**ISTANBUL TECHNICAL UNIVERSITY ⋆ INSTITUTE OF SCIENCE AND TECHNOLOGY**

# SPHERICAL FINITE TYPE HYPERSURFACES

**MSc Thesis by**

**Selin Taşkent**

**Department : Mathematics**

**Programme : Mathematical Engineering**

**JANUARY 2007**

İSTANBUL TECHNICAL UNIVERSITY ★ INSTITUTE OF SCIENCE AND TECHNOLOGY

# SPHERICAL FINITE TYPE HYPERSURFACES

**MSc. Thesis by**
**Selin TAŞKENT**

**(509041005)**

**Date of submission :** 25 December 2006

**Date of defence examination:** 30 January 2007

**Supervisor (Chairman):** Assoc. Prof. Dr. Uğur DURSUN

**Members of the Examining Committee** Prof. Dr. Abdülkadir ÖZDEĞER (KHÜ)

Assist. Prof. Dr. Elif CANFES (İTÜ)

**JANUARY 2007**

**<u>İSTANBUL TEKNİK ÜNİVERSİTESİ ★ FEN BİLİMLERİ ENSTİTÜSÜ</u>**

# SONLU TİPTEN KÜRESEL HİPERYÜZEYLER

**Yüksek Lisans Tezi**

**Selin TAŞKENT**
**(509041005)**

**Tezin Enstitüye Verildiği Tarih : 25 Aralık 2006**
**Tezin Savunulduğu Tarih : 30 Ocak 2007**

**Tez Danışmanı :** **Doç. Dr. Uğur DURSUN**

**Diğer Jüri Üyeleri** **Prof. Dr. Abdülkadir ÖZDEĞER (KHÜ)**

**Yard. Doç. Dr. Elif CANFES (İTÜ)**

**OCAK 2007**

# ACKNOWLEDGEMENTS

I would like to pay my gratitude to my supervisor Assoc. Prof. Dr. Uğur DURSUN for his support and encouragement during the preparation of this thesis. I also would like to thank to my family for their considerable support.

December, 2006 Selin TAŞKENT

CONTENTS

**SYMBOL LIST**

$\bar{\nabla}$ : Riemannian connection of the ambient space

$\nabla$ : Riemannian Connection of the Submanifold

$\omega_B^A$ : Connection Forms

$h$ : Second Fundamental Form

$A$ : Weingarten Map

$D$ : Normal Connection of the Submanifold

$\Delta$ : Laplace Operator

$\Delta^D$ : Laplace Operator of the Normal Connection

$H$ : Mean Curvature Vector

$\mathcal{A}(H)$ :Allied Mean Curvature Vector

$TM$ : Tangent Vector Bundle

$TM^{\perp}$ : Normal Vector Bundle

**SONLU TİPTEN KÜRESEL HİPERYÜZEYLER**

**ÖZET**

Bu tez çalışmasında, Öklid uzayının hiperkürelerinin sonlu tipten alt manifoldlarının (özellikle hiperyüzeylerinin) sınıflandırılması kısa bir şekilde incelenmiştir. Öklid uzayının bir hiperküresinin kompakt bir hiperyüzeyinin, küçük hiperküre olmaması durumunda, 2-tipinden ve kütlesel-simetrik olması için gerek ve yeter koşul, ortalama ve skaler eğriliklerinin sabit olmasıdır. Bu sonuç bize, bir hiperkürenin sonlu tipten kompakt izoparametrik bir hiperyüzeyinin, 1 veya 2 tipinden olması gerektiğini gösterir. Öklid uzayının bir hiperküresinin kompakt 2-tipinden bir hiperyüzeyi, ancak ve ancak kütlesel-simetrik ise sabit ortalama eğriliğe sahip olabilir. Bu genel sonuç kullanılarak, bir hiperkürenin Dupin hiperyüzeyleri ve izoparametrik hiperyüzeyleri arasında, 2-tipinden olmayı içeren bir bağlantı verilmiştir. Bunlara ek olarak, bir hiperkürenin kompakt 2-tipinden bir Dupin hiperyüzeyinin sabit ortalama eğriliğe sahip olacağı gösterilmiştir.

En fazla iki asal eğriliğe sahip küresel bir hiperyüzeyin 2-tipinden olması için gerek ve yeter koşul, uygun yarıçaplı iki kürenin çarpımı olarak yazılabilmesidir. Bunun yanı sıra, 2-tipinden küresel birçok hiperyüzeyin kütlesel-simetrik olduğu, ve 2-tipinden kütlesel simetrik küresel bir hiperyüzeyin umbilik noktasının bulunmadığı gösterilmiştir.

Öklid uzayının paralel ortalama eğrilik vektörüne sahip 2-tipinden bir alt manifoldu, küresel veya sıfırlı tipinden olmak zorundadır. Bu sonucu kullanarak, paralel eğrilik vektörüne sahip 2-tipinden yüzeylerin tam bir sınıflandırması verilmiştir.

**SPHERICAL FINITE TYPE HYPERSURFACES**


**SUMMARY**

In this thesis, we give a short survey on the classification of finite type submanifolds (especially hypersurfaces) of hyperspheres of a Euclidean space. A compact hypersurface of a hypersphere of a Euclidean space $\mathbb{R}^m$ is mass-symmetric and is of 2-type if and only if it has constant mean curvature and constant scalar curvature unless it is a small hypersphere. This result shows that a compact isoparametric hypersurface of a hypersphere is either of 1-type or of 2-type. A compact, 2-type hypersurface of a hypersphere in $\mathbb{R}^m$ has constant mean curvature if and only if it is mass-symmetric. Using this general result, a relation between Dupin hypersurfaces and isoparametric hypersurfaces in a hypersphere involving 2-typeness is given. Moreover, it is shown that a compact 2-type Dupin hypersurface of a hypersphere has constant mean curvature.

A hypersurface of a hypersphere with at most two distinct principal curvatures is of 2-type if and only if it is the product of two spheres of appropriate radii. It is shown that many 2-type hypersurfaces of a hypersphere are mass-symmetric and that, mass-symmetric, 2-type hypersurfaces of a hypersphere have no umbilical point.

Furthermore, a 2-type submanifold (not necessarily compact) in $\mathbb{R}^m$ with parallel mean curvature vector, is either spherical or null. By applying this result a complete classification of 2-type surfaces with parallel mean curvature vector is given.

# 1. INTRODUCTION

The notion of finite type submanifolds and maps in Euclidean space was introduced by B.Y. Chen in the late seventies, and it has become a useful tool for investigation of submanifolds. A submanifold $M$ of a Euclidean space $\mathbb{R}^m$ is said to be of finite type if the position vector of $M$ in $\mathbb{R}^m$ can be expressed as a finite sum of $\mathbb{R}^m$-valued maps on $M$, such that for each one of these maps, every component function of the map lays in the same eigenspace of the Laplacian $\Delta$, which acts on smooth functions on $M$. If one of the nonconstant maps is harmonic, then the submanifold $M$ is said to be of null finite type.

The first results on the finite type submanifolds were collected in the book [1] more than twenty years ago. Since that time the subject has had a rapid development. In a survey article [8], B.Y. Chen reported the progress made by various geometers on the subject up to year 1996. The most of the references on this subject can be seen in [8].

The concept of finite type is the natural extension of minimal submanifolds. The class of submanifolds of finite type consists of nice submanifolds of the Euclidean space. For example, all minimal submanifolds of a Euclidean space and all minimal submanifolds of hyperspheres of a Euclidean space are of 1-type and vice versa. Also, all parallel submanifolds of a Euclidean space are of finite type. Furthermore, circular cylinders and helical cylinders are of null 2-type, and results on null 2-type submanifolds can be seen in [6, 7, 9].

The purpose of this thesis is to give a short survey on finite type compact submanifolds of hyperspheres of a Euclidean space. The second chapter is devoted to prelimineries and some results on submanifolds.

In Chapter 3, we give the evaluation of the Laplacian of the mean curvature vector of a spherical submanifold because it plays an important role in finite type theory.

In Chapter 4, we are concerned with compact mass-symmetric 2-type submanifolds of a hypersphere of a Euclidean space. It is shown that, for a compact hypersurface $M$ of $\mathbb{S}^{m-1}$ in $\mathbb{R}^m$, if $M$ has nonzero constant mean curvature and constant scalar curvature, then both the mean and scalar curvatures are completely determined by the eigenvalues of the Laplacian.

Moreover, it is proved that, a compact hypersurface $M$ of a hypersphere with at most two distinct principal curvatures is of 2-type if and only if $M$ is the product of two spheres with appropriate radii.

In Chapter 5, we give some results on isoparametric and Dupin hypersurfaces involving finite typeness.

In Chapter 6, we study 2-type submanifolds of Euclidean spaces. A 2-type submanifold $M$ of $\mathbb{R}^m$ with parallel mean curvature vector is either spherical or it is of null 2-type. Also, it is shown that there are no spherical hypersurfaces of null 2-type. Applying these results, a complete classification of 2-type surfaces with parallel mean curvature vector is given.

In Chapter 7, we give results and discussion.

## 2. PRELIMINERIES

Let $M$ be an n-dimensional manifold embedded in an m-dimensional Riemannian manifold $\bar{M}$. Then the submanifold $M$ is also a Riemannian manifold with the induced Riemannian metric. Let $X$ and $Y$ be two vector fields on $M$. Then we have

$$\overline{\nabla}_X Y = \nabla_X Y + h(X,Y), \tag{2.1}$$

where $\overline{\nabla}$ is the Riemannian connection defined on $\bar{M}$, $\nabla$ is the induced Riemannian connection on $M$, and $h$ is the second fundamental form of the submanifold $M$. Let $\xi$ be a normal vector field on $M$ and $X$ be a tangent vector field on $M$. Then $\overline{\nabla}_X \xi$ can be decomposed as

$$\overline{\nabla}_X \xi = -A_\xi X + D_X \xi \tag{2.2}$$

where $A_\xi$ and $D$ are the Weingarten map of $M$ with respect to $\xi$, and the normal connection in the normal bundle $TM^\perp$ of $M$ in $\bar{M}$, respectively. The equations (2.1) and (2.2) are called the Gauss and Weingarten formulas.

The curvature tensor of the Riemannian manifold $\bar{M}$ is given by

$$\overline{R}(X,Y)Z = \overline{\nabla}_X \overline{\nabla}_Y Z - \overline{\nabla}_Y \overline{\nabla}_X Z - \overline{\nabla}_{[X,Y]} Z \tag{2.3}$$

for any vector fields $X$, $Y$ and $Z$ on the submanifold $M$. Similarly, the curvature tensor of the submanifold $M$ is given by

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

For any vector field $W$ on $M$, if we write $\bar{R}(X,Y,Z,W) = \langle \bar{R}(X,Y)Z, W\rangle$ and $R(X,Y,Z,W) = \langle R(X,Y)Z, W\rangle$, then we have

$$\bar{R}(X,Y,Z,W) = R(X,Y,Z,W) + \langle h(X,Z), h(Y,W)\rangle - \langle h(X,W), h(Y,Z)\rangle, \tag{2.4}$$

which is called the Gauss equation. Let $\xi_1, \ldots, \xi_{m-n}$ be an orthonormal normal basis for the normal bundle of $M$ in $\bar{M}$ and let $h^\alpha$ $(\alpha = 1, \ldots, m-n)$ be the corresponding second fundamental forms, that is, $h(X,Y) = \sum_\alpha h^\alpha(X,Y)\xi_\alpha$. The normal component of $\overline{R}(X,Y)Z$ is given by

$$\left(\overline{R}(X,Y)Z\right)^N = (\overline{\nabla}_X h)(Y,Z) - (\overline{\nabla}_Y h)(X,Z) \tag{2.5}$$

where the covariant derivative of $h$, denoted by $(\overline{\nabla}_X h)$, is defined as

$$\begin{aligned} (\overline{\nabla}_X h)(Y,Z) &= \sum_\alpha \left(D_X(h^\alpha(Y,Z)\xi_\alpha) - h^\alpha(\nabla_X Y, Z)\xi_\alpha - h^\alpha(Y, \nabla_X Z)\xi_\alpha\right) \\ &= \sum_\alpha (\nabla_X h^\alpha)(Y,Z)\xi_\alpha + \sum_\alpha h^\alpha(Y,Z) D_X \xi_\alpha \end{aligned} \tag{2.6}$$

The equation (2.5) is called the Codazzi equation. If the ambient space $\bar{M}$ is a space of constant curvature, then Codazzi equation can be written as

$$(\nabla_X A_\xi)Y - A_{D_X\xi}Y = (\nabla_Y A_\xi)X - A_{D_Y\xi}X. \tag{2.7}$$

Given two vectors $X$ and $Y$ in $T_p(M)$ and an orthonormal basis $e_1, \ldots, e_n$ of $T_p(M)$, we define the Ricci tensor $S$ and the scalar curvature $\rho$ by

$$\begin{aligned} S(X,Y) &= \sum_{i=1}^n \langle R(e_i, X)Y, e_i \rangle, \\ \rho &= \frac{1}{n(n-1)} \sum_{i=1}^n S(e_i, e_i). \end{aligned}$$

The mean curvature vector $H$ of $M$ in $\bar{M}$ is given by

$$H = \frac{1}{n} \mathrm{tr} h, \tag{2.8}$$

and if $e_{n+1}, \ldots, e_m$ are orthogonal unit normal vector fields of $M$ in $\bar{M}$ such that $e_{n+1}$ is parallel to the mean curvature vector $H$ of $M$, the allied mean curvature vector $\mathcal{A}(H)$ of $M$ in $\bar{M}$ is defined as

$$\mathcal{A}(H) = \sum_{\beta=n+2}^m \mathrm{tr} A_H A_\beta e_\beta.$$

Let $M$ be an n–dimensional, oriented Riemannian manifold. We choose an orthonormal local basis $e_1, \ldots, e_n$ whose orientation is the same with that of

$M$. Denote by $w^1, \dots, w^n$ the dual basis of $e_1, \dots, e_n$. Then $w^1 \wedge \ldots \wedge w^n$ is the volume element of $M$. Since $w^1, \dots, w^n$ form a local basis of $\bigwedge^1(M)$, every $p$-form $\alpha$ on $M$ can be expressed locally as

$$\alpha = \sum_{1 \leq i_1 < \cdots < i_p \leq n} a_{i_1 \cdots i_p} w^{i_1} \wedge \cdots \wedge w^{i_p}. \tag{2.9}$$

The Hodge star isomorphism, $* : \bigwedge^p(M) \to \bigwedge^{n-p}(M)$, from $p$-forms into $(n-p)$-forms is defined as

$$*\alpha = \sum_{1 \leq j_1 < \cdots . < j_{n-p} \leq n} \epsilon_{i_1 \ldots i_p\, j_1 \ldots j_{n-p}} a_{i_1 \ldots i_p} w^{j_1} \wedge \ldots \wedge w^{i_{n-p}}, \tag{2.10}$$

where $\epsilon_{i_1 \ldots i_p\, j_1 \ldots j_{n-p}}$ is zero if $i_1 \ldots i_p\ j_1 \ldots j_{n-p}$ do not form a permutation of $\{1, \dots, n\}$, and is equal to 1 or $-1$ according to whether the permutation is even or odd. The form $*\alpha$ is called the adjoint of the form $\alpha$. The adjoint of 1 is just the volume element, $*1 = w^1 \wedge \ldots \wedge w^n$, and adjoint of any function is its product with the volume element.

Let $\alpha$ and $\beta$ be $p$-forms given by

$$\alpha = \sum a_{i_1 \ldots i_p} w^{i_1} \wedge \ldots \wedge w^{i_p} \qquad \text{and} \qquad \beta = \sum b_{j_1 \ldots j_p} w^{j_1} \wedge \ldots \wedge w^{j_p}.$$

Then we have

$$\begin{aligned}
\alpha \wedge *\beta &= \Big( \sum_{i_1 < \cdots < i_p} a_{i_1 \ldots i_p} w^{i_1} \wedge \ldots \wedge w^{i_p} \Big) \\
&\qquad \wedge \Big( \sum_{k_1 < \cdots . < k_{n-p}} \epsilon_{j_1 \ldots j_p\, k_1 \ldots k_{n-p}}\, b_{j_1 \ldots j_p} w^{k_1} \wedge \ldots \wedge w^{k_{n-p}} \Big) \\
&= \sum_{k_1 < \cdots . < k_{n-p}} \epsilon_{i_1 \ldots i_p\, k_1 \ldots k_{n-p}}\, a_{i_1 \ldots i_p}\, b_{i_1 \ldots i_p} w^{i_1} \wedge \ldots \wedge w^{i_p} \wedge w^{k_1} \wedge \ldots \wedge w^{k_{n-p}} \\
&= \sum_{i_1 < \cdots < i_p} a_{i_1 \ldots i_p}\, b_{i_1 \ldots i_p}\ w^1 \wedge w^2 \wedge \ldots \wedge w^n \\
&= \sum_{i_1 < \cdots . < i_p} a_{i_1 \ldots i_p}\, b_{i_1 \ldots i_p} * 1.
\end{aligned} \tag{2.11}$$

For any two $p$-forms $\alpha$ and $\beta$ on $M$, a (global) scalar product of $\alpha$ and $\beta$ is defined by

$$(\alpha, \beta) = \int_M \alpha \wedge *\beta, \tag{2.12}$$

whenever the integral converges. Two forms are orthogonal if their scalar product is zero. Using the star operator and the differential operator, the co-differential operator $\delta : \bigwedge^p(M) \to \bigwedge^{p-1}(M)$ is defined as

$$\delta\alpha = (-1)^{np+n+1} * d * \alpha,$$

where $\alpha$ is a $p$-form on $M$. It follows from straightforward computation that,

$$*\delta\alpha = (-1)^p d * \alpha \qquad \text{and} \qquad * d\alpha = (-1)^{p+1}\delta * \alpha. \tag{2.13}$$

Using the operators $d$ and $\delta$ we define an operator $\Delta$ by

$$\Delta = d\delta + \delta d.$$

Then $\Delta$ maps $p$-forms into $p$-forms. The operator $\Delta$ is called the Laplacian of $M$. A $p$-form $\alpha$ on $M$ is called harmonic if $\Delta\alpha = 0$.

**Proposition 2.1.** [1] If $M$ is a compact, oriented Riemannian manifold, and $\alpha$ and $\beta$ are two forms of degree $p$ and $p+1$ respectively, then we have

$$(d\alpha, \beta) = (\alpha, \delta\beta), \tag{2.14}$$

i.e., the operator $\delta$ is the adjoint of $d$. Consequently the Laplacian $\Delta$ is self-adjoint.

**Proof :** The manifold M is compact without boundary, so the Stokes theorem implies

$$\int_M d(\alpha \wedge *\beta) = 0$$

Since $d(\alpha \wedge *\beta) = d\alpha \wedge *\beta + (-1)^p \alpha \wedge d(*\beta)$, we have

$$\int_M d\alpha \wedge *\beta = (-1)^{p-1} \int_M \alpha \wedge d(*\beta).$$

The two forms $d\alpha$ and $\beta$ are of degree $p+1$, so we can write $\int_M d\alpha \wedge *\beta = (d\alpha, \beta)$. On the other hand, from (2.13), we have $d(*\beta) = (-1)^{p+1} * \delta\beta$, which yields

$$(-1)^{p-1} \int_M \alpha \wedge d(*\beta) = (-1)^{p-1}(-1)^{p-1} \int_M \alpha \wedge *(\delta\beta) = (\alpha, \delta\beta).$$

Combining these results, we obtain (2.14). Moreover, we have

$$(\Delta\alpha, \beta) = (d\delta\alpha, \beta) + (\delta d\alpha, \beta) = (\alpha, d\delta\beta) + (\alpha, \delta d\beta) = (\alpha, \Delta\beta),$$

hence $\Delta$ is self-adjoint. $\square$

**Theorem 2.2.** [1] On a compact, oriented Riemannian manifold $M$, a form $\alpha$ is harmonic if and only if $d\alpha = \delta\alpha = 0$.

**Proof :** Let $\alpha$ be a $p$-form on $M$. Then Proposition 2.1 implies

$$(\Delta\alpha, \alpha) = (d\delta\alpha, \alpha) + (\delta d\alpha, \alpha) = (\delta\alpha, \delta\alpha) + (d\alpha, d\alpha).$$

Since for any form $\gamma$ we have $(\gamma, \gamma) \geq 0$ and we have $(\gamma, \gamma) = 0$ if and only if $\gamma = 0$, we conclude that $\alpha$ is harmonic if and only if $d\alpha = \delta\alpha = 0$. $\square$

**Corollary 2.3.** [1] Let $f \in \bigwedge^0(M)$, where $M$ is a compact Riemannian manifold. Then $f$ is harmonic if and only if it is constant.

