MATHEMATICAL MODELING OF ARTERIAL TISSUE

M.Sc. Thesis by
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SYMBOL LIST

\[ u, x, v, w \ldots \] : Vector
I : Identity tensor
O : Zero tensor
F : Second order tensor
W : Skew-symmetric tensor
\( \tau \) : Kirchhoff stress tensor
C : Cauchy-Green tensor
E : Euler-almansi strain tensor
P : The first piola-kirchoff stress tensor
S : The second piola-kirchoff stress tensor
L : Velocity gradient tensor
D : Strain rate tensor
\( \sigma \) : Cauchy stress tensor
Q : Orthogonal tensor
\( \Psi \) : Strain energy function
B : Left cauchy green tensor
n : Outward normal
T : Cauchy traction vector
i,j : Index parameters
\( x \) : Cross product
\( \otimes \) : Tensor product
\( T^{(n)} \) : Traction vector
\( \beta_0 \) : Boundary surface
J : Volume ratio
\( \omega \) : Current domain
\( \Omega_o \) : Reference domain
\( \varepsilon \) : Internal energy density
\( \nabla_o \) : The referential del operator
\( \delta \) : Kronocker delta
\( q_0 \) : The referential heat flux vector
\( D_{int} \) : Inner energy
p : Intermediate lagrange multiplier
\( \lambda \) : Stretch ratio
\( \mu_i, \alpha_i \) : Ogden material parameter
G : Neo-hookean material coefficient
\( C_{10}, C_{01} \ldots \) : Material model coefficients.
MATHEMATICAL MODELING OF ARTERIAL TISSUE

SUMMARY

In this study, we examined the constitutive relations for mechanical response of arterial tissues. This work could be investigated in two parts as mathematical and experimental requirements for arterial mechanics.

The preliminaries of the continuum mechanics theory were explained in detail to prepare a background for the reader. The artery is modeled as a nonlinearly elastic circular cylindrical tube which consists of a single orthotropic layer. It is also important to note that the mathematical theory of arterial elastostatics is not only based on the experimental approach which is limited with the experimental capabilities.

In the experimental part, we used a testing system including universal testing machine, load cell, strain gages, water boiler, 3D image correlation system. During the experiments we encountered with three main problems. We used sheep, cow and pig arteries for arterial testing. The artery was subjected to different loading conditions namely extension, inflation and torsion.

The data which was obtained from experiments were applied to the constitutive relations in the literature and mechanical response of the tissue was examined.

It must be noticed that experimental results affect directly to the material parameters and stability in the theoretical model. Therefore, the experimental system and its capabilities are the main parameters while determining the mechanical behavior of soft tissues.
DAMAR DOKUSUNUN MATEMATİKSEL MODELLENMESİ

ÖZET

Bu çalışmada damar dokusunun mekanik özellikleri ve bu özellikleri ifade eden denklemler incelenmiştir. Çalışma teorik ve deneysel olarak iki kısımdan meydana gelmektedir.

Sürekli ortamlar mekaniğinin temel yaklaşımları okuyucuya bir altyapı oluşturması için ayrıntılı olarak verilmiştir. Damar tek bir ortotropik katmandan oluşan nonlineer elastik dairesel silindir olarak modellenmiştir. Çalışmanın matematiksel teorisi deneysel yaklaşım baz alınarak oluşturulmuştur. Deneysel uygulamalar ise tamamen teknolojik imkanlarla sınırlıdır.

Deneysel çalışmalar sırasında universal test makinası, kuvvet ve basınç ölçüm aletleri, sıcaklık kontrolü su ısıtıcı, 3 boyutlu kamera ölçüm sistemleri kullanılmıştır. Deneyler sırasında üç ana promlemle karşılaşılmıştır. Testler sırasında koyun damarları kullanılmış ve bu damalar çekme, şişirme ve burulma yüklerine maruz bırakılmıştır.

Deneylerden elde edilen veriler damar dokusu için belirlenen teorik ifadelerde kullanılarak malzemenin davranışı incelenmiştir.

Deneysel sonuçların teorik modeldeki malzeme parametrelerine etkisi dikkate alınmalıdır. Kullanılan deneysel sistemler ve bu sistemlerin yetenekleri yumuşak dokuların mekanik özelliklerinin belirlenmesindeki en önemli unsurlardan birisidir.
1. INTRODUCTION

Appropriate physical models of biological soft tissue organs and associated numerical algorithms are of great interest for clinicians and medical product suppliers; in particular, in the fields of orthopedics and cardiovascular medicine. However, numerous existing models of soft tissues suffer either from lack of experimental data, do not fulfill mathematical and mechanical requirements, or are not suitable for practical finite element implementations. In order to achieve clinically meaningful results an engineering approach requires,

- A comprehensive experimental data base of the material to be modeled,
- A physical model that captures the essential mechanical characteristics
- An efficient numerical model with the aim to study the effects of medical treatments subject to certain procedural parameters in a robust and reliable way since as in other fields of applied mechanics the computational approach offers an essential alternative in situations where experiments are either too costly or even impossible.

The great majority of diseases in the (western) world, such as atherosclerosis and degeneration of intervertebral discs are diseases of soft tissues. Hence, the multidisciplinary field of soft tissue research is of crucial scientific, medical and socioeconomic importance. The fast progress in the developments of hardware and software make it possible to thoroughly investigate biological soft tissues and their pathologies on a computational basis. Since soft tissues are biological materials, which fulfill mechanical purposes and adapt to their mechanical environment (growth, remodeling and morphogenesis), it is of fundamental importance to identify the complex interactions of mechanical and biological responses. This challenging field of mechanobiology bears the potential to be an important contributor to tissue engineering and clinical medicine [1].
2. BIOMECHANICS OF SOFT TISSUE

2.1. Soft Tissue Physiology

A primary group of tissue called binds, supports and protects our human body and structures such as organs is soft connective tissue. In contrary to other tissues, it is a wide-ranging biological material in which the cells are separated by extra cellular material [1]. Soft tissues may be distinguished from other body tissues like bones for their flexibility, their soft mechanical properties. Examples for soft tissues are skeletal muscles, tendons, ligaments, blood vessels, skin or articular cartilage among many others. Skeletal muscles are responsible for generating forces to move the skeleton. The role of tendons is to transmit these forces to the bones, whereas that of ligaments is to handle the stability of joints and restrict their ranges of motion. Blood vessels are prominent organs composed of soft tissues which have to distend in response to pulse waves. Among other functions, skin supports internal organs and protects them from injury while allowing considerable mobility [2].

Figure 2.1: Schema of the "musculo-elastic fascicle" that was proposed by Clark & Glagov to be the basic structural and functional unit of the media in an elastic artery; E denotes elastin, Ce denotes smooth muscle cells, and F denotes collagen bundles which exist between the elastin sheets [3]
Figure 2.2: Fibrous structure of tendon [4]

Soft tissues are all essentially composed of collagen. A study on this issue, made by Elliot in 1965 [5], had reported that collagen represents among 75% dry weight in human tendons. The remaining weight is shared between elastin, reticulin, and a hydrophilic gel called ground substance. However, soft tissues exhibit large differences in their mechanical properties. This observation has led to the conclusion that the mechanical properties of soft tissues are due to their structure rather than to the relative amount of their constituents. As described in Fig. 2.2, the tendon fascicles are organized in hierarchical bundles of fibers arranged in a more or less parallel fashion in the direction of the effort handled. A close look at the fiber networks shows that this parallel arrangement is more irregular and distributed in more directions for ligaments than for tendons [5].

In addition, skin is organized into two biaxial membranes: a relatively thin layer of stratified epithelium called the epidermis, and a thicker layer of disordered wavy coiled collagen and elastin fibers called the dermis (Fig. 2.3). The elastin fibers play an
important role in the skin's response at low strains: they are the first stretched when the tissue is strained whereas the collagen fibers are still crimped.

Figure 2.3: The skin layers
They are also considerably more pliant but can be reversibly stretched to more than 100 per cent. The epidermis and dermis are also connected by collagen fibers to a subcutaneous fatty tissue, called the hypodermis. This layer is sometimes considered as a third layer of skin. It appears as a honeycomb fat container structure connected to the fascia which surrounds the muscle bundles. It is the hypodermis that provides the skin's loose flexible connection with the other internal soft tissues, whereas the upper layers are more resistant to protect from injuries.

2.2. Mechanical Properties

As aforementioned biological tissues are roughly divided into: hard tissues like bone and tooth, and soft tissues such as skin, muscle, blood vessel, and lung. Hard tissues contain mineral, whereas soft tissues do not. Because of this, they have very different mechanical properties. One of the major differences in mechanical properties is that soft tissues are much more deformable than hard tissues. Therefore, infinitesimal deformation theories that are applied to metals and hard plastics cannot be used for soft
tissues; instead, finite (large) deformation theories that are basically developed for rubber elasticity are often used to describe the mechanical behavior of soft tissues [6].

2.2.1. Inhomogeneous structure

Biological soft tissues being composed mainly of cells and intercellular substances, the latter consisting of connective tissues such as collagen and elastin, and ground substance (hydrophilic gel), which have different physical and chemical properties. Thus their contents differ from tissue to tissue and even from location to location within a tissue. Leading the mechanical properties depend both on tissue and on site.

Collagen, which is defined as a protein containing sizeable domains of triple-helical conformation (tropo-collagen) and is synthesized by fibroblasts, vascular smooth muscle cells, and so on, is a basic structural element for biological tissues in animals. It gives mechanical integrity and strength to our bodies and is present in a variety of structural forms in different tissues and organs.

Elastin is also a long-chained protein similar to collagen, but has a much less integrated structure than collagen. Its mechanical properties are greatly different from those of collagen and it has much less strength and much more flexibility than collagen [6].

2.2.2. Nonlinear elasticity

The main property of soft tissues may be outlined as referring to their nonlinear elasticity. Kwan described the phenomenon as follows: "Under uniaxial tension, parallel-fibered collagenous tissues exhibit a non-linear stress-strain relationship characterized by an initial low modulus region, an intermediate region of gradually increasing modulus, a region of maximum modulus which remains relatively constant, and a final region of decreasing modulus before complete tissue rupture occurs. The low modulus region is attributed to the removal of the undulations of collagen fibrils that normally exist in a relaxed tissue. As the fibrils start to resist the tensile load, the modulus of the tissue increases. When all the fibrils become taut and loaded, the tissue modulus reaches a maximum value, and thereafter, the tensile stress increases linearly with increasing strain. With further loading, groups of fibrils begin to fail, causing the decrease in modulus until complete tissue rupture occurs." A typical tensile curve is shown in Fig. 2.4. From a
functional point of view, the first parts of the curve are more useful since they correspond to the physiological range in which the tissue normally functions [2].

Also, Holzapfel emphasizes the intermolecular cross links of collagen gives the connective tissues the strength which varies with age, pathology, etc. Table 2.1 shows the correlation between the collagen content in the tissue, % dry weight, and its ultimate tensile strength [7].

Table 2.1. Mechanical properties [8,5,9] and associated biochemical data [10] of some representative organs mainly consisting of soft connective tissues.

<table>
<thead>
<tr>
<th>Material</th>
<th>Ultimate tensile strength [MPa]</th>
<th>Ultimate tensile strain [%]</th>
<th>Collagen (% dry weight)</th>
<th>Elastin (% dry weight)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Tendon</td>
<td>50-100</td>
<td>10-15</td>
<td>75-85</td>
<td>&lt;3</td>
</tr>
<tr>
<td>Ligament</td>
<td>50-100</td>
<td>10-15</td>
<td>70-80</td>
<td>10-15</td>
</tr>
<tr>
<td>Aorta</td>
<td>0.3-0.8</td>
<td>50-100</td>
<td>25-35</td>
<td>40-50</td>
</tr>
<tr>
<td>Skin</td>
<td>1-20</td>
<td>30-70</td>
<td>60-80</td>
<td>5-10</td>
</tr>
<tr>
<td>Articular Cartilage</td>
<td>9-40</td>
<td>60-120</td>
<td>40-70</td>
<td>-</td>
</tr>
</tbody>
</table>

2.2.3. Viscoelasticity

The aforementioned experiment reveals the relationship between stress and strain in the static case. However, when the equilibrium is not reached, a history-dependent component exists in the mechanical behavior of living tissues. When measured in dynamic extension, the stress values appear higher than those at equilibrium, for the same strain. The resulting tensile curve appears steeper than the one at equilibrium (Fig. 2.4). When a tissue is suddenly extended and maintained at its new length, the stress gradually decreases slowly against time. This phenomenon is (uniaxial) stress relaxation (Fig. 2.5a). When the tissue is suddenly submitted to a constant tension, its elongation velocity decreases against time until equilibrium. This phenomenon is called creep (Fig. 2.5b). Under cyclic loading, the stress-strain curve shows two distinct paths corresponding to the loading and unloading trajectories. This phenomenon is named hysteresis (Fig. 2.5c). As a global statement, the stress at any instant of time depends not only on the strain at that time, but also on the history of the deformation. These mechanical properties, observed for all living tissues, are common features of a physical phenomenon named viscoelasticity [5].
2.2.4. Anisotropy

Since collagen and elastin are long-chained high polymers, they are intrinsically anisotropic. Moreover, not only their fibers but also cells are oriented in tissues and organs in order that they function most effectively. Inevitably, almost all biological tissues are mechanically anisotropic. For example, skin has very different properties in two orthogonal directions; see Figure 4 (Lanir and Fung (1974) [13] and Tong and Fung (1976) [14]); it cannot deform much in the direction of Langer's line (Ridge and Wright (1966) [15]), but can deform much more in the perpendicular direction. Collagen fibers in the articular cartilage are preferentially oriented to the split line and, therefore, this tissue also shows anisotropic behavior [6].

Figure 2.4: Load-extension curve and influence of the train rate [5,11]

Figure 2.5: Viscoelastic behaviors [11,12]
2.2.5. **Strain rate insensitivity**

Because of the viscoelastic characteristics of soft tissues, they show different properties under different strain rates (test speed). In fact, higher strain rates give higher stresses. However, such a strain rate effect is not very large in biological soft tissues; there are not very large differences in the stress-extension ratio curves across three orders of magnitude of the tensile speed. In addition, the area of the hysteresis loop does not depend upon strain rate [6].

2.2.6. **Incompressibility**

Most biological soft tissues have a water content of more than 70%. Therefore, they hardly change their volume even if load is applied, and they are almost incompressible. The incompressibility assumption is applicable to most biological soft tissues; it has been confirmed experimentally in arterial wall [16]. However, for example, articular cartilage, because the tissue is micro-porous and, therefore, water can enter and leave pores depending upon load [4]. The incompressibility assumption is very important in the formulation of constitutive laws for soft tissues because the product of all principal stretches is always zero.
3. CONTINUUM MECHANICS

3.1. Basic Algebra of Vectors and Tensors

3.1.1. Direct notation

A vector is a mathematical quantity possessing characteristics of magnitude and direction. For this reason, vectors are often represented by arrows, the length of which denotes the magnitude [17].

In other words, a vector designated by \( \mathbf{u}, \mathbf{v}, \mathbf{w} \ldots \) is a directed line element in space. It is a model for physical quantities having both direction and length, for example, force, velocity or acceleration. The two vectors that have the same direction and length are said to be equal [18].

The sum of vectors yields a new vector, based on the parallelogram law of addition. The following properties,

\[
\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u} \quad \text{(3-1)}
\]

\[
(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w}), \quad \text{(3-2)}
\]

\[
\mathbf{u} + \mathbf{0} = \mathbf{u}, \quad \text{(3-3)}
\]

\[
\mathbf{u} + (-\mathbf{u}) = \mathbf{0}, \quad \text{(3-4)}
\]

Hold, where “\( \mathbf{0} \)” denotes the unique zero vector with unspecified direction and zero length [18].

Besides addition and subtraction, which can be accomplished using the parallelogram law with the arrow representation, three “vector operations” of utmost importance are the scalar (or, dot) product,
\[ \mathbf{u} \cdot \mathbf{v} = \mathbf{a} \quad \text{where} \quad \mathbf{a} = \| \mathbf{u} \| \| \mathbf{v} \| \cos \theta \quad (3-5) \]

The vector (or, cross) product,

\[ \mathbf{u} \times \mathbf{v} = \mathbf{w} \quad \text{where} \quad \mathbf{w} = \| \mathbf{u} \| \| \mathbf{v} \| \sin(\theta) \mathbf{e} \quad (3-6) \]

And the tensor (or, dyadic) product,

\[ \mathbf{u} \otimes \mathbf{v} = \mathbf{T} \quad (3-7) \]

Herein, \( \theta \) is the angle between vectors \( \mathbf{u} \) and \( \mathbf{v} \), \( \| \cdot \| \) denotes the magnitude of a vector, \( \mathbf{e} \) is a unit vector (i.e., \( |\mathbf{e}| = 1 \)) perpendicular to the plane containing \( \mathbf{u} \) and \( \mathbf{v} \), \( \mathbf{T} \) is a second-order tensor. The magnitude of the vector \( \mathbf{w} \) is found by \( |\mathbf{w}| = (\mathbf{w} \cdot \mathbf{w})^{1/2} \), and a unit vector \( \mathbf{e} \) in the direction of \( \mathbf{w} \) can be found via \( \mathbf{e} = \frac{\mathbf{w}}{|\mathbf{w}|} \). Two vectors, \( \mathbf{u} \) and \( \mathbf{v} \) are aid to be orthogonal if \( \mathbf{u} \cdot \mathbf{v} = 0 \).

Collectively these equations above reveal that two vectors can “operate” on one another to yield a scalar, a new vector, or a second order tensor. Higher order tensors, as, for example, the third order tensor \( \mathbf{u} \otimes \mathbf{v} \otimes \mathbf{w} \), are equally easy to obtain [17].

Recall that the dot product commutes, that is

\[ \mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u} \quad (3-8) \]

In contrast,

\[ \mathbf{u} \times \mathbf{v} = -\mathbf{v} \times \mathbf{u} \quad (3-9) \]

and in general,

\[ \mathbf{u} \otimes \mathbf{v} \neq \mathbf{v} \otimes \mathbf{u} \quad (3-10) \]

Also note that,

\[ (\mathbf{w} \otimes \mathbf{u}) \cdot \mathbf{v} = \mathbf{w} \cdot (\mathbf{u} \cdot \mathbf{v}) = \mathbf{w} \cdot \mathbf{a} \quad (3-11) \]
which shows, for example, that a “dot product” between a second-order tensor \( w \otimes u \) and a vector \( v \), yields a vector in the direction of \( w \) that has a different magnitude. Moreover, the “.” operation takes precedence over the “\( \otimes \)”; thus the parenthesis can be deleted.

The last two equations reveal, therefore, that a second order tensor transforms a vector into a new vector, which is why tensors are called linear transformations.

Many of the basic operations for second order tensors, say \( S \) and \( T \), are similar to those for vectors. For example, recall the basic associative and distributive laws for vectors,

\[
(a.u)v = a(\mathbf{u} \cdot \mathbf{v}) = u.(a.v), \tag{3-13}
\]

\[
(u + v)w = u.w + v.w. \tag{3-14}
\]

These laws are similar for second order tensors, thus

\[
(aS)v = a(S.v) = S.(a.v) \tag{3-15}
\]

and,

\[
(S + T)v = S.v + T.v \tag{3-16}
\]

Satisfaction of these two equations ensures that the set of all second order tensors from a vector space. Likewise,

\[
(au + bv) \otimes w = a(u \otimes w) + b(v \otimes w) \tag{3-17}
\]

Additional operations important for second order tensors include the transpose \((...)^T\), trace \((...)\) and determinant \((...)\). In particular,

\[
(u \otimes v)^T = (v \otimes u) \tag{3-18}
\]
which is to say that the transpose interchanges the order of the vectors that constitute the dyad;

\[ tr(u \otimes v) = u \cdot v \] (3-19)

Thus the trace of a tensor yields the scalar product of the vectors constituting the dyad; and

\[ \det T = \det[T] \] (3-20)

Where \((\ldots)\) denotes a matrix representation of \(T\). The determinant of a tensor thereby yields a scalar, one that equals the determinant of the matrix of components of the tensor. Another scalar measure of a second order tensor is its magnitude, given as

\[ |T| = \sqrt{tr(T.T^T)} \] (3-21)

A second order tensor, say \(w \otimes u\), can also act on another second order tensor, say \(v \otimes x\), to yield a second order tensor, viz;

\[ w \otimes u \cdot v \otimes x = (u \cdot v)w \otimes x = a(w \otimes x) \] (3-22)

or either of two scalars,

\[ w \otimes u : v \otimes x = (w \cdot v)(u \cdot x) \text{ or } \]

\[ w \otimes u \cdot v \otimes x = (w \cdot x)(u \cdot v) \] (3-23)

Note the order of these two operations, each of which is called a double-dot (or scalar) product [38].

Other important relations involving the transpose are

\[ (S + T)^T = S^T + T^T , \] (3-25)

\[ (S \cdot T)^T = T^S S^T , \] (3-26)
\( (S^T)^T \), \hspace{1cm} (3-27) 

and likewise for the trace,

\[
tr(aS + b.T) = atr(S) + btr(T),
\]

\[
(3-28)
\]

\[
tr(S.T) = tr(TS),
\]

\[
(3-29)
\]

\[
tr(S^T) = tr(S),
\]

\[
(3-30)
\]

And for the determinant,

\[
det(aS) = a^3 \det(S),
\]

\[
(3-31)
\]

\[
det(S.T) = \det(S)\det(T),
\]

\[
(3-32)
\]

\[
det(S^T) = \det(S).
\]

\[
(3-33)
\]

Here, it should be noted that a tensor is said to be symmetric or skew-symmetric if, respectively,

\[
U = U^T, \quad W = -W^T.
\]

\[
(3-34)
\]

Every skew-symmetric tensor \( W \) has an associated axial vector \( w \) such that

\[
W \cdot v = w \times v \text{ for all vectors } v. \ [17]
\]

Further, every second order tensor \( T \) can be written as the sum of asymmetric tensor \( U \) and skew-symmetric tensor \( W \), that is,

\[
T = U + W, \text{ where } U = \frac{1}{2}(T + T^T), \quad W = \frac{1}{2}(T - T^T).
\]

\[
(3-35)
\]

It is easy to show, therefore, that

\[
tr(W) = 0, \quad \det(W) = 0.
\]

\[
(3-36)
\]

The square, cube, etc. of a tensor are given by
\[ S^2 = S S, \quad S^3 = S S^2. \] (3-37)

There are two special second order tensors of importance, namely the zero tensor \( \mathbf{O} \) and the identity tensor \( \mathbf{I} \), where

\[ \mathbf{O} \cdot \mathbf{v} = \mathbf{0}, \quad \mathbf{I} \cdot \mathbf{v} = \mathbf{v}. \] (3-38)

That is, the zero tensor transforms all vectors into the zero vectors and the identity tensor transforms all vectors into themselves. Likewise,

\[ \mathbf{O} \mathbf{S} = \mathbf{O}, \quad \mathbf{I} \mathbf{S} = \mathbf{S}. \] (3-39)

The trace and the determinant of the identity tensor arise often. They are

\[ tr(\mathbf{I}) = 3, \quad \det(\mathbf{I}) = 1 \] (3-40)

The inverse of a tensor \((\ldots)^{-1}\) is defined by

\[ \mathbf{S} \mathbf{S}^{-1} = \mathbf{I}, \quad \mathbf{S}^{-1} \mathbf{S} = \mathbf{I}. \] (3-41)

Important relations for the inverse are

\[ (a\mathbf{S})^{-1} = \frac{1}{a} \mathbf{S}^{-1}, \] (3-42)

\[ (\mathbf{S} \mathbf{T})^{-1} = \mathbf{T}^{-1} \mathbf{S}^{-1}. \] (3-43)

Moreover, the transpose and determinant of the inverse of a tensor are given by

\[ (\mathbf{S}^{-1})^T = (\mathbf{S}^T)^{-1}, \quad \det(\mathbf{S}^{-1}) = \frac{1}{\det \mathbf{S}}. \] (3-44)

Note, too, that \((\mathbf{S}^{-1})^T\) is often denoted by \(\mathbf{S}^{-T}\).

