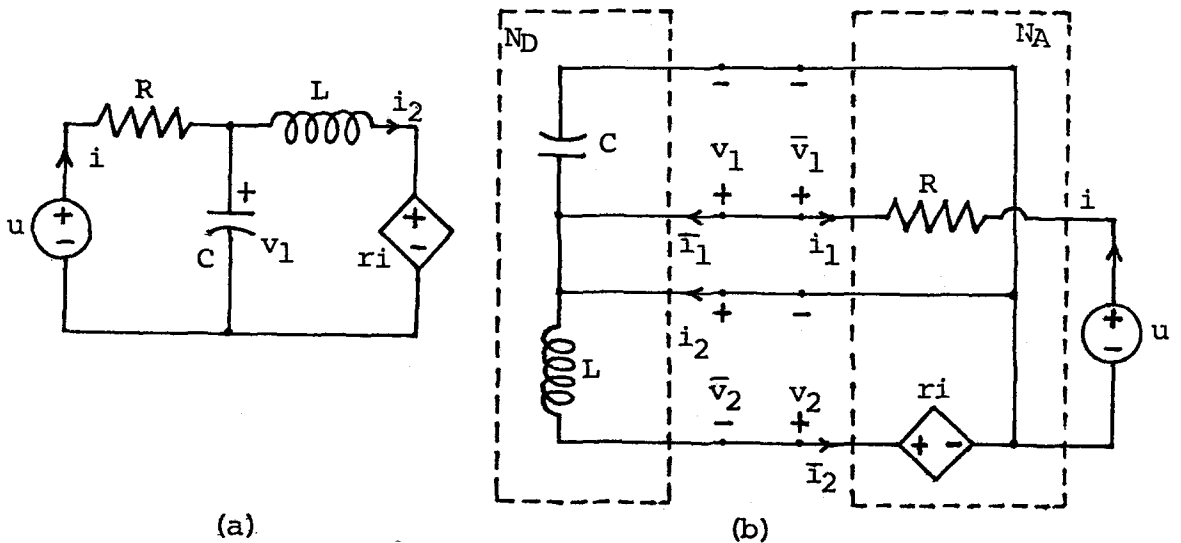


Fig.6.2. (a) Electrical network

(b) Decomposition of the network in Fig.6.2a into algebraic and dynamic parts.

While the decomposed system is introduced here and in the references [71-75] with an application to circuit theory in mind, there is still another interpretation of equations (6.1)-(6.4): It is possible to view (6.1) as an algebraic system with exogenous inputs u, q and exogenous outputs y, z and (6.2) as auxiliary integrators with inputs $\bar{w}$ and outputs x which are incorporated in the feedback path as shown in Fig.6.3. In this case the representation (6.4) fits into the general theory of dynamic compensation introduced in section 3.3.

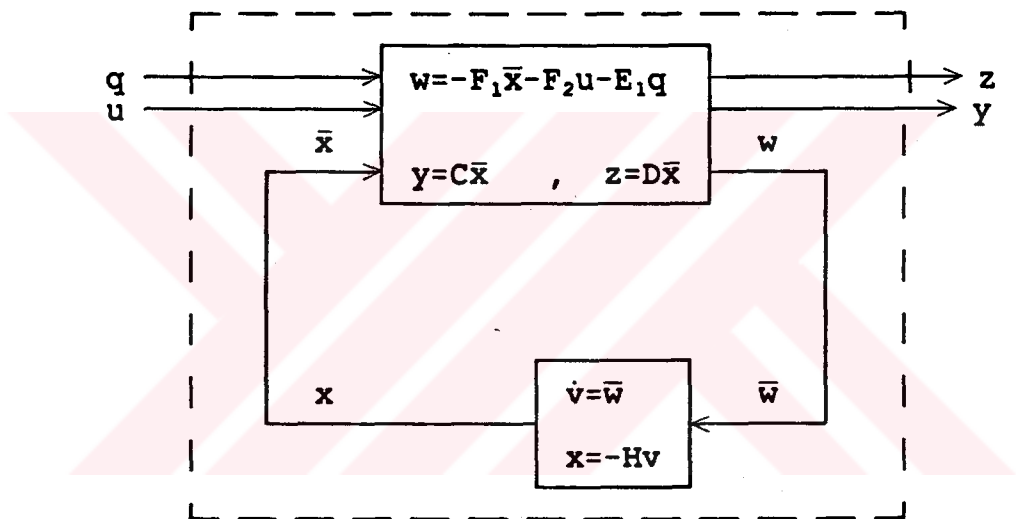


Fig.6.3. Decomposed system viewed as a dynamic compensation of the algebraic system N_A

In (6.4) and (6.5) $u(t) \in U = R^m$, $y(t) \in Y = R^p$, $z(t) \in Z = R^l$, $x(t), v(t) \in X = R^n$ denote the input, measured output, the controlled output and the state respectively. $q(t) \in F = R^r$ represents the disturbance which enters through the algebraic subsystem N_A . HF_1, HF_2 and HE_1 will be denoted by A, B and E respectively when H is fixed in the discussion. In this case the system (6.4) is indistinguishable from an undecomposed (regular) state space system.

6.2 DISTURBANCE DECOUPLING IN DECOMPOSED SYSTEMS

The disturbance decoupling problem which is studied in Chapter 3 is formulated for a decomposed system as follows.

1. Open loop decoupling (Filtering problem): is the problem of finding a dynamic subsystem (a nonsingular matrix H) such that the composite system (6.4) is disturbance decoupled.
2. Closed loop decoupling (Feedback problem): is the problem of finding a dynamic subsystem (a nonsingular matrix H) and a feedback control law from the measured outputs y to the control inputs u such that the closed loop composite system is disturbance decoupled.

Both problems can be investigated under the additional condition of stability or with the restriction to passive and lossless dynamic subsystems. The filtering problem and the problem of finding all dynamic subsystems which yield a given transfer function matrix have been solved in [72-74]. On the other hand, the feedback problem has been studied for regular state space systems as reviewed in Chapter 3. In fact an analog of filtering problem does not exist for regular systems. This extra degree of freedom is gained because of the variable structure of a decomposed system.

In theorem 6.1 it is shown that under reasonable conditions the problems of filtering and feedback are equivalent as far as their solvability is concerned.

Theorem 6.1: Assume that $\text{im}F_2 \subset \text{im}E_1$ (This is the case if the control input u is of the form $u=s+n$ with n representing a noise term for example.) then feedback

problem is solvable if and only if filtering problem is solvable.

Proof: «If» This is trivial. If there exists a dynamic subsystem which makes the composite system (6.4) disturbance decoupled then apply no feedback at all ie., zero feedback. Thus feedback problem is solved.

«Only if» Let H define the dynamic subsystem which solves the feedback problem; then there exists a subspace $V \in V(HF_1, HF_2; \ker D)$ such that $\text{im} HE_1 \subset V$. By definition $HF_1 V \subset V + \text{im} HF_2 \subset V$ where the last inclusion holds because of the assumption $\text{im} F_2 \subset \text{im} E_1$. Thus V is HF_1 -invariant and H solves the filtering problem.

The following lemmas will be needed for the solution of filtering and feedback problems by dynamic subsystems defined by nonsingular H matrices.

Lemma 6.1: Let S and V be subspaces of X . There exists a nonsingular matrix $H: X \rightarrow X$ such that $HS \subset V$ if and only if $d(S) \leq d(V)$.

Proof: «Only if» Assume that $HS \subset V$ then $d(HS) = d(S) \leq d(V)$.

«If» Assume that $j = d(S) \leq d(V) = k$. Choose bases $\{s_1 \dots s_j\}$ and $\{v_1 \dots v_k\}$ for S and V respectively and extend them to bases for X as $\{s_1 \dots s_j s_{j+1} \dots s_n\}$ and $\{v_1 \dots v_k v_{k+1} \dots v_n\}$ in an arbitrary way. Define $H: X \rightarrow X$ by $HS_i = v_i$ for $i=1, 2, \dots, n$ then H is nonsingular and satisfies $HS \subset V$.