**Proof :** If $f$ is harmonic, from Theorem 2.2 we have $df = 0$, which implies that $f$ is constant. Conversely, let $f$ be constant and $w^1 \wedge \ldots \wedge w^n$ be the volume element on $M$. Then we have $df = 0$ and

$$\delta f \;=\; (-1)^{n+1} * d * f = (-1)^{n+1} * d(fw^1 \wedge \ldots \wedge w^n) = 0.$$

From Theorem 2.2, we see that $f$ is harmonic. $\square$

**Corollary 2.4.** [1] If $f$ is a differentiable function on a compact, oriented Riemannian manifold $M$, then we have

$$\int_M \Delta f * 1 = 0.$$

**Proof :** Since $f$ is a 0-form, $\Delta f = \delta df$. Then using Proposition 2.1, we have

$$\int_M \Delta f * 1 = \int_M \delta df * 1 = (\delta df, 1) = (df, d1) = 0.$$

$\square$

**Proposition 2.5.** [1] Let $x : M \to \mathbb{R}^m$ be an isometric immersion of a compact $n$-dimensional Riemannian manifold $M$ into $\mathbb{R}^m$. Then we have

$$\int_M (1 + \langle x, H\rangle) * 1 = 0.$$

**Proof :** Because $\Delta x = -nH$, we have

$$\begin{aligned}
n\int_M \langle x, H\rangle * 1 &= -\int_M \langle x, \Delta x\rangle * 1 = -\int_M \langle (x_1, \ldots, x_m), (\Delta x_1, \ldots, \Delta x_m)\rangle * 1 \\
&= -\sum_{i=1}^{m}\int_M x_i \Delta x_i * 1 = -\sum_{i=1}^{m}\int_M \Delta x_i \wedge * x_i = -\sum_{i=1}^{m}(\Delta x_i, x_i) \\
&= -\sum_{i=1}^{m}(\delta d x_i, x_i) = -\sum_{i=1}^{m}(dx_i, dx_i), \\
n\int_M \langle x, H\rangle * 1 &= -\sum_{i=1}^{m}(dx_i, dx_i). \qquad (2.15)
\end{aligned}$$

The 1-form $dx_i$ can be written in terms of the dual basis $w^1, \ldots, w^n$ as $dx_i = \sum_{j=1}^{n} dx_i(e_j)w^j = \sum_{j=1}^{n} e_j(x_i)w^j$. Substituting this in $(dx_i, dx_i)$ we have

$$\begin{aligned}
(dx_i, dx_i) &= \int_M dx_i \wedge *(dx_i) = \int_M \big(\sum_{k=1}^{n} e_k(x_i)w^k\big) \wedge *\big(\sum_{j=1}^{n} e_j(x_i)w^j\big) \\
&= \int_M \big(\sum_{k=1}^{n} e_k(x_i)w^k\big) \wedge \big(\sum_{l_1<\ldots<l_{n-1}} \epsilon_{j\, l_1 \ldots l_{n-1}} e_j(x_i) \\
&\qquad\qquad w^1 \wedge \ldots \wedge \widetilde{w^j} \wedge \ldots \wedge w^n\big),
\end{aligned}$$

where $\widetilde{w^j}$ means that $w^j$ is missing. Hence,

$$\begin{aligned}
(dx_i, dx_i) &= \int_M \big(\sum_{k=1}^{n} e_k(x_i)w^k\big) \wedge \big(\sum_{j=1}^{n}(-1)^{j-1} e_j(x_i) w^1 \wedge \cdots \wedge \widetilde{w^j} \wedge \cdots \wedge w^n\big) \\
&= \int_M \big(\sum_{k,\, j}(-1)^{j-1} e_k(x_i) e_j(x_i) w^k \wedge (w^1 \wedge \cdots \wedge \widetilde{w^j} \wedge \cdots \wedge w^n)\big) \\
&= \int_M \big(\sum_{j=1}^{n}(-1)^{j-1} (e_j(x_i))^2 w^j \wedge (w^1 \wedge \cdots \wedge \widetilde{w^j} \wedge \cdots \wedge w^n)\big) \\
&= \int_M \big(\sum_{j=1}^{n}(-1)^{j-1}(-1)^{j-1} (e_j(x_i))^2 w^1 \wedge \cdots \wedge w^n\big) \\
(dx_i, dx_i) &= \int_M \sum_{j=1}^{n}(e_j(x_i))^2 * 1. \qquad (2.16)
\end{aligned}$$

Let $y_1, \ldots, y_n$ be a local coordinate system on $M$, $\frac{\partial}{\partial y_1}, \ldots, \frac{\partial}{\partial y_n}$ be the local coordinate base field and $dy_1, \ldots, dy_n$ be the corresponding dual base field. Then any 1-form $w$ on $M$ can be expressed locally in this basis as $w = \sum_{j=1}^{n} w(\frac{\partial}{\partial y_j})dy_j$. So we can write

$$dx_i = \sum_{j=1}^{n} dx_i(\frac{\partial}{\partial y_j})dy_j = \sum_{j=1}^{n} \frac{\partial x_i}{\partial y_j} dy_j.$$

We denote by $g_{ij} = \langle \frac{\partial}{\partial y_i}, \frac{\partial}{\partial y_j} \rangle$ the components of the metric tensor with respect to the coordinates $y_1, \ldots, y_n$, and by $g^{ij}$ the inverse of $g_{ij}$. Since the basis $e_1, \ldots, e_n$ is orthonormal, we have $\langle e_k, e_l \rangle = \delta_{kl}$. Let $\eta$ and $\gamma$ be two 1-forms on $M$. Then we can write $\eta = \sum_k \eta(\frac{\partial}{\partial y_k})dy_k = \sum_k \eta(e_k)w^k$ and $\gamma = \sum_l \gamma(\frac{\partial}{\partial y_l})dy_l = \sum_l \gamma(e_l)w^l$. We know that an inner product of two 1-forms is defined as

$$\langle \eta, \gamma \rangle = \sum_{k,\,l} g^{kl} \eta(\frac{\partial}{\partial y_k})\gamma(\frac{\partial}{\partial y_l}),$$

so we see that $\langle \eta, \gamma \rangle = \sum_{k,\,l} \delta^{kl} \eta(e_k)\gamma(e_l) = \sum_k \eta(e_k)\gamma(e_k)$. Therefore we find

$$\langle dx_i, dx_i \rangle = \sum_{k,l} g^{kl} dx_i(\frac{\partial}{\partial y_k})dx_i(\frac{\partial}{\partial y_l}) = \sum_k (e_k(x_i))^2. \tag{2.17}$$

Combining (2.15), (2.16) and (2.17) we obtain

$$\begin{aligned}
n\int_M \langle x, H\rangle dV &= -\sum_{i=1}^{m} \int_M \big(\sum_{k=1}^{n} (e_k(x_i))^2\big) * 1 \\
&= -\sum_{i=1}^{n} \int_M \big(\sum_{k,l} g^{kl} dx_i(\frac{\partial}{\partial y_k})dx_i(\frac{\partial}{\partial y_l})\big) * 1 = -\sum_{i=1}^{m} \int_M \big(\sum_{k,l} g^{kl} \frac{\partial x_i}{\partial y_k}\frac{\partial x_i}{\partial y_l}\big) * 1 \\
&= -\int_M \left(\sum_{k,l} g^{kl} \langle (\frac{\partial x_1}{\partial y_k}, \frac{\partial x_2}{\partial y_k}, \cdots, \frac{\partial x_m}{\partial y_k}), (\frac{\partial x_1}{\partial y_l}, \frac{\partial x_2}{\partial y_l}, \cdots, \frac{\partial x_m}{\partial y_l}) \rangle \right) * 1 \\
&= -\int_M \big(\sum_{k,l} g^{kl} g_{kl}\big) * 1 = -n\int_M *1
\end{aligned}$$

and the proof is completed. $\square$

Let $M$ be a connected (not necessarily compact) submanifold of $\mathbb{R}^m$ and $x : M \to \mathbb{R}^m$ be an isometric immersion of $M$ into $\mathbb{R}^m$. If $e_1, \ldots, e_n$ is an orthonormal local frame field tangent to $M$, then the Laplacian $\Delta$ of $M$, which acts on smooth functions $C^\infty(M)$ on $M$, is given by

$$\Delta = \sum_{i=1}^{n} (\nabla_{e_i} e_i - e_i e_i).$$

If the position vector $x$ of a submanifold $M$ in $\mathbb{R}^m$ can be written as

$$x = x_0 + \sum_{t=p}^{q} x_t, \qquad \Delta x_t = \lambda_t x_t, \tag{2.18}$$

where $q$ is finite, then $M$ is said to be of finite type, or of order [p, q]. Here $x_0$ is a constant map and $x_t$'s are non-constant maps. A submanifold $M$ is said to be of $k$-type, if there are $k$ nonzero $x_t$'s in (2.18), where $k$ is a natural number. A $k$-type submanifold is null if one of the $x_t$'s is harmonic. If a submanifold $M$ is compact, then every eigenvalue $\lambda_t$ of the Laplacian is nonnegative, and on $M$ only the constant functions are harmonic. So a compact submanifold $M$ can not be null. In this case, the constant vector $x_0$ in (2.18) becomes the center of mass of $M$ in $\mathbb{R}^m$. A submanifold $M$ is of null 1-type if and only if it is a minimal submanifold of $\mathbb{R}^m$. Moreover, a submanifold of $\mathbb{R}^m$ is non-null 1-type if and only if it is a minimal submanifold of a hypersphere of $\mathbb{R}^m$.

We give the main theorems about finite typeness of an isometric immersion of a compact Riemannian manifold $M$ into $\mathbb{R}^m$ omitting their proofs.

**Theorem 2.6.** [1] Let $x : M \to \mathbb{R}^m$ be an isometric immersion of a compact Riemannian manifold $M$ into $\mathbb{R}^m$. Then $M$ is of finite type if and only if there is a non-trivial polynomial $P$ such that $P(\Delta)H = 0$ (or $P(\Delta)(x - x_0) = 0$). In other words, $M$ is of finite type if and only if the mean curvature vector $H$ satisfies a differential equation of the form

$$\Delta^k H + c_1 \Delta^{k-1} H + \cdots + c_{k-1} \Delta H + c_k H = 0$$

for some integer $k \geq 1$ and some real numbers $c_1, \ldots, c_k$.

**Theorem 2.7.** [12] Let $M$ be a finite type submanifold of $\mathbb{R}^m$. Denote by $P_m(t)$ a monic polynomial of least degree with $P_m(\Delta)H = 0$. Then we have

(a) the polynomial $P_m(t)$ is unique,

(b) if $Q$ is a polynomial with $Q(\Delta)H = 0$, then $P_m(t)$ is a factor of $Q$, and

(c) $M$ is of k-type if and only if $\deg P_m = k$.

**Example.** (A Flat Torus in $\mathbb{R}^6$) We consider the immersion $x : T^2 \to \mathbb{R}^6$ defined by

$$x(s,t) = (a\sin s, b\sin s\sin\frac{t}{b}, b\sin s\cos\frac{t}{b}, a\cos s, b\cos s\sin\frac{t}{b}, b\cos s\cos\frac{t}{b}). \tag{2.19}$$

Assume that $a^2 + b^2 = 1$ and $a, b > 0$. The coordinate base fields of the tangent bundle are

$$\begin{aligned} e_1 &= (a\cos s, b\cos s\sin\frac{t}{b}, b\cos s\cos\frac{t}{b}, -a\sin s, -b\sin s\sin\frac{t}{b}, -b\sin s\cos\frac{t}{b}), \\ e_2 &= (0, \sin s\cos\frac{t}{b}, -\sin s\sin\frac{t}{b}, 0, \cos s\cos\frac{t}{b}, -\cos s\sin\frac{t}{b}). \end{aligned}$$

We can see that $e_1, e_2$ form an orthonormal basis. We have

$$\begin{aligned} \overline{\nabla}_{e_1}e_1 &= (-a\sin s, -b\sin s\sin\frac{t}{b}, -b\sin s\cos\frac{t}{b}, -a\cos s, -b\cos s\sin\frac{t}{b}, \\ &\qquad -b\cos s\cos\frac{t}{b}), \\ \overline{\nabla}_{e_1}e_2 &= \overline{\nabla}_{e_2}e_1 = (0, \cos s\cos\frac{t}{b}, -\cos s\sin\frac{t}{b}, 0, -\sin s\cos\frac{t}{b}, \sin s\sin\frac{t}{b}), \\ \overline{\nabla}_{e_2}e_2 &= (0, -\frac{1}{b}\sin s\sin\frac{t}{b}, -\frac{1}{b}\sin s\cos\frac{t}{b}, 0, -\frac{1}{b}\cos s\sin\frac{t}{b}, -\frac{1}{b}\cos s\cos\frac{t}{b}). \end{aligned}$$

Using $\nabla_{e_i}e_j = \langle\overline{\nabla}_{e_i}e_j, e_1\rangle e_1 + \langle\overline{\nabla}_{e_i}e_j, e_2\rangle e_2$, we find that $\nabla_{e_i}e_j = 0$ for $i, j = 1, 2$. In local coordinates, the Laplace operator takes the form

$$\Delta = -(\frac{\partial^2}{\partial s^2} + \frac{\partial^2}{\partial t^2}).$$

From (2.1) and (2.8) we obtain

$$H = -\frac{x}{2} - \frac{1}{2b}(0,\ \sin s\sin\frac{t}{b},\ \sin s\cos\frac{t}{b},\ 0, \cos s\sin\frac{t}{b},\ \cos s\cos\frac{t}{b}).$$

Applying the Laplacian we have

$$\begin{aligned} \Delta H &= -(1 + \frac{1}{b^2})H + \frac{a}{2b^2}(\sin s, 0, 0, \cos s, 0, 0), \\ \Delta^2 H &= -(1 + \frac{1}{b^2})^2 H + \frac{a}{2b^2}(2 + \frac{1}{b^2})(\sin s, 0, 0, \cos s, 0, 0). \end{aligned}$$

Consequently we have

$$\Delta^2 H - (2 + \frac{1}{b^2})\Delta H + (1 + \frac{1}{b^2})H = 0.$$

This shows that $T^2$ is of 2-type in $\mathbb{R}^6$.

Let $v_1$ and $v_2$ be two $\mathbb{R}^m$-valued functions on $M$. An inner product of $v_1$ and $v_2$ is defined by

$$(v_1, v_2) = \int_M \langle v_1, v_2 \rangle dV, \tag{2.20}$$

where $\langle v_1, v_2 \rangle$ denotes the Euclidean inner product of $v_1$ and $v_2$.

**Lemma 2.8.** [1] Let $x : M \to \mathbb{R}^m$ be an isometric immersion of a compact Riemannian manifold $M$ into $\mathbb{R}^m$. Then we have $(x_t, x_s) = 0$ for $t \neq s$, where $x_t$ is given as in (2.18).

**Proof :** We can write (2.18) in vector form as

$$(x_1, \ldots, x_m) = (x_{01}, \ldots, x_{0m}) + \sum_{t=p}^{q} (x_{t1}, \ldots, x_{tm}).$$

Since $\Delta$ is self adjoint, using (2.20) and (2.12) we have

$$\begin{aligned} \lambda_t (x_t, x_s) &= \lambda_t \int_M \big(\sum_i x_{ti} x_{si}\big) dV = \lambda_t \sum_i (x_{ti}, x_{si}) \\ &= \sum_i (\lambda_t x_{ti}, x_{si}) = \sum_i (\Delta x_{ti}, x_{si}) = \sum_i (x_{ti}, \Delta x_{si}) \\ &= \sum_i (x_{ti}, \lambda_s x_{si}) = \lambda_s \int_M \big(\sum_i x_{ti} x_{si}\big) dV = \lambda_s (x_t, x_s). \end{aligned}$$

But we have $\lambda_t \neq \lambda_s$ for $t \neq s$, hence $(x_t, x_s) = 0$. $\square$

We give some theorems without proofs for later use.

**Lemma 2.9.** [1] Let $M$ be a compact minimal submanifold of a hypersphere $\mathbb{S}^{m-1}(r)$ in $\mathbb{R}^m$. Then $M$ is mass-symmetric in $\mathbb{S}^{m-1}(r)$.

An $m$-dimensional complete Riemannian manifold of constant curvature $k$ is called a space form and it is denoted by $R^m(k)$.

**Proposition 2.10.** [1] An n-dimensional totally umbilical submanifold $M$ in the real space form $R^m(k)$ is either totally geodesic in $R^m(k)$ or contained in a small hypersphere of an (n+1)-dimensional totally geodesic submanifold of $R^m(k)$.

**Proposition 2.11.** [1] Let $M$ be a pseudo-umbilical submanifold of the real space form $R^m(k)$. If $M$ has parallel mean curvature vector, then either $M$ is a minimal submanifold of $R^m(k)$, or $M$ is a minimal submanifold of a small hypersphere of $R^m(k)$.

**Theorem 2.12.** [2] Let $M$ be a surface in an m-dimensional space form $R^m(k)$ of curvature $k$. If the mean curvature vector $H$ is parallel in the normal bundle, then $M$ is one of the following surfaces;

(a) minimal surfaces of $R^m(k)$,

(b) minimal surfaces of a small hypersphere of $R^m(k)$, or

(c) surfaces with constant mean curvature $|H|$ in a 3-sphere of $R^m(k)$.

**Proposition 2.13.** [2] Let $M$ be a surface in a 3-dimensional space form $R^3(k)$ with constant mean curvature $|H|$. If $M$ has nonzero constant Gaussian curvature, then $M$ is contained in a hypersphere of $R^3(k)$.

**Proposition 2.14.** [2] The minimal surfaces of a small hypersphere of a Euclidean m-space $\mathbb{R}^m$, the open pieces of the product of two plane circles, and the open pieces of a circular cylinder are the only nonminimal surfaces in $\mathbb{R}^m$ with parallel mean curvature vector and constant Gaussian curvature.

## 3. THE LAPLACIAN OF THE MEAN CURVATURE VECTOR

In this section we give the evaluation of the Laplacian of the mean curvature vector of a spherical submanifold.

**Lemma 3.1.** [1] Let $M$ be an $n$-dimensional submanifold of a hypersphere $\mathbb{S}^{m-1}(r)$ of radius r in $\mathbb{R}^m$ centered at the origin. Then we have

$$\Delta H = \Delta^{D'} H' + \mathcal{A}'(H') + \mathrm{tr}(\overline{\nabla} A_H) + \alpha'(\|A_\xi\|^2 + \frac{n}{r^2})\xi - \frac{n\alpha^2}{r^2}x, \tag{3.1}$$

where $\overline{\nabla} A_H = \nabla A_H + A_{DH}$.

**Proof:** We denote by $\nabla'$ and $\nabla$ the connections of $\mathbb{S}^{m-1}$ and $M$. Let $H,\ h,\ A$ and $D$ be the mean curvature vector, second fundamental form, the Weingarten map and the normal connection of $M$ in $\mathbb{R}^m$; $H',\ h',\ A'$ and $D'$ be those of $M$ in $\mathbb{S}^{m-1}$, respectively. Let $\alpha$ and $\alpha'$ be the lengths of $H$ and $H'$ respectively and $\xi$ be the unit normal vector field $\xi = \dfrac{H'}{\alpha'}$. Then we have $A_\xi = A'_\xi$ and $D\xi = D'\xi$. For an $n$-dimensional submanifold $M$ of $\mathbb{R}^m$ we have

$$\Delta H = \Delta^D H + \|A_{\frac{H}{\alpha}}\|^2 H + \mathcal{A}(H) + \mathrm{tr}(\overline{\nabla} A_H). \tag{3.2}$$

Let $\{\eta_{n+1}, \eta_{n+2}, \ldots, \eta_m\}$ be an orthonormal normal basis of $M$ in $\mathbb{R}^m$ such that $\eta_m = \dfrac{x}{r}$. Then we have

$$H = \frac{1}{n}\sum_{\beta=n+1}^{m} \mathrm{tr}A_\beta \eta_\beta = \frac{1}{n}\big(\mathrm{tr}A_{\frac{x}{r}}\frac{x}{r} + \sum_{\beta=n+1}^{m-1}\mathrm{tr}A_\beta\eta_\beta\big) = H' - \frac{x}{r^2}. \tag{3.3}$$

If we apply the Laplacian of the normal bundle of $M$ in $\mathbb{R}^m$ to the mean curvature vector $H$, we write

$$\Delta^D H = \sum_{i=1}^{n} D_{\nabla_{e_i}e_i}H - D_{e_i}D_{e_i}H, \tag{3.4}$$

where $e_1, \ldots, e_n$ is an orthonormal tangent basis of $M$. Since ${D'}_{e_i}H'$ is a vector normal to $M$ and tangent to $\mathbb{S}^{m-1}$, we have $D_{e_i}{D'}_{e_i}H' = {D'}_{e_i}{D'}_{e_i}H'$. We also see from $Dx = 0$ that $D_{\nabla_{e_i}e_i}H = D_{\nabla_{e_i}e_i}(H' - \frac{x}{r^2}) = D_{\nabla_{e_i}e_i}H' = D'_{\nabla_{e_i}e_i}H'$. Applying these in (3.4), we find $\Delta^D H = \Delta^{D'} H'$.

