İSTANBUL TECHNICAL UNIVERSITY ★ INSTITUTE OF SCIENCE AND TECHNOLOGY

QUANTUM MECHANICAL SYSTEMS WITH NONCOMMUTATIVE PHASE SPACE VARIABLES

M.Sc. Thesis by

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Department : **Physics Engineering**

Programme : **Physics Engineering**

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Last words are the name of a story ,written by Edgar Allan POE, which means so much to me, "Never Bet the Devil Your Head".

June 2008 Mahmut ELBİSTAN

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ABBREVATIONS

WWGM	:	Weyl-Wigner-Groenewold-Moyal
HE	:	Hall Effect
SHE	:	Spin Hall Effect

LIST OF SYMBOLS

h	:	Planck's constant
e	:	Electron charge
P_w	:	Wigner distribution function
$\star$	:	Star product
S	:	Symbol map
W	:	Weyl map
δ_{ij}	:	Kronecker delta

QUANTUM MECHANICAL SYSTEMS WITH NONCOMMUTATIVE PHASE SPACE VARIABLES

SUMMARY

The generalization of quantum mechanics involving noncommutative space-time is originally introduced by Snyder. A few decades later Connes studied Yang Mills theories in noncommutative space. Applications of noncommutative theories can be found in condensed matter physics, for instance quantum Hall effect or Landau problem. It was found that a noncommutative geometry lies under the semiclassical dynamics of electrons in semiconductors. Moreover noncommutative geometry can be seen in the physics of spinning particles. Foldy-Wouthuysen transformation of the Dirac equation changes the position operator, adding a spin-orbit contribution which turns out to be a Berry gauge potential, making the coordinates noncommuting. In classical physics the dynamics of particles are studied with the help of Poisson brackets and the passage to analogue quantum mechanical system is a well known procedure called canonical quantization. All dynamical variables of the system turns to be quantum operators and Poisson brackets to quantum commutators. In his work Wigner studied quantum mechanics as a statistical theory and used classical functions those are derived from quantum mechanical analogues. Weyl, Moyal, Groenewold also studied in this area. WWGM introduces an alternative approach to study quantum mechanical systems. In this type of quantization one uses the symbols of operators which are classical functions and change the ordinary product with star product. The way back to quantum phase space can be taken with associative operator ordering. In this work, with the help of the first order lagrangian and the gauge fields we studied on quantum mechanical systems with the symbols of operators and star product. Definition of canonical momenta leads to some constraints so we deal with a constrained hamiltonian system. We study spin dynamics, our observables turn to be matrices whose elements are classical functions. In order to explain spin dynamics Dayı expand the Moyal bracket up to $\hbar$ order. It is this semiclassical approach and the existence of second class constraints those lead us to use semiclassical Dirac brackets in order to explain the dynamics of observables. In this approach the coordinates become noncommuting. We deformed the space with the parameter θ and in this deformed space we studied Hall effect. Then we studied spin Hall effect with two different type of formulations.

UYUŞUMLU OLMAYAN FAZ UZAYI DEĞİŞKENLERİ İLE KUANTUM MEKANİKSEL SİSTEMLER

ÖZET

İçerisinde uyuşumlu olmayan uzay-zaman koordinatlarını barındırarak kuantum mekaniğinin genişletilmesi ilk olarak Snyder tarafından gerçekleştirilmiştir. Yıllar sonra Connes Yang-Mills teorilerini uyuşumlu olmayan uzayda incelemiştir. Uyuşumsuz teorilerin uygulamaları yoğun madde fiziğinde örneğin kuantum Hall etkisinde veya Landau probleminde görülebilir. Yarıiletkenlerde de elektronların yarıklasik dinamiğinin altında yatan uyuşumlu olmayan bir geometri bulunmuştur. Dahası uyuşumsuz bir geometri parçacıkların spinleri göz önüne alındığında da ortaya çıkmaktadır. Dirac denklemine Foldy-Wouthuysen dönüşümü yapıldığında konum operatörüne aslında Berry ayar potansiyeli olan bir spin-yörünge etkileşim terimi gelmekte ve bu terim koordinatları uyuşumsuz hale çevirmektedir. Klasik fizikte parçacıkların dinamiği Poisson parantezleri yardımı ile ifade edilebilmekte ve benzeri kuantum mekaniksel sisteme geçiş yapmak amacı ile kanonik kuantizasyon yöntemi uygulanmaktadır. Bu yöntemle sistemin tüm dinamik değişkenleri kuantum işlemcilerine ve Poisson parantezleri de kuantum komütatörlerine dönüşmektedir. Wigner çalışmasında kuantum mekaniğini istatistiksel bir kuram olarak ele almış ve kuantum işlemcilerden benzeri klasik fonksiyonlar türetmiştir. Weyl, Moyal, Groenewold de bu alanda çalışmalar yapmışlardır. WWGM kuantum mekaniksel sistemleri incelemek için farklı bir yol önermişlerdir. Bu tip kuantizasyonda herbiri klasik birer fonksiyon olan işlemci sembolleri kullanılmakta ve bu sembollerin çarpımları yıldız çarpımı ile tanımlanmaktadır. Kuantum mekniksel faz uzayına dönüş ise asosiyatif işlemci sıralaması ile mümkündür. Bu çalışmada, birinci mertebe lagrange fonksiyoneli ve içerisinde ayar alanları kullanılarak kuantum mekaniksel sistemler üzerine çalışılmış ve cebir işlemci sembolleri ve yıldız çarpımı ile verilmiştir. Kanonik momentumun tanımı sistemin bağlarına işaret etmiş ve esasında bağlı bir Hamilton sistemi ile uğraşıldığı anlaşılmıştır. Sistemin spin dinamiği incelendiğinden gözlemlenebilirlerin her biri elemanları klasik fonksiyonlar olan matrisler haline gelmişlerdir. Spin dinamiğini ele alabilmek için Dayı Moyal parantezlerini $\hbar$ mertebesine kadar açmıştı. Bu yarıklasik yaklaşıklık ve sistemin sahip olduğu ikincil bağlar sebebi ile spin dinamiğini açıklayabilmek adına yarıklasik Dirac parantezleri tanımlanmıştır. Böylelikle koordinatlar uyuşumsuz hale gelmişlerdir. θ paratmetresi ile uzay deforme edilerek Hall etkisi ve iki farklı formülasyonu ile spin Hall etkisi tartışılmıştır.

1. INTRODUCTION

The idea that the space-time coordinates may not be commuting was originally introduced by Snyder [1]. Later, this idea becomes popular when Connes [2] analyzed Yang-Mills theories on noncommutative space. Applications of noncommutative theories can be found in condensed matter physics, for instance in quantum Hall effect and the Landau problem. Also, a noncommutative geometry underlies the algebraic structure of all known spinning particles. Berard and Mohrbach [3] showed that in the Foldy Wouthuysen representation of the Dirac equation, the position operator acquires a spin-orbit contribution which turns out to be a gauge potential (Berry connection), making the algebra noncommutative.

Usual probabilistic interpretation of quantum mechanics contrasts with the deterministic structure of classical mechanics. However, there are attempts to interpret quantum mechanics as a statistical theory on the classical phase space. Wigner [4] studied on the quantum corrections to the statistical physics. The expectation values of classical observables can be found via classical-like phase space distribution functions. A well known fact is it is not possible to know the position and the momentum of a quantum mechanical particle simultaneously. Unlike classical case, there is no such a simple distribution function in quantum mechanics. Wigner offered a quasiprobability distribution function called Wigner function which can be used to calculate the averages of quantum mechanical observables in a way very similar to classical mechanics. Together with the Weyl correspondence rule and the Moyal bracket the dynamics of quantum mechanics can be given in terms of classical functions. Moyal bracket corresponds to the quantum commutator of quantum mechanics. Essential aspects of quantum mechanics can be given in a classical formulation using the Moyal brackets. This process is called as WWGM method of quantization, makes it probable to calculate quantum mechanical relations with c-number functions and distributions on classical phase space with deformed products and space. WWGM quantization (deformation quantization) corresponds to the canonical quantization with symbol maps and star product. Operators are mapped by symbol maps into c-number functions however their composition is given by star product which is noncommutative but associative.