Finally, a second order tensor \( \mathbf{Q} \) is called orthogonal if

\[ \mathbf{Q} \mathbf{Q}^T = \mathbf{Q}^T \mathbf{Q} = \mathbf{I} \] (3-45)
That is, if its inverse equals its transpose. Also, the equations 3-31, 3-32 and 3-39 reveal that

\[ \det(Q) = \pm 1. \] (3-46)

An orthogonal tensor is said to be proper if \( \det(Q) = 1 \).

Because many operations on tensors take a special form depending on the type of tensor, it is often useful to introduce the following nomenclature: Let Lin denote all second-order tensors,

Sym all symmetric tensors, Psym all positive-definite symmetric tensors, Skw all skew-symmetric tensors, and Orth all orthogonal tensors with Orth\(^+\) being those that are proper orthogonal. Hence, for example, \( W \in \text{Skw} \) implies that \( W \) is a skew-symmetric tensor.

Vectors \( v \) and scalars \( a \) are similarly denoted by \( v \in V \) and \( a \in R \), which is to say that they are members of the vector space \( V \) or real numbers \( R \), respectively [17].

### 3.1.2. Index notation

So far algebra has been presented in symbolic notation exclusively employing bold face letters. It represents a very convenient and concise tool to manipulate most of the relations used in continuum mechanics. However, particularly in computational mechanics, it is essential to refer vector quantities to a basis. Additionally, to gain more insight in some quantities and to carry out mathematical operations among tensors more readily it is often helpful to refer to components [18].

In order to present coordinate expressions relative to a right-handed and orthonormal system we introduce a fixed set of three basis vectors \( e_1, e_2, e_3 \), (sometimes introduced as \( i, j, k \) called a (Cartesian) basis, with properties

\[
\begin{align*}
  e_1 \cdot e_2 &= e_1 \cdot e_3 = e_2 \cdot e_3 = 0, \\
  e_1 \cdot e_1 &= e_2 \cdot e_2 = e_3 \cdot e_3 = 1
\end{align*}
\] (3-47)

These vectors of unit length which are mutually orthogonal form a so-called orthonormal system. Then any vector \( u \) in the three-dimensional Euclidean space is represented uniquely by a linear combination of the basis vectors \( e_1, e_2, e_3 \), i.e.
\[ \mathbf{u} = u_1 \mathbf{e}_1 + u_2 \mathbf{e}_2 + u_3 \mathbf{e}_3, \quad (3-48) \]

Where the three real numbers \( u_1, u_2, u_3 \) are the uniquely determined Cartesian components of vector \( \mathbf{u} \) along the given directions \( \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3 \), respectively.

Using index notation the vector \( \mathbf{u} \) can be written as \( \mathbf{u} = \sum_{i=1}^{3} u_i \mathbf{e}_i \) or in an abbreviated form by leaving out the summation symbol, simply as

\[ \mathbf{u} = u_i \mathbf{e}_i = u_1 \mathbf{e}_1 + u_2 \mathbf{e}_2 + u_3 \mathbf{e}_3. \quad (3-49) \]

**Figure 3.1**: Vector \( \mathbf{u} \) with its Cartesian components \( u_1, u_2, u_3 \)

The summation convention says that whenever an index is repeated (only once) in the same term, then, a summation over the range of this index is implied unless otherwise indicated [18].

The index \( i \) that is summed over is said to be a dummy index, since a replacement by any other symbol does not affect the value of the sum. An index that is not summed over in a given term is called a free index. Note that in the same equation an index is either dummy or free. Thus, these relations can be written in a more convenient form as
\[ \mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij} \equiv \begin{cases} 1, & \text{if } i = j \\ 0, & \text{if } i \neq j \end{cases} \quad (3-50) \]

Which defines the Kronecker delta \( \delta_{ij} \). The useful properties are

\[ \delta_i^i = 3, \quad \delta_i^j u_j = u_i, \quad \delta_i^j \delta_j^k = \delta_i^k. \quad (3-51) \]

Taking the basis \( \{ \mathbf{e}_i \} \) and the equations above, the component expression for the dot product gives,

\[ \mathbf{u} \cdot \mathbf{v} = u_i \mathbf{e}_i \cdot v_j \mathbf{e}_j = u_i v_j \mathbf{e}_i \cdot \mathbf{e}_j = u_i v_j \delta_{ij} = u_i v_i \quad (3-52) \]

\[ \mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + u_3 v_3 \quad (3-53) \]

In an analogous manner, the component expression for the square of the length of \( \mathbf{u} \), i.e. is

\[ |\mathbf{u}|^2 = u_1^2 + u_2^2 + u_3^2 \quad (3-54) \]

The cross product of \( \mathbf{u} \) and \( \mathbf{v} \), denoted by \( \mathbf{u} \times \mathbf{v} \) produces a new vector. In order to express the cross product in terms of components the permutation symbol is introduced as,

\[ \varepsilon_{ijk} = \begin{cases} 1, & \text{for even permutations of } (i, j, k) \\ -1, & \text{for odd permutations of } (i, j, k), \\ 0, & \text{if there is a repeated index} \end{cases} \quad (3-55) \]

Consider the right-handed and orthonormal basis \( \{ \mathbf{e}_i \} \), then

\[ \mathbf{e}_1 \times \mathbf{e}_2 = \mathbf{e}_3, \quad \mathbf{e}_2 \times \mathbf{e}_3 = \mathbf{e}_1, \quad \mathbf{e}_3 \times \mathbf{e}_1 = \mathbf{e}_2, \quad (3-56) \]

\[ \mathbf{e}_2 \times \mathbf{e}_1 = -\mathbf{e}_3, \quad \mathbf{e}_3 \times \mathbf{e}_2 = -\mathbf{e}_1, \quad \mathbf{e}_1 \times \mathbf{e}_3 = -\mathbf{e}_2. \]

\[ \mathbf{e}_1 \times \mathbf{e}_1 = \mathbf{e}_2 \times \mathbf{e}_2 = \mathbf{e}_3 \times \mathbf{e}_3 = 0 \quad (3-57) \]
Or in more convenient short-hand notation

\[ \mathbf{e}_i \times \mathbf{e}_j = \varepsilon_{ijk} \mathbf{e}_k \]  \hspace{1cm} (3-58)

Then the cross product of \( \mathbf{u} \) and \( \mathbf{v} \) yields,

\[ \mathbf{w} = \mathbf{u} \times \mathbf{v} = u_i \mathbf{e}_i \times v_j \mathbf{e}_j = u_i v_j (\mathbf{e}_i \times \mathbf{e}_j) = \varepsilon_{ijk} u_i v_j \mathbf{e}_k = w_k \mathbf{e}_k \]  \hspace{1cm} (3-59)

Recall the components of the resultant vector \( \mathbf{u} \) relative to the coordinate axes. That is,

\[ \mathbf{u} = u_1 + u_2 + u_3 = u_i \mathbf{e}_i + u_2 \mathbf{e}_2 + u_3 \mathbf{e}_3 \]  \hspace{1cm} (3-60)

This equation also reveals that any vector can be represented in terms of linearly independent vectors. Likewise, any second-order tensor can be represented in terms of linearly independent dyads, as, for example, \( \mathbf{e}_1 \otimes \mathbf{e}_1, \mathbf{e}_1 \otimes \mathbf{e}_2 \ldots \) in Cartesian components.

Hence, for the second-order tensor \( \mathbf{T} \) we can write

\[ \mathbf{T} = T_{11} \mathbf{e}_1 \otimes \mathbf{e}_1 + T_{12} \mathbf{e}_1 \otimes \mathbf{e}_2 + T_{13} \mathbf{e}_1 \otimes \mathbf{e}_3 \\
+ T_{21} \mathbf{e}_2 \otimes \mathbf{e}_1 + T_{22} \mathbf{e}_2 \otimes \mathbf{e}_2 + T_{23} \mathbf{e}_2 \otimes \mathbf{e}_3 \\
+ T_{31} \mathbf{e}_3 \otimes \mathbf{e}_1 + T_{32} \mathbf{e}_3 \otimes \mathbf{e}_2 + T_{33} \mathbf{e}_3 \otimes \mathbf{e}_3 \]  \hspace{1cm} (3-61)

Where \( T_{11}, T_{12}, \) etc. are said to be components of \( \mathbf{T} \) relative to Cartesian axes. The equation above can be written in the more compact Einstein summation convention as

\[ \mathbf{T} = T_{ij} \mathbf{e}_i \otimes \mathbf{e}_j \]  \hspace{1cm} (3-62)

Where the subscripts \( i \) and \( j \) are both repeated, that is "dummy." Note, too, the nine components of \( \mathbf{T} \) with respect to a Cartesian coordinate system, say \( T_{mn} \), can easily be determined, viz.,

\[ T_{mn} = \mathbf{e}_m \cdot (T_{ij} \mathbf{e}_i \otimes \mathbf{e}_j) \mathbf{e}_n , \]  \hspace{1cm} (3-63)

\[ T_{mn} = T_{ij} (\mathbf{e}_m \cdot \mathbf{e}_i) (\mathbf{e}_j \cdot \mathbf{e}_n), \]  \hspace{1cm} (3-64)

\[ T_{mn} = T_{ij} \delta_{mi} \delta_{jn}. \]  \hspace{1cm} (3-65)
wherein we again factored out the scalar components $T_{ij}$ before performing the dot products (on vectors); the replacement property of the Kronecker delta is thus revealed again. Because a second-order tensor has nine components, they can also be written in the form of a $3 \times 3$ matrix as

$$T_{ij} = [T] = \begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{bmatrix}. \quad (3-66)$$

A familiar example of matrix representation is the identity tensor $I$, which has components

$$[I] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (3-67)$$

Relative to Cartesian coordinate axes. Thus Kronecker delta simply represents the components of $I$ relative to Cartesian coordinate system. That is, we can write

$$I = \delta_{ij} e_i \otimes e_j. \quad (3-68)$$

Cartesian component representations for vectors and tensors reveal that the transformation of a vector into another vector via a second-order tensor involves a scalar product between appropriate bases:

$$\left( T_{ij} e_i \otimes e_j \right)(v_k e_k) = T_{ij} v_k e_i \delta_{jk}, \quad (3-69)$$

$$\left( T_{ij} e_i \otimes e_j \right)(v_k e_k) = T_{ij} v_k e_i \delta_{jk}, \quad (3-70)$$

$$\left( T_{ij} e_i \otimes e_j \right)(v_k e_k) = T_{ij} v_k e_i, \quad (3-71)$$

$$\left( T_{ij} e_i \otimes e_j \right)(v_k e_k) = u_i e_i, \quad (3-72)$$
Wherein we again used the replacement property of the Kronecker delta and let $u_i$ represent the term(s) $T_{ij}v_j + T_{ik}v_k + T_{ij}v_j$. The equation above reveals that many tensor manipulations can be reduced to manipulations of the bases; the transpose, trace, determinant, and inverse operations are straightforward

$$T^T = (T_{ij}e_i \otimes e_j)^T = T_{ij}e_j \otimes e_i,$$  \hspace{1cm} (3-73)

$$tr T = tr(T_{ij}e_i \otimes e_j) = T_{ij}(e_i, e_j) = T_{ii},$$  \hspace{1cm} (3-74)

$$\det T = \det[T] = \det(T_{ij}),$$  \hspace{1cm} (3-75)

$$T^{-1} = (T_{ij}e_i \otimes e_j)^{-1} = T_{ij}^{-1}e_j \otimes e_i.$$  \hspace{1cm} (3-76)

The transpose of a vector equals the vector itself, that is $v^T = v$ or $(v_i e_i)^T = v_i e_i$. Moreover, the inverse switches the order of the bases that constitute the dyad, just as the transpose does, but it also modifies the scalar components.

The del operator, relative to Cartesian coordinates is defined by

$$\nabla = e_i \frac{\partial}{\partial x_i}$$  \hspace{1cm} (3-77)

and from which we obtain, the gradient of a scalar $a$,

$$\nabla a = e_i \frac{\partial}{\partial x_i} (a) = \frac{\partial a}{\partial x_i} e_i;$$  \hspace{1cm} (3-78)

The divergence and gradient of a vector $u$, that is,

$$\nabla u = e_i \frac{\partial}{\partial x_i} (u_j e_j) = e_i \left( \frac{\partial u_j}{\partial x_i} e_j + u_j \frac{\partial e_j}{\partial x_i} \right) = \frac{\partial u_j}{\partial x_i} (e_j e_j) = \frac{\partial u_i}{\partial x_i},$$  \hspace{1cm} (3-79)

and
\[ \nabla \mathbf{u} = e_j \frac{\partial}{\partial x_i} (u_j e_j) = \frac{\partial u_i}{\partial x_i} e_i \otimes e_j; \]  

(3-80)

or the divergence tensor \( \mathbf{T} \),

\[ \nabla \mathbf{T} = e_k \frac{\partial}{\partial x_k} (T_{ij} e_i \otimes e_j), \]

(3-81)

\[ \nabla \mathbf{T} = \frac{\partial T_{ij}}{\partial x_k} (e_i e_j) e_j = \frac{\partial T_{ij}}{\partial x_i} e_j . \]  

(3-82)

Hence, \( \nabla \mathbf{u} \) yields a scalar, \( \nabla a \) and \( \nabla \mathbf{T} \) yield vectors, and \( \nabla \mathbf{u} \) yields a tensor.

Another convention arises naturally when one takes a derivative with respect to a vector.

\[ \frac{\partial a}{\partial \mathbf{x}} = \frac{\partial a}{\partial x_i} e_i, \]  

(3-83)

and

\[ \frac{\partial \mathbf{u}}{\partial \mathbf{x}} = \frac{\partial (u_i e_i)}{\partial x_i} \otimes e_j = \frac{\partial e_i}{\partial x_j} e_i \otimes e_j . \]  

(3-84)

Derivatives with respect to a second-order tensor follow a similar convention:

\[ \frac{\partial a}{\partial \mathbf{T}} = \frac{\partial a}{\partial T_{ij}} e_i \otimes e_j \]  

(3-85)

The scalar products between two second-order tensors are

\[ \mathbf{T} : \mathbf{S} = (T_{ij} e_i \otimes e_j) : (S_{mn} e_m \otimes e_n) \]  

(3-86)

\[ \mathbf{T} : \mathbf{S} = T_{ij} S_{mn} (e_i e_m) (e_j e_n) \]  

(3-87)

\[ \mathbf{T} : \mathbf{S} = T_{ij} S_{mn} \delta_{im} \delta_{jn} \]  

(3-88)

\[ \mathbf{T} : \mathbf{S} = T_{ij} S_{ij} \]  

(3-89)
And also an alternative representation is

\[ T : S = tr(T \cdot S^T) = tr(T^T \cdot S) \quad (3-90) \]

### 3.1.3. Coordinate transformations

It is worthwhile to mention that vectors and tensors themselves remain invariant upon a change of basis they are said to be independent of any coordinate system. However, their respective components do depend upon the coordinate system introduced, which is arbitrary. The components change their magnitudes by a rotation of the basis vectors, but are independent of any translation.

We now set up the transformation laws for various components of vectors and tensors under a change of basis.

\[ \tilde{e}_i = Qe_i \text{ and } e_i = Q^T \tilde{e}_i \quad i = 1, 2, 3 \quad (3-91) \]

where \( Q \) denotes the orthogonal tensor, with components \( Q_{ij} \) which are the same in either basis. The components describe the orientation of the two sets of basis vectors relative to each other. In particular, \( Q \) rotates the basis vectors \( e_i \) in to \( \tilde{e}_i \), while \( Q^T \) rotates \( \tilde{e}_i \) back to \( e_i \). Using equations 3-61, 3-11 and 3-49 we find that

\[ Qe_i = Q_{ij}e_j \quad \text{and} \quad Q^T \tilde{e}_i = Q_{ij} \tilde{e}_j \quad (3-92) \]

By comparing the equations above we may extract the orthogonality condition of the cosines, characterized by \( Q^T Q = QQ^T = I \). Equivalently, expressed in index or matrix notation

\[ Q_{ij}Q_{ik} = Q_{ji}Q_{ki} = \delta_{jk} \quad Q_{ij}Q_{ik} = Q_{ji}Q_{ki} = \delta_{jk} \quad (3-93) \]

Where \( [Q] \) contains the collection of the components \( Q_{ij} \). It is an orthogonal matrix which is referred to as the transformation matrix. Note that \( [Q]^T = [Q^T] \). In order to maintain the right-handedness of the basis vectors we have admitted only rotations of the basis vectors, consequently \( \det[Q] = \pm 1 \).
3.1.4. Vectorial transformation law

We consider any vector $\mathbf{u}$ resolved along the two sets $\{\hat{\mathbf{e}}_i\}$ and $\{\mathbf{e}_i\}$ of basis vectors, i.e.

$$\tilde{u}_i = \mathbf{u} \cdot \hat{\mathbf{e}}_i \quad \text{in} \quad \{\hat{\mathbf{e}}_i\} \tag{3-94}$$

$$u_i = \mathbf{u} \cdot \mathbf{e}_i \quad \text{in} \quad \{\mathbf{e}_i\} \tag{3-95}$$

We obtain the vectorial transformation law for the Cartesian components of the vector $\mathbf{u}$, i.e.

$$\tilde{u}_i = \mathbf{u} \cdot \hat{\mathbf{e}}_i = Q_{ji} (\mathbf{u} \cdot \mathbf{e}_j) = Q_{ji} u_j \tag{3-96}$$

$$[\tilde{\mathbf{u}}] = [Q]^T [\mathbf{u}]. \tag{3-97}$$

These equations determine the relationship between the components of a vector associated with the (old) basis $\{\mathbf{e}_i\}$ and the components of the same vector associated with another (new) basis $\{\hat{\mathbf{e}}_i\}$.

3.1.5. Tensorial transformation law

To determine the transformation laws for the Cartesian components of any second-order tensor $\mathbf{A}$, we describe its components along the sets $\{\hat{\mathbf{e}}_i\}$ and $\{\mathbf{e}_i\}$ of basis vectors, i.e.

$$\tilde{A}_{ij} = \hat{\mathbf{e}}_i \cdot \mathbf{A} \cdot \hat{\mathbf{e}}_j \quad \text{in} \quad \{\hat{\mathbf{e}}_i\} \tag{3-98}$$

$$A_{ij} = \mathbf{e}_i \cdot \mathbf{A} \cdot \mathbf{e}_j \quad \text{in} \quad \{\mathbf{e}_i\} \tag{3-99}$$

Combining the equations above with 3-89 and 3-91, then the components $A_{ij}$, $\tilde{A}_{ij}$ are related via the so-called tensorial transformation law.

$$\tilde{A}_{ij} = \hat{\mathbf{e}}_i \cdot \mathbf{A} \cdot \hat{\mathbf{e}}_j = (Q_{ki} \mathbf{e}_k) \cdot \mathbf{A} \cdot (Q_{mj} \mathbf{e}_m) \tag{3-100}$$

$$\tilde{A}_{ij} = Q_{ki} Q_{mj} (\mathbf{e}_k \cdot \mathbf{A} \cdot \mathbf{e}_m) \tag{3-101}$$
\[ \bar{A}_{ij} = Q_{ki} Q_{mj} A_{km} \quad \text{or} \quad [\bar{A}] = [Q]^T [A] [Q] \quad (3-102) \]

Transformation \([\bar{A}] = [Q]^T [A] [Q]\) relates different matrices \([\bar{A}]\) and \([A]\), which have the components of the same tensor \(A\). In analogous manner, we find that

\[ A_{ij} = Q_{ki} Q_{jm} \bar{A}_{km} \quad \text{or} \quad [A] = [Q]^T [\bar{A}] [Q] \quad (3-103) \]

### 3.1.6. Principal values

The scalars \(\lambda_i\) characterize eigenvalues of a tensor \(A\) if there exist corresponding nonzero normalized eigenvectors \(\hat{n}_i\) of \(A\), so that

\[ A \hat{n}_i = \lambda_i \hat{n}_i \quad (i = 1, 2, 3; \text{no summation}) \quad (3-104) \]

To identify the eigenvectors of a tensor, we use subsequently a hat on the vector quantity concerned, for example \(\hat{n}\).

Thus, a set of homogeneous algebraic equations for the unknown eigenvalues \(\lambda_i\), \(i = 1, 2, 3\), and the unknown eigenvectors \(\hat{n}_i\), \(i = 1, 2, 3\) is

\[ (A - \lambda_i \mathbf{I}) \hat{n}_i = 0 \quad (i = 1, 2, 3; \text{no summation}) \quad (3-105) \]

Eigenvalues characterize the physical nature of a tensor. They do not depend on coordinates. For a positive definite symmetric tensor \(A\), all eigenvalues \(\lambda_i\) are (real and) positive since, using \(3-103\), we have \(\lambda_i = \hat{n}_i^T A \hat{n}_i > 0\), \(i = 1, 2, 3\). Moreover, the set of eigenvectors of a symmetric tensor \(A\) form a mutually orthogonal basis \(\{\hat{n}_i\}\).

For the system \(3-104\) to have solutions \(\hat{n}_i \neq 0\)

\[ \det(A - \lambda_i \mathbf{I}) = -\lambda_i^3 + I_1 \lambda_i^2 - I_2 \lambda_i + I_3 = 0 \quad (3-106) \]

The latter called the characteristic polynomial (or equation) for \(A\), the solutions of which are the eigenvalues \(\lambda_i\), \(i = 1, 2, 3\).
Here, \( I_i(A) \) \( i = 1, 2, 3 \), are the so-called principal scalar invariants of \( A \). In terms of \( A \) and its principal values \( \lambda_i \), \( i = 1, 2, 3 \), these are given by

\[
I_1(A) = A_{ii} = \text{tr}A = \lambda_1 + \lambda_2 + \lambda_3 \tag{3-107}
\]

\[
I_2(A) = \frac{1}{2}(A_{ii}A_{jj} - A_{ij}A_{ji}) = \frac{1}{2}[(\text{tr}A)^2 - \text{tr}(A^2)] = \text{tr}A^{-1} \text{det} A \tag{3-108}
\]

\[
I_3(A) = \varepsilon_{ijk}A_{ii}A_{jj}A_{kk} = \text{det} A = \lambda_1\lambda_2\lambda_3 \tag{3-109}
\]

A repeated application of tensor \( A \) to equation 3-103 yields \( A^n\hat{n}_i = \lambda^n_i\hat{n}_i \), \( i = 1, 2, 3 \), for any positive integer \( \alpha \). Using this relation and 3-107 multiplied by \( \hat{n}_i \), we obtain the well-known Cayley-Hamilton equation which it states that every (second-order) tensor \( A \) satisfies its own characteristic equation.

\[
A^3 - I_1A^2 + I_2A - I_3I = 0 \tag{3-110}
\]

### 3.1.7. Spectral decomposition of a tensor

Any symmetric tensor \( A \) may be represented by its eigenvalues \( \lambda_i \), \( i = 1, 2, 3 \), and the corresponding eigenvectors of \( A \) forming an orthonormal basis \( \{\hat{n}_i\} \). Using the unit tensor, by analogy with 3-103, i.e. \( I = \hat{n}_i \otimes \hat{n}_i \) and relations 3-49, 3-103 we obtain an expression which is known as the spectral decomposition of \( A \), i.e.

\[
A = AI = (A\hat{n}_i) \otimes \hat{n}_i = \sum_{i=1}^{3} \lambda_i \hat{n}_i \otimes \hat{n}_i \tag{3-111}
\]

The components \( A_{ij} \) of tensor \( A \) relative to a basis of principal directions follow with 3-61 by replacing \( e_i \) with the three orthonormal basis vectors \( \{\hat{n}_i\} \). With equations 3-103, 3-49 we obtain

\[
A_{ij} = \hat{n}_i.A\hat{n}_j = \hat{n}_i.\lambda_j \hat{n}_j = \lambda_j \delta_{ij} \quad (j = 1, 2, 3; \text{no summation}) \tag{3-112}
\]
This produces a diagonal matrix \([A]\) in the form,

\[
[A] = \begin{bmatrix}
\lambda_1 & 0 & 0 \\
0 & \lambda_2 & 0 \\
0 & 0 & \lambda_3
\end{bmatrix}
\]  

(3-113)

Where the diagonal elements are the eigenvalues of \(A\). This result may be obtained directly from the spectral decomposition 3-112 of \(A\) [18].