Lemma 6.2: Let V_1, V_2 be subspaces of X . Consider the family of subspaces $\underline{S} = \{S: V_1 \subset S + V_2\}$ then 1°) $S \in \underline{S}$ only if $d(S) \geq d(V_1) - d(V_1 \cap V_2)$ and 2°) There exists $S \in \underline{S}$ such that $d(S) = d(V_1) - d(V_1 \cap V_2)$.

Proof: 1°) Assume that $V_1 \subset S + V_2$ then also $V_1 + V_2 \subset S + V_2$ hence $d(V_1 + V_2) = d(V_1) + d(V_2) - d(V_1 \cap V_2) \leq d(S + V_2) \leq d(S) + d(V_2)$ from which it follows that $d(V_1) - d(V_1 \cap V_2) \leq d(S)$.

2°) Let $\{v_1 \dots v_k\}$ be a basis for $V_1 \cap V_2$. Complete this to a basis for V_1 as $\{v_1 \dots v_k, v_{k+1} \dots v_l\}$ where $l = d(V_1)$. Let $S = \text{span}\{v_{k+1} \dots v_l\}$ then S satisfies $V_1 = V_1 \cap V_2 \oplus S \subset V_2 + S$ and $d(S) = d(V_1) - d(V_1 \cap V_2)$.

Necessary and sufficient conditions for the solvability of filtering problem are given in theorem 6.2.

Theorem 6.2: Filtering problem is solvable if and only if there exists a subspace $V \subset \ker D$ such that $d(\text{im} E_1 + F_1 V) \leq d(V)$.

Proof: «Only if» Assume that filtering problem is solvable. There exist a nonsingular matrix H and a subspace $V \subset \ker D$ such that $\text{im} H E_1 \subset V$ and $H F_1 V \subset V$. It follows that $\text{im} H E_1 + H F_1 V \subset V$ or $H(\text{im} E_1 + F_1 V) \subset V$ which implies that $d(\text{im} E_1 + F_1 V) \leq d(V)$.

«If» By lemma 6.1 there exists a nonsingular matrix H such that $H(\text{im} E_1 + F_1 V) \subset V$. Let the dynamical subsystem be characterized by this matrix. Then $\text{im} H E_1 \subset V$ and $H F_1 V \subset V \subset \ker D$ which proves that (6.4) is disturbance decoupled by this choice of H .

It is not quite clear how the condition of theorem 6.2 can be tested constructively. The following corollary provides a solution to this question.

Corollary 6.1: Let $K = \ker D$. Filtering problem is solvable if and only if

$$d(\text{im} E_1 + F_1 K) \leq d(K) \quad (6.6)$$

Proof: The following statements are equivalent:

- (i) There exists $V \subset \ker D$ such that $d(\operatorname{im} E_1 + F_1 V) \leq d(V)$.
- (ii) There exists $V \subset \ker D$ such that $\operatorname{rank} E_1 + d(F_1 V) - d(\operatorname{im} E_1 \cap F_1 V) \leq d(V)$.
- (iii) There exists $V \subset \ker D$ such that $\operatorname{rank} E_1 + d(V) - d(\ker F_1 \cap V) - d(\operatorname{im} E_1 \cap F_1 V) \leq d(V)$.
- (iv) $\operatorname{rank} E_1 \leq d(\ker F_1 \cap \ker D) + d(\operatorname{im} E_1 \cap F_1(\ker D))$

The conclusion follows by applying theorem 6.2.

The next theorem gives conditions for simultaneous decoupling and stabilization of (6.4).

Theorem 6.3: Let C_g denote the open left half complex plane. There exists a dynamic subsystem for which (6.4) is disturbance decoupled and $\sigma(HF_1) \subset C_g$ if and only if

$$(a) \operatorname{rank} F_1 = n \quad \text{and} \quad (b) \operatorname{im} E_1 \subset F_1(\ker D) \quad (6.7)$$

Proof: «Only if» Since $0 \notin \sigma(HF_1)$ and H is nonsingular it follows that $\operatorname{rank} F_1 = n$. (6.7b) follows from corollary 6.1 and (6.7a).

«If» Let $\{v_1 \dots v_k\}$ be a basis for $\ker D$ and $\{v_1 \dots v_k, v_{k+1} \dots v_n\}$ be a basis for X . Define $A: X \rightarrow X$ by $Av_i = s_i v_i$ where $\operatorname{Re}(s_i) < 0$ for $i=1, 2, \dots, n$. Since $\operatorname{rank} F_1 = \operatorname{rank} A = n$ there exists a nonsingular matrix H such that $HF_1 = A$. Then we have $\operatorname{im} HE_1 \subset HF_1(\ker D) = \ker D$ and $\sigma(HF_1) \subset C_g$. Thus filtering problem with stability is solvable for this H .

The subsequent theorems consider feedback problem and extend the work in [73] to dynamic measurement feedback and state feedback with disturbance feedforward. Theorem 6.4 gives a more refined solvability condition and design procedure for

disturbance decoupling with state feedback than the one given in [73].

Theorem 6.4: Feedback problem is solvable by constant state feedback if and only if there exists a subspace $V \subset \ker D$ such that

$$\text{rank} E_1 \leq d(\ker F_1 \cap V) + d(F_1 V \cap (\text{im} E_1 + \text{im} F_2))$$

Proof: The following chain of equivalent statements follows from theorem 3.1, lemma 6.1 and lemma 6.2.

1. Feedback problem is solvable by a control law $u = Kx$.
2. There exist a nonsingular matrix H and a subspace $V \subset \ker D$ such that $\text{im} H E_1 \subset V$, $H F_1 V \subset V + \text{im} H F_2$.
3. There exist subspaces V and S such that $V \subset \ker D$, $d(S) \leq d(V)$, $\text{im} E_1 \subset S$ and $F_1 V \subset S + \text{im} F_2$.
4. There exist subspaces V and W such that $V \subset \ker D$, $d(W) \leq d(V) - \text{rank} E_1$ and $F_1 V \subset W + \text{im} E_1 + \text{im} F_2$.
5. There exists a subspace $V \subset \ker D$ such that $d(F_1 V) - d(F_1 V \cap (\text{im} E_1 + \text{im} F_2)) \leq d(V) - \text{rank} E_1$.

The conclusion follows from 5. by making use of the identity $d(F_1 V) = d(V) - d(\ker F_1 \cap V)$.

Theorem 6.5: Feedback problem is solvable by a control law $u = Kx + Rq$ (state feedback with disturbance feedforward) if and only if there exists a subspace $V \subset \ker D$ such that $d(F_1 V + \text{im} E_1 + \text{im} F_2) \leq \text{rank} F_2 + d(V)$

Proof: The following statements are equivalent:

1. Feedback problem is solvable by a control law $u = Kx + Rq$
2. There exist a nonsingular matrix H and a subspace $V \subset \ker D$ such that $\text{im} H E_1 \subset V + \text{im} H F_2$, $H F_1 \subset V + \text{im} H F_2$
3. There exist a nonsingular matrix H and a subspace $V \subset \ker D$ such that $\text{im} E_1 + F_1 V \subset H^{-1} V + \text{im} F_2$

4. There exist subspaces V and S such that $d(S) \leq d(V)$, $V \subset \ker D$ and $\text{im} E_1 + F_1 V \subset S + \text{im} F_2$
5. There exists a subspace $V \subset \ker D$ such that $d(\text{im} E_1 + F_1 V) - d(\text{im} F_2 \cap (\text{im} E_1 + F_1 V)) \leq d(V)$

The theorem follows by adding $\text{rank} F_2$ to both sides of the inequality in 5.

The constructive forms of theorem 6.4 and theorem 6.5 are given in corollary 6.2. The proof is similar to that of corollary 6.1 and is therefore omitted.