We choose an orthonormal normal basis $\{e_{n+1}, ...., e_m\}$ of $M$ such that

$$e_{n+1} = \frac{H}{\alpha} = \frac{1}{\alpha}(\alpha'\xi - \frac{x}{r^2}) \qquad \text{and} \qquad e_{n+2} = \frac{1}{r\alpha}(\xi + \alpha' x), \tag{3.5}$$

where $\alpha^2 = (\alpha')^2 + \frac{1}{r^2}$. We can see from the definitions of $e_{n+1}$ and $e_{n+2}$ that, at each point of $M$, the two sets of vectors $\{e_{n+1}, e_{n+2}\}$ and $\{\frac{x}{r}, \xi\}$ span the same subspace of the normal space of $M$ in $\mathbb{R}^m$. Hence $\{\frac{x}{r}, \xi, e_{n+3}, ...., e_m\}$ is also an orthonormal normal basis of $M$. The allied mean curvature vector of $M$ in $\mathbb{S}^{m-1}$ is given in this basis by

$$\mathcal{A}'(H') = \sum_{\beta=n+3}^{m} \operatorname{tr}(A_{H'}A_\beta)e_\beta. \tag{3.6}$$

On the other hand, when $e_{n+1}$ and $e_{n+2}$ are chosen as in (3.5), we can write the allied mean curvature vector of $M$ in $\mathbb{R}^m$ as

$$\mathcal{A}(H) = \sum_{\beta=n+2}^{m} \operatorname{tr}(A_H A_\beta)e_\beta. \tag{3.7}$$

Moreover, we have

$$\begin{aligned} \mathcal{A}(H) &= \operatorname{tr}(A_H A_{n+2})e_{n+2} + \sum_{\beta=n+3}^{m} \operatorname{tr}(A_H A_\beta)e_\beta \\ &= \operatorname{tr}(A_H A_{n+2})e_{n+2} + \sum_{\beta=n+3}^{m} \big(\operatorname{tr}(A_{H'}A_\beta) + \frac{1}{r^2}\operatorname{tr}A_\beta\big)e_\beta. \end{aligned}$$

Since $H = \alpha e_{n+1}$, we obtain $\operatorname{tr}A_\beta = 0$ for $\beta = n+3, \ldots, m$. Using (3.6) we find

$$\mathcal{A}(H) = \operatorname{tr}(A_H A_{n+2})e_{n+2} + \mathcal{A}'(H'). \tag{3.8}$$

A direct computation gives

$$
\begin{aligned}
\mathrm{tr}(A_H A_{n+2}) &= \sum_{i=1}^{n} \langle A_H A_{n+2} e_i, e_i \rangle \\
&= \sum_{i=1}^{n} \frac{1}{r\alpha} \big( \alpha' \langle A_\xi A_{\xi+\alpha' x} e_i, e_i \rangle - \frac{1}{r^2} \langle A_x A_{\xi+\alpha' x} e_i, e_i \rangle \big) \\
&= \frac{\alpha'}{r\alpha} \mathrm{tr}(A_\xi^2) + \frac{1}{r\alpha} \sum_{i=1}^{n} (\frac{1}{r^2} - (\alpha')^2) \langle A_\xi e_i, e_i \rangle - \frac{\alpha'}{r^3\alpha} \sum_{i=1}^{n} \langle e_i, e_i \rangle \\
&= \frac{\alpha'}{r\alpha} \|A_\xi\|^2 + \frac{1}{r\alpha} (\frac{1}{r^2} - (\alpha')^2) \mathrm{tr} A_\xi - \frac{n\alpha'}{r^3\alpha}.
\end{aligned}
$$

Since $\mathrm{tr}A_\xi = n\alpha'$, we have

$$
\mathrm{tr}(A_H A_{n+2}) = \frac{\alpha'}{r\alpha} (\|A_\xi\|^2 - n(\alpha')^2). \tag{3.9}
$$

Substituting (3.9) in (3.8) gives

$$
\mathcal{A}(H) = \mathcal{A}'(H') + \frac{\alpha'}{r\alpha} (\|A_\xi\|^2 - n(\alpha')^2) e_{n+2}. \tag{3.10}
$$

Applying (3.10) in (3.2), we obtain

$$
\Delta H = \Delta^{D'} H' + \mathrm{tr}(\overline{\nabla} A_H) + \|A_{\frac{H}{\alpha}}\|^2 H + \mathcal{A}'(H') + \frac{\alpha'}{r\alpha} (\|A_\xi\|^2 - n(\alpha')^2) e_{n+2}. \tag{3.11}
$$

From (3.3) and (3.5), we also find,

$$
\begin{aligned}
\alpha^2 \|A_{\frac{H}{\alpha}}\|^2 &= \alpha^2 \mathrm{tr}(A_{\frac{H}{\alpha}} A_{\frac{H}{\alpha}}) = \sum_{i=1}^{n} \langle A_H A_H e_i, e_i \rangle \\
&= \sum_{i=1}^{n} \langle A_{H' - \frac{x}{r^2}} A_{H' - \frac{x}{r^2}} e_i, e_i \rangle = (\alpha')^2 \|A_\xi\|^2 + \frac{2\alpha'}{r^2} \mathrm{tr} A_\xi + \frac{n}{r^4},
\end{aligned}
$$

$$
\|A_{\frac{H}{\alpha}}\|^2 = \frac{1}{\alpha^2} \big( (\alpha')^2 \|A_\xi\|^2 + \frac{2n(\alpha')^2}{r^2} + \frac{n}{r^4} \big). \tag{3.12}
$$

If we substitute (3.12) in (3.11), we get

$$\begin{aligned}\Delta H &= \Delta^{D'}H' + \mathrm{tr}(\overline{\nabla}A_H) + \mathcal{A}'(H') + \frac{\alpha'}{\alpha^2}\big((\alpha')^2\|A_\xi\|^2 + \frac{2n(\alpha')^2}{r^2} + \frac{n}{r^4}\big)\xi \\ &\quad -\frac{1}{\alpha^2 r^2}\big((\alpha')^2\|A_\xi\|^2 + \frac{2n(\alpha')^2}{r^2} + \frac{n}{r^4}\big)x + \frac{\alpha'}{r\alpha}(\|A_\xi\|^2 - n(\alpha')^2)(\frac{\xi+\alpha' x}{r\alpha}) \\ &= \Delta^{D'}H' + \mathrm{tr}(\overline{\nabla}A_H) + \mathcal{A}'(H') + \frac{\alpha'^3 r^2 + \alpha'}{r^2\alpha^2}\|A_\xi\|^2\xi + (\frac{n(\alpha')^3}{r^2\alpha^2} + \frac{n\alpha'}{\alpha^2 r^4})\xi \\ &\qquad -(\frac{2n(\alpha')^2}{\alpha^2 r^4} + \frac{n}{\alpha^2 r^6} + \frac{n(\alpha')^4}{\alpha^2 r^2})x \\ &= \Delta^{D'}H' + \mathrm{tr}(\overline{\nabla}A_H) + \mathcal{A}'(H') + \alpha'\|A_\xi\|^2\xi + \frac{n\alpha'}{r^2}\xi \\ &\qquad -\frac{n}{r^4\alpha^2}\big(\alpha^2 + (\alpha')^2((\alpha')^2 + \frac{1}{r^2})r^2\big)x \\ &= \Delta^{D'}H' + \mathrm{tr}(\overline{\nabla}A_H) + \mathcal{A}'(H') + \alpha'\|A_\xi\|^2\xi + \frac{n\alpha'}{r^2}\xi - \frac{nr^2\alpha^2}{r^4\alpha^2}((\alpha')^2 + \frac{1}{r^2})x \\ &= \Delta^{D'}H' + \mathrm{tr}(\overline{\nabla}A_H) + \mathcal{A}'(H') + \alpha'\|A_\xi\|^2\xi + \frac{n\alpha'}{r^2}\xi - \frac{n\alpha^2}{r^2}x.\end{aligned}$$

Consequently we have (3.1). □

Moreover, if $r = 1$, then using $H = H' - x$, the equation (3.1) becomes

$$\Delta H = \Delta^{D'}H' + \mathrm{tr}(\overline{\nabla}A_H) + \mathcal{A}'(H') + (\|A_\xi\|^2 + n)H + (\|A_\xi\|^2 - n(\alpha')^2)x. \quad (3.13)$$

**Lemma 3.2.** [3] Let $M$ be an n-dimensional submanifold of $\mathbb{S}^{m-1}(1)$ in $\mathbb{R}^m$. Then we have

$$\mathrm{tr}(\overline{\nabla}A_H) = \frac{n}{2}\,\mathrm{grad}\,\alpha^2 + 2\mathrm{tr}A_{DH'} \quad (3.14)$$

**Proof:** Let $E_1, E_2, \ldots, E_n$ be orthonormal eigenvectors of $A'_\xi = A_\xi$ and $\rho_1, \rho_2, \ldots, \rho_n$ be the corresponding eigenvalues. Since

$$A_{H'}E_i = \alpha'\rho_i E_i \quad (3.15)$$

and

$$\nabla_{E_i}E_j = \sum w_j^k(E_i)E_k, \quad (3.16)$$

we obtain

$$(\nabla_{E_i}A_{H'})E_j = \rho_j(E_i\alpha')E_j + \alpha'(E_i\rho_j)E_j + \sum_k \alpha'(\rho_j - \rho_k)w_j^k(E_i)E_k. \quad (3.17)$$

Combining the Codazzi equation (2.7) and (3.17) we have

$$\begin{aligned} A_{D_{E_i}\xi}E_j - A_{D_{E_j}\xi}E_i &= (\nabla_{E_i}A_\xi)E_j - (\nabla_{E_j}A_\xi)E_i \\ &= (E_i\rho_j)E_j - (E_j\rho_i)E_i \\ &+ \sum_k \big((\rho_j-\rho_k)w_j^k(E_i) - (\rho_i-\rho_k)w_i^k(E_j)\big)E_k, \end{aligned} \tag{3.18}$$

whenever $\alpha' \neq 0$.

If we take the inner product of both sides of (3.18) with $E_i$, we find

$$-(E_j\rho_i) + (\rho_j-\rho_i)w_j^i(E_i) - (\rho_i-\rho_i)w_i^i(E_j) = \langle A_{D_{E_i}\xi}E_j - A_{D_{E_j}\xi}E_i, E_i\rangle,$$

$$(\rho_i-\rho_j)w_i^j(E_i) = (E_j\rho_i) + \langle A_{D_{E_i}\xi}E_j - A_{D_{E_j}\xi}E_i, E_i\rangle \qquad i \neq j. \tag{3.19}$$

Noting that $\overline{\nabla}A_H = \nabla A_H + A_{DH}$ and applying $H = H' - x$, $A_x = -I$ and $Dx = 0$ in $\mathrm{tr}(\overline{\nabla}A_H) = \sum_i((\nabla_{E_i}A_H)E_i + A_{D_{E_i}H}E_i)$, we have

$$\mathrm{tr}(\overline{\nabla}A_H) = \sum_i((\nabla_{E_i}A_{H'})E_i + A_{D_{E_i}H'}E_i) \tag{3.20}$$

If we put $j = i$ in (3.17), we obtain

$$(\nabla_{E_i}A_{H'})E_i = \rho_i(E_i\alpha')E_i + \alpha'(E_i\rho_i)E_i + \sum_k \alpha'(\rho_k-\rho_i)w_k^i(E_i)E_k.$$

Substituting the above equation in (3.20) gives

$$\begin{aligned} \mathrm{tr}(\overline{\nabla}A_H) &= \sum_i \big(\rho_i(E_i\alpha') + \alpha'(E_i\rho_i) + \sum_k \alpha'(\rho_k-\rho_i)w_k^i(E_k)\big)E_i \\ &\qquad + \sum_i A_{\alpha' D_{E_i}\xi + (E_i\alpha')\xi}E_i \\ &= \sum_i \big(\rho_i(E_i\alpha') + \alpha'(E_i\rho_i) + \sum_k \alpha'(\rho_k-\rho_i)w_k^i(E_k)\big)E_i \\ &\qquad + \alpha'\mathrm{tr}A_{D\xi} + \sum_i (E_i\alpha')\rho_i E_i \end{aligned}$$

Consequently we have

$$\mathrm{tr}(\overline{\nabla}A_H) = \alpha'\mathrm{tr}A_{D\xi} + \sum_i \big(2(E_i\alpha')\rho_i + \alpha'(E_i\rho_i) + \sum_k \alpha'(\rho_k-\rho_i)w_k^i(E_k)\big)E_i. \tag{3.21}$$

Substituting (3.19) into (3.21) and making a direct computation we find

$$\begin{aligned}
\mathrm{tr}(\overline{\nabla}A_H) &= \alpha'\mathrm{tr}A_{D\xi} + \sum_i \Big(2(E_i\alpha')\rho_i + \alpha'(E_i\rho_i) \\
&\quad + \sum_{k,\, k\neq i} \alpha'(E_i\rho_k + \langle A_{D_{E_k}\xi}E_i, E_k\rangle - \langle A_{D_{E_i}\xi}E_k, E_k\rangle)\Big)E_i \\
&= \alpha'\mathrm{tr}A_{D\xi} + \sum_i (E_i\alpha')\rho_i E_i + \sum_i \Big((E_i\alpha')\rho_i + \alpha'(E_i\rho_i) \\
&\quad + \sum_{k,\, k\neq i} \alpha'(E_i\rho_k)\Big)E_i + \sum_{i,\, k, k\neq i} \alpha'\langle A_{D_{E_k}\xi}E_i, E_k\rangle E_i \\
&\quad - \sum_{i,\, k, k\neq i} \alpha'\langle A_{D_{E_i}\xi}E_k, E_k\rangle E_i \\
&= \mathrm{tr}A_{DH'} + \sum_i (E_i\alpha')\rho_i E_i + \alpha'\mathrm{tr}A_{D\xi} - \alpha'\sum_i \langle A_{D_{E_i}\xi}E_i, E_i\rangle E_i \\
&\quad + \sum_i \big(\alpha'(E_i\rho_i) + \sum_{k,\, k\neq i} \alpha'(E_i\rho_k)\big)E_i - n\alpha'\sum_i \langle H, D_{E_i}\xi\rangle E_i \\
&\quad + \alpha'\sum_i \langle A_{D_{E_i}\xi}E_i, E_i\rangle E_i \\
&= 2\mathrm{tr}A_{DH'} - n\alpha'\sum_i \langle H, D_{E_i}\xi\rangle E_i + \sum_i \alpha'(E_i\rho_i)E_i + \sum_{i,\, k} \alpha'(E_i\rho_k)E_i
\end{aligned}$$

where we have used the following statements,

$$\sum_{i,\, k,\, k\neq i} \langle A_{D_{E_k}\xi}E_i, E_k\rangle E_i = \mathrm{tr}A_{D\xi} - \sum_i \langle A_{D_{E_i}\xi}E_i, E_i\rangle$$

and

$$\sum_{i,\, k,\, k\neq i} \langle A_{D_{E_i}\xi}E_k, E_k\rangle E_i = \sum_i \langle nH, D_{E_i}\xi\rangle E_i - \sum_i \langle A_{D_{E_i}\xi}E_i, E_i\rangle E_i.$$

Hence, considering

$$\sum_{i,\, k} (E_i\rho_k)E_i = n\sum_i (E_i\alpha')E_i,$$

we obtain

$$\mathrm{tr}(\overline{\nabla}A_H) = 2\mathrm{tr}A_{DH'} - n\alpha'\sum_i \langle H, D_{E_i}\xi\rangle E_i + n\alpha'\sum_i E_i(\alpha')E_i. \tag{3.22}$$

On the other hand, we have

$$\begin{aligned}
\langle H, D_{E_i}\xi\rangle &= \langle H', D'_{E_i}\xi\rangle = \alpha'\langle \xi, -A'_\xi E_i + D'_{E_i}\xi\rangle \\
&= \alpha'\langle \xi, \nabla'_{E_i}\xi\rangle = \frac{\alpha'}{2}E_i\langle \xi, \xi\rangle = 0
\end{aligned}$$

and

$$\begin{aligned} n\alpha' \sum_i (E_i\alpha')E_i &= \frac{n}{2}\sum_i (E_i\alpha^2)E_i = \frac{n}{2}\sum_i d\alpha^2(E_i)E_i \\ &= \frac{n}{2}\sum_i \langle \mathrm{grad}\alpha^2, E_i\rangle E_i = \frac{n}{2}\mathrm{grad}\alpha^2. \end{aligned}$$

Substituting these in (3.22), we obtain (3.14). □

**Corollary 3.3.** [4] Let $M$ be a hypersurface of $\mathbb{S}^{n+1}(1)$ in $\mathbb{R}^{n+2}$. Then we have

$$\Delta H = \Delta^{D'} H' + \frac{n}{2}\,\mathrm{grad}\,\alpha^2 + 2\mathrm{tr}A_{DH'} + (\|A_\xi\|^2 + n)H' - n\alpha^2 x. \tag{3.23}$$

**Lemma 3.4.** [3] If $M$ is a submanifold of $\mathbb{S}^{m-1}$ with parallel mean curvature vector $H'$ (or $H$) then $\mathrm{tr}(\overline{\nabla} A_H) = 0$.

**Proof :** The mean curvature vector is parallel, so we have $DH' = 0$, which implies $\mathrm{tr}A_{DH'} = 0$. On the other hand, since $\alpha'$ is constant, we have $\frac{n}{2}\mathrm{grad}\alpha^2 = \frac{n}{2}\mathrm{grad}((\alpha')^2 + 1) = 0$. □

**Corollary 3.5.** [3] If $M$ is a hypersurface of $\mathbb{S}^{n+1}$ in $\mathbb{R}^{n+2}$ with constant mean curvature, then $\mathrm{tr}(\overline{\nabla} A_H) = 0$.

## 4. COMPACT SPHERICAL SUBMANIFOLDS OF FINITE TYPE

In this section we study properties of compact spherical submanifolds of finite type in $\mathbb{R}^m$. We give some relations between the eigenvalues of the Laplacian and the mean and scalar curvatures of a spherical submanifold of $\mathbb{R}^m$.

**Theorem 4.1.** [3] Let $M$ be a compact, n-dimensional submanifold of a hypersphere $\mathbb{S}^{m-1}$ in $\mathbb{R}^m$ such that $M$ is not of 1-type and the mean curvature vector $H'$ is parallel. Then $M$ is mass-symmetric and of 2-type if and only if $M$ is an $\mathcal{A}$-submanifold of $\mathbb{S}^{m-1}$ and $\|A_{H'}\|$ is constant.

**Proof:** Assume that $\mathbb{S}^{m-1}$ is centered at the origin with radius 1. Since $H'$ is parallel, we have $\triangle^{D'}H' = 0$ and Lemma 3.4 implies that $\mathrm{tr}(\overline{\nabla}A_H) = 0$. So (3.13) becomes

$$\Delta H = \mathcal{A}'(H') + (\|A_\xi\|^2 + n)H + (\|A_\xi\|^2 - n(\alpha')^2)x. \tag{4.1}$$

Let $e_1, e_2, \dots e_n$ be an orthonormal local tangent basis on $M$. If $A_{H'}$ has constant length, then since

$$\|A_{H'}\|^2 = \mathrm{tr}(A^2_{H'}) = (\alpha')^2 \sum_{i=1}^{n} \langle A_\xi e_i, A_\xi e_i\rangle = (\alpha')^2 \|A_\xi\|^2,$$

$(\alpha')^2\|A_\xi\|^2$ is also constant. Here $\alpha'$ is a nonzero constant, because if $\alpha' = 0$, then $M$ is minimal in $\mathbb{S}^{m-1}$ and a compact minimal submanifold of a hypersphere of $\mathbb{R}^m$ is of 1-type. But this contradicts with our assumption, hence $\alpha' \neq 0$ and $\|A_\xi\|^2$ is a constant.

If $M$ is an $\mathcal{A}$-submanifold of $\mathbb{S}^{m-1}$ and $\|A_{H'}\|$ is constant, from (4.1) we have

$$\Delta H = bH + cx, \tag{4.2}$$

where $b = \|A_\xi\|^2 + n$ and $c = \|A_\xi\|^2 - n(\alpha')^2$ are constants. Integrating both sides of (4.2) on $M$, we find

$$\int_M \Delta H dV = b\int_M H dV + c\int_M x dV. \tag{4.3}$$

The integral on the left hand side is zero by Corollary 2.4 and the first integral on the right hand side is zero by Corollary 2.4 since $\Delta x = -nH$. So we obtain

$$c\int_M x dV = 0. \tag{4.4}$$

We know that the center of mass $x_0$ of $M$ in $\mathbb{R}^m$ is given by

$$x_0 = \frac{\int_M x dV}{\int_M dV}.$$

If $c \neq 0$, from (4.4) we see that $x_0 = 0$, which implies $M$ is mass-symmetric.

If $c = 0$, we have $\|A_\xi\|^2 = n(\alpha')^2$. Let $E_1, E_2, ...., E_n$ be the principal vector fields of $A_\xi$ and $\rho_1, \rho_2, ...., \rho_n$ be the corresponding principal curvatures. Then we have

$$\|A_\xi\|^2 = \mathrm{tr}(A_\xi^2) = \sum_{i=1}^{n} \langle A_\xi^2(E_i), E_i\rangle = \sum_{i=1}^{n} {\rho_i}^2.$$

On the other hand we have

$$n(\alpha')^2 = n(\frac{1}{n}\mathrm{tr}A_\xi)^2 = n(\frac{1}{n}\sum_i \langle A_\xi(E_i), E_i\rangle)^2 = \frac{1}{n}(\sum_i \rho_i)^2.$$

Substituting these in $\|A_\xi\|^2 = n(\alpha')^2$, we obtain

$$\rho_1^2 + \rho_2^2 + \cdots + \rho_n^2 = \frac{1}{n}(\rho_1 + \rho_2 + ... + \rho_n)^2$$

which holds if and only if $\rho_1 = \rho_2 = \cdots = \rho_n$. But this implies that $A_{H'} = (\alpha')^2 I$, that is, $M$ is pseudo-umbilical in $\mathbb{S}^{m-1}$. Since $H'$ is parallel, Proposition 2.11 implies that $M$ is of 1-type which is a contradiction to our assumption. Thus $x_0 = 0$, $M$ is mass-symmetric in $\mathbb{S}^{m-1}$. By applying (4.2) and Theorem 2.6, we conclude that $M$ is of 2-type.