3.1.8. Further results in tensor calculus

Because vectors and tensor are defined on linear vector spaces, rules for differentiation are similar to those from elementary calculus. For example, if scalar, vector and tensor fields – say, \(a \in R\) and \(u, v \in V\), and \(S, T \in Lin\) - depend only on the variable \(t \in R\), then

\[
\frac{d}{dt}(av) = \frac{da}{dt}v + a \frac{dv}{dt}
\]

(3-114)

\[
\frac{d}{dt}(uv) = \frac{du}{dt}v + u \frac{dv}{dt}
\]

(3-115)

\[
\frac{d}{dt}(Tv) = \frac{dT}{dt}v + T \frac{dv}{dt}
\]

(3-116)

\[
\frac{d}{dt}(TS) = \frac{dT}{dt}S + T \frac{dS}{dt}
\]

(3-117)

Similarly,

\[
\nabla(u \otimes v) = (\nabla u)v + u \nabla v
\]

(3-118)

\[
\nabla(Su) = (\nabla S)u + S : (\nabla u)
\]

(3-119)

Following is the divergence theorem, which will be used extensively in the formulation of the five basic postulates of continuum mechanics. It is
\[ \iiint (\mathbf{n} \mathbf{T}) dA = \iiint (\nabla \mathbf{T}) dV \]  

(3-120)

Where \( dA \) and \( dV \) are differential areas and volumes, respectively, and \( \mathbf{n} \) is an outward unit normal vector to \( dA \). In Cartesian components, the divergence theorem is

\[ \iiint (n_i T_{ij}) dA = \iiint \left( \frac{\partial T_{ij}}{\partial x_i} \right) dV \]  

(3-121)

### 3.2. Kinematics of (continuum) particles

Kinematics is defined as the study of motion. However, motion not only includes the current movement of a body, but also how the position of a particle within a particular configuration of a body has changed relative to its position in reference configuration. Here, we define a body to be a collection of material particles and configuration of the body to be the specification of the positions of each of the particles in the body at a particular time \( t \). Motion can be defined, therefore, as a sequence of configurations parameterized by time [17].

It is useful to locate a generic particle in a reference configuration \( \beta_0 \), at time \( t = 0 \), via a position vector \( \mathbf{X} \), and likewise the position of the same particle in a current configuration \( \beta_t \), at time \( t \), via a position vector \( \mathbf{x} \). Generally the reference configuration is often taken to be a stress-free, undeformed configuration, it need not be. It is also useful to refer \( \mathbf{X} \) and \( \mathbf{x} \) to different coordinate systems (that are related by a known translation and rotation): for Cartesian components, we refer \( \mathbf{X} \) and \( \mathbf{x} \) to the coordinate systems \( \{O; \mathbf{E}_A\} \) and \( \{o; \mathbf{e}_i\} \), respectively. Hence, the position vectors have representations \( \mathbf{X} = X_A \mathbf{E}_A \) and \( \mathbf{x} = x_i \mathbf{e}_i \), where summation is implied over dummy indices \( A = 1, 2, 3 \) and \( i = 1, 2, 3 \) in \( E^3 \). Without a loss of generality, let the origins \( O \) and \( o \) coincide (Figure). The displacement vector \( \mathbf{u} \) for each material particle is thus given by \( \mathbf{u} = \mathbf{x} - \mathbf{X} \). With the exception of a rigid body motion, each particle constituting a body can experience a different displacement.
The position of a material particle, relative to a common origin, is given by $X$ and $x$ in these two configurations, respectively. The displacement $\mathbf{u} = x - X$ and $E_A$ and $e_i$ are orthonormal bases.

There are four basic approaches to describe the kinematics of a continuum: the material, referential, spatial and relative approaches.

- In the material approach, motion is described via the particles themselves and time; this approach is not particularly useful in solid mechanics \cite{17}.

![Figure 3.2: Schematic illustration of material body in two configurations – an initial reference configuration at time $t = 0$, denoted as $\beta_0$, and a current configuration at time $t$, denoted as $\beta_t$.](image)

- The Lagrangian (referential) description is a characterization of the motion with respect to the material coordinates $(X_1, X_2, X_3)$ and time $t$. In material description attention is paid to a particle, and we observe what happens to the particle as it moves. Traditionally, the material description is often referred to as the Lagrangian description. Note that at $t=0$ we have the consistency condition $X = x$ and $X_A = x_a$.  

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- The Eulerian (spatial) description is a characterization of the motion with respect to the spatial coordinates \((x_1, x_2, x_3)\) and time \(t\). In spatial description attention is paid to a point in space, and we study what happens at the point as the time changes. In fluid mechanics we quite often work in the Eulerian description in which we refer all relevant quantities to the position in space at time \(t\). It is not useful to refer the quantities to the material coordinates \(X_A, A = 1, 2, 3\), at \(t=0\), which are, in general, not known in fluid mechanics. However, in solid mechanics we use both types of description. Due to the fact that the constitutive behavior of solids is often given in terms of material coordinates we often prefer the Lagrangian description [18].

- Finally, in the relative approach one uses independent variables \((x, \tau)\) where \(\tau\) is a measure of time often related to an intermediate configuration; this approach is useful in viscoelasticity [17].

If the positions of material particles at time \(t\) depend on their original positions, viz.,

\[
x = x(X, t), x, X \in V \quad t \in R
\]

The associated displacement field is given by,

\[
u(X, t) = x(X, t) - X
\]

(3-122)

(3-123)

Being interested primarily in the motion of individual material particles, it is useful to consider what happens to generic differential line segments as a body passes from one configuration to another. Hence, let \(dx\) be an oriented differential line segment in \(\beta_i\) that was originally \(dX\) in \(\beta_0\). A fundamental question then is how do we relate these two differential position vectors? Recalling that a second order tensor transforms a vector in to a new vector, in direct and Cartesian component notations, at each time \(t\), let

\[
dx = F \cdot dX, \quad dx_i = F_{iA} \cdot dX_A
\]

(3-124)

Where \(F\) is a second order tensor that accomplishes the desired transformation. The quantify \(F\) is crucial in nonlinear continuum mechanics and is primary measure of
deformation, called the deformation gradient. In general \( F \) has nine (independent) components for all \( t \), and characterizes the behavior of motion in the neighborhood of a point.

Expression 3-125 clearly defines a linear transformation which generates a vector \( dx \) by the action of the second order tensor \( F \) on the vector \( dX \). Hence, equation 3-124 serves as transformation rule. Therefore, \( F \) is said to be a two point tensor involving points in two distinct configurations. One index describes spatial coordinates, \( x_a \) and the other material coordinates, \( X_A \). In summary: material tangent vectors map (i.e. transform) in to spatial tangent vectors via the deformation gradient [18].

Because \( x \) is a function of \( X \), at each fixed time \( t \), the chain rule requires

\[
dx = \frac{\partial x}{\partial X} \cdot dX, \quad dx_i = \frac{\partial x_i}{\partial X_A} \cdot dX_A
\]  

(3-125)

Moreover, comparing equations above reveals that

\[
F = \frac{\partial x}{\partial X} = F_{ia} e_i \otimes e_{A}, \quad \text{where} \quad F_{ia} = \frac{\partial x_i}{\partial X_A}
\]  

(3-126)

This provides a method for computing the components \( F \) given a referential description of the motion relative to a Cartesian coordinate system [17].

Assuming equation 3-125 is invertible, that is \( X \) can be written as a function of \( x \) at a fixed time \( t \), we can alternatively consider

\[
dX = \frac{\partial X}{\partial x} \cdot dx, \quad dX_A = \frac{\partial X_A}{\partial x_i} dx_i
\]  

(3-127)

With

\[
F^{-1} = \frac{\partial X}{\partial x} = F^{-1}_{Ai} E_A \otimes e_i, \quad F^{-1}_{Ai} = \frac{\partial X_A}{\partial x_i}
\]  

(3-128)
Another important point to observe is that position vectors $\text{dx}$ can be mapped from $d\text{X}$ via a rigid body motion (i.e., a translation and/or rotation), a "deformation" (i.e., extension and shear), or a combination of both. Thus, it can be shown that $\text{F}$ can be decomposed via

$$\text{F} = \text{R} \cdot \text{U} = \text{V} \cdot \text{R}$$

(3-129)

Where $\text{R} \in \text{Orth}^+$ (i.e., $\text{R}^{-1} = \text{R}^T$ and $\det \text{R} = 1$) represents the rigid body motion, $\text{U} \in \text{Psym}$ (i.e., $\text{U}^T = \text{U}$ and is positive definite) is defined in the reference configuration $\beta_0$, and $\text{V} \in \text{Psym}$ is defined in the current configuration $\beta_t$. Equation 3-130 can be interpreted, therefore, as "stretch" followed by a "rigid rotation" ($\text{R} \cdot \text{U}$) or a "rigid rotation" followed by "stretch" ($\text{V} \cdot \text{R}$); it is called the polar decomposition theorem. Referred to Cartesian coordinates,

$$\text{R} = R_{iA}e_i \otimes E_A = R_{Ai}E_A \otimes e_i, \quad \text{U} = U_{AB}E_A \otimes E_B, \quad \text{V} = V_{ij}e_i \otimes e_j.$$

(3-130)

Hence, $\text{R}$ is a two-point tensor, whereas $\text{U}$ and $\text{V}$ are one-point tensors. $\text{U}$ and $\text{V}$ represent the complete deformation (extension and shear), but are called right and left

Figure 3.3: Decomposition in rotational and stretching part
"stretch" tensors, respectively, because their principal values are the principal stretches (e.g., current divided by reference lengths) experienced by the body at a point.

At first, we will introduce the right Cauchy-Green tensor $C$ defined by

$$C = F^T F.$$  \hfill (3-131)

Unlike $F$, $C$ is symmetric and positive definite and, therefore, holds

$$C = F^T F = (F^T F)^T = C^T.$$  \hfill (3-132)

Further, we will define

$$\det C = (\det F)^2 = J^2 > 0$$  \hfill (3-133)

with $J$ as the determinant of $F$ called the volume ratio.

A commonly used strain measurement is the Green-Lagrange strain tensor $E$ defined by

$$E = (F^T F - I) = \frac{1}{2} (C - I)$$  \hfill (3-134)

which is based on the observation of the change of squared lengths of line elements. Since $C$ and $I$ are symmetric $E$ is also symmetric. $C$ and $E$ are defined on the undeformed reference configuration and are, therefore, referred to as material (Lagrangian) strain tensors.

An important strain measure in terms of spatial (Eulerian) coordinates is the left Cauchy-Green tensor $b$ defined by

$$b = F^T F$$  \hfill (3-135)

The second order tensor $b$ is, as $C$, symmetric and positive definite

$$b = F^T F = (F^T F)^T = b^T$$  \hfill (3-136)

It can also be shown that
\[ \det \mathbf{b} = (\det \mathbf{F})^2 = J^2 > 0 \quad (3-137) \]

holds.

An observation of the change of squared lengths of line elements defined in the current configuration leads to the spatial counterpart of \( \mathbf{e} \), namely the Euler-Almansi strain tensor \( \mathbf{e} \) defined by [19]

\[ \mathbf{e} = \frac{1}{2} (\mathbf{I} - \mathbf{F}^{-T} \mathbf{F}^{-1}) = \frac{1}{2} (\mathbf{I} - \mathbf{b}^{-1}) \quad (3-138) \]

It can be shown that the right Cauchy-Green tensor \( \mathbf{C} \) and left Cauchy-Green tensor \( \mathbf{b} \) can be expressed as

\[ \mathbf{C} = \mathbf{F}^T \mathbf{F} = \mathbf{U}^T \mathbf{R}^T \mathbf{R} \mathbf{U} = \mathbf{U}^2 \quad (3-139) \]

\[ \mathbf{b} = \mathbf{F}^T \mathbf{F} = \mathbf{V}^T \mathbf{R}^T \mathbf{V} = \mathbf{V}^2 \quad (3-140) \]

\( \mathbf{C} \) and \( \mathbf{b} \) are both one-point, symmetric tensors that are independent of rigid body motion, \( \mathbf{C} \) being defined in the reference configuration \( \beta_0 \) and \( \mathbf{b} \) in the current configuration \( \beta_t \).

When referred to Cartesian coordinates [17],

\[ \mathbf{C} = C_{AB} \mathbf{E}_A \otimes \mathbf{E}_B , \quad \text{where} \quad C_{AB} = \frac{\partial x_i}{\partial X_A} \frac{\partial x_j}{\partial X_B} , \quad (3-141) \]

\[ \mathbf{b} = b_{ij} \mathbf{e}_i \otimes \mathbf{e}_j , \quad \text{where} \quad b_{ij} = \frac{\partial x_i}{\partial X_A} \frac{\partial x_j}{\partial X_A} \quad (3-142) \]

### 3.3. The Concept of Stress

Force is considered by many to be an intuitive concept, yet its precise definition is not necessarily straightforward. Nonetheless, we consider force to simply be the action of one body on another, which is a vectorial push or pull. There are two general types of forces that we shall be interested in:
- Body forces, such as gravity or electromagnetic forces, which act on all material particles in a body without physical contact,

- Surface forces, such as a pressure or frictional forces, which act through physical contact on a body through its bounding surface.

![Figure 3.4: Traction vectors](image)

Because many forces act on a material through a surface area, it is very useful to define a traction vector $T^{(n)}$ as follows:

$$ T^{(n)} = \lim_{\Delta a \to 0} \left( \frac{\Delta f}{\Delta a} \right) = \frac{df}{da} \quad (3-143) $$

Where $df$ is a differential force vector and $da$ a differential area, both defined in $\beta_i$, with $da$ having an orientation given by the outward unit normal vector $n$. That $df = T^{(n)} da$ will prove convenient below in the definition of multiple measure of stress.

Now we introduce Cauchy’s postulate of the form

$$ t = t(x,t,n) \quad T = T(X,t,N) \quad (3-144) $$
There $\mathbf{t}$ represents the Cauchy traction vector (force measured per unit surface area defined in the current configuration) with $\mathbf{n}$ as the unit outward normal. The vector $\mathbf{T}$ represents the first Piola-Kirchhoff traction vector which represents a force measured per unit surface area defined in the reference configuration. The pseudo traction vector $\mathbf{T}$ does not describe the actual intensity. It acts on the region $\mathbf{t}$, i.e. the current configuration, but it is described in terms of the reference position $\mathbf{X}$ and the outward normal $\mathbf{N}$ to the boundary surface $\beta_0$.

Using the introduced traction vectors we can write Cauchy’s stress theorem as

$$
\mathbf{t}(x,t,n) = \sigma(x,t)\mathbf{n}
$$

(3-145)

$$
\mathbf{T}(X,t,\mathbf{N}) = \mathbf{P}(X,t)\mathbf{N}
$$

(3-146)

There $\sigma$ denotes a symmetric spatial tensor field called the Cauchy stress tensor. $\mathbf{P}$ is called the first Piola-Kirchhoff stress tensor. Further, we can introduce Newton’s third law of action and reaction of the form

$$
\mathbf{t}(x,t,n) = -\mathbf{t}(x,t,-\mathbf{n})
$$

(3-147)

$$
\mathbf{T}(X,t,\mathbf{N}) = -\mathbf{T}(X,t,-\mathbf{N})
$$

(3-148)

Now we can derive a relation between $\sigma$ and $\mathbf{P}$ using Nanson’s formula connecting line elements in the different configurations

$$
da = J\mathbf{F}^{-T}dA
$$

(3-149)

This leads to

$$
\mathbf{t}(x,t,n)da = \mathbf{T}(X,t,\mathbf{N})dA,
$$

(3-150)

$$
\sigma(x,t)n da = \mathbf{P}(X,t)\mathbf{N}dA,
$$

(3-151)

$$
\sigma(x,t)da = \mathbf{P}(X,t)dA,
$$

(3-152)

$$
\sigma = J^{-1}\mathbf{P}\mathbf{F}^{T},
$$

(3-153)
\[ \mathbf{P} = J\sigma \mathbf{F}^{-T} . \] (3-154)

It can be shown that \( \sigma \) is symmetric (see balance laws) and \( \mathbf{P} \) is asymmetric in general and follows the rule

\[ \mathbf{P} \mathbf{F}^T = \mathbf{F} \mathbf{P}^T \] (3-155)

In nonlinear analysis two more definitions of stress tensors are frequently used. First there is a tensor called the Kirchhoff stress tensor \( \tau \) which differs from the Cauchy stress tensor by the volume ratio \( J \) and is defined by

\[ \tau = J\sigma \] (3-156)

Another important quantity is the second Piola-Kirchhoff stress tensor \( \mathbf{S} \) which does not admit a physical interpretation in terms of surface tractions. It is defined by material coordinates and is suitable for the formulation of constitutive equations, especially for solid materials. The second Piola-Kirchhoff stress tensor can be obtained by a so called pull-back operation on the contravariant tensor field \( \tau \). So we get the relation

\[ \mathbf{S} = \mathbf{F}^{-1} \tau \mathbf{F}^{-T} \] (3-157)

We can further derive relations between \( \mathbf{S}, \sigma \) and \( \mathbf{P} \) like

\[ \mathbf{S} = J\mathbf{F}^{-1}\sigma \mathbf{F}^{-T} = \mathbf{F}^{-1}\mathbf{P} = \mathbf{S}^T \] (3-158)

\[ \sigma = J^{-1} \mathbf{F} \mathbf{S} \mathbf{F}^T \] (3-159)

\[ \mathbf{P} = \mathbf{F} \mathbf{S} . \] (3-160)

3.4. Basic Postulates

There are five basic postulates in continuum mechanics:

1. Conservation of mass
2. Conservation of linear momentum
3. Conservation of angular momentum
4. Conservation of energy
5. The entropy inequality

3.4.1. Conservation of mass

In non-relativistic physics mass cannot be produced or destroyed. It is assumed that during a motion there are either mass sources or mass sinks, so that the mass $m$ of a body is a conserved quantity. Hence, if a particle has certain mass in the reference configuration it must stay the same during a motion [18].

$$ \int_{\beta_0} \rho_0 dV = \int_{\beta_t} \rho dV $$

(3-161)

Where $\beta_0$ and $\beta_t$ simply denote reference and current domains (i.e. volumes). Property $\rho_0$ is called the reference mass density and $\rho$ is called the spatial mass density during a motion. The spatial mass density, also known as the density in the motion, depends on place $x \in \beta_t$ and time $t$ throughout the body. Note that, $\rho_0$ is time-independent and intrinsically associated with the reference configuration of the body. Hence, $\rho_0$ depends only on the position $X$ chosen in configuration $\beta_0$. If the density does not depend on $X \in \beta_0$, i.e. $\text{Grad}\rho_0 = 0$, the configuration is said to be homogeneous [18].

Using the relation,

$$ dv = J dV = \det F dV $$

(3-162)

the local formulation of the conservation of mass can be written as

$$ \rho_0 = J\rho = \rho \det F $$

(3-163)

We can further derive dealing with the reference density as a time independent quantity the continuity equation

$$ \frac{D}{Dt} (J\rho) = \rho \text{Grad} \cdot \text{Grad} \rho = 0 $$

(3-164)
There $v = \dot{\mathbf{x}}$ denotes the velocity and we have used $\mathbf{f} = J \text{div} \mathbf{v}$ [19]

### 3.4.2. Balance of linear momentum

Conservation of linear momentum requires that the time rate of change of the linear momentum (i.e., mass times velocity for all particles) of a body must balance all the forces (body plus surface) that act on the body. By definition, forces act on a deformed (current) configuration, not an undeformed (reference) configuration. Hence, the linear momentum equation is most usually derived using spatial approach, though it is often more convenient to use the referential form in pseudoelasticity. For completeness, and because of the importance of understanding interrelations between these two approaches, we drive the linear momentum balance equation both ways. In a spatial approach, we have [17]

$$
\frac{d}{dt} \int_{V} \rho v dV = \int_{V} \rho \mathbf{b} dV + \int_{\partial V} T^{(n)} d\mathbf{a} \quad (3-165)
$$

Where $v$ is the velocity, and $\mathbf{b}$ and $T$ are the actual body force (defined per unit mass since mass remains constant) and traction vector that act on the body in the current configuration. We desire to write our equation in the form $\int (\ldots) dV = 0$ for all $\beta$, which will yield a local form. To accomplish this, the order of the time differentiation and volume integration on the left hand side must be interchanged. Because the current differential volume $dv$ varies with time, in general, we eliminate $dv$ in favor of $JdV$. Thus observe that [17]

$$
\frac{d}{dt} \int_{\Omega_{\beta}} \rho v d\Omega = \int_{\Omega_{\beta}} \frac{d}{dt} (\rho v J) dV = \ldots = \int_{\Omega_{\beta}} \rho \mathbf{a} dV \quad (3-166)
$$

Wherein we used the result that, from equation 3-154

$$
\int_{\Omega_{\beta}} \left( \frac{d\rho}{dt} \mathbf{v} J + \rho \mathbf{v} \frac{dJ}{dt} \right) dV = \int_{\Omega_{\beta}} \left( \frac{d\rho}{dt} + \rho tr \mathbf{L} \right) \mathbf{v} J dV = 0 \quad (3-167)
$$

Because the term in $(\ldots)$ on the right hand side is zero by the spatial mass balance equation. Next, recognize that the second term on the right hand side of equation 3-176
can be written as a volume integral by using the definition of the Cauchy stress and the divergence theorem. That is,

\[ \int_{\beta} T^{(n)} da = \int_{\beta} nt da = \int_{\beta} \nabla t dv. \]  

(3-168)

Hence, the spatial form of the linear momentum equation becomes

\[ \int_{\beta} (\rho a - \nabla t - \rho b) dv = 0 \quad \forall \ \beta, \]  

(3-169)

And thereby the point wise field equation, in direct and Cartesian notation, is

\[ \nabla t + \rho b = \rho a, \quad \frac{\partial t_{ij}}{\partial x_i} + \rho b_j = \rho a_j. \]  

(3-170)

Again, this is the most natural and thus the most familiar form of the linear momentum equation; it is typically called the equation of motion. If the acceleration is zero (or negligible), then the equation above is called the equilibrium equation.