WWGM method works well for observables possessing a classical limit. However, it is not clear how to deal with spin degrees of freedom. In order to embrace spin, Dayı [5] make a semiclassical expansion of Moyal bracket up to $\hbar$ and observables turn into matrices whose elements are c-number functions. Time evolution of matrix valued observables are then described in terms of this semiclassical bracket. We began with a semiclassical hamiltonian system that has second class constraints. We used semiclassical Dirac brackets in order to describe our system completely. We studied Hall effect and Spin Hall effect with both the deformation of Drude type formulation [6] and deformation of extension of Hall effect that was introduced by Dayi [5]

2. WWGM METHOD OF QUANTIZATION

2.1 Hamilton's Formalism and Canonical Quantization

Equations of motion for a physical system can be obtained with the lagrangian of that system. Lagrangian is defined in terms of generalized coordinates and generalized velocities.

$$L = L(q,\dot{q}) \tag{2.1}$$

The mathematical passage from one set of independent variables to another is called Legendre's transformation. The total differential of the lagrangian is,

$$dL = \sum_i \frac{\partial L}{\partial q_i} dq_i + \sum_i \frac{\partial L}{\partial \dot{q}_i} d\dot{q}_i \tag{2.2}$$

by definition $\partial L/\partial \dot{q}_i$ is generalized momentum p_i and $\partial L/\partial q_i$ is $\dot{p}_i$. So equation (2.2) can be written as,

$$dL = \sum_i \dot{p}_i dq_i + \sum_i p_i d\dot{q}_i \tag{2.3}$$

one can easily obtain from the above equation:

$$d(\sum_i p_i\dot{q}_i - L) = \sum_i \dot{q}_i dp_i - \sum_i \dot{p}_i dq_i \tag{2.4}$$

The term $\sum_i p_i\dot{q}_i - L$ is called the Hamilton's function or hamiltonian of the system. So the total differential of the Hamilton's function can be written as:

$$dH = \sum_i \dot{q}_i dp_i - \sum_i \dot{p}_i dq_i \tag{2.5}$$

Here the independent variables are coordinates and momenta those having equations of motion as,

$$\dot{q}_i = \frac{\partial H}{\partial p_i} \ , \ \dot{p}_i = -\frac{\partial H}{\partial q_i} \tag{2.6}$$

The equations (2.6) are called Hamilton's equations.

Let an arbitrary function $f = f(q,p,t)$, the total time derivative of this function is;

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \sum_i \left(\frac{\partial f}{\partial q_i}\dot{q}_i + \frac{\partial f}{\partial p_i}\dot{p}_i\right) \tag{2.7}$$

Using (2.6) we can simplify the above expression;

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \{f,H\} \tag{2.8}$$

$\{,\}$ is called the Poisson bracket and defined as

$$\{f,g\} = \sum_i \left(\frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i} \right) \tag{2.9}$$

If arbitrary function f is not depend explicitly on time, the total derivative finally turns to be;

$$\frac{df}{dt} = \{f,H\} \tag{2.10}$$

so the time derivatives of canonical variables are defined as follows:

$$\dot{p}_i = \{p_i,H\} = -\frac{\partial H}{\partial q_i}\ ,\ \dot{q}_i = \{q_i,H\} = \frac{\partial H}{\partial p_i} \tag{2.11}$$

and the Poisson algebra is the Poisson brackets of coordinates and momenta:

$$\{q_i, q_j\} = 0 \tag{2.12}$$

$$\{q_i, p_j\} = \delta_{ij} \tag{2.13}$$

$$\{p_i, p_j\} = 0 \tag{2.14}$$

Various quantum mechanical relations can be obtained from the corresponding classical ones just by replacing Poisson brackets by commutators and classical canonical variables by quantum operators. So in the case of classical functions $f(q,p)$ and $g(q,p)$ one can quantize the system defining analogue quantum operators $\hat{f}(\hat{q},\hat{p})$ and $\hat{g}(\hat{q},\hat{p})$ and following the rule,

$$\{f(q,p), g(q,p)\} \rightarrow \frac{-i}{\hbar}[\hat{f}(\hat{q},\hat{p}), \hat{g}(\hat{q},\hat{p})] \tag{2.15}$$

$[,]$ is quantum commutator. The process described in (2.15) is called as canonical quantization. Poisson algebra defined in (2.12), (2.13) and (2.14) turns to be Heisenberg algebra which is expressed in terms of quantum mechanical operators:

$$[\hat{q}_i, \hat{q}_j] = 0 \tag{2.16}$$

$$[\hat{q}_i, \hat{p}_j] = ih\delta_{ij} \tag{2.17}$$

$$[\hat{p}_i, \hat{p}_j] = 0 \tag{2.18}$$

2.2 Quantum Mechanics in Terms of Classical Phase Space Elements

Heisenberg's uncertainty principle lies in the heart of quantum mechanics. A quantum mechanical particle does not have a well defined position q and momentum p; that makes the phase space of quantum mechanics problematic. One can not define a true phase space distribution function for a quantum mechanical particle. However, there are quasiprobability distribution functions and with the help of these distribution functions quantum mechanical averages can be expressed in a similar way of classical averages.

Consider the average of an arbitrary observable $A(q,p)$ of a classical particle in one dimension which has q and p as its coordinate and conjugate momentum respectively, in its phase space.

$$\langle A(q,p)\rangle_{cl} = \int_{-\infty}^{+\infty} dq \int_{-\infty}^{+\infty} dp A(q,p) P_{cl}(q,p) \tag{2.19}$$

In equation (2.19) P_{cl} is the distribution function of our classical particle over the classical phase space. On the other hand, the expectation value of an arbitrary observable of a quantum mechanical particle can be written in terms of its density matrix $\hat{\rho}$,

$$\langle \hat{A}(\hat{q},\hat{p})\rangle_{qm} = Tr(\hat{A}\hat{\rho}) \tag{2.20}$$

here Tr is the trace operator.

The use of the quasiprobability distribution function P_Q gives rise to the expression of the quantum mechanical averages in terms of classical functions:

$$\langle \hat{A}(\hat{q},\hat{p})\rangle_{qm} = \int_{-\infty}^{+\infty} dq \int_{-\infty}^{+\infty} dp A(q,p) P_Q(q,p) \tag{2.21}$$

In (2.21) classical function $A(q,p)$ can be derived from the operator $\hat{A}(\hat{q},\hat{p})$ by a well defined correspondence rule. With the quasiprobability distribution P_Q and this correspondence rule one can get the quantum mechanical results in a form which resemble classical ones.

First of these quasiprobability functions is introduced by Wigner [4] in order to study quantum mechanical corrections to classical statistical mechanics and it is known as Wigner distribution. In this case quasiprobability distribution function P_Q turns into P_w and then the correspondence rule between the function $A(q,p)$ and the operator $\hat{A}(\hat{q},\hat{p})$ is proposed by Weyl [7]. Wigner's distribution function gives the same expectation value for every function of coordinates q and momenta p or the expectation value of their products as does the corresponding operators those proposed by Weyl.

2.2.1 Wigner Distribution

In 1932 Wigner proposed the distribution function,

$$P_w(q,p) = \frac{1}{\pi\hbar}\int_{-\infty}^{+\infty} dy \langle q-y|\hat{\rho}|q+y\rangle e^{\frac{2ipy}{\hbar}} \tag{2.22}$$

for a quantum mechanical system which is in mixed state and represented by a density matrix $\hat{\rho}$. Here, we must mention that this expression is constructed over two dimensional phase space that has independent variables as q and p. Extension to n- dimensional case can be done via considering the scalar product of $\vec{y}$ and $\vec{p}$ and calculating the integral with respect to $d\vec{y}$ and replacing $\pi\hbar$ with $(\pi\hbar)^n$. It must be mentioned that this particular choice of distribution function is not unique.

Consider two quasiprobability distributions P_ψ and P_ϕ corresponding to the states $\psi(q)$ and $\phi(q)$ respectively. These distributions have the property below:

$$2\pi\hbar \int dq \int dp P_\psi(q,p) P_\phi(q,p) = \left| \int dq \psi^*(q)\phi(q) \right|^2 \tag{2.23}$$

If $\psi(q)$ and $\phi(q)$ are equal then,

$$\int dq \int dp [P_\psi(q,p)]^2 = \frac{1}{2\pi\hbar} \tag{2.24}$$

if one chooses states $\psi(q)$ and $\phi(q)$ such that they are orthogonal, he or she comes to a solution,

$$\int dq \int dp P_\psi(q,p) P_\phi(q,p) = 0 \tag{2.25}$$

that means (2.22) is not a true probability distribution function as it can not be positive everywhere. Such a distribution also satisfies the properties:

$$\int dp P(q,p) = |\psi(q)|^2 = \langle q|\hat{\rho}|q\rangle \tag{2.26}$$

$$\int dq P(q,p) = |\psi(p)|^2 = \langle p|\hat{\rho}|p\rangle \tag{2.27}$$

$$\int dp \int dq P(q,p) = Tr(\hat{\rho}) = 1 \tag{2.28}$$

The classical function $A(q,p)$ corresponding to the quantum mechanical operator $\hat{A}(\hat{q},\hat{p})$ is defined as,

$$A(q,p) = \int_{-\infty}^{+\infty} dz e^{\frac{ipz}{\hbar}} \langle q - \frac{1}{2}z|\hat{A}|q + \frac{1}{2}z\rangle \tag{2.29}$$

so that,

$$\int_{-\infty}^{+\infty} dq \int_{-\infty}^{+\infty} dp A(q,p) = 2\pi\hbar Tr(\hat{A}) \tag{2.30}$$

The trace operation of the product of two operators namely $\hat{A}(\hat{q},\hat{p})$ and $\hat{B}(\hat{q},\hat{p})$ is,

$$\int_{-\infty}^{+\infty} dq \int_{-\infty}^{+\infty} dp A(q,p)B(q,p) = 2\pi\hbar Tr(\hat{A}\hat{B}) \tag{2.31}$$

and using the above equation one can choose $\hat{B} = \hat{\rho}$ and reaches the expectation value of a quantum mechanical observable in terms of its function correspondence:

$$\int_{-\infty}^{+\infty} dq \int_{-\infty}^{+\infty} dp A(q,p) P_w(q,p) = Tr(\hat{\rho}\hat{A}) = \langle\hat{A}\rangle_{qm} \tag{2.32}$$

2.2.2 Operator Ordering

Equation (2.29) says beginning with a quantum mechanical operator how one can obtain a classical correspondence of that operator. If one has a classical function of q and p, he or she can obtain the quantum mechanical operator via associated operator ordering.