Conversely, if $M$ is mass-symmetric and of 2-type, then by Theorem 2.6, there exists constants b and c such that $\Delta H = bH + cx$. Combining this with (4.1), we have

$$\Delta H = \mathcal{A}'(H') + (\|A_\xi\|^2 + n)H + (\|A_\xi\|^2 - n(\alpha')^2)x = bH + cx$$

Since $H = H' - x$ and $\mathcal{A}'(H')$ is normal to $H'$ and tangent to $\mathbb{S}^{m-1}$, we see that $\mathcal{A}'(H') = 0$. From the above equation , we also see that $bH' = (\|A_\xi\|^2 + n)H'$. As $M$ is of 2-type, we have $H' \neq 0$ and $\|A_\xi\|^2 = b - n$, which is a constant. From $\|A_\xi\|^2 - n(\alpha')^2 = c$, we have $\alpha'$=constant. Consequently $\|A_{H'}\|^2 = (\alpha')^2\|A_\xi\|^2$ is constant. $\square$

The second fundamental form $h$ of a submanifold $M$ of a Riemannian manifold $\bar{M}$ is parallel if we have

$$(\overline{\nabla}_Z h)(X, Y) = D_Z h(X, Y) - h(\nabla_Z X, Y) - h(X, \nabla_Z Y) = 0,$$

for any vector fields $X, Y$ and $Z$ tangent to M. A submanifold is called a parallel submanifold if it has parallel second fundamental form. Let $M$ be a parallel submanifold of $\bar{M}$ and let $e_1, e_2, \ldots, e_n$ be an orthonormal local tangent basis of $M$. Using $\nabla_{e_k} e_i = \sum_{j=1}^{n} w_i^j(e_k)e_j$, we have

$$\begin{aligned}
D_{e_k} H &= D_{e_k}\big(\frac{1}{n}\sum_{i=1}^{n} h(e_i, e_i)\big) \\
&= \frac{1}{n}\sum_{i=1}^{n}(h(\nabla_{e_k} e_i, e_i) + h(e_i, \nabla_{e_k} e_i)) = \frac{2}{n}\sum_{i,j=1}^{n} w_i^j(e_k)h(e_j, e_i) \\
&= \frac{2}{n}\sum_{i<j}(w_i^j(e_k) + w_j^i(e_k))h(e_j, e_i) = 0
\end{aligned}$$

Therefore, a parallel submanifold has parallel mean curvature vector. If we write $H = \beta\eta$ where $\eta$ is the unit vector in the direction of $H$, then $\|A_H\|^2 = \beta^2\|A_\eta\|^2$ is constant because $\beta$ is constant and for any vector field $X$ tangent to $M$ we have,

$$
\begin{aligned}
X\|A_\eta\|^2 &= X\sum_i \langle A_\eta^2 e_i, e_i\rangle = X\sum_{i,j}\langle A_\eta e_i, e_j\rangle^2 \\
&= X\sum_{i,j}\langle h(e_i,e_j),\eta\rangle^2 = X\sum_{i,j}(h_{ij}^\eta)^2 = 2\sum_{i,j} h_{ij}^\eta X(h_{ij}^\eta) \\
&= 2\sum_{i,j} h_{ij}^\eta \langle D_X h(e_i,e_j),\eta\rangle = 2\sum_{i,j} h_{ij}^\eta\langle h(\nabla_X e_i, e_j) + h(e_i,\nabla_X e_j),\eta\rangle \\
&= 2\sum_{i,j,k} h_{ij}^\eta \langle w_i^k(X) h(e_k,e_j),\eta\rangle + 2\sum_{i,j,k} h_{ij}^\eta\langle w_j^k(X) h(e_i,e_k),\eta\rangle \\
&= 4\sum_{i,j,k} h_{ij}^\eta w_i^k(X)\langle h(e_j,e_k),\eta\rangle = 4\sum_{j,\, i<k} h_{ij}^\eta h_{jk}^\eta (w_i^k(X) + w_k^i(X)) = 0.
\end{aligned}
$$

**Corollary 4.2.** [3] If $M$ is a compact parallel submanifold of $\mathbb{S}^{m-1}$, then $M$ is an $\mathcal{A}$-submanifold of $\mathbb{S}^{m-1}$ if and only if $M$ is mass-symmetric and $M$ is either of 1 or 2-type in $\mathbb{R}^m$.

**Proof :** Let $M$ be an $\mathcal{A}$-submanifold of $\mathbb{S}^{m-1}$. Since $M$ is a parallel submanifold, the mean curvature vector is parallel and $\|A_{H'}\|^2 = (\alpha')^2\|A_\xi\|^2$ is constant. If $\alpha' = 0$, then M is minimal in $\mathbb{S}^{m-1}$, hence it is of 1-type. Also, by Lemma 2.9, M is mass-symmetric in $\mathbb{S}^{m-1}$. Let $\alpha' \neq 0$, then from the proof of Theorem 4.1, we see that $\Delta H = bH + cx$, where $b$ and $c$ are constants. Thus from (4.3), we have

$$c\int_M x dV = 0.$$

If c=0, $M$ is pseudoumbilical. Since the mean curvature vector is parallel, by Proposition 2.11, $M$ is a minimal submanifold of a small hypersphere $\mathbb{S}^{m-2}(c_0)$, $(c_0 < 1)$ of $\mathbb{S}^{m-1}(1)$, which implies that $M$ is mass-symmetric in $\mathbb{S}^{m-2}(c_0)$. On the other hand, if $c \neq 0$, since we have $\mathcal{A}'(H') = 0$ and $\|A_{H'}\|$ is constant, by Theorem 4.1, $M$ is mass-symmetric and is of 2-type. Conversely, let M be mass-symmetric in $\mathbb{S}^{m-1}$ and be of 1 or 2-type in $\mathbb{R}^m$. If $M$ is of 1-type, then we have $\Delta H = aH$ for some constant $a$. Then from (4.1), we get $\mathcal{A}'(H') = 0$. Finally, if $M$ is of 2-type, it follows from Theorem 4.1 that $\mathcal{A}'(H') = 0$. □

**Theorem 4.3.** [3] Let $M$ be a compact hypersurface of a hypersphere $\mathbb{S}^{n+1}$ such that $M$ is not a small hypersphere of $\mathbb{S}^{n+1}$. Then, $M$ is mass-symmetric and of

2-type if and only if $M$ has nonzero constant mean curvature $\alpha'$ and constant scalar curvature.

**Proof :** Without loss of generality we may assume that $\mathbb{S}^{n+1}$ in $\mathbb{R}^{n+2}$ is centered at the origin with radius 1. Assume that $M$ is of 2-type in $\mathbb{R}^{n+2}$ and mass-symmetric in $\mathbb{S}^{n+1}(1)$. As $M$ is a hypersurface of $\mathbb{S}^{n+1}$, $\mathcal{A}'(H') = 0$ on $M$. $\{\xi, x\}$ is an orthonormal normal basis of $M$ in $\mathbb{R}^{n+2}$ so, for any vector fields X and Y tangent to $M$, we can write the second fundamental form of $M$ as

$$h(X,Y) = \langle h(X,Y), \xi\rangle \xi + \langle h(X,Y), x\rangle x = h^{\xi}(X,Y)\xi + h^{x}(X,Y)x.$$

Let $e_1, e_2, \ldots, e_n$ be an orthonormal tangent basis of $M$ and $\|h\|$ denote the length of the second fundamental form which is given by $\|h\|^2 = \sum\limits_{i,j}((h^{\xi}_{ij})^2 + (h^{x}_{ij})^2)$ where $h^{\xi}_{ij} = h^{\xi}(e_i, e_j)$ and $h^{x}_{ij} = h^{x}(e_i, e_j)$. Then we have

$$\|h\|^2 = \sum_{i,j}((\langle A_{\xi}e_i, e_j\rangle)^2 + (\langle A_x e_i, e_j\rangle)^2) = \|A_{\xi}\|^2 + n.$$

Using $\alpha^2 = (\alpha')^2 + 1$ and $\|h\|^2 = \|A_{\xi}\|^2 + n$, we can write

$$\|A_{\xi}\|^2 - n(\alpha')^2 = \|h\|^2 - n\alpha^2. \tag{4.5}$$

Since $M$ is of 2-type we have $\Delta H = bH + c(x - x_0)$, where b and c are constants and $x_0$ is the center of mass of $M$ in $\mathbb{R}^{n+2}$, Moreover we have $x_0 = 0$ because $M$ is mass-symmetric in $\mathbb{S}^{n+1}$.

Substituting $\mathcal{A}'(H') = 0$ and (4.5) in (3.13), and combining with $\Delta H = bH + cx$, we obtain

$$\Delta^{D'} H' + \mathrm{tr}(\overline{\nabla} A_H) + \|h\|^2 H + (\|h\|^2 - n\alpha^2)x = bH + cx. \tag{4.6}$$

Since $\mathrm{tr}(\overline{\nabla} A_H)$ is tangent to $M$ and other terms in (4.6) are normal to $M$, we get $\mathrm{tr}(\overline{\nabla} A_H) = 0$. If we put $H = H' - x$ in (4.6) we have

$$\Delta^{D'} H' + \|h\|^2 H' - n\alpha^2 x = bH' + (c - b)x. \tag{4.7}$$

$\Delta^{D'}H'$ is parallel to $H'$ so we find $n\alpha^2 = b - c$, which implies that $\alpha'$ is constant. Therefore $H'$ is parallel and $\Delta^{D'}H' = 0$. Because $M$ is of 2-type, $\alpha'$ is nonzero, so (4.7) yields $\|h\|^2 = b$. Since the scalar curvature satisfies

$$n(n-1)\rho = n^2\alpha^2 - \|h\|^2, \tag{4.8}$$

$\rho$ is also constant.

Conversely if $M$ has constant scalar curvature and nonzero constant mean curvature $\alpha'$, then the mean curvature vector $H'$ is parallel, hence $\Delta^{D'}H' = 0$ and $\mathrm{tr}(\overline{\nabla}A_H) = 0$ by Lemma 3.4. Also $M$ is an $\mathcal{A}$-submanifold of $\mathbb{S}^{n+1}$ because $M$ is a hypersurface. From (4.8), we see that $\|h\|^2$ is constant. As a result we have

$$\Delta H = \|h\|^2 H + (\|h\|^2 - n\alpha^2)x = bH + cx$$

where b and c are constants. From the proof of Theorem 4.1, we know that if $c \neq 0$, $M$ is mass-symmetric in $\mathbb{S}^{n+1}$ and is of 2-type in $\mathbb{R}^{n+2}$. If $c = 0$, then $M$ is of 1-type, which implies that it is a minimal submanifold of a hypersphere $\bar{\mathbb{S}}^{n+1}$ in $\mathbb{R}^{n+2}$ and $M$ lies in the intersection of $\bar{\mathbb{S}}^{n+1}$ and $\mathbb{S}^{n+1}$. Then $M$ is either a great or a small hypersphere of $\mathbb{S}^{n+1}$. $M$ can not be a great hypersphere of $\mathbb{S}^{n+1}$ because great hyperspheres of a space form are totally geodesic, thus minimal in $\mathbb{S}^{n+1}$, but we assumed that $\alpha'$ is nonzero. Finally, $M$ can not be a small hypersphere of $\mathbb{S}^{n+1}$, because this contradicts to our assumption. $\square$

**Theorem 4.4.** [1] Let $M$ be an $n$-dimensional compact submanifold of a hypersphere $\mathbb{S}^m(r)$ of radius r in $\mathbb{R}^{m+1}$. Then

(1) if $M$ is of finite type, then $\lambda_q \geq \dfrac{n}{r^2}$ and $\lambda_q = \dfrac{n}{r^2}$ if and only if $M$ is of 1-type,

(2) if $M$ is mass-symmetric in $\mathbb{S}^m(r)$, then $\lambda_1 \leq \lambda_p \leq \dfrac{n}{r^2}$ and $\lambda_p = \dfrac{n}{r^2}$ if and only if $M$ is minimal in $\mathbb{S}^m(r)$ and hence $M$ is of 1-type.

**Proof :** It is known that for a compact submanifold $M$ of $\mathbb{R}^{m+1}$, we have

$$\int_M |H|^k dV \leq (\frac{\lambda_q}{n})^{\frac{k}{2}} \mathrm{vol}(M) \qquad k = 1, 2, 3 \quad \text{or} \quad 4. \tag{4.9}$$

equality holding for some $k = 1, 2, 3$ or 4, if and only if $M$ is of order $q$. (for the proof, see [1], p 296) We also have

$$|H|^2 = |H'|^2 + \frac{1}{r^2}. \tag{4.10}$$

From (4.9) and (4.10), we find

$$(\frac{1}{r^2})\mathrm{vol}(M) \leq \int_M |H|^2 dV \leq (\frac{\lambda_q}{n})\mathrm{vol}(M). \tag{4.11}$$

This shows that $\lambda_q \geq \frac{n}{r^2}$. If $\lambda_q = \frac{n}{r^2}$, then (4.10) and (4.11) implies $H' = 0$. So $M$ is minimal in $\mathbb{S}^m$ and is of 1-type. The converse of this is clear.

For statement (2), we assume that the centroid of $M$ is the center of $\mathbb{S}^m(r)$ and without loss of generality we may assume that $\mathbb{S}^m(r)$ is centered at the origin. Then we have $x = \sum_{t=p}^{q} x_t$. From $\Delta x = -nH$, Proposition 2.5 and Lemma 2.8 we have

$$\begin{aligned}
n\mathrm{vol}(M) &= -n \int_M \langle x, H \rangle dV = (x, \Delta x) \\
&= (\sum_{t=p}^{q} x_t, \sum_{s=p}^{q} \lambda_s x_s) = \sum_{t=p}^{q} \lambda_t (x_t, x_t) \\
&\geq \lambda_p \sum_{t=p}^{q} (x_t, x_t) = \lambda_p (\sum_{t=p}^{q} x_t, \sum_{s=p}^{q} x_s) = \lambda_p (x, x) = \lambda_p \|x\|^2, \\
n\mathrm{vol}(M) &\geq \lambda_p \|x\|^2. \qquad (4.12)
\end{aligned}$$

Since $M$ lies in $\mathbb{S}^m(r)$, using (2.20) we find

$$\|x\|^2 = (x, x) = \int_M \langle x, x \rangle dV = \int_M (\sum_{i=1}^{m} x_i^2) dV = r^2 \mathrm{vol}(M).$$

Thus, by (4.12) we obtain

$$\frac{n}{r^2} \geq \lambda_p. \tag{4.13}$$

If the equality of (4.13) holds, then the inequality of (4.12) becomes equality and M is of 1-type. The converse of this is clear. □

Combining Theorem 4.3 and Theorem 4.4 we obtain the following:

**Corollary 4.5.** [3] Let $M$ be a compact hypersurface of $\mathbb{S}^{n+1}(1)$. If $M$ has constant mean curvature and constant scalar curvature, then either $M$ is a small hypersphere or the eigenvalue $\lambda_p$ of $\Delta$ on $M$ satisfies $\lambda_p \leq n$. Equality holds if and only if $M$ is minimal in $\mathbb{S}^{n+1}$ and $M$ is of 1-type in $\mathbb{R}^{n+2}$.

**Theorem 4.6.** [3] Let $M$ be a compact hypersurface of $\mathbb{S}^{n+1}(1)$ such that $M$ is not a small hypersphere. If $M$ has nonzero constant mean curvature $\alpha'$ and constant scalar curvature $\rho$, then both $\alpha'$ and $\rho$ are completely determined by the order of $M$. We have

$$(\alpha')^2 = (1 - \frac{\lambda_p}{n})(\frac{\lambda_q}{n} - 1), \tag{4.14}$$

$$\rho = \frac{1}{n}(\lambda_p + \lambda_q) - \frac{\lambda_p \lambda_q}{n(n-1)}, \tag{4.15}$$

$$\|h\|^2 = \lambda_p + \lambda_q. \tag{4.16}$$

**Proof :** Without loss of generality we may assume that $\mathbb{S}^{n+1}(1)$ is centered at the origin. It follows from Theorem 4.1 that $M$ is mass-symmetric and is of 2-type. Since $M$ is of 2-type, by Theorem 2.6, there exists a unique monic polynomial $P(t) = t^2 + \bar{b}t + \bar{c}$ such that $P(\Delta)x = 0$ and $\Delta H = -\bar{b}H + \frac{\bar{c}}{n}x$. Combining this with (3.1), we have

$$\Delta^{D'} H' + \mathcal{A}'(H') + \mathrm{tr}(\overline{\nabla} A_H) + (\|A_\xi\|^2 + n)H' - n\alpha^2 x + \bar{b}H' - \bar{b}x - \frac{\bar{c}}{n}x = 0. \tag{4.17}$$

The two distinct real roots of the polynomial $P(t)$ correspond to the two eigenvalues $\lambda_p$ and $\lambda_q$ of the Laplacian of $M$ which satisfy $\Delta x_p = \lambda_p x_p$ and $\Delta x_q = \lambda_q x_q$. Hence we can write $P(t) = t^2 + \bar{b}t + \bar{c} = (t - \lambda_p)(t - \lambda_q)$ and we obtain

$$\bar{b} = -(\lambda_p + \lambda_q) \qquad \text{and} \qquad \bar{c} = \lambda_p \lambda_q. \tag{4.18}$$

The sum of the terms normal to $\mathbb{S}^{n+1}$ in (4.17) vanishes, so we have

$$(\alpha')^2 = -\frac{\bar{b}}{n} - \frac{\bar{c}}{n^2} - 1 = \frac{\lambda_p + \lambda_q}{n} - \frac{\lambda_p \lambda_q}{n^2} - 1 = (1 - \frac{\lambda_p}{n})(\frac{\lambda_q}{n} - 1).$$

We have $\mathrm{tr}(\overline{\nabla} A_H) = 0$ because it is the only term tangent to $M$ in (4.17). Since $M$ is a hypersurface in $\mathbb{S}^{n+1}$ we have $\mathcal{A}'(H') = 0$. Moreover $\alpha'$ is constant on $M$, so we get $D'H' = 0$, hence $\Delta^{D'}H' = 0$. Applying these in (4.17) gives

$$(\|A_\xi\|^2 + n)H' + \bar{b}H' = (\|A_\xi\|^2 + n)H' - (\lambda_p + \lambda_q)H' = 0$$

Consequently we have $\|h\|^2 = \|A_\xi\|^2 + n = \lambda_p + \lambda_q$. On the other hand using (4.8), we find

$$\begin{aligned}\rho = \frac{1}{n(n-1)}(n^2(\alpha')^2 + n^2 - \|h\|^2) &= \frac{1}{n(n-1)}((n-1)(\lambda_p + \lambda_q) - \lambda_p\lambda_q), \\ &= \frac{\lambda_p + \lambda_q}{n} - \frac{\lambda_p\lambda_q}{n(n-1)},\end{aligned}$$

which completes the proof. □

Assume that $M$ is a compact 2-type submanifold of $\mathbb{S}^{m-1}(1)$. Then we have

$$\Delta H = bH + c(x - x_0) \tag{4.19}$$

where $x_0$ is the center of mass of $M$ in $\mathbb{R}^m$ and b and c are constants given by $b = \lambda_p + \lambda_q$ and $c = \dfrac{\lambda_p\lambda_q}{n}$.

Since the mean curvature vectors of $M$ in $\mathbb{R}^m$ and $\mathbb{S}^{m-1}(1)$ are related by $H = H' - x$, then (3.1) and (4.19) yield

$$\Delta^{D'}H' + \mathrm{tr}(\overline{\nabla}A_H) + \mathcal{A}'(H') + (\|A_\xi\|^2 + n)H' - bH' = n\alpha^2 x - bx + c(x - x_0).$$

Taking the inner product of both sides of the above equation with $x$, we have

$$c\langle x_0, x\rangle = n\alpha^2 - b + c. \tag{4.20}$$

If $\alpha$ is constant, then $\langle x_0, x\rangle = |x_0||x|\cos\theta$ is a constant, where $\theta$ denotes the angle between the vectors $x_0$ and $x$ in $\mathbb{R}^m$. If $\langle x_0, x\rangle = 0$, then we can have $|x_0| = 0$, which implies that $M$ is mass-symmetric in $\mathbb{S}^{m-1}$. If $|x_0| \neq 0$ and $|x|\cos\theta = 0$, since $x$ is the position vector of $M$ in $\mathbb{R}^m$, $|x|$ is nonzero and we have $\cos\theta = 0$, that is, the point $x$ is contained in the hyperplane which passes through the origin and is normal to the vector $x_0$. This implies that $M$ is contained in this hyperplane, but since $|x_0| \neq 0$, $x_0$ is not in this hyperplane, which contradicts with $x_0$ being the center of mass of $M$ in $\mathbb{R}^m$. Thus, if $\langle x_0, x\rangle = 0$, then $M$ is mass-symmetric in $\mathbb{S}^{m-1}$. On the other hand, if $\langle x_0, x\rangle \neq 0$, then $M$ lies in the hyperplane which is defined by (4.20). This hyperplane is normal to the vector $x_0$. Moreover, $x_0$ becomes the centroid of the small hypersphere which is the intersection of this hyperplane and $\mathbb{S}^{m-1}$. Thus $M$ is mass-symmetric in this hypersphere.