Similarly, balance of linear momentum in referential approach is given by

\[ \frac{d}{dt} \int_{\beta_0} \rho_0 v dV = \int_{\beta_0} \rho_0 b dV + \int_{\beta_0} T^{(n)} dA, \]  

(3-171)

\( T^{(n)} \) is the traction vector defined in reference configuration in terms of the actual surface forces that act on the body in the current configuration. In contrast to the left hand side of the equation 3-176 the differentiation and integration can be interchanged directly because the reference volume \( dV \) is constant. Hence, using the definition for the first Piola-Kirchhoff stress and the divergence theorem, equation 3-182 can be written as

\[ \int_{\beta_0} (\rho_0 a - \nabla_0 P - \rho_0 b) dv = 0 \quad \forall \ \beta_0 \]  

(3-172)

From which the local form of the equation is

\[ \nabla_0 P + \rho_0 b = \rho_0 a, \quad \frac{\partial P}{\partial X_A} + \rho_0 b_j = \rho_0 a_j \]  

(3-173)
We see therefore that the referential approach is not as “natural” as the spatial approach, but the formulation is simpler because the reference configuration does not change with time; moreover, the two relations are similar, and indeed can be obtained one from the other [17].

3.4.3. Conservation of angular momentum

Conservation of angular momentum requires that the time rate of change of the total momentum of the body must balance the applied moments. Hence, in referential form,

\[
\frac{d}{dt} \int_{\mathcal{V}_0} (\mathbf{x} \times \rho_0 \mathbf{v}) dV = \int_{\mathcal{V}_0} (\mathbf{x} \times \rho_0 \mathbf{v}) dV + \int_{\partial \mathcal{V}_0} (\mathbf{x} \times \mathbf{T}^{(S)}) dA \tag{3-174}
\]

Although this is a referential form, the moment is obtained using the current position vector \( \mathbf{x} \) since both force and velocity are defined in \( \mathbf{\beta} \). [38] Equation 3-184 can be shown to yield

\[
\mathbf{F} \mathbf{P} = \mathbf{P}^T \mathbf{F}^T \tag{3-175}
\]

Considering the continuity mass equation and the local form of equilibrium we get as result of this equation the symmetry of the Cauchy stress tensor

\[
\mathbf{\sigma} = \mathbf{\sigma}^T \tag{3-176}
\]

Following the definition of the second Piola-Kirchhoff stress tensor \( \mathbf{S} \) we get also symmetry for this tensor.

\[
\mathbf{S} = \mathbf{S}^T \tag{3-177}
\]

Contrary to the above the first Piola-Kirchhoff stress tensor \( \mathbf{P} \) is asymmetric in general as can be seen in equation 3-186 [19].

In practice, therefore, balance of angular momentum provides little information for the formulation of a boundary value problem. Rather it provides restrictions on constitutive relations that are written in terms of \( \mathbf{P}, \mathbf{\sigma} \text{ or } \mathbf{S} \); that is, it requires particular symmetries to be respected in the stress strain relations [17].
3.4.4. Conservation of energy

Conservation of energy is also known as the first law of thermodynamics. Simply, the first law asserts that the time rate of change of the total energy of a body (kinetic plus potential being accounted for via body forces) must balance the rate at which work is done on the body (via volumetric and surface heating). Because we focus on isothermal processes throughout this work, we shall not need to invoke the energy equation. Nevertheless, for completeness the referential form of the energy equation is

\[
\rho_0 \frac{d\epsilon}{dt} = P^T : \frac{d\mathbf{F}}{dt} - \nabla_0 q_0 + \rho_0 g \tag{3-178}
\]

Where \( \epsilon \) is the internal energy density (defined per unit mass), \( \nabla_0 \) the referential del operator, \( q_0 \) the referential heat flux vector, and \( g \) a heat addition defined per unit mass.

3.4.5. Entropy inequality

Entropy inequality is also known as the second law of thermodynamics. The referential form of the equation is

\[-\rho_0 \left( \frac{d\Psi}{dt} + \mu \frac{dT}{dt} \right) + P^T : \frac{d\mathbf{F}}{dt} - \frac{1}{T} q_0 \nabla_0 T \geq 0 \tag{3-179} \]

For an isothermal process with no heat transfer, the equation above reduces to

\[-\rho_0 \frac{d\Psi}{dt} + P^T : \frac{d\mathbf{F}}{dt} \geq 0 \tag{3-180} \]

This reveals that the second law of the thermodynamics can provide important information even for isothermal, mechanical processes. It is also essential to the development of a general constitutive relation for all hyperelastic materials [17].
3.5. Hyperelastic Materials

3.5.1. General hyperelasticity

A hyperelastic material postulates the existence of a Helmholtz free-energy function $\Psi$ defined by unit reference volume. If $\Psi$ only depends on the deformation gradient $F$, then it is called a strain-energy function. We will use such a function $\Psi(F)$ and assume it to be continuous. Further, we will restrict our considerations to homogeneous materials, i.e. every particle of the material behaves in the same manner. Using a function $\Psi(F)$ we can define hyperelastic material by the material model [19]

$$ P = \frac{\partial \Psi(F)}{\partial F} $$  \hspace{1cm} (3-181)

This result can be obtained by defining the inner energy $D_{int}$ with the Claudius-Planck form of the second law of thermodynamics under the assumption of perfectly elastic material (no entropy production). We get

$$ D_{int} = P : \dot{\varepsilon} - \Psi = 0 $$  \hspace{1cm} (3-182)

$$ \dot{\varepsilon} = \frac{\partial \Psi}{\partial t} = \frac{\partial \Psi}{\partial F} : \frac{\partial F}{\partial t} = \frac{\partial \Psi}{\partial F} : \dot{F} $$  \hspace{1cm} (3-183)

$$ D_{int} = \left( P - \frac{\partial \Psi}{\partial F} \right) : \dot{F} = 0 $$  \hspace{1cm} (3-184)

leading to the material model 3-192. The strain energy function has to fulfill several conditions:

- Normalization condition $\Psi(I) = 0$; the strain energy function has to vanish in the stress free reference configuration.
- $\Psi(F) \geq 0$ The strain energy function has to increase with deformation
- Growth conditions; the strain energy function has to grow to infinity if $J = \det F$ approaches $\infty$ or 0. Physically this means that it would enquire an
infinite amount of strain energy to expand the continuum body to infinite range or to compress it to a point of vanishing volume [19].

The strain energy function $\Psi$ has to be objective, i.e. observer independent. This means after a translation or rotation the amount of stored energy has to be unchanged

$$\Psi(F) = \Psi(QF)$$ \hspace{1cm} (3-185)

with $Q$ as an orthogonal tensor having the property $Q^TQ = I$. Setting $Q = R^T$ as a special choice and knowing that $F$ can be decomposed in a rotational and a stretching part using the orthogonal tensor $R$ and the material stretch tensor $U$ or the spatial stretch tensor $\nu$ like

$$F = RU = \nu R$$ \hspace{1cm} (3-186)

we get

$$\Psi(F) = \Psi(R^TF) = \Psi(R^TRU) = \Psi(U)$$ \hspace{1cm} (3-187)

This means that is independent of the rotational part of the deformation. Knowing $C = U^2$ and $E = \frac{1}{2}(C - I)$ we can express the strain energy in terms of $F$, $C$ or $E$

$$\Psi(F) = \Psi(C) = \Psi(E)$$ \hspace{1cm} (3-188)

We now want to develop a relation between $\Psi(F)$ and $\Psi(C)$, i.e. how to express $\Psi(F)$ in terms of $C$. We will derive the relation by using the index notation. From $C = F^TF$ we get

$$C_{ij} = (F_{ik})^TF_{.j}^k = F_{ki}F_{.j}^k = F_{ki}F_{.j}G^{sk}$$ \hspace{1cm} (3-189)

using the contravariant metric coefficients $G^{sk}$. So we are able to write using the chain rule and the rules of tensor derivation [19]
\[
\frac{\partial \Psi(F)}{\partial F} = \frac{\partial \Psi}{\partial F_{mn}} G_m \otimes G_n
\] (3-190)

\[
\frac{\partial \Psi(F)}{\partial F} = \frac{\partial \Psi}{\partial C_{ij}} \frac{\partial C_{ij}}{\partial F_{mn}} G_m \otimes G_n
\] (3-191)

\[
\frac{\partial C_{ij}}{\partial F_{mn}} = \frac{\partial (F_{ki} F_{sj} G^{sk})}{\partial F_{mn}}
\] (3-192)

\[
\frac{\partial C_{ij}}{\partial F_{mn}} = \frac{\partial F_{ki}}{\partial F_{mn}} F_{sj} G^{sk} + F_{ki} \frac{\partial F_{sj}}{\partial F_{mn}} G^{sk}
\] (3-193)

\[
\frac{\partial C_{ij}}{\partial F_{mn}} = \delta_k^m \delta_i^n F_{sj} G^{sk} + F_{ki} \delta_s^m \delta_j^n G^{sk}
\] (3-194)

\[
\frac{\partial C_{ij}}{\partial F_{mn}} = \delta_j^n F_{sj} G^{sm} + \delta_j^n F_{ki} G^{mk}
\] (3-195)

\[
\frac{\partial C_{ij}}{\partial F_{mn}} = \delta_i^n F_{mj} + \delta_j^n F_{im}
\] (3-196)

\[
\frac{\partial \Psi(F)}{\partial F} = \frac{\partial \Psi}{\partial C_{ij}} \left( \delta_i^n F_{mj} + \delta_j^n F_{im} \right) G_m \otimes G_n
\] (3-197)

\[
\frac{\partial \Psi(F)}{\partial F} = F_{mj} \frac{\partial \Psi}{\partial C_{ij}} G_m \otimes G_n + F_{im} \frac{\partial \Psi}{\partial C_{ij}} G_m \otimes G_j
\] (3-198)

Using now the symmetry of the right Cauchy-Green deformation tensor \( C = C^T \) (in index form \( C_{ij} = C_{ji} \)) we finally get

\[
\frac{\partial \Psi(F)}{\partial F} = 2 F_{mj} \frac{\partial \Psi}{\partial C_{ij}} G_m \otimes G_i
\] (3-199)

\[
\frac{\partial \Psi(F)}{\partial F} = 2 F \frac{\partial \Psi(C)}{\partial C}
\] (3-200)
With this relation we are able to write

\[ P = \frac{\partial \Psi(F)}{\partial F} = 2F \frac{\partial \Psi(C)}{\partial C} \]  
(3-201)

\[ S = F^{-1}P = 2 \frac{\partial \Psi(C)}{\partial C} = \frac{\partial \Psi(E)}{\partial E} \]  
(3-202)

\[ \sigma = J^{-1}PF^T = 2J^{-1}F \frac{\partial \Psi(C)}{\partial C} F^T = J^{-1}F \frac{\partial \Psi(E)}{\partial E} F^T \]  
(3-203)

Which we extensively use throughout this work.

### 3.5.2. Isotropic hyperelastic materials

Isotropy is based on the physical idea that the response of the material is the same in all directions. An example of an isotropic material is rubber, which has a wide range of applications. It can be shown that the important constraint on isotropy can be expressed by

\[ \Psi(F) = \Psi(F^*) = \Psi(FQ^T) \]  
(3-204)

This means that a material is isotropic if we can show that a motion of an elastic body superimposed on a particularly translated and/or rotated reference configuration leads to the same strain energy function [19].

Knowing that we can express the strain energy function in terms of the right Cauchy-Green tensor \( C \) we can write:

\[ \Psi(C) = \Psi(C^*) = \Psi(F^{*T}F^*) \]  
(3-205)

\[ \Psi(C) = \Psi(QF^TFQ^T) \]  
(3-206)

\[ \Psi(C) = \Psi(QCQ^T) \]  
(3-207)

If the last equation is fulfilled for all symmetric \( C \) and orthogonal \( Q \) then \( \Psi \) is called an scalar-valued isotropic tensor function. It can further be shown that for isotropic response
the strain energy function can be expressed in terms of the left Cauchy-Green tensor $b = FF^T$ like

$$\Psi(C) = \Psi(b)$$

(3-208)

It can be shown that at isotropic material behavior the scalar-valued tensor function $\Psi$ is an invariant of $C$ and, therefore, can be expressed in terms of the principal invariants $I_1$, $I_2$ and $I_3$ of $C$ which are defined by [19]

$$I_1(C) = tr(C) \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad
Now constitutive equations for the second Piola-Kirchhoff stress tensor $S$ and the Cauchy stress tensor $\sigma$ can be obtained using 3-214 and 3-215

$$S = 2 \left[ \left( \frac{\partial \Psi}{\partial I_1} + I_1 \frac{\partial \Psi}{\partial I_2} \right) I - \frac{\partial \Psi}{\partial I_2} C + I_3 \frac{\partial \Psi}{\partial I_3} C^{-1} \right]$$  \hspace{1cm} (3-217)$$

$$\sigma = 2J^{-1}F \left[ \left( \frac{\partial \Psi}{\partial I_1} + I_1 \frac{\partial \Psi}{\partial I_2} \right) I - \frac{\partial \Psi}{\partial I_2} C + I_3 \frac{\partial \Psi}{\partial I_3} C^{-1} \right] F^T$$  \hspace{1cm} (3-218)$$

$$\sigma = 2J^{-1} \left[ I_3 \frac{\partial \Psi}{\partial I_3} I + \left( \frac{\partial \Psi}{\partial I_1} + I_1 \frac{\partial \Psi}{\partial I_2} \right) b - \frac{\partial \Psi}{\partial I_2} b^2 \right]$$  \hspace{1cm} (3-219)$$

Finally, we have derived constitutive equations which are suitable for a representation of the strain energy function in terms of invariants, like a Mooney-Rivlin material or the Neo-Hookean material law [19].

### 3.5.3. Incompressible hyperelastic materials

Numerous polymeric materials can sustain finite strains without noticeable volume changes. These materials can be considered as incompressible, meaning that only isochoric motions are possible. Incompressible materials are characterized by the incompressibility constraint

$$J = \lambda_1 \lambda_2 \lambda_3 = 1$$  \hspace{1cm} (3-220)$$

In order to derive constitutive equations for the common incompressible materials an Lagrange multiplier has to be introduced and so we get strain energy functions of the constrained form [19]

$$\Psi = \Psi(F) - p(J - 1)$$  \hspace{1cm} (3-221)$$

The scalar $p$ serves as an indeterminate Lagrange multiplier, which can be identified as a hydrostatic pressure. Note that the scalar $p$ may only be determined from the equilibrium equations and the boundary conditions. It represents a workless reaction to the kinematics constraint on the deformation field [18].
Differentiating equation 3-258 with respect to the deformation gradient $F$ and using $\frac{\partial J}{\partial F} = JF^{-T}$, we arrive at a general constitutive equation for the first Piola-Kirchoff stress tensor $P$. Hence, it will be in the form

$$P = -pF^{-T} + \frac{\partial \Psi(F)}{\partial F}$$  \hspace{1cm} (3-222)

Multiplying equation 3-259 by $F^{-1}$ from the left hand side, we conclude that the second Piola-Kirchoff stress tensor $S$ takes on the form

$$S = -pF^{-1}F^{-T} + F^{-1} \frac{\partial \Psi(F)}{\partial F} = -pC^{-1} + 2 \frac{\partial \Psi(F)}{\partial C}$$ \hspace{1cm} (3-223)

However, multiplying equation with $F^T$ from the right hand side, we conclude that the symmetric Cauchy stress tensor $\sigma$ may be expressed as

$$\sigma = -pI + \frac{\partial \Psi(F)}{\partial F} F^T = -pI + F\left( \frac{\partial \Psi(F)}{\partial F} \right)^T$$ \hspace{1cm} (3-224)

The fundamental constitutive equations 3-259, 3-260, 3-261 are the most general forms used to define incompressible Hyperelastic materials at finite strains.

3.5.4. Anisotropic hyperelastic materials

Anisotropic materials belong to a very important class of materials which are employed frequently in a variety of industrial applications. The simplest form of material anisotropy is represented by transversely isotropic materials which have only a single preferred principal (fiber) direction. The material response is isotropic to arbitrary rotations about this preferred fiber direction. The descriptions of the hyperelastic and viscoelastic response of fiber-reinforced composites at finite strains are represented by Holzapfel et al. [25] in which a matrix material is reinforced by two families of fibers. The crucial goal of this section is propose an orthotropic hyperelastic constitutive model which can be applied to simulate of the response of the nonlinear anisotropic hyperelastic material
and some types of fiber-reinforced composite material in the large strain and also application in soft tissue mechanics [20].

3.5.5. Transversely isotropic materials

The materials are composed of a matrix material and one or more families of fibers, are called composite materials (or fiber-reinforced composites). The types of composites have strong directional properties and their mechanical responses are regarded as anisotropic. The simplest anisotropic material is reinforced by only one family of fibers and has single preferred direction is called transversely isotropic material. The stiffness of this type of composites material in the fiber direction is typically much greater than in the directions orthogonal to the fibers. These composite materials are used in a many applications of engineering and medicine.

For the transversely isotropic material the stress at a material point depends not only on the deformation gradient $F$ but also on the single preferred direction, namely, fiber direction. The direction of a fiber at point $P \in C^0$ with the position vector $X$ is defined by a unit vector field $a_0(X), |a_0| = 1$. The new fiber direction at the associated point $P' \in C'$ with the position vector $x$ is defined by a unit vector field $a(x,t), |a| = 1$.

The length changes of the fibers are characterized by fiber stretch $\lambda$ along the fiber direction $a_0$. This fiber stretch which is defined the ratio between the length of a fiber element in the deformed and undeformed configuration depends on the fiber direction of the undeformed configuration $a_0$ and strain measure, the right Cauchy-Green tensor $C$.

$$\lambda^2 = a_0 \cdot F^T F a_0 = a_0 \cdot C a_0$$

(3-225)
Figure 3.5: The undeformed and deformed configuration under large strains noticing the preferred directions.

Because of the directional dependence on the deformation the Helmholtz free energy function (strain energy function) depends explicitly on both the right Cauchy-Green tensor $C$ and the fiber direction $a_0$ in the reference configuration [20]

$$\Psi = \Psi(C, a_0 \otimes a_0)$$  \hspace{1cm} (3-226)

As above mentioned an isotropic hyperelastic material may be represented by the three invariants $I_1$, $I_2$ and $I_3$ of $C$, characterized in 3-143 or 3-144. For a transversely hyperelastic material, apart from three invariants $I_1$, $I_2$ and $I_3$, must be two additional invariants $I_4$ and $I_5$ to the strain energy function in order to describe the properties of the fiber family and its interaction with the other material constituents

$$\Psi = \Psi[I_1(C), I_2(C), I_3(C), I_4(C, a_0), I_5(C, a_0)]$$  \hspace{1cm} (3-227)

Where $I_4$, $I_5$ are so-called pseudo-invariants, which given by
\[ I_4(C, a_0) = a_0 \cdot Ca_0 = \lambda^2 \]  
\[ I_5(C, a_0) = a_0 \cdot C^2 a_0 \]  

(3-228)  

(3-229)  

The two pseudo-invariants \( I_4 \) and \( I_5 \) arise directly from the anisotropy and contribute to the strain energy. The second Piola-Kirchhoff stress tensor is derived from the strain energy as

\[
S = 2 \frac{\partial \Psi}{\partial C} = 2 \sum a \frac{\partial \Psi}{\partial I_a} \frac{\partial I_a}{\partial C}
\]

(3-230)  

The derivatives of \( I_1 \), \( I_2 \) and \( I_3 \) with respect to \( C \) were derived from 3-245 – 3-247. The remaining derivatives of \( I_4 \) and \( I_5 \) are derived from 3-265 and 3-266

\[
\frac{\partial I_4}{\partial C} = a_0 \otimes a_0
\]

(3-231)  

\[
\frac{\partial I_5}{\partial C} = a_0 \otimes Ca_0 + a_0 C \otimes a_0
\]

(3-232)  

By replacing 3-245 – 3-247, 3-267 – 3-269 the second Piola-Kirchhoff stress tensor will be obtained in the form

\[
S = 2 \left[ \left( \frac{\partial \Psi}{\partial I_1} + I_1 \frac{\partial \Psi}{\partial I_2} \right) \mathbf{I} - \frac{\partial \Psi}{\partial I_2} C + I_3 \frac{\partial \Psi}{\partial I_3} C^{-1} + \frac{\partial \Psi}{\partial I_4} a_0 \otimes a_0 \right]
\]

\[
+ \frac{\partial \Psi}{\partial I_5} \left( a_0 \otimes Ca_0 + a_0 C \otimes a_0 \right)
\]

(3-233)  

Similarly to the (3.15) the Cauchy stress tensor will be obtained in the following form

\[
\sigma = 2J^{-1} \left[ I_3 \frac{\partial \Psi}{\partial I_1} \mathbf{I} + \left( \frac{\partial \Psi}{\partial I_1} + I_1 \frac{\partial \Psi}{\partial I_2} \right) \mathbf{B} - \frac{\partial \Psi}{\partial I_2} \mathbf{B}^2 + \frac{\partial \Psi}{\partial I_4} a \otimes a \right]
\]

\[
+ I_5 \frac{\partial \Psi}{\partial I_5} \left( a \otimes Ca + a C \otimes a \right)
\]

(3-234)
3.5.6. Incompressible transversely isotropic materials

The transversely isotropic materials which have an incompressible isotropic matrix material, i.e. \( I_3 = 1 \), are here considered. If the embedded fibers are extensible, the strain energy function is expressed in terms of remaining four independent invariants

\[
\Psi = \Psi[I_1(C), I_2(C), I_4(C,a_0), I_5(C,a_0)] - \frac{1}{2} p(I_3 - 1)
\]  
(3-235)

Where \( p/2 \) is an indeterminate Lagrange multiplier.

In the case the matrix material is incompressible isotropic and the reinforced fibers are inextensible. This mean that \( \lambda = 1 \) and the fourth invariant equals to one (\( I_4 = 1 \)). The strain energy function is expressed in terms of remaining three independent invariants

\[
\Psi = \Psi[I_1(C), I_2(C), I_5(C,a_0)] - \frac{1}{2} p(I_3 - 1) - \frac{1}{2} q(I_4 - 1)
\]  
(3-236)

Where \( q/2 \) is an additional indeterminate Lagrange multiplier [20].

The constitutive equations for transversely isotropic materials with an incompressible isotropic matrix material and inextensible fibers are obtained as

\[
S = -pC^{-1} - qa_0 \otimes a_0 + 2 \left( \frac{\partial \Psi}{\partial I_1} + I_1 \frac{\partial \Psi}{\partial I_2} \right) - 2 \frac{\partial \Psi}{\partial I_2} C + 2 \frac{\partial \Psi}{\partial I_5} (a_0 \otimes Ca_0 + a_0 C \otimes a_0)
\]  
(3-237)

\[
\sigma = -pC^{-1} - qa_0 \otimes a_0 + 2 \frac{\partial \Psi}{\partial I_1} \mathbf{B} - 2 \frac{\partial \Psi}{\partial I_2} \mathbf{B}^{-1} + 2 \frac{\partial \Psi}{\partial I_5} (a \otimes \mathbf{B}a + \mathbf{aB} \otimes a)
\]  
(3-238)

According to Weiss (Weiss, 1995) if the composite structure will be assumed incompressible, the strain energy may then be written as

\[
\Psi = F_1(I_1, I_2) + F_2(\lambda) + F_3(I_1, I_2, \lambda)
\]  
(3-239)

The function \( F_1 \) represents the material response of the isotropic ground substance matrix, \( F_2 \) represents the contribution from the fiber family, and \( F_3 \) is the contribution
from interaction between fibers and matrix, this interaction can take several forms. Stretch along the fiber direction could cause stress to develop in the matrix.