Corollary 6.2: Let $K = \ker D$. 1) Feedback problem is solvable by constant state feedback if and only if

$$\text{rank} E_1 \leq d(\ker F_1 \cap K) + d(F_1 K \cap (\text{im} E_1 + \text{im} F_2)) \quad (6.8)$$

2) Feedback problem is solvable by a control law $u = Kx + Rq$ if and only if

$$d(F_1 K + \text{im} E_1 + \text{im} F_2) \leq \text{rank} F_2 + d(K) \quad (6.9)$$

Remark 6.2: It is clear from the foregoing theorems that solvability of filtering problem $\Rightarrow$ Solvability of feedback problem with $u = Kx \Rightarrow$ Solvability of feedback problem with $u = Kx + Rq$ as expected where the first implication turns into an equivalence under the condition $\text{im} F_2 \subset \text{im} E_1$ and the second implication becomes an equivalence if $\text{im} E_1 \cap \text{im} F_2 = 0$.

Remark 6.3: Some easy to test conditions are useful to decide quickly whether the feedback problems are solvable or not for a given algebraic subsystem. By writing the condition (6.8) in the form:

$$\text{rank} E_1 + d(F_1 K + \text{im} E_1 + \text{im} F_2) \leq d(K) + \text{rank}[E_1, F_2] \quad (6.10)$$

it is seen that a necessary condition for the solvability of feedback problem by constant state feedback is: $\text{rank} E_1 + \text{rank} D \leq n$. On the other hand for state feedback with disturbance feedforward $\text{rank} D \leq \text{rank} F_2$ is sufficient for solvability. An interpretation of these rank conditions in terms of control input, controlled output and disturbance input numbers will be given in theorem 6.7.

In the next theorem a sufficient condition is given for the solvability of feedback problem by dynamic measurement feedback.

Theorem 6.6: Let $K = \ker D$. Feedback problem is solvable by dynamic measurement feedback if

$$\text{rank} E_1 \leq d(\ker F_1 \cap K) + d(F_1 K \cap (\text{im} E_1 + \text{im} F_2)) \quad (6.11)$$

$$\text{rank} E_1 + \text{rank} D \leq \text{rank} \begin{bmatrix} C \\ D \end{bmatrix} \quad (6.12)$$

Proof: Because of assumption (6.11) feedback problem is solvable by constant state feedback. From theorem 6.4 it follows that there exists a subspace S such that $F_1 K \subset S + \text{im} F_2$, $\text{im} E_1 \subset S$ and $d(S) \leq d(K)$. Let $\{x_1 \dots x_r, x_{r+1} \dots x_s\}$ be a basis for S such that the first r vectors form a basis for $\text{im} E_1$. Next, choose a basis $\{v_1 \dots v_j, v_{j+1} \dots v_k\}$ for K such that $\ker C \cap K = \text{span}\{v_{j+1} \dots v_k\}$ and write $Q = \text{span}\{v_1 \dots v_j\}$. By this construction of Q and from (6.12) it can be verified that $Q \cap \ker C = 0$ and $d(Q) \geq \text{rank} E_1$. Define a nonsingular matrix $H: X \rightarrow X$ such that $Hx_i = v_i$, $i=1, 2, \dots, k$ then we have $HF_1 K \subset K + \text{im} HF_2$, $\text{im} HE_1 \subset K$ and $\text{im} HE_1 \cap \ker C \subset Q \cap \ker C = 0$. Hence $\text{im} HE_1 = Q + (C, HF_1; \text{im} HE_1) \subset V^*(HF_1, HF_2; K) = K$ and disturbance decoupling by dynamic measurement feedback is solvable for the system (6.4).

Remark 6.4: The order of the dynamic compensator that can be designed for the decomposed system constructed in theorem 6.6 is at most $d(\ker D) - d(Q) = d(\ker D \cap \ker C)$. This number is smaller than the order of the compensator that could be designed by the procedure of theorem 3.5. It follows that feedback problem is solvable by constant measurement feedback i.e., by zero order dynamic compensation if $d(\ker D \cap \ker C) = 0$ and (6.11) is satisfied. This result can also be inferred from theorem 3.4.

The improvement on solvability of disturbance decoupling problem obtained by theorems 6.4 and 6.6 is remarkable. For a fair comparison with regular state space systems the following definitions are adopted from [1].

Definition 6.1: Consider finite number of polynomials $p_i(z_1, z_2, \dots, z_N)$ $i=1, 2, \dots, k$ with coefficients in R . The set $V = \{x \in R^N : p_i(x) = 0 \text{ } i=1, 2, \dots, k\}$ is called a variety. V is said to be proper if $V \neq R^N$.

Definition 6.2: List elements of F_1, F_2, E_1, C, D in arbitrary order and regard them as a data point $d \in R^N$, $N = n(n+m+r+p+1)$. A property associated with the algebraic subsystem (6.1) is said to hold generically if there exists a proper variety $V \subset R^N$ such that the property fails only if $d \in V$.

It is well known that [1,23] the disturbance decoupling problem is generically unsolvable in the space of data points A, B, E, C, D . Certain structural constraints must be satisfied by the system matrices in order for the problems to have a solution. From a practical point of view this constitutes an unpleasant situation for only the nominal values of the system parameters can be known and it is desired that a problem

which is solvable at the nominal parameter set $(A_0, B_0, \text{etc.})$ should also be solvable for sufficiently small variations from the nominal values $(A_0 + \delta A, B_0 + \delta B \text{ etc.})$. By corollary 6.2 and theorem 6.6 we are able to obtain generic solvability conditions for disturbance decoupling problem by considering system decomposition. The precise conditions for generic solvability are given in theorem 6.7.

Theorem 6.7: Let r =number of disturbance inputs, m =number of control inputs, l =number of controlled outputs, p =number of measured outputs and $n=d(X)$. Assume that $0 < r \leq n$, $0 \leq m \leq n$, $0 \leq l \leq n$ and $0 \leq p \leq n$. Feedback problem is generically solvable by

1. Constant state feedback if and only if
(a) $r+l \leq n$, (b) $l \leq m$ (6.13)

2. State feedback with disturbance feedforward
if and only if $l \leq m$ (6.14)

3. Dynamic measurement feedback if and only if
(a) $r+l \leq n$, (b) $l \leq m$, (c) $r \leq p$ (6.15)

Proof: If S is a subspace which depends on the parameters d of (6.1) the notation $d(S)=s$ (g) means that the generic dimension of S is s ie., that $d(S) \neq s$ only for d in some fixed proper variety. It is clear that the generic ranks of the matrices E_1, D, F_2 and C are r, l, m and p respectively since the rank evaluations fail only if all $r \times r$, $l \times l$, $m \times m$ and $p \times p$ minors of the corresponding matrices vanish.

1) «Only if» (6.13a) follows from the necessity of $\text{rank} E_1 + \text{rank} D \leq n$. By reasoning similar to [1, pp.279] we have $\text{rank}[E_1, F_2] = \min(r+m, n)$ (g) and $d(F_1 K + i m E_1 + i m F_2) = \min(n-l+r+m, n)$ (g). Thus it follows from Remark 6.3 that $r + \min(n-l+r+m, n) \leq n-l + \min(r+m, n)$. Now, if $r+m < n$ (6.13b)

results immediately from the last inequality and if $r+m \geq n$ this together with (6.13a) gives (6.13b)

«If» First consider the case $r+m \geq n$ then $d(F_1K + \text{im}E_1 + \text{im}F_2) = n(g)$ and $\text{rank}[E_1, F_2] = n(g)$ thus (6.10) generically holds under condition (6.13a). If $r+m < n$ then using the generic dimensions of subspaces as before the sufficiency of (6.13b) is established.

2) «If» When $l \leq m$ the condition $\text{rank}D \leq \text{rank}F_2$ is generically satisfied. Hence by (6.9) feedback problem is solvable by a control law $u = Kx + Rq$.

«Only if» Using the generic dimensions of subspaces as in the proof of 1) it follows from (6.9) that $\min(n-l+r+m, n) \leq m+n-l$ from which (6.14) follows.