Consequently we obtain the following.

**Lemma 4.7.** [4] Let $M$ be a compact 2-type submanifold of $\mathbb{S}^{m-1}(1)$ in $\mathbb{R}^m$. If $M$ has constant mean curvature, then either $M$ is mass-symmetric in $\mathbb{S}^{m-1}(1)$ or $M$ is contained in a small hypersphere of $\mathbb{S}^{m-1}(1)$ as a mass-symmetric submanifold.

**Theorem 4.8.** [4] Let $M$ be a compact hypersurface of $\mathbb{S}^{n+1}$ in $\mathbb{R}^{n+2}$. Then we have

(1) if $M$ is of 2-type, then mean curvature of $M$ is constant if and only if $M$ is mass-symmetric in $\mathbb{S}^{n+1}$;

(2) if $M$ is of 3-type then either $M$ is non-mass-symmetric in $\mathbb{S}^{n+1}$ or $M$ has nonconstant mean curvature.

**Proof :** Let $M$ be a compact hypersurface of a hypersphere $\mathbb{S}^{n+1}$ in $\mathbb{R}^{n+2}$. Without loss of generality, we may assume that $\mathbb{S}^{n+1}$ is the unit hypersphere centered at the origin. If $M$ is of 2-type and it has constant mean curvature, then by Lemma 4.7, either $M$ is mass-symmetric in $\mathbb{S}^{n+1}$ or it is a small hypersphere of $\mathbb{S}^{n+1}$. We have $H = H' - x$ and since $M$ is a hypersurface in $\mathbb{S}^{n+1}$, at each point of $M$, the vectors $H'$ and $x$ span the normal space of $M$ in $\mathbb{R}^{n+2}$. If $M$ is a small hypersphere of $\mathbb{S}^{n+1}$, then $M$ is the intersection of $\mathbb{S}^{n+1}$ with a hyperplane of $\mathbb{R}^{n+2}$. Denote the normal vector of this hyperplane by $a$. Since $M$ also lies in this hyperplane, $a$ is in the normal space of $M$ in $\mathbb{R}^{n+2}$, therefore we can write $a = c_1 x + c_2 H'$. Then we get

$$H = H' - x = \frac{a}{c_2} - (\frac{c_1}{c_2} + 1)x.$$

Applying the Laplace operator to this equation, we obtain

$$\Delta H = -(\frac{c_1}{c_2} + 1)\Delta x = (\frac{c_1}{c_2} + 1)nH,$$

which implies that a small hypersphere of $\mathbb{S}^{n+1}$ is of 1-type. But this contradicts with the assumption that $M$ is of 2-type, hence $M$ can not be a small hypersphere of $\mathbb{S}^{n+1}$. Therefore, a compact 2-type hypersurface $M$ of $\mathbb{S}^{n+1}$ with constant mean curvature is mass-symmetric. Conversely if $M$ is a mass-symmetric 2-type hypersurface, then Theorem 4.3 implies that $M$ has constant mean curvature.

Now we prove statement (2). Assume that $M$ is of 3-type, the mean curvature $\alpha'$ of $M$ in $\mathbb{S}^{n+1}$ is constant, and $M$ is mass-symmetric in $\mathbb{S}^{n+1}$. Then it is clear that $\alpha'$ is nonzero. Since $\alpha'$ is constant and $M$ is a hypersurface in $\mathbb{S}^{n+1}$, we have $D'H' = 0$ and $\mathcal{A}'(H') = 0$. Thus Corollary 3.3 and $\|h\|^2 = \|A_\xi\|^2 + n$ imply

$$\Delta H = \|h\|^2 H' - n\alpha^2 x. \tag{4.21}$$

Applying the Laplace operator to (4.21) yields

$$\Delta^2 H = \Delta(\|h\|^2 H' - n\alpha^2 x) = \Delta(\|h\|^2 H + \|h\|^2 x - n\alpha^2 x). \tag{4.22}$$

We want to find an expression for $\Delta(fH)$ where $f \in C^\infty(M)$. Let $a$ be a constant vector in $\mathbb{R}^{n+2}$, then $\langle fH, a\rangle$ becomes a smooth function on $M$ and we can write

$$\Delta\langle fH, a\rangle = \sum_{i=1}^{n}((\nabla_{e_i}e_i)\langle fH, a\rangle - e_i e_i\langle fH, a\rangle).$$

We have

$$(\nabla_{e_i}e_i)\langle fH, a\rangle = \langle -A_{fH}(\nabla_{e_i}e_i) + D_{\nabla_{e_i}e_i} fH, a\rangle,$$

and therefore,

$$\begin{aligned}
\Delta fH &= \sum_{i=1}^{n}(D_{\nabla_{e_i}e_i} fH - D_{e_i}D_{e_i} fH) \\
&\qquad + \sum_{i=1}^{n}((\nabla_{e_i}A_{fH})e_i + A_{D_{e_i}fH}e_i + h(e_i, A_{fH}e_i)) \\
&= \sum_{i=1}^{n}\Big((\nabla_{e_i}e_i)fH + fD_{\nabla_{e_i}e_i}H - e_i(e_i f)H - (e_i f)D_{e_i}H \\
&\quad -(e_i f)D_{e_i}H - fD_{e_i}D_{e_i}H + (e_i f)A_H e_i + f\nabla_{e_i}(A_H e_i) - fA_H(\nabla_{e_i}e_i) \\
&\qquad +(e_i f)A_H e_i + fA_{D_{e_i}H}e_i + fh(e_i, A_h e_i)\Big)
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=1}^{n}\Big(((\nabla_{e_i}e_i)f - e_i(e_if))H + f(D_{\nabla_{e_i}e_i}H - D_{e_i}D_{e_i}H + \nabla_{e_i}(A_He_i)\\
&\quad -A_H(\nabla_{e_i}e_i) + A_{D_{e_i}H}e_i + h(e_i, A_He_i))\\
&\quad\quad +(e_if)(-D_{e_i}H - D_{e_i}H + A_He_i + A_He_i)\Big)
\end{aligned}
$$

$$
\Delta fH = (\Delta f)H + f(\Delta H) + \sum_{i=1}^{n} 2(e_if)(A_He_i - D_{e_i}H). \tag{4.23}
$$

Substituting $\|h\|^2$ for $f$ in (4.23) and using $D_{e_i}H = D_{e_i}H' = D'_{e_i}H' = 0$, we obtain

$$
\begin{aligned}
\Delta(\|h\|^2H) &= (\Delta\|h\|^2)H + \|h\|^2(\Delta H) + \sum_{i=1}^{n} 2(e_i\|h\|^2)A_He_i\\
&= (\Delta\|h\|^2)H + \|h\|^2(\Delta H) + 2A_H(\sum_{i=1}^{n}(e_i\|h\|^2)e_i)\\
&= (\Delta\|h\|^2)H + \|h\|^2(\Delta H) + 2A_H(\mathrm{grad}\|h\|^2). \qquad (4.24)
\end{aligned}
$$

On the other hand for any $f\epsilon\, C^\infty(M)$, we have

$$
\begin{aligned}
\Delta(fx) &= \sum_{i=1}^{n}((\nabla_{e_i}e_i)fx + fD_{\nabla_{e_i}e_i}x - D_{e_i}((e_if)x + fD_{e_i}x)\\
&\quad +\nabla_{e_i}(A_{fx}e_i) - A_{fx}(\nabla_{e_i}e_i) + A_{(e_if)x+fD_{e_i}x}e_i + fh(e_i, A_xe_i)).
\end{aligned}
$$

Substituting $D_{e_i}x = 0$ and $A_x = -I$ above, we obtain

$$
\Delta(fx) = (\Delta f)x - 2\sum_{i=1}^{n}(e_if)e_i - f\sum_{i=1}^{n}h(e_i, e_i) = (\Delta f)x - 2\mathrm{grad}f - nfH.
$$

As a result we get

$$
\Delta(\|h\|^2x) = (\Delta\|h\|^2)x - 2\mathrm{grad}\|h\|^2 - \|h\|^2nH. \tag{4.25}
$$

Lastly, since $\alpha^2$ is constant and $\Delta$ is linear, we have

$$
\Delta(n\alpha^2x) = n\alpha^2\Delta x = n\alpha^2(-nH) = -n^2\alpha^2H. \tag{4.26}
$$

Noting that

$$
A_H(\mathrm{grad}\|h\|^2) = A_{H'}(\mathrm{grad}\|h\|^2) - A_x(\mathrm{grad}\|h\|^2) = A_{H'}(\mathrm{grad}\|h\|^2) + \mathrm{grad}\|h\|^2
$$

and substituting (4.23), (4.24) and (4.25) in (4.22), we get

$$\begin{aligned}
\Delta^2 H &= (\Delta\|h\|^2)H + \|h\|^2(\Delta H) + 2A_H(\text{grad}\|h\|^2) + (\Delta\|h\|^2)x \\
&\quad -2\text{grad}\|h\|^2 - \|h\|^2 nH + n^2\alpha^2 H \\
&= (\Delta\|h\|^2)H' - (\Delta\|h\|^2)x + \|h\|^2(\Delta H) + 2A_{H'}(\text{grad}\|h\|^2) + 2\text{grad}\|h\|^2 \\
&\quad +(\Delta\|h\|^2)x - 2\text{grad}\|h\|^2 - \|h\|^2 nH + n^2\alpha^2 H \\
&= (\Delta\|h\|^2)H' + \|h\|^2(\|h\|^2 H + \|h\|^2 x - n\alpha^2 x) + 2A_{H'}(\text{grad}\|h\|^2) \\
&\quad -\|h\|^2 nH + n^2\alpha^2 H,
\end{aligned}$$

and hence,

$$\Delta^2 H = (\Delta\|h\|^2 + \|h\|^4 - n\|h\|^2 + n^2\alpha^2)H' - (n\|h\|^2(\alpha')^2 + n^2\alpha^2)x + 2A_{H'}(\text{grad}\|h\|^2). \tag{4.27}$$

Moreover, since $M$ is of 3-type and mass-symmetric, there exists a polynomial $P(t) = t^3 + a_1t^2 + a_2t + a_3$ such that $P(\Delta)x = 0$. If we put $\Delta x = -nH$, we obtain

$$\Delta^2 H = c_1\Delta H + c_2 H + c_3 x, \tag{4.28}$$

where $c_1, c_2$ and $c_3$ are constants. Substituting (4.21) into (4.28) we have

$$\begin{aligned}
\Delta^2 H &= c_1(\|h\|^2 H' - n\alpha^2 x) + c_2 H' - c_2 x + c_3 x, \\
&= (c_1\|h\|^2 + c_2)H' + (-c_1 n\alpha^2 - c_2 + c_3)x.
\end{aligned} \tag{4.29}$$

Equating the terms normal to $\mathbb{S}^{n+1}$ in (4.27) and (4.29) gives

$$-n(\alpha')^2\|h\|^2 = n(n - c_1)\alpha^2 - c_2 + c_3. \tag{4.30}$$

So $\|h\|^2$ is constant, hence from (4.8), we see that $M$ has constant scalar curvature. Therefore, by applying Theorem 4.3 we conclude that $M$ is of 2-type. This is a contradiction. □

**Corollary 4.9.** [3] If $M$ is a 2-type compact hypersurface of $\mathbb{S}^{n+1}$ and $M$ has constant mean curvature, then $M$ has constant scalar curvature.

**Proof:** By Theorem 4.8 we see that $M$ is mass-symmetric. According to Theorem 4.3 $M$ has constant scalar curvature. $\square$

Let $M$ be a compact 2-type hypersurface of $\mathbb{S}^{n+1}$ in $\mathbb{R}^{n+2}$. We assume that $\mathbb{S}^{n+1}$ is of radius one and is centered at the origin. From Corollary 3.3 we have

$$\langle \Delta H, x\rangle = -n\alpha^2. \tag{4.31}$$

Also, taking the inner product of both sides of (4.19) with the position vector $x$ gives

$$\langle \Delta H, x\rangle = -b + c - c\langle x, x_0\rangle. \tag{4.32}$$

If we take the inner product of both sides of (4.19) with a vector field $X$ tangent to $M$, we get

$$\langle \Delta H, X\rangle = -c\langle x_0, X\rangle. \tag{4.33}$$

We denote by $(\Delta H)^T$ the component of $\Delta H$ tangent to $M$. Let $e_1, \ldots, e_n$ be an orthonormal tangent basis of $M$. Then we can write $(\Delta H)^T = \sum_{i=1}^{n}\langle \Delta H, e_i\rangle e_i$. Applying (4.33), we find $(\Delta H)^T = -\sum_{i=1}^{n} c\langle x_0, e_i\rangle e_i$. On the other hand , from (4.20) we see that $e_i(c\langle x, x_0\rangle) = e_i(n\alpha^2 + c - b)$. So we have

$$ne_i(\alpha^2) = c\langle \overline{\nabla}_{e_i} x, x_0\rangle = c\langle e_i, x_0\rangle.$$

Combining our results we get

$$(\Delta H)^T = -\sum_{i=1}^{n} c\langle x_0, e_i\rangle e_i = -n\sum_{i=1}^{n} e_i(\alpha^2)e_i = -n\mathrm{grad}(\alpha^2) = -n\mathrm{grad}(\alpha^2). \tag{4.34}$$

The only terms tangent to $M$ on the right hand side of (3.23) are $\frac{n}{2}\,\mathrm{grad}(\alpha^2)$ and $2\mathrm{tr}A_{DH'}$ . So from (4.34) we can write

$$\mathrm{tr}A_{DH'} = -\frac{3n}{4}\,\mathrm{grad}\,\alpha^2. \tag{4.35}$$

Let $E_1, E_2, \ldots, E_n$ be orthonormal principle directions of $A_\xi$ with principal curvatures $\mu_1, \mu_2, \ldots, \mu_n$ respectively. Then from (4.35) we find

$$\mathrm{tr}A_{DH'} = -\frac{3n}{4}\mathrm{grad}\alpha^2 = -\frac{3n}{4}\sum E_i(\alpha^2)E_i = -\frac{3n}{2}\sum \alpha' E_i(\alpha')E_i. \tag{4.36}$$

On the other hand, from the definition of $\mathrm{tr}A_{DH'}$, we write

$$\mathrm{tr}A_{DH'} = \sum_{i=1}^{n} A_{D_{E_i}H'}E_i = \sum_{i=1}^{n}(A_{E_i(\alpha')\xi}E_i + A_{\alpha' D_{E_i}\xi}E_i) = \sum_{i=1}^{n}(E_i\alpha')\mu_i E_i. \quad (4.37)$$

Combining (4.36) and (4.37) we find

$$(2\mu_i + 3n\alpha')(E_i\alpha') = 0 \qquad (i = 1, ....n) \quad (4.38)$$

**Lemma 4.10.** [4] Let $M$ be a compact 2-type hypersurface of $\mathbb{S}^{n+1}(1)$. Then $\mathrm{grad}(\alpha')^2$ is a principal direction of $A_\xi$ with principal curvature $-\dfrac{3n\alpha'}{2}$ on the set $U = \{u \,\epsilon\, M \,|\, \mathrm{grad}(\alpha')^2 \neq 0 \;\; at\, u\}$.

**Proof :** Since $\alpha'$ is not constant on $U$, we can not have $E_i\alpha' = 0$ for all $i_1 \ldots n$. Therefore, using (4.38) we get

$$\begin{aligned} A_\xi(\mathrm{grad}(\alpha')^2) &= \sum_{i=1}^{n} E_i(\alpha')^2 A_\xi E_i = \sum_{i=1}^{n} 2\alpha'(E_i\alpha')\mu_i E_i \\ &= -\sum_{i=1}^{n} \alpha' 3n\alpha'(E_i\alpha')E_i = -\frac{3n\alpha'}{2}\mathrm{grad}(\alpha')^2. \end{aligned}$$

□

Now we give a general lemma on 2-type hypersurfaces.

**Lemma 4.11.** [4] Let $M$ be a compact 2-type hypersurface of $\mathbb{S}^{n+1}$ in $\mathbb{R}^{n+2}$. Then either $M$ has constant mean curvature or $U = \{u \,\epsilon\, M \,|\, \mathrm{grad}(\alpha')^2 \neq 0 \;\; \mathrm{at}\, u\}$ is dense in $M$, that is, the closure of $U$ is $M$.

**Proof :** Let $M$ be a 2-type hypersurface of $\mathbb{S}^{n+1}$. We assume that $U$ is neither empty nor dense in $M$. Since $U$ is nonempty, the mean curvature $\alpha'$ is not constant on $M$ and hence, by Theorem 4.8, $M$ is not mass-symmetric in $\mathbb{S}^{n+1}$, that is, $x_0 \neq 0$, and $M - U$ has nonempty interior. Let $V$ be a component of $\mathrm{int}(M - U)$. Then $\alpha'$ is constant on $V$. From (4.20), we see that $\langle x, x_0\rangle = |x||x_0|\cos\theta$ is constant on V, where $\theta$ is the angle between the vectors $x$ and $x_0$ in $\mathbb{R}^{n+2}$. Here $x_0$ is nonzero because $M$ is not mass-symmetric in $\mathbb{S}^{n+1}$. We see that $V$ is contained in a hyperplane of $\mathbb{R}^{n+2}$ which is normal to the vector $x_0$. Therefore $V$ is an open portion of a small hypersphere of $\mathbb{S}^{n+1}$ and consequently $V$ is totally umbilical in $\mathbb{S}^{n+1}$. For any vector field $X$ tangent to $M$, we have $A_\xi X = \mu X$ on V. So, on V we have $\alpha' = \dfrac{1}{n}\mathrm{tr}A_\xi = \mu$ which implies that $\mu_1 = \mu_2 = \cdots = \mu_n = \alpha'$ on V. Thus, by considering Lemma 4.10, we have $-\dfrac{3n\alpha'}{2} = \mu_1 = \mu_2 = \cdots = \mu_n = \alpha'$ on $U$. This is a contradiction and Lemma 4.11 is proved. □

Let $M$ be a compact 2-type hypersurface of $\mathbb{S}^{n+1}(1)$ in $\mathbb{R}^{n+2}$. Then from (4.19), (4.35) and Corollary 3.3, we find

$$cx_0 = -\Delta^{D'}H' + n\mathrm{grad}(\alpha^2) + (b - \|h\|^2)H' + (n\alpha^2 + c - b)x. \tag{4.39}$$

Thus we have

$$-c\langle x_0, H\rangle = \langle \Delta^{D'}H', H'\rangle + (\|h\|^2 - b)(\alpha')^2 + n\alpha^2 + c - b. \tag{4.40}$$

Since

$$\Delta\langle x, x_0\rangle = \langle \Delta x, x_0\rangle = -n\langle H, x_0\rangle,$$

Using (4.20), we find

$$-c\langle x_0, H\rangle = \Delta\alpha^2. \tag{4.41}$$

Consequently, (4.40) and (4.41) yield

$$\Delta\alpha^2 = \langle \Delta^{D'}H', H'\rangle + (\|h\|^2 - b)(\alpha')^2 + n\alpha^2 + c - b. \tag{4.42}$$

As before, let $\alpha'$ be the local function defined by $H' = \alpha'\xi$ and let $e_1, e_2, \ldots, e_n$ form an orthonormal tangent basis for $M$. Then we have $\alpha^2 = (\alpha')^2 + 1$, and

$$\begin{aligned}
\Delta\alpha^2 &= \Delta(\alpha')^2 = \sum_{i=1}^{n}(\nabla_{e_i}e_i - e_ie_i)(\alpha')^2 = \sum_{i=1}^{n}\big(2\alpha'(\nabla_{e_i}e_i\alpha') - e_i(2\alpha' e_i\alpha')\big) \\
&= \sum_{i=1}^{n}(2\alpha'(\nabla_{e_i}e_i\alpha' - e_i(e_i\alpha')) - 2(e_i\alpha')^2) \\
&= 2\alpha'\Delta\alpha' - 2|\mathrm{grad}\alpha'|^2, \qquad (4.43)
\end{aligned}$$

where we have put

$$\sum_{i=1}^{n}(e_i\alpha')^2 = \langle\sum_{i=1}^{n}(e_i\alpha')e_i, \sum_{j=1}^{n}(e_j\alpha')e_j\rangle = \langle \mathrm{grad}\alpha', \mathrm{grad}\alpha'\rangle = |\mathrm{grad}\alpha'|^2.$$

We also have

$$\begin{aligned}\langle H', \Delta^{D'}H'\rangle &= \langle H', \sum_{i=1}^{n}(D'_{\nabla_{e_i}e_i}H' - D'_{e_i}D'_{e_i}H')\rangle\\ &= \sum_{i=1}^{n}\langle H', (\nabla_{e_i}e_i\alpha')\xi + \alpha' D'_{\nabla_{e_i}e_i}\xi - (e_i\alpha')D'_{e_i}\xi - (e_ie_i\alpha')\xi\\ &\qquad\qquad -\alpha' D'_{e_i}D'_{e_i}\xi - (e_i\alpha')D'_{e_i}\xi\rangle\\ &= \langle \alpha'\xi, \alpha'\Delta\alpha' + \sum_{i=1}^{n}\big(\alpha'(D'_{\nabla_{e_i}e_i}\xi - D'_{e_i}D'_{e_i}\xi) - 2e_i\alpha' D'_{e_i}\xi\big)\rangle.\end{aligned}$$