Consider a quantum mechanical operator $\hat{A}(\hat{q},\hat{p})$ and related classical function $A(q,p)$ and the state ket of the system $|\psi\rangle$; then,

$$\langle\psi|\hat{A}|\psi\rangle = \int_{-\infty}^{+\infty} dq \int_{-\infty}^{+\infty} dp P_w(q,p) A(q,p) \tag{2.33}$$

In order to prove (2.33) one needs to have the Fourier expansion of the classical function $A(q,p)$ and quantum mechanical operator $\hat{A}(\hat{q},\hat{p})$;

$$A(q,p) = \int_{-\infty}^{+\infty} d\sigma \int_{-\infty}^{+\infty} d\tau \alpha(\sigma,\tau) e^{i(\sigma q + \tau p)} \tag{2.34}$$

$$\hat{A}(\hat{q},\hat{p}) = \int_{-\infty}^{+\infty} d\sigma \int_{-\infty}^{+\infty} d\tau \alpha(\sigma,\tau) e^{i(\hat{\sigma} q + \hat{\tau} p)} \tag{2.35}$$

by replacing (2.34) and (2.35) into (2.33), the validity of (2.33) can be seen as follows:

$$\langle\psi|\exp\{i(\sigma\hat{q} + \tau\hat{p})\}|\psi\rangle = \int dq \int dp P_w(q,p) \exp\{i(\sigma q + \tau p)\} \tag{2.36}$$

After the integration over p, right hand side becomes,

$$\int dq\psi^*(q-\frac{1}{2}\tau)\psi(q+\frac{1}{2}\tau)e^{i\sigma q} \tag{2.37}$$

in order to evaluate the left hand side Baker-Hausdorff formula is used,

$$e^{\hat{A}+\hat{B}} = e^{\hat{A}} + e^{\hat{B}} + e^{-[\hat{A},\hat{B}]/2} \tag{2.38}$$

and that leads to,

$$e^{i(\sigma\hat{q}+\tau\hat{p})} = e^{i\sigma\hat{q}}e^{i\tau\hat{p}}e^{i\sigma\tau/2} \tag{2.39}$$

then using the fact,

$$e^{i\tau\hat{p}}|\psi(x)\rangle = |\psi(x+\tau)\rangle \tag{2.40}$$

one can obtain the relation;

$$e^{i\sigma\tau/2}\langle e^{-i\sigma x}\psi(x)|\psi(x+\tau)\rangle = \int dxe^{i(\sigma x+\sigma\tau/2)}\psi^*(x)\psi(x+\tau) \tag{2.41}$$

and with a change of variable $x = q - 1/2\tau$ the proof of (2.33) is complete.
In summary a classical function

$$A(q,p) = \int_{-\infty}^{+\infty} d\sigma \int_{-\infty}^{+\infty} d\tau\alpha(\sigma,\tau)e^{i(\sigma q+\tau p)} \tag{2.42}$$

corresponds to a quantum mechanical operator

$$\hat{A}(\hat{q},\hat{p}) = \int_{-\infty}^{+\infty} d\sigma \int_{-\infty}^{+\infty} d\tau\alpha(\sigma,\tau)e^{i(\hat{\sigma}q+\hat{\tau}p)} \tag{2.43}$$

and the relation between them is given by (2.29).
Passage from classical phase space to quantum mechanics requires Weyl correspondence. According to Bayen and Flato [8] this type of passage is a deformation of classical Poisson manifold and there is only one formal function of the Poisson bracket (up to a constant factor and a linear change of variable) that generates a formal deformation of the associative algebra by the usual product: it is the exponential function. Let A and B are classical functions that are derived from the operators $\hat{A}$ and $\hat{B}$ respectively. The algebra of the deformed manifold is defined in terms of star product ($\star$) and Moyal bracket,

$$[A,B]_\star = Ae^{(i\hbar/2)\Delta}B \tag{2.44}$$

where $\hbar$ is the deformation parameter.
According to [9], the expression of the Weyl correspondence of the operator $\hat{F} =$

$\hat{A}\hat{B}$ in terms of Weyl correspondences of $\hat{A}$ and $\hat{B}$ those are A and B respectively, is;

$$\hat{A}\hat{B} = \hat{F} \rightarrow F(q,p) = A(q,p)e^{(\hbar\lambda/2i)}B(q,p) \tag{2.45}$$

where

$$\lambda = \frac{\overrightarrow{\partial}}{\partial p}\frac{\overleftarrow{\partial}}{\partial q} - \frac{\overrightarrow{\partial}}{\partial q}\frac{\overleftarrow{\partial}}{\partial q} \tag{2.46}$$

So let the classical functions to be q and p and the quantum operators correspond to them $\hat{q}$ and $\hat{p}$ respectively. Taking the arbitrary powers of these classical functions operator ordering W maps them to the following quantum mechanical operators:

$$q^m p^n \rightarrow W(q^m p^n) = \frac{1}{2^n}\sum_{k=0}^{n}\left(\begin{array}{c} n \\ k \end{array}\right)\hat{p}^{n-k}\hat{q}^m\hat{p}^k \tag{2.47}$$

By following this rule one sums over all the permutations of q and p having an hermitean operator. Above rule can be restated as,

$$W(q^m p^n) = \left[\exp\left[-\frac{1}{2}i\hbar\left(\frac{\partial^2}{\partial p\partial q}\right)\right]q^m p^n\right]_{q\rightarrow\hat{q},p\rightarrow\hat{p}} \tag{2.48}$$

Taking care of all permutations of canonical variables this way is also called as symmetric ordering.

2.3 WWGM Method of Quantization and Star Product

Dynamics of classical observables is described in terms of classical phase space elements and with the help of Poisson brackets (2.9). When one wants to observe the quantum mechanical analogue of the system, he or she may recall the canonical quantization method (2.15) and uses quantum operators and commutators. However there is an alternative way to study the quantum dynamics of the system without using the quantum mechanical phase space. WWGM proposes us to use the symbols of operators those are the classical correspondences proposed by Weyl instead of themselves. Product between the symbols of operators is called the star product. For a quantum mechanical operator $\hat{f}(\hat{x},\hat{p})$ symbol map effects and carries this operator to the set of c-number functions as follows:

$$S(\hat{f}(\hat{x},\hat{p})) = f(x,p) \tag{2.49}$$

Here $f(x,p)$ is the classical correspondence of the operator $\hat{f}(\hat{x},\hat{p})$. The quantum mechanical product of observables $\hat{f}$ and $\hat{g}$ is,

$$\hat{f}(\hat{x},\hat{p})\hat{g}(\hat{x},\hat{p}) = \hat{h}(\hat{x},\hat{p}) \tag{2.50}$$

With the usage of symbol map the operator product is defined in terms of classical phase space elements and star product. Star product satisfies the below property.

$$S\left(\hat{f}(\hat{x},\hat{p})\hat{g}(\hat{x},\hat{p})\right) = S\left(\hat{h}(\hat{x},\hat{p})\right) = S\left(\hat{f}(\hat{x},\hat{p})\right) \star S\left(\hat{g}(\hat{x},\hat{p})\right) \tag{2.51}$$

Star product is associative and defined in terms of coordinates x_μ and momenta p_μ as

$$\star = \exp\left[\frac{i\hbar}{2}\left(\frac{\overleftarrow{\partial}}{\partial x^\mu}\frac{\overrightarrow{\partial}}{\partial p_\mu} - \frac{\overleftarrow{\partial}}{\partial p^\mu}\frac{\overrightarrow{\partial}}{\partial x_\mu}\right)\right] \tag{2.52}$$

Above equation is Einstein's convention adopted and this convention will be used in the remaining.