Note that the dependence on $I_4$ has been replaced by an equivalent dependence on the stretch along the fiber direction, $\lambda = \sqrt{I_4}$ 3-268 and 3-269. This facilitates the fitting of experimental data to the function. The dependence on $I_3$ has been omitted because of the incompressibility constraint ($I_3 = I_2 = 1$). The dependence on $I_5$ has been omitted as well as many of the effects governed by $I_5$ can be introduced into the function through the derivatives of the strain energy with respect to $I_4$.

The Cauchy stress for an incompressible material with strain energy given by 3-276 can be written as

$$\sigma = 2\left[(\Psi_1 + I_1 \Psi_2)B - \Psi_2 B^2\right] + \lambda \Psi_2 a \otimes a + pI \quad (3-240)$$

where

$$\Psi_1 = \frac{\partial F_1}{\partial I_1} + \frac{\partial F_2}{\partial I_1}, \quad \Psi_2 = \frac{\partial F_1}{\partial I_2} + \frac{\partial F_3}{\partial I_2}, \quad \Psi_\lambda = \frac{\partial F_2}{\partial \lambda} + \frac{\partial F_3}{\partial \lambda} \quad (3-241)$$

The previously described transversely isotropic constitutive model was evaluated as a model tissue.

### 3.5.7. Orthotropic hyperelastic materials

In this section the constitutive equations for composite material with two families of fibers are presented. The matrix material (rubber) is assumed to be hyperelastic and reinforced by two families of fibers (textile cords). The preferential fiber directions in the reference and the current configuration are denoted by the unit vector fields $a_0$, $b_0$ and $a$, $b$ respectively. For notational simplicity we have introduced the abbreviation

$$A_0 = a_0 \otimes a_0, \quad B_0 = b_0 \otimes b_0 \quad (3-242)$$
Similarly to the transversely hyperelastic materials the strain energy function depends explicitly on both the right Cauchy-Green tensor $C$ and the fiber directions $\mathbf{a}_0$ and $\mathbf{b}_0$ in the reference configuration as

$$\Psi = \Psi(C, \mathbf{A}_0, \mathbf{B}_0)$$  \hspace{1cm} (3-243)

According to Holzapfel (Holzapfel, 2000) the strain energy function may be expressed in terms of set of principal invariants $I_1 \rightarrow I_9$. The three invariants are presented in the isotropic case in equation 3-221 – 3-223. The pseudo-invariants $I_4$,...,$I_8$ are associated with the anisotropy generated by the two families of fibers. The invariants $I_4$ and $I_5$ are identical to those from transversely isotropic materials and

$$I_6(C, \mathbf{b}_0) = \mathbf{b}_0 \cdot C\mathbf{b}_0$$  \hspace{1cm} (3-244)

$$I_7(C, \mathbf{b}_0) = \mathbf{b}_0 \cdot C^2\mathbf{b}_0$$  \hspace{1cm} (3-245)

$$I_8(C, \mathbf{a}_0, \mathbf{b}_0) = (\mathbf{a}_0 \cdot \mathbf{b}_0)\mathbf{a}_0 \cdot C\mathbf{b}_0$$  \hspace{1cm} (3-246)

The invariant $I_9$ does not depend on the deformation and is subsequently no longer considered. In one specific case of composite materials with two families of fibers if $\mathbf{a}_0 \cdot \mathbf{b}_0 = 0$, the two families of fibers have orthogonal directions. Then, this material is said to be orthotropic in the reference configuration with respect to the planes normal to the fibers and the surface in which the fibers lie. The Helmholtz free energy function is a function of first seven invariants and has the form $\Psi = \Psi(I_1,\ldots,I_7)$ \cite{20}.

A further case may be found under the assumption that the isotropic matrix material is incompressible, i.e. $I_3 = 1$. Additionally, the families of fibers may be also inextensible in the two fiber directions $\mathbf{a}_0$ and $\mathbf{b}_0$, consequently $I_4 = 1$ and $I_6 = 1$. For this case a suitable Helmholtz free-energy function is given by

$$\Psi = \Psi[I_1(C), I_2(C), I_5(C, \mathbf{a}_0), I_7(C, \mathbf{b}_0)] - \frac{1}{2} p(I_3 - 1) - \frac{1}{2} q(I_4 - 1) - \frac{1}{2} r(I_6 - 1)$$  \hspace{1cm} (3-247)
with the in determinant Lagrange multipliers $p/2$, $q/2$, $r/2$.

The constitutive equations for an orthotropic material composed of an incompressible isotropic matrix material and inextensible fibers are obtained as

$$S = 2 \frac{\partial \Psi}{\partial C} = 2 \left( \frac{\partial \Psi}{\partial I_1} + I_1 \frac{\partial \Psi}{\partial I_2} \right) I - 2 \frac{\partial \Psi}{\partial I_2} C + \frac{\partial \Psi}{\partial I_5} \left( a_0 \otimes C a_0 + a_0 C \otimes a_0 \right) +$$

$$2 \frac{\partial \Psi}{\partial I_7} \left( b_0 \otimes C b_0 + b_0 C \otimes b_0 \right) - p C^{-1} - q a_0 \otimes a_0 - r b_0 \otimes b_0$$

$$\sigma = 2 F \frac{\partial \Psi}{\partial C} F^T = 2 \frac{\partial \Psi}{\partial I_1} B - 2 \frac{\partial \Psi}{\partial I_2} B^{-1} + 2 \frac{\partial \Psi}{\partial I_5} \left( a \otimes B a + a B \otimes a \right) +$$

$$2 \frac{\partial \Psi}{\partial I_7} \left( b \otimes B b + B B \otimes b \right) - p I - q a \otimes a - r b \otimes b$$

(3-248)

(3-249)
4. CONSTITUTIVE FRAMEWORK FOR ARTERIAL BIOMECHANICS

4.1. Arterial Biomechanics

4.1.1. Overview of the arterial system

The vasculature is a complex architecture of blood vessels that carry blood to and from various organs of the body. The blood vessels may be classified based on their sizes, function and proximity to the heart. Typically, they fall under one of the following 7 categories and the path of blood flow is as shown in the following Figure 3.1 [21].

![Classification of blood vessels according to their size and pressure they carry](image)

**Figure 4.1**: Classification of blood vessels according to their size and pressure they carry
Table 4.1. Characteristics of various types of blood vessels

<table>
<thead>
<tr>
<th>Vessel</th>
<th>Aorta</th>
<th>Artery</th>
<th>arteriole</th>
<th>Capillary</th>
<th>Venule</th>
<th>Vein</th>
<th>Vena Cava</th>
</tr>
</thead>
<tbody>
<tr>
<td>Wall Thickness</td>
<td>2mm</td>
<td>1mm</td>
<td>20µm</td>
<td>1µm</td>
<td>2µm</td>
<td>0.5mm</td>
<td>1.5mm</td>
</tr>
<tr>
<td>Lumen</td>
<td>25mm</td>
<td>4mm</td>
<td>30µm</td>
<td>8µm</td>
<td>20µm</td>
<td>5mm</td>
<td>30mm</td>
</tr>
</tbody>
</table>

Figure 4.2: Schematic illustration of blood circulation (A) and a layout of the arterial system in the human body (B) [21]

4.1.2. Arterial histology

This study is concerned with the in vitro passive behavior of arteries. Hence, in vivo effects such as the vasa vasorum, nerve control, humoral control, perivascular connective tissue etc. and effects through boundaries of neighboring organs such as the pulsating heart are not relevant and are not therefore discussed here. In general, arteries are roughly subdivided into two types: elastic and muscular. Elastic arteries have relatively large diameters and are located close to the heart (for example, the aorta and the carotid and iliac arteries), while muscular arteries are located at the periphery (for example, femoral,
celiac, cerebral arteries). However, some arteries exhibit morphological structures of both types. Here we focus attention on the microscopic structure of arterial walls composed of three distinct layers, the intima (tunica intima), the media (tunica media) and the adventitia (tunica externa). We discuss the constituents of arterial walls from the mechanical perspective and emphasize those aspects which are important to researchers interested in constitutive issues. Figure 3.3 shows a model of a healthy elastic artery.

The intima is the innermost layer of the artery. It consists of a single layer of endothelial cells lining the arterial wall and resting on a thin basal membrane (basal lamina). There is also a subendothelial layer whose thickness varies with topography, age and disease. In healthy young muscular arteries, however, the subendothelial layer is almost nonexistent. In healthy young individuals the intima is very thin and makes an insignificant contribution to the solid mechanical properties of the arterial wall. However, it should be noted that the intima thickens and stiffens with age (arteriosclerosis) so that the mechanical contribution may become significant.

It is known that pathological changes of the intimal components may be associated with atherosclerosis, the most common disease of arterial walls. It involves deposition of fatty substances, calcium, collagen fibers, cellular waste products and fibrin (a clotting material in the blood). The resulting build-up is called atherosclerotic plaque. It may be very complex in geometry and biochemical composition. In later stages the media is also affected. These pathological changes are associated with significant alterations in the mechanical properties of the arterial wall. Hence, the mechanical behavior of atherosclerotic arteries differs significantly from that of healthy arteries.

The media is the middle layer of the artery and consists of a complex three-dimensional network of smooth muscle cells, and elastin and collagen fibrils. According to [22] the fenestrated elastic lamina separates the media into a varying number of well-defined concentrically fiber-reinforced medial layers. The number of elastic lamina decreases toward the periphery (as the size of the vessels decreases) so that elastic lamina is hardly present in muscular arteries.
Figure 4.3: Diagrammatic model of the major components of a healthy elastic artery

Figure 4.3 displays a diagrammatic model of major components of a healthy elastic artery which is composed of three layers: intima (I), media (M), and adventitia (A). It is the innermost layer consisting of a single layer of endothelial cells that rests on a thin basal membrane and a subendothelial layer whose thickness varies with topography, age and disease. M is composed of smooth muscle cells, a network of elastic and collagen fibrils and elastic lamina which separate M into a number of fiber-reinforced layers. The primary constituents of A are thick bundles of collagen fibrils arranged in helical structures; A is the outermost layer surrounded by loose connective tissue.
The media is separated from the intima and adventitia by the so-called internal elastic lamina and external elastic lamina (absent in cerebral blood vessels), respectively. In muscular arteries these lamina appear as prominent structures, whereas in elastic arteries they are hardly distinguishable from the regular elastic lamina. The orientation of and close interconnection between the elastic and collagen fibrils, elastic lamina, and smooth muscle cells together constitute a continuous fibrous helix (Faserschraube) [23,24].

The helix has a small pitch so that the fibrils in the media are almost circumferentially oriented. This structured arrangement gives the media high strength, resilience and the ability to resist loads in both the longitudinal and circumferential directions. From the mechanical perspective, the media is the most significant layer in a healthy artery.

The adventitia is the outermost layer of the artery and consists mainly of fibroblasts and fibrocytes (cells that produce collagen and elastin), histological ground substance and thick bundles of collagen fibrils forming a fibrous tissue. The adventitia is surrounded continuously by loose connective tissue. The thickness of the adventitia depends strongly on the type (elastic or muscular) and the physiological function of the blood vessel and its topographical site. For example, in cerebral blood vessels there is virtually no adventitia. The wavy collagen fibrils are arranged in helical structures and serve to reinforce the wall. They contribute significantly to the stability and strength of the arterial wall. The adventitia is much less stiff in the load-free configuration and at low pressures than the media. However, at higher levels of pressure the collagen fibers reach their straightened [25].

4.1.3. Mechanics of arterial wall

Each constitutive framework and its associated set of material parameters require detailed studies of the particular material of interest. Its reliability is strongly related to the quality and completeness of available experimental data, which may come from appropriate in vivo tests or from in vitro tests that mimic real loading conditions in a physiological environment. In vivo tests seem to be preferable because the vessel is observed under real life conditions. However, in vivo tests have major limitations because of, for example, the influence of hormones and neural control. Moreover, data sets from the complex material response of arterial walls subject to simultaneous cyclic inflation, axial extension
and twist can only be measured in an in vitro experiment. Only with such data sets can the anisotropic mechanical behavior of arterial walls be described completely. In addition, in in vitro experiments the contraction state (active or passive) of the muscular media has to be determined. This can be done with appropriate chemical agents.

**Figure 4.4:** Transverse sections of rabbit arteries

In the Figure 4.4, Transverse sections of rabbit arteries (thickness ≈ 5 µm) fixed in situ at a transmural pressure of 100 mm Hg. (A) Descending thoracic aorta, (B) mid-abdominal aorta, (C) common iliac artery, and (D) femoral artery. Sections in the left-hand half of each lettered panel were treated with Masson’s trichrome stain to visualize smooth muscle cell nuclei. In these monochrome images, the nuclei are dark gray and collagen appears as a gray background. Sections in the right-hand halves of each panel were treated with Miller’s elastic stain in which elastin appears dark. The aorta in this rabbit had been subjected to balloon denudation two weeks before the animal was killed in order to induce intimal hyperplasia, clearly seen in the elastic stained sections (I
indicates intima; M, media and A, adventitia.) The number of smooth muscle cells per unit area and collagen content (revealed by increased staining intensity in the left-hand panels) increase with distance from the heart. There is a corresponding fall in the elastin content and the number of elastic lamellae. At the same time, the lamellae become thinner and more fragmented. Scale bar is 50 µm in length and applies to all panels.

For pure inflation tests of straight artery tubes, which is the most common two dimensional test, see the early work [26] (in which shear deformations are not considered). Since arteries do not change their volume within the physiological range of deformation [27], they can be regarded as incompressible materials.

Hence, by means of the incompressibility constraint we may determine the mechanical properties of three-dimensional specimens from two-dimensional tests [28]. It is important to note that uniaxial extension tests on arterial patches (strips) provide basic information about the material [29] but they are certainly not sufficient to quantify completely the anisotropic behavior of arterial walls. Other uniaxial extension tests on small arterial rings (so-called ring tests) are also insufficient [30]. In general, a segment of vessel shortens on removal from the body, as was first reported in [31]. The in vivo pre-stretch in the longitudinal direction must therefore be reproduced within in vitro tests [32]. Each non-axisymmetric arterial segment (such as a bifurcation or a segment with sclerotic changes) under combined inflation and axial extension develops significant shear stresses in the wall. Hence, in order to characterize the shear properties of arterial walls shear tests are required. In shear tests either the angle of twist of an arterial tube subjected to transmural pressure, longitudinal force and torque [33] or the shear deformation of a rectangular arterial wall specimen subjected to shear forces is measured.

Additionally, one can classify mechanical tests according to the strain rates used (quasi-static or dynamic) and to whether the loading is performed cyclically or discontinuously (creep and relaxation tests). It has been known for many years that the load-free configuration of an artery is not a stress-free state [34]. In general, a load-free arterial ring contains residual stresses. It is of crucial importance to identify these in order to predict reliably the state of stress in an arterial wall, and this has been the aim of many
experimental investigations (see, for example, the bending tests on blood vessel walls in [35]).

The mechanical behavior of arteries depends on physical and chemical environmental factors, such as temperature, osmotic pressure, pH, partial pressure of carbon dioxide and oxygen, ionic concentrations and monosaccharide concentration. In ex vivo conditions the mechanical properties are altered due to biological degradation. Therefore, arteries should be tested in appropriate oxygenated, temperature controlled salt solutions as fresh as possible. For an overview of experimental test methods used to verify material parameters, see [36], and the references contained therein.

**Figure 4.5** Schematic diagram of typical uniaxial stress-strain curves for circumferential arterial strips

Figure 4.5 demonstrates schematic diagram of typical uniaxial stress-strain curves for circumferential arterial strips (from the media) in passive condition (based on tension tests performed in the authors' laboratory): cyclic loading and unloading, associated with stress softening effects, lead to a pre-conditioned material which behaves (perfectly) elastically or viscoelastically (nearly repeatable cyclic behavior)- point I. Loading beyond the viscoelastic domain up to point II leads to inelastic deformations. Additional loading and unloading cycles display stress softening again until point III is reached.
Then the material exhibits (perfectly) elastic or viscoelastic response. The thick solid line indicates the (approximate) engineering response of the material.

As indicated in Section 2.1 the composition of arterial walls varies along the arterial tree. Hence, there seems to be a systematic dependence of the shape of the stress-strain curve for a blood vessel on its anatomical site. This fact has been demonstrated several times experimentally; see, for example, the early works [37-39]. Although the mechanical properties of arterial walls vary along the arterial tree, the general mechanical characteristics exhibited by arterial walls are the same. In order to explain the typical stress-strain response of an arterial wall of smooth muscles in the passive state (governed mainly by elastin and collagen fibers), we refer to Figure 2. Note that the curves in Figure 2 are schematic, but based on experimental tension tests performed in the authors' laboratory (some of which is described in a recent paper [40]).

As can be seen, a circumferential strip of the media subjected to uniaxial cyclic loading and unloading typically displays pronounced stress softening, which occurs during the first few load cycles. The stress softening effect diminishes with the number of load cycles until the material exhibits a nearly repeatable cyclic behavior, and hence the biological material is said to be 'pre-conditioned' (compare with, for example, the characteristic passive behavior of a bovine coronary artery in [36], and p.33. Thus, depending on the type of artery considered, the material behavior may be regarded as (perfectly) elastic for proximal arteries of the elastic type, or viscoelastic for distal arteries of the muscular type, often modeled as pseudo elastic (see, for example, [31]).

Healthy arteries are highly deformable composite structures and show a nonlinear stress-strain response with a typical (exponential) stiffening effect at higher pressures. This stiffening effect, common to all biological tissues, is based on the recruitment of the embedded (load carrying) wavy collagen fibrils, which leads to the characteristic anisotropic mechanical behavior of arteries; see the classical works [41], [42]. Early works on arterial anisotropy (see, for example, [43]) considered arterial walls to be cylindrically orthotropic, which is generally accepted in the literature. Loading beyond the viscoelastic domain (indicated by point I in Figure 4.5), far outside the physiological range of deformation, often occurs during mechanical treatments such as percutaneous transluminal angioplasty. This procedure involves dilation of an artery using a balloon.
catheter (see [44]). In the strain range up to point II in Figure 2, the deformation process in an arterial layer is associated with inelastic effects (elastoplastic and/or damage mechanisms) leading to significant changes in the mechanical behavior (see [45-47]). This overstretching involves dissipation, which is represented by the area between the loading and unloading curves. Hence, starting from point II, additional cyclic loading and unloading again displays stress softening, which diminishes with the number of load cycles. At point III the material exhibits a (perfectly) elastic or viscoelastic behavior. However, unloading initiated from point III returns the arterial (medial) strip to an unstressed state with non-vanishing strains remaining, these being responsible for the change of shape. With preconditioning effects neglected, the thick solid line in Figure 4.5 indicates the (approximate) engineering response associated with the actual physical behavior.

4.2. Theoretical Approach

Fundamental to detailed stress analysis is a constitutive relation for the material. There are five general steps in every constitutive formulation. That is;

- Delineate general characteristics of the material
- Establish an appropriate theoretical framework
- Identify a specific functional form of the relation
- Calculate values of the material parameters
- Evaluate the predictive capability of the final relation

Given the general characteristics discussed in previous part, we now establish a theoretical framework for general nonlinear, anisotropic, incompressible, pseudo elastic behavior. Also, as a basis for reporting the performance of different constitutive models for arteries we consider the mechanical response of a thick-walled cylindrical tube under various boundary loads.

For purpose of illustration, a general relation for the Cauchy stress is given by
\[ \sigma = -pI + F \frac{\partial W}{\partial E} F^T \]  

(4-1)

Where \( p \) is a Lagrange multiplier enforcing the incompressibility constraint \( \det F = 1 \), \( F \) is the deformation gradient tensor, \( W \) is the 3D pseudo strain energy function, and \( E = \frac{1}{2}(C - I) \) is the Green strain tensor.

Regarding the boundary value problems, the most common and useful experiment on excised normal arteries is the finite inflation, extension and torsion of a straight cylindrical specimen.

### 4.2.1. The stress-relieving cut

Consider a material particle located at \((R, \Theta, Z)\), in the central region of a radially cut arterial segment, that is mapped to \((\rho, \nu, \xi)\) in the central region of an associated unloaded intact configuration according to

\[
\rho = \rho(R) \quad \nu = \Theta \frac{\pi}{\Theta_0} \quad \xi = \Lambda Z
\]  

(4-2)

Where \( \Theta_0 \) is one of the aforementioned angles and \( \Lambda = \Lambda_\xi \) is an axial stretch ratio associated with the residual stress. Therefore, in the absence of the residual stress \( \Theta_0 = \pi \) and \( \Lambda = 1 \).

According to these values, the physical components of deformation gradient tensor (in cylindrical coordinates) associated with the mapping in equation 7.5 are

\[
[F_1] = \begin{bmatrix}
\frac{\partial \rho}{\partial R} & \frac{\partial \rho}{\partial \theta} & \frac{\partial \rho}{\partial Z} \\
\frac{\partial \rho}{\partial R} & \frac{\partial \rho}{\partial \theta} & \frac{\partial \rho}{\partial Z} \\
\frac{\partial \rho}{\partial \xi} & \frac{\partial \rho}{\partial \xi} & \frac{\partial \rho}{\partial Z}
\end{bmatrix} = \begin{bmatrix}
\frac{\partial \rho}{\partial R} & 0 & 0 \\
0 & \frac{\pi \rho}{\Theta_0 R} & 0 \\
0 & 0 & \Lambda
\end{bmatrix}
\]  

(4-3)

Applying the incompressibility constraint \( \det F_1 = 1 \), yields
\[
\frac{\partial \rho}{\partial R} = \frac{\Theta_0 R \Lambda}{\pi \rho} \tag{4-4}
\]

And integrating this relation with respect to \( R \) to yield,

\[
\rho^2 - \rho_i^2 = \frac{\Theta_0}{\pi \Lambda} \left( R^2 - R_i^2 \right), \quad R \in [R_i, R_a] \tag{4-5}
\]

Where the “\( i \)” and “\( a \)” are subscriptions which denote the intimal and adventitial surfaces. This relation allows determining the location of every point in the wall in either configuration given the corresponding location in other configuration and knowledge of either the inner or outer radius.

Then, the physical components of the various measures of deformation can be easily determined. The right Cauchy-Green tensor;

\[
[C_1] = [F_i] [F_i]^T = \begin{bmatrix}
\left( \frac{\Theta_0 R}{\pi \Lambda \rho} \right)^2 & 0 & 0 \\
0 & \left( \frac{\pi \rho}{\Theta_0 R} \right)^2 & 0 \\
0 & 0 & \Lambda^2
\end{bmatrix} \tag{4-6}
\]

and the Green strain tensor,

\[
[E_1] = \frac{1}{2} (C_1 - I) = \frac{1}{2} \begin{bmatrix}
\left( \frac{\Theta_0 R}{\pi \Lambda \rho} \right)^2 - 1 & 0 & 0 \\
0 & \left( \frac{\pi \rho}{\Theta_0 R} \right)^2 - 1 & 0 \\
0 & 0 & \Lambda^2 - 1
\end{bmatrix}. \tag{4-7}
\]

The residual stress related to the deformation gradient \( F_i \) contains only diagonal terms; thus, components of the associated right stretch tensor \( U_i \) are numerically same as those of \( F_i \) which means \( F_i \) does not contain rigid body motion. (\( F = RU = VR \)).
4.2.2. Inflation, extension and torsion,

Next, consider a subsequent mapping of the same centrally located material particle from its position \((\rho, \nu, \xi)\) in the intact unloaded configuration to a new position \((r, \theta, z)\) in a loaded configuration

\[
r = r(\rho), \quad \theta = \nu + \gamma \xi, \quad z = \lambda \xi
\]  

(4-8)

\(\gamma\) is a twist per unit unloaded length, \(\lambda\) is an axial load induced stretch defined per unit unloaded length.