3) Since the statement: «there exist a nonsingular matrix H and a subspace Q such that $HF_1(Q \cap \ker C) \subset Q$, $\text{im}HE_1 \subset Q \subset \ker D$ » is exactly the dual of the statement 2 in the proof of theorem 6.4, the conditions (6.15a-c) follow from (6.13) by duality. Sufficiency of (6.15) is easily established from theorem 6.6 by noting that (6.11) and (6.12) generically hold when (6.15) is satisfied.

The filtering problem considered in theorem 6.2 is not generically, solvable in the sense of definition 6.2. However in a specific application the parameters of the algebraic subsystem may be constrained because of some definitional relations among the variables. In this case the problem may be generically solvable, in the space of admissible parameter values. Such cases require separate investigation.

Solvability of filtering and feedback problems by using a passive and lossless dynamic subsystem is

considered in theorem 6.8. These results are based on the following lemma which gives a basic property of a positive definite and symmetric operator.

Lemma 6.3: Let S and V be subspaces of X . There exists a positive definite and symmetric matrix $H: X \rightarrow X$ such that $HS \subset V$ if and only if $SNV^\perp = 0$.

Proof: «Only if» Assume that $0 \neq x \in SNV^\perp$ then there exists $y \in V$ such that $y = Hx$. Hence $(x, y) = (x, Hx) = 0$ which contradicts that H is positive definite.

«If» Let $\{s_1, s_2, \dots, s_j\}$ and $\{u_1, u_2, \dots, u_k\}$ be bases for S and $V^\perp$ respectively. By assumption $d(S) \leq d(V)$ hence there exist linearly independent vectors $\{v_1, v_2, \dots, v_j\}$ in V . Define a nonsingular matrix A such that $As_i = v_i$ for $i=1, 2, \dots, j$ and $Au_i = u_i$ for $i=1, 2, \dots, k$. By this construction of A we have $AS \subset V$ and $A'V \subset V$. Let H be defined as $H = A'A$ then H is positive definite, symmetric and $HS \subset V$.

Theorem 6.8: Assume that (6.2) defines a passive and lossless dynamical subsystem. Then

- 1) Filtering problem is solvable if and only if there exists $V \subset \ker D$ such that

$$(imE_1 + F_1V)NV^\perp = 0 \quad (6.16)$$

- 2) Feedback problem is solvable by constant state feedback if and only if there exist subspaces V and W such that

$$(a) V \subset \ker D, \quad (b) F_1V \subset W + imE_1 + imF_2, \quad (c) (imE_1 + W)NV^\perp = 0 \quad (6.17)$$

- 3) Feedback problem is solvable by state feedback and disturbance feedforward if and only if there exist subspaces V and S such that

$$(a) V \subset \ker D, (b) \operatorname{im} E_1 + F_1 V \subset S + \operatorname{im} F_2, (c) S \cap V^\perp = 0 \quad (6.18)$$

Proof: By noting that passive and lossless dynamical subsystems are characterized by positive definite, symmetric H matrices, the theorem follows from lemma 6.3 reasoning as in theorem 6.2, theorem 6.4 and theorem 6.5.

Remark 6.5: Unfortunately the conditions of theorem 6.8 can not be easily tested like the ones given in corollary 6.1 or corollary 6.2. However for the subclass of algebraic subsystems for which filtering (feedback) problems are solvable by a nonsingular matrix H , the conditions (6.16) (6.17, 6.18) are generically satisfied since they happen to be transversality conditions [1, pp-29] in this case. In other words, the easy to test necessary conditions (6.6), (6.8) and (6.9) are generically sufficient for the solvability of the corresponding problems by passive and lossless dynamic subsystems. To justify this statement consider for instance disturbance decoupling by constant state feedback. In the proof of theorem 6.4 it has been shown that under condition (6.8) there exist subspaces V and W such that $V \subset \ker D$, $F_1 V \subset W + \operatorname{im} E_1 + \operatorname{im} F_2$ and $d(W + \operatorname{im} E_1) \leq d(W) + \operatorname{rank} E_1 \leq d(V)$. Hence $d((\operatorname{im} E_1 + W) \cap V^\perp) = \max(0, d(\operatorname{im} E_1 + W) + d(V^\perp) - n) = 0$ (g) which proves that (6.17c) is generically satisfied.

The main results of this section are summarized in Table 6.1. The conclusions apply to the decomposed system (6.4). K stands for $\ker D$.

Table 6.1. Disturbance decoupling in decomposed systems

		Necessary and sufficient condition for solvability	Iff condition for generic solvability	
FILTERING PROBLEM	Naive solution	$d(\text{im}E_1 + F_1K) \leq d(K)$	None	
	Stable solution	$\text{im}E_1 \subset F_1K, \text{rank}F_1 = n$		
	Passive, lossless solution	$\exists V \subset K \ni (\text{im}E_1 + F_1V)NV^\perp = 0$		
	$u = Fx$	Naive solution	$\text{rank}E_1 \leq d(\ker F_1 \cap K) + d(F_1K \cap (\text{im}E_1 + \text{im}F_2))$	$r + l \leq n$ $l \leq m$
		Passive, lossless solution	$\exists V, W \ni V \subset K, F_1V \subset W + \text{im}E_1 + \text{im}F_2, (\text{im}E_1 + W)NV^\perp = 0$	
	$u = Fx + Rq$	Naive solution	$d(F_1K + \text{im}E_1 + \text{im}F_2) \leq \text{rank}F_2 + d(K)$	$l \leq m$
Passive, lossless solution		$\exists V, S \ni V \subset K, \text{im}E_1 + F_1V \subset S + \text{im}F_2, SNV^\perp = 0$		
FEEDBACK PROBLEM	$u = Lx + Ky$ $\dot{w} = Nw + My$	Sufficient condition for solvability	$r + l \leq n$ $l \leq m$ $r \leq p$	
		$\text{rank}E_1 \leq d(\ker F_1 \cap K) + d(F_1K \cap (\text{im}E_1 + \text{im}F_2))$		
		$\text{rank}E_1 + \text{rank}D \leq \text{rank} \begin{bmatrix} C \\ D \end{bmatrix}$		

6.3 DISTURBANCE DECOUPLED ESTIMATION IN DECOMPOSED SYSTEMS

In this section solvability of disturbance decoupled estimation problems will be investigated for the decomposed system (6.4). More precisely, necessary and sufficient conditions will be given for the existence of a dynamic subsystem which makes (6.4) strongly observable, strongly detectable or known-initial state observable. In view of the equivalences derived in section 4.3 these conditions can be used to establish existence conditions of full state PID class of observers for decomposed systems. All results can be specialized to the case where unknown inputs are not present by letting $E_1=0$. In this case the results reported in [71] are recovered. In addition, by duality arguments a number of side results are obtained for the controllability, right invertibility of decomposed systems. Apart from their significance in control theory the concepts of controllability, observability, input observability also play key roles in the circuit theoretic studies of passivity, losslessness [76-77].

The first theorem which treats the case $q=0$ will be the basis for subsequent theorems as well.

Theorem 6.9: Let $q=0$ in (6.4). 1) There exists a dynamic subsystem (a nonsingular matrix H) such that (6.4) is observable if and only if

$$(a) \text{ rank } \begin{bmatrix} F_1 \\ C \end{bmatrix} = n \text{ and} \quad (b) C \neq 0 \quad (6.19)$$

2) There exists a dynamic subsystem such that (6.4) is detectable if and only if

$$\text{rank} \begin{bmatrix} F_1 \\ C \end{bmatrix} = n \quad (6.20)$$

Proof: 1) «Only if» Assume that the pair (C, HF_1) is observable. From theorem 5.7 we have

$$\text{rank} \begin{bmatrix} sI - HF_1 \\ C \end{bmatrix} = n \quad \forall s \in \mathbb{C} \quad (6.21)$$

Setting $s=0$, $\text{rank} \begin{bmatrix} -HF_1 \\ C \end{bmatrix} = \text{rank} \begin{bmatrix} F_1 \\ C \end{bmatrix} = n$. The condition $C \neq 0$ follows since $\langle \ker C / HF_1 \rangle = X$ for all nonsingular H matrices when $C=0$.