In WWGM formalism Moyal bracket corresponds to the quantum commutator:

$$[f(x,p), g(x,p)]_\star = f(x,p) \star g(x,p) - g(x,p) \star f(x,p) \tag{2.53}$$

The Moyal bracket of coordinates and momenta is as follows:

$$[x^\mu, p_\nu]_\star = i\hbar\delta^\mu_\nu \tag{2.54}$$

Classical limit of the Moyal bracket is the Poisson bracket:

$$\lim_{\hbar\to 0}\frac{-i}{\hbar}[f(x,p), g(x,p)]_\star = \{f(x,p), g(x,p)\} \equiv \frac{\partial f}{\partial x^\mu}\frac{\partial g}{\partial p_\mu} - \frac{\partial f}{\partial p_\mu}\frac{\partial g}{\partial x^\mu} \tag{2.55}$$

3. CONSTRAINED HAMILTONIAN SYSTEMS

Constrained hamiltonian systems are studied explicitly in [10]. Considered lagrangian functional may be singular that means there is no unique solution for the velocities in terms of canonical coordinates and momenta such that $\dot{q}_i = \dot{q}_i(q,p)$. Necessary and sufficient condition of an arbitrary lagrangian to be singular is;

$$\det\left(\frac{\partial^2 L}{\partial \dot{q}_i \partial \dot{q}_j}\right) = 0 \tag{3.1}$$

In a singular lagrangian system there exist primary constraints satisfying the condition,

$$\varphi_m(q,p) \approx 0 \ , \ m = 1,...M \tag{3.2}$$

The symbol "≈ 0" is called "weakly zero" and means primary constraints $\varphi_m(q,p)$ may have nonvanishing canonical Poisson brackets with some canonical variables. In order to obtain the most general equations of motion one must replace the canonical Hamiltonian by the effective one,

$$\tilde{H} = H_e = H_c + u_m \varphi_m \approx H_c \tag{3.3}$$

being $u_m = u_m(q,p)$. New equations of motion are generated and they describe the system truly:

$$\dot{q}_i = \{q_i, \tilde{H}\} \approx \frac{\partial H_c}{\partial p^i} + u_m \frac{\partial \varphi_m}{\partial p^i} \tag{3.4}$$

$$\dot{p}^i = \{p^i, \tilde{H}\} \approx -\frac{\partial H_c}{\partial q_i} - u_m \frac{\partial \varphi_m}{\partial q_i} \tag{3.5}$$

In order to have consistent systems it is required that the time derivatives of the primary constraints or linear combinations of them to be zero.

$$\dot{\varphi}_n = \{\varphi_n, \tilde{H}\} \approx \{\varphi_n, H_c\} + u_m \{\varphi_n, \varphi_m\} \approx 0 \tag{3.6}$$

If (3.6) is not true two possibilities occur. First one, equation has no new information but imposes conditions on the form of $u_m(q,p)$. Secondly, it may give a new relation between q's and p's, independent of $u_m(q,p)$. These are so-called secondary constraints. Together with the M primary constraints, these K constraints form an complete set of T constraints.

$$\varphi_a(q,p) \approx 0 \ , \ a = 1,...,K+M = T \tag{3.7}$$

So, effective Hamiltonian $\tilde{H}$ is a function of q's and p's with all $u_m(q,p)$s:

$$\tilde{H} = \tilde{H}(q,p) \tag{3.8}$$

A defined function $R(q,p)$ is a first class quantity if it validates the below equation,

$$\{R, \varphi_a\} \approx 0 \ , \ a = 1,....,T \tag{3.9}$$

otherwise it is second class;

$$\{R, \varphi_a\} \not\approx 0 \tag{3.10}$$

for at least one φ_a.

All constraints can now be separated as first or second class. The set of first class constraints is,

$$\psi_i(q,p) \approx 0 \ , \ i = 1,...I \tag{3.11}$$

and remaining $T - I = N$ constraints form a set:

$$\varphi_\alpha(q,p) \approx 0 \ , \ \alpha = 1,...N \tag{3.12}$$

Dirac has shown that second class constraints form an $N \times N$ nonsingular, antisymmetric matrix defined via Poisson brackets:

$$C_{\alpha\beta} = \{\varphi_\alpha, \varphi_\beta\} \tag{3.13}$$

The determinant of an odd dimensional antisymmetric matrix vanishes, so the number of second class constraints must be even. One can redefine an arbitrary dynamical observable A as A' which has vanishing Poisson brackets with all second class constraints,

$$A' = A - \{A, \varphi_\alpha\} C^{-1}_{\alpha\beta} \varphi_\beta \tag{3.14}$$

and observe;

$$\{A', \varphi_\gamma\} \approx \{A, \varphi_\gamma\} - \{A, \varphi_\alpha\} C^{-1}_{\alpha\beta} C_{\beta\gamma} \tag{3.15}$$

$$= \{A, \varphi_\gamma\} - \{A, \varphi_\gamma\} = 0 \tag{3.16}$$

So, the Poisson brackets of two dynamical variables A, B must be replaced by their primed values A' and B':

$$\{A,B\} \to \{A',B'\} \tag{3.17}$$

Although $A' \approx A$, $B' \approx B$, their Poisson brackets $\{A',B'\}$ is not weakly equal to $\{A,B\}$.

One can define Dirac bracket as:

$$\{A,B\}^* = \{A,B\} - \{A,\varphi_\alpha\}C^{-1}_{\alpha\beta}\{\varphi_\beta,B\} \tag{3.18}$$

It can be seen that,

$$\{A,B\}^* \approx \{A',B'\} \approx \{A',B\} \approx \{A,B'\} \tag{3.19}$$

Dirac brackets are used instead of Poisson brackets in order to set all the second class constraints strongly to the zero because Dirac bracket of any dynamical observable with a second class object vanishes:

$$\{A,\varphi_\gamma\}^* \approx \{A,\varphi_\gamma\} - \{A,\varphi_\alpha\}C^{-1}_{\alpha\beta}C_{\beta\gamma} = 0 \tag{3.20}$$

Jacobi identity is,

$$\{A,\{B,C\}^*\}^* + \{B,\{C,A\}^*\}^* + \{C,\{A,B\}^*\}^* \approx 0 \tag{3.21}$$

weakly satisfied by the Dirac bracket.

Effective Hamiltonian (3.3) can be rewritten as a first class one,

$$\tilde{H} = H' = H_c - \{H_c,\varphi_\alpha\}C^{-1}_{\alpha\beta}\varphi_\beta \tag{3.22}$$

redefining u_β's as:

$$u_\beta(q,p) = -\{H_c,\varphi_\alpha\}C^{-1}_{\alpha\beta} \tag{3.23}$$

$\tilde{H}$ is the physical first class replacement for the canonical hamiltonian H_c, which could have been second class. One can extract all the second class constraints from the physical system with using Dirac brackets instead of Poisson brackets. After the extraction of all kinds of constraints the quantization procedure can be succeeded.

4. A SEMICLASSICAL APPROACH

4.1 Semiclassical Brackets

We will briefly mention the semiclassical approach that was originally introduced in [5]. WWGM method works well for observables possessing a classical limit. It is known that spin is an intrinsic quantum mechanical quantity and has no classical counterpart.

Electron which is interacting with the electromagnetic field is described by the Dirac hamiltonian,

$$\alpha.(\mathbf{p}-e\mathbf{A})+\beta m+e\phi \tag{4.1}$$

where $\mathbf{A}$ and ϕ are called as vector potential and scalar potential respectively. Nonrelativistic limit of Dirac hamiltonian can be obtained via Foldy-Wouthuysen transformation of that hamiltonian as,

$$\begin{aligned} H = \beta\left(m+\frac{(\mathbf{p}-e\mathbf{A})^2}{2m}-\frac{\mathbf{p}^4}{8m^3}\right)+e\phi-\frac{e}{2m}\beta\sigma.\mathbf{p} \\ -\frac{ie}{8m^2}\sigma.(\mathbf{E})-\frac{e}{4m^2}\sigma.(\mathbf{E}\times\mathbf{p})-\frac{e}{8m^2}\nabla\cdot\mathbf{E} \end{aligned} \tag{4.2}$$

acquiring a new position operator naturally,

$$\mathbf{r} = i\partial_{\mathbf{p}}+\frac{\mathbf{p}\wedge\sigma}{2E_p(E_p+m)} \tag{4.3}$$

with $E_p = (p^2+m^2)^{1/2}$ which is noncommutative. The term $\frac{e}{4m^2}\sigma.(\mathbf{E}\times\mathbf{p})$ turns out to be spin orbit interaction term that is,

$$H_{SL} = \frac{e}{4m^2}\frac{1}{r}\frac{\partial V}{\partial r}\sigma.\mathbf{L} \tag{4.4}$$

σ is Pauli spin matrix.

When non-relativistic limit of Dirac hamiltonian or higher spin formalisms are considered still a symbol map can be defined similarly but the observables are no more classical functions; yet they are defined as matrices whose elements are classical functions of coordinates and momenta.

Moyal bracket of two such matrices, namely $M_{ab}(x,p)$ and $N_{ab}(x,p)$, is defined as:

$$([M(x,p),N(x,p)]_\star)_{ab} = M_{ac}(x,p)\star N_{cb}(x,p) - N_{ac}(x,p)\star M_{cb}(x,p) \tag{4.5}$$

In order to incorporate spin degrees of freedom into classical mechanics Dayı extend the Moyal bracket semiclassically up to the two lowest order of $\hbar$ in [5]. So although we deal with the classical phase space elements, we get terms those depend on $\hbar$.