Since $M$ is a hypersurface of $\mathbb{S}^{n+1}$, we have $D'\xi = 0$, thus

$$\langle H', \Delta^{D'}H'\rangle = \alpha'\Delta\alpha'. \tag{4.44}$$

Using (4.42), (4.43) and (4.44), we find

$$\frac{1}{2}\Delta\alpha^2 = |\mathrm{grad}\alpha'|^2 + (\|h\|^2 - b)(\alpha')^2 + n\alpha^2 + c - b. \tag{4.45}$$

Let $U = \{u \,\epsilon\, M \,|\, \mathrm{grad}(\alpha')^2 \neq 0 \text{ at } u\}$ be dense in $M$. Then $\mathrm{grad}(\alpha')^2$ is a principal direction on $U$. Since $\mathrm{grad}(\alpha')^2 = 2\alpha'\mathrm{grad}\alpha'$, then $\mathrm{grad}\alpha'$ is parallel to $\mathrm{grad}(\alpha')^2$. Let $E_1, E_2, \cdots, E_n$ be orthonormal principal directions with principal curvatures $\mu_1, \mu_2, \cdots, \mu_n$ respectively, and $E_1$ is assumed to be in the direction of $\mathrm{grad}\alpha'$. From (4.39), we find

$$\begin{aligned}0 = E_j(cx_0) = n\overline{\nabla}_{E_j}(\mathrm{grad}\alpha^2) + \overline{\nabla}_{E_j}(b - \|h\|^2)\alpha'\xi - \overline{\nabla}_{E_j}(\Delta^{D'}H')\\ +\overline{\nabla}_{E_j}(n\alpha^2 + c - b)x.\end{aligned} \tag{4.46}$$

Since $E_1$ is parallel to $\mathrm{grad}\alpha^2$, we have $\mathrm{grad}\alpha^2 = (E_1\alpha^2)E_1$, and using

$$\nabla_{E_i}E_j = \sum_k w_j^k(E_i)E_k \qquad (i,j,k = 1,\ldots,n), \tag{4.47}$$

we have

$$\begin{aligned}
\overline{\nabla}_{E_j}(\mathrm{grad}\alpha^2) &= \overline{\nabla}_{E_j}((E_1\alpha^2)E_1) = E_j(E_1\alpha^2)E_1 + (E_1\alpha^2)\overline{\nabla}_{E_j}E_1 \\
&= E_j(E_1\alpha^2)E_1 + (E_1\alpha^2)\big(\nabla_{E_j}E_1 + h(E_j, E_1)\big) \\
&= E_j(E_1\alpha^2)E_1 + (E_1\alpha^2)\sum_{k=2}^{n} w_1^k(E_j)E_k \\
&\qquad +(E_1\alpha^2)\big(\langle A_\xi E_j, E_1\rangle\xi + \langle A_x E_j, E_1\rangle x\big) \\
&= E_j(E_1\alpha^2)E_1 + (E_1\alpha^2)\sum_{k=2}^{n} w_1^k(E_j)E_k.
\end{aligned}$$

If we substitute this in (4.46), we get

$$\begin{aligned}
0 &= n\Big(E_j(E_1\alpha^2)E_1 + E_1(\alpha')^2\sum_{k=2}^{n} w_1^k(E_j)E_k\Big) - A_{\alpha'(b-\|h\|^2)\xi}E_j \\
&\quad +D_{E_j}\alpha'(b-\|h\|^2)\xi - \overline{\nabla}_{E_j}\big[\sum_{l=1}^{n}(\alpha' D'_{\nabla_{E_l}E_l}\xi) + (\nabla_{E_l}E_l\alpha')\xi \\
&\quad -D'_{E_l}((E_1\alpha')\xi + \alpha' D_{E_l}\xi)\big] - A_{(n\alpha^2+c-b)x}E_j + D_{E_j}(n\alpha^2+c-b)x
\end{aligned}$$

Because $M$ is a hypersurface in $\mathbb{S}^{n+1}$, we have $D\xi = D'\xi = 0$, and therefore we get

$$\begin{aligned}
0 &= n\Big(E_j(E_1\alpha^2)E_1 + E_1(\alpha')^2\sum_{k=2}^{n} w_1^k(E_j)E_k\Big) - \alpha'(b-\|h\|^2)A_\xi E_j \\
&\quad +E_j(\alpha'(b-\|h\|^2))\xi - \overline{\nabla}_{E_j}\Big(\sum_{l=1}^{n}(\nabla_{E_l}E_l)\alpha'\xi - (E_lE_l\alpha')\xi\Big) \\
&\quad -(n\alpha^2+c-b)A_xE_j + E_j(n\alpha^2+c-b)x + (n\alpha^2+c-b)D_{E_j}x \\
&= n\Big(E_j(E_1\alpha^2)E_1 + E_1(\alpha')^2\sum_{k=2}^{n} w_1^k(E_j)E_k\Big) - \mu_j\alpha'(b-\|h\|^2)E_j \\
&\quad +E_j(\alpha' b - \alpha'\|h\|^2)\xi - \overline{\nabla}_{E_j}(\Delta\alpha'\xi) + (n\alpha^2+c-b)E_j + E_j(n\alpha^2+c-b)x \\
&= n\Big(E_j(E_1\alpha^2)E_1 + E_1(\alpha')^2\sum_{k=2}^{n} w_1^k(E_j)E_k\Big) - \mu_j\alpha'(b-\|h\|^2)E_j \\
&\quad +E_j(\alpha' b - \alpha'\|h\|^2)\xi - (E_j\Delta\alpha')\xi + \Delta\alpha' A_\xi E_j + (n\alpha^2+c-b)E_j \\
&\quad +E_j(n\alpha^2+c-b)x.
\end{aligned}$$

Consequently we obtain

$$
\begin{aligned}
0 &= n\Big(E_j(E_1\alpha^2)E_1 + E_1(\alpha')^2\sum_{k=2}^{n} w_1^k(E_j)E_k\Big) - (\alpha' b - \|h\|^2\alpha' - \Delta\alpha')\mu_j E_j \\
&\quad + E_j(\alpha'(b-\|h\|^2) - \Delta\alpha')\xi + (n\alpha^2 + c - b)E_j + E_j(n\alpha^2+c-b)x \quad j>1
\end{aligned}
\tag{4.48}
$$

on $U$.

By taking the inner product of (4.48), with $E_j$, we obtain

$$0 = 2n(E_1\alpha')\alpha' w_1^j(E_j) - \mu_j(b\alpha' - \|h\|^2\alpha' - \Delta\alpha') + n\alpha^2 + c - b, \qquad j>1 \tag{4.49}$$

on $U$.

**Theorem 4.12.** [12] Let $M$ be a hypersurface in $R^{n+1}(k)$ whose principal curvatures are constant. If exactly two are distinct, then $M$ is locally isometric to the product of two spaces of constant curvature.

**Theorem 4.13.** [4] Let $M$ be a compact hypersurface of $\mathbb{S}^{n+1}(1)$ with at most two distinct principal curvatures. Then $M$ is of 2-type if and only if $M$ is the product of two spheres $\mathbb{S}^p(r_1)\times\mathbb{S}^{n-p}(r_2)$ such that $r_1^2 + r_2^2 = 1$ and $(r_1, r_2) \neq \big(\sqrt{\frac{p}{n}}, \sqrt{\frac{n-p}{n}}\big)$.

**Proof :** We assume that $M$ is a compact 2-type hypersurface in $\mathbb{S}^{n+1}(1)$ with at most two distinct principal curvatures.

If the mean curvature $\alpha'$ of $M$ in $\mathbb{S}^{n+1}$ is non-constant, then according to Lemma 4.11, the open subset $U = \{u \,\epsilon\, M \mid \text{grad}(\alpha')^2 \neq 0 \text{ at } u\}$ is dense in $M$. From Lemma 4.10 we know that $\text{grad}(\alpha')^2$ is a principal direction on $U$ with corresponding principal curvature $-\frac{3n\alpha'}{2}$. Since $\text{grad}(\alpha')^2 = 2\alpha'\text{grad}\alpha'$, we see that $\text{grad}\alpha'$ is also a principal direction with principal curvature $\mu_1 = -\frac{3n\alpha'}{2}$ on $U$. First we will show that the multiplicity of $\mu_1$ is one on $U$.

Since $M$ is a hypersurface of $\mathbb{S}^{n+1}(1)$, from Codazzi equation (2.7) we have

$$(\nabla_X A_\xi)Y = (\nabla_Y A_\xi)X \tag{4.50}$$

for X, Y tangent to $M$. Let $E_1, \ldots, E_n$ be orthonormal principal directions on $M$ such that $E_1 = \mathrm{grad}\alpha'$, and $\mu_1, \ldots, \mu_n$ be the corresponding principal curvatures. We have

$$\begin{aligned}(\nabla_{E_j} A_\xi)E_i &= \nabla_{E_j}(A_\xi E_i) - A_\xi(\nabla_{E_j} E_i) = \nabla_{E_j}(\mu_i E_i) - A_\xi(\sum_k w_i^k(E_j)E_k) \\ &= (E_j \mu_i)E_i + \mu_i \sum_k w_i^k(E_j)E_k - \sum_k w_i^k(E_j)\mu_k E_k \\ &= (E_j \mu_i)E_i + \sum_k w_i^k(E_j)(\mu_i - \mu_k)E_k. \end{aligned} \tag{4.51}$$

Applying (4.50) and (4.51), we obtain

$$(E_j\mu_i)E_i - (E_i\mu_j)E_j = \sum_k \Big(w_j^k(E_i)(\mu_j - \mu_k) - w_i^k(E_j)(\mu_i - \mu_k)\Big)E_k.$$

For $i \neq j$ and $k = i$ we have

$$E_j(\mu_i) = w_i^j(E_i)(\mu_i - \mu_j). \tag{4.52}$$

Since $\mathrm{grad}\alpha' = \sum_i E_i(\alpha')E_i$ and $E_1 = \mathrm{grad}\alpha'$, it is clear that $\mathrm{grad}\alpha' = E_1(\alpha')E_1$ and $E_i\alpha' = 0$ for $i = 2, \ldots, n$. Moreover, we must have $E_1\alpha' \neq 0$ because we have assumed that mean curvature is nonconstant on $M$. Let the multiplicity of $\mu_1 = -\dfrac{3n\alpha'}{2}$ be $\geq 2$ and let $E_2$ be a principal direction with $\mu_2 = \mu_1$. Then (4.52) yields

$$E_1\mu_2 = (\mu_2 - \mu_1)w_2^1(E_2) = 0,$$

which implies that $E_1\mu_2 = E_1\mu_1 = 0$. But this contradicts with the nonconstancy of $\alpha'$. Therefore the multiplicity of $\mu_1$ is one on $U$. Since $M$ has at most two distinct principal curvatures, and the mean curvature is given by $\alpha' = \dfrac{1}{n}\mathrm{tr}A_\xi = \dfrac{1}{n}\sum_{i=1}^{n}\mu_i$, then we have

$$\mu_1 = -\frac{3n\alpha'}{2} \quad \text{and} \quad \mu_2 = \cdots = \mu_n = \frac{5n\alpha'}{2(n-1)}. \tag{4.53}$$

If we put $j = 1$ in (4.52) and substitute (4.53) in (4.52), we have

$$\begin{aligned} E_1(\mu_i) &= w_1^i(E_i)(\mu_1 - \mu_i) \qquad (i = 2, \ldots, n), \\ E_1\Big(\frac{5n\alpha'}{2(n-1)}\Big) &= w_1^i(E_i)\Big(-\frac{3n\alpha'}{2} - \frac{5n\alpha'}{2(n-1)}\Big), \\ 5E_1\alpha' &= -(3n+2)\alpha' w_1^i(E_i) \qquad (i = 2, \ldots, n). \end{aligned} \tag{4.54}$$

on $U$. Consequently by using the equation $|\text{grad}\alpha'| = E_1\alpha'$, (4.49) and (4.54), we get

$$0 = -\frac{10n}{3n+2}|\text{grad}\alpha'|^2 - \frac{5n\alpha'}{2(n-1)}(b\alpha' - \|h\|^2\alpha' - \Delta\alpha') + n\alpha^2 + c - b \quad (4.55)$$

on $U$. Since the right hand side of (4.55) is a well defined continuous function on $M$ and $U$ is dense in $M$, the equation (4.55) holds on the whole hypersurface $M$. Equation (4.55) yields

$$\alpha'\Delta\alpha' = \frac{4(n-1)}{3n+2}|\text{grad}\alpha'|^2 + (b - \|h\|^2)(\alpha')^2 - \frac{2(n-1)}{5n}(n\alpha^2 + c - b).$$

Since from (4.43) we have $\alpha'\Delta\alpha' = |\text{grad}\alpha'|^2 + \frac{1}{2}\Delta(\alpha')^2$, substituting this in the above equation and making a direct computation we get

$$\frac{1}{2}\Delta(\alpha')^2 = \frac{n-6}{3n+2}|\text{grad}\alpha'|^2 + (b - \|h\|^2)(\alpha')^2 - \frac{2(n-1)}{5n}(n\alpha^2 + c - b). \quad (4.56)$$

Combining (4.45) and (4.56), we obtain

$$\Delta(\alpha')^2 = \frac{4(n-1)}{3n+2}|\text{grad}\alpha'|^2 + \frac{3n+2}{5n}(n\alpha^2 + c - b). \quad (4.57)$$

Multiplying both sides of (4.57) with $(3n+2)5n$ and applying Corollary 2.4 gives

$$20n(n-1)\int\limits_M |\text{grad}\alpha'|^2 dV + (3n+2)^2 \int\limits_M (n\alpha^2 + c - b)dV = 0. \quad (4.58)$$

On the other hand, integrating both sides of (4.20) we have

$$\int\limits_M (n\alpha^2 + c - b)dV = \int\limits_M c\langle x, x_0\rangle dV \quad (4.59)$$

where

$$x_0 = \frac{\int\limits_M x dV}{\int\limits_M dV},$$

from which it follows that

$$\langle x_0, x_0\rangle \int\limits_M dV = \int\limits_M \langle x_0, x\rangle dV. \quad (4.60)$$

If we substitute (4.60) in (4.59), we obtain

$$\int_M (n\alpha^2 + c - b)dV = c\langle x_0, x_0\rangle \int_M dV. \tag{4.61}$$

Therefore by using (4.58) and (4.61) we have

$$\int_M \Big(20n(n-1)|\mathrm{grad}\alpha'|^2 + (3n+2)^2 c\langle x_0, x_0\rangle\Big)dV = 0.$$

This gives

$$\int_M c\langle x_0, x_0\rangle dV = 0,$$

which implies that $x_0 = 0$. Since $M$ is a compact 2-type mass-symmetric hypersurface of $\mathbb{S}^{n+1}$, it follows from Theorem 4.3 that, $M$ has nonzero constant mean curvature and constant scalar curvature $\rho$. But this is a contradiction with our assumption that $\alpha'$ is non-constant. Therefore the mean curvature $\alpha'$ must be constant on $M$.

By Theorem 4.8, a compact 2-type hypersurface of $\mathbb{S}^{n+1}$ with constant mean curvature is mass-symmetric and hence, by Theorem 4.3 , it has constant scalar curvature $\rho$. If $M$ has only one principal curvature, then $M$ is totally umbilical in $\mathbb{S}^{n+1}$, hence it is pseudo umbilical. Moreover, constancy of $\alpha'$ implies that $H'$ is parallel because $M$ is a hypersurface of $\mathbb{S}^{n+1}$. So Proposition 2.11 implies that $M$ is of 1-type, therefore $M$ must have exactly two distinct principal curvatures. On the other hand, since the scalar curvature is constant, from (4.8), we see that $\|h\|^2 = \|A_\xi\|^2 + n = k{\mu_1}^2 + l{\mu_2}^2 + n$ (where $k + l = n$) is constant and this, together with the constancy of $\alpha' = \frac{1}{n}(k\mu_1 + l\mu_2)$, imply that $M$ has exactly two distinct constant principal curvatures. So $M$ is the product of two spheres, $M = \mathbb{S}^p(r_1) \times \mathbb{S}^{n-p}(r_2)$. Since $M$ is a nonminimal hypersurface of $\mathbb{S}^{n+1}(1)$, we have $r_1^2 + r_2^2 = 1$ and $(r_1, r_2) \neq \big(\sqrt{\frac{p}{n}}, \sqrt{\frac{n-p}{n}}\big)$.

Conversely, since $\mathbb{S}^p(r_1)$ and $\mathbb{S}^{n-p}(r_2)$ have constant principal curvatures, the product manifold $\mathbb{S}^p(r_1) \times \mathbb{S}^{n-p}(r_2)$ has constant principal curvatures. By a direct computation it can be shown that the condition $(r_1, r_2) \neq \big(\sqrt{\frac{p}{n}}, \sqrt{\frac{n-p}{n}}\big)$

implies that $M = \mathbb{S}^p(r_1) \times \mathbb{S}^{n-p}(r_2)$, $r_1^2 + r_2^2 = 1$ is not minimal in $\mathbb{S}^{n+1}(1)$. Therefore, Theorem 4.3 implies that $M$ is of 2-type. $\square$

**Corollary 4.14.** [1] Let $M$ be a compact, mass-symmetric surface of $\mathbb{S}^3(r)$ in $\mathbb{R}^4$. Then $M$ is of 2-type if and only if $M$ is the product of two plane circles of different radii, that is, $M = \mathbb{S}^1(a) \times \mathbb{S}^1(b)$, $a \neq b$.

Let $M$ be a 2-type hypersurface of a unit hypersphere $\mathbb{S}^{n+1}(1)$. We have

$$\begin{aligned}
\Delta^{D'} H' &= \sum_{i=1}^{n} (D'_{\nabla_{e_i} e_i} H' - D'_{e_i} D'_{e_i} H') \\
&= \sum_{i=1}^{n} (\alpha' D'_{\nabla_{e_i} e_i} \xi + (\nabla_{e_i} e_i)\alpha'\xi - D'_{e_i}(\alpha' D'_{e_i}\xi + (e_i\alpha')\xi)) \\
&= \sum_{i=1}^{n} ((\nabla_{e_i} e_i)\alpha'\xi - (e_i\alpha')D'_{e_i}\xi - e_i e_i \alpha'\xi) \\
&= \sum_{i=1}^{n} ((\nabla_{e_i} e_i)\alpha' - e_i e_i \alpha')\xi = (\Delta\alpha')\xi. \qquad (4.62)
\end{aligned}$$

Hence from (3.23) we get

$$\begin{aligned}
\langle \Delta H, \xi \rangle &= \langle (\Delta\alpha')\xi + \frac{n}{2}\text{grad}\alpha^2 + 2\text{tr}A_{D'H'} + \|h\|^2 H' - n\alpha^2 x, \xi \rangle, \\
&= \langle (\Delta\alpha')\xi + \|h\|^2 H', \xi \rangle.
\end{aligned}$$

On the other hand, taking the inner product of (4.19) with $\xi$ gives

$$\langle \Delta H, \xi \rangle = \langle bH' + (c-b)x - cx_0, \xi \rangle = \langle bH', \xi \rangle - \langle cx_0, \xi \rangle$$

Combining these, we obtain

$$c\langle x_0, \xi \rangle = (b - \|h\|^2)\alpha' - \Delta\alpha'. \qquad (4.63)$$

**Lemma 4.15.** [12] Let $M$ be a compact 2-type hypersurface of $\mathbb{S}^{n+1}$ in $\mathbb{R}^{n+2}$. Then we have

$$\int_M (\|h\|^2 - b)(\alpha')^2 dV + \int_M |\text{grad}\alpha'|^2 dV + c|x_0|^2 \text{vol}M = 0 \qquad (4.64)$$

**Proof :** Let $M$ be a compact 2-type hypersurface of a unit hypersphere $\mathbb{S}^{n+1}(1)$. Applying Corollary 2.4 to (4.43) we obtain

$$\int_M \alpha' \Delta \alpha' dV = \int_M |\text{grad}\alpha'|^2 dV. \tag{4.65}$$

Using (4.60), we get

$$\int_M \langle x_0, H \rangle dV = \int_M \langle x_0, H' - x \rangle dV = \int_M \langle x_0, H' \rangle dV - |x_0|^2 \text{vol}M. \tag{4.66}$$

Moreover by Corollary 2.4, we have

$$\int_M \langle x_0, H \rangle dV = \int_M \langle x_0, -\frac{1}{n} \Delta x \rangle dV = 0. \tag{4.67}$$

It follows from (4.66) and (4.67) that

$$\int_M \langle x_0, H' \rangle dV = |x_0|^2 \text{vol}M. \tag{4.68}$$

From (4.63), (4.65) and (4.68) we find (4.64). □

We also need the following theorem (Proof can be seen in [10]).

**Theorem 4.16.** [10] If $M$ is a compact 2-type hypersurface of a unit hypersphere $\mathbb{S}^{n+1}(1)$ in $\mathbb{R}^{n+2}$, then we have $\lambda_p < n < \lambda_q$.