«If» Let $\{v_1 \dots v_k\}$ be a basis for $\ker F_1$, and $\{v_1 \dots v_k v_{k+1} \dots v_n\}$ its completion to a basis for X . Because of the assumption $C \neq 0$ at least one of vectors v_i , $i=1, 2, \dots, n$ satisfies $Cv_i \neq 0$, say v_1 . Define the new set of vectors x_i , $i=1, 2, \dots, n$ by $x_i = \begin{cases} v_i, & \text{if } Cv_i \neq 0 \\ v_i + v_1, & \text{if } Cv_i = 0 \end{cases} \quad i=1, 2, \dots, n$.

Now the set $\{x_1 \dots x_n\}$ is linearly independent and $Cx_i \neq 0$, $i=1, 2, \dots, n$. Define the map $A: X \rightarrow X$ by its action on the basis vectors as:

$Ax_i = 0$, $i=1, 2, \dots, k$ and $Ax_i = s_i x_i$, $i=k+1, \dots, n$ where $s_i \neq 0$ and arbitrary otherwise. Now the pair (C, A) is observable by construction and there exists a nonsingular matrix H such that $HF_1 = A$ since $\ker F_1 = \ker A$.

2) «Only if» The necessity of (6.20) follows from theorem 5.7 as in part 1.

«If» When $C \neq 0$ a nonsingular matrix H can be constructed so that the pair (C, HF_1) is observable. If $C=0$, since $\text{rank} F_1 = n$ there exists a nonsingular matrix H such that $\sigma(HF_1) \subset C_0$, where C_0 is the open left half complex plane. Thus the pair (C, HF_1) is detectable by definition 2.6.

From theorem 6.9 the dual result on controllability and stabilizability of (6.4) immediately follows.

Theorem 6.10: 1) There exists a dynamic subsystem (a nonsingular matrix H) such that (6.4) is controllable if and only if

$$(a) \text{rank}[F_1, F_2] = n \quad \text{and} \quad (b) F_2 \neq 0 \quad (6.22)$$

2) There exists a dynamic subsystem such that (6.4) is stabilizable if and only if

$$\text{rank}[F_1, F_2] = n \quad (6.23)$$

Proof: Only the proof of statement 1 will be given since the proof of 2 is essentially the same. The following statements are equivalent.

- (i) There exists a nonsingular H such that the pair (HF_1, HF_2) is controllable.
- (ii) There exists a nonsingular H such that the pair $(F_1 H, F_2)$ is controllable.
- (iii) There exists a nonsingular H such that the pair $(F_2', H' F_1')$ is observable.
- (iv) $\text{rank} \begin{bmatrix} F_1' \\ F_2' \end{bmatrix} = n$ and $F_2' \neq 0$.

The equivalence of (i) to (ii) follows from the equivalence of systems (6.4) and (6.5). The second holds

by the well known duality of controllability and observability. The last equivalence follows from the previous theorem with F_1' and F_2' playing the roles of F_1 and C respectively.

Existence conditions of known-initial state PID_k , $k \geq 0$ observers for decomposed systems are given in the next theorem. It is shown that unlike regular state space systems the existence conditions are independent of k .

Theorem 6.11: Assume that $z=x$ in (6.4). Define the sequence of subspaces V^k $k=1,2,\dots$ by $V^1 = \ker C$, $V^{k+1} = \ker C \cap (HF_1)^{-1}(V^k + \text{im} HE_1)$. Then the following statements are equivalent.

- (i) There exists a dynamical subsystem such that a known-initial state PID_{k-1} observer exists for (6.4).
- (ii) There exists a nonsingular matrix H such that $V^k \cap Q_*(C, HF_1; \text{im} HE_1) = 0$.
- (iii) $\text{rank} E_1 \leq \text{rank} C$.

Proof: $\langle (i) \Leftrightarrow (ii) \rangle$ Follows from theorem 4.4 by taking $D=I$.

$\langle (ii) \Rightarrow (iii) \rangle$ Assume that $\text{rank} E_1 > \text{rank} C$ then for all nonsingular matrices H and for all $s \in \mathbb{C}$ $\text{rank} \begin{bmatrix} sI - HF_1 & HE_1 \\ C & 0 \end{bmatrix} < n + \text{rank} HE_1$. It follows from theorem 5.8 that $R^*(HF_1, HE_1; \ker C) \neq 0$ for all nonsingular matrices H which implies that $V^k \cap Q_*(C, HF_1; \text{im} HE_1) \neq 0$ for $k=1,2,\dots$ as well.

$\langle (iii) \Rightarrow (ii) \rangle$ Let S be a subspace such that $S \oplus \ker C = X$. Since $\text{rank} E_1 \leq d(S)$, by lemma 6.1 there exists a nonsingular matrix H such that $\text{im} HE_1 \subset S$. Thus $Q_*(C, HF_1; \text{im} HE_1) = \text{im} HE_1$ and $V^k \cap Q_*(C, HF_1; \text{im} HE_1) = \ker C \cap \text{im} HE_1 = 0$ for $k=1,2,\dots$

Necessary and sufficient conditions for known-initial state-observability and left and right invertibility of decomposed systems immediately follow from theorem 6.11.

Theorem 6.12: There exists a dynamical subsystem such that (6.4) is

- 1) Known-initial state-observable if and only if $\text{rank}E_1 \leq \text{rank}C$.
- 2) Left invertible if and only if $\text{rank}E_1 \leq \text{rank}C$ and $\text{rank}E_1 = r$.
- 3) Right invertible if and only if $\text{rank}D \leq \text{rank}F_2$ and $\text{rank}D = 1$.

Proof: 1) Follows from theorem 4.8 and theorem 6.11 by taking $k > d(\ker C)$.

2) Follows from lemma 4.7 and 1).

3) Consider the dual system $\Sigma': \dot{s} = F_1' H' s + C' v + D' d$, $w = F_2' H' s$, $r = E_1' H' s$ of (6.4) where v is the control input, d is the disturbance input, w is the measured output and r is the controlled output. By theorem 5.5 there exists a nonsingular matrix H such that (6.4) is right invertible if and only if there exists a nonsingular matrix H such Σ' is left invertible. Since left and right invertibility is invariant under nonsingular state space transformations it follows from part 2) that $\text{rank}D' \leq \text{rank}F_2'$ and $\text{rank}D' = 1$ which establishes the conclusion.

Necessary and sufficient conditions for the existence of dynamic subsystems which make (6.4) strongly observable and strongly detectable are presented in theorem 6.13. While this result can be guessed from theorem 5.7 and theorem 6.9 the sufficiency proof of theorem 6.13 is rather intricate and does not easily follow from theorem 6.9.

Theorem 6.13: 1) There exists a dynamic subsystem (a nonsingular matrix H) such that (6.4) is strongly observable if and only if

$$(a) \text{ rank } \begin{bmatrix} F_1 & E_1 \\ C & 0 \end{bmatrix} = n + \text{rank} E_1 \quad \text{and} \quad (b) C \neq 0 \quad (6.24)$$

2) There exists a dynamic subsystem (a nonsingular matrix H) such that (6.4) is strongly detectable if and only if

$$\text{rank} \begin{bmatrix} F_1 & E_1 \\ C & 0 \end{bmatrix} = n + \text{rank} E_1 \quad (6.25)$$

Proof: 1) «Only if» Assume that (6.4) is strongly observable it follows from theorem 5.7 that $\text{rank} \begin{bmatrix} sI - HF_1 & HE_1 \\ C & 0 \end{bmatrix} = n + \text{rank} HE_1$ for all $s \in \mathbb{C}$. Since H is nonsingular setting $s=0$ (6.24a) results. (6.24b) follows

from (6.19b). In fact under condition (6.24a) $C \neq 0$ necessarily holds if $E_1 \neq 0$.