The semiclassical expansion of the Moyal bracket is as follows:

$$\{M(x,p),N(x,p)\}_C \equiv \frac{-i}{\hbar}[M,N] + \frac{1}{2}\{M(x,p),N(x,p)\} - \frac{1}{2}\{N(x,p),M(x,p)\} \tag{4.6}$$

In (4.6) the first term is the commutator of matrices and it is not singular in the limit $\hbar \to 0$ because observables $M(x,p)$, $N(x,p)$ may depend on $\hbar$; second and third ones are the Poisson brackets of matrices:

$$\{M(x,p),N(x,p)\} \equiv \frac{\partial M}{\partial x^\mu}\frac{\partial N}{\partial p_\mu} - \frac{\partial M}{\partial p_\mu}\frac{\partial N}{\partial x^\mu} \tag{4.7}$$

Analogue to the classical case with Poisson brackets, in this semiclassical expansion in order to have the dynamical equations of motion one can use the semiclassical bracket (4.6). Letting the symbol of Hamiltonian is $H(x,p)$, time derivative of the matrix valued semiclassical observable $M(x,p)$ is defined as,

$$\dot{M}(x,p) = \{M(x,p),H(x,p)\}_C \tag{4.8}$$

4.2 First Order Lagrangian

Semiclassical hamiltonian dynamics are given by the usual hamiltonian methods but replacing Poisson brackets with the semiclassical brackets (4.6).

Consider the first order $N \times N$ matrix lagrangian:

$$L = \dot{r}^{\alpha}\left(\frac{1}{2}Ip_{\alpha} + \rho\mathscr{A}_{\alpha}(r,p)\right) - \dot{p}^{\alpha}\left(\frac{1}{2}Ir_{\alpha} - \xi\mathscr{B}_{\alpha}(r,p)\right) - H_0(r,p) \tag{4.9}$$

$\alpha = 1,2,...,n$ and ρ, ξ are coupling constants and they are attached to the $N \times N$ gauge fields $\mathscr{A}$, $\mathscr{B}$ respectively. I is the unit matrix. Canonical momenta is defined as;

$$\Pi_r^{\alpha} = \frac{\partial L}{\partial \dot{r}_{\alpha}}\ ,\ \Pi_p^{\alpha} = \frac{\partial L}{\partial \dot{p}_{\alpha}} \tag{4.10}$$

So all the variables in our system are identified as canonical coordinates r^{α}, p^{α} and canonical momenta Π_r^{α}, Π_p^{α}. However, this relations lead to two primary constraints:

$$\psi^{1\alpha} \equiv (\Pi_r^{\alpha} - \frac{1}{2}p^{\alpha})I - \rho\mathscr{A}^{\alpha}, \tag{4.11}$$

$$\psi^{2\alpha} \equiv (\Pi_p^{\alpha} + \frac{1}{2}r^{\alpha})I - \xi\mathscr{B}^{\alpha} \tag{4.12}$$

These are the primary constraints those are to be seen in the extended hamiltonian:

$$H_e = H_0 + \lambda_z^{\alpha}\psi_{\alpha}^{z} \tag{4.13}$$

Here $z = 1,2$ and λs are called as Lagrange multipliers. Semiclassical brackets between the constraints are as follows:

$$\{\psi_{\alpha}^{1}\ ,\psi_{\beta}^{1}\}_C = \rho\mathscr{F}_{\alpha\beta} \tag{4.14}$$

$$\{\psi_{\alpha}^{1}\ ,\psi_{\beta}^{2}\}_C = -g_{\alpha\beta} + M_{\alpha\beta} \tag{4.15}$$

$$\{\psi_{\alpha}^{2}\ ,\psi_{\beta}^{2}\}_C = \xi\mathscr{G}_{\alpha\beta} \tag{4.16}$$

Expressions (4.14), (4.15) and (4.16) are not completely determined unless the fields strengths are defined:

$$\mathscr{F}_{\alpha\beta} = \frac{\partial\mathscr{A}_{\beta}}{\partial r^{\alpha}} - \frac{\partial\mathscr{A}_{\alpha}}{\partial r^{\beta}} - \frac{i\rho}{\hbar}[\mathscr{A}_{\alpha},\mathscr{A}_{\beta}], \tag{4.17}$$

$$M_{\alpha\beta} = \xi\frac{\partial\mathscr{B}_{\beta}}{\partial r^{\alpha}} - \rho\frac{\partial\mathscr{A}_{\alpha}}{\partial p^{\beta}} - \frac{i\xi\rho}{\hbar}[\mathscr{A}_{\alpha},\mathscr{B}_{\beta}] \tag{4.18}$$

$$\mathscr{G}_{\alpha\beta} = \frac{\partial \mathscr{B}_\beta}{\partial p^\alpha} - \frac{\partial \mathscr{B}_\alpha}{\partial p^\beta} - \frac{i\xi}{\hbar}[\mathscr{B}_\alpha, \mathscr{B}_\beta], \tag{4.19}$$

Since the semiclassical brackets of the constraints do not vanish, they are all so-called second class constraints. These constraints form the matrix $C_{\alpha\beta}$ mentioned in (3.13).

$$C^{zz'}_{\alpha\beta} = \{\psi^z_\alpha, \psi^{z'}_\beta\}_C \tag{4.20}$$

$N \times N$ matrix $C_{\alpha\beta}$ and its inverse satisfies the equation:

$$C^{zz''}_{\alpha\gamma} C^{-1\gamma\beta}_{z'z''} = \delta^\beta_\alpha \delta^z_{z'} \tag{4.21}$$

Preserving second class constraints in time,

$$\{\psi^z_\alpha, H_e\}_C \approx 0 \tag{4.22}$$

leads us to solve the λs as follows:

$$\lambda^\alpha_z = -\{\psi^{z'}_\beta, H_0\}_C C^{-1\alpha\beta}_{zz'} \tag{4.23}$$

In order to set all the second class constraints (4.11) and (4.12) effectively equal to zero a semiclassical Dirac bracket is introduced:

$$\{M,N\}_{CD} \equiv \{M,N\}_C - \{M,\psi^z\}_C C^{-1}_{zz'}\{\psi^{z'},N\}_C \tag{4.24}$$

Coordinates (r_α, p_α) satisfy the semiclassical Dirac brackets;

$$\{r^\alpha, r^\beta\}_{CD} = C^{-1\alpha\beta}_{11}, \tag{4.25}$$

$$\{r^\alpha, p^\beta\}_{CD} = C^{-1\alpha\beta}_{12}, \tag{4.26}$$

$$\{p^\alpha, p^\beta\}_{CD} = C^{-1\alpha\beta}_{22}, \tag{4.27}$$

thus once after defining the semiclassical Dirac brackets (4.25), (4.26) and (4.27) and effectively eliminating the constraints (4.11) and (4.12), r_α and p_α should be considered as coordinates and the corresponding momenta, respectively.

Equation of motion for a given observable $O(r,p)$ is defined with the extended hamiltonian H_e:

$$\dot{O}(r,p) = \{O(r,p), H_e\}_C \tag{4.28}$$

4.3 Equations of Motion

The semiclassical dynamics of spinning particles those are defined with the first order lagrangian (4.9) in terms of gauge fields $\mathscr{A}_\alpha$, $\mathscr{B}_\alpha$ are to be considered. Using (4.28) and (4.23) the time derivatives of r^α and p^α is stated as:

$$\dot{r}^\alpha = \left(\frac{\partial H_0}{\partial r^\beta} - \frac{i\rho}{\hbar}[\mathscr{A}_\beta, H_0]\right) C_{11}^{-1\alpha\beta} + \left(\frac{\partial H_0}{\partial p^\beta} - \frac{i\xi}{\hbar}[\mathscr{B}_\beta, H_0]\right) C_{12}^{-1\alpha\beta} \tag{4.29}$$

$$\dot{p}^\alpha = \left(\frac{\partial H_0}{\partial r^\beta} - \frac{i\rho}{\hbar}[\mathscr{A}_\beta, H_0]\right) C_{21}^{-1\alpha\beta} + \left(\frac{\partial H_0}{\partial p^\beta} - \frac{i\xi}{\hbar}[\mathscr{B}_\beta, H_0]\right) C_{22}^{-1\alpha\beta} \tag{4.30}$$

The matrix $C^{zz'}_{\alpha\beta}$ is defined as,

$$C^{zz'}_{\alpha\beta} = \begin{pmatrix} \rho\mathscr{F}_{\alpha\beta} & -g_{\alpha\beta} + \mathscr{M}_{\alpha\beta} \\ g_{\alpha\beta} - \mathscr{M}_{\beta\alpha} & \xi\mathscr{G}_{\alpha\beta} \end{pmatrix} \tag{4.31}$$