Regardless, the associated deformation gradient, \(F_2\) becomes

\[
[F_2] = \begin{bmatrix}
\frac{\partial r}{\partial \rho} & \frac{\partial r}{\partial \nu} & \frac{\partial r}{\partial \xi} \\
\frac{\partial r}{\partial \theta} & \frac{\partial r}{\partial \nu} & \frac{\partial r}{\partial \xi} \\
\frac{\partial r}{\partial \rho} & \frac{\partial r}{\partial \nu} & \frac{\partial r}{\partial \xi}
\end{bmatrix} = \begin{bmatrix}
\frac{\partial r}{\partial \rho} & 0 & 0 \\
0 & \frac{r}{\rho} & r \gamma \\
0 & 0 & \lambda
\end{bmatrix}
\]  

(4-9)

Enforcing the incompressibility of this second motion

\[
\det(F_2) = 1 \quad \Rightarrow \quad \frac{\partial r}{\partial \rho} = \frac{\rho}{r \lambda}
\]  

(4-10)

and integration according to \(r\)

\[
r^2 - r_i^2 = \frac{1}{\lambda} \left( \rho^2 - \rho_i^2 \right), \quad R \in [\rho_i, \rho_a]
\]  

(4-11)

Likewise the associated physical components of \(C_2\) and \(E_2\) are
Comparing the deformation gradient of second deformation with the first, in the second deformation there are non-diagonal terms and the principal directions are not \((r, \theta, z)\).

Anymore, combining these two situations, we can find the total deformation gradient according to the zero stress state as;

\[
[C_T] = [F_T]^T [F_T] = \begin{bmatrix}
\left(\frac{\rho}{r\lambda}\right)^2 & 0 & 0 \\
0 & \left(\frac{r}{\rho}\right)^2 & \frac{r^2 \gamma}{\rho} \\
0 & \frac{r^2 \gamma}{\rho} & (\lambda^2 + r^2 \gamma^2)
\end{bmatrix},
\]  
\(4-12\)

\[
[E_T] = \frac{1}{2} (C_T - I) = \frac{1}{2} \begin{bmatrix}
\left(\frac{\rho}{r\lambda}\right)^2 - 1 & 0 & 0 \\
0 & \left(\frac{r}{\rho}\right)^2 - 1 & \frac{r^2 \gamma}{\rho} \\
0 & \frac{r^2 \gamma}{\rho} & (\lambda^2 + r^2 \gamma^2) - 1
\end{bmatrix}.
\]  
\(4-13\)

\[
[F_T] = [F_2].[F_1] = \begin{bmatrix}
\frac{\partial r}{\partial R} & 0 & 0 \\
0 & \frac{r \pi}{\Theta_0 R} & r \lambda \gamma \\
0 & 0 & \lambda \lambda
\end{bmatrix},
\]  
\(4-14\)

\[
[C_T] = [F_T]^T [F_T] = \begin{bmatrix}
\left(\frac{\partial r}{\partial R}\right)^2 & 0 & 0 \\
0 & \left(\frac{r \pi}{\Theta_0 R}\right)^2 & \frac{r^2 \pi \lambda \gamma}{\Theta_0 R} \\
0 & \frac{r^2 \pi \lambda \gamma}{\Theta_0 R} & \lambda^2 (\lambda^2 + r^2 \gamma^2)
\end{bmatrix},
\]  
\(4-15\)
\[
\begin{bmatrix}
    \frac{\partial r}{\partial R}^2 - 1 & 0 & 0 \\
    0 & \left(\frac{r\pi}{\Theta_0 R}\right)^2 - 1 & \frac{r^2 \pi \lambda y}{\Theta_0 R} \\
    0 & \frac{r^2 \pi \lambda y}{\Theta_0 R} & \Lambda^2 \left(\lambda^2 + r^2 y^2\right) - 1
\end{bmatrix}
\]  

(4-16)

Then the stretch ratios become;

\[
\lambda_\theta = \frac{\pi r}{\Theta_0 R}, \quad \lambda_z = \lambda \Lambda \quad \text{and due to incompressibility} \quad \lambda_R = \frac{1}{\lambda_\theta \lambda_z} = \frac{\Theta_0 R}{r \pi \lambda \Lambda}
\]

(4-17)

By using these stretch ratios

\[
E_{rr} = \frac{1}{2} \left(\lambda_R^2 - 1\right), \quad E_{\theta \theta} = \frac{1}{2} \left(\lambda_\theta^2 - 1\right), \quad E_{zz} = \frac{1}{2} \left(\lambda_z^2 + \left(r \gamma \Lambda\right)^2 - 1\right), \quad E_{xz} = \frac{1}{2} \left(\lambda_\theta \left(r \gamma \Lambda\right)\right)
\]

(4-18)

can be calculated.

It is useful to record the general expressions for the associated physical components of the Cauchy stress.

Using the general form of the equation

\[
\sigma = -pI + F \frac{\partial W}{\partial E} F^T
\]

(4-19)

\[
\sigma_{rr} = -p + \lambda_R^2 \frac{\partial W}{\partial E_{rr}} = -p + \left(\frac{\pi r}{\Theta_0 R}\right)^2 \frac{\partial W}{\partial E_{rr}},
\]

(4-20)

\[
\sigma_{\theta \theta} = -p + \lambda_\theta^2 \frac{\partial W}{\partial E_{\theta \theta}} + r \gamma \Lambda \lambda_\theta \left(\frac{\partial W}{\partial E_{\theta \theta}} + \frac{\partial W}{\partial E_{zz}}\right) + \gamma^2 \lambda_z^2 \frac{\partial W}{\partial E_{zz}},
\]

(4-21)

\[
\sigma_{zz} = -p + \lambda_z^2 \frac{\partial W}{\partial E_{zz}}
\]

(4-22)

are the normal components of the Cauchy stress tensor where,
\[ \sigma_{r\theta} = \lambda_R \left( \lambda_0 \frac{\partial W}{\partial E_{r\theta}} + r\gamma \Lambda \right) = \frac{1}{\lambda \Lambda} \frac{\partial W}{\partial E_{r\theta}} + \left( \frac{\Theta_0 R \gamma}{\pi \lambda} \right) \frac{\partial W}{\partial E_{rz}}, \]  \hspace{1cm} (4-23)

\[ \sigma_{rz} = \lambda_R \lambda_z \frac{\partial W}{\partial E_{rz}} = \left( \frac{\Theta_0 R \gamma}{\pi \lambda} \right) \frac{\partial W}{\partial E_{rz}}, \]  \hspace{1cm} (4-24)

\[ \sigma_{zt} = \lambda_z \left( \lambda_0 \frac{\partial W}{\partial E_{tz}} + r\gamma \Lambda \right) \frac{\partial W}{\partial E_{zz}} = \frac{r \pi \lambda \Lambda}{\Theta_0 R} \frac{\partial W}{\partial E_{ez}} + \left( r \gamma \Lambda^2 \right) \frac{\partial W}{\partial E_{zz}}, \]  \hspace{1cm} (4-25)

are the shear components.

Applying the global equilibrium equations regarding the conditions;

- The total deformation \( F \) (thus \( C \) and \( E \)) and the Lagrange multiplier varies only with \( r \)
- Assuming both axisymmetry \( \frac{\partial}{\partial \theta} = 0 \)
- Neglecting the body forces

yield,

\[ \frac{\partial \sigma_{rr}}{\partial r} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} = 0 \]  \hspace{1cm} (4-26)

\[ \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \sigma_{r\theta} \right) = 0 \]  \hspace{1cm} (4-27)

\[ \frac{1}{r} \frac{\partial}{\partial r} (r \sigma_{rz}) = 0 \]  \hspace{1cm} (4-28)

Note that integrating the last two equations yield,

\[ \sigma_{r\theta} = \frac{d_1}{r^2} \quad \text{and} \quad \sigma_{rz} = \frac{d_2}{r} \]  \hspace{1cm} (4-29)

where \( d_1 \) and \( d_2 \) are constants. If \( W \) is known, these equations determine what tractions are required on the \( n = \pm e \) faces to maintain the prescribed deformation. Or, if the
circumferential and axial components of the traction vector \( \mathbf{T}^{(o)} \), with \( n = \pm e_r \) are known at \( r = r_i \) and \( r = r_a \), then the associated values of \( d_i \) can be calculated. Whereas it is nearly impossible to ascertain the tractions on the adventitial surface in vivo, which result from perivascular coupling, the tractions on the intimal surface can be calculated from the hemodynamics. Indeed, \( \sigma_{rz} \) at the intimal must be equal and opposite of the axially directed shear stress in the blood at the wall, a quantity that is known to be very important in vascular mechanics. Conversely, tractions on the intimal surface due to endovascular balloon catheters are problematic to evaluate. Finally, in most in vitro experiments designed to study wall properties, the shearing components of the traction vector are negligible, hence \( d_i = 0 \). In this case, the only nonzero equilibrium equation 4-26 can be integrated to yield,

\[
\sigma_{rr}(r) - \sigma_{rr}(r_i) = \int_i^r \left( \sigma_{\theta \theta} - \sigma_{rr} \right) \frac{1}{r} \, dr \tag{4-30}
\]

applying \( \sigma_{rr}(r_i) = -P_i \) as the third traction boundary condition on the intimal surface

\[
p(r) = \left( \frac{R \Theta_0}{r \pi \lambda \Lambda} \right)^2 \frac{\partial W}{\partial E_{RR}} + P_i - \int_i^r \left( \sigma_{\theta \theta} - \sigma_{rr} \right) \frac{1}{r} \, dr \tag{4-31}
\]

The variation of the Lagrange multiplier can be calculated. Again using the condition \( \sigma_{rr}(r_i) = -P_i \) we obtain the transmural pressure

\[
P_i = \int_i^a \left( \sigma_{\theta \theta} - \sigma_{rr} \right) \frac{1}{r} \, dr \tag{4-32}
\]

Substituting the equations and changing \( r_a = a \) and \( r_i = b \)

\[
P = \int_a^b \left( \lambda_q^2 \frac{\partial W}{\partial E_{\theta \theta}} + \gamma^2 \lambda_z^2 \frac{\partial W}{\partial E_{zz}} - \lambda_r^2 \frac{\partial W}{\partial E_{rr}} \right) \frac{1}{r} \left( \frac{\partial W}{\partial E_{\theta z}} + \frac{\partial W}{\partial E_{z \theta}} \right) \, dr \tag{4-33}
\]

Note that the equation of transmural pressure does not contain the Lagrange multiplier. Thus; it is only related to the inner and outer radius and strain energy.
Likewise, the axial force can be obtained as

$$ F_{AX} = 2\pi \int_{a}^{b} \sigma_{zz} r dr $$  \hspace{1cm} (4-34)

Integrating by parts the equation, to avoid from Lagrange multiplier yields

$$ F_{AX} = \pi \int_{a}^{b} (2\sigma_{zz} - \sigma_{rr} - \sigma_{\theta \theta}) r dr , $$  \hspace{1cm} (4-35)

And substituting the stress equations in to the relation yields

$$ F_{AX} = \pi \int_{a}^{b} \left( 2\lambda_{Z} \frac{\partial W}{\partial E_{ZZ}} - \lambda_{R} \frac{\partial W}{\partial E_{RR}} - \lambda_{\theta} \frac{\partial W}{\partial E_{\theta \theta}} - 2r\gamma\lambda\lambda_{\theta} \frac{\partial W}{\partial E_{\theta Z}} - (r\gamma\lambda)^{2} \frac{\partial W}{\partial E_{ZZ}} \right) r dr . $$  \hspace{1cm} (4-36)

A similar expression for the twisting moment M is given directly by

$$ M_{b} = 2\pi \int_{a}^{b} \sigma_{\theta \theta} r^{2} dr $$  \hspace{1cm} (4-37)

which has the open form,

$$ M_{b} = 2\pi \int_{a}^{b} \left( \lambda_{Z} \left( \lambda_{\theta} \frac{\partial W}{\partial E_{\theta Z}} + r\gamma\lambda \right) \frac{\partial W}{\partial E_{ZZ}} \right) r^{2} dr . $$  \hspace{1cm} (4-38)

### 4.3. Experimental Approach

It is an optimization process involving extracting the experimental data in to the theoretical approach and finding the material parameters. Therefore, either during the experiment or in the theoretical approach there are some key points which should be taken in to account as

- Obtaining pressure, axial force and torsional moment simultaneously
- Obtaining deformations in 3D simultaneously
- Obtaining Geometric dimensions simultaneously
• Using non-contact measurement systems because of the structure of the soft tissue
• Applying appropriate loading and boundary conditions
• Protecting the tissue physiology before the test and during the test
• The measurements should be taken from the most homogeneous loading part of the material

According to these important points and also regarding to the recent works in the literature, optical measurement systems are the most appropriate one. Thus, in this work, a new optical measurement system will be used.

In the experiment, the measured and geometrical values are going to be given according to outer radius (adventitia). Therefore, the expressions of physical quantities such as transmural pressure, axial force and torsional moment should be in terms of the parameters which are measured due to the outer radius. Then these expressions have the form:

\[
P = \int_0^L \left( \lambda_0^2 \frac{\partial W}{\partial E_{\theta\theta}} + r\gamma\Lambda_0 \left( \frac{\partial W}{\partial E_{\theta\theta}} + \frac{\partial W}{\partial E_{\varphi\varphi}} \right) + \gamma^2 \lambda_z^2 \frac{\partial W}{\partial E_{zz}} - \lambda_R^2 \frac{\partial W}{\partial E_{rr}} \right) \frac{dr}{r},
\]

\[
F_{AX} = \pi \int_0^L \left( 2\lambda_z^2 \frac{\partial W}{\partial E_{zz}} - \lambda_R^2 \frac{\partial W}{\partial E_{rr}} - \lambda_0^2 \frac{\partial W}{\partial E_{\theta\theta}} - 2r\gamma\Lambda_0 \lambda_0 \frac{\partial W}{\partial E_{\varphi\varphi}} - (r\gamma\lambda)^2 \frac{\partial W}{\partial E_{zz}} \right) r dr,
\]

\[
M_b = 2\pi \int_0^L \left( \lambda_z \left( \lambda_0 \frac{\partial W}{\partial E_{\varphi\varphi}} + r\gamma\Lambda_0 \frac{\partial W}{\partial E_{zz}} \right) \right) r^2 dr.
\]

Regarding to these equations, the following steps should be realized:

• Obtaining the term \( r\gamma\lambda \) in terms of measured strains
• Expressing inner and outer radius in terms of the measured strains
• The geometrical parameter “\( r \)” in the expressions should be shifted in terms non-dimensionless stretch parameters

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• Noting that the problem is containing two different steps and therefore, changing the measured strain values in to the total strains.

Examining the radius expressions regarding equation 4-11 and incompressibility

\[ b^2 - r^2 = \frac{\Theta_0}{\pi \Lambda} \left( b_0^2 - R^2 \right) \]  \hspace{1cm} (4-42)

Tangential stretch

\[ \lambda_\theta = \lambda_{\theta_1} \lambda_{\theta_2} = \frac{\pi \rho}{\theta_0 R} \frac{r}{\rho} \Rightarrow \frac{\pi b_1 b_0}{\theta_0 b_0} = \lambda_{\theta_1} (b_1) \beta = \lambda_{\theta_1} \beta \Rightarrow b = \lambda_{\theta_1} \beta b_0 \]  \hspace{1cm} (4-43)

\[ \lambda_\varphi = \frac{r \pi}{R \Theta_0} \Rightarrow R = \frac{r \pi}{\lambda_\varphi \Theta_0} \]  \hspace{1cm} (4-44)

Using both expressions

\[ \left( \lambda_{\theta_1} \beta b_0 \right)^2 - r^2 = \frac{\Theta_0}{\pi \Lambda} \left( b_0^2 - \left( \frac{r \pi}{\lambda_\varphi \Theta_0} \right)^2 \right) \]  \hspace{1cm} (4-45)

Thus the “r” term becomes

\[ r^2 = b_0^2 \left( \left( \frac{\lambda_{\theta_1} \beta}{\Theta_0} \right)^2 - \frac{\Theta_0}{\pi \Lambda \lambda_\varphi^2} \right) \]  \hspace{1cm} (4-46)

Taking derivative of both sides

\[ 2rdr = \left( -2b_0^2 \right) \frac{\left( \lambda_{\theta_1} \beta \right)^2 - \frac{\Theta_0}{\pi \Lambda \lambda_\varphi^2} \left( \frac{\pi}{\Theta_0 \lambda_\Lambda} \lambda_\varphi^{-3} \right) \lambda_\Lambda^{-1}}{1 - \frac{\pi}{\Theta_0 \lambda_\Lambda} \lambda_\varphi^{-2}} d\lambda_\varphi \]  \hspace{1cm} (4-47)

and dividing with \( r^2 \)
\[
\int r^2 dr = b_0^3 \left( \beta^2 - \frac{\Theta_0}{\pi \lambda \Lambda} \right)^2 \frac{\pi}{\Theta_0 \lambda^2 \lambda^{-3}} \lambda^{-1} d\lambda
\]  

(4-48)

can be calculated easily.

Likewise, upper and lower limit can be calculated with parameter transformation as follows,

\[
r^2 = \left( \lambda_{ab} \beta b_0 \right)^2 - \frac{\Theta_0}{\pi \lambda \Lambda} \left( b_0^2 - \left( \frac{r \pi}{\lambda_0 \Theta_0} \right)^2 \right),
\]  

(4-49)

\[
b^2 = \left( \lambda_{ab} \beta b_0 \right)^2 - \frac{\Theta_0}{\pi \lambda \Lambda} \left( b_0^2 - \left( \frac{b \pi}{\lambda_0 \Theta_0} \right)^2 \right),
\]  

(4-50)

\[
\left( \lambda_{ab} \beta b_0 \right)^2 = \left( \lambda_{ab} \beta b_0 \right)^2 - \frac{\Theta_0}{\pi \lambda \Lambda} \left( b_0^2 - \left( \frac{\lambda_{ab} \beta b_0}{\lambda_0 \Theta_0} \right)^2 \right),
\]  

(4-51)

\[
0 = b_0^2 - \left( \frac{\lambda_{ab} \beta b_0}{\lambda_0 \Theta_0} \right)^2,
\]  

(4-52)

\[
\lambda_p = \left( \frac{\pi}{\Theta_0} \lambda_{ab} \right) \beta.
\]  

(4-53)

Also, repeating the same manner,

\[
b^2 - a^2 = \frac{\Theta_0}{\pi \lambda \Lambda} \left( b_0^2 - a_0^2 \right),
\]  

(4-54)

\[
a^2 = \left( \lambda_{ab} \beta b_0 \right)^2 - \frac{\Theta_0}{\pi \lambda \Lambda} \left( b_0^2 - a_0^2 \right),
\]  

(4-55)
and shifting \( r \) with a

\[
\lambda_a = \frac{\pi}{\Theta_0} \sqrt{\left(\lambda_{\alpha \beta}, \beta\right)^2 \left(\frac{b_0}{a_0}\right)^2 - \frac{\Theta_0}{\pi \lambda \Lambda} \left(\frac{b_0}{a_0}\right)^2 - 1},
\]

(4-58)
can be found.

The arterial wall mechanics is forming from two parts. Therefore, total deformation expressions should be written in terms of both parameters. In the following expressions these quantities are obtained.

\[
F_T = F_2 F_1
\]

(4-59)

\[
C_T = F^T F = (F_2, F_1)^T (F_2, F_1) = F_1^T F_2 F_2^T F_1 = F_1^T C_2 F_1
\]

(4-60)

\[
E_r = \frac{1}{2} (C_T - I)
\]

(4-61)

\[
E_r = \frac{1}{2} \left(F_1^T C_2 F_1 - I\right)
\]

(4-62)

\[
E_2 = \frac{1}{2} (C_2 - I)
\]

(4-63)

\[
C_2 = 2E_2 + I
\]

(4-64)

\[
E_r = \frac{1}{2} \left(F_1^T (2E_2 + I) F_1 - I\right)
\]

(4-65)
\( E_r = \frac{1}{2} (2F_i^T E_2 F_1 + F_i^T \mathbf{F} - \mathbf{I}) \)  
\[ (4-66) \]

\( E_r = \frac{1}{2} (2F_i^T E_2 F_1 + 2E_i) \)  
\[ (4-67) \]

\( E_r = F_i^T E_2 F_1 + E_i \)  
\[ (4-68) \]

\[
E_r = \frac{1}{2} \begin{bmatrix}
\left( \lambda_R^2 - 1 \right) & 0 & 0 \\
0 & \left( \lambda_\theta^2 - 1 \right) & r_\gamma \Lambda \lambda_\theta \\
0 & r_\gamma \Lambda \lambda_\theta & \left( \lambda_Z^2 + (r_\gamma \Lambda)^2 - 1 \right)
\end{bmatrix}
\]  
\[ (4-69) \]

\[
F_i^T E_2 F_1 = \begin{bmatrix}
\Theta_0 R \\
\pi \rho
\end{bmatrix}
\begin{bmatrix}
0 & 0 & E_{pp} & 0 & 0 \\
0 & E_{vv} & E_{\psi v} & 0 & 0 \\
0 & 0 & \Lambda & 0 & 0
\end{bmatrix}
\begin{bmatrix}
\Theta_0 R \\
\pi \rho
\end{bmatrix}
\]  
\[ (4-70) \]

\[
E_i = \frac{1}{2} \begin{bmatrix}
\left( \frac{\Theta_0 R}{\pi \rho} \right)^2 & 0 & 0 \\
0 & \left( \frac{\pi \rho}{\Theta_0 R} \right)^2 & 0 \\
0 & 0 & \left( \Lambda^2 - 1 \right)
\end{bmatrix}
\]  
\[ (4-71) \]

or in different notation

\[
F_i^T E_2 F_1 = \begin{bmatrix}
\frac{1}{\lambda_{Z_i} \lambda_{\theta_i}} & 0 & 0 \\
0 & \lambda_{\theta_i} & 0 \\
0 & 0 & \lambda_{Z_i}
\end{bmatrix}
\begin{bmatrix}
E_{pp} & 0 & 0 \\
0 & E_{vv} & E_{\psi v} \\
0 & 0 & E_{\psi v}
\end{bmatrix}
\begin{bmatrix}
\frac{1}{\lambda_{Z_i} \lambda_{\theta_i}} & 0 & 0 \\
0 & \lambda_{\theta_i} & 0 \\
0 & 0 & \lambda_{Z_i}
\end{bmatrix}
\]  
\[ (4-72) \]
\[
E_i = \frac{1}{2} \begin{bmatrix}
\left( \frac{1}{\lambda_{Z_i} \lambda_{\theta_i}} \right)^2 - 1 & 0 & 0 \\
0 & \lambda_{\theta_i}^2 - 1 & 0 \\
0 & 0 & \lambda_{Z_i}^2 - 1
\end{bmatrix}
\]

\[
F_i^T E_2 F_i + E_i = \frac{1}{2} \begin{bmatrix}
0 & 0 & \left( \lambda_{\theta_i}^2 - 1 + 2\lambda_{\theta_i}^2 E_{\theta \theta} \right) \\
0 & 0 & \left( \lambda_{Z_i} \lambda_{\theta_i} E_{\phi \phi} \right) \\
\left( \lambda_{Z_i} \lambda_{\theta_i} E_{\phi \phi} \right) & \left( \lambda_{Z_i} \lambda_{\theta_i} E_{\phi \phi} \right) & \left( \lambda_{Z_i}^2 - 1 + 2\lambda_{Z_i}^2 E_{\phi \phi} \right)
\end{bmatrix}
\]