«If» Without loss of generality, it is assumed that E_1 is full column rank and C is full row rank otherwise the construction given below can be applied to matrices E_1^*, C^* which are obtained from E_1 and C by eliminating linearly dependent columns and rows respectively. Next, assume that (6.24) is satisfied, then by theorem 6.9 the pair (C, F_1) can be made observable. Let H_0 be a nonsingular matrix for which the pair $(C, H_0 F_1)$ is observable. Write $A_0 = H_0 E_1$ and $E_0 = H_0 E_1$. By state space, output space and output injection transformations T, S and G respectively on the triple (C, A_0, E_0) bring (A_0', C') to feedback canonical form (A_c', C_c') [10] ie.,

$$\begin{bmatrix} T^{-1}(A_0+GC)T & T^{-1}E_0 \\ SCT & 0 \end{bmatrix} = \begin{bmatrix} A_c & E \\ C_c & 0 \end{bmatrix} \quad (6.26)$$

where $A_c = \text{block diag } [A_1, A_2, \dots, A_p]$ $C_c = \text{block diag } [C_1, C_2, \dots, C_p]$

$$A_i = \begin{bmatrix} 0 & 0 & \vdots & \vdots & \vdots & 0 & 0 \\ 1 & 0 & \vdots & \vdots & \vdots & 0 & 0 \\ 0 & 1 & \vdots & \vdots & \vdots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \vdots & \vdots & \vdots & 1 & 0 \end{bmatrix}_{n_i \times n_i} \text{ and } C_i = [0 \ 0 \ \dots \ 0 \ 1]_{1 \times n_i}$$

Define the $n \times r$ matrix E_c such that

$$E_c = \text{block diag } [E_1, E_2, \dots, E_r], \quad E_i = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}_{n_i \times 1}$$

The triple (C_c, A_c, E_c) has the following properties:

$$1) \ker[A_c, E_c] = \ker[A_c, E] \quad 2) V^*(A_c, E_c; \ker C_c) = 0$$

Property 1) can be easily established from assumption (6.24a) by noting that the null spaces of $[A_c, E_c]$ and $[A_c, E]$ are both spanned by the p unit vectors corresponding to the zero columns of A_i , $i=1, 2, \dots, p$. Property 2) follows from the canonical structure of the triple (C_c, A_c, E_c) either by applying algorithm (2.9) or theorem 5.7. Because of property 1) a nonsingular matrix H_1 exists such that $H_1 A_c = A_c$ and $H_1 E = E_c$. Let $H_2 = TH_1 T^{-1}$ and $H = H_2 H_0$. This choice of the nonsingular matrix H meets the requirement of part 1 for, the transformations (6.26) (with the new output injection $G^* = H_2 G$) applied to the triple $(C, H F_1, H E_1)$ results in a canonical system

(C_c, A_c, E_c) which is strongly observable and this property is invariant under transformations (6.26).

2) «Only if» The necessity of (6.25) follows from theorem 5.7 as in the proof of part 1

«If» When $C \neq 0$ the construction in part 1 can be applied to yield a system which is strongly observable and hence strongly detectable. If $C=0$ H can be chosen as in theorem 6.9 to make (6.4) detectable.

The foregoing theorems complete the development of unknown input observability of decomposed systems by giving conditions for the three fundamental concepts: Strong observability, strong detectability and known initial state observability. It is interesting to note that the implications established in section 4.3 between the three types of observability are preserved for decomposed systems. More precisely we have $(6.24) \Rightarrow (6.25) \Rightarrow \text{rank } E_1 \leq \text{rank } C$ where the first implication turns into an equivalence when $C \neq 0$. The generic solvability conditions for the three types of observability coincides and is given by: $r \leq p$.

Since strong observability is equivalent to the absence of finite zeros, the construction given in theorem 6.13 can be used for the synthesis of transfer functions of all pole type. The possibility of assigning the poles as well can be investigated in future research.

As pointed out in section 4.3 strong observability or strong detectability is not sufficient for the existence of full state PID_k , $k \geq 0$ observers. The conditions for the existence and the construction of a dynamic subsystem satisfying the requirements of theorem 4.9 are presented in theorem 6.14. As in the case of known-initial state observers it is shown that the

conditions are independent of the number of differentiators employed.

Theorem 6.14: Let $z=x$ in (6.4). Define the sequence of subspaces V^k $k=1,2,\dots$ as in theorem 6.11. Then the following statements are equivalent.

- (i) There exists a dynamic subsystem such that a PID_{k-1} observer exists for (6.4)
- (ii) There exists a nonsingular matrix H such that $\text{im}HE_1 \cap V^k = 0$ and (6.4) is strongly detectable
- (iii) $\text{rank} \begin{bmatrix} F_1 & E_1 \\ C & 0 \end{bmatrix} = n + \text{rank} E_1$

Proof: «(i) $\Leftrightarrow$ (ii)» Follows from theorem 4.10.

«(ii) $\Rightarrow$ (iii)» Proved in theorem 6.13.

«(iii) $\Rightarrow$ (ii)» If $C=0$ since $E_1=0$ as well, the conditions can be satisfied by choosing H such that $\sigma(HF_1) \subset C_0$. Thus assume that $C \neq 0$. The proof is in two parts. First, consider the case $\text{rank} E_1 < \text{rank} C$. Choose nonsingular matrix H_0 such that the pair $(C, H_0 F_1)$ is observable and obtain the canonical form (6.26). Define the $n \times r$ matrix $E_c = [e_1, e_2, \dots, e_r]$ such that the k^{th} column

$$e_k = \begin{bmatrix} e_k^{(1)} \\ \vdots \\ e_k^{(p)} \end{bmatrix} \text{ of } E_c \text{ is partitioned according to the}$$

partitioning of A_c as

$$e_k^{(i)} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}_{n_k \times 1}, \quad e_k^{(k+1)} = C'_{k+1} \text{ and } e_k^{(i)} = 0 \text{ if } i \neq k \text{ or } i \neq k+1.$$

The intergers $n_i=1,2,\dots,p$ are defined by the canonical form (6.26). In this case we still have 1) $\ker[A_c \ E_c]=\ker[A_c \ E]$, 2) $V^*(A_c, E_c; \ker C_c)=0$ and moreover 3) $\ker C_c \cap \text{im} E_c=0$. Thus the matrix $H=H_2 H_0$ constructed as in the proof of theorem 6.13 makes (6.4) strongly observable and satisfies the requirement $V^* \cap \text{im} H E_1=0 \ k \geq 1$.

Next, assume that $\text{rank} E_1 = \text{rank} C$. Choose nonsingular matrix H_0 such that $\ker C \cap \text{im} H_0 E_1=0$ then $R^*(H_0 F_1, H_0 E_1; \ker C)=0$ and $V^*(H_0 F_1, H_0 E_1; \ker C)=\ker C$. By theorem 2.2 $\sigma(H_0 F_1 + H_0 E_1 F; \ker C)$ is independent of $F \in \underline{F}(\ker C)$. Introduce the complexification X_c of X [1, pp.2]. Let $s_i \in \mathbb{C}$, $x_i \in \ker C$ $i=1, \dots, d(\ker C)$ be the eigenvalues and associated generalized eigenvectors of $H_0 F_1 + H_0 E_1 F$, $F \in \underline{F}(\ker C)$. By assumption $s_i \neq 0$ for $i=1 \dots d(\ker C)$. Let $\{z_1, z_2 \dots z_r\}$ be a basis for $\text{im } H_0 E_1$. Define a nonsingular matrix $H_1: X_c \rightarrow X_c$ such that $H_1 x_i = \begin{cases} -x_i, & \text{Re}(s_i) > 0 \\ x_i, & \text{Re}(s_i) < 0 \end{cases}$ and if $\text{Re}(s_i)=0$ $H_1 x_i = \begin{cases} j x_i, & \text{Im}(s_i) > 0 \\ -j x_i, & \text{Im}(s_i) < 0 \end{cases}$ for $i=1 \dots d(\ker C)$ and $H_1 z_i = z_i$ for $i=1, 2 \dots r$. Since eigenvalues occur in complex conjugate pairs a real matrix H_1 can be defined in this way. Finally, let $H=H_1 H_0$ then $V^* \cap \text{im} H E_1 = \ker C \cap \text{im} H_0 E_1=0 \ k \geq 1$ and $V_b^*(H F_1, H E_1; \ker C)=0$ since $\sigma(H F_1 + H E_1 F; \ker C) \subset \mathbb{C}_0$ for all $F \in \underline{F}(\ker C)$. This completes the construction of H when $\text{rank} E_1 = \text{rank} C$.