Inverse of (4.31) can be calculated up to the first order in $\hbar$:

$$C^{-1}_{11\alpha\beta} = \xi\mathscr{G}_{\alpha\beta} - \xi(\mathscr{M}\mathscr{G})_{\alpha\beta} + \xi(\mathscr{M}\mathscr{G})_{\beta\alpha} - \rho\xi^2(\mathscr{G}\mathscr{F}\mathscr{G})_{\alpha\beta} + \ldots \tag{4.32}$$

$$C^{-1}_{12\alpha\beta} = g_{\alpha\beta} + \mathscr{M}_{\beta\alpha} - \rho\xi(\mathscr{G}\mathscr{F})_{\alpha\beta} - (\mathscr{M}\mathscr{M})_{\alpha\beta} + \ldots \tag{4.33}$$

$$C^{-1}_{21\alpha\beta} = -g_{\alpha\beta} - \mathscr{M}_{\alpha\beta} + \rho\xi(\mathscr{F}\mathscr{G})_{\alpha\beta} + (\mathscr{M}\mathscr{M})_{\alpha\beta} + \ldots \tag{4.34}$$

$$C^{-1}_{22\alpha\beta} = \rho\mathscr{F}_{\alpha\beta} - \rho(\mathscr{M}\mathscr{F})_{\alpha\beta} + \rho(\mathscr{M}\mathscr{F})_{\beta\alpha} - \rho^2\xi(\mathscr{F}\mathscr{G}\mathscr{F})^{\alpha\beta} + \ldots \tag{4.35}$$

5. QUANTUM MECHANICS IN NONCOMMUTATIVE COORDINATES

Deformation quantization, known as Weyl-Wigner-Groenewold-Moyal method of quantization is although developed as an alternative to the operator quantization became one of the main approaches of incorporating noncommutative coordinates into quantum mechanics. Having established a star product of coordinates in terms of deformation parameter θ,

$$\star \equiv \exp\frac{i\theta}{2}\left(\frac{\overleftarrow{\partial}}{\partial x}\frac{\overrightarrow{\partial}}{\partial y}-\frac{\overleftarrow{\partial}}{\partial y}\frac{\overrightarrow{\partial}}{\partial x}\right) \tag{5.1}$$

in a two dimensional xy plane one can employ the deformed hamiltonian in order to define the energy eigenvalue problem. The commutation relation of the coordinates via this star product is defined as,

$$[x,y]_\star = x\star y - y\star x = i\theta \tag{5.2}$$

which implies a Heisenberg relation

$$\Delta x.\Delta y \sim \theta \tag{5.3}$$

that is completely consistent with the standard rules of quantum mechanics. In quantum phase space given by $(\hat{x}_\mu, \hat{p}_\mu)$ this procedure is equivalent to shift the coordinates in the related hamiltonian as,

$$\hat{x}_\mu \to \hat{x}_\mu - \frac{1}{2\hbar}\theta\varepsilon_{\mu\nu}\hat{p}_\nu = \hat{x}' \tag{5.4}$$

New shifted coordinates satisfy the relation (5.2):

$$[\hat{x}',\hat{y}'] = i\theta \tag{5.5}$$

Noncommutativity parameter θ and the noncommutative algebra defined in (5.2) arises in the theory as a postulate. However when gauge fields are present in the theory this procedure depends explicitly on the chosen gauge. Moreover, it is not suitable to envisage Dirac particles in noncommutative coordinates. Hence, it is desirable to establish a systematic method of introducing noncommutative coordinates which does not depend on a particular gauge as well as embraces spin dependent systems. Dayı [5] presented a general formulation of spin dependent dynamics as a semiclassical hamiltonian system.

We will study Hall effect in noncommutative coordinates using the recipe that has mentioned in chapter four. Then, we will consider two simple models of spin Hall effect. First one is the extension of Drude model and the second one is the generalization of Hall effect.

We must mention that in the following sections we will study in flat Euclidean space so metric tensor $g_{\alpha\beta}$ turns into Kronecker delta δ_{ij} being i, j=1,2,3.

5.1 Hall Effect in Noncommutative Coordinates and θ Deformation

Electrons moving in a thin slab in the presence of an external magnetic field that is perpendicular to the plane will experience Lorentz force. Hence they will be pushed on a side of the slab producing a potential difference between the two sides. This is known the Hall effect. If one applies electric field that will balance the potential difference, electrons move without deflection. This approach gives a simple derivation of Hall conductivity [5].

We would derive Hall conductivity in noncommuting coordinates. To do this let us take the H_0 as ,

$$H_0 = \frac{p_1^2 + p_2^2}{2m} + V(\mathbf{r}) \tag{5.6}$$

with m is the electron mass and e is the charge of electron.

Scalar potential $V(\mathbf{r})$ in the hamiltonian is given in terms of electric field components E_i with i is 1,2 as:

$$V(\mathbf{r}) = -e.\mathbf{E}\cdot\mathbf{r} \tag{5.7}$$

Consider the coupling constant ξ is equal to θ and the gauge field $\mathscr{B}_i$ is linear in p_i such that it forms the curvature ε_{ij} which is antisymmetric in two dimensions.

$$\mathscr{G}_{ij} = \varepsilon_{ij} \tag{5.8}$$

In order to have such a curvature, one can have the gauge field $\mathscr{B}$ as;

$$\mathscr{B} = (-p_2, 0, 0) \tag{5.9}$$

or;

$$\mathscr{B}_i = -\varepsilon_{ij}\frac{p_j}{2} \tag{5.10}$$

although its specific form is not needed. Considering an electron that is moving on $r_1 r_2$ plane, we let the uniform magnetic field B is in the r_3 direction. We choose the gauge field $\mathscr{A}_i$ such that its field strength $\mathscr{F}_{ij}$ takes the form:

$$\mathscr{F}_{ij} = B\varepsilon_{ij} \tag{5.11}$$

The coupling constant $\rho = e/c$ in the first order lagrangian (4.9). Keeping the terms first order in θ and eB/c equations (4.25), (4.26) and (4.27) are rewritten as

$$\{r_i, r_j\}_{CD} = \theta\varepsilon_{ij}, \tag{5.12}$$

$$\{r_i, p_j\}_{CD} = (1 + \frac{e}{c}\theta B)\delta_{ij}, \tag{5.13}$$

$$\{p_i, p_j\}_{CD} = \frac{eB}{c}\varepsilon_{ij} \tag{5.14}$$

Wee see that the existence of the curvature $\mathscr{G}_{ij}$ together with the noncommutativity parameter θ causes the semiclassical Dirac brackets of coordinates (5.12) not to be vanished.
Also, it can be seen from (5.14) that in this formalism p_i acts as kinematic momenta. Although we have not deformed momentum space, (5.13) has a θ dependent term because p_i is kinematic momentum and have a r dependent part. Equations of motion from (4.29) and (4.30) with keeping only the first order terms of θ and eB/c takes the form:

$$\dot{r}_i = -e\theta\varepsilon_{ij}E_j + \left(1 + \frac{e}{c}\theta B\right)\frac{p_i}{m} \tag{5.15}$$

$$\dot{p}_i = eE_i\left(1 + \frac{e}{c}\theta B\right) + \frac{eB}{mc}\varepsilon_{ij}p_j \tag{5.16}$$

With the use of equations (5.15), and (5.16) one can get the force as,

$$F_i = m\ddot{r}_i = \left(1 + 2\frac{e}{c}\theta B\right)eE_i + \frac{eB}{mc}\varepsilon_{ij}(v_j + e\theta\varepsilon_{jk}E_k) \tag{5.17}$$

and in order to obtain the motion without deflection we solve the equation,

$$F_i = 0 \tag{5.18}$$

we get $v_i = \dot{r}_i$,

$$v_i = \frac{c}{B}\left(1 + \frac{eB\theta}{c}\right)\varepsilon_{ij}E_j \tag{5.19}$$

the velocity of an electron. In order to express the system of electrons we use the electric current density j:

$$\mathbf{j} = e\kappa\mathbf{v} \tag{5.20}$$

Here, κ is the density of electrons. The relation between the current density and the Hall conductivity is,

$$j_i = -\sigma_H(\theta)\varepsilon_{ij}E_j \tag{5.21}$$

which gives us the deformed Hall conductivity as:

$$\sigma_H(\theta) = -\frac{c\kappa e}{B}\left(1 + \frac{e}{c}\theta B\right) \tag{5.22}$$

Since the result found in (5.22) depends only on the field strength (5.11), it does not depend on the explicit choice of the gauge field $\mathscr{A}_i$.