**Theorem 4.17.** [12] Let $M$ be a compact hypersurface of a hypersphere $\mathbb{S}^{n+1}$ in $\mathbb{R}^{n+2}$. If $M$ is of 2-type and

$$(\lambda_p + \lambda_q) - \frac{9n+16}{(3n+2)^2} \lambda_p \lambda_q \geq n,$$

then $M$ is mass-symmetric.

**Proof :** We assume that $M$ is a compact 2-type hypersurface of a hypersphere $\mathbb{S}^{n+1}$ in $\mathbb{R}^{n+2}$ and it is not mass-symmetric. Then by Theorem 4.8, $M$ has non-constant mean curvature. From Lemma 4.11 we see that the open set $U = \{u \,\epsilon\, M \,|\, \text{grad}(\alpha')^2 \neq 0 \text{ at } u\}$ is dense in $M$. By Lemma 4.10, grad$\alpha'$ is a principal direction with principal curvature $-\dfrac{3n\alpha'}{2}$ on $U$. Let $E_1, E_2, \ldots, E_n$ be orthonormal principal directions with principal curvatures $\mu_1, \mu_2, \ldots, \mu_n$

respectively, and $E_1$ is assumed to be in the direction of grad$\alpha'$. From the proof of Theorem 4.13 we know that the multiplicity of $\mu_1$ is one on $U$. Therefore we get

$$\|h\|^2 - n = \|A_\xi\|^2 = \sum_{i=1}^{n} \mu_i^2 = (-\frac{3n\alpha'}{2})^2 + \sum_{i=2}^{n} \mu_i^2 = \frac{9n^2(\alpha')^2}{4} + \sum_{i=2}^{n} \mu_i^2.$$

On the other hand we have

$$(n-1)\sum_{i=2}^{n} \mu_i^2 \geq (\sum_{i=2}^{n} \mu_i)^2 = (\mathrm{tr}A_\xi - \mu_1)^2 = (n\alpha' + \frac{3n\alpha'}{2})^2 = \frac{25n^2(\alpha')^2}{4}.$$

Thus we obtain

$$\|h\|^2 - n \geq \frac{9n^2(\alpha')^2}{4} + \frac{25n^2(\alpha')^2}{4(n-1)} = \frac{9n+16}{4(n-1)} n^2(\alpha')^2,$$

from which it follows

$$\begin{aligned} \int_M (\|h\|^2 - b)(\alpha')^2 dV &= \int_M (\|h\|^2 - n + (n-b))(\alpha')^2 dV \\ &\geq \int_M \Big(\frac{9n+16}{4(n-1)} n^2 (\alpha')^2 + n - b\Big)(\alpha')^2 dV \\ &= \frac{9n+16}{4(n-1)} n^2 \int_M (\alpha')^4 dV + (n-b)\int_M (\alpha')^2 dV, \end{aligned}$$

$$\int_M (\|h\|^2 - b)(\alpha')^2 dV \geq \frac{9n+16}{4(n-1)} n^2 \int_M (\alpha')^4 dV + (n-b)\int_M (\alpha')^2 dV. \quad (4.69)$$

From (4.20), (4.60) and $\alpha^2 = (\alpha')^2 + 1$, we get

$$\begin{aligned} c|x_0|^2 \mathrm{vol}M &= \int_M c\langle x_0, x\rangle dV = \int_M (n\alpha^2 + c - b) dV \\ &= n\int_M (\alpha')^2 dV + (n + c - b)\mathrm{vol}M, \end{aligned} \quad (4.70)$$

Expanding the left hand side of $[n(\alpha')^2 + (n+c-b)]^2 \geq 0$ and integrating it on $M$ with use of (4.70), we get

$$\int_M (n^2(\alpha')^4 + 2n(\alpha')^2(n+c-b) + (n+c-b)^2) dV \geq 0,$$

$$\begin{aligned} n^2 \int_M (\alpha')^4 dV &\geq -(n+c-b)\big(2n\int_M (\alpha')^2 dV + (n+c-b)\mathrm{vol}M\big), \\ n^2 \int_M (\alpha')^4 dV &\geq (b-n-c)(2c|x_0|^2 + (b-n-c))\mathrm{vol}M. \end{aligned} \quad (4.71)$$

From (4.69), (4.70) and (4.71), we see that

$$\int_M (\|h\|^2 - b)(\alpha')^2 dV \geq \frac{(3n+2)^2}{4(n-1)}\Big(b - n - \frac{9n+16}{(3n+2)^2} nc\Big)\frac{1}{n}(c|x_0|^2 + b - n - c)\mathrm{vol}M$$

$$+\frac{9n+16}{4(n-1)}(b-n-c)c|x_0|^2 \mathrm{vol}M. \qquad (4.72)$$

Using Theorem 4.16, we see that

$$b - c - n = (\lambda_p + \lambda_q) - \frac{\lambda_p \lambda_q}{n} - n = \frac{1}{n}(n - \lambda_p)(\lambda_q - n) > 0. \qquad (4.73)$$

By the hypothesis, we have

$$n \leq (\lambda_p + \lambda_q) - \frac{9n+16}{(3n+2)^2}\lambda_p\lambda_q = b - \frac{9n+16}{(3n+2)^2} cn,$$

$$0 \leq b - n - \frac{9n+16}{(3n+2)^2} cn.$$

By combining this with (4.72) and (4.73) we may find

$$\int_M (\|h\|^2 - b)(\alpha')^2 dV > 0.$$

But this contradicts with Lemma 4.15, so $M$ has to be mass-symmetric. □

**Theorem 4.18.** [12] Let $M$ be a compact and mass-symmetric hypersurface of a hypersphere $\mathbb{S}^{n+1}(1)$ in $\mathbb{R}^{n+2}$. If $M$ is of 2-type, then $M$ has no umbilical point.

**Proof :** If a point $p$ in $M$ is an umbilical point, for any vector $X$ tangent to $M$ at p, we have $A_\xi X = \mu X$ where $\mu$ is a constant. Since $\alpha' = \mu$ at $p$ we have

$$\|h\|^2 - n = \|A_\xi\|^2 = \sum_{i=1}^{n} \mu_i^2 = n\mu^2 = n(\alpha')^2 \qquad (4.74)$$

at $p$. On the other hand, by Theorem 4.6, we know that

$$n(\alpha')^2 = \lambda_p + \lambda_q - \frac{\lambda_p\lambda_q}{n} - n$$

and

$$\|h\|^2 = \lambda_p + \lambda_q,$$

from which together with (4.74), we obtain

$$\lambda_p + \lambda_q - n = \lambda_p + \lambda_q - \frac{\lambda_p\lambda_q}{n} - n$$

which implies that $\dfrac{\lambda_p \lambda_q}{n} = 0$. This is a contradiction because both $\lambda_p$ and $\lambda_q$ must be nonzero since $M$ is not of null 2-type. $\square$

In Theorem 4.8, it was proved that, there is no compact, mass-symmetric hypersurface of constant mean curvature in $\mathbb{S}^{n+1}$ which is of 3-type. Now we give a more restrictive theorem.

**Theorem 4.19.** [12] There is no compact hypersurface of constant mean curvature in $\mathbb{S}^{n+1}$ which is of 3-type.

**Proof :** Let $M$ be a compact hypersurface of a hypersphere $\mathbb{S}^{n+1}$ which is of 3-type and has constant mean curvature $\alpha'$. Since $\alpha'$ is constant and $M$ is a hypersurface, we have $D'H' = \Delta^{D'}H' = \mathrm{tr}A_{D'H'} = 0$, and $\mathrm{grad}\alpha' = 0$. Then Corollary 3.3 gives

$$\Delta H = \|h\|^2 H' - n\alpha^2 x. \tag{4.75}$$

On the other hand, from Theorem 2.6 and Theorem 2.7, there exist nonzero constants $c_1$, $c_2$ and $c_3$ such that

$$\Delta^2 H = c_1 \Delta H + c_2 H + c_3(x - x_0) \tag{4.76}$$

Substituting (4.75) and $H = H' - x$ in (4.76) we have

$$\begin{aligned} \Delta^2 H &= c_1(\|h\|^2 H' - n\alpha^2 x) + c_2(H' - x) + c_3(x - x_0), \\ &= (c_1\|h\|^2 + c_2)H' + (-c_1 n\alpha^2 - c_2 + c_3)x - c_3 x_0. \end{aligned} \tag{4.77}$$

Taking the inner product of (4.27) and (4.77) with $\xi$ and equalizing them, we have

$$\alpha'(\Delta\|h\|^2 + \|h\|^4 - n\|h\|^2 + n^2\alpha^2) = \alpha'(c_1\|h\|^2 + c_2) - c_3\langle x_0, \xi\rangle. \tag{4.78}$$

Similarly, taking the inner product of (4.27) and (4.77) with $x$ and equalizing them, we obtain

$$n(\alpha')^2\|h\|^2 + n^2\alpha^2 = c_1 n\alpha^2 + c_2 - c_3 + c_3\langle x_0, x\rangle. \tag{4.79}$$

Applying the Laplace operator to both sides of (4.79) gives

$$\begin{aligned} n(\alpha')^2\Delta\|h\|^2 &= c_3\Delta\langle x_0, x\rangle = c_3\langle x_0, \Delta x\rangle = c_3\langle x_0, -nH\rangle, \\ &= -c_3\alpha'\langle x_0, \xi\rangle + c_3\langle x_0, x\rangle. \end{aligned} \tag{4.80}$$

From (4.78) we see that

$$(\alpha')^2\Delta\|h\|^2 = -(\alpha')^2(\|h\|^4 - n\|h\|^2 + n^2\alpha^2) + (\alpha')^2(c_1\|h\|^2 + c_2) - \alpha' c_3\langle x_0, \xi\rangle.$$

Combining this with (4.80), we have

$$(\alpha')^2(\|h\|^4 - c_1\|h\|^2 - c_2 - n\|h\|^2 + n^2\alpha^2) = -c_3\langle x_0, x\rangle. \tag{4.81}$$

On the other hand, from (4.79) we have

$$-n(\alpha')^2\|\|^2 - n^2\alpha^2 + c_1 n\alpha^2 + c_2 - c_3 = -c_3\langle x_0, x\rangle.$$

If we combine this with (4.81) we get

$$(\alpha')^2\|h\|^4 - c_1(\alpha')^2\|h\|^2 + n^2\alpha^4 - c_1 n\alpha^2 - c_2\alpha^2 + c_3 = 0$$

So $(\alpha')^2\|h\|^4 - c_1(\alpha')^2\|h\|^2$ is a constant. Since $M$ is of 3-type, $\alpha'$ is a nonzero constant and $h$ has constant length. Hence the scalar curvature of $M$ is also constant. By Theorem 4.3, $M$ is of 2-type. This is a contradiction, so $\alpha'$ can not be constant. □

# 5. FINITE TYPE ISOPARAMETRIC AND DUPIN HYPERSURFACES OF A HYPERSPHERE

A hypersurface $M$ of $\mathbb{S}^{n+1}$ is called an isoparametric hypersurface if $M$ has constant principal curvatures. Isoparametric hypersurfaces have constant mean curvature and constant scalar curvature. A hypersurface $M$ of $\mathbb{S}^{n+1}$ is called a Dupin hypersurface if the multiplicities of the principal curvatures are constant and each principal curvature is constant along its principal direction.

**Lemma 5.1.** [3] If $M$ is a compact isoparametric hypersurface of $\mathbb{S}^{n+1}(1)$, then we have

(a) M is mass-symmetric in $\mathbb{S}^{n+1}(1)$ or a small hypersphere,

(b) $M$ is either of 1 or 2-type,

(c) the mean curvature, the scalar curvature and the length of the second fundamental form are completely determined by the order of $M$ in $\mathbb{R}^{n+2}$, and

(d) if $M$ is not a small hypersphere of $\mathbb{S}^{n+1}(1)$, then $\lambda_p \leq n$, equality holding when and only when $M$ is of 1-type.

**Proof :** For the proof of statements (a) and (b), let $\rho_1, \ldots, \rho_n$ be the principal curvatures of $M$ in $\mathbb{S}^{n+1}(1)$. As $M$ is an isoparametric hypersurface, the mean curvature $\alpha' = \dfrac{1}{n}\sum_i \rho_i$ is a constant, and hence the mean curvature vector $H'$ is parallel. Since $\|A_\xi\|^2 = \dfrac{1}{n}\sum_i {\rho_i}^2$ and $\|h\|^2 = \|A_\xi\|^2 + n$, from (4.8), we see that the scalar curvature is also constant. As $M$ is a hypersurface, it is an $\mathcal{A}$-submanifold. If we have $\alpha' = 0$, then $M$ is mass-symmetric in $\mathbb{S}^{n+1}(1)$ and it is of 1-type. If $\alpha' \neq 0$, then from (3.23), we can write $\Delta H = bH + cx$ where $b$ and $c$ are constants. From the proof of Theorem 4.1 we see that if $c \neq 0$, then $M$ is mass-symmetric in $\mathbb{S}^{n+1}(1)$ and is of 2-type, and $c = 0$ implies that $M$ is pseudo umbilical and of 1-type. In the latter case, $M$ is minimal in another hypersphere $\bar{\mathbb{S}}^{n+1}$, and hence it is a small hypersphere in $\mathbb{S}^{n+1}(1)$.

For the proof of statement (c), if $\alpha' \neq 0$ and $M$ is not a small hypersphere, then by Theorem 4.6, $\alpha'$ and $\rho$ are completely determined by the order of $M$. On the other hand, if $M$ is of 1-type, we have $\Delta H = \lambda_p H$, hence (3.23) gives $\|h\|^2 = \|A_\xi\|^2 + n = \lambda_p$ and $\|A_\xi\|^2 = n(\alpha')^2$. If $\alpha' = 0$, (4.8) implies $\rho = \dfrac{1}{n(n-1)}(n^2 - \lambda_p^2)$. If $\alpha' \neq 0$ and $M$ is a small hypersphere in $\mathbb{S}^{n+1}(1)$, then it is totally umbilical and hence $\rho_1 = \cdots = \rho_n$. In this case since $\|A_\xi\|^2 = n\rho_1^2$ we have $n\rho_1^2 + n = \|h\|^2 = \lambda_p$. So we obtain $\rho_1 = \sqrt{\dfrac{\lambda_p - n}{n}}$, which implies that $\alpha' = \rho_1 = \sqrt{\dfrac{\lambda_p - n}{n}}$. Using (4.8), we see that the scalar curvature can also be written in terms of $\lambda_p$. Statement (d) follows from statement (a) and Theorem 4.4. $\square$

**Theorem 5.2.** [3] Let $M$ be a compact Dupin hypersurface of $\mathbb{S}^{n+1}$ such that $M$ is not of 1-type and it has at most 3 distinct principal curvatures. Then $M$ is isoparametric if and only if $M$ is of 2-type and it is mass-symmetric in $\mathbb{S}^{n+1}$.

**Proof :** Let $M$ be a compact Dupin hypersurface of $\mathbb{S}^{n+1}$ such that $M$ is not of 1-type and it has at most 3 distinct principal curvatures. Then $M$ has 2 or 3 distinct principal curvatures. Assume that M has 3 distinct principal curvatures $\rho_1$, $\rho_2$ and $\rho_3$ with multiplicities $m_1$, $m_2$ and $m_3$, respectively. Then we have

$$n\alpha' = m_1\rho_1 + m_2\rho_2 + m_3\rho_3, \tag{5.1}$$

$$\|A_\xi\|^2 = m_1{\rho_1}^2 + m_2{\rho_2}^2 + m_3{\rho_3}^2. \tag{5.2}$$

If $M$ is of 2-type and mass-symmetric in $\mathbb{S}^{n+1}$, then Theorem 4.3 implies that both $\alpha'$ and $\|A_\xi\|$ are constant. Let $E_1$ be an eigenvector with eigenvalue $\rho_1$. Then by the definition of Dupin hypersurface, we have $E_1\rho_1 = 0$. Thus (5.1) and (5.2) imply

$$E_1(n\alpha') = m_2(E_1\rho_2) + m_3(E_1\rho_3) = 0, \tag{5.3}$$

and

$$E_1(\|A_\xi\|^2) = 2m_2\rho_2(E_1\rho_2) + 2m_3\rho_3(E_1\rho_3) = 0. \tag{5.4}$$

Since $\rho_2$ and $\rho_3$ are different, (5.3) and (5.4) give $E_1\rho_2 = E_1\rho_3 = 0$. Similarly if $E_2$ and $E_3$ are eigenvectors with eigenvalues $\rho_2$ and $\rho_3$ respectively, we have $E_2\rho_1 = E_2\rho_3 = E_3\rho_1 = E_3\rho_2 = 0$. Because $E_1\rho_1 = E_2\rho_2 = E_3\rho_3 = 0$, we conclude that $\rho_1$, $\rho_2$ and $\rho_3$ are constant. Therefore $M$ is isoparametric. If $M$ has 2 distinct principal curvatures and it is of 2-type, then by Theorem 4.3, it is clear that $M$ is isoparametric. The converse of this follows from Lemma 5.1. $\square$

**Theorem 5.3.** [4] If $M$ is a compact 2-type Dupin hypersurface of $\mathbb{S}^{n+1}$, then $M$ has constant mean curvature. And hence, it is mass-symmetric in $\mathbb{S}^{n+1}$.

**Proof :** If $M$ is a Dupin hypersurface of $\mathbb{S}^{n+1}$, then the multiplicities of the principal curvatures are constant and the principal curvatures are constant along their principal directions. We define $E_1 = \dfrac{\mathrm{grad}(\alpha')^2}{|\mathrm{grad}(\alpha')^2|}$ on $U$, where $U$ is given in Lemma 4.10. Then according to Lemma 4.10, $E_1$ is a principal direction on $U$ with $A_\xi E_1 = -\dfrac{3n\alpha'}{2}E_1$, and since $M$ is a Dupin hypersurface, we have $E_1(\alpha') = 0$. So we get

$$E_1(\alpha') = \frac{\mathrm{grad}(\alpha')^2}{|\mathrm{grad}(\alpha')^2|}(\alpha') = \frac{1}{|\mathrm{grad}(\alpha')^2|}\sum_{i=1}^{n} E_i(\alpha')^2 E_i(\alpha') = 0.$$

Since $\mathrm{grad}(\alpha')^2$ is parallel to $E_1$, from the definition of $\mathrm{grad}(\alpha')^2$, we see that $E_2(\alpha') = E_3(\alpha') = \cdots = E_n(\alpha') = 0$ on $U$. But this implies that $\mathrm{grad}(\alpha')^2 = 0$ on $U$, which is a contradiction . Consequently, the subset $U$ is empty. Thus $M$ has constant mean curvature. And hence $M$ is mass-symmetric in $\mathbb{S}^{n+1}$. $\square$

## 6. 2-TYPE SUBMANIFOLDS OF $\mathbb{R}^m$

A 2-type submanifold of $\mathbb{R}^m$ with parallel mean curvature vector is either spherical or null. Using this result, we give a complete classification of 2-type surfaces with parallel mean curvature vector.

If $M$ is an $n$-dimensional null 2-type submanifold of $\mathbb{R}^m$. Then we can write the position vector $x$ of $M$ in $\mathbb{R}^m$ as,

$$x = x_0 + x_p + x_q, \qquad \Delta x_p = 0, \qquad \Delta x_q = \lambda_q x_q \tag{6.1}$$

where $x_0$ is a constant vector, and $x_p$ and $x_q$ are nonconstant maps from $M$ into $\mathbb{R}^m$.

**Theorem 6.1.** [11] Let $M$ be a 2-type submanifold of $\mathbb{R}^m$. If $M$ has parallel mean curvature vector, then one of the following two cases occurs;

(a) $M$ is spherical,

(b) $M$ is of null 2-type.

In particular, if $M$ is compact then $M$ is spherical and mass-symmetric.