Although strong observability, strong detectability and the set of zeros of (6.4) are invariant under linear state variable feedback and output injection transformations they can be altered by the transformation H which corresponds to the change of the dynamic subsystem as demonstrated in the foregoing theorems. This shows that the idea of system decomposition can be used as a powerful design technique.

The discussion in this section is limited to full state PID class of observers. Theorem 6.11 and theorem 6.14 completely give existence conditions of full state PID_k , $k \geq 0$ observers for decomposed systems. The proof of theorem 6.14 shows that under the additional condition $\text{rank} E_1 < \text{rank} C$ it is possible to design an observer with arbitrarily assignable error poles. These results can also be extended to observers estimating an arbitrary function, $z = Dx$, of the state vector. In fact existence conditions of known-initial state I and PI observers immediately follow from the solvability conditions given for DDP with state feedback and DDP with state feedback and disturbance feedforward using the duality relationships established in section 5.1. On the other hand for $q=0$ case it follows from the proof of theorem 6.9 that the smallest undetectable subspace which can be attained by changing the nonsingular matrix H is given by $\ker F_1 \cap \ker C$. Further study is needed to generalize this result to the $q \neq 0$ case.

The main results of this section are displayed in Table 6.2.

Table 6.2. Disturbance decoupled estimation in decomposed systems

	The system (6.4) can be made	Necessary and sufficient conditions
UNKNOWN INPUTS	Strongly observable	$\text{rank} \begin{bmatrix} F_1 & E_1 \\ C & 0 \end{bmatrix} = n + \text{rank} E_1, C \neq 0$
	Strongly detectable	$\text{rank} \begin{bmatrix} F_1 & E_1 \\ C & 0 \end{bmatrix} = n + \text{rank} E_1$
	Known-initial state-observable	$\text{rank} E_1 \leq \text{rank} C$
KNOWN INPUTS	Observable	$\text{rank} \begin{bmatrix} F_1 \\ C \end{bmatrix} = n, C \neq 0$
	Detectable	$\text{rank} \begin{bmatrix} F_1 \\ C \end{bmatrix} = n$

6.4 CONCEPTUAL ALGORITHMS AND NUMERICAL EXAMPLES

In order to emphasize the constructive character of the results presented in the previous sections two conceptual algorithms are given which can be used to construct dynamic subsystems satisfying the desired design objectives. Since there are similarities in the procedures given for different problems the algorithms are selected as being representative of the class of problems which they belong to. They can be easily translated into matrix language since they require nothing more than elementary row and column operations and solving linear matrix equations. The input of the algorithms is the matrices of the algebraic subsystem (6.1); the output is the nonsingular matrix H defining the dynamic subsystem (6.2).

Algorithm 6.1: Disturbance decoupling by state feedback:
The proofs of lemma 6.1-lemma 6.3 theorem 6.5 and theorem 6.8 lead to the following construction steps.

Step 1: Let $K = \ker D$. Proceed if (6.8) is satisfied otherwise the problem is not solvable.

Step 2: Determine a subspace W satisfying $F_1 K = F_1 K \cap (\text{im} E_1 + \text{im} F_2) \oplus W$.

Step 3: Compute a nonsingular matrix H such that $H(\text{im} E_1 + W) \subset K$.

Note 6.1: The construction of a nonsingular matrix H is completed in steps 1-3. If a positive definite and symmetric matrix is desired the construction can be continued by steps 4-6. However failure of the test in step 4 does not indicate insolvability of the problem. As noted in remark 6.5 it is generically satisfied under the assumption (6.8).

Step 4: Check whether $(\text{im} E_1 + W) \cap K^\perp = 0$. If this condition holds a positive definite, symmetric matrix H can be determined by lemma 6.3.

Step 5: Find a nonsingular matrix A such that $A(\text{im} E_1 + W) \subset K$ and $AK^\perp \subset K^\perp$.

Step 6: Let $H = A'A$

The next example illustrates solvability of disturbance decoupling problem in decomposed systems.

Example 6.2: Consider the algebraic system defined by

$$F_1 = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad F_2 = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} \quad D = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$E_1 = \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} \quad C = [1 \quad 0 \quad 0] \quad (6.27)$$

In order to verify the generic solvability conditions of theorem 6.7 we investigate solvability of disturbance decoupling problem by state feedback in the parameter space consisting of the data points $E_1 = [e_1 \ e_2 \ e_3]' \in \mathbb{R}^3$. In fact in this space condition (6.8) fails only on the variety $e_1=0$. Hence DDP by state feedback is generically solvable as predicted by theorem 6.7. Assuming $e_1 \neq 0$ it is found that the space W at step 2 of Algorithm 6.1 is given by $W=0$. Since the condition at step 4 is satisfied a positive definite matrix can be computed following steps 5 and 6. This gives

$$A = \begin{bmatrix} e_1 & 0 & 0 \\ -e_2 & e_1 & 0 \\ -e_3 & 0 & e_1 \end{bmatrix} \quad \text{and} \quad H = A'A = \begin{bmatrix} e_1^2 + e_2^2 + e_3^2 & -e_1 e_2 & -e_1 e_3 \\ -e_1 e_2 & e_1^2 & 0 \\ -e_1 e_3 & 0 & e_1^2 \end{bmatrix}$$

The computation of the feedback control law is performed using the methods described in Chapter 3. Since $\ker C \cap \ker D = 0$ is HF_1 -invariant theorem 3.4 is applicable in this case. In fact $\ker D$ is $(HF_1 + HF_2 KC)$ -invariant for $K = \frac{1}{e_1} \begin{bmatrix} e_2 - e_1 \\ e_3 - e_2 + e_1 \end{bmatrix}$. Hence $u = Ky$ is the desired feedback control law.

For comparison purposes we also note that for the present example filtering problem is solvable if and only if $e_1 = e_2$ and $e_3 = 0$. This indicates how solvability boundaries are enhanced by considering feedback.

Algorithm 6.2: Design of dynamic subsystem to make the composite system strongly observable The proof of theorem 6.13 justifies the following construction steps.

Step 1: Continue if (6.24) is satisfied, otherwise stop since the problem is not solvable.

Step 2: Eliminate any linearly dependent columns (rows) in the matrices $E_1(C)$ and continue to the construction with the new matrices $E_1^*(C^*)$ obtained in this way.

Step 3: Following the sufficiency proof of theorem 6.9 find a nonsingular matrix H_0 such that the pair $(C^*, H_0 F_1)$ is observable.

Step 4: Write $A_0 = H_0 F_1$ and $E_0 = H_0 E_1^*$

Step 5: By state space, output space and output injection transformations T , S and G respectively on the triple (C^*, A_0, E_0) bring it to the canonical form (C_c, A_c, E) described by (6.26). Store T for future use.

Step 6: Construct the $n \times r$ matrix E_c as described in theorem 6.13 and determine nonsingular matrix H_1 satisfying $H_1 [A_c \ E] = [A_c \ E_c]$.

Step 7: Let $H_2 = T H_1 T^{-1}$ and $H = H_2 H_0$.