On the other hand, within the semiclassical approach, one can deal with the hamiltonian,

$$H_0 = \frac{1}{2m}\left(p_i - (e/c)A_i\right)^2 + V(\mathbf{r}) \tag{5.23}$$

where the scalar potential is given by (5.7) and the noncommutativity of coordinates is still coming from the field strength $\mathscr{G}_{ij} = \varepsilon_{ij}$ and the related coupling constant θ. But this time we have no such a gauge field $\mathscr{A}_i$. In that way, p_i behaves as canonical momenta and the deformed brackets have the form,

$$\{r_i, r_j\}_{CD} = \theta\varepsilon_{ij} \tag{5.24}$$

$$\{r_i, p_j\}_{CD} = \delta_{ij} \tag{5.25}$$

$$\{p_i, p_j\}_{CD} = 0 \tag{5.26}$$

unlike (5.14), p_i in (5.26) acts as canonical momentum and that is the reason why we can not see any θ dependence in (5.25). If one chooses the symmetric gauge,

$$A_i = -\frac{B}{2}\varepsilon_{ij}r_j \tag{5.27}$$

he/she reaches the equations of motion,

$$\dot{r}_i = -e\theta\varepsilon_{ij}E_j + \left(1 + \frac{eB\theta}{2c}\right)\left(\frac{p_i}{m} - \frac{e}{mc}A_i\right) \tag{5.28}$$

$$\dot{p}_i = eE_i + \frac{eB}{2c}\varepsilon_{ij}\left(\frac{p_j}{m} - \frac{e}{mc}A_j\right) \tag{5.29}$$

Following the same procedure described before, deformed Hall conductivity can be found as,

$$\sigma_H(\theta) = -\left(1 - \frac{eB\theta}{4c}\right)\frac{ec\kappa}{B} \tag{5.30}$$

which is the result that was found in [11] up to a scaling factor $\hbar$. But this result is gauge dependent. Indeed, if one chooses the gauge,

$$A_i = (-Br_2, 0) \tag{5.31}$$

or,

$$A_i = (0, Br_1) \tag{5.32}$$

and follows the same procedure; although the coordinates are still non-commuting Hall conductivity appears as θ independent:

$$\sigma_H(\theta) = -\frac{ec\kappa}{B} \tag{5.33}$$

Fractional quantum Hall effect is an electron-electron interacting system and it can be obtained within our noninteracting, non-commuting theory.

$$\sigma_H^F = \nu\frac{e^2}{\hbar} \tag{5.34}$$

can be obtained as putting

$$\theta_F = -\frac{\nu}{\kappa h} - \frac{c}{eB} \tag{5.35}$$

in (5.22) where $\nu = 1/3, 2/3, 1/5, \ldots$.
Being an interacting and complicated theory fractional quantum Hall effect can be obtained from noninteracting Hall effect in noncommutative space which is a simpler effective theory just by tuning the parameter θ.

5.2 Spin Hall Effect in Noncommutative Space

Spin Hall effect basically occurs due to the spin currents those are produced by spin orbit coupling terms in the presence of electric field. In this work we deal with two semiclassical models which are suitable to investigate spin Hall effect in noncommuting coordinates.
We will deal with non-abelian gauge field $\mathscr{A}_i$ whose explicit form depends on the formalism that will be considered. We will set the curvature $\mathscr{G}_{ij} = \varepsilon_{ij}$ and the related coupling constant $\xi = \theta$.

5.2.1 Deformation of the Drude Type Formulation

This part is the semiclassical extension of the Drude model of spin Hall effect which is discussed by Chudnovsky [6]. Let $\rho = -\hbar$ and the gauge field $\mathscr{A}$ as,

$$\mathscr{A}_i = \frac{\varepsilon_{ijk}\sigma_j}{4mc^2}\frac{\partial V}{\partial r_k} \tag{5.36}$$

where $i = 1,2,3$ and σ_i are the Pauli spin matrices. Gauge field $\mathscr{A}$ yields the field strength;

$$\mathscr{F}_{ij} = \frac{\varepsilon_{jmn}\sigma_m}{4mc^2}\frac{\partial^2 V}{\partial r_i \partial r_n} - \frac{\varepsilon_{imn}\sigma_m}{4mc^2}\frac{\partial^2 V}{\partial r_j \partial r_n} - \frac{\varepsilon_{ijm}}{8m^2c^4}\sigma\cdot\nabla V\frac{\partial V}{\partial r_m} \tag{5.37}$$

We deal with the external potential being $V(\mathbf{r}) = -e\mathbf{E}\cdot\mathbf{r}$ and the noncommutative r_1r_2 plane. We set $\xi = \theta$ and $G_{ij} = \varepsilon_{ij}$. Ignoring the $\hbar^2$ order terms classical variables satisfy,

$$\{r_i, r_j\}_{CD} = \theta\varepsilon_{ij} \tag{5.38}$$

$$\{r_i, p_j\}_{CD} = \delta_{ij} + \hbar\theta\varepsilon_{ik}\mathscr{F}_{kj} \tag{5.39}$$

$$\{p_i, p_j\}_{CD} = -\hbar\mathscr{F}_{ij} \tag{5.40}$$

As before p_i is the kinematic momenta and this is the reason that why deformation parameter θ appears in (5.39).

Equations of motion have the form:

$$\dot{r}_i = \frac{p_i}{m} + \frac{\hbar\theta}{m}\varepsilon_{ij}\mathscr{F}_{jk}p_k + \theta\varepsilon_{ij}\frac{\partial V}{\partial r_j} \tag{5.41}$$

$$\dot{p}_i = -\frac{\partial V}{\partial r_i} - \hbar\theta\mathscr{F}_{ij}\varepsilon_{jk}\frac{\partial V}{\partial r_k} - \frac{\hbar}{m}\mathscr{F}_{ij}p_j \tag{5.42}$$

We add the term $-p_i/\tau$ drag force in order to be consistent with Drude model where τ is the relaxation time. Retaining the terms linear in the velocity v_i force becomes,

$$\begin{aligned} F_i = m\ddot{r}_i - \frac{p_i}{\tau} &= -\frac{\partial V}{\partial r_i} - \hbar\theta\mathscr{F}_{ij}\varepsilon_{jk}\frac{\partial V}{\partial r_k} - \hbar\mathscr{F}_{ij}v_j \\ &+\theta\varepsilon_{ij}v_k\frac{\partial^2 V}{\partial r_j \partial r_k} - \frac{m}{\tau}v_i + \frac{m\hbar\theta}{\tau}\varepsilon_{ij}\mathscr{F}_{jk}v_k + \frac{m\theta}{\tau}\varepsilon_{ij}\frac{\partial V}{\partial r_j} \end{aligned} \tag{5.43}$$

where we extract momenta from equation (5.41) as,

$$\frac{p_i}{m} = v_i - \theta\varepsilon_{ij}\frac{\partial V}{\partial r_j} - \hbar\theta\varepsilon_{ij}\mathscr{F}_{jk}v_k \tag{5.44}$$

For $\theta = 0$ the force (5.43) contracted to the force in [12] and also in [5].

$$V = V_c + V_e \tag{5.45}$$

in terms of the crystal potential V_c and the external potential V_e which is in harmony with [6].

We replace the terms in the (5.43) with their volume averages. Like Chudnovsky, we consider the cubic lattice in which the average of the second derivative of the potential is given in the terms of the lattice constant A,

$$\langle \frac{\partial^2 V}{\partial r_i \partial r_j} \rangle = -eA\delta_{ij} \tag{5.46}$$

Because of the external field $\mathbf{E}$,

$$\langle \frac{\partial V}{\partial r_i} \rangle = -eE_i \tag{5.47}$$

The average value of the field strength $\mathscr{F}_{ij}$ is,

$$\langle \mathscr{F}_{ij} \rangle = -\left(\frac{eA}{2mc^2}\sigma_3 + \frac{e^2}{8m^2c^4}\sigma \cdot \mathbf{E}E_3 \right) \varepsilon_{ij} \tag{5.48}$$

In order to have the constant velocity the total force acting on an electron should vanish,

$$F_i = 0 \tag{5.49}$$

Using averages and putting them with in (5.43) we get,

$$\begin{aligned} eE_i - \frac{mv_i}{\tau} + e\theta\varepsilon_{ij}\left(Av_j - \frac{mE_j}{\tau} \right) \\ + \left(\frac{e\hbar A}{2mc^2}\sigma_3 + \frac{e^2\hbar}{8m^2c^4}\sigma \cdot \mathbf{E}E_3 \right)\left(e\theta E_i + \varepsilon_{ij}v_j + \frac{m\theta v_i}{\tau} \right) = 0 \end{aligned} \tag{5.50}$$

solving velocities perturbatively,

$$v_i = v_i^0 + v_i^I \tag{5.51}$$

where

$$v_i^0 = \frac{\tau e}{m}E_i \tag{5.52}$$

and;

$$\begin{aligned} v_i^I = \frac{2e\tau\theta}{m}\left(\frac{e\hbar A}{2mc^2}\sigma_3 + \frac{e^2\hbar}{8m^2c^4}\sigma \cdot \mathbf{E}E_3 \right) E_i \\ - \left[e\theta\left(1 + \frac{\tau^2 eA}{m} \right) - \frac{\tau^2 e}{m}\left(\frac{e\hbar A}{2mc^2}\sigma_3 + \frac{e^2\hbar}{8m^2c^4}\sigma \cdot \mathbf{E}E_3 \right) \right] \varepsilon_{ij}E_j \end{aligned} \tag{5.53}$$