According to the equivalence of tensors, stretch ratios can be obtained in terms of measurable quantities. For the outer surface,

\[
\lambda_R^2 - 1 = \left( \frac{1}{\lambda_{Z_i} \lambda_{\theta_i}} \right)^2 - 1 + \frac{2E_{\theta \theta}}{(\lambda_{Z_i} \lambda_{\theta_i})^2}
\]

\[
\lambda_R = \sqrt{1 + 2E_{\theta \theta}}
\]

\[
\lambda_{\theta_i}^2 - 1 = \left( \lambda_{\theta_i}^2 - 1 + 2\lambda_{\theta_i}^2 E_{\theta \theta} \right)
\]

\[
\lambda_{\theta_i} = \lambda_{\theta_i} \sqrt{1 + 2E_{\theta \theta}}
\]

The \( r\gamma \Delta \) in the outer surface conditions, parameter transformation should be made as

\[
(r\gamma \Delta \lambda)_{r=b} = \lambda_{Z_i} \lambda_{\theta_i} E_{\phi \phi}
\]

\[
(r\gamma \Delta) = \frac{\lambda_{Z_i} \lambda_{\theta_i} E_{\phi \phi}}{\lambda_{\theta_i}}
\]
\[ r\gamma\Lambda = \frac{\lambda_{z1} E_{vz}}{\sqrt{1 + 2E_{vv}}} \]  \hspace{2cm} (4-81)

\[ r = b_0 \sqrt{\left( \frac{\left( \lambda_{00}, \beta \right) - \Theta_0}{\pi \lambda \Lambda} \right)^2 + \left( 1 - \frac{\Theta_0}{\pi \lambda \Lambda} \frac{\pi^2}{\Theta_0^2} \left( \lambda_{00}, \beta \right)^2 \right)} \]  \hspace{2cm} (4-82)

\[ b = b_0 \sqrt{\left( \frac{\left( \lambda_{00}, \beta \right) - \Theta_0}{\pi \lambda \Lambda} \right)^2 + \left( 1 - \frac{\Theta_0}{\pi \lambda \Lambda} \frac{\pi^2}{\Theta_0^2} \left( \lambda_{00}, \beta \right)^2 \right)} \]  \hspace{2cm} (4-83)

(Outer surface)

Thus,

\[ r\gamma\Lambda = \frac{\lambda_{z1} E_{vz}(r)}{\sqrt{1 + 2E_{vv}(r)}} \]  \hspace{2cm} (4-84)

\[ b\gamma\Lambda = \frac{\lambda_{z1} E_{vz}}{\sqrt{1 + 2E_{vv}}} \]  \hspace{2cm} (4-85)

\[ \gamma = \frac{\lambda_{z1} E_{vz}}{b\lambda_{z1} \sqrt{1 + 2E_{vv}}} \]  \hspace{2cm} (4-86)

\[ \gamma = \frac{E_{vz}}{b_0 \sqrt{\left( \frac{\left( \lambda_{00}, \beta \right) - \Theta_0}{\pi \lambda \Lambda} \right)^2 + \left( 1 - \frac{\Theta_0}{\pi \lambda \Lambda} \frac{\pi^2}{\Theta_0^2} \left( \lambda_{00}, \beta \right)^2 \right)} \sqrt{1 + 2E_{vv}}} \]  \hspace{2cm} (4-87)
\[ r\gamma \Lambda = \frac{b_0 E_v \lambda_{Zi}}{\sqrt{1 + 2E_v}} \sqrt{\frac{1 - \frac{\Theta_0}{\pi \lambda \Lambda} \left( \frac{1}{\Theta_0} \left( \lambda_{\phi_i} \beta \right) \right)^2}{1 - \frac{\Theta_0}{\pi \lambda \Lambda} \left( \frac{1}{\Theta_0} \left( \lambda_{\phi_i} \beta \right) \right)^2}} \]  

\[ (4-88) \]

\[ r\gamma \Lambda = \frac{E_v \lambda_{Zi}}{\sqrt{1 + 2E_v}} \sqrt{\frac{1 - \frac{\pi}{\Theta_0} \frac{1}{\lambda \Lambda} \left( \frac{1}{\Theta_0} \left( \lambda_{\phi_i} \beta \right) \right)^2}{1 - \frac{\pi}{\Theta_0} \frac{1}{\lambda \Lambda} \left( \frac{1}{\Theta_0} \left( \lambda_{\phi_i} \beta \right) \right)^2}} \]  

\[ (4-89) \]

\[ \lambda_z^2 + (r\gamma \Lambda)^2 - 1 = \lambda_z^2 - 1 + \lambda_{Zi}^2 \left( 1 + 2E_v \right) \]  

\[ (4-90) \]

\[ \lambda_z^2 = \lambda_{Zi}^2 \left( 1 + 2E_v \right) - \lambda_{Zi}^2 \frac{E_v \lambda_{Zi}^2}{1 + 2E_v} \]  

\[ (4-91) \]

\[ \lambda_z = \lambda_{Zi} \sqrt{1 + 2E_v} - \frac{E_v \lambda_{Zi}^2}{1 + 2E_v} \]  

\[ (4-92) \]

Finally, it is seen that; by substitution the expressions of pressure, axial force and moment are obtained in terms of measurable quantities.

### 4.4. Some Constitutive Models for Arterial Walls

The active mechanical behavior of arterial walls is governed mainly by the intrinsic properties of elastin and collagen fibers and by the degree of activation of smooth muscles. An adequate constitutive model for arteries which incorporates the active state (contraction of smooth muscles) was proposed recently by Rachev and Hayashi [48].

The passive mechanical behavior of arterial walls is quite different and is governed mainly by the elastin and collagen fibers. The passive state of the smooth muscles may also contribute to the passive arterial behavior but the extent of this contribution is not yet known. Most constitutive models proposed for arteries are valid for the passive state of
smooth muscles and are based on a phenomenological approach which describes the artery as a macroscopic system. Furthermore, most of these models are designed to capture the response near the physiological state and in this respect they have been successfully applied in fitting experimental data. The most common potentials (strain-energy functions) are of exponential type, although polynomial and logarithmic forms are also used.

Some of the constitutive models proposed use the biphasic theory to describe arterial walls as hydrated soft tissues; see, for example, the works by Simon and co-workers [49,3]. Less frequently used are models which account for the specific morphological structure of arterial walls. One attempt to model the helically wound fibrous structure is provided by Tozeren [50], which is based on the idea that the only wall constituent is the fiber structure. However, this is a significant simplification of the histological structure.

Another structural model due to Wuyts et al. [51] assumes that the wavy collagen fibrils are embedded in concentrically arranged elastin/smooth-muscle membranes, which is in agreement with the histological situation [52]. The model in [51] assumes that the collagen fibrils have a statistically distributed initial length. Each fibril may be stretched initially with a very low force but thereafter its behavior is linearly elastic. Only the media is considered as (solid) mechanically relevant. Although the model proposed in [51] attempts to incorporate histological information, which is a very promising approach, it is only possible to represent the deformation behavior of axially-symmetric thick-walled vessels: Another drawback is the fact that the artery is considered as a tube reinforced by circularly oriented collagen and elastin fibers, which does not model the real histological situation.

Most of the constitutive models treat the arterial wall as a single layer, but a number of two-layer models have been proposed in the literature. Two-layer models which include anisotropy are those due to, for example, Von Maltzahn et al. [53], Demiray [54] and Rachev [55]. However, the emphasis of the latter paper is on stress-dependent remodeling.

Here are some constitutive strain energy equations which are summarized from the recent cardiovascular mechanical works.
The Mooney and Ogden models are first isotropic hyperelastic models which are commonly used for elastomers and soft tissues. These models are also integrated with some commercial finite element software. Ogden and Mooney type models are based on stretch ratios. However, for arterial tissues, these models are failed to determine material behavior.

\[ W = c_0 (I_1 - 3) + c_1 (I_2 - 3) + c_2 (I_1 - 3)(I_2 - 3) \quad (4-93) \]

\[ I_1 = \lambda_1^2 + \lambda_2^2 + \lambda_3^2 \quad (4-94) \]

\[ I_2 = \lambda_1^{-2} + \lambda_2^{-2} + \lambda_3^{-2} \quad (4-95) \]

Ogden;

\[ W = \sum_{i=1}^{n} \left( \lambda_1^{\alpha_i} + \lambda_2^{\alpha_i} + \lambda_3^{\alpha_i} - 3 \right) \quad (4-96) \]

Vaishnav et al (1973) [17] proposed and compared three different multi axial forms of \( W \) to describe the 3D anisotropic behavior of canine thoracic aorta. They explicitly enforce incompressibility rather than via a Lagrange multiplier approach. Briefly, each \( W \) was taken to be a polynomial function of the Green strains, but contained different number of material parameters. Overall a seven parameter relation was deemed best. It is

\[ W = c_1 E_{\theta \theta}^2 + c_2 E_{\theta \theta} E_{ZZ} + c_3 E_{ZZ}^2 + c_4 E_{\theta \theta}^3 + c_5 E_{\theta \theta}^2 E_{ZZ} + c_6 E_{\theta \theta} E_{ZZ}^2 + c_7 E_{ZZ}^3 \quad (4-97) \]

In an earlier paper they determined “best-fit” values of the parameters from mean stress-strain data, whereas in the 1973 paper they determined the values from global equilibrium and thereby accounted for transmural variations in stress and strains. Differences in the two sets of parameter values prompted Vaishnav and colleagues to conclude that the thick walled approach was preferred, a conclusion that is generally held today.

Holzapfel and Wiewsacker (1998) [17] proposed a combined polynomial exponential form of \( W \) for passive aorta. Specifically, they suggested that
\[ W = c_0 (I_C - 3) + \frac{1}{2} c (e^0 - 1) \quad (4-98) \]

\[ I_C = trC = 2trE + 3 \quad (4-99) \]

\[ Q = c_1 E_{RR}^2 + c_2 E_{\theta\theta}^2 + c_3 E_{\theta\theta}^2 + 2c_4 E_{RR} E_{\theta\theta} + 2c_5 E_{\theta\theta} E_{ZZ} + 2c_6 E_{ZZ} E_{RR} \quad (4-100) \]

Here \( c_0 \) and \( c \) values are parameters having the units of stress, whereas the other parameters are nondimensional.

Demiray and Vito (1991) [54] presented such results for passive aorta. Using a Lagrange multiplier approach and based on biaxial data, they suggested the following form of \( W \)

\[ W_{Media} = \frac{c}{2} (\exp [c_1 (C_{11} - 1) + c_2 (C_{22} - 1) + c_3 (C_{33} - 1)] - 1) \quad (4-101) \]

for the media, where the subscripts 1 and 2 on the components of \( C \) denote in-plane components (\( \Theta, Z \)) however, the third one denotes the out of plane component \( R \). In contrast they suggested

\[ W_{Adv} = \frac{\beta}{2\alpha} (\exp [\alpha (I_C - 3)] - 1) \quad (4-102) \]

For the adventitia, where \( \alpha, \beta \) are material parameters and

\[ I_C = trC = 2trE + 3 \quad (4-103) \]

Fung et al (1979) [5] proposed a 2D exponential pseudo strain-energy function \( W \) for various passive elastic rabbit arteries which they suggested was a better descriptor of their data than polynomial form of \( W \) proposed by Vaishnav et al for aorta. It appears however that Fung et al treated the Vaishnav relation as 2D even tough it was 3D with incompressibility enforced explicitly. Nonetheless, Choung and Fung (1983) [16] later suggested a 3D exponential form of \( W \). It is

\[ W = \frac{1}{2} c e^0 \quad (4-104) \]
where $W$ is defined per unit initial volume and $Q$ is

$$
Q = c_1 E_{rr}^2 + c_2 E_{\theta \theta}^2 + c_3 E_{zz}^2 + 2c_4 E_{rr} E_{\theta \theta} + 2c_5 E_{\theta \theta} E_{zz} + 2c_6 E_{zz} E_{rr} + c_7 (E_{rr}^2 + E_{\theta \theta}^2) + c_8 (E_{\theta \theta}^2 + E_{zz}^2) + c_9 (E_{zz}^2 + E_{rr}^2)
$$

(4-105)

Von Malzahn et al (1984) and Takamizawa and Hayashi (1987) [29] reported that Fung type exponential forms of $W$ describe better the passive behavior of carotid arteries than polynomial forms. They both used 3D relations with incompressibility enforced explicitly as suggested by Vaishnav. Also including residual stresses in their data analysis, Takamizawa and Hayashi suggested a logarithmic form of $W$

$$
W = -c \ln(1 - Q)
$$

(4-106)

$$
Q = \frac{1}{2} c_1 E_{\theta \theta}^2 + \frac{1}{2} c_2 E_{zz}^2 + c_3 E_{\theta \theta} E_{zz}
$$

(4-107)

was a slightly better descriptor of carotid behavior than an exponential form.

Von Malzahn et al (1984) [53] showed that the following exponential pseudo strain-energy function described well the behavior of both the adventitia and media of bovine carotid arteries

$$
W = \frac{1}{2} c(\exp[c_1 E_{\theta \theta}^2 + c_2 E_{zz}^2 + 2c_3 E_{\theta \theta} E_{zz}] - 1)
$$

(4-108)

Kasyanov and Rachev (1980) [48] presented a very nice paper on the mechanical behavior of human carotid arteries tested via finite inflation, extension and torsion. Using the approach advocated by Vaishnav, they enforced incompressibility directly and proposed a combined polynomial exponential form of $W$

$$
W = c_1 [e^Q - 1] + [c_7 E_{\theta \theta} e^{c_8 e_{\text{str}}} + c_9 E_{zz} + c_{10} (E_{\theta \theta}^2 + E_{zz}^2)]
$$

(4-109)

$$
Q = c_2 E_{zz}^2 + c_3 E_{\theta \theta} E_{zz} + c_4 E_{\theta \theta}^2 + c_5 E_{\theta \theta} E_{zz}^2 + c_6 E_{\theta \theta}^2 E_{zz}
$$

(4-110)

\begin{equation}
W(C, A_1, A_2) = W_{iso}(I_1) + W_{aniso}(I_4, I_6) \tag{4-111}
\end{equation}

\begin{equation}
W_{iso} = \frac{c}{2}(I_1 - 3) \tag{4-112}
\end{equation}

\begin{equation}
W_{aniso} = \frac{k_1}{2k_2} \sum_{i=4,6} \exp[k_2(I_i - 1)^2] - 1 \tag{4-113}
\end{equation}

\begin{equation}
I_1(C) = trC \tag{4-114}
\end{equation}

\begin{equation}
I_4(C, a_{01}) = C : A_1 \tag{4-115}
\end{equation}

\begin{equation}
I_6(C, a_{02}) = C : A_2 \tag{4-116}
\end{equation}

\begin{equation}
A_1 = a_{01} \otimes a_{01} \tag{4-117}
\end{equation}

\begin{equation}
[a_{01}] = \begin{bmatrix} 0 \\ Cos\beta \\ Sin\beta \end{bmatrix} \tag{4-118}
\end{equation}

\begin{equation}
A_2 = a_{02} \otimes a_{02} \tag{4-119}
\end{equation}

\begin{equation}
[a_{02}] = \begin{bmatrix} 0 \\ Cos\beta \\ -Sin\beta \end{bmatrix} \tag{4-120}
\end{equation}

In summary, many different constitutive relations have been proposed for the passive aorta and carotid arteries. Although none can be considered complete, the foundation is laid for doing so. It is remarkable, however, that there are no multiaxial constitutive relations available for the active behavior of these vessels, which is different and important.
5. THE EXPERIMENTAL WORK

5.1. Experimental Development on the Issue

In the past few decades, studies on the behavior of soft tissues were involving constitutive equations which hadn’t been supported with a complex experimental approach. These works also hadn’t been fulfilled with neither mathematical nor medical practice.

Y.C. Fung, recalled as the father of the biomechanics, has been working on modeling soft tissue behavior. In his first studies (in 70s & 80s) on arterial tissues, he experienced stability problems on his constitutive equation. It was mainly a mathematical problem which was due to the form of the constitutive equation. However, it is important to note that the characteristics demonstrated by experimental data is an effective parameter while formulizing the constitutive expression.

The experimental techniques, used in these studies, contain deficiencies. As an example, in the work by Mohan and Melvin 1982, the mechanical properties of arterial tissues were observed by uniaxial and biaxial tension tests like the many other researchers made. Although their experimental setup was well qualified, their specimens were planar samples cut out from vessel wall and continuous collagen structure was disturbed.

However, following the late 90s the technological developments in the measurement systems have been ushered in a new epoch. The developments in the optical measurement systems have been helpful for non-contacting displacement and strain measurements in the 3D domain. Thus, these developments enforced new theoretical and experimental techniques to occur.

J.D. Humphrey, Holzapfel and Ogden are among the raising names in the biomechanics of soft tissues. Holzapfel and Ogden had the most recent and updated work on the subject. In their work, they examine the mechanical properties of the arterial layers
separately which allows installing different parameters to define the effects of these layers in the constitutive relation. In addition, they also worked on the anisotropy of arterial tissue and created a concept involving the actual fiber directions. Humphrey has also been working on arterial subjects and is experienced in both the theoretical and experimental part. In his works, although he has used the same mathematical & experimental approaches by Holzapfel and Ogden, his papers and his books are relatively simple and explanatory. Another researcher like Larry A. Taber has also works on the issue in the same echoless as Humphrey in his publications.

5.2. The Experimental Work in This Study

Regarding all the literature and these works, we follow the theoretical procedure as Humphrey, Holzapfel and Ogden did. The experimental approach is based on the theoretical assumptions with feedbacks.

The experiments were first carried on a commercial balloon. They were practice experiments to check whether the system is working or not. These practices gave us many information and as well as warnings. Nevertheless, leaving the problems as could be encountered in all custom experiments, we struggled with three main points;

A. The first thing considered was the fixing apparatus for the specimens because either keeping the connection too tight would harm the specimen at the fixing point or it would be too loose and the specimen would just disengage.

B. The next point was obtaining a random spackle pattern on the arterial wall; a basic requirement for 3D correlation. Considering that the material used for spackling should be in the optimum size (according to parameters such as the camera resolution, distance, measuring area…), stick on the tissue, do not make any damage and would not be affected from water leak.

C. At last, during the experiment torsional moment of in the order of [Nmm] should be measured. In a paper by X. Lu et al., the shear modulus of porcine coronary artery has been obtained a tri-axial testing machine including a motor, encoder and torque transducer where this system could measure torque in µNm sizes.
However, there were unclear issues about the calibration of this system and made a custom designed bending bear system with strain gages.

After practicing with balloons, the experimental system was switched so as to test an artery.

We had sheep aortas obtained from two different slaughterhouses. We cared to take tissues from the animals having approximately same gender, weight and age according to provide experimental and statistical stability.

The experiments were carried on ten sheep aortas which were subjected to extension, inflation and torsional loading. Noting that, in the experiments the artery and water in it was heated up to 37°C with in a bath. These loading conditions were applied with respect to an experimental procedure which was important for providing a systematic approach.

In the pre-measurement stage, the tissues and the measurement system were prepared with the following steps:

- Arterial tissues are taken out from fridge and heated up to 37°C
- Camera system is setup regarding to the light sources
- Prepared arteries are mounted to the experimental system
- A random spackle pattern is generated on the arterial wall
- Camera resolution and focus are controlled according to these spackle pattern and needed adjustments are made
- ESAM data acquisition system is set uppded. All the channels are controlled and balanced.

In the measurement part, there were two main stages; applying loads and recording the data.

- In the protocol of applying loads, the axial force was varied from 0 to 20 N. The transmural pressure was set at 0, 5, 10, 15, 20 kPa at every force. Also, the artery was set to 5, 10 and 15 N and the transmural pressure was then varied from 0 to 20 kPa. In addition, at some certain loading points, a ramp of twist was performed from 0° to 20° and 0° to -20°.
Figure 5.1: The blood vessel during inflation and extension loading

Figure 5.2: The blood vessel during torsional loading

- The data was restored in two systems which were the camera 3D correlation and the ESAM data acquisition system. Camera system saved the images during the measurement while ESAM took the data of load cell, torque transducer, pressure transducer and trigger signals. Each data of specimen has been recorded separately to ensure simplicity in processing.

Data processing part briefly includes; calculation of strain values, importing data, calibration of two separate measurement system and obtaining theoretical material parameters.

- Strain values and as well as the displacement fields were calculated in the 3D image correlation software. (Vic3D)
- The data in the measurement systems were exported
- Synchronous data values with the camera and analogue signals have been extracted.

The related test results, the material parameters and discussion about the data processing are presented in the results part. This experimental work has become a combination of the literature and the experience gathered during tests. However, it should be noticed that it
has a theoretical background which is supported with technological equipments such as load cell, pressure transducer, and optics and camera system.

5.3. Experimental Setup

The experimental setup consisted of the following items

5.3.1. Hardware (devices / apparatus / machines)

- Custom design universal testing machine
- Constant temperature bath
- Positive displacement pump
- 3D image correlation camera system and illumination
- Load cell
  - ESIT 10 kg Load Cell, 1.9 mV/V
  - Custom design strain gage system (for torque measurement)
  - Linear 120 ohm strain gage x2 connected in half bridge
- Pressure transducer system; HBM; 100 kPa
- Signal generator (TTL)
- ESAM Traveler Data Acquisition System (ESA Messtechniche)

5.3.2. Software

- 3D image correlation software for Image Processing & strain Calculation
- ESAM Data Acquisition Software for Filtering & Post Processing Analogue Data
- MATLAB
  - Synchronous Extraction of Camera and ESAM recordings
  - Calculation of material parameters
**Figure 5.3:** Schematic view of arterial test system

Arterial Test System

[1] Flow Control Valve from Artery
[5] Pressure Transducer
[6] By-Pass Valve
[8] Constant Temp. Water Bath
[9] Stainless Steel Collector Base
[10] Arterial Specimen
The experimental setup has been realized in the laboratory of Strength of Materials, Mechanical Engineering Department of Istanbul Technical University.

Figure 5.4: Real view of arterial test system (ITU Dept. of Mech. Eng., Laboratory of Strength of Materials)
5.4. Data Acquisition

5.4.1. Data Acquisition Systems

Recalling the aforementioned knowledge; it is evident that the data was collected in two different data acquisition systems.

The ESAM data acquisition system was used to collect and process transducers, strain gage and trigger signals. The force, pressure and torque signals are collected from digital inputs where displacement signals are collected from analog input. All these data are collected in the ESAM data folder and then exported as a ASCII file to a new folder for the data correlation process. In the figure 5.4, an axial force data at constant pressure is shown as example.

![Figure 5.5: Change of axial force with time at constant pressure](image)

The 3D strain measurement system has two CMOS cameras with external trigger, light sources and 3 different soft wares. The system has two cameras to perform 3D strain measurement like the natural human eye. However, you can use just one camera and measure displacement in 2D plane. The snapshot software enables you to take pictures from the experimental area to perform a calibration and also to gather a reference image.
The Vic3D software is used for simultaneous data collection in a desired frequency and processing these images for 3D strains or 3D displacements. In the arterial test system we used Vic3D software in order to obtain strain measures either in 3D or a contour plot for the calculated results or an ASCII export. The figure 5.5 gives a quick demonstration of these different plotting styles.