**Proof :** Let $e_{n+1}, \ldots, e_m$ be an orthonormal normal basis of $M$ such that $e_{n+1}$ is parallel to H. If H is parallel, we see that $\Delta^D H = 0$ and by Lemma 3.4, $\mathrm{tr}(\overline{\nabla} A_H) = 0$. Then (3.2) becomes

$$\Delta H = \|A_{\frac{H}{\alpha}}\|^2 H + \mathcal{A}(H). \tag{6.2}$$

If $M$ is of 2-type in $\mathbb{R}^m$, then the position vector $x$ of $M$ in $\mathbb{R}^m$ can be written as

$$x - x_0 = x_p + x_q, \quad \Delta x_p = \lambda_p x_p, \quad \Delta x_q = \lambda_q x_q. \tag{6.3}$$

We have $\Delta x = \Delta(x_p + x_q) = \lambda_p x_p + \lambda_q x_q$ and $\Delta^2 x = \lambda_p^2 x_p + \lambda_q^2 x_q$. So we see that

$$\begin{aligned}(\lambda_p + \lambda_q)\Delta x - \lambda_p \lambda_q (x - x_0) &= \lambda_p^2 x_p + \lambda_q^2 x_q + \lambda_p \lambda_q (x_p + x_q) - \lambda_p \lambda_q (x - x_0) \\ &= \lambda_p^2 x_p + \lambda_q^2 x_q = \Delta^2 x.\end{aligned}$$

Hence,

$$\Delta^2 x = (\lambda_p + \lambda_q)\Delta x - \lambda_p \lambda_q (x - x_0). \tag{6.4}$$

Since $\Delta x = -nH$, we find

$$\Delta H = (\lambda_p + \lambda_q) H + \frac{\lambda_p \lambda_q}{n}(x - x_0). \tag{6.5}$$

Combining this with (6.2), we see that a 2-type submanifold $M$ of $\mathbb{R}^m$ with parallel mean curvature vector satisfies

$$\|A_{\frac{H}{\alpha}}\|^2 H + \mathcal{A}(H) = (\lambda_p + \lambda_q) H + \frac{\lambda_p \lambda_q}{n}(x - x_0). \tag{6.6}$$

The terms on the left hand side of (6.6) are linearly independent and are normal to $M$. We have either $\lambda_p \lambda_q = 0$, which implies that $M$ is of null 2-type, or $x - x_0$ is normal to $M$. If the second case occurs, then for any vector field $Y$ tangent to $M$ we have $Y\langle x - x_0, x - x_0\rangle = 2\langle \overline{\nabla}_Y (x - x_0), x - x_0\rangle = 2\langle Y, x - x_0\rangle = 0$. Hence $\langle x - x_0, x - x_0\rangle$ is a positive constant, and $M$ is contained in a hypersphere $\mathbb{S}^{m-1}$ centered at $x_0$. In particular if $M$ is compact, $M$ can not be null and the term $x_0$ in (6.3) corresponds to the center of mass of $M$ in $\mathbb{R}^m$. So the constancy of $\langle x - x_0, x - x_0\rangle$ implies that $M$ is mass-symmetric in $\mathbb{S}^{m-1}$. $\square$

**Corollary 6.2.** [11] Every 2-type compact hypersurface of $\mathbb{R}^m$ has non-constant mean curvature.

**Proof :** For a hypersurface $M$ of $\mathbb{R}^m$, the constancy of mean curvature is equivalent to the parallelism of the mean curvature vector. A compact 2-type submanifold of $\mathbb{R}^m$ with $DH = 0$ is contained in a hypersphere $\mathbb{S}^{m-1}$. Since $M$ is a hypersurface in $\mathbb{R}^m$, $M$ is an open portion of $\mathbb{S}^{m-1}$, and so $M$ is of 1-type. This is a contradiction. $\square$

**Proposition 6.3.** [10] There is no spherical hypersurface of null 2-type.

**Proof :** Let $M$ be a hypersurface of the unit sphere $\mathbb{S}^{n+1}$ in $\mathbb{R}^{n+2}$. If $M$ is of 2-type, then substituting $\lambda_p + \lambda_q = b$ and $\dfrac{\lambda_p \lambda_q}{n} = c$ in (6.5), we get $\Delta H = bH + c(x - x_0)$. Using this, (4.35) and Corollary 3.3, we can obtain equation (4.39) for the case when $M$ is not necessarily compact. Combining (4.39) and (4.62), we find

$$c(x - x_0) = -n\text{grad}\alpha^2 + \big(\Delta\alpha' + \|h\|^2\alpha' - b\alpha'\big)\xi + (b - n\alpha^2)x. \tag{6.7}$$

If $M$ is of null 2-type, we have $c = 0$. So the coefficient of $x$ in (6.7) vanishes and $\alpha$ is constant. Since $M$ is a hypersurface, the constancy of $\alpha'$ implies that the mean curvature vector is parallel. Then it follows from Theorem 6.1 that $M$ can not be spherical, which contradicts to our assumption. $\square$

**Theorem 6.4.** [11] Let $M$ be a compact 2-type surface in $\mathbb{R}^m$. Then $M$ has parallel mean curvature vector if and only if $M$ is the product of two plane circles with different radii.

**Proof :** If $M$ is a surface in a space form $R^m(k)$ with parallel mean curvature vector, then by Theorem 2.12, $M$ is one of the following:

(i) a minimal surface of $R^m(k)$,

(ii) a minimal surface of a small hypersphere of $R^m(k)$,

(iii) a surface with constant mean curvature $|H|$ in a 3-sphere of $R^m(k)$ (great or small).

Here k=0 and $R^m(k) = \mathbb{R}^m$, so if $M$ is a minimal surface of $\mathbb{R}^m$, then $M$ is of null 1-type. A small hypersphere of $\mathbb{R}^m$ is a usual hypersphere of $\mathbb{R}^m$, so if $M$ is a minimal surface of a hypersphere, then $M$ is of non-null 1-type. Hence $M$ can not be one of (i) or (ii). A great 3-sphere of $\mathbb{R}^m$ is a linear 3-dimensional subspace $\mathbb{R}^3$ of $\mathbb{R}^m$, and a small 3-sphere of $\mathbb{R}^m$ is a hypersphere $\mathbb{S}^3$ of $\mathbb{R}^4$ in $\mathbb{R}^m$. Therefore $M$ is a surface in either $\mathbb{R}^3$ or $\mathbb{S}^3$. If $M$ lies in $\mathbb{R}^3$ and $M$ is compact and is of 2-type with parallel mean curvature, then by Theorem 6.1, $M$ is spherical, and hence it is a 2-sphere of $\mathbb{R}^m$. In this case $M$ is of 1-type, which is a contradiction. If $M$ lies in $\mathbb{S}^3$, then since $M$ is compact and is of 2-type with parallel mean curvature vector, $M$ is mass-symmetric in $\mathbb{S}^3$ by Theorem 4.8. According to Corollary 4.14, $M$ is the product surface of two plane circles with different radii. $\square$

**Lemma 6.5.** [11] If $M$ is a null 2-type submanifold of $\mathbb{R}^m$, then we have $\mathrm{tr}(\overline{\nabla} A_H) = 0$ and $\Delta^D H = (\lambda_p - \|A_{\underset{\alpha}{H}}\|^2)H + \mathcal{A}(H)$

**Proof :** If $M$ is of null 2-type, from (6.1) we find $-nH = \Delta x = \Delta x_q = \lambda_q x_q$ and $\Delta H = -\dfrac{1}{n}\lambda_q \Delta x_q = -\dfrac{1}{n}\lambda_q \Delta x$. So we have

$$\Delta H = \lambda_q H \tag{6.8}$$

Therefore, by applying formula (3.2), we obtain

$$\lambda_q H = \Delta^D H + \|A_{\underset{\alpha}{H}}\|^2 H + \mathcal{A}(H) + \mathrm{tr}(\overline{\nabla} A_H). \tag{6.9}$$

Because $\mathrm{tr}(\overline{\nabla} A_H)$ is tangent to $M$ and all other terms in (6.9) are normal to $M$, formula (6.9) implies the lemma. $\square$

**Theorem 6.6.** [11] Let $M$ be a 2-type submanifold in $\mathbb{R}^m$ with parallel mean curvature vector. Then either $M$ is spherical and non-null or $M$ is a 2-type submanifold with $\|A_{\underset{\alpha}{H}}\|^2 = \lambda_q$ which is a nonzero constant.

**Proof :** This theorem follows from Theorem 6.1 and Lemma 6.5, since the parallelism of $H$ implies $\Delta^D H = 0$. $\square$

The following theorem is a generalization of Theorem 6.4 which gives a complete classification of 2-type surfaces with parallel mean curvature vector.

**Theorem 6.7.** [11] Let $M$ be a surface in $\mathbb{R}^m$ with parallel mean curvature vector. Then $M$ is of 2-type if and only if $M$ is one of the following two surfaces:

(a) an open portion of the product surface of two plane circles with different radii;

(b) an open portion of a circular cylinder.

**Proof :** Let $M$ be a 2-type surface in $\mathbb{R}^m$ with parallel mean curvature vector. Then by Theorem 2.12, $M$ must lie either in a 3-dimensional linear subspace $\mathbb{R}^3$ with constant mean curvature or in a hypersphere $\mathbb{S}^3$ in a 4-dimensional linear subspace $\mathbb{R}^4$ of $\mathbb{R}^m$ with constant mean curvature. According to Theorem 6.1, $M$ is either spherical or null. We consider these two cases separately.

Case (1): *$M$ is null*. In this case Theorem 6.1 implies that $M$ can not be spherical, so $M$ can not lie in a hypersphere $\mathbb{S}^3$ in $\mathbb{R}^4$ and it must lie in a 3-dimensional linear subspace $\mathbb{R}^3$. Let $e_3$ be the unit normal to $M$ in $\mathbb{R}^3$ and $\{e_1, e_2\}$ be the principle directions for $A_3$ with corresponding principle curvatures $\{\mu_1, \mu_2\}$. From Theorem 6.6, we see that $\|A_3\|^2 = \lambda_q$, where $\lambda_q$ is a nonzero constant. Since $M$ is a hypersurface in $\mathbb{R}^3$, we have $h(X, Y) = \langle h(X, Y), e_3\rangle e_3 = h^3(X, Y)e_3$ for any vector fields X and Y tangent to $M$. Then $\|h\|^2 = \|A_3\|^2$ is constant. From the equation of Gauss (2.4), we have

$$\overline{R}(X, Y, Y, X) = R(X, Y, Y, X) + \langle h(X, Y), h(X, Y)\rangle - \langle h(X, X), h(Y, Y)\rangle. \quad (6.10)$$

If X and Y are orthogonal unit vectors, the sectional curvature of $M$ is given by $K(X, Y) = R(X, Y, Y, X)$. Because $M$ is a surface, sectional curvature at each point of $M$ is equal to its Gaussian curvature at that point. Moreover we have $h(e_i, e_j) = \langle h(e_i, e_j), e_3\rangle e_3 = \langle A_3 e_i, e_j\rangle e_3 = \mu_i \langle e_i, e_j\rangle e_3$. Substituting $X = e_1$ and $Y = e_2$ in (6.10) we have

$$\begin{aligned} \overline{K}(e_1, e_2) &= K(e_1, e_2) + \langle h(e_1, e_2), h(e_1, e_2)\rangle - \langle h(e_1, e_1), h(e_2, e_2)\rangle. \\ &= K(e_i, e_j) - \mu_i \mu_j \end{aligned}$$

The sectional curvature of $\mathbb{R}^3$ is zero, so we have $K(e_1, e_2) = \mu_1 \mu_2$. On the other hand,

$$\|h\|^2 = \|A_3\|^2 = \mathrm{tr}(A_3^2) = \sum_{i=1}^{2} \langle A_3^2 e_i, e_i\rangle = \mu_1^2 + \mu_2^2$$

and the constancy of $\|h\|^2$ implies that $\mu_1^2 + \mu_2^2$ is constant. $M$ has constant mean curvature, so we have $2|H| = \mathrm{tr} A_3 = \mu_1 + \mu_2 = \textit{constant}$. Consequently, we see that $\mu_1$ and $\mu_2$ are constants and therefore the Gaussian curvature

$K(e_1, e_2) = \mu_1\mu_2$ of $M$ is a constant. $M$ is nonminimal in $\mathbb{R}^m$, because minimal surfaces of $\mathbb{R}^m$ are null 1-type. By Proposition 2.14, since $M$ is a nonminimal surface of $\mathbb{R}^m$ with parallel mean curvature, it can be either a minimal surface of a small hypersphere of $\mathbb{R}^m$, or an open piece of the product of two plane circles, or an open piece of a circular cylinder. But minimal surfaces of $\mathbb{S}^{m-1}$ are of non-null 1-type, hence $M$ is either an open piece of the product of two plane circles, or an open piece of a circular cylinder. In the first case, the radii of the two plane circles must be different, since $M$ is of 2-type.

Case (2): *$M$ is spherical and non-null.* In this case $M$ can not lie in a 3-dimensional linear subspace $\mathbb{R}^3$ of $\mathbb{R}^m$. Because if it does, $M$ lies in the intersection of a hypersphere $\mathbb{S}^{m-1}$ and $\mathbb{R}^3$, in other words, $M$ lies in a small hypersphere of the totally geodesic submanifold $\mathbb{R}^3$ of $\mathbb{R}^m$. Then, since $M$ is a surface, it becomes a 2-sphere of $\mathbb{R}^m$ and consequently a small hypersphere of $\mathbb{R}^3$, which is of 1-type. Therefore $M$ is non-null and lies in a 3-sphere $\mathbb{S}^3$. Without loss of generality we may assume that $\mathbb{S}^3$ is of radius one and is centered at the origin. Since the mean curvature vector $H$ of $M$ in $\mathbb{R}^m$ and the mean curvature vector $H' = \alpha'\xi$ of $M$ in $\mathbb{S}^3$ are related by $H = H' - x$, the constancy of $\alpha$ implies the constancy of $\alpha'$. Let $\{e_3, e_4\}$ be an orthonormal normal basis of $M$ in $\mathbb{R}^4$, where $e_3 = \dfrac{H}{\alpha}$ and $e_4 = \dfrac{\xi + \alpha' x}{\alpha}$. We have

$$\begin{aligned}
\|A_3\|^2 &= \operatorname{tr}(A_3^2) = \sum_{i=1}^{2}\langle A_3^2 e_i, e_i\rangle = \frac{1}{\alpha^2}\sum_{i=1}^{2}\langle A_H e_i, A_H e_i\rangle \\
&= \frac{1}{\alpha^2}\sum_{i=1}^{2}\Big((\alpha')^2\langle A_\xi e_i, A_\xi e_i\rangle + 2\alpha'\langle A_\xi e_i, e_i\rangle + \langle e_i, e_i\rangle\Big) \\
&= \frac{1}{\alpha^2}((\alpha')^2\|A_\xi\|^2 + 4(\alpha')^2 + 2)
\end{aligned}$$

and

$$\|A_3\|^2 H = \frac{1}{\alpha^2}\big((\alpha')^2\|A_\xi\|^2 + 4(\alpha')^2 + 2\big)(H' - x) \tag{6.11}$$

By using (3.10) we obtain

$$\begin{aligned}
\mathcal{A}(H) = \mathcal{A}'(H') + (\operatorname{tr}A_H A_4)e_4 &= \frac{\alpha'}{\alpha}(\|A_\xi\|^2 - 2(\alpha')^2)e_4 \\
&= \frac{\alpha'}{\alpha}(\|A_\xi\|^2 - 2(\alpha')^2)\frac{\xi + \alpha' x}{\alpha},
\end{aligned}$$

$$\mathcal{A}(H)=\frac{1}{\alpha^2}\|A_\xi\|^2H'-2\frac{(\alpha')^2}{\alpha^2}H'+\frac{(\alpha')^2}{\alpha^2}\|A_\xi\|^2x-2\frac{(\alpha')^2}{\alpha^2}x. \tag{6.12}$$

Taking the sum of (6.11) and (6.12), we get

$$\begin{aligned}\|A_3\|^2H+\mathcal{A}(H) &= \Big(\frac{(\alpha')^2+1}{\alpha^2}\|A_\xi\|^2+2\frac{(\alpha')^2}{\alpha^2}+\frac{2}{\alpha^2}\Big)H' \\ &\quad +\frac{1}{\alpha^2}\big(-2(\alpha')^4-4(\alpha')^2-2\big)x \\ &= \|h\|^2H'-\frac{2}{\alpha^2}\big((\alpha')^4+2(\alpha')^2+1\big)x=\|h\|^2H'-2\alpha^2\end{aligned}$$

Substituting the above result in (6.6) yields

$$\|h\|^2H'-2\langle H,H\rangle x=(\lambda_p+\lambda_q)(H'-x)+\frac{\lambda_p\lambda_q}{2}(x-x_0). \tag{6.13}$$

Because $M$ is non-null, we have $\lambda_p\lambda_q\neq 0$, so taking the inner product of both sides of (6.13) with $x$ we get

$$\begin{aligned}\langle\|h\|^2H'-2\langle H,H\rangle x,x\rangle &= \langle(\lambda_p+\lambda_q)(H'-x)+\frac{\lambda_p\lambda_q}{2}(x-x_0),x\rangle, \\ -2\langle H,H\rangle &= -(\lambda_p+\lambda_q)+\frac{\lambda_p\lambda_q}{2}-\langle x,x_0\rangle.\end{aligned}$$

Since the mean curvature of $M$ is constant, $\langle x,x_0\rangle$ is constant. We assume that $x_0\neq 0$. Then we have $\langle x,x_0\rangle=|x||x_0|\cos\theta=\mathit{constant}$, where $\theta$ is the angle between the vectors $x$ and $x_0$ in $\mathbb{R}^4$. Hence $|x|\cos\theta$ is constant and this implies that $M$ is contained in a hyperplane of $\mathbb{R}^4$ which is normal to $x_0$. $M$ lies in the intersection of $\mathbb{S}^3$ and this hyperplane, so $M$ is a small hypersphere of $\mathbb{S}^3$ which implies that $M$ is of 1-type. But this is a contradiction, so we must have $x_0=0$. Applying in (6.13) yields $\|h\|^2=\lambda_p+\lambda_q$. On the other hand, we have

$$\|h\|^2=\|A_\xi\|^2+2=\mathrm{tr}(A_\xi^2)+2=\sum_{i=1}^{2}\langle A_\xi e_i,A_\xi e_i\rangle+2=\mu_1^2+\mu_2^2+2,$$

As a result $\mu_1^2+\mu_2^2$ is constant, and together with the constancy of $\mu_1+\mu_2=2\alpha'$, it implies that $\mu_1$ and $\mu_2$ are also constants. If we let $\overline{K}(e_1,e_2)$ be the sectional curvature of $\mathbb{S}^3$ and $K(e_1,e_2)$ be the Gaussian curvature of $M$, then from $\overline{K}(e_1,e_2)=K(e_1,e_2)-\mu_1\mu_2$, we see that Gaussian curvature of $M$ is a constant. By Proposition 2.13, if the Gaussian curvature of $M$ is nonzero, then $M$ is a hypersphere of $\mathbb{S}^3$, but this implies that $M$ is of 1-type, which can not be the case. The Gaussian curvature of $M$ is zero and hence the curvature tensor of $M$ vanishes and $M$ is a flat surface. Therefore, $M$ is an open portion of the product of two plane circles with different radii ([2], p. 69, problem 8).

The converse follows from Corollary 4.14. $\square$

# 7. RESULTS AND DISCUSSION

Let $M$ be a compact hypersurface of a hypersphere $\mathbb{S}^{n+1}$ such that $M$ is not a small sphere of $\mathbb{S}^{n+1}$. Then, $M$ is mass-symmetric and of 2-type if and only if $M$ has nonzero constant mean curvature and constant scalar curvature. Also, if $M$ has nonzero constant mean curvature and constant scalar curvature, then the mean and scalar curvatures of $M$ are completely determined by the eigenvalues of the Laplacian of $M$.

Let $M$ be a compact hypersurface of $\mathbb{S}^{n+1}$ with at most two distinct principal curvatures. Then, $M$ is of 2-type if and only if $M$ is a product of two spheres with appropriate radii.

There are no compact hypersurfaces of constant mean curvature in $\mathbb{S}^{n+1}$ which are of 3-type and there are no spherical hypersurfaces of null 2-type.

Some of the results on hypersurfaces of a hypersphere can be generalized to submanifolds of a hypersphere with codimension two in the hypersphere under some conditions. Especially, a classification of submanifolds $M^n$ of the hypersphere $\mathbb{S}^{n+2}$ with at most two distinct principal curvatures can be studied.

## REFERENCES


[1] **Chen, B.Y.**, 1984. Total Mean Curvature and Submanifolds of Finite Type, World Scientific, Singapore and New Jersey.

[2] **Chen, B.Y.**, 1973. Geometry of Submanifolds, Mercel Dekker, New York.

[3] **Chen, B.Y.**, 1986. 2-Type Submanifolds and Their Applications, Chinese Journal of Mathematics, 14, 1-14.

[4] **Chen, B.Y., Barros, M., Garay, O.J.**, 1987. Spherical Finite Type Hypersurfaces, Algebras, Groups and Geometries, 4, 58-72.

[5] **Chen, B.Y., Lue, H.S.**, 1988. Some 2-Type Submanifolds and Applications, Annales de la faculte des sciences de Toulouse $5^e$ serie, 9, 121-131.

[6] **Chen, B.Y.**, 1988. Null 2-Type Surfaces In $E^3$ Are Circular Cylinders, Kodai Math. J., 11, 295-299.

[7] **Chen, B.Y.**, 1989. Null 2-Type Surfaces In Euclidean Spaces, World Sci. Publishing, 1-18.

[8] **Chen, B.Y.**, 1996. A report on submanifolds of finite type, Soochow Math. J., 22, 117-337.

[9] **Dursun, U.**, 2005. Null 2-Type Submanifolds of the Euclidean Space $E^5$ with parallel Normalized Mean Curvature Vector, Kodai Math. J., 28, 191-198.

[10] **Ferrandez, A., Lucas, P.**, 1991. Null Finite-Type Hypersurfaces in Space Forms, Kodai Math., 14, 406-419.

[11] **Li, S.T.**, 1995. Null 2-type Chen Surfaces, Glasgow Math J. 37, 233-242.

[12] **Nagatomo, Y.**, 1991. Finite Type Hypersurfaces of a Sphere, Tokyo J. Math., 14, 85-92.

[12] **Ryan, P.J.**, 1969. Homogeneity and Some Curvature Conditions for Hypersurfaces, Tohoku Math. J., 21, 363-388.

## CURRICULUM VITAE

Selin TAŞKENT was born in Istanbul in 1980. She graduated from Kadıköy Anatolian High School in 1999. In the period 1999-2004, she studied at Istanbul Technical University (ITU) and graduated as a Mathematical Engineer. She began her M.Sc. in Mathematical Engineering Programme of Mathematics Department of ITU in 2004, and has been working as a research assistant in this department since 2005.