Example 6.3: Suppose that for the algebraic system (6.27) given in Example 6.2 it is required to find a dynamic subsystem which gives rise to a strongly observable overall system. Applying Algorithm 6.2 it is readily found that condition (6.24) is satisfied if and only if $e_3 \neq 0$. Thus assume that $e_3 \neq 0$. We choose

$H_0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ for the nonsingular matrix which makes the pair $(C, H_0 F_1)$ observable. The transformations T and G

which bring the triple (C, A_0, E_0) to the canonical form (6.26) are given by $T = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ and $G = \begin{bmatrix} -1 \\ 0 \\ -1 \end{bmatrix}$. The nonsingular matrix H_1 satisfying $H_1[A_c \ E] = [A_c \ E_c]$ is $H_1 = \frac{1}{e_3} \begin{bmatrix} 1 & 0 & 0 \\ -e_2 & e_3 & 0 \\ -e_1 & 0 & e_3 \end{bmatrix}$. Finally, we have $H_2 = TH_1T^{-1}$ and $H = H_2H_0 = \frac{1}{e_3} \begin{bmatrix} e_3 & 0 & -e_1 \\ 0 & 0 & 1 \\ 0 & e_3 & -e_2 \end{bmatrix}$.

For the present example it is also possible to find a nonsingular matrix H which satisfies $\ker C \cap \text{im } HE_1 = 0$ and $V_b^*(HF_1, HE_1; \ker C) = 0$ simultaneously. Following the sufficiency proof of theorem 6.14 such a matrix is found to be $H = \begin{bmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{bmatrix}$.

CONCLUSION

Disturbance decoupling and disturbance decoupled estimation problems have been studied for finite dimensional, linear, time invariant systems using the geometric approach. When concluding it is appropriate to point out some of the original results and suggest possible extensions and related areas for future research.

The mathematical background required for the geometric approach has been summarized in Chapter 2 and in Chapter 3 disturbance decoupling problem has been reviewed for regular (=undecomposed) state space systems. Disturbance decoupled estimation problem has been investigated in Chapter 4. Existence conditions of PID class of observers have been given in section 4.1 which are specialized to full state observers in section 4.3 in order to express existence conditions in terms of strong detectability and strong observability. Specifically, theorem 4.7 shows that strong detectability is equivalent to the existence of full state PID observers and in theorem 4.9 additional conditions beside strong detectability have been presented for the existence of full state PID_k , $k \geq 0$ observers. These results extend similar conditions given for systems subject to only control inputs to unknown input systems. A new type of observability called «known-initial state-observability» has been defined (Definition 4.6) and it has been shown that this concept is related to the existence of observers when the initial state is known. Geometric and frequency domain conditions for known-initial state-observability have been derived in theorem 4.8 and theorem 5.8.

The concept of undetectable subspaces has been introduced in section 4.2. It has been shown that the class of undetectable subspaces is closed under subspace intersection (property 4.2) hence possesses a smallest element. The smallest undetectable subspaces for known and unknown initial state observers have been established in theorem 4.5 and theorem 4.6. These results are completely general and are also valid for nonlinear or infinite dimensional observers. The PID class of observers have been compared in terms of their undetectable subspaces. It is clear from theorem 4.3, 4.4 that the undetectable subspaces get smaller as the number of differentiators increase and the lower bounds of undetectable subspaces can be attained by using a PID observer. However because of the obvious disadvantage of using differentiators, it is important to know whether it is worth to employ PID observers and how much benefit is gained by using a PID observer instead of a PI observer. This question has been precisely answered in Remark 4.4 which gives the relationship between the undetectable subspaces of PI and PID observers. In order to circumvent the drawback of using differentiators design of observers based on delays instead of differentiators can be investigated in future research.

Design of minimal order observers has been considered in section 4.4. A structure consisting of a reduced order PI observer driven by a PD class of observer has been suggested to realize a minimal order PID class of observer. (Fig.4.4) Existence conditions and design procedures of minimal order observers have been presented in theorem 4.10. It has been shown that the minimal orders are given by the dimensions of the smallest undetectable subspaces of PD class of observers and a trade off exists between the observer order and the number of differentiators employed (theorem 4.11).

Since the minimal orders have been derived for full state observers they are only upper bounds for observers estimating an arbitrary function of the state vector. However the idea of realizing a minimal order observer as a cascade of a PD class and PI observers can also be applied to this more general case. As a side result an easy method has been given for the computation of (A,E) -invariant and internally stabilizable (A,E) -invariant subspaces in $\ker C$ (lemma 4.8).

In section 4.5 the design procedures developed for minimal order observers using geometric approach have been translated into matrix language suitable for computer calculations. The algorithm summarized in corollary 4.4 enables the computation of observer matrices directly in terms of the submatrices of a system obtained from the original one by state space transformations. By incorporating numerically reliable linear equation solving and eigenvalue placement routines the algorithm can be turned into an efficient software package.

The purpose of Chapter 5 is to provide the link between the geometric and the frequency domain approaches to disturbance decoupling and disturbance decoupled estimation problems. With this objective the equivalence of disturbance decoupling and disturbance decoupled estimation problems to solvability of rational matrix equations has been established in theorem 5.1, 5.4. As a by-product of this analysis new methods have been suggested for the design of feedforward control laws and inverse systems.

In Chapter 6 disturbance decoupling and disturbance decoupled estimation problems have been investigated for decomposed systems consisting of dynamic and algebraic

parts. It is shown that the solvability range of the problems is enlarged by considering system decomposition. Open loop and closed loop disturbance decoupling problems have been defined in section 6.2. The open loop disturbance decoupling, which is called filtering problem has been solved in theorem 6.2 while theorem 6.3 gives the solution under the additional condition of internal stability. The closed loop disturbance decoupling has been investigated by constant state feedback (theorem 6.4), constant state feedback with disturbance feedforward (theorem 6.5) and by dynamic measurement feedback (theorem 6.6). All results are completely constructive and the sufficiency parts of the theorems give design procedures for the dynamic subsystems as demonstrated in Algorithm 6.1. In theorem 6.8 a set of necessary and sufficient conditions have been obtained for the solvability of filtering and feedback problems by passive and lossless dynamic subsystems. However these conditions are not easily verifiable like the ones given in corollary 6.1 and corollary 6.2. Further study is needed to get more refined conditions. Lemma 6.3 which establishes a basic property of positive definite and symmetric operators may be a good starting point for further studies in this direction. The results have been compared with regular state space systems in terms of their generic solvability. Unlike regular systems generic solvability conditions have been obtained for all feedback problems.

Disturbance decoupled estimation in decomposed systems have been discussed in section 6.3. Necessary and sufficient conditions for the existence of a dynamic subsystem which gives rise to a strongly observable, strongly detectable and known-initial state-observable overall system have been given in theorem 6.12 and theorem 6.13. These results have been utilized in

theorem 6.11 and theorem 6.14 to establish existence conditions of full state PID class of observers for decomposed systems. A number of side results have also been obtained for the controllability, stabilizability (theorem 6.10) and invertibility (theorem 6.12) of decomposed systems. Some of the synthesis procedures developed in this section have been illustrated with Algorithm 6.2.

The advantage of considering system decomposition is in the additional degree of freedom gained by the variable structure of the dynamic subsystem. The ability to vary the dynamic subsystem by changing the nonsingular matrix H gives great flexibility in design problems. To assign eigenvalues and eigenvectors, to reject disturbances from the output or to realize a given transfer function matrix are only a few of the design objectives which can be achieved by system decomposition. In this sense the proposed methods may be compared with the commonly used methods like linear state variable feedback, output feedback etc. In most cases the idea of system decomposition either improves the solvability range of the problem or makes solvable an otherwise unsolvable problem. The results presented and techniques used in this work constitute a basis for the investigation of related problems like noninteraction [9], model matching [52, Chapter 8] or almost disturbance decoupling [26-28] which have been studied by the classical methods before.

It may be interesting to investigate practical ways to implement the decomposition for any given dynamical system. Although most physical systems can be viewed as the interconnection of dynamic and algebraic subsystems as described in section 6.1 there may be constraints in practice not taken into account in the present work. For

example the matrix H defining the dynamic subsystem may not be completely arbitrary because of some topological constraints or due to only limited number of variable elements in the dynamic subsystem. In this case a submatrix of H may represent the variable part of the dynamic subsystem. Nevertheless it is worth to spend effort to circumvent such difficulties in view of the solutions offered by system decomposition to practical synthesis problems.



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