Figure 5.6: 3D and Contour plot of the strain values

These two acquisition systems are working separately therefore, to correlate these systems we used an external trigger which helped to put together separate data in different platforms and this merged data then will be used in the nonlinear curve fitting operation to obtain material parameters.

Thus, the main problem with these systems is the integration failure of camera or ESAM systems in to other in means of software and hardware. In addition, 3D strain measurements force the computer’s capabilities therefore during continuous measurements using just one computer even with different software could cause damage in the collected data as the frequency increases.
5.4.2. Experimental results

Figure 5.7: Loading conditions over the axial direction

Figure 5.6 displays the loading conditions on the test specimen. The figure represents force, pressure and moment as measured values and strain energy as calculated from experimental data. Regarding to the experimental procedure the artery was first extended to a certain stretch ratio around 1.2, then inflated to 10 kPa. Extension was continued to 1.65 stretch ratios. At this point a torsion moment was applied from +10Nmm to -10Nmm. As a last condition the artery was extended to a stretch ratio of 1.8, again the same torsion moment was applied and one experimental cycle finishes.
Figure 5.8: Change of axial stretch ratio

It is noted that as either under inflation or extension loading conditions the arterial specimen becomes stiffer as load increases. This is contrary to standard rubber behavior. As in figure 5.8, the axial stretch ratio increases sharply after the pressure was increased 0 to 10 kPa.

At low pressure values (around zero Pa), the arterial tissue has a resonance which was also observed in the experiment and represented in the figure 5.8. This issue has been told in [57] briefly.

The following figures 5.9, 5.10 and 5.11 represents the loading conditions in the experiment procedure with the contour plots of the deformed body over the axial strain. The figures are listed in the same order of loading conditions. First image of figure 5.9 displays the reference position and condition of the arterial segment.
Figure 5.9: Contour plots of different loading conditions case 1

- **Case 1:**
  - **Condition:** \( F = 0 \text{ N} \), \( P = 0 \text{ Pa} \), \( M = 0 \text{ Nmm} \)

- **Condition:** \( F = 1 \text{ N} \), \( P = 0 \text{ Pa} \), \( M = 0 \text{ Nmm} \)

- **Condition:** \( F = -1 \text{ N} \), \( P = 10000 \text{ Pa} \), \( M = 0 \text{ Nmm} \)
Figure 5.10: Contour plots of different loading conditions case 2
5.5. Calculation of Total Strain Energy from the Experimental Measurable Quantities

The change in the strain energy of a hyperelastic material is the stored energy in the material because of the external effects. In this part, the strain energy of the arterial specimen will be obtained in terms of the measurable quantities.

The total strain energy of the arterial material is stored in the elastic system with radial, axial and circumferential strain and stress values which are caused by the loading conditions as extension, inflation and torsion.

Therefore, the strain energy could be divided into three main parts and becomes

$$\psi_T = \psi_V + \psi_F + \psi_M$$  \hspace{1cm} (5-1)

This equation briefly displays the parameters or factors which are producing the total energy. Also, it is possible to calculate the energy from the sum of energies produced by these external effects. However, it should be noted that in a nonlinear regime, even a change in the torsional moment could affect the energy in the $\psi_F$.

The stored energy caused by transmural pressure is

$$\psi_V = \int PdV = \sum P_i \Delta V_i$$  \hspace{1cm} (5-2)

where $V$ is the volume of inside artery not the arterial wall.

$$V_i = \pi \left( a_z^2 L \right)$$  \hspace{1cm} (5-3)

For the whole measured points, it will become

$$\Delta V_i = \pi \left( a_z^2 \right)_i L_i - \pi \left( a_z^2 \right)_{i-1}^2 L_{i-1}$$  \hspace{1cm} (5-4)

Non-dimensionalizing the equation results
\[ \Delta V = \pi a_1^2 L \left[ \frac{(a_2)^2}{a_1^2} \frac{L}{L_1} - \frac{(a_2)^2}{a_1^2} \frac{L_{i+1}}{L_{i+1}} \right] \]  

\[ (5-5) \]

Where, \( a_1 \) is the initial inner radius of the artery, \( a_2 \) is the inner radius after loading, \( L_1 \) is the initial length of the artery, \( L_2 \) is the length after loading.

In addition, most hyperelastic formulations use the total strain energy per unit volume. Therefore, these calculated values should be proportioned to the material volume and density.

\[ W_v = \frac{\psi_v}{\rho_{wall} V_{wall}} \]  

\[ (5-6) \]

where

\[ V_{cidar} = \pi (b_1^2 - a_1^2) L_1 \]  

\[ (5-7) \]

and substituting these to equation *1 gives

\[ W_v = \sum_i P_i \sum_n \frac{1}{\rho_{cidar}} \frac{a_i^2}{b_1^2 - a_1^2} \omega_n \left( \frac{(a_2)^2}{a_1^2} (\lambda_2) \right)_n \]  

\[ (5-8) \]

So the energy equation becomes to be numerically calculated in every data points. Where \( \omega_n \) is the Gauss Quadrature multiplier.

The same procedure is implied to the stored energy coming from force and torsion. The stored energy caused by axial force is calculated as

\[ \psi_f = \int F dX = \sum F_{i} \Delta X_{i} \]  

\[ (5-9) \]

\[ \Delta X_{i} = L_{i} - L_{i-1} \]  

\[ (5-10) \]

\[ W_f = \frac{\psi_f}{\rho_{cidar} V_{cidar}} \]  

\[ (5-11) \]
\[ V_{cidar} = \pi (b_1^2 - a_1^2) L_1 \]  
\[ W_r = \sum_i F_i \sum_n \frac{1}{\rho_{cidar} \pi (b_1^2 - a_1^2)} \alpha_n \Delta (\lambda_{cr})_n \]

\[ 5.6. \text{Calculating the Total Strain Energy Theoretically} \]

In theory, the artery is modeled as a thick walled tube so that the strain, stress and so the stored energy values are changing according to the radial position. Therefore, the total energy should be calculated through an integration process

\[ W_r = \int_{a_z}^{b_z} dW = \int_{a_z}^{b_z} \frac{dW}{dr} dr = \int_{a_z}^{b_z} \left( \frac{\partial E}{\partial E_{ij}} \frac{\partial E_{ij}}{\partial E_{ij}} + \frac{\partial E}{\partial E_{RR}} \frac{\partial E_{RR}}{\partial E_{RR}} + \frac{\partial E}{\partial E_{ZZ}} \frac{\partial E_{ZZ}}{\partial E_{ZZ}} + \ldots \right) dr \]

Where \( a \) is the inner diameter, \( b \) is the outer diameter, \( Z \) is the axial, \( Q \) is the circumferential and \( R \) is the radial coordinates of the system.
6. DATA FITTING

6.1. Theory: The Least Squares Fitting Method

The Curve fitting operations mostly uses the method of least squares when fitting data. The fitting process requires a model that relates the response data to the predictor data with one or more coefficients. The result of the fitting process is an estimate of the "true" but unknown coefficients of the model.

To obtain the coefficient estimates, the least squares method minimizes the summed square of residuals. The residual for the “i” th data point r_i is defined as the difference between the observed response value y_i and the fitted response value ŷ_i and is identified as the error associated with the data [9].

\[ e_i = y_i - ŷ_i \]  \hspace{1cm} (6-1)

Residual = data - fit

The summed square of residuals is given by

\[ S = \sum_{i=1}^{n} e_i^2 = \sum_{i=1}^{n} (y_i - ŷ_i)^2 \]  \hspace{1cm} (6-2)

where n is the number of data points included in the fit and S is the sum of squares error estimate. The curve fitting procedures have been computed on MATLAB environment. The theory for both software has mostly 3 types [9].

- Linear least squares
- Weighted linear least squares
- Nonlinear least squares
6.2. Linear Least Squares

A linear model is defined as an equation that is linear in the coefficients. For example, polynomials are linear but Gaussians are not. To illustrate the linear least squares fitting process, suppose you have n data points that can be modeled by a first-degree polynomial [9].

\[ y = p_1 x + p_2 x \]  

(6-3)

To solve this equation for the unknown coefficients \( p_1 \) and \( p_2 \), you write \( S \) as a system of n simultaneous linear equations in two unknowns. If \( n \) is greater than the number of unknowns, then the system of equations is over determined [9].

\[ S = \sum_{i=1}^{n} (y_i - (p_1 x_i + p_2))^2 \]  

(6-4)

Because the least squares fitting process minimizes the summed square of the residuals, the coefficients are determined by differentiating \( S \) with respect to each parameter, and setting the result equal to zero.

\[ \frac{\partial S}{\partial p_1} = -2 \sum_{i=1}^{n} x_i (y_i - (p_1 x_i + p_2)) = 0 \]  

(6-5)

\[ \frac{\partial S}{\partial p_2} = -2 \sum_{i=1}^{n} (y_i - (p_1 x_i + p_2)) = 0 \]  

(6-6)

The estimates of the true parameters are usually represented by \( b \). Substituting \( b_1 \) and \( b_2 \) for \( p_1 \) and \( p_2 \), the previous equations become [9]

\[ \sum x_i (y_i - (b_1 x_i + b_2)) = 0 \]  

(6-7)

\[ \sum (y_i - (b_1 x_i + b_2)) = 0 \]  

(6-8)

where the summations run from \( i = 1 \) to \( n \). [9]
6.3. Weighted Linear Least Squares

As described in basic assumptions about the error, it is usually assumed that the response data is of equal quality and, therefore, has constant variance. If this assumption is violated, your fit might be unduly influenced by data of poor quality. To improve the fit, you can use weighted least squares regression where an additional scale factor (the weight) is included in the fitting process. Weighted least squares regression minimizes the error estimate [9].

\[ S = \sum_{i=1}^{n} (w_i (y_i + \hat{y}_i))^2 \]  

(6-9)

where \( w_i \) are the weights. The weights determine how much each response value influences the final parameter estimates. A high-quality data point influences the fit more than a low-quality data point. Weighting your data is recommended if the weights are known, or if there is justification that they follow a particular form [9].

From an engineering point of view; multi scale fitting problems should be conducted via “Weighted Linear Least Squares Method”; since the residual magnitudes of different orders should be homogenized [9]. For an example from this work,

6.4. Nonlinear Least Squares

The nonlinear least squares formulation to fit a nonlinear model to data. A nonlinear model is defined as an equation that is nonlinear in the coefficients or a combination of linear and nonlinear in the coefficients. For example, Gaussians, ratios of polynomials, and power functions are all nonlinear.

In matrix form, nonlinear models are given by the formula

\[ y = f(X, \beta) + \varepsilon \]  

(6-10)

where

\( y \) is an n-by-1 vector of responses.

\( f \) is a function of \( \beta \) and \( X \).
\( \beta \) is a m-by-1 vector of coefficients.

\( X \) is the n-by-m design matrix for the model.

\( \varepsilon \) is an n-by-1 vector of errors.

Nonlinear models are more difficult to fit than linear models because the coefficients cannot be estimated using simple matrix techniques. Instead, an iterative approach is required that follows these steps:

Start with an initial estimate for each coefficient. For some nonlinear models, a heuristic approach is provided that produces reasonable starting values. For other models, random values on the interval [0,1] are provided.

Produce the fitted curve for the current set of coefficients. The fitted response value is given by

\[
y = f(X, \beta) + \varepsilon
\]

and involves the calculation of the Jacobian of \( f(X, b) \), which is defined as a matrix of partial derivatives taken with respect to the coefficients.

Adjust the coefficients and determine whether the fit improves. The direction and magnitude of the adjustment depend on the fitting algorithm. The toolbox provides these algorithms:

- Trust-region -- This is the default algorithm and must be used if you specify coefficient constraints. It can solve difficult nonlinear problems more efficiently than the other algorithms and it represents an improvement over the popular Levenberg-Marquardt algorithm.

- Levenberg-Marquardt -- This algorithm has been used for many years and has proved to work most of the time for a wide range of nonlinear models and starting values. If the trust-region algorithm does not produce a reasonable fit, and you do not have coefficient constraints, you should try the Levenberg-Marquardt algorithm.
7. DISCUSSION

7.1. On Theoretical Models

In this study, responses to experimental data of three different material models proposed for the arterial structure has been investigated. These models are namely

- Fung’s Model
- Vaishnaw’s Model
- Holzapfel’s Model

which basically differs from each other as

- being a robust combination of sum of squares of strain components; the sum being taken to an exponent and being purely phenomenological
- being a quasi-linear model in form of a polynomial of strain components
- being a model incorporating the physical basics of the arterial structure, such as the collagen orientation to incorporate the stiffening effects separately from the base matrix (elastin).

Over the experimental data acquired, the models’ responses are compared in the following headlines with respect to

- their correspondence to experimental data
- the variation of stress versus strain
- the variation of stress versus thickness
7.1.1. Fung’s model

The form of the strain energy for Fung’s Model is a modified form of the original model

\[ W = \frac{1}{2} c e^q \]  

(7-1)

that revokes the zero strain energy at zero loading condition so as to satisfy the curve fitting process, in the form of

\[ W = \frac{1}{2} C (e^q - 1) \]  

(7-2)

where \( W \) is defined per unit initial volume and \( Q \) is

\[
Q = \sum_{i,j} c_i \left( E_{ij}^2 + E_{ij}^2 \right) + \left( \sum_{i,j} c_i \left( E_{ij}^2 + E_{ij}^2 \right) \right) + c_9 \left( E_{Zj}^2 + E_{Rj}^2 \right)
\]

(7-3)

Figure 7.1: Comparison of Fung’s model with the experimental data. The pink color represents the experimental data where the blue one represents theoretical result.

Notice that the difference offered does not alter the derivatives, thus, the stress components. The fits to experimental data are given in Figure 7.1
The evolution of stress components with respect to axial strain values are given in Figure 7.2.

![Stress components with respect to axial strain on outer surface](image)

**Figure 7.2**: Stress components with respect to axial strain on outer surface. The colors dark blue, purple, yellow, blue represent in the order from inside to outside of the wall thickness.

The evolution of stress components with respect to tangential strain values are given in Figure 7.3.

Fung model is still the most famous model in the literature which includes all the strain components in an exponential form. Many researchers used this model in their experimental approaches. Fung model could be used in different kind of soft tissue applications.
Figure 7.3: Stress components with respect to tangential strain values. The colors dark blue, purple, yellow, blue represent in the order from inside to outside of the wall thickness.

Since it is an order different in value, and demonstrates and includes a characteristic other than a normal stress component, the plot of shear stress component with respect to shear strain is given in Figure 7.4. The structure is observed to behave quite well in a linear fashion.

Figure 7.4: Shear stress distribution with respect to shear strain. The colors dark blue, purple, yellow, blue represent in the order from inside to outside of the wall thickness.
7.1.2. Vaishnav’s model

The form of the strain energy equation for Vaishnav is a quasi linear mode in form of a polynomial of strain components. This model only includes the tangential and axial directional strain components with seven material parameters.

\[
W = c_1 E_{\theta\theta}^2 + c_2 E_{\theta\theta} E_{ZZ} + c_3 E_{ZZ}^2 + c_4 E_{\theta\theta}^3 + c_5 E_{\theta\theta}^2 E_{ZZ} + c_6 E_{\theta\theta} E_{ZZ}^2 + c_7 E_{ZZ}^3 \tag{7-4}
\]

**Figure 7.5**: Comparison of Vaishnav’s model with the experimental data. The blue color represents the theoretical data where the pink one represents experimental one.

Figure 7.5 displays the convergence of theoretical model with the experimental data. It is seen that the Vaisnav’s model is as accurate as Fung’s type. However, in the literature it is told that Fung’s model is more successful. In the following section we will compare and discuss these models behaviors under different loading conditions.
Figure 7.6: Stress components with respect to axial strain. The colors dark blue, purple, yellow, blue represent in the order from inside to outside of the wall thickness.

Figure 7.6, 7.7 and 7.8 give stress distributions according to the related strains over the arterial wall thickness.

Figure 7.7: Stress components with respect to tangential strain. The colors dark blue, purple, yellow, blue represent in the order from inside to outside of the wall thickness.
Figure 7.8: Shear stress with respect to shear strain. The colors dark blue, purple, yellow, blue represent in the order from inside to outside of the wall thickness.

7.1.3. Holzapfel’s model

As aforementioned Holzapfel’s model is the most recent work which models the artery as two separated layers namely media and adventitia with substituting the fiber orientations directly to the constitutive relation. The model has two parts namely isotropic and anisotropic parts. The isotropic part is a basic form of the Rivlin model which is characterizing the elastin behavior. The anisotropic part has two material parameters and invariants which are enforcing the fiber orientations. The anisotropic part of the model is basically characterizing the collagen behavior.

\[ W(C, A_1, A_2) = W_{iso}(I_1) + W_{aniso}(I_4, I_6) \]  \hspace{1cm} (7-5)

\[ W_{iso} = \frac{c}{2}(I_1 - 3) \]  \hspace{1cm} (7-6)

\[ W_{aniso} = \frac{k_1}{2k_2} \sum_{i=4,6} \{ \exp[k_2(I_i - 1)^2] - 1 \} \]  \hspace{1cm} (7-7)
**Figure 7.9:** Comparasion of Holzapfel’s model with the experimental data

In the figure 7.10-12 the stress distributions with respect to the related strains are given.

**Figure 7.10:** Stress components with respect to axial strain on outer surface. The colors dark blue, purple, yellow, blue represent in the order from inside to outside of the wall thickness.
7.2. Comparison of the Theoretical Models

In this study, we examined three different material models namely Vaishnav, Fung and Holzapfel which are characterizing the general basis of the other works in the literature. Figure 7.13 displays the comparison of theoretical models in terms of loading conditions.

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**Figure 7.11:** Shear stress with respect to shear strain. The colors dark blue, purple, yellow, blue represent in the order from inside to outside of the wall thickness.

**Figure 7.12:** Stress components with respect to tangential strain. The colors dark blue, purple, yellow, blue represent in the order from inside to outside of the wall thickness.
and strain energy. In terms of strain energy and axial force, Vaishnav and Fung models have a better fitting on the experimental data. In the pressure graphic, at low pressure values Fung model and at high pressure points Holzapfel model diverged from the experimental data. However, the arterial experiments and theoretical considerations at zero pressure values would not be meaningful regarding the arterial physiology.

![Figure 7.13: Theoretical models with respect to loading conditions and strain energy on the outer surface.](image)

### 7.3. Optimization Problem

The experimental data was substitute in to the theoretical models with the nonlinear curve fitting code of MATLAB program. This is a complicated process which involves many parameters, nonlinear equations and numerical integrations. In this process, a control loop iterates the equations by calculating the error values and comparing them with the convergence criteria until it reaches the optimized values. The convergence criteria in this study are gathered according to the literature and the experimental work. During the optimization process MATLAB code does not only use this criteria, it will also notice the initial conditions, the upper and lower boundaries and the convergence theory which is used in MATLAB code by default. Therefore, forcing convergence criteria externally will not be a direct effect to the program. This limitation causes and
enforces the user to apply different boundaries with different initial guesses and even applying different criteria for each model. A researcher could adjust the whole code or develop a new one to take the whole control under his limitations. Furthermore, the experimental data could change, for example, according to the kind of animal or age of human, and then it should be calculated to see the consistency of the code’s optimization process. Additionally, in the literature majority of the works in this issue are carried on rabbit aortas and carotids however none of them prove the symmetry of these with the human aorta. As a result of these, optimization is one of main part of nonlinear material modeling and thus a mathematical help in this issue should be gathered from a mathematician. This proves that the arterial wall modeling needs strong mathematical performance and formulation next to the mechanical and medical parts.

7.4. The Effects of Thick Walled Tube Theory

In this study, the artery is modeled as a thick walled tube which is now a major consensus on the issue. The theory enforces a numerical integration over the wall thickness causing an increase in solving time and produce a radially change material behavior and load response. In the figure 7.14 radial stress distributions with respect to axial force and transmural pressure are given. The figure displays the change of radial stress with wall thickness with different colors.

![Figure 7.14](image)

**Figure 7.14:** Radial stress distribution with respect to axial force and transmural pressure from the Fung type material model. The colors dark blue, purple, yellow, blue represent in the order from inside to outside of the wall thickness.
7.5. The Arterial Models with Two Layers

In this study the arterial wall is modeled as a one-layer thick walled tube. Whereas as aforementioned the arterial wall has two main layers which are affecting the arterial mechanics. These two layers have different collagen structure and fiber orientations. It will be a future work to separate these layers, test them individually and get the constitutive relations for both parts. This procedure will enable to show the mechanical effects of these layers separately by either visually in the experiment or mathematically in the theory. Furthermore, it will also enforce a new original material model.

7.6. Experimental and Theoretical Works in the Literature

The literature review on arterial modeling has shown that there are no available experimental data to provide insight to data acquisition and for comparison purposes in the literature. Mostly, the convergence of experimental data with the theory is not given in a mathematical manner but it is displayed graphically. Furthermore, the mathematical expressions, terms and formulations in the literature are given in a reverse, confusing
and deficient manner. Therefore, these works in the literature should be examined carefully before application to one’s own model.

7.7. Experimental System Response is Linear with Respect to Moment

The experiments in this work indicate that the system response to the torsional moment is quite linear. Figure 7.16 displays the linear relation between shear stress over the moment.

7.8. New Constitutive Relations

Regarding to the literature and with the outcome of this study, it should be denoted that there is a need to have a new constitutive relation. Applying high technological developments to the experimental system, using appropriate arterial specimens which are separated in two layers, addition of viscoelastic terms and reliable convergence criteria supported with an approved optimization code to find material parameters will result a new successful model.

7.9. The Addition of Viscoelastic Effects

The experiments on the arterial tissue prove that the tissue has a mechanical behavior changing with as time (rate effects) arise. Therefore, some kinds of experiments namely stress relaxation and creep should be carried on the tissue. Furthermore, using experimental data obtained with respect to time arise new parameters which are covering the viscoelastic effects could be substituted in to the constitutive relation.

7.10. Using Human Artery in the Experiments

In the literature, the majority of the works are carried on rabbit aortas or carotids. We have not met to a theoretical work in which human arteries are extensively investigated. Even porcine substitutes that there are well-known to be the most likely of human tissues are rest of quite interest and abundance with respect to available data and research.
CONCLUSION

This study covers a work including experimental and theoretical approaches in the arterial biomechanics. The arterial wall is modeled as an elastic thick walled tube. The mechanical properties are determined experimentally and three main theoretical models are applied to the experimental data in order to expose the differences between them.

The theoretical models are compared according to the convergence criteria namely, total strain energy, transmural pressure, axial force and torsional moment. In the low pressure regions Fung model while at high pressure values Holzapfel model diverges from the experimental data. In the shear stress regions, the Vaishnav model has the least stress values which is caused from the lack of the shear terms in the theory.

The experimental works in the literature are generally carried on different (rabbit, canine…) arteries and with different technologies (camera, software…). Therefore, although optimization of curve fitting seems to be a big problem, using different kind of species could make it complicated to compare different works.

This study constitutes briefly the experimental and theoretical background on the issue. In future works, additional studies could be made by determining time dependent material behaviors, using pig arteries, examining residual stress, obtaining a new optimization code and finally generating a new own material model.
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RESUME


As a student of Mechanical Engineering Faculty, I have attended to Mizah student’s club and organized faculty festivals, concerts, symposiums and congress.

I am interested in latin dance, history of ancient civilizations and football.