Introducing the density matrix as,

$$N = \frac{1}{2}n(1+\vec{\xi}.\vec{\sigma}) \tag{5.54}$$

$n = n^{\uparrow} + n^{\downarrow}$ is the total concentration of spins and $\vec{\xi}$ is the spin polarization vector with the norm,

$$\xi = \frac{n^{\uparrow} - n^{\downarrow}}{n^{\uparrow} + n^{\downarrow}} \tag{5.55}$$

where $n^{\uparrow}$ and $n^{\downarrow}$ are the concentrations of spins along the $\hat{\xi}$ and $-\hat{\xi}$. We choose the spin polarization to point in the third direction:

$$\xi = \xi \hat{r}_3 \tag{5.56}$$

Defining the spin current as

$$j_i^{\hat{\xi}} = eTr(Nv_i) = \sigma_C(\theta)E_i - \sigma_{SH}^D(\theta)\xi\varepsilon_{ij}E_j \tag{5.57}$$

where $n = n^{\uparrow} + n^{\downarrow}$ and using the velocities we solve θ deformed Hall conductivity as:

$$\sigma_C(\theta) = \left(1 + \frac{e\hbar A\theta}{mc^2} + \frac{e^2\hbar\theta E_3^2}{4m^2c^4}\right)\frac{ne^2\tau}{m} \tag{5.58}$$

and θ deformed spin hall conductivity is

$$\sigma_{SH}^D(\theta) = -\frac{n\hbar\tau^2e^3A}{2m^3c^2} - \frac{n\hbar\tau^2e^4E_3^2}{8m^4c^4} + \frac{ne^2\theta}{\xi}\left(1 + \frac{e\tau^2A}{m}\right) \tag{5.59}$$

When $\theta = 0$ and $1/c^4$ terms are ignored the conductivities are the same as [6].

5.2.2 Deformation of the Hall Effect Type Formulation

Another model for deriving spin Hall conductivity was developed in [5]. In this approach, consider the gauge field,

$$\mathscr{A}_i = \varepsilon_{ij}\sigma_j \tag{5.60}$$

consistent with Rashba spin-orbit coupling term. Related field strength $\mathscr{F}_{ij}$ is estimated as:

$$\mathscr{F}_{ij} = \frac{2\rho}{\hbar}\sigma_3\varepsilon_{ij} \tag{5.61}$$

We deal with the r_1r_2 plane with the noncommutativity parameter $\xi = \theta$ and

$$\mathscr{G}_{ij} = \varepsilon_{ij} \tag{5.62}$$

Hamiltonian of the system is the same with (5.6). Keeping only the first order terms in θ and ρ^2, variables satisfy;

$$\{r_i, r_j\}_{CD} = \theta \varepsilon_{ij} \tag{5.63}$$

$$\{r_i, p_j\}_{CD} = \left(1 + \frac{2\rho^2\theta}{\hbar}\sigma_3\right)\delta_{ij} \tag{5.64}$$

$$\{p_i, p_j\}_{CD} = \frac{2\rho^2}{\hbar}\sigma_3\varepsilon_{ij} \tag{5.65}$$

once more p_i is kinematic momenta. Equations of motion are,

$$\dot{r}_i = \left(1 + \frac{2\rho^2\theta}{\hbar}\sigma_3\right)\frac{p_i}{m} - e\theta\varepsilon_{ij}E_j \tag{5.66}$$

$$\dot{p}_i = \left(1 + \frac{2\rho^2\theta}{\hbar}\sigma_3\right)eE_i + \frac{2\rho^2}{\hbar m}\sigma_3\varepsilon_{ij}p_j \tag{5.67}$$

In the first order in θ with $v_i = \dot{r}_i$, momentum of the particle can be extracted from (5.66) is:

$$\frac{p_i}{m} = \left(1 - \frac{2\rho^2\theta}{\hbar}\sigma_3\right)v_i + e\theta\varepsilon_{ij}E_j \tag{5.68}$$

The force acting on the particle up to the order θ and ρ^2 is,

$$F_i = m\ddot{r}_i = eE_i + \frac{2\rho^2}{\hbar}\sigma_3\varepsilon_{ij}v_j + \frac{2\rho^2 e\theta}{\hbar}\sigma_3 E_i \tag{5.69}$$

In order to study the motion without deflection we set $F_i = 0$ and solve the velocities as:

$$v_i^{\uparrow} = \left(\frac{e\hbar}{2\rho^2} + e\theta\right)\varepsilon_{ij}E_j \tag{5.70}$$

$$v_i^{\downarrow} = -\left(\frac{e\hbar}{2\rho^2} - e\theta\right)\varepsilon_{ij}E_j \tag{5.71}$$

Arrows $\uparrow$ and $\downarrow$ corresponds to the positive and negative eigenvalues of σ_3. Spin current is defined as,

$$j_i^z = \frac{\hbar}{2}(n^{\uparrow}v_i^{\uparrow} - n^{\downarrow}v_i^{\downarrow}) \tag{5.72}$$

where $n^{\uparrow}$ and $n^{\downarrow}$ are the concentrations of spins along the $\hat{z}$ and $-\hat{z}$, respectively. Employing (5.70) and (5.71) into (5.72) provides us to write spin current as,

$$\mathbf{j}_i^z = -\sigma_{SH}(\theta)\hat{z} \times \mathbf{E} \tag{5.73}$$

where θ deformed spin Hall conductivity defined as,

$$\sigma_{SH}(\theta) = \frac{-e\hbar^2 n}{4\rho^2} - \frac{1}{2}e\hbar n\xi\theta \tag{5.74}$$

with $n = n^{\uparrow} + n^{\downarrow}$ is the concentration of states occupying the lower energy state of the Rashba hamiltonian times a constant l,

$$n = \frac{\rho^2 l}{\pi \hbar^2} \tag{5.75}$$

and coupling constant ρ and Rashba spin orbit coupling constant α is related to each other as $\rho = \frac{-\alpha m}{\hbar}$.

θ deformed spin Hall conductivity is obtained as:

$$\sigma_{SH}(\theta) = -\frac{el}{4\pi} - \frac{e\hbar\tilde{\theta}}{2} \tag{5.76}$$

where $\tilde{\theta} \equiv (n^{\uparrow} - n^{\downarrow})\theta$.

In the limit $\theta = 0$ (5.76) agrees with [13] for the value of $l = 1/2$.

6. RESULTS AND DISCUSSION

The difference in the ξ dependence in the two formalisms is due to the fact that in the former we use the density matrix to define the spin current but in the latter we avoided it. Both of the formalisms lead to a deformed Hall conductivity which can be stated as,

$$\sigma_{SH}(\theta) = \Sigma_0 + \theta \Sigma_1. \tag{6.1}$$

which yield spin Hall conductivity when the noncommutativity is switched off $\theta = 0$. We will focus on the spin Hall conductivity of the latter one (5.76). In the sprit of interpreting the noncommutativity as a link between similar physical phenomena θ can be fixed to obtain other formulations of spin Hall effect. We will illustrate this point of view considering spin Hall conductivities obtained by inclusion of impurities, the Rashba type spin orbit couplings with higher order momenta and the quantum spin Hall effect.

When impurity effects included into the Rashba hamiltonian which is linear in momenta, the universal behavior of spin Hall conductivity [13] is swept out [14], [15], [16]. This case is given with fixing the value of the deformation parameter as

$$\tilde{\theta}_0 = -\frac{l}{2\pi\hbar}. \tag{6.2}$$

However, dealing with Rashba type hamiltonian with higher order momenta

$$H = \varepsilon_k - \frac{1}{2} b_i(\mathbf{k})\sigma_i + V(\mathbf{r}) \tag{6.3}$$

one finds a non-vanishing spin Hall conductivity [17] where $\mathbf{k}$ is the kinematic momentum, ε_k is the energy dispersion in the absence of spin-orbit coupling and v is the velocity, $b_1 + ib_2 \equiv b_0(k)\exp(iN\theta)$ and

$$\tilde{N} = \frac{d\ln|b_0|}{d\ln k}, \; 1+\zeta = \frac{d\ln v}{d\ln k}. \tag{6.4}$$

Spin Hall conductivity results:

$$\sigma_{SH}^{HR} = -\frac{eN}{4\pi}\left(\frac{N^2-1}{N^2+1}\right)(\tilde{N} - \zeta - 2). \tag{6.5}$$

This can be achieved from (5.76) by setting $l = N$ and fixing the deformation parameter as

$$\tilde{\theta}_{HR} = \frac{N}{2\pi\hbar}\left[-1+\left(\frac{N^2-1}{N^2+1}\right)(\tilde{N}-\zeta-2)\right] \tag{6.6}$$

Quantization of spin Hall conductance in units of $\frac{e}{2\pi}$ was predicted by [18]. Hence, the quantized spin Hall conductivity can be written as

$$\sigma_{SH}^{Q} = -\frac{e}{2\pi}\mu \tag{6.7}$$

where μ is a number depending on the physical system considered. This can be obtained from (5.76) by fixing the deformation parameter as:

$$\tilde{\theta}_{Q} = \frac{1}{2\pi\hbar}(-l+2\mu). \tag{6.8}$$

Hence the spin Hall effect in noncommutative coordinates can be considered as the master formulation such that fixing the noncommutativity parameter θ yields different manifestations of the same physical phenomenon.

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